In-Class Problem Solutions for Session 7

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Contents

1 Problem 1

R is	iff	R^{-1} is
total		a surjection
a function		an injection
a surjection		total
an injection		a function
a bijection		a bijection

2 Problem 2

Let $A = \{a_0, a_1, ..., a_{n-1}\}$ be a set of size n, and and $B = \{b_0, b_1, ..., b_{m-1}\}$ be a set of size m. Prove that

$$|A \times B| = mn \tag{1}$$

Proof. Let

$$\begin{aligned} &f((a_i, b_j)) ::= mi + j \\ &C ::= \{\ z \in \mathbb{N} \mid 0 \leq z \leq (mn - 1)\ \} \end{aligned}$$

For every $c \in C$, there is exactly 1 $s \in A \times B$ such that $f(s) = c$. Therefore, f is a bijection from $A \times B$ to C. From part 3 of Lemma 4.5.3 on page 116 of the book, it follows that

$$|A \times B| = |C| = mn$$

Hence, (1) must be true. $\square$

3 Problem 3

3.1 (a)

$$|f(A)| \leq |B|$$

3.2 (b)

$$|A| \geq |B|$$

3.3 (c)

$$|f(A)| = |B|$$

3.4 (d)

$$|f(A)| = |A|$$

3.5 (e)

$$|A| = |B|$$

4 Problem 4

Prove that

$$|X| \geq |R(X)| \tag{2}$$

Proof. Let n be the number of arrows that originate from an element of X. Since R is a function, at most 1 arrow originates from each member of X. Therefore,

$$|X| \geq n \tag{3}$$

From the definition of image, we know that $\forall r \in R(x)$ there exists an arrow originating from a member of X that terminates at r. Therefore,

$$|R(X)| \leq n \tag{4}$$

From (3) and (4), it follows that

$$|X| \geq n \geq |R(X)|$$

Hence, (2) must be true.

$\square$

5 Problem 5

5.1 (a)

Prove that

$$A \text{ surj } B \wedge B \text{ surj } C \implies A \text{ surj } C \tag{5}$$

Proof.

$$\begin{aligned} A \text{ surj } B &\implies \exists \text{ a surjective function } f : A \to B \\ B \text{ surj } C &\implies \exists \text{ a surjective function } g : B \to C \end{aligned}$$

Since g is a surjective function, $\forall c \in C\ \exists b \in B.g(b) = c$. Since f is a surjective function, $\forall b \in B\ \exists a \in A.f(a) = b$. Therefore, $\forall c \in C\ \exists a \in A.g(f(a)) = c$. In other words, $g \circ f : A \to C$ is a surjective function. Hence, (5) must be true. □

5.2 (b)

Prove that

$$A \text{ surj } B \iff B \text{ inj } A \tag{6}$$

Proof. We will proceed by proving the following lemmas:

$$A \text{ surj } B \implies B \text{ inj } A \tag{7}$$
$$B \text{ inj } A \implies A \text{ surj } B \tag{8}$$

Proof of (7). The proof is by contradiction. Suppose A surj B is true but B inj A is false. Then $\exists$ a surjective function $f : A \to B$.

Suppose that $f^{-1}B \to A$ isn't total. Then $\exists b \in B$ such that no arrows of f^{-1} originate from b. However, this would imply that no arrows of f terminate at b. Therefore, we've reached a contradiction where f isn't surjective. Hence, f^{-1} must be total.

Suppose that $f^{-1}B \to A$ isn't injective. Then $\exists a \in A$ such that more than 1 arrow of f^{-1} terminate at a. However, this would imply that more than 1 arrow of f originates at a. Therefore, we've reached a contradiction where f isn't a function. Hence, f^{-1} must be injective.

Since f^{-1} is both total and injective, f^{-1} is an injective total relation. Therefore, we've reached a contradiction where B inj A is true. Hence, (7) must be true. ■

Proof of (8). The proof is by contradiction. Suppose that B inj A is true but A surj B is false. Then $\exists$ an injective total relation $g : B \to A$.

Suppose that $g^{-1} : A \to B$ isn't surjective. Then $\exists b \in B$ such that no arrows of g^{-1} terminate at B. However, this would imply that no arrows of g originate at b. Therefore, we've reached a contradiction where g isn't total. Hence, g^{-1} must be surjective.

Suppose that g^{-1} isn't a function. Then $\exists a \in A$ such that more than 1 arrow of g^{-1} originates at a. However, this would imply that more than 1 arrow of g terminates at a. Therefore, we've reached a contradiction where g isn't injective. Hence, g^{-1} must be a function.

Since g^{-1} is both surjective and a function, g^{-1} is a surjective function. Therefore, we've reached a contradiction where A surj B is true. Hence, (8) must be true. ■

We've proved that both (7) and (8) are true. Hence, (6) must be true. □

5.3 (c)

Prove that

$$A \text{ inj } B \land B \text{ inj } C \implies A \text{ inj } C \tag{9}$$

Proof.

$$\begin{aligned}
&A \text{ inj } B \land B \text{ inj } C && \iff \\
&B \text{ surj } A \land C \text{ surj } B && \text{(apply (6))} \implies \\
&C \text{ surj } A && \text{(apply (5))} \iff \\
&A \text{ inj } C && \text{(apply (6))}
\end{aligned}$$

Hence, (9) must be true. □

5.4 (d)

Prove that

$$A \text{ inj } B \iff \exists \text{ a total injective function f} : A \to B \tag{10}$$

Proof. From part 2 of Definition 4.5.2, we can rewrite (10) as

$$\begin{aligned}
&\exists \text{ a total injective relation g} : A \to B \iff \\
&\exists \text{ a total injective function f} : A \to B
\end{aligned} \tag{11}$$

We will proceed by proving the following lemmas:

$$\begin{aligned}
&\exists \text{ a total injective function f} : A \to B \implies \\
&\exists \text{ a total injective relation g} : A \to B
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\exists \text{ a total injective relation g} : A \to B \implies \\
&\exists \text{ a total injective function f} : A \to B
\end{aligned} \tag{13}$$

Proof of Lemma (12). (12) is trivially true since a function is a type of relation. In other words, f is also a relation. ■

Proof of Lemma (13). f can be constructed from g by eliminating arrows of g until there is exactly one arrow originating from every $a \in A$. Hence, (13) must be true. ■

We've proved that (12) and (13) are true. Hence, (11) must be true and (10) must be true.

□