



— Page references correspond to locations of Extra Examples icons in the textbook.

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**p.238, icon at Example 2**

**#1.** Certain rules allow us to determine by inspection when a positive integer  $n$  is divisible by a positive integer  $k$ . For example,  $5|n$  if and only if  $n$  ends in the digit 5 or 0. Similarly,  $2|n$  if and only if  $n$  ends in one of the digits 0, 2, 4, 6, 8.

There is also a rule to determine divisibility by 3:

$$3|n \text{ if and only if the sum of the digits in } n \text{ is divisible by 3.}$$

For example  $3|478,125$  because the sum of the six digits in 478,125 is 27, which is divisible by 3. Why does the rule work?

**Solution:**

We want to show that if  $n$  is written as the string of digits  $a_k a_{k-1} \dots a_2 a_1 a_0$ , then  $3|n$  if and only if  $3|(a_k + a_{k-1} + \dots + a_2 + a_1 + a_0)$ .

Begin by writing

$$\begin{aligned} n &= a_k 10^k + a_{k-1} 10^{k-1} + \dots + a_2 10^2 + a_1 10 + a_0 \\ &= [(10^k - 1) + 1]a_k + [(10^{k-1} - 1) + 1]a_{k-1} + \dots + [(10^2 - 1) + 1]a_2 + [(10 - 1) + 1]a_1 + a_0 \\ &= \underbrace{[(10^k - 1)a_k + (10^{k-1} - 1)a_{k-1} + \dots + (10^2 - 1)a_2 + (10 - 1)a_1]}_A + \underbrace{[a_k + a_{k-1} + \dots + a_2 + a_1 + a_0]}_B. \end{aligned}$$

Note that in  $A$  each summand is of the form  $(10^i - 1)a_i$  and is divisible by 3 because each coefficient is a power of 10, with 1 subtracted, that is, each summand is a string of 9's. Therefore  $A$  is divisible by 3 because  $A$  is the sum of numbers of this form.

Now suppose that  $3|n$ . Write  $n - A = B$ . The numbers  $n$  and  $A$  are both divisible by 3. Therefore  $B$  is divisible by 3 because  $B$  is the difference of two numbers divisible by 3. But  $B$  is the sum of the digits in  $n$ . Therefore, if  $3|n$ , then the sum of the digits of  $n$  is divisible by 3.

Conversely, suppose 3 divides the sum of the digits in  $n$ . Then  $3|(a_k + a_{k-1} + \dots + a_2 + a_1 + a_0)$ , which is  $B$ . But we already know that  $3|A$ . Therefore 3 divides their sum  $A + B$ , which is  $n$ .

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**p.238, icon at Example 2**
**#2.**

- (a) Find the number of positive integer divisors of  $648 = 2^3 3^4$ .
- (b) Find the sum of all positive integer divisors of 648.

**Solution:**

(a) Each divisor must have the form  $2^i 3^j$  where  $0 \leq i \leq 3$  and  $0 \leq j \leq 4$ . Hence, there are  $4 \cdot 5 = 20$  divisors of 648.

(b) The sum of the divisors is

$$\begin{aligned}
& 2^0 3^0 + 2^0 3^1 + 2^0 3^2 + 2^0 3^3 + 2^0 3^4 + \\
& 2^1 3^0 + 2^1 3^1 + 2^1 3^2 + 2^1 3^3 + 2^1 3^4 + \\
& 2^2 3^0 + 2^2 3^1 + 2^2 3^2 + 2^2 3^3 + 2^2 3^4 + \\
& 2^3 3^0 + 2^3 3^1 + 2^3 3^2 + 2^3 3^3 + 2^3 3^4 = \\
& 2^0(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
& 2^1(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
& 2^2(3^0 + 3^1 + 3^2 + 3^3 + 3^4) + \\
& 2^3(3^0 + 3^1 + 3^2 + 3^3 + 3^4) = \\
& 2^0 \cdot 121 + 2^1 \cdot 121 + 2^2 \cdot 121 + 2^3 \cdot 121 = 15 \cdot 121 = 1815.
\end{aligned}$$

If you are familiar with sigma notation (covered in Section 2.4), this summation process can be more compactly written as

$$\begin{aligned}
\sum_{i=0}^3 \sum_{j=0}^4 2^i 3^j &= 2^0 \sum_{j=0}^4 3^j + 2^1 \sum_{j=0}^4 3^j + 2^2 \sum_{j=0}^4 3^j + 2^3 \sum_{j=0}^4 3^j \\
&= (2^0 + 2^1 + 2^2 + 2^3) 121 \\
&= 1815.
\end{aligned}$$


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#### p.240, icon at Example 4

#1. For each pair of numbers, when the division algorithm is used to divide  $a$  by  $d$ , what are the quotient  $q$  and remainder  $r$ ?

- (a)  $a = 88$ ,  $d = 11$ .
- (b)  $a = -29$ ,  $d = 9$
- (c)  $a = 58^{237}$ ,  $d = 58^{168}$

#### Solution:

- (a) Because  $88 = 11 \cdot 8 + 0$ , we have  $q = 8$ ,  $r = 0$ . (The fact that  $r = 0$  says that  $11|88$ .)
  - (b) Because  $-29 = 9 \cdot (-4) + 7$ , we have  $q = -4$  and  $r = 7$ . (Note that although we can write  $-29 = 9 \cdot (-3) + (-2)$ , we cannot use  $-2$  as  $r$  because  $r$  is not allowed to be negative.)
  - (c) We do not need to perform the exponentiations to find  $a$  and  $d$ . We need only observe that  $a$  is a multiple of  $d$ :  $58^{237} = 58^{168} \cdot 58^{69}$  (recall the rule for exponents:  $a^b a^c = a^{b+c}$ ). Therefore  $58^{237} = 58^{168} \cdot 58^{69} + 0$  and we have  $q = 58^{69}$  and  $r = 0$ .
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