Chapter 1 Section 3 Exercise Solutions

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Contents

1 Exercise 5

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the truth values of the compound propositions $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ agree for all possible combinations of the truth values of p, q, and r, said compound propositions are logically equivalent.

2 Exercise 6

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the truth values of the compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ agree for all possible combinations of the truth values of p and q, said compound propositions are logically equivalent.

3 Exercise 9

3.1 (a)

$$\begin{aligned}
p \implies \neg q &\equiv \\
\neg p \lor \neg q &
\end{aligned}$$

3.2 (b)

$$\begin{aligned}
(p \implies q) \implies r &\equiv \\
\neg(p \implies q) \lor r &\equiv \\
\neg(\neg p \lor q) \lor r &\equiv \\
(p \land \neg q) \lor r &
\end{aligned}$$

3.3 (c)

$$\begin{aligned}
(\neg q \implies p) \implies (p \implies \neg q) &\equiv \\
\neg(\neg q \implies p) \lor (p \implies \neg q) &\equiv \\
\neg(q \lor p) \lor (\neg p \lor \neg q) &\equiv \\
(\neg q \land \neg p) \lor (\neg p \lor \neg q) &\equiv \\
(\neg p \land \neg q) \lor (\neg p \lor \neg q) &\equiv \\
((\neg p \land \neg q) \lor \neg p) \lor \neg q &\equiv \\
(\neg p \lor (\neg p \land \neg q)) \lor \neg q &\equiv \\
\neg p \lor \neg q &
\end{aligned}$$

4 Exercise 10

4.1 (a)

$$
\begin{aligned}
\neg p \implies \neg q &\equiv \\
\neg(\neg p) \lor \neg q &\equiv \\
p \lor \neg q
\end{aligned}
$$

4.2 (b)

$$
\begin{aligned}
p \lor q \implies \neg p &\equiv \\
\neg(p \lor q) \lor \neg p &\equiv \\
(\neg p \land \neg q) \lor \neg p &\equiv \\
\neg p \lor (\neg p \land \neg q) &\equiv \\
\neg p
\end{aligned}
$$

4.3 (c)

$$
\begin{aligned}
(p \implies \neg q) \implies (\neg p \implies q) &\equiv \\
\neg(p \implies \neg q) \lor (\neg p \implies q) &\equiv \\
\neg(\neg p \lor \neg q) \lor (p \lor q) &\equiv \\
(p \land q) \lor (p \lor q) &\equiv \\
((p \land q) \lor p) \lor q &\equiv \\
(p \lor (p \land q)) \lor q &\equiv \\
p \lor q
\end{aligned}
$$

5 Exercise 15

5.1 (a)

$$\begin{aligned}
&(p \land q) \implies p &\equiv\\
&\neg(p \land q) \lor p &\equiv\\
&(\neg p \lor \neg q) \lor p &\equiv\\
&\neg q \lor (p \lor \neg p) &\equiv\\
&\neg q \lor T &\equiv\\
&T
\end{aligned}$$

5.2 (b)

$$\begin{aligned}
&p \implies (p \lor q) &\equiv\\
&\neg p \lor (p \lor q) &\equiv\\
&q \lor (p \lor \neg p) &\equiv\\
&q \lor T &\equiv\\
&T
\end{aligned}$$

5.3 (c)

$$\begin{aligned}
&\neg p \implies (p \implies q) &\equiv\\
&\neg p \implies (\neg p \lor q) &\equiv\\
&p \lor (\neg p \lor q) &\equiv\\
&q \lor (p \lor \neg p) &\equiv\\
&q \lor T &\equiv\\
&T
\end{aligned}$$

5.4 (d)

$$
\begin{aligned}
(p \wedge q) \implies (p \implies q) & \quad \equiv \\
(p \wedge q) \implies (\neg p \vee q) & \quad \equiv \\
\neg(p \wedge q) \vee (\neg p \vee q) & \quad \equiv \\
(\neg p \vee \neg q) \vee (\neg p \vee q) & \quad \equiv \\
(\neg p \vee \neg p) \vee (q \vee \neg q) & \quad \equiv \\
\neg p \vee T & \quad \equiv \\
T &
\end{aligned}
$$

5.5 (e)

$$
\begin{aligned}
\neg(p \implies q) \implies p & \quad \equiv \\
\neg(\neg p \vee q) \implies p & \quad \equiv \\
(\neg p \vee q) \vee p & \quad \equiv \\
q \vee (p \vee \neg p) & \quad \equiv \\
q \vee T & \quad \equiv \\
T &
\end{aligned}
$$

5.6 (f)

$$
\begin{aligned}
\neg(p \implies q) \implies \neg q & \quad \equiv \\
\neg(\neg p \vee q) \implies \neg q & \quad \equiv \\
(\neg p \vee q) \vee \neg q & \quad \equiv \\
\neg p \vee (q \vee \neg q) & \quad \equiv \\
\neg p \vee T & \quad \equiv \\
T &
\end{aligned}
$$

6 Exercise 16

6.1 (a)

$$\begin{aligned}
&[\neg p \wedge (p \vee q)] \implies q && \equiv \\
&\neg[\neg p \wedge (p \vee q)] \vee q && \equiv \\
&p \vee \neg(p \vee q) \vee q && \equiv \\
&p \vee (\neg p \wedge \neg q) \vee q && \equiv \\
&(p \vee q) \vee (\neg p \wedge \neg q) && \equiv \\
&((p \vee q) \vee \neg p) \wedge ((p \vee q) \vee \neg q) && \equiv \\
&(q \vee (p \vee \neg p)) \wedge (p \vee (q \vee \neg q)) && \equiv \\
&(q \vee T) \wedge (p \vee T) && \equiv \\
&T \wedge T && \equiv \\
&T
\end{aligned}$$

6.2 (b)

$$\begin{aligned}
&[(p \implies q) \wedge (q \implies r)] \implies (p \implies r) && \equiv \\
&\neg[(p \implies q) \wedge (q \implies r)] \vee (p \implies r) && \equiv \\
&\neg[(\neg p \vee q) \wedge (\neg q \vee r)] \vee (\neg p \vee r) && \equiv \\
&[\neg(\neg p \vee q) \vee \neg(\neg q \vee r)] \vee (\neg p \vee r) && \equiv \\
&[(p \wedge \neg q) \vee (q \wedge \neg r)] \vee (\neg p \vee r) && \equiv \\
&[((p \wedge \neg q) \wedge q) \vee ((p \vee \neg q) \wedge \neg r)] \vee (\neg p \vee r) && \equiv \\
&[(p \wedge (q \wedge \neg q)) \vee ((p \vee \neg q) \wedge \neg r)] \vee (\neg p \vee r) && \equiv \\
&[(p \wedge F) \vee (\neg r \wedge (p \vee \neg q))] \vee (\neg p \vee r) && \equiv \\
&[F \vee ((\neg r \wedge p) \vee (\neg r \wedge \neg q))] \vee (\neg p \vee r) && \equiv \\
&((\neg r \wedge p) \vee (\neg r \wedge \neg q)) \vee (\neg p \vee r) && \equiv \\
&((\neg p \vee r) \vee (\neg r \wedge p)) \vee (\neg r \wedge \neg q) && \equiv \\
&(((\neg p \vee r) \vee \neg r) \wedge ((\neg p \vee r) \vee p)) \vee (\neg r \wedge \neg q) && \equiv \\
&((\neg p \vee T) \wedge (r \vee T)) \vee (\neg r \wedge \neg q) && \equiv \\
&(T \wedge T) \vee (\neg r \wedge \neg q) && \equiv \\
&(\neg r \wedge \neg q) \vee T && \equiv \\
&T
\end{aligned}$$

6.3 (c)

$$
\begin{aligned}
&[p \land (p \implies q)] \implies q && \equiv \\
&\neg[p \land (\neg p \lor q)] \lor q && \equiv \\
&[\neg p \lor \neg(\neg p \lor q)] \lor q && \equiv \\
&[\neg p \lor (p \land \neg q)] \lor q && \equiv \\
&(\neg p \lor q) \lor (p \land \neg q) && \equiv \\
&[((\neg p \lor q) \lor p) \land ((\neg p \lor q) \lor \neg q)] && \equiv \\
&[(q \lor T) \land (\neg p \lor T)] && \equiv \\
&T \land T \equiv \\
&T
\end{aligned}
$$

6.4 (d)

$$
\begin{aligned}
&[(p \lor q) \land (p \implies r) \land (q \implies r)] \implies r && \equiv \\
&\neg[(p \lor q) \land (\neg p \lor r) \land (\neg q \lor r)] \lor r && \equiv \\
&[\neg(p \lor q) \lor \neg(\neg p \lor r) \lor \neg(\neg q \lor r)] \lor r && \equiv \\
&[(\neg p \land \neg q) \lor (p \land \neg r) \lor (q \land \neg r)] \lor r && \equiv \\
&[(\neg p \land \neg q) \lor (p \land \neg r)] \lor [r \lor (q \land \neg r)] && \equiv \\
&[(\neg p \land \neg q) \lor (p \land \neg r)] \lor [(r \lor q) \land (r \lor \neg r)] && \equiv \\
&[(\neg p \land \neg q) \lor (p \land \neg r)] \lor (r \lor q) && \equiv \\
&[(r \lor q) \lor (\neg p \land \neg q)] \lor (p \land \neg r) && \equiv \\
&[((r \lor q) \lor \neg p) \land ((r \lor q) \lor \neg q)] \lor (p \land \neg r) && \equiv \\
&((r \lor q) \lor \neg p) \lor (p \land \neg r) && \equiv \\
&(((r \lor q) \lor \neg p) \lor p) \land (((r \lor q) \lor \neg p) \lor \neg r) && \equiv \\
&((r \lor q) \lor (p \lor \neg p)) \land ((\neg p \lor q) \lor (r \lor \neg r)) && \equiv \\
&((r \lor q) \lor T) \land ((\neg p \lor q) \lor T) && \equiv \\
&T \land T && \equiv \\
&T
\end{aligned}
$$

7 Exercise 17

7.1 (a)

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	**T**
T	F	F	**T**
F	T	F	**F**
F	F	F	**F**

Since the truth values of $p \vee (p \wedge q)$ and p agree for all possible combinations of truth values for p and q, $p \vee (p \wedge q)$ and p are logically equivalent. I.e. $p \vee (p \wedge q) \equiv p$ is true.

7.2 (b)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**F**
F	F	F	**F**

Since the truth values of $p \wedge (p \vee q)$ and p agree for all possible combinations of truth values for p and q, $p \wedge (p \vee q)$ and p are logically equivalent. I.e. $p \wedge (p \vee q) \equiv p$ is true.

8 Exercise 18

$$
\begin{aligned}
(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q && \equiv \\
\neg(\neg p \wedge (\neg p \vee q)) \vee \neg q && \equiv \\
(p \vee \neg(\neg p \vee q)) \vee \neg q && \equiv \\
(p \vee (p \wedge \neg q)) \vee \neg q && \equiv \\
p \vee \neg q && \equiv \\
\neg p \rightarrow \neg q
\end{aligned}
$$

Hence, $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is not a tautology.

9 Exercise 19

$$
\begin{aligned}
(\neg q \wedge (p \to q)) \to \neg q && \equiv \\
\neg(\neg q \wedge (\neg p \vee q)) \vee \neg q && \equiv \\
(q \vee \neg(\neg p \vee q)) \vee \neg q && \equiv \\
(q \vee (p \wedge \neg q)) \vee \neg q && \equiv \\
(p \wedge \neg q) \vee (q \vee \neg q) && \equiv \\
(p \wedge \neg q) \vee T && \equiv \\
T &&
\end{aligned}
$$

Hence, $(\neg q \wedge (p \to q)) \to \neg q$ is a tautology.

10 Exercise 20

Let

$$p \leftrightarrow q \tag{1}$$
$$(p \wedge q) \vee (\neg p \wedge \neg q) \tag{2}$$

p	q	(1)	$p \wedge q$	$\neg p \wedge \neg q$	(2)
T	T	**T**	T	F	**T**
T	F	**F**	F	F	**F**
F	T	**F**	F	F	**F**
F	F	**T**	F	T	**T**

Since the truth values of (1) and (2) agree for all possible combinations of truth values for p and q, (1) and (2) are logically equivalent.

11 Exercise 21

Let

$$\neg(p \leftrightarrow q) \tag{3}$$
$$p \leftrightarrow \neg q \tag{4}$$

p	q	$p \leftrightarrow q$	(3)	$\neg q$	(4)
T	T	T	**F**	F	**F**
T	F	F	**T**	T	**T**
F	T	F	**T**	F	**T**
F	F	T	**F**	T	**F**

Since the truth values of (3) and (4) agree for all possible combinations of truth values for p and q, (3) and (4) are logically equivalent.

12 Exercise 22

Let

$$p \to q \tag{5}$$
$$\neg q \to \neg p \tag{6}$$

p	q	(5)	$\neg q$	$\neg p$	(6)
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

Since the truth values of (5) and (6) agree for all possible combinations of truth values for p and q, (5) and (6) are logically equivalent.

13 Exercise 23

Let

$$\neg p \leftrightarrow q \tag{7}$$

$$p \leftrightarrow \neg q \tag{8}$$

p	q	$\neg p$	(7)	$\neg q$	(8)
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	T	F	T
F	F	T	F	T	F

Since the truth values of (7) and (8) agree for all possible combinations of truth values for p and q, (7) and (8) are logically equivalent.

14 Exercise 24

$\neg(p \oplus q)$ is true when $p \oplus q$ is false, which means that p and q share the same truth value. This is exactly when $p \leftrightarrow q$ is true. Hence, $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent.

15 Exercise 25

$\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false, which means that p and q have different truth values. This is exactly when $\neg p \leftrightarrow q$ is true. Hence, $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

16 Exercise 26

$(p \to q) \wedge (p \to r)$ is true when both $(p \to q)$ and $(p \to r)$ are true, which means either $p = F$ or both $q = T$ and $r = T$. This is exactly when $p \to (q \wedge r)$ is true. Hence, $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

17 Exercise 27

$(p \to r) \wedge (q \to r)$ is true when both $(p \to r)$ and $(q \to r)$ are true, which means either $r = T$ or both $p = F$ and $q = F$. This is exactly when $(p \vee q) \to r$ is true. Hence, $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent.

18 Exercise 28

$(p \to q) \vee (p \to r)$ is true when either $(p \to q)$ or $(p \to r)$ is true, which means either $p = F$, $q = T$, or $r = T$. This is exactly when $p \to (q \vee r)$ is true. Hence, $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

19 Exercise 29

$(p \to r) \lor (q \to r)$ is true when either $(p \to r)$ or $(q \to r)$ is true, which means either $p = F$, $q = F$, or $r = T$. This is exactly when $(p \land q) \to r$ is true. Hence, $(p \to r) \lor (q \to r)$ and $(p \land q) \to r$ are logically equivalent.

20 Exercise 30

$\neg p \rightarrow (q \rightarrow r)$ is true when either $\neg p$ is false or $(q \rightarrow r)$ is true, which means that either $p = T$, $q = F$, or $r = T$. This is exactly when $q \rightarrow (p \vee r)$ is true. Hence, $\neg p \rightarrow (q \rightarrow r)$ and $q \implies (p \vee r)$ are logically equivalent.

21 Exercise 31

$p \leftrightarrow q$ is true when p and q share the same truth value. This is exactly when $(p \to q) \wedge (q \to p)$ is true. Hence, $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are logically equivalent.

22 Exercise 32

$p \leftrightarrow q$ is true when p and q share the same truth value. This is exactly when $\neg p \leftrightarrow \neg q$ is true. Hence, $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent.

23 Exercise 34

$$
\begin{aligned}
&(p \lor q) \land (\neg p \lor r) \to (q \lor r) &\equiv\\
&\neg[(p \lor q) \land (\neg p \lor r)] \lor (q \lor r) &\equiv\\
&[\neg(p \lor q) \lor \neg(\neg p \lor r)] \lor (q \lor r) &\equiv\\
&[(\neg p \land \neg q) \lor (p \land \neg r)] \lor (q \lor r) &\equiv\\
&(\neg p \land \neg q) \lor [(q \lor r) \lor (p \land \neg r)] &\equiv\\
&(\neg p \land \neg q) \lor [((q \lor r) \lor p) \land ((q \lor r) \lor \neg r)] &\equiv\\
&(\neg p \land \neg q) \lor [((q \lor r) \lor p) \land T] &\equiv\\
&((q \lor r) \lor p) \lor (\neg p \land \neg q) &\equiv\\
&(((q \lor r) \lor p) \lor \neg p) \land (((q \lor r) \lor p) \lor \neg q) &\equiv\\
&T \land T &\equiv\\
&T
\end{aligned}
$$