Quantum Information Theory

Jian Hou Lai

January 12, 2026

Contents

1 Mathematics Preliminaries

1.1 Group Theory

example: Pauli group

1.2 Tensor Product

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{1.1}$$

Idea 1.1. Tensor Product

Tensor product $V \otimes W$ of two vector spaces V and W is defined as a vector space to which is aossociated a bilinear map $V \times W \to V \otimes W$ that maps a pair ...

2 Introduction

Idea 2.1. Quantum Bit

A quantum bit (qubit) is a superposition (linear combination) of two basis states, $|0\rangle$ and $|1\rangle$:

$$|\Psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle \ , \tag{2.1}$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{2.2}$$

ψ_0 and ψ_1 are complex numbers called **probability amplitudes**.

Idea 2.2. Postulates in Quantum Mechanics

testing

Idea 2.3. Outer Product

Suppoese that $|\Psi\rangle$ and $|\Phi\rangle$ are two state vectors, then the **outer product** between $|\Psi\rangle$ and $|\Phi\rangle$ is defined as

$$|\Psi\rangle\langle\Phi| = |\Psi\rangle \otimes \langle\Phi| \ . \tag{2.3}$$

It bahaves like a matrix multiplication if $|\Psi\rangle$ is a $m \times 1$ column vector while $\langle\Phi|$ is a $1 \times n$ row vector, then the resulting outer product $|\Psi\rangle\langle\Phi|$ is a $m \times n$ matrix.

Idea 2.4. Hermitian Operator

An operator A is said to be **Hermitian** if it satisfies

$$A^\dagger = A \,, \tag{2.4}$$

where $\dagger$ means **Hermitian adjoint** or **conjugate transpose**.

Idea 2.5. Unitary Transformation

An unitary transformation (operator) U satisfies

$$UU^\dagger = \mathbb{1} \implies U^{-1} = U^\dagger \,. \tag{2.5}$$

In contrast with the normal quantum mechanics which the states was described by complex state vector, in quantum information, we use **density matrix** to describe our states. Density matrices formalism

Idea 2.6. Density Operator/ Density Matrix

Sort of repeated with the "pure states" idea Suppose that X is a system and Σ is its classical state set. **Density matrix**, ρ describing state X is a matrix[a] with complex number entries whose rows and columns have been placed in correspondence with Σ.
Properties of density matrices:

1. $\mathrm{Tr}(\rho) = 1$.
2. ρ is positive semidefinite: $\rho \geq 0$[b].
3. Diagonal entries are the probabilities for each classical state to appear from a standard basis measurement.

4. Off-diagonal entries describes how the two corresponding classical states are in quantum superposition. (coherences/ how pure or how mixed of states)

Here are some motivation with this formalism for QI:

- Able to represent broader class of quantum states, like quantum states with uncertainty or randomness.
- Can describe states of isolated parts of the systems, like one system that's in entangled with another system.
- Able to describe classical and quantum information together within single mathematical framework.

Idea 2.7. Pure States

Suppose we have a single, isolated quantum state vector $|\Psi\rangle$, which is a column vector having Euclidean norm 1. With Spectral Theorem (Theorem 2.1), the state vector can be written as the linear combination of the basis:

$$|\Psi\rangle = \sum_{i=1}^{n} \psi_i |i\rangle = \sum_{i=0}^{n-1} \psi_i |i\rangle \ . \tag{2.6}$$

The form in the second equality is normally adapted in QIT so it can start with computational basis state, $|0\rangle$.
Then, the density matrix representation of the same state is called **pure state** (Notice that pure state is a projector on the state $|\Psi\rangle$):

$$\rho = |\Psi\rangle\langle\Psi| \ . \tag{2.7}$$

ρ has these properties:

1. ρ is Hermitian ($\rho^\dagger = \rho$).
2. $\mathrm{Tr}(\rho) = \mathrm{Tr}\big(\rho^2\big) = 1$.

Proof : Properties in Idea 2.7

1. To start with, we take Hermitian on Equation (2.7):

$$\begin{aligned}
\rho^\dagger &= (|\Psi\rangle\langle\Psi|)^\dagger \\
&= \langle\Psi|^\dagger \, |\Psi\rangle^\dagger \\
&= \left(\sum_{i=1}^{n} \psi_i^* \langle i|\right)^\dagger \left(\sum_{j=1}^{n} \psi_j |j\rangle\right)^\dagger \\
&= \left(\sum_{i=1}^{n} \psi_i |i\rangle\right) \left(\sum_{j=1}^{n} \psi_j^* \langle j|\right) \\
&= |\Psi\rangle\langle\Psi| = \rho \, .
\end{aligned}$$

Because of $\rho^\dagger = \rho$, we conclude that ρ is Hermitian. Intuitively, we can think of the ket is a column matrix, and bra is a row matrix, hence $|\Psi\rangle^\dagger = \langle\Psi|$. □

[a] A linear mapping.
[b] If you diagonalise this matrix, the eigenvalues will non-negative. It implies ρ is Hermitian ?

2. Equation (2.7) can be written as

$$\begin{aligned}\rho = |\Psi\rangle\langle\Psi| &= \sum_{i,j=1}^{n} \psi_i \psi_j^* \, |i\rangle\langle j| \\ &= \begin{pmatrix} |\psi_1|^2 & \psi_1\psi_2^* & \dots & \psi_1\psi_n^* \\ \psi_2\psi_1^* & |\psi_2|^2 & \dots & \psi_2\psi_n^* \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n\psi_1^* & \psi_n\psi_2^* & \dots & |\psi_n|^2 \end{pmatrix}\end{aligned}$$

Hence,

$$\mathrm{Tr}(\rho) = \sum_{i=1}^{n} |\psi_i|^2 = 1\,, \tag{2.8}$$

which is equal to total probability, by definition. Next,

$$\mathrm{Tr}\big(\rho^2\big) = \mathrm{Tr}(|\Psi\rangle\,\langle\Psi|\Psi\rangle\,\langle\Psi|) = \mathrm{Tr}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}(\rho) = 1\,, \tag{2.9}$$

where we used the inner product of itself is equal to 1 because the state is normalised. □

Idea 2.8. Multiqubit States

For n qubits, we can represent the state as

$$|\Psi\rangle = \sum_{i=0}^{N} \psi_i \, |i\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}\,, \tag{2.10}$$

where $N = 2^n - 1$. Then, the density matrix of n qubits is

$$\begin{aligned}\rho = |\Psi\rangle\langle\Psi| &= \sum_{i,j=0}^{n} \psi_i \psi_j^* \, |i\rangle\langle j| \\ &= \begin{pmatrix} |\psi_0|^2 & \psi_0\psi_1^* & \dots & \psi_0\psi_N^* \\ \psi_1\psi_0^* & |\psi_1|^2 & \dots & \psi_1\psi_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N\psi_1^* & \psi_N\psi_2^* & \dots & |\psi_N|^2 \end{pmatrix}\,.\end{aligned}$$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.11}$$

???

Idea 2.9. Equation of Motion

ρ satisfies equation of motion, or known as Liouville-von Neumann equation:

no global phase degeneracy

Idea 2.10. Mixed States

Mixed states is an ensemble of pure states which associated with classical probability of occurrence. Let $\{\rho_i\}_{i=1}^n = \{\rho_1, \rho_2, \ldots, \rho_n\}$ be the set of density matrices of the states, and $\{p_i\}_{i=1}^n = \{p_1, p_2, \ldots, p_n)$ be the set of probabilities. Suppose we prepare a system in state ρ_k with probability p_k, then the mixed state is represented by this density matrix,

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i \left|\Psi_i\right\rangle\!\left\langle\Psi_i\right| . \tag{2.12}$$

It's also known as the convex combination of density matrices. The density matrix for mixed state has these properties:

1. $\mathrm{Tr}(\rho) = 1$.
2. $\mathrm{Tr}\left(\rho^2\right) < 1$.

Density matrices don't describe how the system was prepared, but it tells you how system changed after some action performed.

convex combination in maths

Proof : Properties of Idea 2.10

Let ρ be the density matrix of the mixed state:

$$\rho = \sum_{i=1}^{n} p_i \rho_i , \tag{2.13}$$

where

$$\rho_i = \sum_{j=1}^{m} \sum_{k=1}^{m} \psi_j^{(i)} \psi_k^{(i)*} \left|j\right\rangle\!\left\langle k\right| . \tag{2.14}$$

1. Then,

$$\mathrm{Tr}(\rho) = \tag{2.15}$$

Idea 2.11. Expectation Values

$$\begin{aligned}
\langle A \rangle &= \langle \Psi | \hat{A} | \Psi \rangle \\
&= \sum_{i=1}^{n} \langle \Psi | \hat{A} | i \rangle \langle i | \Psi \rangle \\
&= \sum_{i=1}^{n} \langle i | \Psi \rangle \langle \Psi | \hat{A} | i \rangle \\
&= \sum_{i=1}^{n} \langle i | \rho \hat{A} | i \rangle \\
&= \mathrm{Tr}\left(\rho \hat{A}\right) .
\end{aligned}$$

So, we can interpret it as finding the trace of the matrix multiplication between density matrix and matrix (operator) A.

Idea 2.12. Probabilistic States

Idea 2.13. Probabilistic Measurement

normal matrices

Theorem 2.1. Spectral Theorem

Theorem 2.2. Spectral Theorem for Positive Semidefinite Matrices

Idea 2.14. Bloch Sphere/ Poincaré Sphere

Idea 2.15. Bloch Ball

gemetrical representation beyond single qubit

Idea 2.16. Pauli Matrices

Pauli matrices is defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.16}$$

More compactly, we can write Pauli matrices as σ_j, where $j = 1, 2, 3$, corresponds to x, y, z respectively. Pauli matrices have some properties:

1. The anti-commutator of Pauli matrices satisfies $\{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbb{1}$.
2. The commutator (commutation relation) of Pauli matrices satisfies $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$.
3. Pauli matrices are traceless matrices. i.e., $\mathrm{Tr}(\sigma_j) = 0$.
4. Pauli matrices are self-inverse matrices. i.e., $\sigma_j^{-1} = \sigma_j$.

Proof : Properties of Idea 2.16

Notice that

$$\sigma_x\sigma_y = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \tag{2.17}$$

$$\sigma_y\sigma_x = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}, \tag{2.18}$$

$$\sigma_y\sigma_z = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \tag{2.19}$$

$$\sigma_z\sigma_y = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}, \tag{2.20}$$

$$\sigma_x\sigma_z = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{2.21}$$

$$\sigma_z\sigma_x = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{2.22}$$

$$\sigma_x\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}, \tag{2.23}$$

$$\sigma_y\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}, \tag{2.24}$$

$$\sigma_z\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}. \tag{2.25}$$

1. Hence, from equations above, we will get

$$\sigma_x\sigma_y + \sigma_y\sigma_x = 0 \Rightarrow \{\sigma_x, \sigma_y\} = 0, \tag{2.26}$$
$$\sigma_x\sigma_z + \sigma_z\sigma_x = 0 \Rightarrow \{\sigma_x, \sigma_z\} = 0, \tag{2.27}$$
$$\sigma_y\sigma_z + \sigma_z\sigma_y = 0 \Rightarrow \{\sigma_y, \sigma_z\} = 0, \tag{2.28}$$

and

$$\sigma_x\sigma_x + \sigma_x\sigma_x = 2\mathbb{1}, \tag{2.29}$$
$$\sigma_y\sigma_y + \sigma_y\sigma_y = 2\mathbb{1}, \tag{2.30}$$
$$\sigma_z\sigma_z + \sigma_z\sigma_z = 2\mathbb{1}. \tag{2.31}$$

Thus, we can summarise it as

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbb{1}. \quad \square \tag{2.32}$$

2. From Equations (2.17) to (2.25), we will get

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 2i\sigma_z, \tag{2.33}$$

$$[\sigma_y, \sigma_z] = \sigma_y\sigma_z - \sigma_z\sigma_y = 2i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 2i\sigma_x, \tag{2.34}$$

$$[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = 2i\sigma_y, \tag{2.35}$$

Hence,

$$[\sigma_y, \sigma_x] = -[\sigma_x, \sigma_y] = -2i\sigma_z, \tag{2.36}$$
$$[\sigma_z, \sigma_y] = -[\sigma_y, \sigma_z] = -2i\sigma_x, \tag{2.37}$$
$$[\sigma_x, \sigma_z] = -[\sigma_z, \sigma_x] = -2i\sigma_y. \tag{2.38}$$

With the equations above, and the fact that Pauli matrices (operators in quantum mechanics) are always commute with itself, we conclude that

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \quad \square \tag{2.39}$$

3. Obvious. $\square$

4. Obvious from Equations (2.23) to (2.25). $\square$

Notice that Pauli matrices satisfies the equation

$$\sigma_j \sigma_k = \delta_{jk} \mathbb{1} + i\epsilon_{jkl}\sigma_l \,. \tag{2.40}$$

Idea 2.17. Reduced States and Partial Trace for Bipartite Systems

Suppose systems A and B are in a joint pure state $|\Psi\rangle$, so the composite density matrix is $\rho_{AB} = |\Psi\rangle\langle\Psi|$. Then, the reduced state for A is obtained by tracing over B and is given by

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_b (\mathbb{1}_A \otimes \langle b|)\rho_{AB}(\mathbb{1}_A \otimes |b\rangle)\,, \tag{2.41}$$

where $\{|b\rangle\}$ is an orthonormal basis for system B. Similarly, the reduced state for B is:

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_a (\langle a| \otimes \mathbb{1}_B)\rho_{AB}(|a\rangle \otimes \mathbb{1}_B)\,, \tag{2.42}$$

where $\{|a\rangle\}$ is an orthonormal basis for system A. Note that, for example, $\mathrm{Tr}_A(\rho)$ means taking partial trace of ρ. prove this

For example, suppose we have a joint two-qubit pure state, generally (not necessary to be entangled) described as $|\Psi\rangle$, then

$$|\Psi\rangle = \sum_{i=0}^{1}\sum_{j=0}^{1} \psi_{ij} |ij\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle \,. \tag{2.43}$$

Note that: Two qubits is said to be entangled to each other iff the state can be expressed as $|\Psi\rangle = |\Phi_A\rangle \otimes |\Phi_B\rangle$. i.e., ψ_{ij} can be written as $\psi_{ij} = \phi_i\phi_j$, so

$$|\Psi\rangle = \sum_{ij} \psi_{ij}|ij\rangle = \left(\sum_i \phi_i |i\rangle\right) \otimes \left(\sum_j \phi_j |j\rangle\right) = |\Phi_A\rangle \otimes |\Phi_B\rangle \,. \tag{2.44}$$

Then, we can prepare the density matrix for this joint state as

$$\begin{aligned}
\rho_{AB} &= |\Psi\rangle\langle\Psi| \\
&= \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} \begin{pmatrix} \psi_{00}^* & \psi_{01}^* & \psi_{10}^* & \psi_{11}^* \end{pmatrix} \\
\rho_{AB} &= \begin{pmatrix} |\psi_{00}|^2 & \psi_{00}\psi_{01}^* & \psi_{00}\psi_{10}^* & \psi_{00}\psi_{11}^* \\ \psi_{01}\psi_{00}^* & |\psi_{01}|^2 & \psi_{01}\psi_{10}^* & \psi_{01}\psi_{11}^* \\ \psi_{10}\psi_{00}^* & \psi_{10}\psi_{01}^* & |\psi_{10}|^2 & \psi_{10}\psi_{11}^* \\ \psi_{11}\psi_{00}^* & \psi_{11}\psi_{01}^* & \psi_{11}\psi_{10}^* & |\psi_{11}|^2 \end{pmatrix} \,.
\end{aligned} \tag{2.45}$$

Or, generally,

$$\rho_{AB} = \sum_{ij} \psi_{ij}|ij\rangle \left(\sum_{jk} \psi_{kl}^* \langle kl|\right) = \sum_{ijkl} \psi_{ij}\psi_{kl}^* |ij\rangle\langle kl| = \sum_{ijkl} \psi_{ij}\psi_{kl}^* \left(|i\rangle\langle k| \otimes |j\rangle\langle l|\right) \tag{2.46}$$

Then, using Equation (2.41) and Equation (2.46), we can get the reduced state for system A as

$$
\begin{aligned}
\rho_A = \mathrm{Tr}_B(\rho_{AB)} &= \sum_b (\mathbb{1}_A \otimes \langle b|)\rho_{AB}(\mathbb{1}_A \otimes |b\rangle) \\
&= \sum_b (\mathbb{1}_A \otimes \langle b|)\left(\sum_{ijkl} \psi_{ij}\psi^*_{kl}\,|i\rangle\!\langle k| \otimes |j\rangle\!\langle l|\right)(\mathbb{1}_A \otimes |b\rangle) \\
&= \sum_{bijkl} \psi_{ij}\psi^*_{kl}(|i\rangle\!\langle k|)(\langle b|j\rangle\,\langle l|b\rangle) \\
&= \sum_{bijkl} \psi_{ij}\psi^*_{kl}(|i\rangle\!\langle k|)\delta_{bj}\delta_{lb} \\
&= \sum_{bik} \psi_{ib}\psi^*_{kb}\,|i\rangle\!\langle k| \\
&= \sum_{ik} (\psi_{i0}\psi^*_{k0}\,|i\rangle\!\langle k| + \psi_{i1}\psi^*_{k1}\,|i\rangle\!\langle k|) \\
&= (|\psi_{00}|^2|\psi_{01}|^2)\,|0\rangle\!\langle 0| + (\psi_{00}\psi^*_{10} + \psi_{01}\psi_1 1^*)\,|0\rangle\!\langle 1| + (\psi_{10}\psi^*_{00} + \psi_{11}\psi^*_{01})\,|1\rangle\!\langle 0| + (|\psi_{10}|^2 + |\psi_{11}|^2)\,|1\rangle\!\langle 1| \\
&= \begin{pmatrix} |\psi_{00}|^2 + |\psi_{01}|^2 & \psi_{00}\psi^*_{10} + \psi_{01}\psi^*_{11} \\ \psi_{10}\psi^*_{00} + \psi_{11}\psi^*_{01} & |\psi_{10}|^2 + |\psi_{11}|^2 \end{pmatrix}
\end{aligned}
$$

Compare to Equation (2.45),

$$
\rho_{AB} = \begin{pmatrix} |\psi_{00}|^2 & \psi_{00}\psi^*_{01} & \psi_{00}\psi^*_{10} & \psi_{00}\psi^*_{11} \\ \psi_{01}\psi^*_{00} & |\psi_{01}|^2 & \psi_{01}\psi^*_{10} & \psi_{01}\psi^*_{11} \\ \psi_{10}\psi^*_{00} & \psi_{10}\psi^*_{01} & |\psi_{10}|^2 & \psi_{10}\psi^*_{11} \\ \psi_{11}\psi^*_{00} & \psi_{11}\psi^*_{01} & \psi_{11}\psi^*_{10} & |\psi_{11}|^2 \end{pmatrix}, \tag{2.47}
$$

the partial trace over B simply can be done by

$$
\mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \mathrm{Tr}\begin{pmatrix} |\psi_{00}|^2 & \psi_{00}\psi^*_{01} \\ \psi_{01}\psi^*_{00} & |\psi_{01}|^2 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} \psi_{00}\psi^*_{10} & \psi_{00}\psi^*_{11} \\ \psi_{01}\psi^*_{10} & \psi_{01}\psi^*_{11} \end{pmatrix} \\ \mathrm{Tr}\begin{pmatrix} \psi_{10}\psi^*_{00} & \psi_{10}\psi^*_{01} \\ \psi_{11}\psi^*_{00} & \psi_{11}\psi^*_{01} \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} |\psi_{10}|^2 & \psi_{10}\psi^*_{11} \\ \psi_{11}\psi^*_{10} & |\psi_{11}|^2 \end{pmatrix} \end{pmatrix}. \tag{2.48}
$$

Similarly using Equation (2.42), we can get the reduced state for system B as

$$
\begin{aligned}
\rho_B = \mathrm{Tr}_A(\rho_{AB}) &= \sum_a \big(\langle a| \otimes \mathbb{1}_B\big)\rho_{AB}\big(|a\rangle \otimes \mathbb{1}_B\big) \\
&= \sum_{aijkl} \psi_{ij}\psi^*_{kl}\big(\langle a| \otimes \mathbb{1}_B\big)\big(|i\rangle\,\langle k| \otimes |j\rangle\!\langle l|\big)\big(|a\rangle \otimes \mathbb{1}_B\big) \\
&= \sum_{aijkl} \psi_{ij}\psi^*_{kl}\;\langle a|i\rangle\,\langle k|a\rangle\;|j\rangle\!\langle l| \\
&= \sum_{aijkl} \psi_{ij}\psi^*_{kl}\,\delta_{ai}\delta_{ka}\;|j\rangle\!\langle l| \\
&= \sum_{ajl} \psi_{aj}\psi^*_{al}\,|j\rangle\!\langle l| \\
&= \sum_{jl} \psi_{0j}\psi^*_{0l}\,|j\rangle\!\langle l| + \sum_{jl} \psi_{1j}\psi^*_{1l}\,|j\rangle\!\langle l| \\
&= \begin{pmatrix} |\psi_{00}|^2 + |\psi_{10}|^2 & \psi_{00}\psi^*_{01} + \psi_{10}\psi^*_{11} \\ \psi_{01}\psi^*_{00} + \psi_{11}\psi^*_{10} & |\psi_{01}|^2 + |\psi_{11}|^2 \end{pmatrix}.
\end{aligned}
$$

Again, compare to Equation (2.45),

$$\rho_{AB} = \begin{pmatrix} |\psi_{00}|^2 & \psi_{00}\psi_{01}^* & \psi_{00}\psi_{10}^* & \psi_{00}\psi_{11}^* \\ \psi_{01}\psi_{00}^* & |\psi_{01}|^2 & \psi_{01}\psi_{10}^* & \psi_{01}\psi_{11}^* \\ \psi_{10}\psi_{00}^* & \psi_{10}\psi_{01}^* & |\psi_{10}|^2 & \psi_{10}\psi_{11}^* \\ \psi_{11}\psi_{00}^* & \psi_{11}\psi_{01}^* & \psi_{11}\psi_{10}^* & |\psi_{11}|^2 \end{pmatrix} \tag{2.49}$$

the partial trace over A simply can be done by

$$\mathrm{Tr}_A(\rho_{AB}) = \begin{pmatrix} |\psi_{00}|^2 & \psi_{00}\psi_{01}^* \\ \psi_{01}\psi_{00}^* & |\psi_{01}|^2 \end{pmatrix} + \begin{pmatrix} |\psi_{10}|^2 & \psi_{10}\psi_{11}^* \\ \psi_{11}\psi_{10}^* & |\psi_{11}|^2 \end{pmatrix} \tag{2.50}$$

.

Idea 2.18

Pending
Equivalent statement: Tr_A and Tr_B are the unique linear mappings for which these equations are always true:

$$\mathrm{Tr}_A(\mathcal{M} \otimes \mathcal{N}) = \mathrm{Tr}(\mathcal{M})\mathcal{N} \tag{2.51}$$
$$\mathrm{Tr}_B(\mathcal{M} \otimes \mathcal{N}) = \mathrm{Tr}(\mathcal{N})\mathcal{M} \tag{2.52}$$

- Pauli errors
- Cluster states
- Bures length
- Weighted distances in quantum information (e.g. Weighted Bures length)
- Weighted Hilbert-Schmidt distance
- Pauli rotation

Idea 2.19. Evolution of State Through Unitary Operators

For example, Hadamard state:

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{2.53}$$

continue

Idea 2.20. Stabilizer Formalism

Pending

Idea 2.21. Bell States

Why Bell states defined in this way Bell states are defined as

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.54}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.55}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.56}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\,. \tag{2.57}$$