CS5489 - Machine Learning

Lecture 2a - Bayes Classifier

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Outline

1. Bayes Classification and Generative Models
2. Parameter Estimation
3. Bayesian Decision Rule

Classification Examples

- Given an email, predict whether it is spam or not spam.
    - **Email 1:**

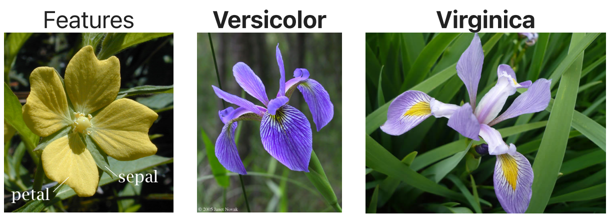
> There was a guy at the gas station who told me that if I knew Mandarin

and Python I could get a job with the FBI.

- **Email 2:**

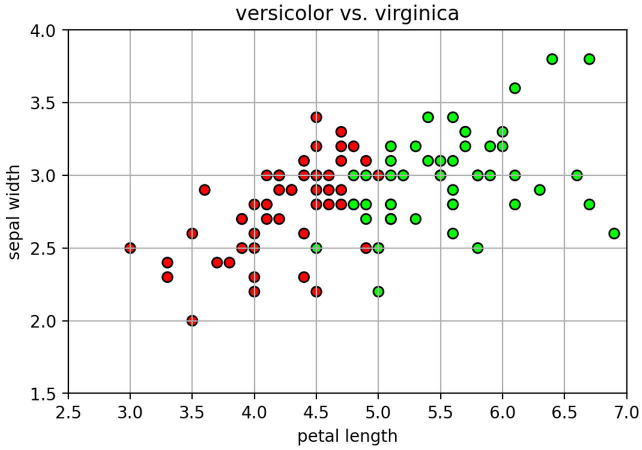
> A home based business opportunity is knocking at your door. Don't be rude and let this chance go by. You can earn a great income and find your financial life transformed. Learn more Here. To Your Success. Work From Home Finder Experts

- Classification Examples
    - Given the *petal length* and *sepal width*, predict the type of iris flower.



In [3]:
```
irisfig
```

Out[3]:



General Classification Problem

- Observation $\mathbf{x}$ (i.e., features)
  - typically a real vector, $\mathbf{x} \in \mathbb{R}^d$.
  - **Example**: a 2-dim vector containing the petal length and sepal width.
    - $\mathbf{x} = \begin{bmatrix} \text{petal length} \\ \text{sepal width} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- Class y
  - takes values from a set of possible class labels $\mathcal{Y}$.
  - **Example:** $\mathcal{Y} = \{\text{"versicolor"}, \text{"virginica"}\}$.
    - or equivalently as numbers, $\mathcal{Y} = \{1, 2\}$.
- **Goal**: given an observed features $\mathbf{x}$, predict its class y.

Probabilistic model

- To build a classifier we need to model the relationship between observations and classes.
- Model *how* the data is generated using probability distributions.
  - called a **generative model**.
  - build our assumptions about the world into the model.
- Generative model
  - 1. The world has objects of various classes.
  - 2. The observer measures features/observations from the objects.
  - 3. Each class of objects has a particular probability distribution of features.

- Need to define probability models for:
  1. the classes
  2. the features for each class

Class model

- Set of possible classes are $\mathcal{Y}$
  - For example, $\mathcal{Y} = \{\text{"versicolor"}, \text{"virginica"}\}$.
    - or more generally, $\mathcal{Y} = \{1, 2\}$.
- In the world, the frequency that class y occurs is given by the probability distribution $p(y)$.
  - $p(y)$ is called the **prior distribution**.

- **Example**: Bernoulli class distribution
  - $p(y = 1) = 0.4$

- $p(y=2) = 0.6$
- "In the world of iris flowers, there are 40% that are Class 1 (versicolor) and 60% that are Class 2 (virginica)"
- distribution: $p(y) = \pi^{1(y=1)}(1-\pi)^{1(y=2)}$
  - π is the parameter (e.g., 0.4)
  - *Indicator function*: $1(q) = \begin{cases} 1, & q \text{ is true} \\ 0, & \text{otherwise} \end{cases}$

Learn from our data

- How to get the parameter $p(y=1) = \pi$ for our model?
  - Assume we have collected some data, $\mathcal{D} = \{y_1, \cdots, y_N\}$.
- **Maximum Likelihood Estimation (MLE)**
  - find the parameter that maximizes the likelihood (log-likelihood) of observing the data.
  - $\pi^* = \mathrm{argmax}_\pi \sum_{i=1}^N \log p(y_i)$
    - sum over the log-likelihoods of each sample (assumes samples are independent)

- if $y=1$, then the log-likelihood is $\log(\pi)$, and if $y=2$ the log-likelihood is $\log(1-\pi)$.
- Sum over each sample:

$$\ell(\pi) = \sum_i 1(y_i=1)\log\pi + 1(y_i=2)\log(1-\pi)$$

  - Then, $\ell(\pi) = N_1 \log\pi + N_2\log(1-\pi)$
    - where $N_1 = \sum_i 1(y_i=1)$ = Number of 1's observed.
    - and $N_2 = \sum_i 1(y_i=2)$ = Number of 2's observed.

- Now solve for π by maximizing $\ell(\pi)$.

  - Take derivative and set to 0 to find the maximum.

  $$\begin{aligned}\frac{d \ell(\pi)}{d\pi} = \frac{N_1}{\pi} - \frac{N_2}{1-\pi} = 0 \ N_1(1-\pi) - N_2 \pi = 0 \ N_1 - (N_1+N_2) \pi = 0 \

\Rightarrow \pi = \frac{N_1}{N_1+N_2}\end{aligned}$$

- $p(y=1) = \frac{\text{number of examples of Class 1}}{\text{total number of examples}}$
- $p(y=2) = \frac{\text{number of examples of Class 2}}{\text{total number of examples}}$

In [4]:
```
N1 = count_nonzero(y==1)  # number of Class 1 examples
N2 = count_nonzero(y==2)  # number of Class 2 examples
N  = len(y)               # total
py = [double(N1)/N, double(N2)/N] # note: avoids integer division!
print(py)
```

```
[0.5, 0.5]
```

Observation model

- We measure/observe a feature x
  - the value of the feature x *depends* on the class.
- The observation is drawn according to the distribution $p(x|y)$.
  - $p(x|y)$ is called the **class conditional distribution**
    - "probability of observing a particular feature value x *given* the object is class y"
  - Each class has its own class conditional:
    - $p(x|y=1)$ = distribution of features when its class 1
    - $p(x|y=2)$ = distribution of features when its class 2

Learn from the data

- Histograms for feature "petal length" for each class

In [6]:
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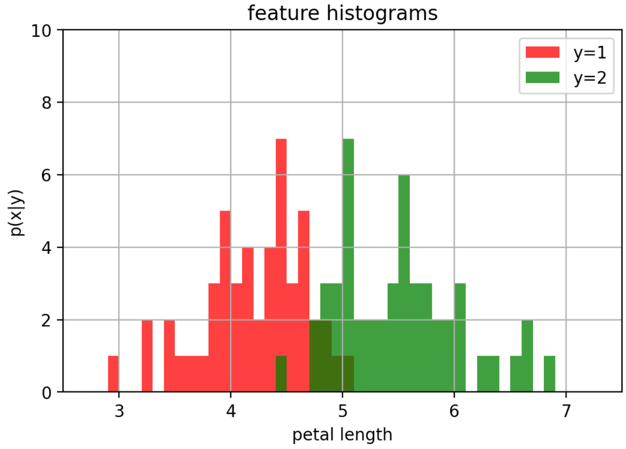
ccdhist
```

Out[6]:



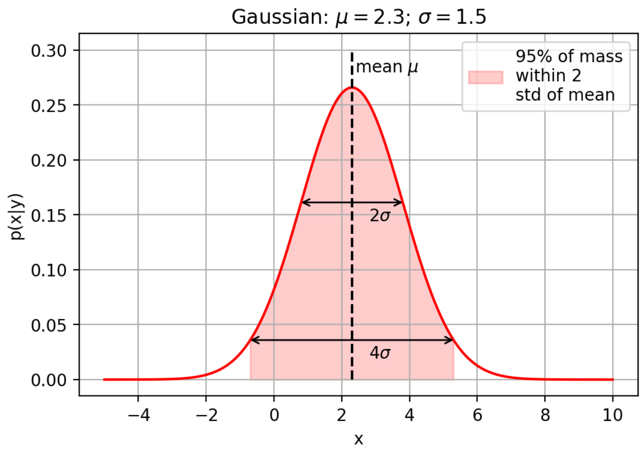
- **Problem:** looks a little bit noisy.
- **Solution:** assume a probability model for the class conditional $p(x|y)$

Gaussian distribution (normal distribution)

- Each class is modeled as a separate Gaussian distribution of the feature value
  - $p(x|y=c) = \frac{1}{\sqrt{2\pi\sigma_c^2}} e^{-\frac{1}{2\sigma_c^2}(x-\mu_c)^2}$
  - Each class has its own mean and variance parameters (μ_c, σ_c^2).

In [8]:
```
gfig
```

Out[8]:



MLE for Gaussian

- Set the parameters (μ, σ^2) to maximize the log-likelihood of the samples for that class.
  - Let $\{x_i\}_{i=1}^N$ be the observed features for class 1:

$$(\hat{\mu}, \hat{\sigma}^2) = \underset{\mu,\sigma^2}{\operatorname{argmax}} \sum_{i=1}^N \log p(x_i|y_i = 1)$$

  - Then, the objective is $\ell(\mu) = \sum_{i=1}^N -\frac{1}{2\sigma^2}(x_i - \mu)^2 - \frac{1}{2}\log 2\pi\sigma^2$

- take derivative and set to 0

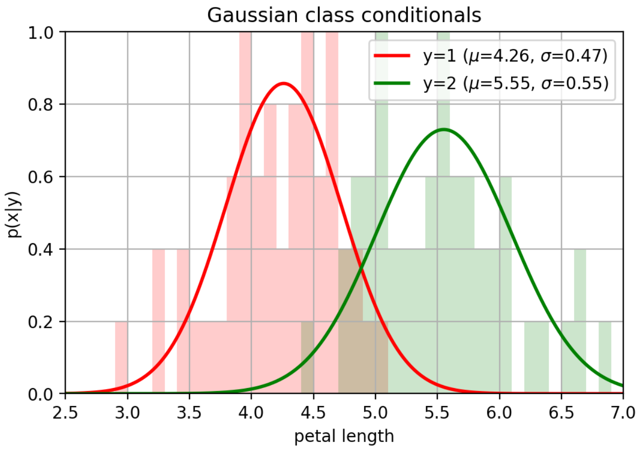
$$\sum_{i=1}^{N} \frac{1}{\sigma^2}(x_i - \mu) = 0$$
$$\sum_{i=1}^{N} x_i - N\mu = 0$$
$$\Rightarrow \mu = \frac{1}{N}\sum_i x_i$$

- Solution:
  - sample mean: $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$
  - sample variance:

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$$

In [11]:
```
gcd
```

Out[11]:



Bayesian Decision Rule

- The Bayesian decision rule (BDR) makes the optimal decisions on problems involving probability (uncertainty).
  - minimizes the *probability of making a prediction error*.
- **Bayes Classifier**
  - Given observation x, pick the class c with the *largest posterior probability*, $p(y = c|x)$.
    - Probability of the class given observed x.
  - **Example:**
    - if $p(y = 1|x) > p(y = 2|x)$, then choose Class 1
    - if $p(y = 1|x) < p(y = 2|x)$, then choose Class 2

- Problem: we don't have $p(y|x)$!
  - we only have $p(y)$ and $p(x|y)$.

Bayes' Rule

- The posterior probability can be calculated using Bayes' rule:

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)}$$

- The denominator is the probability of feature x, regardless of its class.
  - $p(x) = \sum_{y \in \mathcal{Y}} p(x|y)p(y)$
- The denominator makes $p(y|x)$ sum to 1.

- Bayes' rule:

$$p(y = 1|x) = \frac{p(x|y = 1)p(y = 1)}{p(x|y = 1)p(y = 1) + p(x|y = 2)p(y = 2)}$$

- **Example**:

In [13]:
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iris1dpost
```
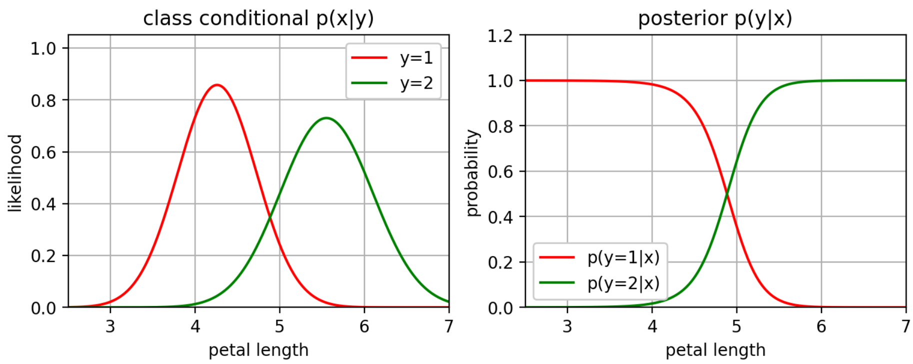
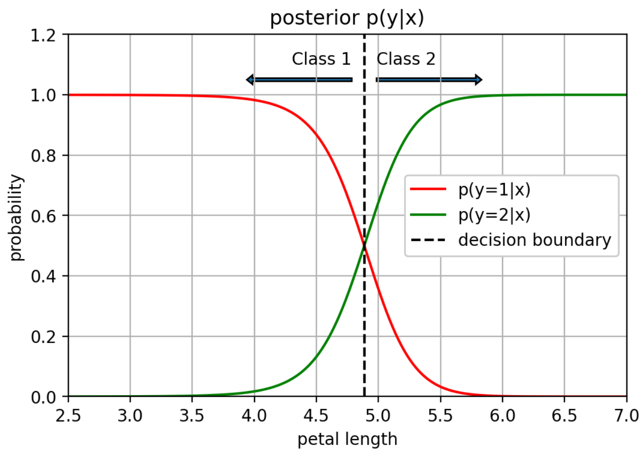

Out[13]:



- The *decision boundary* is where the two posterior probabilites are equal
  - $p(y = 1|x) = p(y = 2|x)$

In [15]:
```
iris1dpost2
```

Out[15]:



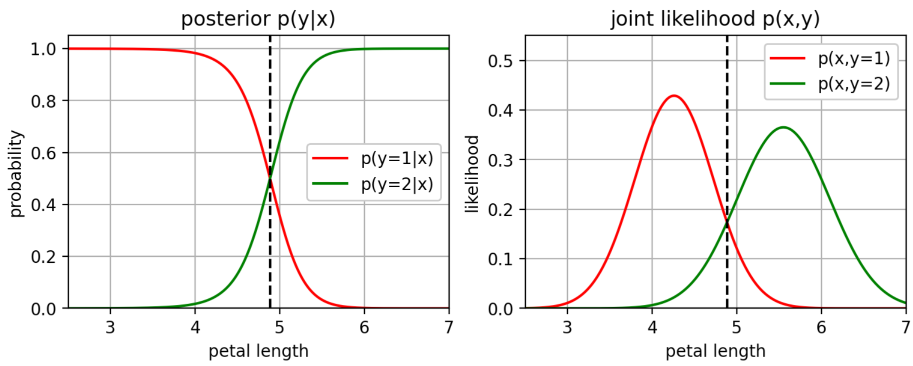
Bayes rule revisited

- Bayes' rule: $p(y|x) = \frac{p(x|y)p(y)}{p(x)}$
- Note that the denominator is the same for each class y.
  - hence, we can compare just the numerators $p(x|y)p(y)$.
  - This also called the *joint likelihood* of the observation and class
    - $p(x, y) = p(x|y)p(y)$

- **Example:**
  - BDR using joint likelihoods:
    - if $p(x|y = 1)p(y = 1) > p(x|y = 2)p(y = 2)$, then choose Class 1
    - otherwise, choose Class 2

In [17]:
```
iris1djoint
```

Out[17]:



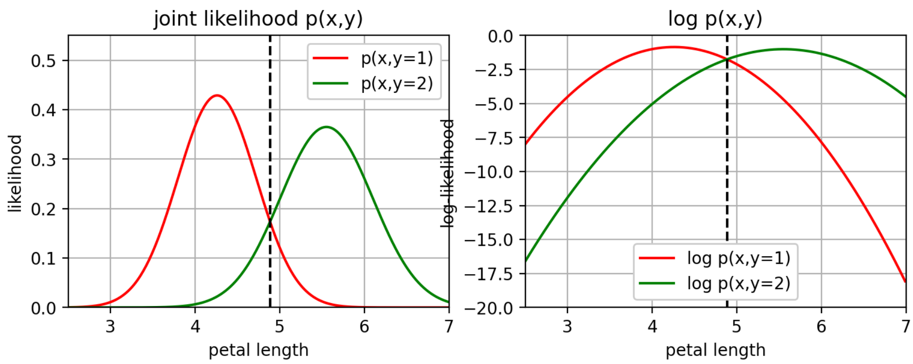
- Can also apply a monotonic increasing function (like $\log$) and do the comparison.
  - Using log likelihoods:

$$\log p(x|y=1) + \log p(y=1) > \log p(x|y=2) + \log p(y=2)$$

  - This is more numerically stable when the likelihoods are small.

In [19]:
```
iris1dLL
```

Out[19]:



Bayes Classifier Summary

- **Training:**
  1. Collect training data from each class.
  2. For each class c, estimate the class conditional densities $p(x|y=c)$:
  3. select a form of the distribution (e.g. Gaussian).
  4. estimate its parameters with MLE.
  5. Estimate the class priors $p(y)$ using MLE.
- **Classification:**
  1. Given a new sample x^*, calculate the likelihood $p(x^*|y=c)$ for each class c.
  2. Pick the class c with largest posterior probability $p(y=c|x^*)$.
  - (equivalently, use $p(x^*|y=c)p(y=c)$ or $\log p(x^*|y=c) + \log p(y=c)$)