



— Page references correspond to locations of Extra Examples icons in the textbook.

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**p.140, icon at Example 3**

#1. Determine if the following describes a function with the given domain and codomain.

$f: \mathbf{N} \rightarrow \mathbf{N}$  where  $f(n)$  is equal to the sum of the digits in  $n$ .

**Solution:**

For each input value  $n$  (a nonnegative integer), there is one number that is the sum of the digits of  $n$ . Thus, this is a function.

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**p.140, icon at Example 3**

#2. Determine if each of the following describes a function with the given domain and codomain.

(a)  $f: \mathbf{N} \rightarrow \mathbf{N}$  where  $f(n) = 7 - n$ .

(b)  $f: \mathbf{N} \rightarrow \mathbf{Z}$  where  $f(n) = 7 - n$ .

**Solution:**

(a) This is not a function with codomain  $\mathbf{N}$  because  $f(8) = 7 - 8 = -1$ , which is not an element of  $\mathbf{N}$ .

(b) (Note that we have taken part (a) and changed the codomain.) If we take any natural number and subtract it from 7, we have an integer. Therefore, this is a function.

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**p.140, icon at Example 3**

#3. Determine if each of the following describes a function with the given domain and codomain.

(a)  $f: \mathbf{N} \rightarrow \mathbf{N}$  where  $f(n) = \frac{1}{n-\pi}$ .

(b)  $f: \mathbf{N} \rightarrow \mathbf{R}$  where  $f(n) = \frac{1}{n-\pi}$ .

(c)  $f: \mathbf{R} \rightarrow \mathbf{R}$  where  $f(n) = \frac{1}{n-\pi}$ .

**Solution:**

(a) This is not a function because  $f(0) = -1/\pi$ , which is not a natural number.

(b) This is a function because every input integer produces a real number as output. Note that no integer will produce a 0 in the denominator.

(c) This is not a function because  $f(\pi)$  is not defined. (It yields a denominator of 0.)

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**p.140, icon at Example 3**

#4. Determine if the following describes a function with the given domain and codomain.

$$f: \mathbf{N} \rightarrow \mathbf{N} \text{ where } f(n) = \begin{cases} x+4, & \text{if } n < 7 \\ x^2, & \text{if } n > 11. \end{cases}$$

**Solution:**

This is not a function because  $f(7)$  is not defined (neither case covers the values  $n = 7, 8, 9, 10, 11$ ).

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**p.140, icon at Example 3**

#5. Determine if the following describes a function with the given domain and codomain.

$$f: \mathbf{N} \rightarrow \mathbf{N} \text{ where } f(n) = \begin{cases} x+4, & \text{if } n < 7 \\ x^2, & \text{if } n > 4. \end{cases}$$

**Solution:**

This is not a function because  $f(5)$  is equal to both 9 (using the first case) and 25 (using the second case). Note: some programming languages will accept this as a function by using the first applicable case to define the function; in this case the programming language would give  $f(5) = 5 + 4 = 9$ .

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**p.142, icon at Example 8**

#1. Let  $f: \mathbf{N} \rightarrow \mathbf{Z}$  be defined by the two-part rule  $f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$

Determine whether  $f$  is one-to-one.

**Solution:**

Suppose  $m \neq n$  are two integers. There are three cases to consider, depending on whether  $m$  or  $n$  are even or odd.

Case 1:  $m$  and  $n$  are even. Then  $f(m) = m/2$  and  $f(n) = n/2$ . But  $m/2 \neq n/2$  (because  $m \neq n$ ). Therefore  $f(m) \neq f(n)$ .

Case 2:  $m$  and  $n$  are odd. Then  $f(m) = -(m+1)/2$  and  $f(n) = -(n+1)/2$ . Because  $m \neq n$ ,  $-m \neq -n$ . Therefore  $-m-1 \neq -n-1$ , and  $-(m+1)/2 \neq -(n+1)/2$ . Therefore  $f(m) \neq f(n)$ .

Case 3: one of  $m$  and  $n$  is even and the other is odd. (Say  $m$  is even and  $n$  is odd.) Therefore  $f(m) \geq 0$  and  $f(n) < 0$ , and hence  $f(m) \neq f(n)$ .

These are the only three possibilities. Therefore  $f$  is one-to-one.

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**p.143, icon at Example 12**

#1. Let  $f: \mathbf{N} \rightarrow \mathbf{Z}$  be defined by the two-part rule  $f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ -(n+1)/2, & \text{if } n \text{ is odd.} \end{cases}$

Determine whether  $f$  is onto  $\mathbf{Z}$ .

**Solution:**

Let  $y \in \mathbf{Z}$ . We need to try to find an  $n \in \mathbf{N}$  such that  $f(n) = y$ . There are two cases to consider, depending on whether  $y \geq 0$  or  $y < 0$ .

Case 1:  $y \geq 0$ . Let  $n = 2y$ . Because  $n$  is even we use the first case in the definition of  $f$ . We have  $f(2y) = (2y)/2 = y$ .

Case 2:  $y < 0$ . Let  $n = -2y - 1$ . Then  $f(-2y - 1) = -(-2y - 1 + 1)/2 = -(-2y)/2 = y$ .

Therefore, for each  $y \in \mathbf{Z}$  there is an  $n \in \mathbf{N}$  such that  $f(n) = y$ . Hence  $f$  is onto  $\mathbf{Z}$

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**p.143, icon at Example 12**

#2. Find a function  $f: \mathbf{Z} \rightarrow \mathbf{N}$  that is one-to-one but not onto.

**Solution:**

We can take  $f(n) = \begin{cases} n^2 & n < 0 \\ n^2 + 2 & n \geq 0. \end{cases}$

The function is not onto because there is no  $n$  such that  $f(n) = 5$  (there is no integer  $n$  such that either  $n^2$  or  $n^2 + 2$  is equal to 5).

The function is one-to-one. If  $m$  and  $n$  are nonequal nonnegative integers, then  $m^2 + 2$  cannot be equal to  $n^2 + 2$ . Likewise, if  $m$  and  $n$  are nonequal negative integers, then  $m^2$  cannot be equal to  $n^2$ . Finally, suppose  $m < 0$  and  $n \geq 0$ . Then the function value  $m^2$  cannot equal the function value  $n^2 + 2$  (because no two squares differ by 2).

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**p.143, icon at Example 12**

#3. Find a function  $f: \mathbf{Z} \rightarrow \mathbf{N}$  that is one-to-one and onto.

**Solution:**

For example, we can take the function  $f(n) = \begin{cases} -2n & n \leq 0 \\ 2n - 1 & n > 0. \end{cases}$

The function is one-to-one because no two function values of the form  $-2n$  ( $n \leq 0$ ) can be equal, no two function values of the form  $2n - 1$  ( $n > 0$ ) can be equal, and no function value of the form  $2n$  (which is an even integer) can equal a function value of the form  $2n - 1$  (which is an odd integer).

The function is onto. If  $n \in \mathbf{N}$  is even, then  $f(-n/2) = n$ ; if  $n \in \mathbf{N}$  is odd, then  $f((n+1)/2) = n$ .

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**p.150, icon at Example 29**

#1. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $f(x) = \lfloor 3x \rfloor - 1$ . Find  $f(S)$  where  $S = [1, 3]$ .

**Solution:**

Because we have the expression  $\lfloor 3x \rfloor$ , we need to examine  $x$ -values where  $x$  has the form  $k/3$  (that is  $x = \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3}, \frac{9}{3}$ ), because at these numbers  $f(x) = \lfloor 3x \rfloor - 1$  changes value. We obtain  $\{2, 3, 4, 5, 6, 7, 8\}$ .

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**p.150, icon at Example 29**

#2. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $f(x) = \lfloor 3x \rfloor - 1$ . Find  $f^{-1}(S)$  where  $S = \{0\}$

**Solution:**

$f^{-1}(\{0\})$  is equal to the set of all  $x$  such that  $\lfloor 3x \rfloor - 1 = 0$ , or  $\lfloor 3x \rfloor = 1$ . Any number  $x \in [1/3, 2/3]$  will work.

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**p.150, icon at Example 29**

#3. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $f(x) = \lfloor 3x \rfloor - 1$  and  $g: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $g(x) = x/3$ . Find  $f \circ g(T)$  where  $T = [-3, 3.5]$ .

**Solution:**

We first find  $g(T)$ . The function  $g$  takes each number and divides it by 3. When we divide the numbers in the interval  $[-3, 3.5]$  by 3 we have the interval  $[-1, \frac{3.5}{3}]$ . Now we apply the function  $f$  to each of these numbers. Applying  $f(x) = \lfloor 3x \rfloor - 1$  to each number in the interval  $[-1, \frac{3.5}{3}]$  we have  $\{-4, -3, -2, -1, 0, 1, 2\}$ .

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**p.150, icon at Example 29**

#4. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $f(x) = \lfloor 3x \rfloor - 1$ .

(a) Find  $(f \circ f)(1)$ .

(b) Find  $(f \circ f)(U)$  where  $U = [2, 3]$ .

**Solution:**

(a)  $(f \circ f)(1) = f(f(1)) = f(\lfloor 3 \cdot 1 \rfloor - 1) = f(2) = \lfloor 3 \cdot 2 \rfloor - 1 = 5$ .

(b) Paying careful attention to the numbers  $2, \frac{7}{3}, \frac{8}{3}, 3$  (because there are the values of  $x$  at which the graph of  $f$  jumps), we have  $(f \circ f)(U) = f(f(U)) = f(\{5, 6, 7, 8\}) = \{14, 17, 20, 23\}$ .

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**p.150, icon at Example 29**

#5. Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $f(x) = \lfloor 3x \rfloor - 1$  and  $g: \mathbf{R} \rightarrow \mathbf{R}$  have the rule  $g(x) = x/3$ .

(a) Find  $(f \circ g)^{-1}(\{2.5\})$ .

(b) Find  $(f \circ g)^{-1}(\{2\})$ .

**Solution:**

(a) We are looking for numbers  $x$  such that  $(f \circ g)(x) = 2.5$ . But  $(f \circ g)(x) = f(g(x))$  and the range of  $f$  consists only of integers, and hence cannot include 2.5. Therefore we cannot have such an  $x$ , so  $(f \circ g)^{-1}(V) = \emptyset$ . That is, 2.5 is not an element of the range of  $f \circ g$ .

(b) Note that  $(f \circ g)^{-1}(\{2\}) = g^{-1} \circ f^{-1}(\{2\}) = g^{-1}(f^{-1}(\{2\}))$ . But  $f^{-1}(\{2\}) = [1, 4/3)$ . Therefore  $g^{-1}([1, 4/3)) = [3, 4)$ . Hence,  $(f \circ g)^{-1}(\{2\}) = [3, 4)$ .

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**p.150, icon at Example 29**

#6. Find all solutions to  $\lceil x \rceil + \lfloor x \rfloor = 2x$ .

**Solution:**

$\lceil x \rceil + \lfloor x \rfloor$ , the sum of two integers, must be an integer. Hence  $2x$  must be an integer, which means that either  $x$  itself is an integer, or  $x + 0.5$  is an integer.

If  $x$  is an integer, then  $\lceil x \rceil + \lfloor x \rfloor = x + x = 2x$ .

If  $x + 0.5$  is an integer, then  $\lceil x \rceil + \lfloor x \rfloor = (x + 0.5) + (x - 0.5) = 2x$ .

Thus, the solution set is  $\{x : x \text{ or } x + 0.5 \text{ is an integer}\} = \{\frac{k}{2} : k \text{ an integer}\}$ .

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**p.150, icon at Example 29**

#7. Find all solutions to  $\lfloor x \rfloor \lceil x \rceil = x^2$ .

**Solution:**

We first observe that every integer is a solution because in this case  $\lfloor x \rfloor = x$  and  $\lceil x \rceil = x$ .

Now suppose that  $x$  is not an integer. Therefore, there is an integer  $n$  such that  $\lfloor x \rfloor = n$  and  $\lceil x \rceil = n + 1$ . Hence, in this case the original equation becomes  $n(n + 1) = x^2$ , or  $x = \pm \sqrt{n(n + 1)} = \pm\sqrt{2}, \pm\sqrt{6}, \pm\sqrt{12}, \pm\sqrt{20}$ , etc. Therefore, the solutions to the equation  $\lfloor x \rfloor \lceil x \rceil = x^2$  are all integers and all numbers of the form  $\pm\sqrt{n(n + 1)}$ .

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**p.150, icon at Example 29**

#8. Use the floor and/or ceiling function to find a formula for computing the units' digit of a positive integer  $n$ .

**Solution:**

For example, the units' digit of 547 is 7, and can be obtained as follows:  $547 - 540 = 7$ . This indicates that the units' digit of  $n$  can be obtained by rounding down  $n$  to the nearest multiple of 10 and subtracting this rounded-down number from  $n$ . The expression  $10 - \frac{n}{10}$  rounds  $n$  down to the nearest multiple of 10. Hence

$$n - 10 - \frac{n}{10} = \text{units' digit of } n.$$

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