Black-Scholes Option Pricing Model

1,It^o's Lemma

Suppose:

S :Price of a particular asset.
t :Time.

$\frac{dS}{S}$:Return on the asset.

μ :Drift,the average rate of growth of the asset price, A predictable,deterministic component.
σ :*volatility,* the standard deviation of the returns.
dX :A random variable having a normal distribution with mean 0 and variance dt .

$$dX \sim N\left(0,\left(\sqrt{dt}\right)^2\right) = \varepsilon\sqrt{dt}$$

the stochastic differential equation:

$$\frac{dS}{S} = \mu dt + \sigma dX$$

$f\left(S,t\right)$:Fuction by Price of a particular asset and Time.

Proof:

With:

Taylor's Theorem

$$\Delta f = \frac{\partial f}{\partial S}\Delta S + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\left(\frac{\partial^2 S}{\partial S^2}(\Delta S)^2 + 2\frac{\partial^2 S}{\partial S \partial t}(\Delta S \Delta t) + \frac{\partial^2 S}{\partial t^2}(\Delta t)^2\right) + \ldots$$

With:

$$dS = S(\mu dt + \sigma dX)$$

$$\begin{aligned}\Delta f &= \frac{\partial f}{\partial S} S\left(\mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}\right) + \frac{\partial f}{\partial t}\Delta t \\ &+ \frac{1}{2}\left(\begin{array}{l}\frac{\partial^2 S}{\partial S^2} S^2\left(\mu^2(\Delta t)^2 + 2\mu\sigma\Delta t\varepsilon\sqrt{\Delta t} + \sigma^2\left(\varepsilon\sqrt{\Delta t}\right)^2\right) \\ +2\frac{\partial^2 S}{\partial S \partial t} S\left(\mu\Delta t\Delta t + \sigma\varepsilon\sqrt{\Delta t}\Delta t\right) + \frac{\partial^2 S}{\partial t^2}(\Delta t)^2\end{array}\right) + \ldots\end{aligned}$$

With:

$$\Delta t \to dt$$

$$(\Delta X)^2 \to dt$$

$$\begin{aligned}df &= \frac{\partial f}{\partial S} S\left(\mu dt + \sigma\varepsilon\sqrt{dt}\right) + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 S}{\partial S^2}\left(S^2\sigma^2\varepsilon^2 dt\right) \\ &= \left(\frac{\partial f}{\partial S} S\mu + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 S}{\partial S^2} S^2\sigma^2\right)dt + \frac{\partial f}{\partial S} S\sigma\varepsilon\sqrt{dt}\end{aligned}$$

Then:

$$df = \left(\frac{\partial f}{\partial S} S\mu + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 S}{\partial S^2} S^2\sigma^2\right)dt + \frac{\partial f}{\partial S} S\sigma dX$$

2,The Black-Scholes PDE

Suppose:

$V(S,t)$:the value of an option, $C(S,t)$ for a call and $P(S,t)$ for a put.

r:the interest rate.

$\prod$: Portfolio, containing one option and $-\Delta$ units of the underlying stock.

$$\prod = V - \Delta S$$

Proof:

With:

It^o's Lemma

$$\begin{aligned} d\prod &= dV - \Delta dS \\ &= \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt - \Delta S \mu dt - \Delta S \sigma dX \\ &= \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt \end{aligned}$$

With:

$$\Delta = \frac{\partial V}{\partial S}$$

$$d\prod = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt$$

With:

Invested $\prod$ in riskless assets with interest rate r.

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt = r \prod dt$$

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = r \left(V - \frac{\partial V}{\partial S} S \right)$$

Then:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

3,Solve The Black-Scholes PDE

Suppose:

$$x = \ln\left(\frac{S}{K}\right)$$

$$\tau = \frac{(T-t)\sigma^2}{2}$$

$$v(x,\tau) = \frac{V(S,t)}{K}$$

Solve:

With:

Black-scholes PDE;

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\
&= \frac{\partial (Kv)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 (Kv)}{\partial S^2} + rS\frac{\partial (Kv)}{\partial S} - r(Kv) \\
&= K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial S}\right) + rS\left(K\frac{\partial v}{\partial S}\right) - r(Kv) \\
&= -K\frac{\partial v}{\partial \tau}\frac{\sigma^2}{2} + \frac{1}{2}\sigma^2 S^2 \frac{\partial}{\partial S}\left(K\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}\right) + rS\left(K\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}\right) - Kvr \\
&= \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v - \frac{\partial v}{\partial \tau}
\end{aligned}$$

$$\frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v = \frac{\partial v}{\partial \tau}$$

With:

$$a = \left(\frac{2r}{\sigma^2} - 1\right);$$

$$b = -\frac{2r}{\sigma^2} = -1 - a\ ;$$

$$\frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv = \frac{\partial v}{\partial \tau}$$

With:

$$v(x,\tau) = f(\tau)\, g(x)\, h(x,\tau);$$

$$0 = \frac{\partial^2 v}{\partial x^2} + a\frac{\partial v}{\partial x} + bv - \frac{\partial v}{\partial \tau}$$
$$= f\frac{\partial^2 g}{\partial x^2}h + 2f\frac{\partial g}{\partial x}\frac{\partial h}{\partial x} + fg\frac{\partial^2 h}{\partial x^2} + a\left(f\frac{\partial g}{\partial x}h + fg\frac{\partial h}{\partial x}\right) + bv - \frac{\partial f}{\partial \tau}gh - fg\frac{\partial h}{\partial \tau}$$

$$\left(f\frac{\partial^2 g}{\partial x^2} + af\frac{\partial g}{\partial x} + bfg - \frac{\partial f}{\partial \tau}g\right)h + \left(2f\frac{\partial g}{\partial x} + afg\right)\frac{\partial h}{\partial x} + fg\frac{\partial^2 h}{\partial x^2} = fg\frac{\partial h}{\partial \tau}$$

With:

$$g(x) = e^{-\frac{a}{2}x}$$

$$f(\tau) = e^{\left(b-\frac{a^2}{4}\right)\tau}$$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial \tau}$$

With:

Fourier transform

Gaussian integral

$$\tilde{h} = \tilde{h}_1\tilde{h}_2$$

$$\frac{\partial \tilde{h}}{\partial \tau} = F\left(\frac{\partial h}{\partial \tau}\right) = F\left(\frac{\partial^2 h}{\partial x^2}\right) = -k^2 F(h) = -k^2\tilde{h}$$

$$\tilde{h} = \tilde{h}_1\tilde{h}_2 = Ae^{-k^2\tau} = \tilde{h}(k,0)$$

$$h = h_1 * h_2 = F^{-1}\left(\tilde{h}_1\right) * F^{-1}\left(\tilde{h}_2\right) = F^{-1}(A) * F^{-1}\left(e^{-k^2\tau}\right)$$

$$h = F^{-1}(A) * \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(k^2\tau - i\lambda x\right)} d\lambda$$

$$h = F^{-1}(A) * \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(k^2\tau - i\lambda x\right)} d\lambda$$

$$= F^{-1}(A) * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4\tau}} \int_{\mathbb{R}} dk \exp\left[-\tau\left(k - i\frac{x}{2\tau}\right)^2\right]$$

$$= F^{-1}(A) * \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4\tau}} \frac{1}{\sqrt{\tau}} \int_{\mathbb{R}} d\xi \, \mathrm{e}^{-\xi^2}$$

$$= F^{-1}(A) * \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{4\tau}} \sqrt{\pi}$$

$$= F^{-1}(A) * \frac{1}{\sqrt{2\tau}} e^{-\frac{x^2}{4\tau}}$$

$$F^{-1}(A) * \frac{1}{\sqrt{2\tau}} e^{-\frac{y^2}{4\tau}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy h_1(x-y) h_2(y)$$

$$= \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} F^{-1}(A)(x-y) e^{-\frac{y^2}{4\tau}} dy$$

$$= \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} h_1(x-y) e^{-\frac{y^2}{4\tau}} dy$$

$$= \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} h(\xi, 0) e^{-\frac{(x-\xi)^2}{4\tau}} d\xi$$

Then:

$$V(S,t) = Ke^{\left(b - \frac{a^2}{4}\right)\tau} e^{-\frac{a}{2}x} \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} h(\xi, 0) e^{-\frac{(x-\xi)^2}{4\tau}} d\xi$$

4,Application To European Options

Suppose:

$V(S,T) = max[\varepsilon(S-K),0]$,European options

$\varepsilon = 1$ for a call option
$\varepsilon = -1$ for a put option

Proof:

$$h(x,0) = \frac{1}{K} e^{\left(\frac{a}{2}\right)x} V\left(Ke^x, T\right)$$
$$= \frac{1}{K} e^{\left(\frac{a}{2}\right)x} \max\left[\varepsilon\left(Ke^x - K\right), 0\right]$$
$$= \max\left\{\varepsilon\left[e^{\left(\frac{a}{2}+1\right)x} - e^{\left(\frac{a}{2}\right)x}\right], 0\right\}$$

$$h(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} d\xi \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] \max\left\{\varepsilon\left[e^{\left(\frac{a}{2}+1\right)\xi} - e^{\left(\frac{a}{2}\right)\xi}\right], 0\right\}$$

$$h(x,\tau) = \varepsilon \frac{1}{\sqrt{4\pi\tau}} \int_0^{\infty} d\eta \exp\left[-\frac{(x-\varepsilon\eta)^2}{4\tau}\right]\left[e^{\varepsilon\left(\frac{a}{2}+1\right)\eta} - e^{\varepsilon\left(\frac{a}{2}\right)\eta}\right]$$

With:

$$c_1 = \frac{1}{4\tau};$$
$$c_2 = x + 2\tau a;$$
$$c_3 = a\left(x + a\tau\right);$$

$$I_a = \frac{1}{\sqrt{4\pi\tau}} \int_0^{\infty} d\xi \exp\left[-\frac{(x-\varepsilon\xi)^2}{4\tau} + a\varepsilon\xi\right] = e^{c_3} \frac{1}{\sqrt{4\pi\tau}} \int_0^{\infty} d\xi e^{-c_1(c_2-\varepsilon\xi)^2}$$

With:

$$\eta = \left(c_2 - \varepsilon\xi\right)\sqrt{2c_1}, d\eta = -d\xi\varepsilon\sqrt{2c_1}$$

$$I_a = \frac{e^{c_3}}{\sqrt{2c_1}} \frac{1}{\sqrt{4\pi\tau}} \left(-\varepsilon \int_{c_2\sqrt{2c_1}}^{-\varepsilon\infty} d\eta e^{-\frac{\eta^2}{2}}\right)$$

With:

$$\Phi(\zeta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta}d\eta e^{-\frac{\eta^2}{2}}$$ cumulative standard normal distribution function.

$$I_a(\varepsilon=1)=e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c_2\sqrt{2c_1}}d\eta e^{-\frac{\eta^2}{2}}=e^{c_3}\Phi\left(c_2\sqrt{2c_1}\right)$$

$$I_a(\varepsilon=-1)=e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{c_2\sqrt{2c_1}}^{\infty}d\eta e^{-\frac{\eta^2}{2}}\xrightarrow{\zeta=-\eta,d\zeta=-d\eta}e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-c_2\sqrt{2c_1}}d\zeta e^{-\frac{\zeta^2}{2}}=e^{c_3}\Phi\left(-c_2\sqrt{2c_1}\right)$$

$$I_a=e^{c_3}\Phi\left(\varepsilon c_2\sqrt{2c_1}\right)=e^{a(x+a\tau)}\Phi\left(\varepsilon\frac{x+2\tau a}{\sqrt{2\tau}}\right)$$

$$\begin{aligned}h(x,\tau)&=\varepsilon\,\mathrm{I}_{a/2+1}-\varepsilon\,\mathrm{I}_{a/2}\\&=\varepsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-\varepsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\end{aligned}$$

$$V(S,t)=Ke^{\left(-1-a-\frac{a^2}{4}\right)\tau}e^{-\frac{a}{2}x}\left[\varepsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-\varepsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right]$$

$$\begin{aligned}V(S,t)&=\varepsilon K\left[e^{x}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-e^{-\tau(1+a)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right]\\&=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)r+\left(\frac{(T-t)\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)r-\left(\frac{(T-t)\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]\\&=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]\\&=\varepsilon S\Phi(\varepsilon d_1)-\varepsilon Ke^{-r(T-t)}\Phi(\varepsilon d_2)\end{aligned}$$

Then:

$$V(S,t)=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]$$

Reference

Taylor's Theorem

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y$$

Fourier transform

$$F(f(x))(k) = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{-ikx} f(x)$$

$$F^{-1}(\tilde{f}(k))(x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dk e^{ikx} \tilde{f}(k)$$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dy f(x-y) g(y)$$

$$F\left(f^{(n)}(x)\right)(k) = (ik)^n F(f(x))(k)$$

With:

$$x = u + y$$
$$dx \mid y = du \mid y$$

$$F(f)F(g) = \frac{1}{\sqrt{2\pi}} \int_{\Re} du e^{-iku} f(u) \frac{1}{\sqrt{2\pi}} \int_{\Re} dy e^{-iky} g(y)$$
$$= \frac{1}{2\pi} \int_{\Re^2} dudy e^{-ik(u+y)} f(u) g(y)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\Re} dx e^{-ikx} \frac{1}{\sqrt{2\pi}} \int_{\Re} dy f(x-y) g(y)$$

$$F(f * g) = F(f)F(g)$$

Gaussian integral

$$\int_{\mathbb{R}} d\xi\, \mathrm{e}^{-\xi^2}$$