# In-Class Problem Solutions for Session 2

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# 1 Problem 1

Prove that if $a \cdot b = n$, then either $a$ or $b$ must be $\leq \sqrt{n}$ where $a$, $b$, and $c$ are nonnegative real numbers.

*Proof.* We use proof by contradiction. Suppose $a \cdot b = n$ but $a > \sqrt{n}$ and $b > \sqrt{n}$. Then

$$a \cdot b > \sqrt{n}\sqrt{n} = n$$

This is a contradiction. Hence, the claim must be true. $\square$

## 2 Problem 2

We can generalize Theorem 1.8.1 to any prime number $p$. However, we should first prove a more general version of Problem 14 from Chapter 1.

Given a prime number $p$, prove that if $n^2$ is a multiple of $p$, then $n$ must be a multiple of $p$.

*Proof.* We use proof by contradiction. Suppose $n^2$ is a multiple of $p$ but $n$ is not a multiple of $p$. Therefore, $p$ is not a factor of $n$. Since $p$ is prime, we can deduce by the *Fundamental Theorem of Arithmetic* that $p$ is not a factor of $n^2$. This leads us to the contradiction that $n^2$ is not a multiple of $p$. Hence, the claim must be true. □

With that proof out of the way, we are now ready to generalize Theorem 1.8.1.

Given a prime number $p$, prove that $\sqrt{p}$ is irrational.

*Proof.* We use proof by contradiction. Suppose that $\sqrt{p}$ is ration. Then $\sqrt{p} = \frac{n}{d}$ where $n$ and $d$ are integers such that $d > 0$ and $n$ and $d$ have no common factors. Consider the following:

$$p = \frac{n^2}{d^2}$$

$$pd^2 = n^2$$

So $p$ is a factor of $n^2$. From our previous proof, we know that this is only possible if $p$ is also a factor of $n$. Therefore, $n = pk$ for some nonzero integer $k$ and:

$$n^2 = (pk)^2 = p^2k^2$$

$$pd^2 = p^2k^2$$

$$d^2 = pk^2$$

I.e. $p$ is a factor of $d^2$. From our previous proof, we know that this is only possible if $p$ is also a factor of $d$. We have reached a contradiction since $n$ and $d$ share a common factor of $p$. Hence, the claim must be true. □

## 3 Problem 3

Prove that raising an irrational number to an irrational power can result in a rational number.

*Proof.* The proof is by case analysis. Let us consider two cases:

1. $\sqrt{2}^{\sqrt{2}}$ is rational.
2. $\sqrt{2}^{\sqrt{2}}$ is irrational.

Since $\sqrt{2}^{\sqrt{2}}$ must be either rational or irrational, we can prove that the claim is true by showing that the claim holds in both cases.

**Case 1:** The claim holds since $\sqrt{2}$ is irrational.

**Case 2:** Consider the following:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^{2} = 2$$

The claim holds since $\sqrt{2}$ is irrational and 2 is rational.

The claim holds in all cases. Hence, we can conclude that the claim is true. $\square$

## 4 Problem 4

Prove that $2\log_2 3$ is irrational.

*Proof.* We use proof by contradiction. Suppose that $\log_2 3$ is rational. Then $\log_2 3 = \frac{n}{d}$ where $n$ and $d$ are integers, $d > 0$, and $n$ and $d$ do not share any common factors. Consider the following:

$$2^{\log_2 3} = 2^{\frac{n}{d}}$$

Apply the definition of the log function:

$$3 = 2^{\frac{n}{d}}$$

$$3^d = 2^n$$

Since both 3 and 2 are prime numbers, we conclude from the *Fundamental Theorem of Arithmetic* that the above equation can only satisfied if both sides equal 1. I.e. $d = n = 0$. This is a contradiction since $d > 0$. Hence, the claim must be true. $\square$