Black-Scholes 期權定價模型

(Black-Scholes Option Pricing Model)

1,伊藤引理(It^o's Lemma)

設:

S:資產價格。

t:時間。

$\frac{dS}{S}$:資產回報率。

μ:資產固定增長率。

σ:回報率標準方差。

dX:正態分佈隨機變量(均值0,方差dt)。

$$dX \sim N\left(0,\left(\sqrt{dt}\right)^2\right) = \varepsilon\sqrt{dt}$$

隨機微分方程:

$$\frac{dS}{S} = \mu dt + \sigma dX$$

$f\left(S,t\right)$:關於資產價格及時間的函數。

演算:

由：

泰勒公式

得：

$$\Delta f=\frac{\partial f}{\partial S}\Delta S+\frac{\partial f}{\partial t}\Delta t+\frac{1}{2}\left(\frac{\partial^2 S}{\partial S^2}\left(\Delta S\right)^2+2\frac{\partial^2 S}{\partial S\partial t}\left(\Delta S\Delta t\right)+\frac{\partial^2 S}{\partial t^2}\left(\Delta t\right)^2\right)+\ldots$$

由：

$$dS=S\left(\mu dt+\sigma dX\right)$$

得：

$$\begin{aligned}\Delta f&=\frac{\partial f}{\partial S}S\left(\mu\Delta t+\sigma\varepsilon\sqrt{\Delta t}\right)+\frac{\partial f}{\partial t}\Delta t\\&+\frac{1}{2}\left(\begin{array}{l}\frac{\partial^2 S}{\partial S^2}S^2\left(\mu^2\left(\Delta t\right)^2+2\mu\sigma\Delta t\varepsilon\sqrt{\Delta t}+\sigma^2\left(\varepsilon\sqrt{\Delta t}\right)^2\right)\\+2\frac{\partial^2 S}{\partial S\partial t}S\left(\mu\Delta t\Delta t+\sigma\varepsilon\sqrt{\Delta t}\Delta t\right)+\frac{\partial^2 S}{\partial t^2}\left(\Delta t\right)^2\end{array}\right)+\ldots\end{aligned}$$

由：

$$\Delta t\rightarrow dt$$

$$\left(\Delta X\right)^2\rightarrow dt$$

得：

$$\begin{aligned}df&=\frac{\partial f}{\partial S}S\left(\mu dt+\sigma\varepsilon\sqrt{dt}\right)+\frac{\partial f}{\partial t}dt+\frac{1}{2}\frac{\partial^2 S}{\partial S^2}\left(S^2\sigma^2\varepsilon^2 dt\right)\\&=\left(\frac{\partial f}{\partial S}S\mu+\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 S}{\partial S^2}S^2\sigma^2\right)dt+\frac{\partial f}{\partial S}S\sigma\varepsilon\sqrt{dt}\end{aligned}$$

結果:

$$df=\left(\frac{\partial f}{\partial S}S\mu+\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 S}{\partial S^2}S^2\sigma^2\right)dt+\frac{\partial f}{\partial S}S\sigma dX$$

2,布萊克斯克爾斯方程

設:

$V(S,t)$：期權價格, $C(S,t)$ 為看漲期權 $P(S,t)$ 為看跌期權。

r:利息率。

$\prod$:資產組合, 包括一單位的期權及$-\Delta$單位的標的資產。

$$\prod = V - \Delta S$$

演算:

由：

伊藤引理

得：

$$\begin{aligned} d\prod &= dV - \Delta dS \\ &= \sigma S\frac{\partial V}{\partial S}dX + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt - \Delta S\mu dt - \Delta S\sigma dX \\ &= \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dX + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right)dt \end{aligned}$$

由：

$$\Delta = \frac{\partial V}{\partial S}$$

得：

$$d\prod = \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt$$

由：

以利率 r 投資$\prod$單位的無風險資產。

得：

$$\left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt = r\prod dt$$

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = r\left(V - \frac{\partial V}{\partial S}S\right)$$

結果:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

3,求解布萊克斯克爾斯方程

求解:

由：

$$x=\ln\left(\frac{S}{K}\right)$$

$$\tau=\frac{(T-t)\sigma^2}{2}$$

$$v(x,\tau)=\frac{V(S,t)}{K}$$

布萊克斯克爾斯方程

得：

$$\begin{aligned}
&\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV\\
&=\frac{\partial(Kv)}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2(Kv)}{\partial S^2}+rS\frac{\partial(Kv)}{\partial S}-r(Kv)\\
&=K\frac{\partial v}{\partial\tau}\frac{\partial\tau}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial S}\right)+rS\left(K\frac{\partial v}{\partial S}\right)-r(Kv)\\
&=-K\frac{\partial v}{\partial\tau}\frac{\sigma^2}{2}+\frac{1}{2}\sigma^2S^2\frac{\partial}{\partial S}\left(K\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}\right)+rS\left(K\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}\right)-Kvr\\
&=\frac{\partial^2v}{\partial x^2}+\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v-\frac{\partial v}{\partial\tau}
\end{aligned}$$

$$\frac{\partial^2v}{\partial x^2}+\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v=\frac{\partial v}{\partial\tau}$$

由：

$$a=\left(\frac{2r}{\sigma^2}-1\right)$$

$$b=-\frac{2r}{\sigma^2}=-1-a$$

得：

$$\frac{\partial^2v}{\partial x^2}+a\frac{\partial v}{\partial x}+bv=\frac{\partial v}{\partial\tau}$$

由：

$$v(x,\tau)=f\left(\tau\right)g\left(x\right)h\left(x,\tau\right)$$

得：

$$\begin{aligned}0&=\frac{\partial^2 v}{\partial x^2}+a\frac{\partial v}{\partial x}+bv-\frac{\partial v}{\partial \tau}\\&=f\frac{\partial^2 g}{\partial x^2}h+2f\frac{\partial g}{\partial x}\frac{\partial h}{\partial x}+fg\frac{\partial^2 h}{\partial x^2}+a\left(f\frac{\partial g}{\partial x}h+fg\frac{\partial h}{\partial x}\right)+bv-\frac{\partial f}{\partial \tau}gh-fg\frac{\partial h}{\partial \tau}\end{aligned}$$

$$\left(f\frac{\partial^2 g}{\partial x^2}+af\frac{\partial g}{\partial x}+bfg-\frac{\partial f}{\partial \tau}g\right)h+\left(2f\frac{\partial g}{\partial x}+afg\right)\frac{\partial h}{\partial x}+fg\frac{\partial^2 h}{\partial x^2}=fg\frac{\partial h}{\partial \tau}$$

由：

$$g\left(x\right)=e^{-\frac{a}{2}x}$$

$$f\left(\tau\right)=e^{\left(b-\frac{a^2}{4}\right)\tau}$$

得：

$$\frac{\partial^2 h}{\partial x^2}=\frac{\partial h}{\partial \tau}$$

由：

傅里葉變換

高斯積分

$$\tilde{h}=\tilde{h}_1\tilde{h}_2$$

得：

$$\frac{\partial \tilde{h}}{\partial \tau}=F\left(\frac{\partial h}{\partial \tau}\right)=F\left(\frac{\partial^2 h}{\partial x^2}\right)=-k^2F\left(h\right)=-k^2\tilde{h}$$

$$\tilde{h}=\tilde{h}_1\tilde{h}_2=Ae^{-k^2\tau}=\tilde{h}\left(k,0\right)$$

$$h=h_1*h_2=F^{-1}\left(\tilde{h}_1\right)*F^{-1}\left(\tilde{h}_2\right)=F^{-1}\left(A\right)*F^{-1}\left(e^{-k^2\tau}\right)$$

$$h=F^{-1}\left(A\right)*\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(k^2\tau-i\lambda x\right)}d\lambda$$

$$h=F^{-1}(A)*\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(k^2\tau-i\lambda x\right)}d\lambda$$

$$=F^{-1}(A)*\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4\tau}}\int_{\mathbb{R}}dk\exp\left[-\tau\left(k-i\frac{x}{2\tau}\right)^2\right]$$

$$=F^{-1}(A)*\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4\tau}}\frac{1}{\sqrt{\tau}}\int_{\mathbb{R}}d\xi\,\mathrm{e}^{-\xi^2}$$

$$=F^{-1}(A)*\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{4\tau}}\sqrt{\pi}$$

$$=F^{-1}(A)*\frac{1}{\sqrt{2\tau}}e^{-\frac{x^2}{4\tau}}$$

$$F^{-1}(A)*\frac{1}{\sqrt{2\tau}}e^{-\frac{y^2}{4\tau}}=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}dyh_1(x-y)h_2(y)$$

$$=\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}F^{-1}(A)(x-y)e^{-\frac{y^2}{4\tau}}dy$$

$$=\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}h_1(x-y)e^{-\frac{y^2}{4\tau}}dy$$

$$=\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}h(\xi,0)e^{-\frac{(x-\xi)^2}{4\tau}}d\xi$$

求得：

$$V(S,t)=Ke^{\left(b-\frac{a^2}{4}\right)\tau}e^{-\frac{a}{2}x}\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}h(\xi,0)e^{-\frac{(x-\xi)^2}{4\tau}}d\xi$$

4，運用於歐式期權

設:

$V(S,T) = max[\varepsilon(S-K),0]$，歐式期權

$\varepsilon=1$ 為看漲期權

$\varepsilon=-1$ 為看跌期權

演算:

$$h(x,0)=\frac{1}{K}e^{\left(\frac{a}{2}\right)x}V\left(Ke^{x},T\right)$$
$$=\frac{1}{K}e^{\left(\frac{a}{2}\right)x}\max\left[\varepsilon\left(Ke^{x}-K\right),0\right]$$
$$=\max\left\{\varepsilon\left[e^{\left(\frac{a}{2}+1\right)x}-e^{\left(\frac{a}{2}\right)x}\right],0\right\}$$

$$h(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}d\xi\exp\left[-\frac{(x-\xi)^{2}}{4\tau}\right]\max\left\{\varepsilon\left[e^{\left(\frac{a}{2}+1\right)\xi}-e^{\left(\frac{a}{2}\right)\xi}\right],0\right\}$$

$$h(x,\tau)=\varepsilon\frac{1}{\sqrt{4\pi\tau}}\int_{0}^{\infty}d\eta\exp\left[-\frac{(x-\varepsilon\eta)^{2}}{4\tau}\right]\left[e^{\varepsilon\left(\frac{a}{2}+1\right)\eta}-e^{\varepsilon\left(\frac{a}{2}\right)\eta}\right]$$

由：

$$c_1=\frac{1}{4\tau};$$
$$c_2=x+2\tau a;$$
$$c_3=a\left(x+a\tau\right);$$

得：

$$I_a=\frac{1}{\sqrt{4\pi\tau}}\int_{0}^{\infty}d\xi\exp\left[-\frac{(x-\varepsilon\xi)^{2}}{4\tau}+a\varepsilon\xi\right]=e^{c_3}\frac{1}{\sqrt{4\pi\tau}}\int_{0}^{\infty}d\xi e^{-c_1(c_2-\varepsilon\xi)^{2}}$$

由：

$$\eta=\left(c_2-\varepsilon\xi\right)\sqrt{2c_1},d\eta=-d\xi\varepsilon\sqrt{2c_1}$$

得：

$$I_a=\frac{e^{c_3}}{\sqrt{2c_1}}\frac{1}{\sqrt{4\pi\tau}}\left(-\varepsilon\int_{c_2\sqrt{2c_1}}^{-\varepsilon\infty}d\eta e^{-\frac{\eta^{2}}{2}}\right)$$

由：

$\Phi(\zeta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\zeta}d\eta e^{-\frac{\eta^{2}}{2}}$ 標準正態分佈累計函數

$$I_a(\varepsilon=1)=e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c_2\sqrt{2c_1}}d\eta e^{-\frac{\eta^2}{2}}=e^{c_3}\Phi\left(c_2\sqrt{2c_1}\right)$$

$$I_a(\varepsilon=-1)=e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{c_2\sqrt{2c_1}}^{\infty}d\eta e^{-\frac{\eta^2}{2}}\xrightarrow{\zeta=-\eta,d\zeta=-d\eta}e^{c_3}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-c_2\sqrt{2c_1}}d\zeta e^{-\frac{\zeta^2}{2}}=e^{c_3}\Phi\left(-c_2\sqrt{2c_1}\right)$$

得：

$$I_a=e^{c_3}\Phi\left(\varepsilon c_2\sqrt{2c_1}\right)=e^{a(x+a\tau)}\Phi\left(\varepsilon\frac{x+2\tau a}{\sqrt{2\tau}}\right)$$

$$\begin{aligned}h(x,\tau)&=\varepsilon\,\mathrm{I}_{a/2+1}-\varepsilon\,\mathrm{I}_{a/2}\\&=\varepsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-\varepsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\end{aligned}$$

$$V(S,t)=Ke^{\left(-1-a-\frac{a^2}{4}\right)\tau}e^{-\frac{a}{2}x}\left[\varepsilon e^{\left(\frac{a}{2}+1\right)\left(x+\frac{a\tau}{2}+\tau\right)}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-\varepsilon e^{\left(\frac{a}{2}\right)\left(x+\frac{a\tau}{2}\right)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right]$$

$$\begin{aligned}V(S,t)&=\varepsilon K\left[e^{x}\Phi\left(\varepsilon\frac{x+\tau a+2\tau}{\sqrt{2\tau}}\right)-e^{-\tau(1+a)}\Phi\left(\varepsilon\frac{x+\tau a}{\sqrt{2\tau}}\right)\right]\\&=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)r+\left(\frac{(T-t)\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)r-\left(\frac{(T-t)\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]\\&=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]\\&=\varepsilon S\Phi(\varepsilon d_1)-\varepsilon Ke^{-r(T-t)}\Phi(\varepsilon d_2)\end{aligned}$$

結果:

$$V(S,t)=\varepsilon\left[S\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)-Ke^{-(T-t)r}\Phi\left(\varepsilon\frac{\ln\left(\frac{S}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{(T-t)}}\right)\right]$$

参考

泰勒公式

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y$$

傅里葉變換

$$F(f(x))(k)=\tilde{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx e^{-ikx} f(x)$$

$$F^{-1}(\tilde{f}(k))(x)=f(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dk e^{ikx} \tilde{f}(k)$$

$$(f*g)(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dy f(x-y)g(y)$$

$$F\left(f^{(n)}(x)\right)(k)=(ik)^n F\left(f(x)\right)(k)$$

由：

$x=u+y$

$dx\,|\,y=du\,|\,y$

得：

$$F(f)F(g)=\frac{1}{\sqrt{2\pi}}\int_{\Re} du e^{-iku} f(u)\frac{1}{\sqrt{2\pi}}\int_{\Re} dy e^{-iky} g(y)$$

$$=\frac{1}{2\pi}\int_{\Re^2} dudy e^{-ik(u+y)} f(u)g(y)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\Re} dx e^{-ikx}\frac{1}{\sqrt{2\pi}}\int_{\Re} dy f(x-y)g(y)$$

$$F(f*g)=F(f)F(g)$$

高斯積分

$$\int_{\mathbb{R}} d\xi\, \mathrm{e}^{-\xi^2}=\sqrt{\pi}$$