Question 1:

a) $T(n) = T(n-1) + \frac{1}{n}$

Expand the equation:

$$T(n) = T(n-1) + \frac{1}{n} = T(n-2) + \frac{1}{n-1} + \frac{1}{n}$$
$$= T(n-3) + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}$$

....

Based on the observation, we can predict the formula:

$$T(n) = T(1) + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n}$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$
$$= \sum_{k=1}^{n} \frac{1}{k}$$

I will prove this formula by induction:

$$T(n+1) = T(n) + \frac{1}{n+1}$$
$$= T(1) + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n} + \frac{1}{n+1} \quad \text{by inductive hypothesis.}$$
$$= \sum_{k=1}^{n+1} \frac{1}{k} \quad \therefore \text{proved.}$$

using the knowledge of caculus, $\sum_{k=1}^{n} \frac{1}{k}$ is in $\Theta(\ln(n))$

$\Rightarrow T(n) = \Theta(\log n)$

b) $T(n) = 3T(n/2) + n\log n \quad (n \le 2)$

Applying master theorem case 1 where $a=3$, $b=2$.

$f(n) = n\log n$, $\quad n^{\log_b a} = n^{\log_2 3} \approx n^{1.58}$

Take $\varepsilon = 0.01$, $f(n) = O(n^{\log_2 3 - 0.01})$ holds (This is because $\log(n) = O(\sqrt{n})$)

As a result, $T(n) = \Theta(n^{\log_2 3})$

Which means $\begin{cases} T(n) = O(n^{\log_2 3}) & \text{[upper bond].} \\ T(n) = \Omega(n^{\log_2 3}) & \text{[lower bond].} \end{cases}$

c) $T(n) = T(\sqrt{n}) + 1 \quad (n \le 2)$

Assume 2^{2^k}, $T(2) = 1$.

$$\begin{aligned} T(2^{2^k}) &= T(2^{2^{k-1}}) + 1 \\ &= T(2^{2^{k-2}}) + 2 \\ &\cdots \\ &= T(2^{2^0}) + k \\ &= T(2) + k \\ &= k+1 \end{aligned}$$

Prove this formula by induction:

$$\begin{aligned} T(2^{2^{k+1}}) &= T(2^{2^k}) + 1 \\ &= k+1+1 \\ &= k+2 \end{aligned}$$

[by inductive hypotheis]

Then caculate k to find bond: (number of recursions)

$n = 2^{2^k} \Rightarrow k = \log\log(n)$ and this is the tight bond

because $T(n) = \log\log(n) + 1$

$\therefore T(n) = \Theta(\log\log(n))$ ∎

d)

$$T(n) = 3T\left(\frac{n}{3}+5\right) + \frac{n}{2}$$

① Guess: $T(n) = \Omega(n\log n)$, using substitution:

$$T(n) = 3T\left(\frac{n}{3}+5\right) + \frac{n}{2}$$
$$\geq 3c\left(\frac{n}{3}+5\right)\log\left(\frac{n}{3}+5\right) + \frac{n}{2}$$
$$\geq 3c\left(\frac{n}{3}\right)\log\left(\frac{n}{3}\right) + \frac{n}{2}$$
$$= cn\log\left(\frac{n}{3}\right) + \frac{n}{2}$$
$$= cn\log n - cn\log 3 + \frac{n}{2}$$
$$\geq cn\log n \quad \text{if we pick } 0 < c \leq \frac{1}{2\log 3}$$

② Guess: $T(n) = O(n\log n)$, using substitution:

$$T(n) = 3T\left(\frac{n}{3}+5\right) + \frac{n}{2}$$
$$\leq 3c\left(\frac{n}{3}+5\right)\log\left(\frac{n}{3}+5\right) + \frac{n}{2}$$
$$\leq (cn+15c)\log\left(\frac{n}{2}\right) + \frac{n}{2} \quad \text{for } n \geq 30$$
$$= (cn+15c)(\log n - \log 2) + \frac{n}{2}$$
$$= cn\log n - cn\log 2 + 15c\log n - 15c\log 2 + \frac{n}{2}$$
$$\leq cn\log n + 15c\log n - cn\log 2 + \frac{n}{2}$$

because we know ~~1 = O(logn)~~ $\log n = O(n)$

$$\leq cn\log n + \left[n\left(\frac{1}{2} - c\log 2\right)\right] + kn$$
$$= cn\log n + n\left(\frac{1}{2} - c\log 2 + k\right)$$

pick c such that $\left(\frac{1}{2} + k - c\log 2\right) \leq 0$

$$k + \frac{1}{2} \leq c\log 2$$
$$c \geq \frac{k+\frac{1}{2}}{\log 2} = \text{constant}$$

As a result, we can find $c \geq \frac{k+\frac{1}{2}}{\log 2}$ for some constant k and $n \geq 30$

To show that $T(n) \leq cn\log n$.

$$\begin{cases} T(n) = \Omega(n\log n) \\ T(n) = O(n\log n) \end{cases} \Rightarrow T(n) = \Theta(n\log n)$$