# Analysis of the Cosine Map

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**Abstract**

The Cosine Map exhibits chaotic dynamics under a change of parameters. It approximates the logistic map for large $a$ and has similar dynamics under a change in $r$. This paper is intended to be read along with the code in *cosine_map_analysis.ipynb*.

# 1 Cosine Map approaches the Logistic Map

## 1.1 Change of Variables

The cosine-map is defined as:

$$x_{n+1} = \frac{r}{4}\left((a+1)\cos\left[k\left(x_n - \frac{1}{2}\right)\right] - a\right),$$
$$\text{with } k = 2\arccos\left(\frac{a}{a+1}\right), \quad a > 0.$$

The Logistic map is defined as:

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n)$$

In this section, we will prove that the cosine-map approaches the logistic map as $a \to \infty$.

Let's consider a change of variables: $a = \frac{1}{T}$ hence, considering the limit as $T \to 0$.

We can rewrite $\frac{a}{a+1}$ as $\frac{1}{1+T}$.

Given that $T > 0$, we have:

$$k = 2\arccos\left(\frac{1}{1+T}\right)$$

From this, as $T \to 0$

$$k \to 2\arccos(1) = 0$$

we will use this later.

## 1.2 Definitions

Here are 3 definitions that we will use in our proof.

**1.** $$\frac{d}{dT}\left(\arccos\left(\frac{1}{T+1}\right)\right) = \frac{-1}{\sqrt{1-\left(\frac{1}{T+1}\right)^2}}\left(\frac{-1}{(T+1)^2}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{1}{(T+1)^2}\right)}}\left(\frac{1}{(T+1)^2}\right)$$
$$= \frac{(T+1)^2}{\sqrt{(T+1)^2-1}}\left(\frac{1}{(T+1)^2}\right)$$
$$= \frac{1}{\sqrt{T^2+2T}}$$

$$\therefore \frac{d}{dT}\left(\arccos\left(\frac{1}{T+1}\right)\left(x_n-\frac{1}{2}\right)\right) = \frac{x_n-\frac{1}{2}}{\sqrt{T^2+2T}(T+1)} \quad (1)$$

**2.** $$\frac{d}{dT}\left(T^2+2T\right)^{\frac{1}{2}} = \frac{1}{2}\left(T^2+2T\right)^{-\frac{1}{2}}(2T+2)$$
$$= \frac{T+1}{(T^2+2T)^{\frac{1}{2}}} \quad (2)$$

## 1.3 Taking the Limit

$$\text{take } \lim_{T\to 0}\frac{r}{4}\left[\frac{(T+1)}{T}\cos\left(\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right)-\frac{1}{T}\right]$$
$$= \lim_{T\to 0}\frac{r}{4}\left[\frac{(T+1)\cos\left(\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right)-1}{T}\right]$$
$$= \frac{r}{4}\left[\frac{\cos[0]-1}{0}\right]$$
$$= \frac{r}{4}\left[\frac{0}{0}\right] \quad \text{Not useful: Apply L'Hopital's Rule}$$

Using (1)

$$= \lim_{T\to 0}\frac{r}{4}\left[\frac{\cos\left[\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right]\left(\frac{x_n-\frac{1}{2}}{(T^2+2)^{\frac{1}{2}}}\right)\sin\left[\arccos\left(\frac{1}{1+T}\left(x_n-\frac{1}{2}\right)\right)\right]}{1}\right]$$
$$= \lim_{T\to 0}\frac{r}{4}\left[\frac{2(T^2+2T)^{\frac{1}{2}}\cos\left(\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right)-(2x_n-1)\sin\left(\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right)}{2(T^2+2T)^{\frac{1}{2}}}\right]$$
$$= \frac{r}{4}\left[\frac{0-0}{0}\right]$$

Not useful again: Apply L'Hopital's Rule for the 2nd time

$$= \lim_{T\to 0} \frac{r}{4}\left[\frac{2(T+1)\cos\left[\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right]-(2x_n-1)\sin\left[\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right]}{2(T+1)(T^2+2T)^{\frac{1}{2}}}\right.$$

$$\left.-\frac{\frac{(2x_n+1)^2}{(T^2+2T)^{\frac{1}{2}}(T+1)}\cos\left[\arccos\left(\frac{1}{1+T}\right)\left(x_n-\frac{1}{2}\right)\right]}{2(T+1)(T^2+2T)^{\frac{1}{2}}}\right]$$

$$= \lim_{T\to 0} \frac{r}{4}\left[\frac{\frac{2(T+1)^2-(2x_n-1)^2}{(T+1)(T^2+2T)^{\frac{1}{2}}}\cos\left(\arccos\left(\frac{1}{T+1}\right)\left(x_n-\frac{1}{2}\right)\right)}{2(T+1)(T^2+2T)^{\frac{1}{2}}}\right.$$

$$\left.-\frac{\frac{(2x_n-1)}{(T+1)}\sin\left(\arccos\left(\frac{1}{T+1}\right)\left(x_n-\frac{1}{2}\right)\right)}{2(T+1)(T^2+2T)^{\frac{1}{2}}}\right]$$

$$= \lim_{T\to 0} \frac{r}{4}\left[\frac{(T+1)^2-(2x_n-1)^2}{(T+1)^2}\cos\left(\arccos\left(\frac{1}{T+1}\right)\left(x_n-\frac{1}{2}\right)\right)\right.$$

$$\left.-\frac{(2x_n-1)\,(T^2+2T)^{\frac{1}{2}}}{(T+1)^2}\sin\left(\arccos\left(\frac{1}{T+1}\right)\left(x_n-\frac{1}{2}\right)\right)\right]$$

$$= \frac{r}{4}\left[\left(1-(2x_n-1)^2\right)\cos[0]-0\right]$$

$$= \frac{r}{4}\left[1-4x_n^2+4x_n-1\right]$$

$$= x_n(1-x_n)$$

Therefore the cosine map approaches the logistic map as $T\to 0$ (as $a\to\infty$).

# 2 Cosine Map Dynamics ($a = 1$)

## 2.1 Introduction

### 2.1.1 Defining the Cosine Map

For $a = 1$ the Cosine Map is defined as:

$$x_{n+1} = \frac{r}{4}\left(2\cos\left(x_n - \frac{1}{2}\right) - a\right),$$
$$\text{where } a = 2\arccos\left(\frac{1}{2}\right) = \frac{3\pi}{2}.$$

so,

$$x_{n+1} = \frac{r}{4}\left(2\cos\left[\frac{2\pi}{3}\left(x_n - \frac{1}{2}\right)\right] - 1\right).$$

### 2.1.2 Introduction to Dynamical Systems and Chaos

Dynamical systems define a state space and a set of rules or equations that describe how a system's state changes with respect to time. These systems can be represented in continuous time by differential equations, or in discrete time steps in discrete time by maps. A map is an equation that maps the state at one time step to the state at another time step.

A defining characteristic of dynamical systems is their sensitivity to initial conditions. Small perturbations in initial conditions can result in significant divergence in the system behaviour over time.

Non-linear dynamical systems often show interesting dynamics when studying how the system changes under a change in a parameter that governs the system. Bifurcations happen at values of this parameter, which change the dynamics of this system as it evolves over time. For example, fixed points or periodic orbits can be created or destroyed.

In 3D systems, chaotic behaviour can arise from the presence of bifurcations and instabilities in the system's dynamics

Chaos, within the context of dynamical systems theory, refers to a state of complex and unpredictable behavior exhibited by certain nonlinear systems. Despite being governed by deterministic laws, chaotic systems display sensitive dependence on initial conditions, meaning that small variations in the starting state can lead to vastly different trajectories over time.

There are 3 principal routes to chaos:

1. Period doubling route
2. Torus breakdown
3. Pomeau-Maneville intermittency

The Cosine-Map exhibits a period-doubling route to chaos, which we will study in further detail

## 2.2 Designing the system

Refer to *cosine_map_analysis.ipynb* to see how the map was initialised and analysed. Figures are taken directly from this notebook unless otherwise stated.

Firstly, we initialise the cosine map from a random ($[0, 1]$) value. We disregard at least the first 200 iterations of the map because this is transient behavior. We aim to analyse the system at a 'steady state equilibrium'. Note the quotation marks because the behavior of the system in a chaotic regime is not a steady state.

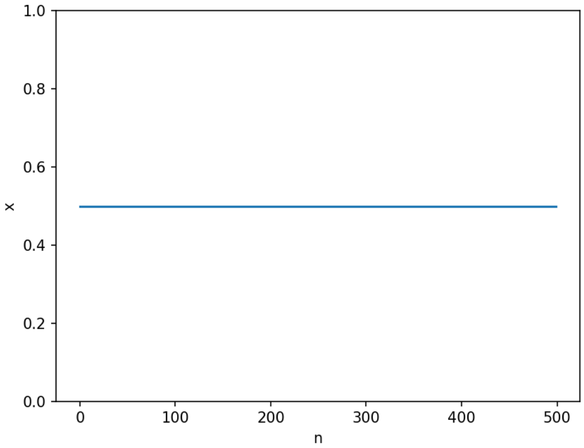



Figure 1: Cosine Map Time Series for $r = 2$

The (a=1, r=2) Cosine Map time series of iteration 4500-5000 after a random initialisation.

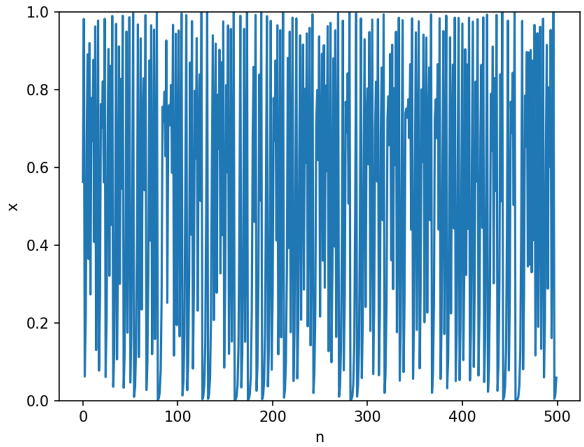



Figure 2: Cosine Map Time Series for $r = 4$

The (a=1, r=4) Cosine Map time series of iteration 4500-5000 after a random initialisation.

Time series for the cosine map at 'steady state' are shown in Figures 1 and 2. We can see in Figure 1 that for $r = 2$ the map approaches a stable fixed point of $x = 0.5$ but in Figure 2 ($r = 4$) there doesn't appear to be any predictable system dynamics. This is indeed chaotic behavior, which we will discuss in more detail later.

In Figure 3, we plot the possible values for $x$ (after initial transient behavior) as on an orbit diagram. This looks very similar to the logistic map (Figure 4). However, we notice something interesting happening when we change the x scale to include negative values (Figure 5).

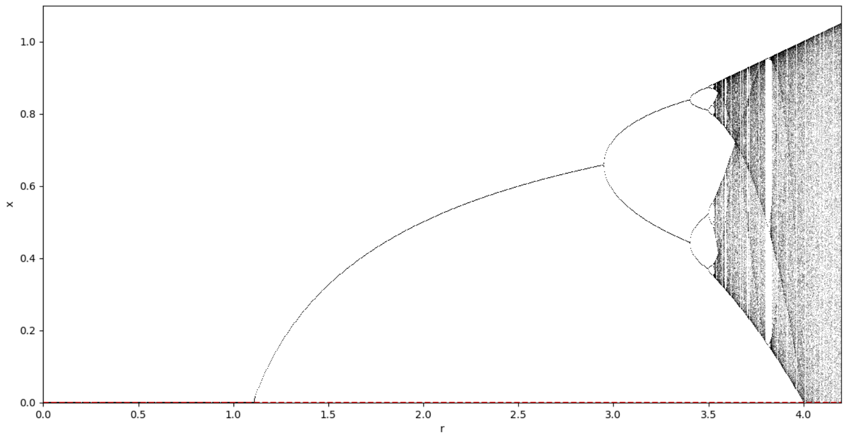



Figure 3: Cosine Map Orbit Diagram

The $(a = 1)$ Cosine Map Orbit Diagram for $r$ for $0 \leq r \leq 4.2$.

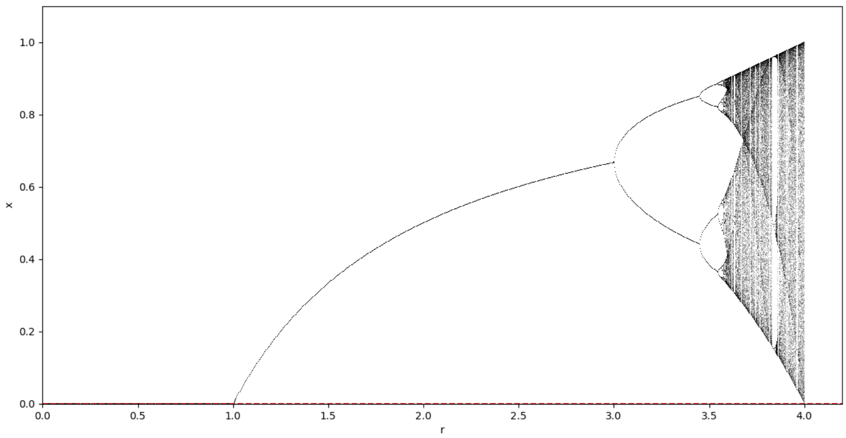



Figure 4: Logistic Map Orbit Diagram

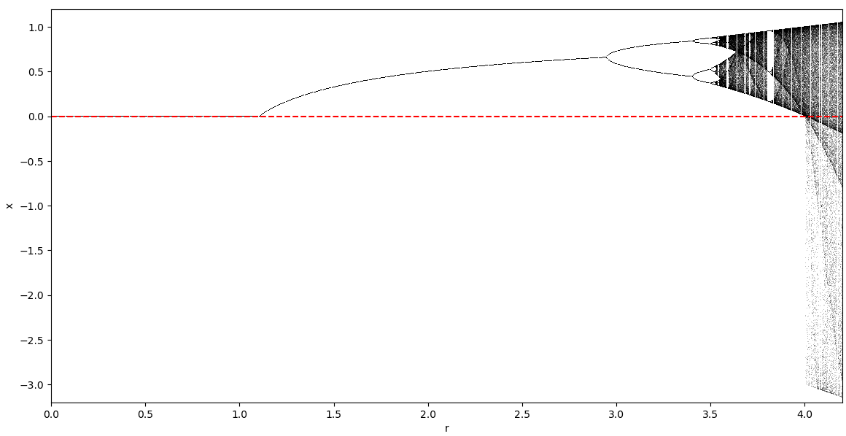



Figure 5: Cosine Map Orbit Diagram revealing negative realisations

The $(a = 1)$ Cosine Map Orbit Diagram for $r$ for $0 \leq r \leq 4.2$.

We understand that the Logistic map for $r \geq 4$ $x_n$ diverges to infinity, therefore, is not meaningful and cannot be plotted on an orbit diagram.

In contrast to this, the Cosine map exhibits meaningful dynamics past $r \geq 4$. In particular, negative values of $x$ appear in the dynamics of the system. This is interesting and it prompts us to explore what happens when we initialise with a negative $x$ value.

In Figure 6, we plot a negative $x$ initialisation in red, on top of the positive $x$ initialisation we saw earlier

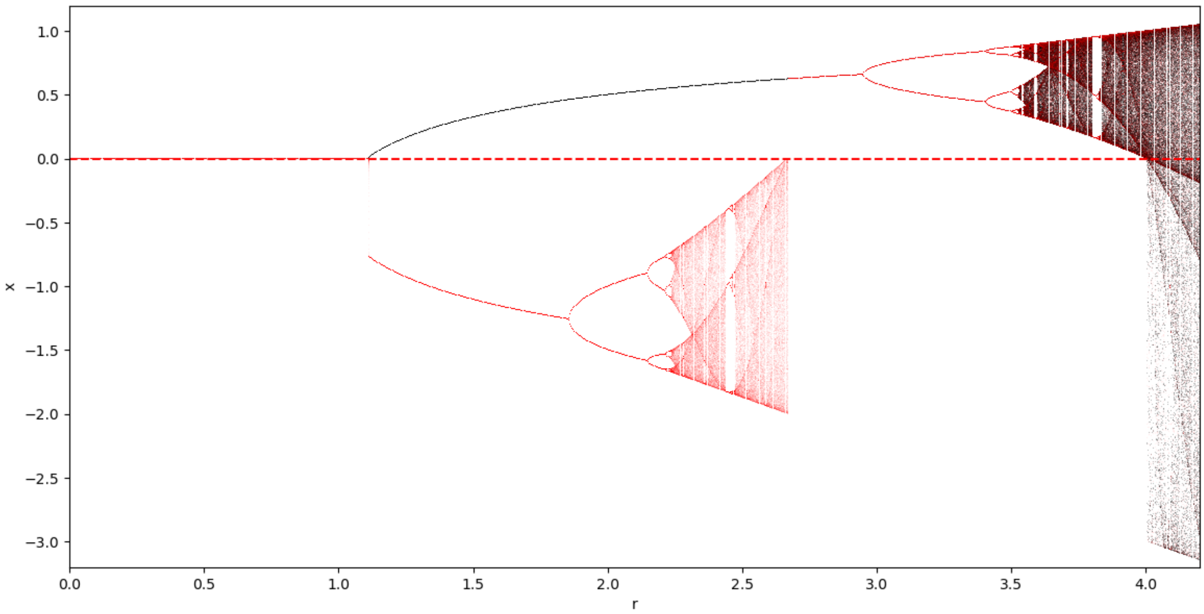



Figure 6: Cosine Map Orbit Diagram with negative initialisation

The ($a = 1$) Cosine Map Orbit Diagram for $r$ for $0 \leq r \leq 4.2$. Negative initalisation in red, Positive initialisations in black as previously.

Here we have set up the map and are ready to analyse it. This covers Section 1 of *cosine_map_analysis.ipynb*. We will numerically analyse the map in Section 2.4 in unison with the following sections of *cosine_map_analysis.ipynb*.

## 2.3 Dynamics of the Cosine Map

### 2.3.1 Period Doubling route to Chaos

From Figure 3, we can see that the cosine map exhibits a period-doubling route to chaos as r increases. We use the notation of $r_n$ being the point at which a stable $2^n$-period cycle is born. Here, we consider only the case where the initiation of x is positive for simplicity.

The long-term behavior of the map is a stable fixed point of $x = 0$ from $0 < r < 1.1$. At $r_0 = 1.1$ (values found in section 2.3.2) the fixed point starts increasing from 0, until there is a bifurcation.

At $r_1 = 2.94$, a stable period-2 cycle is created. This is called a **period-doubling bifurcation** and also involves the creation of an unstable point that is between the two periodically stable values of $x$. From here, there are further period-doubling bifurcations into period $2^n$ cycles at decreasing changes of $r$ until the next bifurcation.

The bifurcations asymptotically approach a value, $r_\infty$ where the system exhibits chaotic dynamics. Therefore, we examine a 'period doubling route to chaos'.

### 2.3.2 The Lyapunov Exponent and Chaos

The derivative of the $a = 1$ cosine-map is defined as:

$$\frac{df}{dx_n}(x_{n+1}) = -\frac{r\pi}{3} \sin\left(\frac{2\pi}{3}\left(x_n - \frac{1}{2}\right)\right)$$

The Lyapunov exponent $\lambda$ for a dynamical system with a map $f(x)$ and a parameter $r$ is defined as:

$$\lambda(r) = \lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} \log\left|\frac{df}{dx}(x_n)\right|$$

The Lyapunov exponent measures the average exponential divergence or convergence of nearby trajectories in the phase space and is used to characterize the chaotic behavior of the system. A positive Lyapunov exponent typically indicates chaos, while a negative exponent indicates stability.

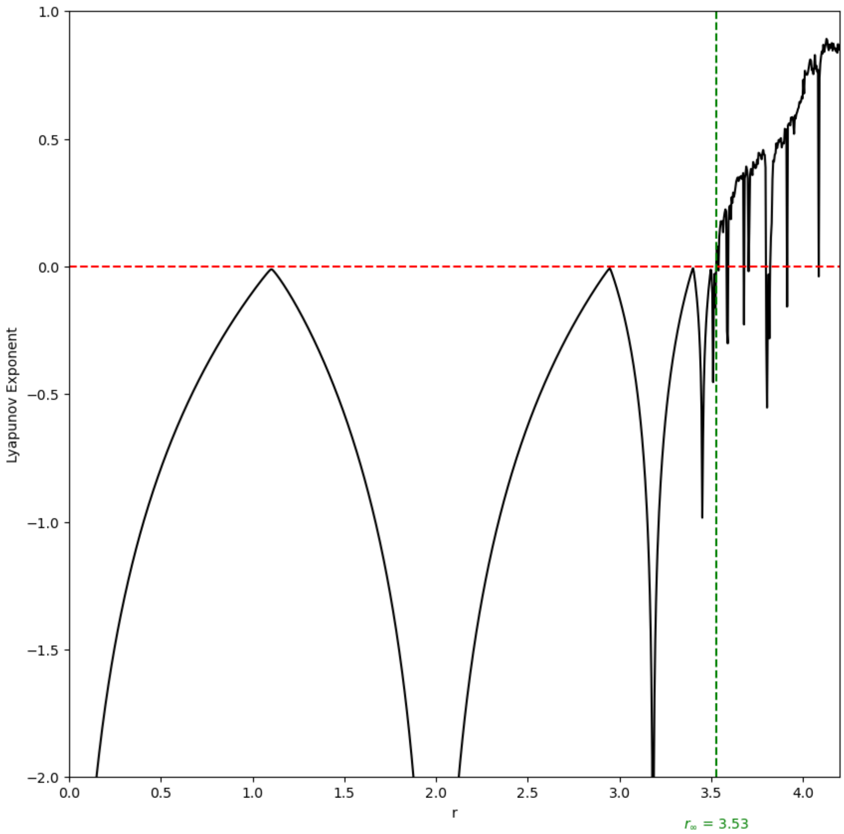



Figure 7: Lyapunov Exponent for Cosine Map

The Lyapunov Exponent for the (a=1) Cosine Map after a positive initialisation.

In Figure 7, we can see the Lyapunov Exponent goes positive at $r_\infty = 3.53$, indicating the onset of chaotic behavior. We can see that at $r > 3.53$ the Lyapunov Exponent dips below 0 briefly, before returning positive. We will explore this in Section 2.3.3.

We can also explore the Lyapunov Exponent for a negative initialisation. In Figure 8 (which can be cross-checked against Figure 6), we see that the system experiences 2 regions where it is mostly in the chaotic regime.

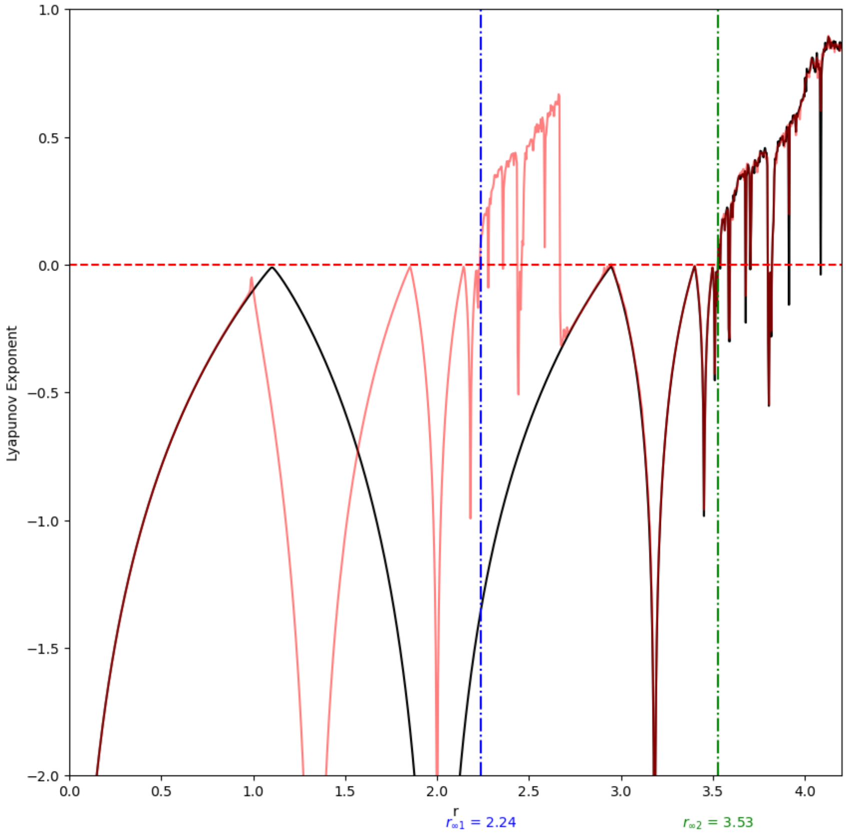



Figure 8: Lyapunov Exponent for Cosine Map (both initialisation cases)

The Lyapunov Exponent for the (a=1) Cosine Map after a positive (black) and negative (red) initialisation. $r_{\infty 1}$ represents the point at which the negative initialisation case goes into chaos, $r_{\infty 2}$ represents the point at which the positive initialisation case goes into chaos.

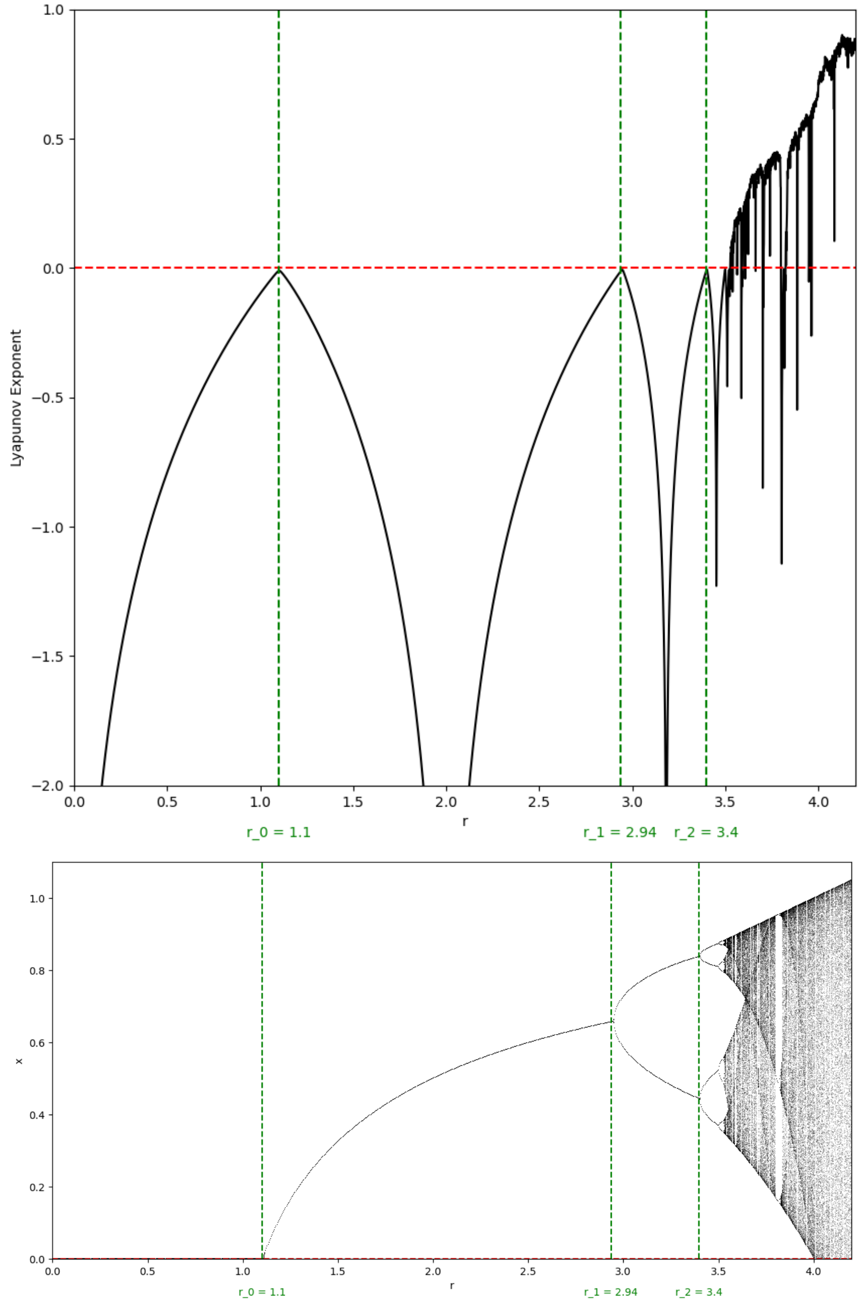



Figure 9: Bifurcation points on Cosine Map

The Lyapunov Exponent and Orbit Diagram for the (a=1) Cosine Map after a positive initialisation showing the first 3 bifurcation points.

One more interesting analysis is the ratio of $r_n$'s, that is, the ratio of values at which period-doubling bifurcations occur. We can find the $r_n$'s numerically by analysing points at which the Lyapunov exponent is near 0 (see 2.1.2 in *cosine_map_analysis.ipynb* for further details).
In Figure 9, we plot the first 3 bifurcations (remember they asymptotically approach $r_\infty$.

We can also use this to numerically calculate Feigenbaum's constant which is the rate at which period-doubling occurs:

$$\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

Calculating this for the numerically estimated $r_n$'s we get $\delta_1 \approx 4$ Theoretically, no matter the map or choice of $n$, $\delta = 4.6692...$ always.

Note that our estimation is inaccurate due to the lack of accuracy in the calculated $r_n$'s.

### 2.3.3 Crisis induced intermittency

Intermittency refers to the occurrence of irregular, alternating phases of periodic (predictable) and chaotic (unpredictable) dynamics.

This is what we referred to when we talked about the spikes down into negative Lyapunov exponents in Figure 7. Within the Chaotic regime, windows of periodicity form, as shown by the white stripes in Figure 3.

These windows occur when the chaotic attractor collides with an unstable periodic orbit, leading to periodic behavior.

For example, let's analyse the largest window on $0 \leq r \leq 4.2$. Using the interactive Orbit Diagram from *cosine_map_analysis.ipynb*, we examine a scaled form of this window in Figure 10. // Clearly, the chaotic behavior turns into a period-3 cycle at $r$ 3.799. This period-3 cycle undergoes period-doubling until it approaches chaos, forming period $3 \times 2^n$ cycles.
We also note that the system is self-similar! That is, each one of these 3 attractors undergoes period-doubling bifurcations and are scaled versions of the whole orbit diagram.

We note it has been proven that within this orbit diagram, all stable period-n cycles exist. This is believable when we see that the Orbit diagram appears fractal-like and windows exist within themselves.

In Figure 10, we zoomed into a period-3 cycle window. However, we could have chosen any period-n cycle window to zoom in. Other large windows are period 5 and period 7, which by extension exhibit period doubling via $5 \times 2^n$ cycles and $7 \times 2^n$ cycles to chaos.

Interestingly, we note from lectures, that a stable period-3 cycle (like the one we have here) implies that the system exhibits chaotic behavior.

Lastly, referring to Figure 10, at a value of $r$ 3.83 the window appears to 'end'. When observing the end of a periodic window within a chaotic regime, the periodic attractor typically undergoes a crisis—a sudden structural change. This crisis often occurs due to the collision of the periodic attractor with an unstable periodic orbit or the boundary of its basin of attraction. As a result of this collision, the system loses its periodic stability and reverts to chaotic behavior.

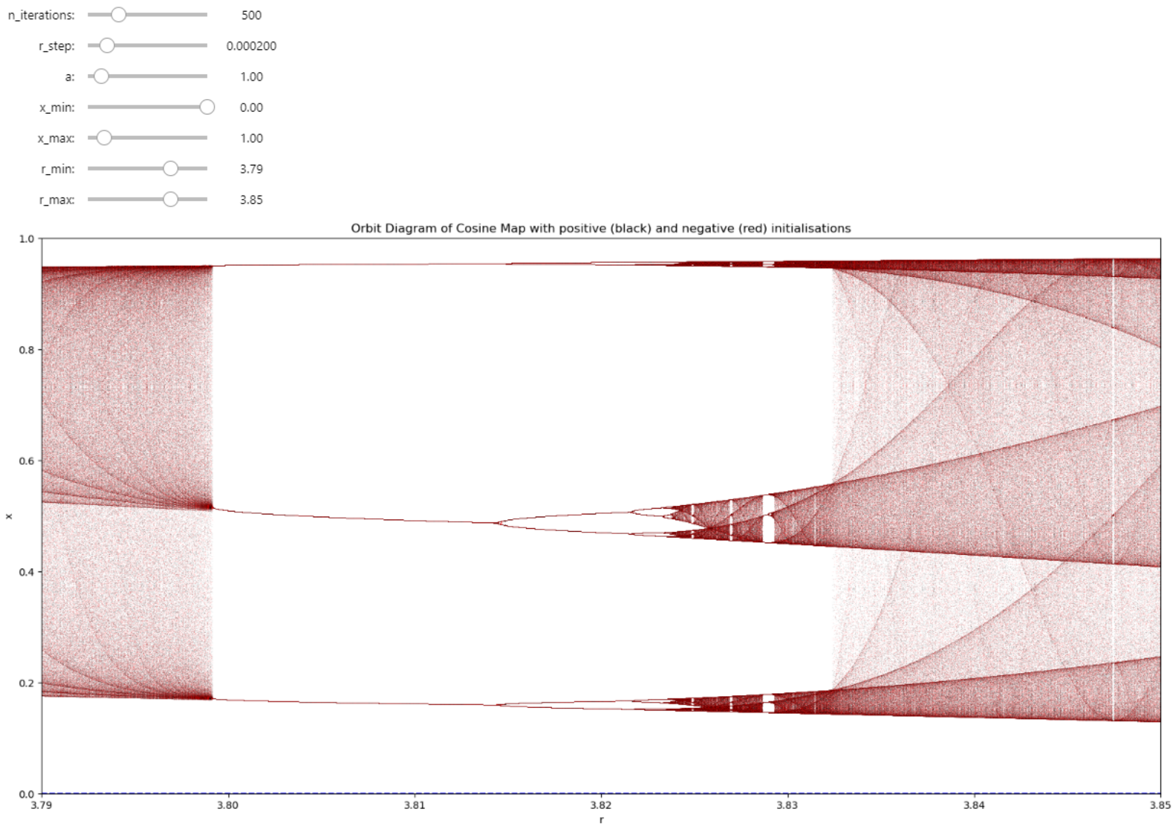



Figure 10: Period-3 window within Chaotic Regime

The Orbit Diagram for the (a=1) Cosine Map after a positive initialisation scaled to the largest periodic window within the chaotic regime.

### 2.3.4 Cobweb Diagrams

Cobweb diagrams illustrate the iterative process of a dynamical system by plotting points iterative along the map's trajectory. This helps visualize how the system evolves over time and whether it converges to a fixed point, cycles, or exhibits chaotic behavior.

For example in Figures 11, 12 and 13, we can visually see how the cosine map evolves. In Figure 11 we are at a value of $r$ that corresponds to a stable fixed point. In Figure 12, we increase $r$ until the system undergoes its first period-doubling bifurcation. We can clearly see the convergence to a period-2 cycle. Finally, we increase $r$ to a point that appears to bring the system to chaotic behavior, there is no analysable convergence to periodicity.

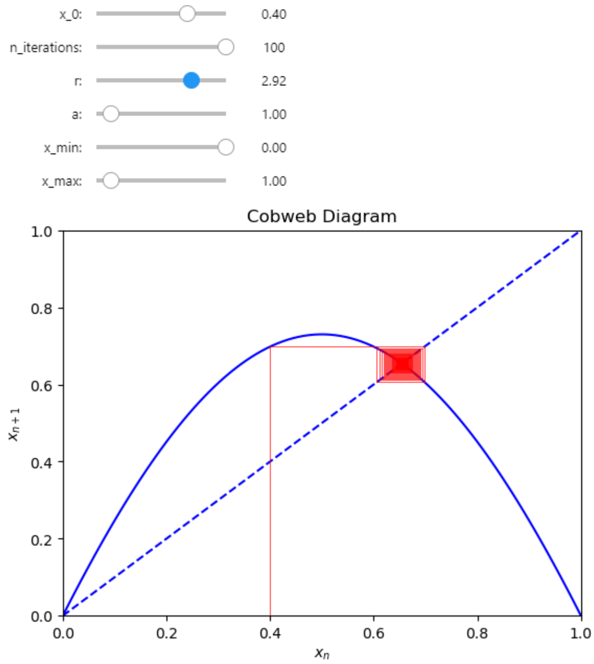



Figure 11: Fixed Point on the Cobweb Diagram

The Cobweb Diagram for the (a=1) Cosine Map after a positive initialisation showing convergence to a fixed point.

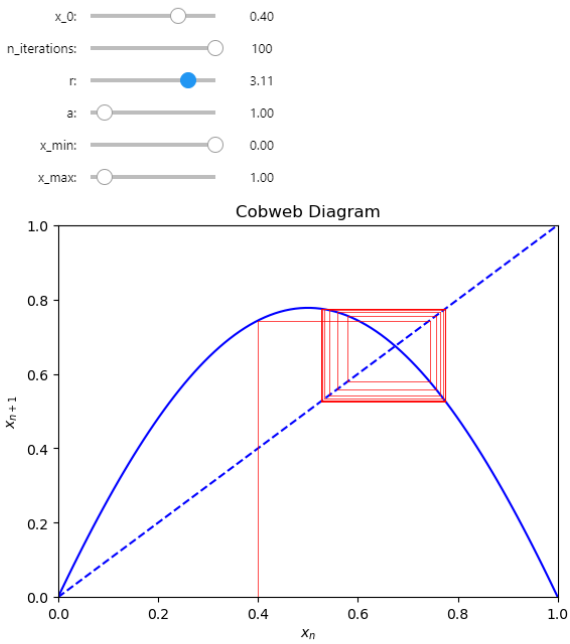



Figure 12: Period-2 Orbit on the Cobweb Diagram

The Cobweb Diagram for the (a=1) Cosine Map after a positive initialisation showing convergence to a Period-2 Orbit.

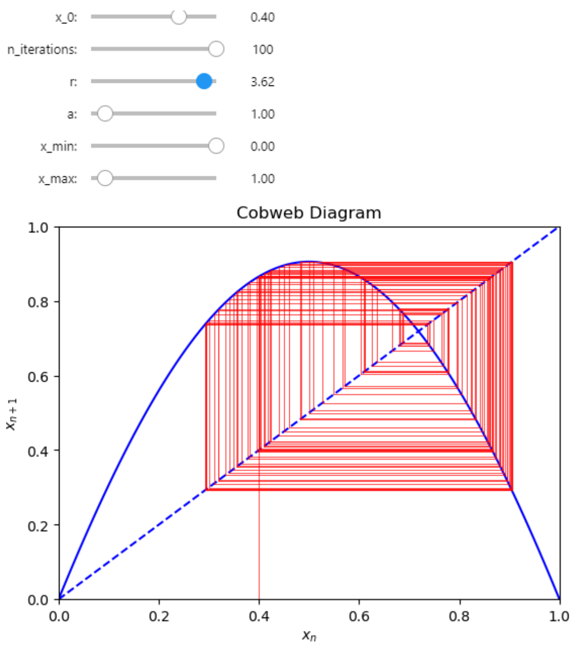



Figure 13: Chaos on the Cobweb Diagram

The Cobweb Diagram for the (a=1) Cosine Map after a positive initialisation with no analysable periodicity.

Briefly, we will mention some interesting facts about Cobweb Diagrams (This touches on the Feigenbaum's Normalisation work):

1. We can clearly see a max of the map on the Cobweb Diagram. When we shift $r$ so that we create a fixed point or cycle with a point on this max, the dynamics is called a "superstable p-cycle", where the period of the cycle is p. We can analyse the values at which this superstable cycle occurs ($R_n$).
2. We can draw the Cobweb Diagram by taking the map $2^p$ times ($f^{(2^p)}(x, R_n)$)). We find that this map contains a scaled version of the original map within it. The scaling is such that:

$$f(x, R_0) \approx \alpha^n f^{(2^p)}(\frac{x}{\alpha^n}, R_n))$$

3. $\alpha = -2.5029$ is also a universal constant independent of the map used

Lastly, we can change the scales and sign of the initial condition to see how the map evolves to negative values, as we saw with the Orbit diagram. An example of this is shown in Figure 14. It corresponds to a slice of the negative attractor in bright red in Figure 6.

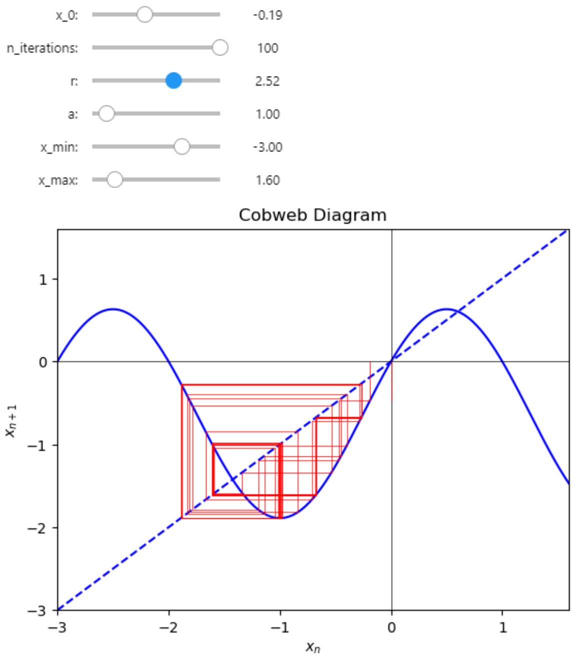



Figure 14: Negative values on the Cobweb Diagram

The Cobweb Diagram for the (a=1) Cosine Map after a negative initialisation with no analysable periodicity.

# 3 Changes in dynamics of the Cosine Map

From Section 1, we see that The Cosine Map is similar to the Lorenz map (equal when $a = 1$). In this section, we will discuss 2 changes in the dynamics of the Cosine Map

1. Changes in the map with a fixed $a$ (say $a = 1$)
2. Changes in the map when varying $a$

## 3.1 Changes in map with fixed $a$

When $a = 1$, the dynamics of the map exhibit similarities to maps we have encountered previously. This similarity arises from the fact that the Cosine Map approaches the Logistic Map when $a \to \infty$ when $x_0$ is positive.

The dynamics of this map include:

1. Period doubling Route to chaos, as described in section 2.3.1. The map exhibits a route to chaos where stable points undergo bifurcations that produce a period-doubling an unstable point. This property is common among some of the maps we have seen; notably the Ikeda system, and the Rossler Equations.
2. Crisis induced intermittency, as described in section 2.3.3. Windows of periodicity appear within the chaotic regimes. This is also occurs in the Ikeda and Rossler systems
3. Self-similarity is a key property of the Cosine Map as the period doubling route to chaos can be found at all length scales. This also seems to be a characteristic of other systems that exhibit periodic windows.
4. Lastly, we have not touched on it yet, but the system appears to exhibit high-density curves within the attractor, that appear to be remnants of periodic orbits. We also find them in the maps spoken about.
5. Finally, as exhibited by Figures 3 and 4, the Orbit Diagrams of the Cosine Map and Logistic Map look identical for $0 \leq r \leq 4.0$ and $0 \leq x \leq 1$. Therefore, they exhibit similarities in all of the above. Therefore, it appears that the maps we have encountered have similar properties. Maybe it can be deduced that there exists a set of defining characteristics of non-linear systems.

## 3.2 Changes in map as $a$ changes

For interest sake, let's look at the dynamics of the map as we vary the parameter $a$.

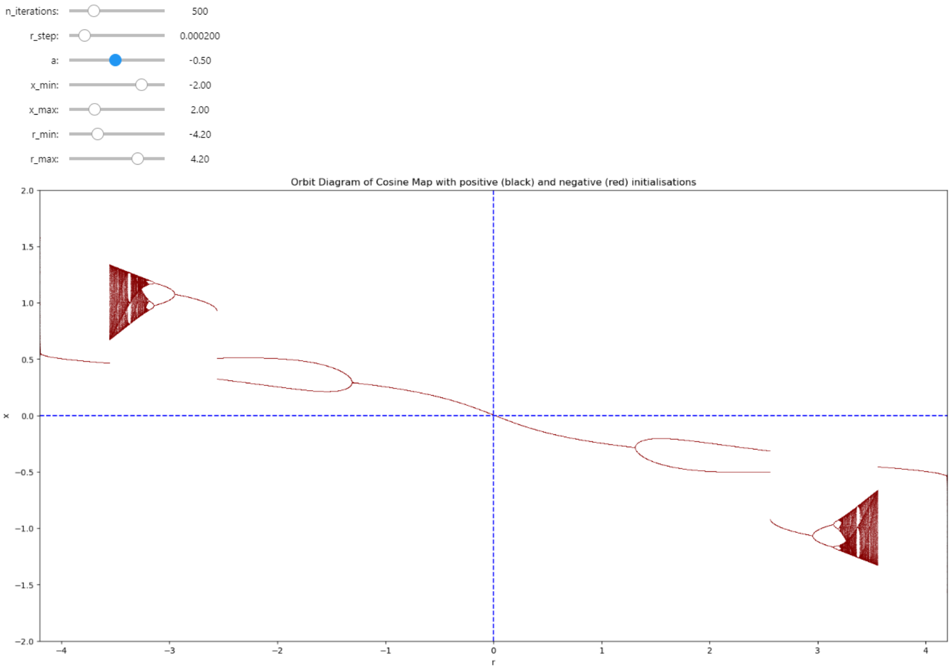
In Figure 16, we plot multiple values of $a$ on a similar axis. We note that values of $a < -0.5$ result in the map being undefined.

Here are some interesting comments:

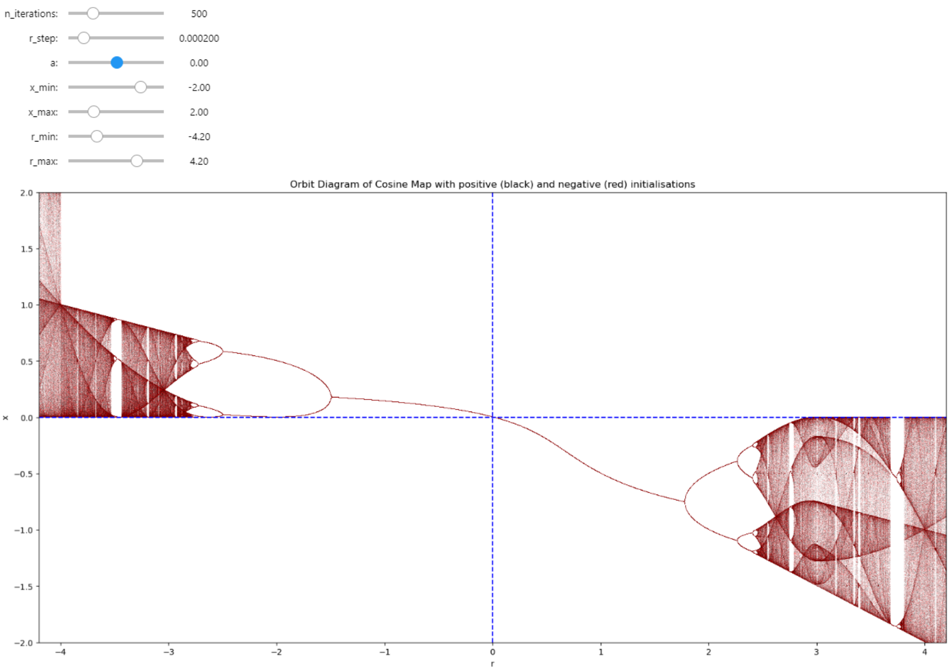
1. The difference in map dynamics between negative and positive initialisations only exist for the special case we examined, $a = 0$.
2. The has different dislocations in fixed points depending on the value of $a$.
3. The overall density distribution of the orbit diagram seems to undergo a transformation akin to compression along the $r$ axis and expansion along the $x$ axis, under an increasing $a$
4. The density regions cross over with respect to the $r$ axis as $a$ increases

Discuss the changes in the dynamics of the cosine–map. How typical are these changes compared to other non–linear systems you know?

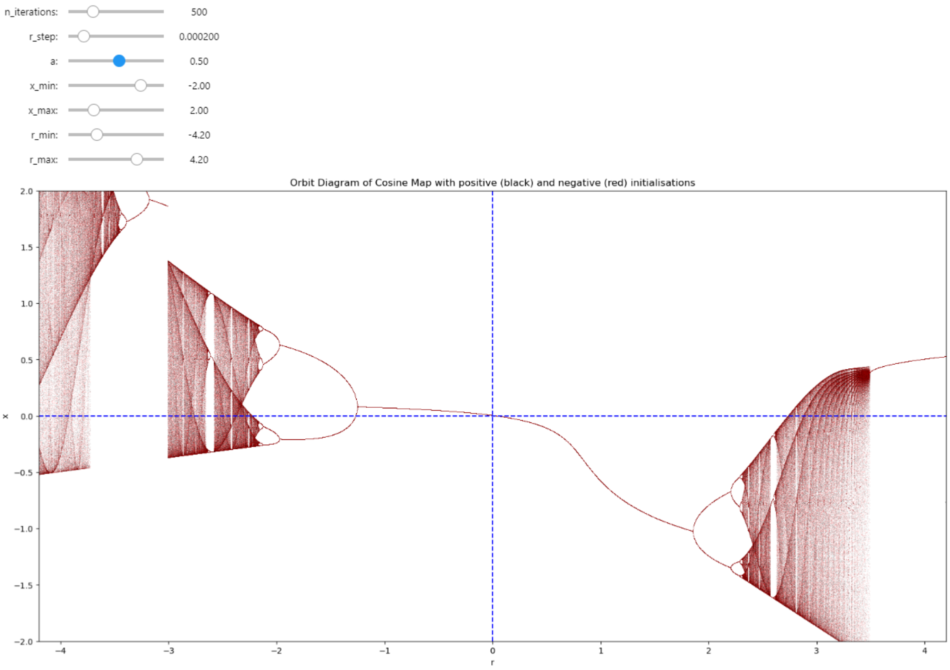
Changes in dynamics - period doubling route to chaos


n_iterations: 500
r_step: 0.000200
a: -0.50
x_min: -2.00
x_max: 2.00
r_min: -4.20
r_max: 4.20
Orbit Diagram of Cosine Map with positive (black) and negative (red) initialisations
x
r

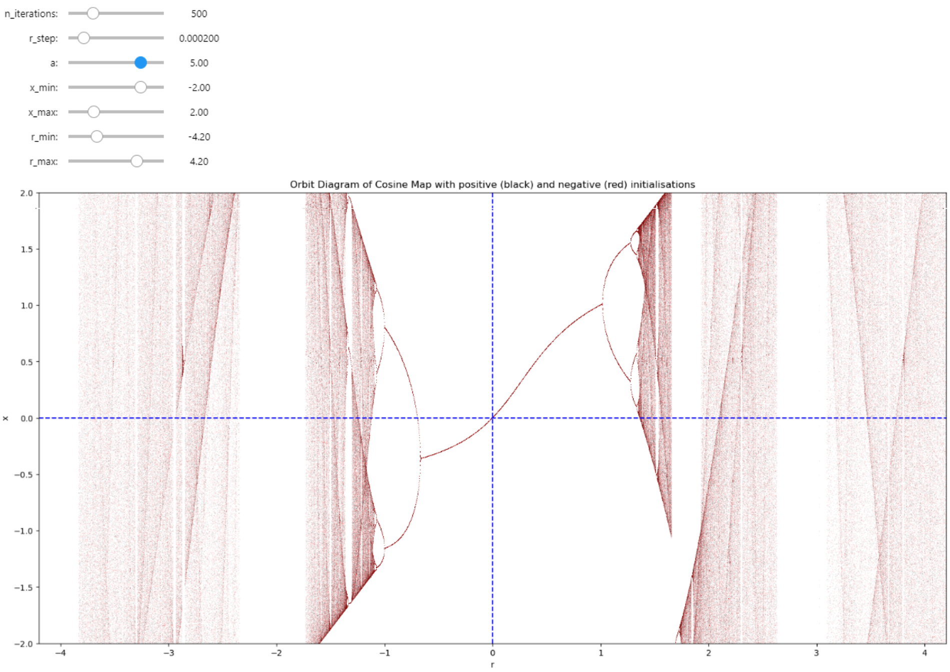
(a) $a = -0.5$


n_iterations: 500
r_step: 0.000200
a: 0.00
x_min: -2.00
x_max: 2.00
r_min: -4.20
r_max: 4.20
Orbit Diagram of Cosine Map with positive (black) and negative (red) initialisations
x
r

(b) $a = 0$




(a) $a = 0.5$




(b) $a = 5$

Figure 16: Cosine Map Orbit Diagram as $a$ varies

The Cosine Map Orbit Diagram for different values of $a$. Positive and negative initializations are non-distinguishable.