In-Class Problem Solutions for Session 6

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1 Problem 1

1.1 (a)

Prove that

$$(P \wedge \neg Q) \vee (P \wedge Q) \equiv P \tag{1}$$

Proof. The proof is by truth table.

P	Q	$(P \wedge \neg Q)$	$\vee$	$(P \wedge Q)$
T	T	F	**T**	T
T	F	T	**T**	F
F	T	F	**F**	F
F	F	F	**F**	F

The truth table for $(P \wedge \neg Q) \vee (P \wedge Q)$ is identical to the truth table for P. Hence, (1) must be true. □

1.2 (b)

Prove that

$$A = (A - B) \cup (A \cap B) \tag{2}$$

Proof.

$$\begin{aligned}
&x \in (A - B) \cup (A \cap B) && \iff \\
&x \in (A - B) \vee x \in (A \cap B) && \iff \\
&(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in B) && \iff \\
&x \in A && \text{(apply (1) taking } P ::= x \in A, Q ::= x \in B\text{)}
\end{aligned}$$

Hence, (2) must be true. □

2 Problem 2

2.1 (a)

$$(x = \emptyset) ::= \forall z.(z \notin x)$$

2.2 (b)

$$(x = \{y, z\}) ::= \forall a.(a \in x \iff a \in \{y, z\})$$

2.3 (c)

$$(x \subseteq y) ::= \forall z.(z \in x \implies z \in y)$$

2.4 (d)

$$(x = y \cup z) ::= \forall a.(a \in x \iff a \in y \vee a \in z)$$

2.5 (e)

$$(x = y - z) ::= \forall a.(a \in x \iff a \in y \wedge a \notin z)$$

2.6 (f)

$$(x = \mathrm{pow}(y)) ::= \forall z.(z \in x \iff z \subseteq y)$$

2.7 (g)

$$(x = \bigcup_{z \in y} z) ::= \forall a.(a \in x \iff \exists z \in y.(a \in z))$$

3 Problem 3

3.1 (a)

Representing (a, b) by $\{a, b\}$ doesn't work because distinct pairs would be represented by the same set. E.g. $(1, 2)$ and $(2, 1)$ would both be represented by $\{1, 2\}$.

3.2 (b)

Representing (a, b) by $\{a, \{b\}\}$ also doesn't work because distinct pairs would be represented by the same set. E.g. $(\{1\}, 2)$ and $(\{2\}, 1)$ would both be represented by $\{\{1\}, \{2\}\}$.

3.3 (c)

$pair(a, b)$ uniquely determines (a, b) because this particular combination of a scalar and a set element removes any ambiguity regarding which item comes first. a, the first item, appears in both the scalar and set elements. b, the second item, only appears in the set element.

4 Problem 4

The second player still has a winning strategy when A has four elements since the second player can always choose the complement of whatever subset the first player chooses on his/her first turn. Consider the following cases:

1. Player 1 chooses a subset containing 1 element of A on his/her first turn.

2. Player 1 chooses a subset containing 2 elements of A on his/her first turn.

3. Player 1 chooses a subset containing 3 elements of A on his/her first turn.

Case 1: Player 2 chooses the subset containing the remaining 3 elements of A on his/her first turn. Player 1 has no legal moves on his/her second turn so Player 2 wins.

Case 2: Player 2 chooses the subset containing the remaining 2 elements of A on his/her first turn. Player 1 has no legal moves on his/her second turn so Player 2 wins.

Case 3: Player 2 chooses the subset containing the remaining element of A on his/her first turn. Player 1 has no legal moves on his/her second turn so Player 2 wins.