De-singularity Subgradient for the q-th-Powered ℓ_p-Norm Weber Location Problem

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Abstract

The Weber location problem is widely used in several artificial intelligence scenarios. However, the gradient of the objective does not exist at a considerable set of singular points. Recently, a de-singularity subgradient method has been proposed to fix this problem, but it can only handle the q-th-powered ℓ_2-norm case ($1 \leqslant q < 2$), which has only finite singular points. In this paper, we further establish the de-singularity subgradient for the q-th-powered ℓ_p-norm case with $1 \leqslant q \leqslant p$ and $1 \leqslant p < 2$, which includes all the rest unsolved situations in this problem. This is a challenging task because the singular set is a continuum. The geometry of the objective function is also complicated so that the characterizations of the subgradients, minimum and descent direction are very difficult. We develop a q-th-powered ℓ_p-norm Weiszfeld Algorithm without Singularity (qPpNWAWS) for this problem, which ensures convergence and the descent property of the objective function. Extensive experiments on six real-world data sets demonstrate that qPpNWAWS successfully solves the singularity problem and achieves a linear computational convergence rate in practical scenarios.

1 Introduction

The Weber location problem is a fundamental problem that is extensively investigated in artificial intelligence (Lai et al. 2024), machine learning (Li, Sahoo, and Hoi 2016; Lai et al. 2018c, 2020), financial engineering (Lai and Yang 2023), computer vision (Aftab, Hartley, and Trumpf 2015), and operations research (Ostresh 1978). For a general definition, it seeks a point $\boldsymbol{x}_*$ that minimizes the weighted sum of the q-th power of the ℓ_p distances to m fixed data points $\{\boldsymbol{x}_i\}_{i=1}^m \subseteq \mathbb{R}^d$ (Weber 1909; Morris 1981; Chen 1984; Brimberg and Love 1993), defined as the q-th-powered ℓ_p-norm Weber location problem (**qPpNWLP**):

$$\boldsymbol{x}_* \in \arg\min_{\boldsymbol{y}\in\mathbb{R}^d} C_{p,q}(\boldsymbol{y}) := \sum_{i=1}^m \xi_i \|\boldsymbol{y} - \boldsymbol{x}_i\|_p^q, \tag{1}$$

where ξ_i denotes the weight for the i-th data point, $\|\cdot\|_p$ denotes the ℓ_p norm, $C_{p,q}(\cdot)$ denotes the cost function related to the q-th power of the ℓ_p norm, $1 \leqslant q \leqslant p$ and $p \geqslant 1$. Due to the same reasons as (Lai et al. 2024), we can assume that the data points $\{\boldsymbol{x}_i\}_{i=1}^m$ are distinct and non-collinear in the rest of the paper.

1.1 The Singularity Problem

There is no closed-form solution to (1), and the gradient-type method is an intuitive and tractable approach. This approach includes those do not explicitly show a gradient form, like the Weiszfeld algorithm (Brimberg and Love 1993). To take a glimpse of the singularity problem, we first compute the gradient of the objective function $C_{p,q}$ as

$$\begin{aligned}&\big(\nabla C_{p,q}(\boldsymbol{y})\big)^{(t)}\\ :=&\sum_{i=1}^m q\xi_i \|\boldsymbol{y} - \boldsymbol{x}_i\|_p^{q-p} |y^{(t)} - x_i^{(t)}|^{p-2}(y^{(t)} - x_i^{(t)}),\end{aligned} \tag{2}$$

where $\big(\nabla C_{p,q}(\boldsymbol{y})\big)^{(t)}$ denotes the t-th dimension ($1 \leqslant t \leqslant d$) of the gradient. Note that if $q < p$ or $p < 2$, the singularity problem can occur if $\boldsymbol{y}$ hits the following singular set:

$$\begin{cases}\mathcal{S}_p{:=}\big\{\boldsymbol{y}{\in}\mathbb{R}^d | \exists i{\in}\{1,\dots,m\}, t{\in}\{1,\dots,d\} \text{ s.t. } y^{(t)}{=}x_i^{(t)}\big\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 1 \leqslant p < 2, \\ \mathcal{S}_2{:=}\{\boldsymbol{x}_i\}_{i=1}^m \qquad\qquad\quad \text{if } 1 \leqslant q < 2, p = 2.\end{cases} \tag{3}$$

This singularity problem occurs frequently and unexpectedly. Chandrasekaran and Tamir (1989); Vardi and Zhang (2000) indicate that the "bad" points that can yield such singularity in a gradient-type method may constitute a continuum set that can be dense in an open region of $\mathbb{R}^d$ ($d \geqslant 2$) or even the entire $\mathbb{R}^d$. Moreover, this problem cannot be circumvented by straightforward treatments like perturbations and random restarts. More details can be found in (Lai et al. 2024).

If $q \geqslant p$ or $p \geqslant 2$, the term $\|\boldsymbol{y} - \boldsymbol{x}_i\|_p^{q-p}$ or $|y^{(t)} - x_i^{(t)}|^{p-2}$ in (2) will not be singular. For $p \geqslant 2$ but $1 \leqslant q < p$, the singularity problem can be solved by (Lai et al. 2024). Hence this paper mainly focuses on the rest unsolved singular cases with the two key hyperparameters $1 \leqslant q \leqslant p$ and $1 \leqslant p < 2$. Users can select any values of p and q within this range to suit their specific needs. As p approaches 2, the distance between data points becomes closer to the Euclidean distance.

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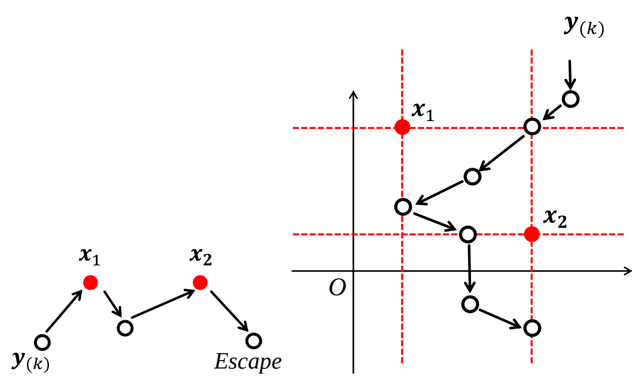
[**]The supplementary material and codes for this paper are available at https://github.com/laizhr/qPpNWAWS.

Conversely, as p approaches 1, the distance between data points becomes closer to the Manhattan distance, representing a typical non-Euclidean geometry. On the other hand, as q approaches p, the distance between data points gets higher power, which can be advantageous in certain computer vision tasks (Aftab, Hartley, and Trumpf 2015). In summary, allowing for a range of values for p and q enhances the geometrical representation of the Weber location problem.

1.2 Continuum Singular Set for $1 \leqslant p < 2$

(3) indicates that when $p = 2$, the singularity only occurs at the m fixed data points $\{\boldsymbol{x}_i\}_{i=1}^m$. This case has already been completely solved by (Vardi and Zhang 2000; Lai et al. 2024). However, **when $1 \leqslant p < 2$, the singularity occurs in at most $m \cdot d$ hyperplanes that encompass a continuum set of points, which is much more difficult than the $p = 2$ case and no solutions have been developed (see Figure 1)**. The difficulties lie in the following three aspects:

1. An effective gradient-type algorithm may visit each singular point for only once. But since there are infinite singular points when $1 \leqslant p < 2$, the algorithm may visit the singular set for infinite times.
2. Based on the first point, there is no unified step size for the escape from the singular set, which affects the convergence property.
3. The geometry of the objective function is complicated when $1 \leqslant q \leqslant p$ and $1 \leqslant p < 2$ so that the characterizations of the subgradients, minimum and descent direction are very difficult, which also affects the convergence property.



(a) $p = 2$ and $1 \leqslant q < 2$ (b) $1 \leqslant p < 2$

Figure 1: (a) When $p = 2$, the singular set $\mathcal{S}_2$ (the red dots) is finite and an effective gradient-type algorithm visits each singular point for only once. Hence the iterate $\boldsymbol{y}_{(k)}$ (the circle) can finally escape from all the singular points. (b) When $1 \leqslant p < 2$, the singular set $\mathcal{S}_p$ is a continuum (the red dashed lines), hence the iterate $\boldsymbol{y}_{(k)}$ may revisit $\mathcal{S}_p$ for infinite times and may not escape from $\mathcal{S}_p$.

To address the above difficulties, we develop a complete de-singularity subgradient methodology for qPpNWLP with $1 \leqslant q \leqslant p$ and $1 \leqslant p < 2$, including all the rest unsolved situations in this problem. **Our main contributions are summarized as follows.**

1. We develop a de-singularity subgradient of the cost function $C_{p,q}$ on the singular set $\mathcal{S}_p$. It can replace the ordinary gradient without increasing computational complexity.
2. We develop a q-th-Powered ℓ_p-Norm Weiszfeld Algorithm without Singularity (q**P**p**NWAWS**). It can identify whether the current iterate is a minimum point; If not, it can further reduce the cost function, no matter whether this iterate is singular or nonsingular. By this way, qPpNWAWS solves the singularity problem.
3. We develop a complete proof for the convergence of qPpNWAWS.
4. We demonstrate that qPpNWAWS achieves a linear computational convergence rate in practical scenarios.

2 Related Works

We review some closely related works on qPpNWLP in this section.

2.1 The ℓ_p-Norm Weiszfeld Algorithm

The ℓ_p-norm Weiszfeld algorithm (pNWA) can be derived by the first-order optimal condition of the qPpNWLP with $1 \leqslant p \leqslant 2$ and $q = 1$ (Brimberg and Love 1993). For a **non-singular optimal point $\boldsymbol{x}_* \notin \mathcal{S}_p$**, setting $\left(\nabla C_{p,1}(\boldsymbol{x}_*)\right)^{(t)} = 0$ in (2) yields

$$x_*^{(t)} = \frac{\sum_{i=1}^m \xi_i \|\boldsymbol{x}_* - \boldsymbol{x}_i\|_p^{1-p} |x_*^{(t)} - x_i^{(t)}|^{p-2} x_i^{(t)}}{\sum_{i=1}^m \xi_i \|\boldsymbol{x}_* - \boldsymbol{x}_i\|_p^{1-p} |x_*^{(t)} - x_i^{(t)}|^{p-2}}, \forall 1 \leqslant t \leqslant d. \tag{4}$$

These are fixed-point equations, which can be converted to a fixed-point algorithm with the following operator $\mathbf{T}_{p,1}$:

$$\begin{aligned} y_{(k+1)}^{(t)} &:= \left(\mathbf{T}_{p,1}(\boldsymbol{y}_{(k)})\right)^{(t)} \\ &:= \frac{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{1-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} x_i^{(t)}}{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{1-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2}}, \end{aligned} \tag{5}$$

where $y_{(k)}^{(t)}$ denotes the t-th dimension of the k-th iterate. **When $\boldsymbol{y}_{(k)}$ hits the singular set $\mathcal{S}_p$, pNWA cannot solve it but just terminates at $\boldsymbol{y}_{(k)}$.**

2.2 q-th Power Weiszfeld Algorithm without Singularity

Lai et al. (2024) propose a q-th power Weiszfeld algorithm without singularity (qPWAWS) to handle the singularity problem in the special case $p = 2, 1 \leqslant q < 2$:

$$\boldsymbol{y}_{(k+1)} = \begin{cases} \mathbf{T}_{2,q}(\boldsymbol{y}_{(k)}) := \frac{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_2^{q-2} \boldsymbol{x}_i}{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_2^{q-2}} \\ \qquad \text{if } \boldsymbol{y}_{(k)} \notin \{\boldsymbol{x}_i\}_{i=1}^m, \\ \mathbf{T}_s(\boldsymbol{y}_{(k)}) := \boldsymbol{y}_{(k)} - \lambda_* \nabla D_{2,q}(\boldsymbol{y}_{(k)}) \\ \qquad \text{if } \boldsymbol{y}_{(k)} = \boldsymbol{x}_l \text{ for some } l \in \{1, \ldots, m\}, \end{cases} \tag{6}$$

where

$$\nabla D_{2,q}(\boldsymbol{y}_{(k)}) = \sum_{i \neq l} q \xi_i \|\boldsymbol{y}_{(k)} - \mathbf{x}_i\|_2^{q-2} (\boldsymbol{y}_{(k)} - \mathbf{x}_i) \tag{7}$$

is the q-th-powered ℓ_2-norm de-singularity subgradient and $\lambda_* > 0$. Intuitively, $\nabla D_{2,q}$ removes the singular component corresponding to $\boldsymbol{x}_l$ and serves as a conventional gradient in the subgradient descent step $\mathbf{T}_s$.

qPWAWS can escape from each singular point and will not revisit it again. Since there are finite singular points when $p = 2$, qPWAWS can eventually escape from the singular set $\mathcal{S}_2$ (see Figure 1a). **However, it cannot be directly adopted in the $1 \leqslant p < 2$ case, because the singular set $\mathcal{S}_p$ is a continuum and qPWAWS may not eventually escape from $\mathcal{S}_p$. Moreover, the step size λ_* in $\mathbf{T}_s$ cannot be uniformly chosen with respect to (w.r.t.) $\mathcal{S}_p$ due to its infinite elements. The characterizations of the subgradients, minimum and descent direction are also difficult. These three problems affect the convergence property of qPWAWS.** Therefore, a new methodology should be developed to overcome these new challenges for the $1 \leqslant p < 2$ case.

3 q-th-Powered ℓ_p-Norm Weiszfeld Algorithm without Singularity

In this section, we present qPpNWAWS for solving qPpNWLP with $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$. For a convenient illustration, we let $\eta_i := \xi_i^{\frac{1}{q}}$ and reformulate (1) as

$$\boldsymbol{x}_* \in \underset{\boldsymbol{y}\in\mathbb{R}^d}{\arg\min}\, C_{p,q}(\boldsymbol{y}) := \sum_{i=1}^{m} \eta_i^q \|\boldsymbol{y} - \boldsymbol{x}_i\|_p^q. \tag{8}$$

$C_{p,q}$ is strictly convex and there exists a unique minimum point $\boldsymbol{x}_*$ for (8) with $1 < p < 2$, while $C_{p,q}$ is convex and there exists one or more minimum points for (8) with $p = 1$. The construction of qPpNWAWS consists of 4 steps:

1. The update at a nonsingular iterate $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$ is designed as a Weiszfeld-style one, which makes $C_{p,q}(\boldsymbol{y}_{(k)})$ non-increasing.
2. Define the q-th-powered ℓ_p-norm de-singularity subgradient $\nabla D_{p,q}(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathcal{S}_p$ to characterize the subgradients of $C_{p,q}(\boldsymbol{y})$ and the minimum of $C_{p,q}$.
3. Adopt $\nabla D_{p,q}(\boldsymbol{y}_{(k)})$ to construct the iterative update at the singular iterate $\boldsymbol{y}_{(k)} \in \mathcal{S}_p$ to reduce the cost function.
4. Establish the convergence proof for qPpNWAWS.

3.1 Update Formula at Nonsingular Iterates

First, we consider the simplest case where the current iterate $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$. The corresponding update formula is:

$$\begin{aligned} y_{(k+1)}^{(t)} &:= \left(\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)})\right)^{(t)} \\ &:= \frac{\sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} x_i^{(t)}}{\sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2}}. \end{aligned} \tag{9}$$

The following descent property guarantees that (9) will reduce $C_{p,q}$ at any non-minimum nonsingular iterate.

Theorem 1 (Descent Property at Nonsingular Iterates)**.** *Let the cost function $C_{p,q}$ and the operator $\mathbf{T}_{p,q}$ be defined in* (8) *and* (9)*, respectively. For $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$, if $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$, then $C_{p,q}(\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)})) \leqslant C_{p,q}(\boldsymbol{y}_{(k)})$ with equality holds only when $\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)}) = \boldsymbol{y}_{(k)}$.*

The proof is provided in Supplementary B.1. The following corollary characterizes a minimum point $\boldsymbol{x}_*$ of (8) if $\boldsymbol{x}_* \notin \mathcal{S}_p$.

Corollary 2. *If $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$, then $\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)}) = \boldsymbol{y}_{(k)} \Leftrightarrow \boldsymbol{y}_{(k)}$ is a minimum point of model* (8)*, i.e., $\boldsymbol{y}_{(k)} = \boldsymbol{x}_*$.*

The proof is provided in Supplementary B.2. We then turn to the singular case.

3.2 Characterization of Subgradients and Minimum

Before we derive the iterative update for the singular iterate $\boldsymbol{y}_{(k)} \in \mathcal{S}_p$, we first introduce the de-singularity subgradient of $C_{p,q}(\boldsymbol{y}_{(k)})$ and characterize the minimum point(s) of (8).

Definition 3 (Subgradient, Rockafellar and Wets 2009)**.** *Let $C_{p,q} : \mathbb{R}^d \to \mathbb{R}$ be a convex function. A vector $\boldsymbol{v} \in \mathbb{R}^d$ is called a subgradient of $C_{p,q}$ at $\boldsymbol{y} \in \mathbb{R}^d$ if for all $\boldsymbol{z}\in\mathbb{R}^d$,*

$$C_{p,q}(\boldsymbol{z}) - C_{p,q}(\boldsymbol{y}) \geqslant \boldsymbol{v}^\top(\boldsymbol{z} - \boldsymbol{y}). \tag{10}$$

The set of all subgradients at $\boldsymbol{y}$ is denoted by $\partial C_{p,q}(\boldsymbol{y})$. If $C_{p,q}$ is differentiable at $\boldsymbol{y}$, then $\partial C_{p,q}(\boldsymbol{y})$ reduces to the gradient $\nabla C_{p,q}(\boldsymbol{y})$.

To construct $\partial C_{p,q}(\boldsymbol{y}_{(k)})$, we need to identify the singular component(s) of $\boldsymbol{y}_{(k)}$.

Definition 4 (Singular Component(s))**.** *For $\boldsymbol{y} \in \mathcal{S}_p$, each $t \in \{1, \ldots, d\}$ and each $i \in \{1, 2, \ldots, m\}$, let*

$$U_i(\boldsymbol{y}) := \{t \in \{1, \ldots, d\} : \ y^{(t)} = x_i^{(t)}\}, \tag{11}$$

$$V_t(\boldsymbol{y}) := \{i \in \{1, \ldots, m\} : \ y^{(t)} = x_i^{(t)}\}, \tag{12}$$

which represent the index sets of the dimensions and the data points such that $y^{(t)} = x_i^{(t)}$, respectively.

Figure 2 shows an intuitive example for $U_i(\boldsymbol{y}_{(k)})$ and $V_t(\boldsymbol{y}_{(k)})$.

t	$\boldsymbol{y}_{(k)}$	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$
1	1.1	1.1	1.1	2
2	1.7	1.3	2	1.6
3	1.4	1.5	1.5	1.4
4	1.7	1.7	1.6	1.7
5	1.5	1.9	1.4	1.8
6	1	0.9	1	1
7	0.6	0.7	1.5	0.8
8	0.8	0.5	0.8	0.6

Figure 2: An intuitive example for $U_i(\boldsymbol{y}_{(k)})$ and $V_t(\boldsymbol{y}_{(k)})$: $U_1(\boldsymbol{y}_{(k)}) = \{1, 4\}$, $U_2(\boldsymbol{y}_{(k)}) = \{1, 6, 8\}$, $U_3(\boldsymbol{y}_{(k)}) = \{3, 4, 6\}$, and $V_1(\boldsymbol{y}_{(k)}) = \{1, 2\}$, $V_2(\boldsymbol{y}_{(k)}) = \emptyset$, $V_3(\boldsymbol{y}_{(k)}) = \{3\}$, $V_4(\boldsymbol{y}_{(k)}) = \{1, 3\}$, $V_6(\boldsymbol{y}_{(k)}) = \{2, 3\}$, $V_8(\boldsymbol{y}_{(k)}) = \{2\}$.

Definition 5 (**q-th-Powered ℓ_p-Norm De-singularity Subgradient**)**.** *By removing the singular term(s), we define the de-singularity part $D_{p,q}: \mathbb{R}^d \to \mathbb{R}$ of $C_{p,q}$ and the q-th-powered ℓ_p-norm de-singularity subgradient as follows:*

$$D_{p,q}(\boldsymbol{y}) := \sum_{i=1}^{m} \eta_i^q \Big(\sum_{t \notin U_i(\boldsymbol{y})} |y^{(t)} - x_i^{(t)}|^p \Big)^{\frac{q}{p}}, \tag{13}$$

$$\big(\nabla D_{p,q}(\boldsymbol{y})\big)^{(t)} := \sum_{i \notin V_t(\boldsymbol{y})} q\eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p} |y^{(t)}-x_i^{(t)}|^{p-2}(y^{(t)}-x_i^{(t)}). \tag{14}$$

Theorem 6 (Characterization of Subgradients and Minimum)**.** *For $\boldsymbol{y} \in \mathcal{S}_p$,*

$$\partial C_{p,q}(\boldsymbol{y}) = \begin{cases} \{\nabla D_{1,1}(\boldsymbol{y}) + \boldsymbol{u}\} \text{ where } -a^{(t)} \leqslant u^{(t)} \leqslant a^{(t)}, \\ \qquad \forall t, \quad \text{if } p = q = 1, \\ \{\nabla D_{p,1}(\boldsymbol{y})\}, \text{ if } q{=}1, 1{<}p{<}2, \\ \qquad \boldsymbol{y} \in \mathcal{S}_p \backslash \{\boldsymbol{x}_i\}_{i=1}^m, \\ \{\nabla D_{p,1}(\boldsymbol{x}_l) + \eta_l \boldsymbol{b}\} \quad \text{where} \quad \|\boldsymbol{b}\|_r \leqslant 1, \\ \qquad \text{if } q{=}1, 1{<}p{<}2, \boldsymbol{y} = \boldsymbol{x}_l, \\ \{\nabla D_{p,q}(\boldsymbol{y})\}, \text{ if } 1 < q \leqslant p, 1 < p < 2, \end{cases} \tag{15}$$

where $a^{(t)} = \sum_{i \in V_t(\boldsymbol{y})} \eta_i$ and $\|\cdot\|_r$ is the conjugate norm of $\|\cdot\|_p$ such that $\frac{1}{r} + \frac{1}{p} = 1$.

The proof is provided in Supplementary B.3. According to Fermat's rule, $\boldsymbol{y} \in \mathcal{S}_p$ is a minimum point of (8) if and only if $\mathbf{0}_d \in \partial C_{p,q}(\boldsymbol{y})$, which is easy to verify. If $\boldsymbol{y}_{(k)}$ is not a minimum point, the following theorem shows a descent direction of $C_{p,q}(\boldsymbol{y}_{(k)})$.

Theorem 7 (Descent Property at Singular Iterates)**.** *For $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$, define the following direction*

$$\mathscr{D}_{p,q}(\boldsymbol{y}) = \begin{cases} \big(\nabla D_{p,1}(\boldsymbol{x}_l)\big)^{\frac{r}{p}} \text{ where } \frac{1}{r} + \frac{1}{p} = 1, \\ \qquad \text{if } q{=}1, 1{<}p{<}2, \boldsymbol{y} = \boldsymbol{x}_l, \\ \nabla D_{p,q}(\boldsymbol{y}), \qquad \text{else}, \end{cases} \tag{16}$$

where $(\cdot)^{\frac{r}{p}}$ denotes the element-wise signed power. If $\boldsymbol{y} \in \mathcal{S}_p$ is not a minimum point of (8), then there exists some $\lambda_* > 0$ such that for any $0 < \lambda \leqslant \lambda_*$, $C_{p,q}(\boldsymbol{y} - \lambda \mathscr{D}_{p,q}(\boldsymbol{y})) < C_{p,q}(\boldsymbol{y})$.

The proof is provided in Supplementary B.4. The key point is to verify that $-\mathscr{D}_{p,q}(\boldsymbol{y})$ is a descent direction if $\boldsymbol{y}$ is not a minimum point. To determine the step size λ_* in practice, we can start with an initial value of $\lambda_0 = \|\mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})\|_p$ and implement a line search $\lambda_{w+1} = \rho\lambda_w$ with $0 < \rho < 1$, until we find a value of λ_* such that $C_{p,q}(\boldsymbol{y}_{(k)} - \lambda_* \mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})) < C_{p,q}(\boldsymbol{y}_{(k)})$. Then we can construct the iterative update at the singular point $\boldsymbol{y}_{(k)}$ as

$$\boldsymbol{y}_{(k+1)} := \mathbf{T}_s(\boldsymbol{y}_{(k)}) := \boldsymbol{y}_{(k)} - \lambda_* \mathscr{D}_{p,q}(\boldsymbol{y}_{(k)}). \tag{17}$$

Combining (9) and (17), the whole qPpNWAWS is given by:

$$\boldsymbol{y}_{(k+1)} := \mathbf{T}(\boldsymbol{y}_{(k)}) := \begin{cases} \mathbf{T}_{p,q}(\boldsymbol{y}_{(k)}) & \text{if } \boldsymbol{y}_{(k)} \notin \mathcal{S}_p, \\ \mathbf{T}_s(\boldsymbol{y}_{(k)}) & \text{if } \boldsymbol{y}_{(k)} \in \mathcal{S}_p. \end{cases} \tag{18}$$

Theorems 1 and 7 indicate that $C_{p,q}(\boldsymbol{y}_{(k+1)}) < C_{p,q}(\boldsymbol{y}_{(k)})$ if $\boldsymbol{y}_{(k)}$ is not a minimum point (which can be characterized by qPpNWAWS). Moreover, since $C_{p,q}(\boldsymbol{y}) \geqslant 0$ for all $\boldsymbol{y} \in \mathbb{R}^d$, we can conclude that the sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ converges.

3.3 Convergence Theorem

To analyze the convergence of qPpNWAWS, we first provide several lemmas on some properties of the operators involved. The first lemma indicates that the operator $\mathbf{T}_{p,q}$ designed for the nonsingular iterates can be extended to the singular iterates.

Lemma 8. *Let the operator $\mathbf{T}_{p,q}$ be defined in (9). Then for $1 \leqslant q \leqslant p$, $1 \leqslant p < 2$,*

$$\lim_{\boldsymbol{y}\to\boldsymbol{x}, \boldsymbol{y}\notin\mathcal{S}_p} \mathbf{T}_{p,q}(\boldsymbol{y}) = \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathcal{S}_p. \tag{19}$$

The proof is provided in Supplementary B.5. Therefore, we can define

$$\mathbf{T}_{p,q}(\boldsymbol{x}) := \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathcal{S}_p,\ 1 \leqslant q \leqslant p,\ 1 \leqslant p < 2. \tag{20}$$

The second lemma confirms that the sequence $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ generated by qPpNWAWS is bounded, which is based on the descent property of $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$.

Lemma 9. *The sequence $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ generated by qPpNWAWS is bounded.*

The proof is provided in Supplementary B.6. As emphasized above, the singular set $\mathcal{S}_p$ contains infinite points in the case $1 \leqslant p < 2$, hence the operator $\mathbf{T}_{p,q}$ may not escape from $\mathcal{S}_p$. This is a major unavoidable obstacle for convergence analysis. To overcome this obstacle, we combine Lemma 9 and the Bolzano-Weierstrasz theorem, then there exists at least one limit point y_* and a subsequence $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}$ such that $\lim_{v\to\infty} \boldsymbol{y}_{(k_v)} = \boldsymbol{y}_*$. The following lemma indicates that there are only a finite number of limit points of $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ in the strictly convex case $1 < p < 2$, even though $\mathcal{S}_p$ has infinite points. This is a novel and significant theoretical result that plays a crucial part in the convergence proof.

Lemma 10. *If $1 < p < 2$, there are only a finite number of limit points of $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$. Moreover, all these limit points have the same cost function value.*

The proof is provided in Supplementary B.7.

Theorem 11 (Convergence Theorem)**.** *Let $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ be the iteration sequence generated by qPpNWAWS in (18). If $\boldsymbol{y}_{(k)}$ hits the minimum point $\boldsymbol{x}_*$ of model (8), the characterization of minimum (Corollary 2 and Theorem 6) ensures that this could be recognized and the algorithm will be stopped. Otherwise, the cost function sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ converges. Assume $1 < p < 2$ in addition. If $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ hits $\mathcal{S}_p$ for a finite number of times, then $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ also converges. On the other hand, if $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ has a nonsingular cluster point, then this cluster point is exactly the minimum point $\boldsymbol{x}_*$ and the entire sequence $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ converges to $\boldsymbol{x}_*$.*

The proof is provided in Supplementary B.8. We provide a practical way to verify the conditions of this theorem. When the algorithm reaches the convergence tolerance, we can check whether the last few iterates are singular points. If not, we can consider that $y_{(k)}$ converges to exactly the minimum point $\boldsymbol{x}_*$.

Data Set	Region	Time	Periods	Frequency	# Assets
CSI300	CN	Mar/16/2015- May/19/2017	534	Daily	47
NYSE(N)	US	Jan/1/1985 - Jun/30/2010	6431	Daily	23
FTSE100	UK	Nov/07/2002 - Nov/04/2016	717	Weekly	83
NASDAQ100	US	Mar/11/2004 - Nov/04/2016	596	Weekly	82
FF100	US	Jul/1971 - May/2023	623	Monthly	100
FF100MEOP	US	Jul/1971 - May/2023	623	Monthly	100

Table 1: Profiles of six benchmark data sets.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	4.92 ± 0.29	2.56 ± 1.07	2.15 ± 0.42	2.08 ± 0.36	2.04 ± 0.30	2.02 ± 0.28	1.99 ± 0.27	1.98 ± 0.25	1.97 ± 0.24	1.97 ± 0.24
1.1	-	2.83 ± 1.27	2.11 ± 0.39	2.07 ± 0.33	2.02 ± 0.29	2.00 ± 0.26	1.99 ± 0.26	1.97 ± 0.24	1.97 ± 0.24	1.96 ± 0.23
1.2	-	-	2.24 ± 0.63	2.04 ± 0.30	2.01 ± 0.27	2.00 ± 0.25	1.98 ± 0.24	1.97 ± 0.23	1.96 ± 0.23	1.96 ± 0.23
1.3	-	-	-	2.05 ± 0.36	2.00 ± 0.26	1.98 ± 0.24	1.97 ± 0.22	1.96 ± 0.22	1.96 ± 0.23	1.96 ± 0.23
1.4	-	-	-	-	2.00 ± 0.25	1.98 ± 0.23	1.96 ± 0.22	1.96 ± 0.22	1.96 ± 0.23	1.95 ± 0.23
1.5	-	-	-	-	-	1.98 ± 0.22	1.96 ± 0.23	1.96 ± 0.23	1.96 ± 0.22	1.95 ± 0.22
1.6	-	-	-	-	-	-	1.96 ± 0.23	1.96 ± 0.22	1.95 ± 0.22	1.95 ± 0.22
1.7	-	-	-	-	-	-	-	1.96 ± 0.21	1.95 ± 0.22	1.95 ± 0.22
1.8	-	-	-	-	-	-	-	-	1.95 ± 0.22	1.95 ± 0.22
1.9	-	-	-	-	-	-	-	-	-	1.95 ± 0.22

Table 2: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on CSI300 (mean$\pm$STD).

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	3.91 ± 0.60	1.90 ± 0.98	1.74 ± 0.81	1.68 ± 0.78	1.64 ± 0.74	1.60 ± 0.68	1.57 ± 0.64	1.54 ± 0.61	1.53 ± 0.60	1.52 ± 0.59
1.1	-	2.07 ± 1.16	1.72 ± 0.81	1.67 ± 0.78	1.63 ± 0.73	1.60 ± 0.70	1.57 ± 0.66	1.54 ± 0.61	1.53 ± 0.60	1.52 ± 0.59
1.2	-	-	1.75 ± 0.86	1.66 ± 0.77	1.62 ± 0.72	1.59 ± 0.68	1.56 ± 0.64	1.53 ± 0.61	1.52 ± 0.59	1.51 ± 0.58
1.3	-	-	-	1.65 ± 0.77	1.61 ± 0.70	1.58 ± 0.66	1.55 ± 0.63	1.53 ± 0.60	1.51 ± 0.58	1.50 ± 0.57
1.4	-	-	-	-	1.60 ± 0.68	1.57 ± 0.66	1.55 ± 0.63	1.52 ± 0.59	1.50 ± 0.57	1.49 ± 0.55
1.5	-	-	-	-	-	1.57 ± 0.65	1.55 ± 0.63	1.51 ± 0.58	1.49 ± 0.55	1.47 ± 0.53
1.6	-	-	-	-	-	-	1.54 ± 0.63	1.50 ± 0.57	1.47 ± 0.53	1.46 ± 0.51
1.7	-	-	-	-	-	-	-	1.48 ± 0.54	1.46 ± 0.50	1.45 ± 0.50
1.8	-	-	-	-	-	-	-	-	1.46 ± 0.50	1.45 ± 0.50
1.9	-	-	-	-	-	-	-	-	-	1.45 ± 0.50

Table 3: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on NYSE(N) (mean$\pm$STD).

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.0271	0.0193	0.0180	0.0169	0.0161	0.0155	0.0148	0.0143	0.0135	0.0130
	$Iter$	35.93 ± 15.04	15.02 ± 2.79	13.89 ± 2.07	13.38 ± 2.49	12.83 ± 2.68	12.34 ± 2.78	11.90 ± 2.82	11.51 ± 2.83	11.19 ± 2.81	10.92 ± 2.83
1.1	$Time$	-	0.0184	0.0173	0.0162	0.0153	0.0144	0.0136	0.0131	0.0123	0.0118
	$Iter$	-	15.01 ± 2.37	13.63 ± 1.89	12.95 ± 2.14	12.28 ± 2.25	11.73 ± 2.35	11.27 ± 2.37	10.84 ± 2.37	10.51 ± 2.37	10.24 ± 2.36
1.2	$Time$	-	-	0.0170	0.0157	0.0145	0.0136	0.0127	0.0122	0.0114	0.0109
	$Iter$	-	-	13.65 ± 1.86	12.66 ± 1.89	11.86 ± 1.87	11.27 ± 1.96	10.75 ± 1.98	10.32 ± 2.01	10.01 ± 2.03	9.74 ± 2.03
1.3	$Time$	-	-	-	0.0151	0.0140	0.0128	0.0119	0.0114	0.0107	0.0102
	$Iter$	-	-	-	12.45 ± 1.67	11.52 ± 1.55	10.82 ± 1.65	10.30 ± 1.64	9.90 ± 1.69	9.60 ± 1.74	9.32 ± 1.73
1.4	$Time$	-	-	-	-	0.0135	0.0122	0.0113	0.0107	0.0100	0.0095
	$Iter$	-	-	-	-	11.20 ± 1.24	10.44 ± 1.34	9.91 ± 1.38	9.48 ± 1.40	9.18 ± 1.48	8.90 ± 1.48
1.5	$Time$	-	-	-	-	-	0.0114	0.0109	0.0100	0.0092	0.0088
	$Iter$	-	-	-	-	-	10.09 ± 1.04	9.55 ± 1.09	9.12 ± 1.15	8.75 ± 1.24	8.49 ± 1.27
1.6	$Time$	-	-	-	-	-	-	0.0101	0.0094	0.0086	0.0081
	$Iter$	-	-	-	-	-	-	9.25 ± 0.86	8.77 ± 0.91	8.38 ± 1.05	8.08 ± 1.09
1.7	$Time$	-	-	-	-	-	-	-	0.0086	0.0079	0.0073
	$Iter$	-	-	-	-	-	-	-	8.46 ± 0.73	7.97 ± 0.88	7.63 ± 1.02
1.8	$Time$	-	-	-	-	-	-	-	-	0.0072	0.0070
	$Iter$	-	-	-	-	-	-	-	-	7.66 ± 0.71	7.45 ± 0.69
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.0064
	$Iter$	-	-	-	-	-	-	-	-	-	7.19 ± 0.41

Table 4: Average computational time (in seconds) and average number of iterations (mean$\pm$STD) for qPpNWAWS on CSI300.

To end this section, we indicate that a general constant-step subgradient descent method cannot guarantee convergence in the objective value for qPpNWLP due to the varying λ_* in Theorem 7. Besides, $\nabla C_{p,q}(\boldsymbol{y})$ in (2) may not be Lipschitz continuous even for $\boldsymbol{y} \notin \mathcal{S}_p$. We raise a counterexample: let $C_{p,q}(\boldsymbol{y}) := \|\boldsymbol{y}\|_p^q$, $\boldsymbol{y}_1 = \varepsilon \mathbf{1}_d > \mathbf{0}_d$, and $\boldsymbol{y}_2 = \frac{\varepsilon}{2}\mathbf{1}_d$. Then

$$\frac{\|\nabla C_{p,q}(\boldsymbol{y}_1) - \nabla C_{p,q}(\boldsymbol{y}_2)\|_2}{\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_2} = (2 - 2^{2-q})qd^{\frac{q-p}{p}}\varepsilon^{q-2}. \tag{21}$$

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.0011	0.0050	0.0071	0.0074	0.0060	0.0064	0.0055	0.0035	0.0033	0.0032
	$Iter$	26.71 ± 15.93	14.84 ± 6.67	13.62 ± 4.51	12.72 ± 3.35	12.07 ± 2.86	11.50 ± 2.73	11.02 ± 2.54	10.62 ± 2.45	10.29 ± 2.44	10.03 ± 2.43
1.1	$Time$	-	0.0067	0.0066	0.0068	0.0052	0.0061	0.0051	0.0032	0.0031	0.0029
	$Iter$	-	14.70 ± 8.64	13.29 ± 4.22	12.29 ± 3.04	11.56 ± 2.55	10.99 ± 2.34	10.52 ± 2.22	10.09 ± 2.11	9.77 ± 2.11	9.52 ± 2.10
1.2	$Time$	-	-	0.0075	0.0059	0.0044	0.0058	0.0048	0.0030	0.0029	0.0027
	$Iter$	-	-	13.05 ± 3.99	11.96 ± 2.84	11.13 ± 2.27	10.53 ± 2.06	10.06 ± 1.95	9.64 ± 1.85	9.33 ± 1.87	9.08 ± 1.90
1.3	$Time$	-	-	-	0.0056	0.0051	0.0057	0.0041	0.0029	0.0027	0.0026
	$Iter$	-	-	-	11.70 ± 2.75	10.77 ± 1.98	10.13 ± 1.80	9.65 ± 1.70	9.24 ± 1.64	8.92 ± 1.68	8.67 ± 1.72
1.4	$Time$	-	-	-	-	0.0048	0.0052	0.0028	0.0027	0.0025	0.0024
	$Iter$	-	-	-	-	10.47 ± 1.85	9.78 ± 1.58	9.29 ± 1.54	8.86 ± 1.47	8.55 ± 1.53	8.29 ± 1.57
1.5	$Time$	-	-	-	-	-	0.0043	0.0026	0.0025	0.0024	0.0023
	$Iter$	-	-	-	-	-	9.47 ± 1.40	8.96 ± 1.43	8.52 ± 1.36	8.17 ± 1.40	7.89 ± 1.46
1.6	$Time$	-	-	-	-	-	-	0.0025	0.0024	0.0022	0.0021
	$Iter$	-	-	-	-	-	-	8.66 ± 1.37	8.17 ± 1.27	7.80 ± 1.34	7.52 ± 1.33
1.7	$Time$	-	-	-	-	-	-	-	0.0022	0.0020	0.0019
	$Iter$	-	-	-	-	-	-	-	7.83 ± 1.16	7.38 ± 1.02	7.10 ± 1.10
1.8	$Time$	-	-	-	-	-	-	-	-	0.0019	0.0018
	$Iter$	-	-	-	-	-	-	-	-	7.07 ± 0.97	6.84 ± 1.03
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.0016
	$Iter$	-	-	-	-	-	-	-	-	-	6.50 ± 0.89

Table 5: Average computational time (in seconds) and average number of iterations (mean$\pm$STD) for qPpNWAWS on NYSE(N).

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.80 ± 0.07	0.64 ± 0.06	0.61 ± 0.06	0.58 ± 0.07	0.55 ± 0.08	0.52 ± 0.09	0.49 ± 0.10	0.47 ± 0.10	0.46 ± 0.11	0.44 ± 0.12
1.1	-	0.63 ± 0.05	0.60 ± 0.05	0.56 ± 0.06	0.52 ± 0.06	0.49 ± 0.07	0.46 ± 0.08	0.44 ± 0.09	0.42 ± 0.10	0.40 ± 0.11
1.2	-	-	0.59 ± 0.05	0.55 ± 0.05	0.51 ± 0.05	0.47 ± 0.06	0.44 ± 0.07	0.41 ± 0.08	0.39 ± 0.09	0.37 ± 0.10
1.3	-	-	-	0.54 ± 0.05	0.50 ± 0.04	0.45 ± 0.05	0.41 ± 0.06	0.38 ± 0.07	0.36 ± 0.08	0.34 ± 0.09
1.4	-	-	-	-	0.48 ± 0.04	0.43 ± 0.04	0.39 ± 0.05	0.35 ± 0.06	0.32 ± 0.07	0.30 ± 0.08
1.5	-	-	-	-	-	0.41 ± 0.03	0.36 ± 0.04	0.32 ± 0.05	0.29 ± 0.06	0.26 ± 0.07
1.6	-	-	-	-	-	-	0.34 ± 0.03	0.29 ± 0.03	0.25 ± 0.05	0.22 ± 0.08
1.7	-	-	-	-	-	-	-	0.26 ± 0.03	0.21 ± 0.04	0.17 ± 0.04
1.8	-	-	-	-	-	-	-	-	0.18 ± 0.02	0.14 ± 0.02
1.9	-	-	-	-	-	-	-	-	-	0.09 ± 0.01

Table 6: Average computational convergence rate (mean$\pm$STD) for qPpNWAWS on CSI300.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.75 ± 0.10	0.64 ± 0.07	0.61 ± 0.08	0.58 ± 0.08	0.55 ± 0.08	0.53 ± 0.09	0.50 ± 0.10	0.48 ± 0.10	0.46 ± 0.10	0.45 ± 0.11
1.1	-	0.63 ± 0.07	0.60 ± 0.07	0.57 ± 0.08	0.53 ± 0.08	0.50 ± 0.08	0.47 ± 0.09	0.45 ± 0.09	0.43 ± 0.10	0.42 ± 0.10
1.2	-	-	0.59 ± 0.07	0.55 ± 0.07	0.51 ± 0.07	0.48 ± 0.08	0.45 ± 0.08	0.42 ± 0.08	0.40 ± 0.09	0.38 ± 0.10
1.3	-	-	-	0.54 ± 0.07	0.50 ± 0.07	0.46 ± 0.07	0.42 ± 0.07	0.39 ± 0.08	0.37 ± 0.09	0.35 ± 0.10
1.4	-	-	-	-	0.48 ± 0.07	0.44 ± 0.07	0.40 ± 0.07	0.36 ± 0.07	0.34 ± 0.09	0.32 ± 0.11
1.5	-	-	-	-	-	0.42 ± 0.06	0.37 ± 0.07	0.33 ± 0.07	0.31 ± 0.09	0.28 ± 0.11
1.6	-	-	-	-	-	-	0.35 ± 0.08	0.30 ± 0.08	0.27 ± 0.10	0.24 ± 0.13
1.7	-	-	-	-	-	-	-	0.27 ± 0.09	0.23 ± 0.11	0.19 ± 0.07
1.8	-	-	-	-	-	-	-	-	0.18 ± 0.07	0.15 ± 0.07
1.9	-	-	-	-	-	-	-	-	-	0.11 ± 0.08

Table 7: Average computational convergence rate (mean$\pm$STD) for qPpNWAWS on NYSE(N).

Since $q<2$, (21) implies $\lim\limits_{\varepsilon \to 0} \frac{\|\nabla C_{p,q}(\boldsymbol{y}_1) - \nabla C_{p,q}(\boldsymbol{y}_2)\|_2}{\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_2} = \infty$. Therefore, a general subgradient descent method cannot achieve the sublinear convergence rate $O(\frac{1}{k})$. However, practical experiments show that qPpNWAWS achieves a linear computational convergence rate, which is efficient to solve qPpNWLP.

The whole procedure of qPpNWAWS is illustrated in Supplementary A. In each iteration, the algorithm first identifies whether the current iterate is a singular point and the dimensions where the singularity occurs. Next, it computes the de-singularity subgradient (or the normal gradient if there is no singularity). The de-singularity subgradient can then be used to determine whether the current iterate is a minimum point. If it is not, the algorithm performs a de-singularity subgradient descent step to reduce the cost function, proceeding to the next iteration until the convergence tolerance is met.

4 Experiment Result

We adopt the evaluating baseline in (Lai et al. 2024) with tests from different aspects to assess the performance of the proposed qPpNWAWS. In the median reversion strategy (Huang et al. 2016) for online portfolio selection (**OPS**, Li, Sahoo, and Hoi 2016; Lai et al. 2018c, 2020; Lai and Yang 2023), an important step is to compute the median of the asset prices in a recent time window. In the context of this

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	2.0603	1.8979	1.9468	1.9223	1.9012	2.0409	**2.0932**	2.0550	2.0468	1.9474
	SR	0.0563	0.0523	0.0538	0.0532	0.0523	0.0561	**0.0574**	0.0563	0.0560	0.0531
1.1	CW	-	1.9107	2.0278	1.8901	1.9175	2.0327	2.0157	2.0334	2.0160	1.9007
	SR	-	0.0526	0.0561	0.0520	0.0527	0.0559	0.0553	0.0557	0.0552	0.0518
1.2	CW	-	-	1.9781	2.0071	1.9788	2.0842	2.0296	2.0181	1.9821	1.8649
	SR	-	-	0.0545	0.0554	0.0545	0.0574	0.0557	0.0553	0.0542	0.0508
1.3	CW	-	-	-	1.9891	1.9682	2.0375	2.0247	1.9860	1.8868	1.7689
	SR	-	-	-	0.0548	0.0542	0.0560	0.0556	0.0544	0.0514	0.0477
1.4	CW	-	-	-	-	1.8620	1.9303	1.9373	1.9106	1.8795	1.7889
	SR	-	-	-	-	0.0509	0.0529	0.0530	0.0521	0.0512	0.0484
1.5	CW	-	-	-	-	-	1.7826	1.7957	1.8503	1.8596	1.8244
	SR	-	-	-	-	-	0.0483	0.0487	0.0503	0.0506	0.0495
1.6	CW	-	-	-	-	-	-	1.7073	1.7543	1.7665	1.8125
	SR	-	-	-	-	-	-	0.0457	0.0473	0.0477	0.0492
1.7	CW	-	-	-	-	-	-	-	1.6923	1.7183	1.7589
	SR	-	-	-	-	-	-	-	0.0452	0.0461	0.0475
1.8	CW	-	-	-	-	-	-	-	-	1.6924	1.7223
	SR	-	-	-	-	-	-	-	-	0.0453	0.0463
1.9	CW	-	-	-	-	-	-	-	-	-	1.7101
	SR	-	-	-	-	-	-	-	-	-	0.0459

Table 8: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on CSI300. The CW and SR for the original setting $(q,p) = (1,2)$ are 1.7750 and 0.0479, respectively.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	$2.0561E+07$	$4.3557E+07$	$6.3893E+07$	$1.7701E+08$	$1.4948E+08$	$2.6522E+08$	$1.8973E+08$	$3.0391E+08$	$3.2967E+08$	$4.1537E+08$
	SR	0.0935	0.0968	0.0980	0.1022	0.1012	0.1036	0.1020	0.1042	0.1042	0.1046
1.1	CW	-	$6.5105E+07$	$1.0056E+08$	$1.9101E+08$	$1.3263E+08$	$2.5663E+08$	$2.5658E+08$	$3.0408E+08$	$4.5008E+08$	$7.1092E+08$
	SR	-	0.0987	0.1002	0.1027	0.1007	0.1034	0.1033	0.1039	0.1051	0.1068
1.2	CW	-	-	$1.4552E+08$	$1.1925E+08$	$1.6110E+08$	$1.5340E+08$	$1.9023E+08$	$3.1231E+08$	$6.0239E+08$	$8.0655E+08$
	SR	-	-	0.1020	0.1006	0.1015	0.1009	0.1017	0.1039	0.1060	0.1072
1.3	CW	-	-	-	$1.1036E+08$	$1.0286E+08$	$1.4315E+08$	$1.7280E+08$	$2.4992E+08$	$6.5705E+08$	$\mathbf{8.5677E+08}$
	SR	-	-	-	0.0998	0.0994	0.1005	0.1012	0.1026	0.1064	**0.1075**
1.4	CW	-	-	-	-	$1.1330E+08$	$1.3560E+08$	$1.6290E+08$	$1.7457E+08$	$5.0738E+08$	$7.7795E+08$
	SR	-	-	-	-	0.0998	0.1001	0.1007	0.1009	0.1053	0.1071
1.5	CW	-	-	-	-	-	$2.2089E+08$	$1.2184E+08$	$2.1543E+08$	$4.6540E+08$	$6.5877E+08$
	SR	-	-	-	-	-	0.1024	0.0993	0.1018	0.1049	0.1064
1.6	CW	-	-	-	-	-	-	$7.5558E+07$	$2.3410E+08$	$3.4107E+08$	$5.0696E+08$
	SR	-	-	-	-	-	-	0.0973	0.1021	0.1035	0.1052
1.7	CW	-	-	-	-	-	-	-	$1.2833E+08$	$3.0709E+08$	$5.0657E+08$
	SR	-	-	-	-	-	-	-	0.0993	0.1030	0.1051
1.8	CW	-	-	-	-	-	-	-	-	$2.8561E+08$	$4.0895E+08$
	SR	-	-	-	-	-	-	-	-	0.1026	0.1041
1.9	CW	-	-	-	-	-	-	-	-	-	$3.2349E+08$
	SR	-	-	-	-	-	-	-	-	-	0.1031

Table 9: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on NYSE(N). The CW and SR for the original setting $(q,p) = (1,2)$ are $3.3183e+08$ and 0.1034, respectively.

paper, it aims to find a q-th-powered ℓ_p-norm median

$$\hat{\boldsymbol{x}} \in \arg\min_{\boldsymbol{y}\in\mathbb{R}^d} \sum_{i=1}^{m} \|\boldsymbol{y} - \boldsymbol{x}_i\|_p^q,\ 1 \leqslant q \leqslant p,\ 1 \leqslant p < 2, \quad (22)$$

where $\mathbf{x}_i \in \mathbb{R}^d$ represents the price vector of d assets on the i-th trading day, and $\{\mathbf{x}_i\}_{i=1}^m$ contains the asset prices for the most recent m trading days.

Experiments are conducted on six data sets: CSI300 (Lai et al. 2024), NYSE(N) (Li et al. 2013), FTSE100, NASDAQ100 (Bruni et al. 2016), FF100, and FF100MEOP (Lin et al. 2024). Profiles of these data sets are shown in Table 1. These data sets cover financial markets from different regions like China, the United States, and the United Kingdom. They also cover different frequencies, including daily, weekly, and monthly. Their dimensionalities range from 23 to 100. CSI300 is extracted by Lai et al. (2024) from the CSI300 constituents[1] of Shanghai Stock Exchange and Shenzhen Stock Exchange in China, while FF100 and FF100MEOP are extracted by Lin et al. (2024) from the Kenneth R. French's Data Library[2]. FF100 is built on ME and BE/ME, while FF100MEOP is built on ME and operating profitability, respectively. All these data sets cover a wide range of practical scenarios that are sufficient to test the performance of the proposed qPpNWAWS. Due to the page limit, the experimental results on CSI300 and NYSE(N) are presented in the main text, while those on other data sets are presented in Supplementary C (Tables A1∼A16). All these results consistently show the effectiveness of qPpNWAWS.

The experiments include four parts:

1. We verify that qPpNWAWS successfully reduces the cost function at singular iterates, which solves the singularity problem.
2. We validate the computational efficiency of qPpNWAWS

[1]http://www.csindex.com.cn

[2]http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

by analyzing the number of iterations and the running time required for convergence.

3. We verify that the computational convergence rate of qPpNWAWS is a linear convergence rate.
4. By assessing the investing metrics in OPS, we demonstrate the advantages of qPpNWAWS with $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$. Hence qPpNWAWS is useful in a practical sense.

We change p and q in $[1, 1.9]$ with $q \leqslant p$, which covers enough situations of $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$. The time window size m is set as 5 by following previous methods (Huang et al. 2016; Lai et al. 2018a,b, 2022). The convergence tolerance thresholds are set as $Tol = 10^{-4}$ and $Tol\_2 = 10^{-14}$, and the reducing factor ρ in the line search is set as 0.1. As the observation window moves from $t = 1$ to $t = T - m + 1$, there are a total of $(T - m + 1)$ sets of data points $\{\mathbf{x}_i\}_{i=1}^m$. Therefore, we evaluate the average performance of qPpNWAWS by conducting the experiments for $(T - m + 1)$ times on each data set. The experiments are carried out on a desktop workstation with an Intel Core i9-14900KF CPU, 64-GB DDR5 6000-MHz memory cards, and an Nvidia RTX 4080 graphics card with 16-GB independent memory.

4.1 Solving the Singularity Problem

We record the average number of iterations required for qPpNWAWS to successfully reduce the cost function at singular iterates. The starting iterate $\boldsymbol{y}_{(0)}$ is set as the singular point $\boldsymbol{x}_1$. For each (q, p) pair, we calculate the mean and the standard deviation (STD) of the number of iterations required on the $(T - m + 1)$ sets of data points $\{\boldsymbol{x}_i\}_{i=1}^m$, shown in Tables 2 and 3. Results show that qPpNWAWS successfully reduces the cost function in only a few iterations, thereby solving the singularity problem. As p and q increase, the average number of iterations shows a decreasing trend, ranging from 4.92 to 1.95 on CSI300 and from 3.91 to 1.45 on NYSE(N). A smaller ρ may lead to even fewer iterations required in the line search of the step size λ_*.

4.2 Computational Cost and Convergence

We record the average number of iterations and the average running time for qPpNWAWS to achieve convergence in Tables 4 and 5. Results show that qPpNWAWS achieves rapid convergence that the average running times are all smaller than 0.03s and the numbers of iterations are no larger than 36. As p and q increase, the average number of iterations also shows a decreasing trend, ranging from 35.93 to 7.19 on CSI300 and from 26.71 to 6.50 on NYSE(N). To summarize, qPpNWAWS successfully converges at a desirable speed.

4.3 Computational Convergence Rate

We use the following formula to assess the computational convergence rate of qPpNWAWS:

$$\frac{1}{Iter - 2} \sum_{o=3}^{Iter} \frac{\|\boldsymbol{y}_{(o-1)} - \boldsymbol{y}_{(Iter)}\|_2}{\|\boldsymbol{y}_{(o-2)} - \boldsymbol{y}_{(Iter)}\|_2}, \tag{23}$$

where $Iter$ and $\boldsymbol{y}_{(Iter)}$ denote the total number of iterations and the final iterate, respectively. Tables 6 and 7 show the mean and STD of the computational convergence rates for qPpNWAWS with different (q, p) pairs. As p and q increase, the average computational convergence rate decreases from 0.8 to 0.09. Since they are all significantly smaller than 1, qPpNWAWS achieves at least a linear computational convergence rate.

4.4 Investing Performance

To further assess the effectiveness of qPpNWAWS in real-world applications, we employ two main investing metrics, the final cumulative wealth (CW) and the daily Sharpe Ratio (SR, Sharpe 1966), to conduct OPS experiments. The final CW indicates the final gain of an investing strategy at the end of the entire investment, while the SR is a kind of risk-adjusted return. We use qPpNWAWS to compute the q-th-powered ℓ_p-norm median in (22), and then adopt the strategy in (Huang et al. 2016) to produce the CW and SR scores. Results with different (q, p) pairs as well as the original setting $(q, p) = (1, 2)$ in (Huang et al. 2016) are given in Tables 8 and 9. They indicate that qPpNWAWS achieves the best results with $(q, p) = (1, 1.6)$ on CSI300 and with $(q, p) = (1.3, 1.9)$ on NYSE(N). Besides, several (q, p) pairs perform better than the original setting $(q, p) = (1, 2)$. These results indicate that qPpNWAWS for solving qPpNWLP is useful and advantageous with $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$.

5 Conclusions and Future Works

This paper proposes a q-th-Powered ℓ_p-Norm Weiszfeld Algorithm without Singularity (qPpNWAWS) for the q-th-Powered ℓ_p-Norm Weber Location Problem (qPpNWLP) with $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$, which includes all the rest unsolved situations in this problem. One main difficulty to solve this problem is that the singular points constitute a continuum set, so that any gradient-type algorithm may visit the singular set for infinite times. qPpNWAWS is able to characterize the subgradients and minimum at any singular or nonsingular point. If it is not a minimum point, qPpNWAWS can further reduce the cost function. Moreover, it guarantees convergence in the objective function value.

Experimental results on six real-world data sets show that qPpNWAWS successfully reduces the cost function in a few iterations at a singular point. It achieves convergence in a few iterations and shows a linear computational convergence rate. Moreover, it performs well in the online portfolio selection task that its final cumulative wealth and its Sharpe ratio with several (q, p) pairs are higher than those with the original setting $(q, p) = (1, 2)$. Thus qPpNWAWS and qPpNWLP with $1 \leqslant p < 2$ and $1 \leqslant q \leqslant p$ are useful and advantageous in practice. In future works, we will extend the de-singularity methodology to the multi-facility location problem.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China under grants 62176103, 62276114, 62206110, in part by Guangdong Basic and Applied Basic Research Foundation under grant 2023B1515120064, in part by the Science and Technology Planning Project of Guangzhou under grants 2024A04J9896, 202206030007, in part by Guangdong-Macao Advanced Intelligent Computing Joint Laboratory under grant 2020B1212030003, in part by the Key Laboratory of Smart Education of Guangdong Higher Education Institutes, Jinan University under grant 2022LSYS003, in part by the Fundamental Research Funds for the Central Universities, JNU under grant 21623202, and in part by the Major Key Project of PCL (No. PCL2024A04).

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Supplementary Material

A Solving Algorithm

To simplify the expressions, we convert the multiplicities from $\{\eta_i\}_{i=1}^m$ back to $\{\xi_i\}_{i=1}^m$.

Algorithm 1 q-th-Powered ℓ_p-Norm Weiszfeld Algorithm without Singularity (qPpNWAWS)

Require: Given m distinct data points $\{\mathbf{x}_i\}_{i=1}^m$, the corresponding multiplicities $\{\xi_i\}_{i=1}^m$, the order of power q and the parameter p of the ℓ_p norm, the line search factor $0 < \rho < 1$ and the tolerance thresholds Tol and Tol_2.

Initialize with a starting point $\mathbf{y}_{(0)}$.

while 1 **do**
  Initialize $Sing = 0$ and $l = 0$.
  for $t = 1$ to d **do**
    Compute $V_t(\boldsymbol{y}_{(k)}) = \{i \in \{1, \ldots, m\}\ s.t.\ y_{(k)}^{(t)} = x_i^{(t)}\}$ and $V_t'(\boldsymbol{y}_{(k)}) = \{1, \ldots, m\} \backslash V_t(\boldsymbol{y}_{(k)})$.
    if $|V_t(\boldsymbol{y}_{(k)})| > 0$ **then**
      $Sing = Sing + 1$.
    end if
  end for
  if $Sing = 0$ **then**
    for $t = 1$ to d **do**
      Compute $y_{(k+1)}^{(t)} = \frac{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} x_i^{(t)}}{\sum_{i=1}^m \xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2}}$.
    end for
    if $\boldsymbol{y}_{(k+1)} = \boldsymbol{y}_{(k)}$ **then**
      Set $\boldsymbol{x}_* = \boldsymbol{y}_{(k)}$ and break.
    end if
  else
    for $t = 1$ to d **do**
      Compute $\left(\nabla D_{p,q}(\boldsymbol{y}_{(k)})\right)^{(t)} = \sum_{i \in V_t'(\boldsymbol{y}_{(k)})} q\xi_i \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} (y_{(k)}^{(t)} - x_i^{(t)})$.
    end for
    if $\|\nabla D_{p,q}(\boldsymbol{y}_{(k)})\|_2 = 0$ **then**
      Set $\boldsymbol{x}_* = \boldsymbol{y}_{(k)}$ and break.
    else if $p = q = 1$ **then**
      Initialize $RE_a = 0$.
      for $t = 1$ to d **do**
        Compute $a^{(t)} = \sum_{i \in V_t(\boldsymbol{y}_{(k)})} \xi_i$.
        if $|\left(\nabla D_{p,q}(\boldsymbol{y}_{(k)})\right)^{(t)}| > a^{(t)}$ **then**
          $RE_a = RE_a + 1$.
        end if
      end for
      if $RE_a = 0$ **then**
        Set $\boldsymbol{x}_* = \boldsymbol{y}_{(k)}$ and break.
      end if
    else if $q = 1$, $1 < p < 2$ **then**
      for $i = 1$ to m **do**
        if $\boldsymbol{y}_{(k)} = \boldsymbol{x}_i$ **then**
          $l = i$. Break.
        end if
      end for
      if $l \neq 0$ and $\|\nabla D_{p,q}(\boldsymbol{y}_{(k)})\|_r \leqslant \xi_l$ **then**
        Set $\boldsymbol{x}_* = \boldsymbol{x}_l$ and break.
      end if
    end if
    Compute $\mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})$ by (16).
    Set $w = 0$, $\lambda_0 = \|\mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})\|_p$.
    while $C_{p,q}(\boldsymbol{y}_{(k)} - \lambda_w \mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})) \geqslant C_{p,q}(\boldsymbol{y}_{(k)})$ **do**
      $\lambda_{w+1} = \rho\lambda_w$. $w \leftarrow w + 1$.
    end while

$\boldsymbol{y}_{(k+1)} = \boldsymbol{y}_{(k)} - \lambda_w \mathscr{D}_{p,q}(\boldsymbol{y}_{(k)})$.
end if
if $\|\boldsymbol{y}_{(k+1)} - \boldsymbol{y}_{(k)}\|_2 / \|\boldsymbol{y}_{(k)}\|_2 \leqslant Tol$ or $\|C_{p,q}(\boldsymbol{y}_{(k+1)}) - C_{p,q}(\boldsymbol{y}_{(k)})\|_2 / \|C_{p,q}(\boldsymbol{y}_{(k)})\|_2 \leqslant Tol\_2$ **then**
Set $\boldsymbol{x}_* = \boldsymbol{y}_{(k+1)}$ and break.
end if
$k \leftarrow k + 1$
end while
Ensure: The output $\boldsymbol{x}_*$.

B Proofs

B.1 Proof of Theorem 1

We need the following lemma from (Beckenbach and Bellman 2012) to prove this Theorem.

Lemma 12. *If $u < 1$, $v < 1$ and $\frac{1}{u} + \frac{1}{v} = 1$, then for $a > 0$, $b > 0$, $a^{\frac{1}{u}} b^{\frac{1}{v}} \geqslant \frac{a}{u} + \frac{b}{v}$.*

Proof of Theorem 1. Let $\tilde{C}_{p,q}^{(t)}(a) := \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} (a - x_i^{(t)})^2$ for $a \in \mathbb{R}$. Then $\tilde{C}_{p,q}^{(t)}(a)$ is strictly convex. We first analyze the case that $1 < p < 2$. Since $\tilde{C}_{p,q}^{(t)'}(a) = 2 \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} (a - x_i^{(t)})$, then $\tilde{C}_{p,q}^{(t)}(a)$ has a unique minimum at $y_{(k+1)}^{(t)}$, which indicates that if $y_{(k+1)}^{(t)} \neq y_{(k)}^{(t)}$,

$$\tilde{C}_{p,q}^{(t)}(y_{(k+1)}^{(t)}) < \tilde{C}_{p,q}^{(t)}(y_{(k)}^{(t)}) = \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^p.$$

Therefore,

$$\sum_{t=1}^d \tilde{C}_{p,q}^{(t)}(y_{(k+1)}^{(t)}) < \sum_{t=1}^d \tilde{C}_{p,q}^{(t)}(y_{(k)}^{(t)}) = \sum_{i=1}^m \sum_{t=1}^d \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^p = \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^q. \tag{24}$$

On the other hand, it follows from Lemma 12 that

$$\begin{aligned}
&\sum_{t=1}^d \tilde{C}_{p,q}^{(t)}(y_{(k+1)}^{(t)}) \\
=&\sum_{t=1}^d \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2} |y_{(k+1)}^{(t)} - x_i^{(t)}|^2 \\
=&\sum_{t=1}^d \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} \big(|y_{(k)}^{(t)} - x_i^{(t)}|^p\big)^{\frac{p-2}{p}} \big(|y_{(k+1)}^{(t)} - x_i^{(t)}|^p\big)^{\frac{2}{p}} \\
\geqslant&\sum_{t=1}^d \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} \big(\frac{p-2}{p} |y_{(k)}^{(t)} - x_i^{(t)}|^p + \frac{2}{p} |y_{(k+1)}^{(t)} - x_i^{(t)}|^p\big) \\
=&(1 - \frac{2}{p}) \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^q + \frac{2}{p} \sum_{i=1}^m \eta_i^q (\|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^q)^{\frac{q-p}{q}} (\|\boldsymbol{y}_{(k+1)} - \boldsymbol{x}_i\|_p^q)^{\frac{p}{q}} \\
\geqslant&(1 - \frac{2}{p}) \sum_{i=1}^m \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^q + \frac{2}{p} \sum_{i=1}^m \eta_i^q \big(\frac{q-p}{q} \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^q + \frac{p}{q} \|\boldsymbol{y}_{(k+1)} - \boldsymbol{x}_i\|_p^q\big) \\
=&(1 - \frac{2p}{pq}) C_{p,q}(\boldsymbol{y}_{(k)}) + \frac{2p}{pq} C_{p,q}(\boldsymbol{y}_{(k+1)}) \\
=&(1 - \frac{2}{q}) C_{p,q}(\boldsymbol{y}_{(k)}) + \frac{2}{q} C_{p,q}(\boldsymbol{y}_{(k+1)}).
\end{aligned} \tag{25}$$

Combining both directions of the inequalities of (24) and (25), we know that

$$(1 - \frac{2}{q}) C_{p,q}(\boldsymbol{y}_{(k)}) + \frac{2}{q} C_{p,q}(\boldsymbol{y}_{(k+1)}) < C_{p,q}(\boldsymbol{y}_{(k)}).$$

Then $C_{p,q}(\boldsymbol{y}_{(k+1)}) \leqslant C_{p,q}(\boldsymbol{y}_{(k)})$ with equality holds only when $\boldsymbol{y}_{(k+1)} = \mathbf{T}_1(\boldsymbol{y}_{(k)}) = \boldsymbol{y}_{(k)}$. ◻

B.2 Proof of Corollary 2

Proof. Since $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$, we can compute that

$$\big(\nabla C_{p,q}(\boldsymbol{y}_{(k)})\big)^{(t)} = \sum_{i=1}^{m} q\eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2}(y_{(k)}^{(t)} - x_i^{(t)}). \tag{26}$$

Combining (9) and (26), we obtain the following equivalence for all t:

$$\big(\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)})\big)^{(t)} = y_{(k)}^{(t)} \Longleftrightarrow \frac{1}{q}\big(\nabla C_{p,q}(\boldsymbol{y}_{(k)})\big)^{(t)} = \sum_{i=1}^{m} \eta_i^q \|\boldsymbol{y}_{(k)} - \boldsymbol{x}_i\|_p^{q-p} |y_{(k)}^{(t)} - x_i^{(t)}|^{p-2}(y_{(k)}^{(t)} - x_i^{(t)}) = 0, \tag{27}$$

which indicates $\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)}) = \boldsymbol{y}_{(k)} \Leftrightarrow \nabla C_{p,q}(\boldsymbol{y}_{(k)}) = \mathbf{0}_d$. Since $C_{p,q}(\boldsymbol{y})$ is convex, then $\nabla C_{p,q}(\boldsymbol{y}_{(k)}) = \mathbf{0}_d \Leftrightarrow \boldsymbol{y}_{(k)}$ is a minimum point $\boldsymbol{x}_*$ satisfying (8). □

B.3 Proof of Theorem 6

Proof. Let $\boldsymbol{y} + \lambda\boldsymbol{z}$ ($\lambda > 0, \boldsymbol{z} \in \mathbb{R}^d$ and $\|\boldsymbol{z}\|_2 = 1$) be a point displaced from $\boldsymbol{y}$ towards an arbitrary direction $\boldsymbol{z}$. Recall the fact that

$$\boldsymbol{v} \in \partial C_{p,q}(\boldsymbol{y}) \Longleftrightarrow \boldsymbol{v}^\top \boldsymbol{z} \leqslant \frac{\mathrm{d}C_{p,q}(\boldsymbol{y} + \lambda\boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} \text{ for all } \boldsymbol{z}. \tag{28}$$

Then we calculate the subgradient(s) $\partial C_{p,q}(\boldsymbol{y})$ by calculating $\frac{\mathrm{d}C_{p,q}(\boldsymbol{y}+\lambda\boldsymbol{z})}{\mathrm{d}\lambda}$ first, which is given by

$$\begin{aligned}\frac{\mathrm{d}C_{p,q}(\boldsymbol{y} + \lambda\boldsymbol{z})}{\mathrm{d}\lambda} =& \sum_{i=1}^{m}\sum_{t=1}^{d} q\eta_i^q z^{(t)} \|\boldsymbol{y} + \lambda\boldsymbol{z} - \boldsymbol{x}_i\|_p^{q-p} |y^{(t)} + \lambda z^{(t)} - x_i^{(t)}|^{p-2}(y^{(t)} + \lambda z^{(t)} - x_i^{(t)}) \\ =& M_{p,q}(\boldsymbol{y}, \lambda) + G_{p,q}(\boldsymbol{y}, \lambda),\end{aligned} \tag{29}$$

where

$$\begin{aligned}M_{p,q}(\boldsymbol{y}, \lambda) &:= \sum_{i=1}^{m}\sum_{t \notin U_i(\boldsymbol{y})} q\eta_i^q z^{(t)} \|\boldsymbol{y} + \lambda\boldsymbol{z} - \boldsymbol{x}_i\|_p^{q-p} |y^{(t)} + \lambda z^{(t)} - x_i^{(t)}|^{p-2}(y^{(t)} + \lambda z^{(t)} - x_i^{(t)}), \\ G_{p,q}(\boldsymbol{y}, \lambda) &:= \sum_{i=1}^{m}\sum_{t \in U_i(\boldsymbol{y})} q\eta_i^q \lambda^{p-1} |z^{(t)}|^p \|\boldsymbol{y} + \lambda\boldsymbol{z} - \boldsymbol{x}_i\|_p^{q-p}.\end{aligned}$$

We compute the subgradient(s) of $C_{p,q}(\boldsymbol{y})$ at $\boldsymbol{y} \in \mathcal{S}_p$ by considering three cases based on the values of p and q: (a) $q = p = 1$; (b) $q = 1, 1 < p < 2$; (c) $1 < q \leqslant p, 1 < p < 2$.

Case (a): We compute the limit of $\frac{\mathrm{d}}{\mathrm{d}\lambda}C_{1,1}(\boldsymbol{y} + \lambda\boldsymbol{z})$ as $\lambda \to 0$, which is given by

$$\frac{\mathrm{d}C_{1,1}(\boldsymbol{y} + \lambda\boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} = M_{1,1}(\boldsymbol{y}, 0) + \sum_{t=1}^{d} |z^{(t)}| \sum_{i \in V_t(\boldsymbol{y})} \eta_i. \tag{30}$$

From Definition 5, $M_{1,1}(\boldsymbol{y}, 0) = \nabla D_{1,1}(\boldsymbol{y})^\top \boldsymbol{z}$. Then (30) can be formulated as

$$\frac{\mathrm{d}C_{1,1}(\boldsymbol{y} + \lambda\boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} = \nabla D_{1,1}(\boldsymbol{y})^\top \boldsymbol{z} + \boldsymbol{a}^\top |\boldsymbol{z}|, \tag{31}$$

where $\boldsymbol{a} \in \mathbb{R}^d$ is defined by $a^{(t)} = \sum_{i \in V_t(\boldsymbol{y})} \eta_i$ for all i. We can easily know from (28) and (31) that the subgradients of $C_{1,1}(\boldsymbol{y})$ at $\boldsymbol{y} \in \mathcal{S}_p$ can be formulated as

$$\partial C_{1,1}(\boldsymbol{y}) = \{\nabla D_{1,1}(\boldsymbol{y}) + \boldsymbol{u}\},$$

where $\boldsymbol{u} \in \mathbb{R}^d$ is an arbitrary vector satisfying $-a^{(t)} \leqslant u^{(t)} \leqslant a^{(t)}$ for all t.

Case (b): In this case, if $\boldsymbol{y} = \boldsymbol{x}_l$ for some $l \in \{1, \dots, m\}$, then the limit of $\frac{\mathrm{d}C_{p,1}(\boldsymbol{x}_l+\lambda\boldsymbol{z})}{\mathrm{d}\lambda}$ as $\lambda \to 0$ is given by

$$\frac{\mathrm{d}C_{p,1}(\boldsymbol{x}_l + \lambda\boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} = M_{p,1}(\boldsymbol{x}_l, 0) + \eta_l\|\boldsymbol{z}\|_p = \nabla D_{p,1}(\boldsymbol{x}_l)^\top \boldsymbol{z} + \eta_l\|\boldsymbol{z}\|_p. \tag{32}$$

Let $\boldsymbol{u} = \nabla D_{p,1}(\boldsymbol{x}_l)^\top + \eta_l\boldsymbol{b}$, where $\boldsymbol{b} \in \mathbb{R}^d$ satisfies $\|\boldsymbol{b}\|_r \leqslant 1$. To prove that

$$\partial C_{p,1}(\boldsymbol{x}_l) = \{\nabla D_{p,1}(\boldsymbol{x}_l)^\top + \eta_l\boldsymbol{b},\ \|\boldsymbol{b}\|_r \leqslant 1\}, \tag{33}$$

we first prove that $\boldsymbol{u} \in \partial C_{p,1}(\boldsymbol{x}_l)$. From the Hölder's inequality, for $p > 1$, $r > 1$ with $\frac{1}{p} + \frac{1}{r} = 1$, we have

$$(\boldsymbol{u} - \nabla D_{p,1}(\boldsymbol{x}_l))^\top \boldsymbol{z} \leqslant \|\boldsymbol{u} - \nabla D_{p,1}(\boldsymbol{x}_l)\|_r \|\boldsymbol{z}\|_p,$$

Then

$$\boldsymbol{u}^\top \boldsymbol{z} \leqslant \nabla D_{p,1}(\boldsymbol{x}_l)^\top \boldsymbol{z} + \|\boldsymbol{u} - \nabla D_{p,1}(\boldsymbol{x}_l)\|_r \|\boldsymbol{z}\|_p \leqslant \nabla D_{p,1}(\boldsymbol{x}_l)^\top \boldsymbol{z} + \eta_l \|\boldsymbol{z}\|_p,$$

which implies that $\boldsymbol{u} \in \partial C_{p,1}(\boldsymbol{x}_l)$. We next prove that all the subgradients of $C_{p,1}(\boldsymbol{x}_l)$ can be formulated as $C_{p,1}(\boldsymbol{x}_l) = \nabla D_{p,1}(\boldsymbol{x}_l)^\top + \eta_l \boldsymbol{b}$ by contradiction. If there exists some $\boldsymbol{v} \in \partial C_{p,1}(\boldsymbol{x}_l)$ such that $\boldsymbol{v} \neq \nabla D_{p,1}(\boldsymbol{x}_l)^\top + \eta_l \boldsymbol{b}$, then $\|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)\|_r > \eta_l$. Let $\boldsymbol{z} = \frac{|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^{\frac{r}{p}}}{\||\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^{\frac{r}{p}}\|_2}$. Then $\|\boldsymbol{z}\|_2 = 1$ and $\boldsymbol{z}^p = \frac{|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^r}{\||\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^{\frac{r}{p}}\|_2}$, which satisfies the condition for equality to hold in the Hölder's inequality. Then

$$|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^\top \boldsymbol{z} = \|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)\|_r \|\boldsymbol{z}\|_p$$

and therefore,

$$|\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^\top \boldsymbol{z} > \eta_l \|\boldsymbol{z}\|_p. \tag{34}$$

Define a vector $\tilde{\boldsymbol{z}} \in \mathbb{R}^d$ such that

$$\tilde{z}^{(t)} := \begin{cases} -z^{(t)} & \text{if } \big(\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)\big)^{(t)} < 0, \\ z^{(t)} & \text{else.} \end{cases}$$

Then $\|\tilde{\boldsymbol{z}}\|_p = \|\boldsymbol{z}\|_p$ and $(\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l))^\top \tilde{\boldsymbol{z}} = |\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l)|^\top \boldsymbol{z}$. It follows from (34) that

$$(\boldsymbol{v} - \nabla D_{p,1}(\boldsymbol{x}_l))^\top \tilde{\boldsymbol{z}} > \eta_l \|\tilde{\boldsymbol{z}}\|_p,$$

which implies $\boldsymbol{v}^\top \tilde{\boldsymbol{z}} > \nabla D_{p,1}(\boldsymbol{x}_l)^\top \tilde{\boldsymbol{z}} + \eta_l \|\tilde{\boldsymbol{z}}\|_p$. This contradicts that $\boldsymbol{v} \in \partial C_{p,1}(\boldsymbol{x}_l)$. Then we can conclude that (33) holds.

If $\boldsymbol{y} \in \mathcal{S}_p \backslash \{\boldsymbol{x}_i\}_{i=1}^m$, then the limit of $\frac{\mathrm{d}}{\mathrm{d}\lambda} C_{p,1}(\boldsymbol{y} + \lambda \boldsymbol{z})$ as $\lambda \to 0$ is given by

$$\frac{\mathrm{d}C_{p,1}(\boldsymbol{y} + \lambda \boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} = M_{p,1}(\boldsymbol{x}_l, 0) = \nabla D_{p,1}(\boldsymbol{y})^\top \boldsymbol{z}, \tag{35}$$

which together with (35) indicates that the subgradient of $C_{p,1}(\boldsymbol{y})$ at $\boldsymbol{y} \in \mathcal{S}_p \backslash \{\boldsymbol{x}_i\}_{i=1}^m$ can be formulated as

$$\partial C_{p,1}(\boldsymbol{y}) = \{\nabla D_{p,1}(\boldsymbol{y})\}. \tag{36}$$

Case (c): We can compute that the limit of $\frac{\mathrm{d}}{\mathrm{d}\lambda} C_{p,q}(\boldsymbol{y} + \lambda \boldsymbol{z})$ as $\lambda \to 0$ is given by

$$\frac{\mathrm{d}C_{p,q}(\boldsymbol{y} + \lambda \boldsymbol{z})}{\mathrm{d}\lambda}|_{\lambda=0} = M_{p,q}(\boldsymbol{y}, 0) = \nabla D_{p,q}(\boldsymbol{y})^\top \boldsymbol{z}. \tag{37}$$

From (28) and (37), we can easily know that the subgradient of $C_{p,q}$ at $\boldsymbol{y} \in \mathcal{S}_p$ can be formulated as

$$\partial C_{p,q}(\boldsymbol{y}) = \{\nabla D_{p,q}(\boldsymbol{y})\}.$$

In conclusion, the subgradient(s) of $C_{p,q}$ at $\boldsymbol{y} \in \mathcal{S}_p$ can be represented by:

$$\partial C_{p,q}(\boldsymbol{y}) = \begin{cases} \{\nabla D_{1,1}(\boldsymbol{y}) + \boldsymbol{u}\} \text{ where } -a^{(t)} \leqslant u^{(t)} \leqslant a^{(t)},\ \forall t, & \text{if } p = q = 1, \\ \{\nabla D_{p,1}(\boldsymbol{y})\}, & \text{if } q=1, 1<p<2, \boldsymbol{y} \in \mathcal{S}_p \backslash \{\boldsymbol{x}_i\}_{i=1}^m, \\ \{\nabla D_{p,1}(\boldsymbol{x}_l) + \eta_l \boldsymbol{b}\} \quad \text{where} \quad \|\boldsymbol{b}\|_r \leqslant 1, & \text{if } q=1, 1<p<2, \boldsymbol{y} = \boldsymbol{x}_l, \\ \{\nabla D_{p,q}(\boldsymbol{y})\}, & \text{if } 1 < q \leqslant p, 1 < p < 2, \end{cases} \tag{38}$$

where $a^{(t)} = \sum_{i \in V_t(\boldsymbol{y})} \eta_i$ and $\|\cdot\|_r$ is the conjugate norm of $\|\cdot\|_p$ such that $\frac{1}{r} + \frac{1}{p} = 1$. From Fermat's rule, we know from (38) that a singular point $\boldsymbol{y} \in \mathcal{S}_p$ is a minimum point satisfying (8) if and only if

$$\begin{cases} |(\nabla D_{1,1}(\boldsymbol{y}))^{(t)}| \leqslant a^{(t)} \text{ for all } t, & \text{if } p = q = 1, \\ \nabla D_{p,1}(\boldsymbol{y}) = \boldsymbol{0}_d, & \text{if } q=1, 1<p<2 \text{ and } \boldsymbol{y} \in \mathcal{S}_p \backslash \{\boldsymbol{x}_i\}_{i=1}^m, \\ \|\nabla D_{p,1}(\boldsymbol{x}_l)\|_r \leqslant \eta_l, & \text{if } q=1, 1<p<2 \text{ and } \boldsymbol{y} = \boldsymbol{x}_l, \\ \nabla D_{p,q}(\boldsymbol{y}) = \boldsymbol{0}_d & \text{if } 1 < q \leqslant p, 1 < p < 2. \end{cases} \tag{39}$$

□

B.4 Proof of Theorem 7

Proof. We prove this theorem by considering three cases based on the values of p and q: (a) $q = p = 1$; (b) $q = 1$, $1 < p < 2$; (c) $1 < q \leqslant p$, $1 < p < 2$. It suffices to show that the directional derivative along the direction $-\mathscr{D}_{p,q}(\boldsymbol{y})$ is negative.

Case (a): By setting $\mathcal{D}_{1,1}(\boldsymbol{y}) = -\frac{\mathscr{D}_{1,1}(\boldsymbol{y})}{\|\nabla D_{1,1}(\boldsymbol{y})\|_2} = -\frac{\nabla D_{1,1}(\boldsymbol{y})}{\|\nabla D_{1,1}(\boldsymbol{y})\|_2}$ in (31), we can deduce that

$$\frac{\mathrm{d}C_{1,1}\big(\boldsymbol{y}+\lambda\mathcal{D}_{1,1}(\boldsymbol{y})\big)}{\mathrm{d}\lambda}|_{\lambda=0} = -\|\nabla D_{1,1}(\boldsymbol{y})\|_2 + \mathcal{D}_{1,1}(\boldsymbol{y})^\top \boldsymbol{a}. \tag{40}$$

If $\boldsymbol{y}$ is not a minimum point, then it follows from (39) that $|\big(\nabla D_{1,1}(\boldsymbol{y})\big)^{(t)}| > a^{(t)}$ for all t, which together with (40) implies that $\frac{\mathrm{d}C_{1,1}(\boldsymbol{y}+\lambda\mathcal{D}_{1,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$. Then $\frac{\mathrm{d}C_{1,1}(\boldsymbol{y}-\lambda\mathscr{D}_{1,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$.

Case (b): If $\boldsymbol{y} \in \mathcal{S}_p\backslash\{\boldsymbol{x}_i\}_{i=1}^m$, then by setting $\mathcal{D}_{p,1}(\boldsymbol{y}) = -\frac{\mathscr{D}_{p,1}(\boldsymbol{y})}{\|\nabla D_{p,1}(\boldsymbol{y})\|_2} = -\frac{\nabla D_{p,1}(\boldsymbol{y})}{\|\nabla D_{p,1}(\boldsymbol{y})\|_2}$ in (35), we can deduce that

$$\frac{\mathrm{d}C_{p,1}(\boldsymbol{y}+\lambda\mathcal{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} = -\|\nabla D_{p,1}(\boldsymbol{y})\|_2. \tag{41}$$

If $\boldsymbol{y}$ is not a minimum point, then from (39), $\|\nabla D_{p,1}(\boldsymbol{y})\|_2 > 0$, which together with (41) implies that $\frac{\mathrm{d}C_{p,1}(\boldsymbol{y}+\lambda\mathcal{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$. Then $\frac{\mathrm{d}C_{p,1}(\boldsymbol{y}-\lambda\mathscr{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$.

If $\boldsymbol{y} = \boldsymbol{x}_l$, we set $\mathcal{D}_{p,1}(\boldsymbol{y}) = -\frac{\mathscr{D}_{p,1}(\boldsymbol{x}_l)}{\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_2} = -\frac{\left(\nabla D_{p,1}(\boldsymbol{x}_l)\right)^{\frac{r}{p}}}{\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_2}$, where $(\boldsymbol{w})^{\frac{r}{p}} := \operatorname{sign}(\boldsymbol{w}) \odot |\boldsymbol{w}|^{\frac{r}{p}}$ and $\odot$ denotes the element-wise multiplication. Then it follows from (32) that

$$\frac{\mathrm{d}C_{p,1}(\boldsymbol{x}_l+\lambda\mathcal{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} = -\frac{\nabla D_{p,1}(\boldsymbol{x}_l)^\top\big(\nabla D_{p,1}(\boldsymbol{x}_l)\big)^{\frac{r}{p}}}{\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_2} + \eta_l\frac{\|\big(\nabla D_{p,1}(\boldsymbol{x}_l)\big)^{\frac{r}{p}}\|_p}{\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_2}. \tag{42}$$

Note that $\big(|\nabla D_{p,1}(\boldsymbol{x}_l)|^{\frac{r}{p}}\big)^p = |\nabla D_{p,1}(\boldsymbol{x}_l)|^r$. Then from the condition for the equality to hold in the Hölder's inequality, if $p > 1$, $r > 1$ and $\frac{1}{p}+\frac{1}{r} = 1$, then

$$\nabla D_{p,1}(\boldsymbol{x}_l)^\top\big(\nabla D_{p,1}(\boldsymbol{x}_l)\big)^{\frac{r}{p}} = |D_{p,1}(\boldsymbol{x}_l)|^\top|\nabla D_{p,1}(\boldsymbol{x}_l)|^{\frac{r}{p}} = \|\big(\nabla D_{p,1}(\boldsymbol{x}_l)\big)^{\frac{r}{p}}\|_p\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_r. \tag{43}$$

If $\boldsymbol{x}_l$ is not a minimum point, then from (39), $\|\nabla D_{p,1}(\boldsymbol{x}_l)\|_r > \eta_l$, which together with (42) and (43) indicates that $\frac{\mathrm{d}C_{p,1}(\boldsymbol{x}_l+\lambda\mathcal{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$. Then $\frac{\mathrm{d}C_{p,1}(\boldsymbol{x}_l-\lambda\mathscr{D}_{p,1}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$.

Case (c): By setting $\mathcal{D}_{p,q}(\boldsymbol{y}) = -\frac{\mathscr{D}_{p,q}(\boldsymbol{y})}{\|\nabla D_{p,q}(\boldsymbol{y})\|_2} = -\frac{\nabla D_{p,q}(\boldsymbol{y})}{\|\nabla D_{p,q}(\boldsymbol{y})\|_2}$ in (37), we can deduce that

$$\frac{\mathrm{d}C_{p,q}(\boldsymbol{y}+\lambda\mathcal{D}_{p,q}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} = -\|\nabla D_{p,q}(\boldsymbol{y})\|_2. \tag{44}$$

If $\boldsymbol{y}$ is not a minimum point, then from (39), $\|\nabla D_{p,q}(\boldsymbol{y})\|_2 > 0$, which together with (44) implies that $\frac{\mathrm{d}C_{p,q}(\boldsymbol{y}+\lambda\mathcal{D}_{p,q}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$. Then $\frac{\mathrm{d}C_{p,q}(\boldsymbol{y}-\lambda\mathscr{D}_{p,q}(\boldsymbol{y}))}{\mathrm{d}\lambda}|_{\lambda=0} < 0$

Summarizing Cases (a)(b)(c), the direction $-\mathscr{D}_{p,q}(\boldsymbol{y})$ defined in (16) is a descent direction. Hence there exists some $\lambda_* > 0$ such that for any $0 < \lambda \leqslant \lambda_*$, $C_{p,q}(\boldsymbol{y}-\lambda\mathscr{D}_{p,q}(\boldsymbol{y})) < C_{p,q}(\boldsymbol{y})$. □

B.5 Proof of Lemma 8

Proof. First, subtracting the left side of (19) by its right side leads to

$$\begin{aligned}
&\big(\mathbf{T}_{p,q}(\boldsymbol{y})\big)^{(t)} - x^{(t)} \\
=&\frac{\sum_{i=1}^m \eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}(x_i^{(t)}-x^{(t)})}{\sum_{i=1}^m \eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}} \\
=&\frac{\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}(x_i^{(t)}-x^{(t)})}{\sum_{i\in V_t(\boldsymbol{x})} \eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}+\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}}, \qquad \forall 1\leqslant t\leqslant d.
\end{aligned} \tag{45}$$

Since $\eta_i > 0$ for all i, $q \leqslant p$ and $p < 2$, then for all t,

$$\lim_{y^{(t)}\to x^{(t)},\boldsymbol{y}\notin\mathcal{S}_p}\sum_{i\notin V_t(\boldsymbol{x})}\eta_i^q\|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}(x_i^{(t)}-x^{(t)})$$

$$
\begin{aligned}
&=\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{x}-\boldsymbol{x}_i\|_p^{q-p} |x^{(t)}-x_i^{(t)}|^{p-2}(x_i^{(t)}-x^{(t)})\\
&=c_{t,1},\\
&\lim_{y^{(t)}\to x^{(t)},\boldsymbol{y}\notin\mathcal{S}_p} \sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p} |y^{(t)}-x_i^{(t)}|^{p-2}\\
&=\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{x}-\boldsymbol{x}_i\|_p^{q-p} |x^{(t)}-x_i^{(t)}|^{p-2}\\
&=c_{t,2},\\
&\lim_{y^{(t)}\to x^{(t)},\boldsymbol{y}\notin\mathcal{S}_p} \sum_{i\in V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p} |y^{(t)}-x_i^{(t)}|^{p-2}\\
&=+\infty,
\end{aligned}
$$

for some fixed constants $c_{t,1}, c_{t,2} \in \mathbb{R}$. Therefore,

$$
\begin{aligned}
&\lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\notin\mathcal{S}_p} \|\mathbf{T}_{p,q}(\boldsymbol{y})-\boldsymbol{x}\|_2\\
=&\lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\notin\mathcal{S}_p} \Big(\sum_{t=1}^{d}\Big(\frac{\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}(x_i^{(t)}-x^{(t)})}{\sum_{i\in V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}+\sum_{i\notin V_t(\boldsymbol{x})} \eta_i^q \|\boldsymbol{y}-\boldsymbol{x}_i\|_p^{q-p}|y^{(t)}-x_i^{(t)}|^{p-2}}\Big)^2\Big)^{\frac{1}{2}}\\
=&0,
\end{aligned}
\tag{46}
$$

which indicates that $\mathbf{T}_{p,q}(\boldsymbol{y}) \to \boldsymbol{x}$ as $\boldsymbol{y}\to\boldsymbol{x}$ and $\boldsymbol{y}\notin\mathcal{S}_p$. ☐

B.6 Proof of Lemma 9

Proof. It can be easily found that $\lim_{\|\boldsymbol{y}\|_2\to\infty} C_{p,q}(\boldsymbol{y}) = \infty$, then we prove this lemma by yielding a contradiction. Suppose the sequence $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ generated by qPpNWAWS is unbounded, then there exists a subsequence $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}$ such that $\|\boldsymbol{y}_{(k_v)}\|_2\to\infty$ and $C_{p,q}(\boldsymbol{y}_{(k_v)})\to\infty$. However, Theorems 1 and 7 indicate that $C_{p,q}(\boldsymbol{y}_{(k_v)}) \leqslant C_{p,q}(\boldsymbol{y}_{(0)}), \forall v$. This yields a contradiction, thus $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ is bounded. ☐

B.7 Proof of Lemma 10

Proof. We know from Lemma 9 that $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ is bounded, then from the Bolzano-Weierstrasz theorem, there exists at least one point y_* and a subsequence $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}$ such that $\lim_{v\to\infty}\boldsymbol{y}_{(k_v)}=\boldsymbol{y}_*$. We then investigate the following two cases regarding whether $\boldsymbol{y}_*\notin\mathcal{S}_p$ or $\boldsymbol{y}_*\in\mathcal{S}_p$.

Case 1. If $\boldsymbol{y}_*\notin\mathcal{S}_p$, then we can assume that $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}\notin\mathcal{S}_p$. Since the operator $\mathbf{T}_{p,q}$ is continuous, then

$$
\lim_{v\to\infty}\mathbf{T}_{p,q}(\boldsymbol{y}_{(k_v)})=\mathbf{T}_{p,q}(\boldsymbol{y}_*). \tag{47}
$$

From Theorems 1 and 7, the sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ is non-increasing. Furthermore, since $C_{p,q}(\boldsymbol{y})\geqslant 0$, the sequence is bounded below. Therefore, we can conclude that the sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ converges. According to the convergence of $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$, it has a limit and any subsequence of $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ should have the same limit. In particular, $C_{p,q}(\boldsymbol{y}_{(k_v)})$ and $C_{p,q}(\mathbf{T}_{p,q}(\boldsymbol{y}_{(k_v)}))$ are two subsequences of $C_{p,q}(\boldsymbol{y}_{(k)})$. Therefore,

$$
\lim_{v\to\infty}C_{p,q}(\mathbf{T}_{p,q}(\boldsymbol{y}_{(k_v)}))=\lim_{v\to\infty}C_{p,q}(\boldsymbol{y}_{(k_v)}). \tag{48}
$$

Since $C_{p,q}$ is continuous, then it follows from (47) and (48) that

$$
C_{p,q}(\mathbf{T}_{p,q}(\boldsymbol{y}_*))=C_{p,q}(\boldsymbol{y}_*). \tag{49}
$$

Since $\boldsymbol{y}_*\notin\mathcal{S}_p$, then Theorem 1 and (49) indicate $\boldsymbol{y}_*=\mathbf{T}_{p,q}(\boldsymbol{y}_*)$. By Corollary 2, $\boldsymbol{y}_*=\mathbf{M}$.

Case 2. If $\boldsymbol{y}_*\in\mathcal{S}_p$, then we denote the limit point set of $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ by $\{\boldsymbol{y}_{j*}\}_{j\in\mathcal{J}}$ with $\mathcal{J}$ denoting some index set. Since $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ converges, we have

$$
C_{p,q}(\boldsymbol{y}_{j*})=a, \qquad \forall j\in\mathcal{J}, \tag{50}
$$

for some $a\geqslant 0$. Summarizing Case 1 and Case 2, $\boldsymbol{y}_*\in\{\mathbf{M}\}\bigcup\big(\mathcal{S}_p\bigcap\{\boldsymbol{y}:C_{p,q}(\boldsymbol{y})=a\}\big)$. In other words, the limit point set of $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ is a subset of $\{\mathbf{M}\}\bigcup\big(\mathcal{S}_p\bigcap\{\boldsymbol{y}:C_{p,q}(\boldsymbol{y})=a\}\big)$.

Now we only need to prove that $\mathcal{C}_a := \{\boldsymbol{y} \in \mathbb{R}^d : C_{p,q}(\boldsymbol{y}) = a\}$ is a finite set, which can be done by yielding a contradiction. Suppose there exist $(n+1)$ points $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_{n+1} \in \mathcal{C}_a$ such that $(\boldsymbol{u}_1 - \boldsymbol{u}_{n+1})$,$(\boldsymbol{u}_2 - \boldsymbol{u}_{n+1})$,$\cdots$,$(\boldsymbol{u}_n - \boldsymbol{u}_{n+1})$ are linear dependent vectors. Then there exist ζ_1',ζ_2',$\cdots$,ζ_n' that are not all zero such that:

$$\zeta_1'(\boldsymbol{u}_1 - \boldsymbol{u}_{n+1}) + \zeta_2'(\boldsymbol{u}_2 - \boldsymbol{u}_{n+1}) + \cdots + \zeta_n'(\boldsymbol{u}_n - \boldsymbol{u}_{n+1}) = 0. \tag{51}$$

We then investigate the following two cases regarding whether $\sum_{o=1}^n \zeta_o' \neq 0$ or $\sum_{o=1}^n \zeta_o' = 0$.

Case 1. If $\sum_{o=1}^n \zeta_o' \neq 0$, then we let $\zeta_j = \frac{\zeta_j'}{\sum_{o=1}^n \zeta_o'}$, $\forall j \in \{1, \cdots, n\}$. Then $\sum_{j=1}^n \zeta_j = 1$ and (51) becomes $\boldsymbol{u}_{n+1} = \zeta_1 \boldsymbol{u}_1 + \zeta_2 \boldsymbol{u}_2 + \cdots + \zeta_n \boldsymbol{u}_n$. It follows from the strict convexity of $C_{p,q}$ and Jensen's inequality that

$$\zeta_1 C_{p,q}(\boldsymbol{u}_1) + \zeta_2 C_{p,q}(\boldsymbol{u}_2) + \cdots + \zeta_n C_{p,q}(\boldsymbol{u}_n) > C_{p,q}(\boldsymbol{u}_{n+1}), \tag{52}$$

which contradicts the fact that $C_{p,q}(\boldsymbol{u}_1) = C_{p,q}(\boldsymbol{u}_2) = \cdots = C_{p,q}(\boldsymbol{u}_{n+1}) = a$. Since there are at most d linear independent vectors in $\mathbb{R}^d$, there are at most $(d+1)$ points in $\mathcal{C}_a$, or else the above contradiction will occur.

Case 2. If $\sum_{o=1}^n \zeta_o' = 0$, then we assume $\zeta_n' \neq 0$ without loss of generality, because ζ_1',ζ_2',$\cdots$,ζ_n' are not all zero. Then (51) becomes

$$\zeta_1' \boldsymbol{u}_1 + \zeta_2' \boldsymbol{u}_2 + \cdots + \zeta_{n-1}' \boldsymbol{u}_{n-1} = -\zeta_n' \boldsymbol{u}_n \quad \text{and} \quad \sum_{o=1}^{n-1} \zeta_o' = -\zeta_n'. \tag{53}$$

Let $\zeta_j = \frac{\zeta_j'}{\sum_{o=1}^{n-1} \zeta_o'}$, $\forall j \in \{1, \cdots, n-1\}$. Then $\sum_{j=1}^{n-1} \zeta_j = 1$ and (53) becomes $\boldsymbol{u}_n = \zeta_1 \boldsymbol{u}_1 + \zeta_2 \boldsymbol{u}_2 + \cdots + \zeta_{n-1} \boldsymbol{u}_{n-1}$. Again by the strict convexity of $C_{p,q}$ and Jensen's inequality,

$$\zeta_1 C_{p,q}(\boldsymbol{u}_1) + \zeta_2 C_{p,q}(\boldsymbol{u}_2) + \cdots + \zeta_{n-1} C_{p,q}(\boldsymbol{u}_{n-1}) > C_{p,q}(\boldsymbol{u}_n), \tag{54}$$

which contradicts the fact that $C_{p,q}(\boldsymbol{u}_1) = C_{p,q}(\boldsymbol{u}_2) = \cdots = C_{p,q}(\boldsymbol{u}_n) = a$. Since there are at most d linear independent vectors in $\mathbb{R}^d$, there are at most $(d+1)$ points in $\mathcal{C}_a$, or else the above contradiction will occur. Summarizing Case 1 and Case 2, $\mathcal{C}_a$ is a finite set and $\{\mathbf{M}\} \bigcup \left(\mathcal{S}_p \bigcap \mathcal{C}_a\right)$ is also a finite set. □

B.8 Proof of Theorem 11

Proof. To verify whether $\boldsymbol{y}_{(k)}$ is a minimum point, we can first check if $\boldsymbol{y}_{(k)}$ belong to $\mathcal{S}_p$. If $\boldsymbol{y}_{(k)} \notin \mathcal{S}_p$, then Corollary 2 implies that $\mathbf{T}_{p,q}(\boldsymbol{y}_{(k)}) = \boldsymbol{y}_{(k)} \Leftrightarrow \boldsymbol{y}_{(k)} = \boldsymbol{x}_*$. Thus, we can verify whether $\boldsymbol{y}_{(k)}$ is a fixed point of $\mathbf{T}_{p,q}$. If $\boldsymbol{y}_{(k)} \in \mathcal{S}_p$, then we can directly check whether $\mathbf{0}_d \in \partial C_{p,q}(\boldsymbol{y}_{(k)})$ by (15).

By Theorems 1 and 7, we know that the sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ is non-increasing. Furthermore, since $C_{p,q}(\boldsymbol{y}) \geqslant 0$, $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ is bounded below. Therefore, we can conclude that the sequence $\{C_{p,q}(\boldsymbol{y}_{(k)})\}_{k\in\mathbb{N}}$ converges.

If $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ hits $\mathcal{S}_p$ for finite times, then we can only consider the rear of the sequence that has already passed all the singular points, still denoted by $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ with $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}} \cap \mathcal{S}_p = \emptyset$. Now we prove that the sequence $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$ converges by showing that there exists only one limit point for $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$. Lemma 10 indicates that there exist only a finite number of limit points for $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$, denoted by $\boldsymbol{y}_{1*}, \boldsymbol{y}_{2*}, \ldots, \boldsymbol{y}_{o*}$. Consider the subsequence $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}$ such that $\lim_{v\to\infty} \boldsymbol{y}_{(k_v)} = \boldsymbol{y}_{1*}$. Let $\delta = \frac{\min_{i,j\in\{1,2,\ldots,o\}, i\neq j} \|\boldsymbol{y}_{i*} - \boldsymbol{y}_{j*}\|_2}{2} > 0$. Then we consider a δ-neighborhood around $\boldsymbol{y}_{1*}$, denoted by $B(\boldsymbol{y}_{1*}, \delta)$. Then there exists only one limit point in $B(\boldsymbol{y}_{1*}, \delta)$. We can choose a subsequence of $\{\boldsymbol{y}_{(k_v)}\}_{v\in\mathbb{N}}$, denoted by $\{\boldsymbol{y}_{(k_{v_n})}\}_{n\in\mathbb{N}}$, such that $\{\mathbf{T}_{p,q}(\boldsymbol{y}_{(k_{v_n})})\}_{n\in\mathbb{N}} \cap B(\boldsymbol{y}_{1*}, \delta) = \emptyset$, because there are other limit points $\boldsymbol{y}_{j*} \neq \boldsymbol{y}_{1*}$. Then

$$\lim_{n\to\infty} \mathbf{T}_{p,q}(\boldsymbol{y}_{(k_{v_n})}) \neq \boldsymbol{y}_{1*}. \tag{55}$$

If $\boldsymbol{y}_{1*} \notin \mathcal{S}_p$, then it follows from the continuity of $\mathbf{T}_{p,q}$ that $\lim_{n\to\infty} \mathbf{T}_{p,q}(\boldsymbol{y}_{(k_{v_n})}) = \mathbf{T}_{p,q}(\boldsymbol{y}_{1*})$, which together with (55) indicates that $\mathbf{T}_{p,q}(\boldsymbol{y}_{1*}) \neq \boldsymbol{y}_{1*}$. Then from Theorem 1, $C_{p,q}(\mathbf{T}_{p,q}(\boldsymbol{y}_{1*})) < C_{p,q}(\boldsymbol{y}_{1*})$, which violates (50). If $\boldsymbol{y}_{1*} \in \mathcal{S}_p$, then from Lemma 8, $\lim_{n\to\infty} \mathbf{T}_{p,q}(\boldsymbol{y}_{(k_{v_n})}) = \boldsymbol{y}_{1*}$, which violates (55). Therefore, there exists only one limit point for $\{\boldsymbol{y}_{(k)}\}_{k\in\mathbb{N}}$. □

C Additional Experimental Results (Tables A1∼A16)

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	3.97 ± 0.29	2.57 ± 1.3	1.35 ± 0.5	1.24 ± 0.42	1.22 ± 0.42	1.21 ± 0.41	1.21 ± 0.41	1.21 ± 0.41	1.21 ± 0.4	1.21 ± 0.41
1.1	-	2.91 ± 1.67	1.53 ± 0.51	1.41 ± 0.49	1.34 ± 0.47	1.28 ± 0.45	1.24 ± 0.43	1.22 ± 0.41	1.2 ± 0.4	1.2 ± 0.4
1.2	-	-	1.61 ± 0.75	1.37 ± 0.48	1.31 ± 0.46	1.27 ± 0.44	1.24 ± 0.43	1.21 ± 0.41	1.19 ± 0.4	1.19 ± 0.39
1.3	-	-	-	1.4 ± 0.6	1.3 ± 0.46	1.26 ± 0.44	1.23 ± 0.42	1.21 ± 0.4	1.19 ± 0.39	1.18 ± 0.39
1.4	-	-	-	-	1.31 ± 0.5	1.25 ± 0.43	1.22 ± 0.42	1.2 ± 0.4	1.18 ± 0.39	1.18 ± 0.38
1.5	-	-	-	-	-	1.26 ± 0.45	1.21 ± 0.41	1.19 ± 0.4	1.18 ± 0.39	1.18 ± 0.38
1.6	-	-	-	-	-	-	1.22 ± 0.43	1.19 ± 0.39	1.18 ± 0.39	1.18 ± 0.38
1.7	-	-	-	-	-	-	-	1.19 ± 0.4	1.18 ± 0.39	1.17 ± 0.38
1.8	-	-	-	-	-	-	-	-	1.19 ± 0.4	1.17 ± 0.38
1.9	-	-	-	-	-	-	-	-	-	1.18 ± 0.38

Table A1: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on FTSE100 (mean$\pm$STD)

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	3.76 ± 0.44	2.63 ± 1.56	1.17 ± 0.38	1.08 ± 0.27	1.08 ± 0.27	1.07 ± 0.26	1.07 ± 0.25	1.07 ± 0.25	1.06 ± 0.24	1.06 ± 0.24
1.1	-	3.18 ± 1.8	1.34 ± 0.49	1.21 ± 0.41	1.13 ± 0.33	1.11 ± 0.31	1.09 ± 0.29	1.08 ± 0.27	1.07 ± 0.25	1.06 ± 0.23
1.2	-	-	1.39 ± 0.66	1.18 ± 0.39	1.12 ± 0.33	1.1 ± 0.3	1.09 ± 0.28	1.07 ± 0.26	1.06 ± 0.23	1.06 ± 0.23
1.3	-	-	-	1.17 ± 0.39	1.12 ± 0.32	1.09 ± 0.29	1.08 ± 0.28	1.07 ± 0.25	1.05 ± 0.23	1.05 ± 0.22
1.4	-	-	-	-	1.11 ± 0.31	1.09 ± 0.29	1.07 ± 0.26	1.06 ± 0.23	1.05 ± 0.22	1.05 ± 0.21
1.5	-	-	-	-	-	1.09 ± 0.28	1.07 ± 0.26	1.06 ± 0.23	1.05 ± 0.22	1.04 ± 0.21
1.6	-	-	-	-	-	-	1.08 ± 0.27	1.05 ± 0.23	1.05 ± 0.21	1.04 ± 0.21
1.7	-	-	-	-	-	-	-	1.05 ± 0.22	1.05 ± 0.21	1.04 ± 0.21
1.8	-	-	-	-	-	-	-	-	1.05 ± 0.21	1.04 ± 0.21
1.9	-	-	-	-	-	-	-	-	-	1.05 ± 0.21

Table A2: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on NASDAQ100 (mean$\pm$STD).

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	2.51 ± 1.19	1.65 ± 1.15	1.2 ± 0.59	1.19 ± 0.58	1.19 ± 0.56	1.14 ± 0.45	1.11 ± 0.42	1.11 ± 0.41	1.11 ± 0.4	1.1 ± 0.36
1.1	-	1.92 ± 1.39	1.2 ± 0.61	1.19 ± 0.58	1.19 ± 0.56	1.17 ± 0.45	1.12 ± 0.4	1.1 ± 0.38	1.1 ± 0.37	1.09 ± 0.33
1.2	-	-	1.21 ± 0.61	1.18 ± 0.57	1.19 ± 0.55	1.17 ± 0.45	1.14 ± 0.42	1.1 ± 0.39	1.09 ± 0.35	1.08 ± 0.32
1.3	-	-	-	1.18 ± 0.56	1.19 ± 0.55	1.17 ± 0.45	1.16 ± 0.44	1.1 ± 0.38	1.09 ± 0.35	1.08 ± 0.29
1.4	-	-	-	-	1.18 ± 0.53	1.17 ± 0.45	1.17 ± 0.45	1.11 ± 0.39	1.09 ± 0.34	1.07 ± 0.28
1.5	-	-	-	-	-	1.16 ± 0.44	1.16 ± 0.44	1.14 ± 0.43	1.1 ± 0.35	1.07 ± 0.26
1.6	-	-	-	-	-	-	1.16 ± 0.44	1.15 ± 0.43	1.14 ± 0.39	1.1 ± 0.29
1.7	-	-	-	-	-	-	-	1.16 ± 0.44	1.16 ± 0.41	1.13 ± 0.34
1.8	-	-	-	-	-	-	-	-	1.15 ± 0.41	1.13 ± 0.34
1.9	-	-	-	-	-	-	-	-	-	1.13 ± 0.34

Table A3: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on FF100 (mean$\pm$STD).

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	2.69 ± 1.1	1.59 ± 1.15	1.09 ± 0.31	1.08 ± 0.28	1.07 ± 0.27	1.07 ± 0.26	1.07 ± 0.27	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.26
1.1	-	1.79 ± 1.24	1.08 ± 0.28	1.08 ± 0.27	1.08 ± 0.27	1.08 ± 0.27	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.25	1.07 ± 0.25
1.2	-	-	1.12 ± 0.44	1.08 ± 0.27	1.08 ± 0.27	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.25	1.07 ± 0.25	1.07 ± 0.25
1.3	-	-	-	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.25	1.07 ± 0.25	1.07 ± 0.25
1.4	-	-	-	-	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.25	1.07 ± 0.25	1.07 ± 0.25
1.5	-	-	-	-	-	1.07 ± 0.26	1.07 ± 0.26	1.07 ± 0.25	1.06 ± 0.25	1.06 ± 0.25
1.6	-	-	-	-	-	-	1.07 ± 0.26	1.06 ± 0.25	1.06 ± 0.24	1.06 ± 0.24
1.7	-	-	-	-	-	-	-	1.06 ± 0.24	1.06 ± 0.24	1.06 ± 0.24
1.8	-	-	-	-	-	-	-	-	1.06 ± 0.24	1.06 ± 0.24
1.9	-	-	-	-	-	-	-	-	-	1.06 ± 0.24

Table A4: Average number of iterates for qPpNWAWS to reduce the cost function at a singular point on FF100MEOP (mean$\pm$STD).

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.0074	0.0598	0.0625	0.0617	0.0582	0.0541	0.0515	0.0484	0.0464	0.0446
	$Iter$	42.23 ± 19.91	16.07 ± 2.14	15.17 ± 2.15	14.85 ± 2.43	14.04 ± 2.5	13.29 ± 2.56	12.65 ± 2.59	12.14 ± 2.6	11.73 ± 2.63	11.41 ± 2.61
1.1	$Time$	-	0.0611	0.0621	0.0584	0.0539	0.0496	0.0465	0.0442	0.0418	0.0404
	$Iter$	-	16.67 ± 3.27	15.21 ± 1.8	14.36 ± 1.91	13.37 ± 2	12.49 ± 2.07	11.84 ± 2.1	11.32 ± 2.13	10.89 ± 2.13	10.55 ± 2.15
1.2	$Time$	-	-	0.0597	0.0562	0.0507	0.0465	0.0431	0.0411	0.0386	0.0373
	$Iter$	-	-	15.04 ± 2.09	13.81 ± 1.48	12.74 ± 1.62	11.83 ± 1.68	11.18 ± 1.74	10.66 ± 1.76	10.22 ± 1.77	9.88 ± 1.83
1.3	$Time$	-	-	-	0.0539	0.0482	0.0441	0.0403	0.0381	0.0358	0.0341
	$Iter$	-	-	-	13.65 ± 2.87	12.23 ± 1.24	11.29 ± 1.35	10.61 ± 1.44	10.11 ± 1.46	9.67 ± 1.49	9.33 ± 1.54
1.4	$Time$	-	-	-	-	0.046	0.0414	0.0377	0.0356	0.0334	0.0315
	$Iter$	-	-	-	-	11.93 ± 1.16	10.81 ± 1.05	10.09 ± 1.15	9.58 ± 1.21	9.16 ± 1.26	8.82 ± 1.31
1.5	$Time$	-	-	-	-	-	0.0398	0.0353	0.0332	0.0309	0.029
	$Iter$	-	-	-	-	-	10.49 ± 0.91	9.62 ± 0.91	9.11 ± 1	8.68 ± 1.06	8.36 ± 1.11
1.6	$Time$	-	-	-	-	-	-	0.0332	0.0309	0.0287	0.0267
	$Iter$	-	-	-	-	-	-	9.32 ± 0.79	8.68 ± 0.83	8.25 ± 0.89	7.91 ± 0.95
1.7	$Time$	-	-	-	-	-	-	-	0.0285	0.0266	0.0246
	$Iter$	-	-	-	-	-	-	-	8.31 ± 0.69	7.85 ± 0.75	7.48 ± 0.84
1.8	$Time$	-	-	-	-	-	-	-	-	0.0242	0.0224
	$Iter$	-	-	-	-	-	-	-	-	7.46 ± 0.75	7.05 ± 0.64
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.0205
	$Iter$	-	-	-	-	-	-	-	-	-	6.75 ± 0.77

Table A5: Average computational time (in seconds) and average number of iterations (mean$\pm$STD) for qPpNWAWS on FTSE100.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.0104	0.0674	0.0925	0.0931	0.0872	0.0566	0.0531	0.0728	0.0479	0.0462
	$Iter$	57.78 ± 27.93	17.13 ± 2.68	16.15 ± 2.41	15.96 ± 2.64	15.13 ± 2.75	14.28 ± 2.81	13.57 ± 2.83	12.99 ± 2.81	12.49 ± 2.77	12.11 ± 2.75
1.1	$Time$	-	0.0836	0.0908	0.0856	0.0784	0.0511	0.048	0.0627	0.0432	0.0415
	$Iter$	-	17.24 ± 3.36	15.9 ± 2.09	15.06 ± 2.09	13.93 ± 2.17	13.14 ± 2.25	12.49 ± 2.25	11.97 ± 2.3	11.52 ± 2.25	11.14 ± 2.26
1.2	$Time$	-	-	0.0882	0.0813	0.0738	0.0472	0.0443	0.0415	0.0397	0.0379
	$Iter$	-	-	15.76 ± 3.05	14.41 ± 1.67	13.24 ± 1.71	12.36 ± 1.79	11.7 ± 1.87	11.2 ± 1.9	10.77 ± 1.91	10.42 ± 1.9
1.3	$Time$	-	-	-	0.0779	0.0602	0.044	0.041	0.0381	0.0368	0.035
	$Iter$	-	-	-	13.99 ± 1.29	12.65 ± 1.31	11.69 ± 1.39	11.04 ± 1.49	10.5 ± 1.55	10.1 ± 1.59	9.76 ± 1.62
1.4	$Time$	-	-	-	-	0.0462	0.0413	0.038	0.0355	0.034	0.0321
	$Iter$	-	-	-	-	12.22 ± 0.94	11.15 ± 1.03	10.43 ± 1.17	9.9 ± 1.26	9.49 ± 1.32	9.15 ± 1.35
1.5	$Time$	-	-	-	-	-	0.0387	0.0438	0.0331	0.0309	0.0294
	$Iter$	-	-	-	-	-	10.7 ± 0.76	9.9 ± 0.87	9.35 ± 1.01	8.92 ± 1.08	8.61 ± 1.13
1.6	$Time$	-	-	-	-	-	-	0.0473	0.0306	0.0285	0.0265
	$Iter$	-	-	-	-	-	-	9.47 ± 0.67	8.85 ± 0.76	8.41 ± 0.87	8.08 ± 0.95
1.7	$Time$	-	-	-	-	-	-	-	0.0278	0.0263	0.0242
	$Iter$	-	-	-	-	-	-	-	8.4 ± 0.59	7.94 ± 0.67	7.61 ± 0.78
1.8	$Time$	-	-	-	-	-	-	-	-	0.0239	0.022
	$Iter$	-	-	-	-	-	-	-	-	7.53 ± 0.59	7.11 ± 0.61
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.02
	$Iter$	-	-	-	-	-	-	-	-	-	6.76 ± 0.53

Table A6: Average computational time (in seconds) and average number of iterations (mean$\pm$STD) for qPpNWAWS on NASDAQ100.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.015	0.1253	0.13	0.1296	0.1261	0.1244	0.1216	0.1197	0.1177	0.117
	$Iter$	68.9 ± 32.57	26.12 ± 27.31	25.54 ± 25.66	25.2 ± 24.24	24.75 ± 23.87	24.21 ± 23.47	23.64 ± 23.02	23.44 ± 23.29	23.29 ± 23.7	23.15 ± 24.11
1.1	$Time$	-	0.1281	0.1274	0.126	0.1211	0.1185	0.1134	0.1106	0.1069	0.1055
	$Iter$	-	26.68 ± 26.77	25.48 ± 25.71	24.79 ± 24.45	23.98 ± 23.56	23.24 ± 23.28	22.55 ± 22.94	22.03 ± 22.93	21.66 ± 23.27	21.37 ± 23.6
1.2	$Time$	-	-	0.1253	0.1222	0.1164	0.111	0.1073	0.103	0.099	0.0982
	$Iter$	-	-	25.22 ± 25.49	24.33 ± 24.71	23.38 ± 24.09	22.25 ± 23.31	21.52 ± 23.11	20.99 ± 23.05	20.59 ± 23.27	20.29 ± 23.51
1.3	$Time$	-	-	-	0.1163	0.1317	0.1051	0.1013	0.0974	0.0943	0.0928
	$Iter$	-	-	-	23.73 ± 24.69	22.68 ± 24.31	21.43 ± 23.4	20.75 ± 23.27	20.21 ± 23.24	19.81 ± 23.39	19.51 ± 23.58
1.4	$Time$	-	-	-	-	0.1047	0.0992	0.0946	0.0921	0.0884	0.0863
	$Iter$	-	-	-	-	21.68 ± 24.14	20.66 ± 23.7	19.97 ± 23.51	19.47 ± 23.45	19.05 ± 23.53	18.75 ± 23.67
1.5	$Time$	-	-	-	-	-	0.0928	0.0898	0.0868	0.0837	0.0813
	$Iter$	-	-	-	-	-	19.87 ± 23.84	19.25 ± 23.75	18.78 ± 23.77	18.35 ± 23.73	18.01 ± 23.81
1.6	$Time$	-	-	-	-	-	-	0.0839	0.0817	0.0789	0.0764
	$Iter$	-	-	-	-	-	-	18.57 ± 23.96	18.16 ± 24.03	17.74 ± 24.07	17.43 ± 24.23
1.7	$Time$	-	-	-	-	-	-	-	0.0763	0.0746	0.0722
	$Iter$	-	-	-	-	-	-	-	17.53 ± 24.24	17.14 ± 24.24	16.76 ± 24.31
1.8	$Time$	-	-	-	-	-	-	-	-	0.0699	0.0678
	$Iter$	-	-	-	-	-	-	-	-	16.61 ± 24.45	16.19 ± 24.52
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.0631
	$Iter$	-	-	-	-	-	-	-	-	-	15.63 ± 24.72

Table A7: Average computational time (in seconds) and average number of iterations (mean$\pm$STD) for qPpNWAWS on FF100.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	$Time$	0.0144	0.0965	0.1022	0.1053	0.105	0.1031	0.0987	0.0965	0.0956	0.0935
	$Iter$	65.22 ± 33.00	16.55 ± 4.04	16.5 ± 3.71	17 ± 3.62	16.88 ± 3.62	16.5 ± 3.72	16.18 ± 3.77	15.86 ± 3.8	15.56 ± 3.83	15.21 ± 3.9
1.1	$Time$	-	0.0938	0.0974	0.0978	0.0954	0.0923	0.088	0.0848	0.0829	0.0799
	$Iter$	-	16.26 ± 2.91	15.81 ± 2.81	15.87 ± 2.8	15.57 ± 2.78	15.11 ± 2.79	14.66 ± 2.83	14.21 ± 2.86	13.8 ± 2.89	13.41 ± 2.95
1.2	$Time$	-	-	0.0961	0.0946	0.0903	0.0857	0.0805	0.0779	0.0751	0.0725
	$Iter$	-	-	15.76 ± 2.93	15.38 ± 2.59	14.82 ± 2.45	14.2 ± 2.46	13.65 ± 2.46	13.15 ± 2.48	12.77 ± 2.49	12.35 ± 2.53
1.3	$Time$	-	-	-	0.089	0.0854	0.0786	0.0749	0.072	0.0687	0.0664
	$Iter$	-	-	-	14.79 ± 2.09	14.04 ± 2.04	13.37 ± 2.05	12.79 ± 2.08	12.32 ± 2.12	11.9 ± 2.14	11.55 ± 2.19
1.4	$Time$	-	-	-	-	0.0774	0.0719	0.0683	0.0666	0.0628	0.0601
	$Iter$	-	-	-	-	13.12 ± 1.5	12.47 ± 1.57	11.9 ± 1.63	11.48 ± 1.7	11.08 ± 1.74	10.72 ± 1.84
1.5	$Time$	-	-	-	-	-	0.0647	0.0617	0.0596	0.0568	0.0539
	$Iter$	-	-	-	-	-	11.57 ± 1.18	11.05 ± 1.24	10.61 ± 1.34	10.25 ± 1.38	9.93 ± 1.48
1.6	$Time$	-	-	-	-	-	-	0.0558	0.0536	0.0504	0.0481
	$Iter$	-	-	-	-	-	-	10.24 ± 0.96	9.84 ± 1.02	9.47 ± 1.09	9.15 ± 1.2
1.7	$Time$	-	-	-	-	-	-	-	0.0481	0.0452	0.0429
	$Iter$	-	-	-	-	-	-	-	9.13 ± 0.77	8.75 ± 0.83	8.45 ± 0.91
1.8	$Time$	-	-	-	-	-	-	-	-	0.0402	0.0379
	$Iter$	-	-	-	-	-	-	-	-	8.07 ± 0.62	7.77 ± 0.71
1.9	$Time$	-	-	-	-	-	-	-	-	-	0.0323
	$Iter$	-	-	-	-	-	-	-	-	-	7.04 ± 0.52

Table A8: Average computational time (in seconds) and average number of iterations (mean±STD) for qPpNWAWS on FF100MEOP.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.84 ± 0.06	0.65 ± 0.04	0.64 ± 0.04	0.62 ± 0.05	0.59 ± 0.05	0.56 ± 0.06	0.53 ± 0.07	0.51 ± 0.08	0.49 ± 0.09	0.47 ± 0.09
1.1	-	0.63 ± 0.05	0.62 ± 0.04	0.59 ± 0.05	0.55 ± 0.06	0.51 ± 0.06	0.48 ± 0.07	0.46 ± 0.08	0.44 ± 0.08	0.42 ± 0.09
1.2	-	-	0.61 ± 0.05	0.57 ± 0.04	0.53 ± 0.05	0.49 ± 0.05	0.46 ± 0.06	0.43 ± 0.07	0.41 ± 0.07	0.39 ± 0.08
1.3	-	-	-	0.56 ± 0.04	0.51 ± 0.04	0.47 ± 0.04	0.43 ± 0.05	0.40 ± 0.06	0.37 ± 0.07	0.35 ± 0.07
1.4	-	-	-	-	0.50 ± 0.03	0.44 ± 0.03	0.40 ± 0.04	0.36 ± 0.05	0.33 ± 0.06	0.31 ± 0.06
1.5	-	-	-	-	-	0.42 ± 0.03	0.37 ± 0.03	0.33 ± 0.04	0.3 ± 0.04	0.27 ± 0.05
1.6	-	-	-	-	-	-	0.35 ± 0.03	0.3 ± 0.02	0.26 ± 0.03	0.23 ± 0.04
1.7	-	-	-	-	-	-	-	0.26 ± 0.02	0.22 ± 0.02	0.18 ± 0.03
1.8	-	-	-	-	-	-	-	-	0.18 ± 0.03	0.14 ± 0.02
1.9	-	-	-	-	-	-	-	-	-	0.09 ± 0.03

Table A9: Average computational convergence rate (mean±STD) for qPpNWAWS on FTSE100.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.88 ± 0.06	0.66 ± 0.04	0.65 ± 0.04	0.63 ± 0.05	0.61 ± 0.06	0.58 ± 0.06	0.55 ± 0.07	0.53 ± 0.08	0.51 ± 0.09	0.49 ± 0.09
1.1	-	0.62 ± 0.05	0.63 ± 0.05	0.6 ± 0.05	0.56 ± 0.06	0.53 ± 0.07	0.5 ± 0.07	0.48 ± 0.08	0.46 ± 0.08	0.44 ± 0.09
1.2	-	-	0.62 ± 0.05	0.58 ± 0.04	0.54 ± 0.05	0.5 ± 0.06	0.47 ± 0.06	0.44 ± 0.07	0.42 ± 0.08	0.41 ± 0.08
1.3	-	-	-	0.57 ± 0.04	0.52 ± 0.04	0.47 ± 0.05	0.44 ± 0.06	0.41 ± 0.06	0.38 ± 0.07	0.36 ± 0.07
1.4	-	-	-	-	0.5 ± 0.03	0.45 ± 0.03	0.41 ± 0.04	0.37 ± 0.05	0.34 ± 0.06	0.32 ± 0.07
1.5	-	-	-	-	-	0.43 ± 0.03	0.38 ± 0.03	0.34 ± 0.04	0.3 ± 0.05	0.28 ± 0.06
1.6	-	-	-	-	-	-	0.35 ± 0.02	0.3 ± 0.03	0.26 ± 0.04	0.23 ± 0.04
1.7	-	-	-	-	-	-	-	0.27 ± 0.02	0.22 ± 0.02	0.19 ± 0.03
1.8	-	-	-	-	-	-	-	-	0.18 ± 0.02	0.14 ± 0.02
1.9	-	-	-	-	-	-	-	-	-	0.09 ± 0.02

Table A10: Average computational convergence rate (mean±STD) for qPpNWAWS on NASDAQ100.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.89 ± 0.08	0.66 ± 0.2	0.67 ± 0.19	0.67 ± 0.17	0.66 ± 0.18	0.65 ± 0.17	0.64 ± 0.16	0.63 ± 0.17	0.62 ± 0.18	0.61 ± 0.18
1.1	-	0.68 ± 0.11	0.68 ± 0.11	0.67 ± 0.12	0.65 ± 0.12	0.63 ± 0.13	0.62 ± 0.13	0.6 ± 0.15	0.58 ± 0.16	0.57 ± 0.17
1.2	-	-	0.67 ± 0.12	0.65 ± 0.12	0.63 ± 0.13	0.6 ± 0.13	0.58 ± 0.14	0.56 ± 0.15	0.55 ± 0.17	0.53 ± 0.19
1.3	-	-	-	0.64 ± 0.12	0.61 ± 0.14	0.58 ± 0.14	0.55 ± 0.15	0.53 ± 0.16	0.51 ± 0.18	0.5 ± 0.2
1.4	-	-	-	-	0.58 ± 0.14	0.55 ± 0.15	0.52 ± 0.16	0.5 ± 0.18	0.48 ± 0.18	0.46 ± 0.22
1.5	-	-	-	-	-	0.52 ± 0.16	0.48 ± 0.17	0.46 ± 0.18	0.43 ± 0.2	0.42 ± 0.22
1.6	-	-	-	-	-	-	0.44 ± 0.18	0.42 ± 0.2	0.39 ± 0.21	0.37 ± 0.22
1.7	-	-	-	-	-	-	-	0.37 ± 0.21	0.34 ± 0.23	0.32 ± 0.23
1.8	-	-	-	-	-	-	-	-	0.3 ± 0.26	0.26 ± 0.25
1.9	-	-	-	-	-	-	-	-	-	0.21 ± 0.27

Table A11: Average computational convergence rate (mean±STD) for qPpNWAWS on FF100.

q \ p	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	0.88 ± 0.09	0.64 ± 0.14	0.64 ± 0.13	0.64 ± 0.11	0.64 ± 0.1	0.62 ± 0.09	0.61 ± 0.1	0.6 ± 0.09	0.59 ± 0.09	0.58 ± 0.1
1.1	-	0.64 ± 0.05	0.63 ± 0.05	0.62 ± 0.05	0.6 ± 0.05	0.59 ± 0.06	0.57 ± 0.06	0.55 ± 0.07	0.54 ± 0.08	0.52 ± 0.09
1.2	-	-	0.62 ± 0.05	0.61 ± 0.04	0.58 ± 0.05	0.56 ± 0.05	0.53 ± 0.06	0.51 ± 0.07	0.5 ± 0.07	0.48 ± 0.08
1.3	-	-	-	0.59 ± 0.04	0.56 ± 0.04	0.53 ± 0.05	0.5 ± 0.06	0.48 ± 0.06	0.46 ± 0.07	0.44 ± 0.08
1.4	-	-	-	-	0.53 ± 0.03	0.49 ± 0.04	0.46 ± 0.05	0.44 ± 0.06	0.41 ± 0.06	0.39 ± 0.07
1.5	-	-	-	-	-	0.45 ± 0.03	0.42 ± 0.04	0.39 ± 0.04	0.37 ± 0.05	0.34 ± 0.07
1.6	-	-	-	-	-	-	0.37 ± 0.02	0.34 ± 0.03	0.31 ± 0.04	0.29 ± 0.05
1.7	-	-	-	-	-	-	-	0.29 ± 0.02	0.26 ± 0.03	0.23 ± 0.04
1.8	-	-	-	-	-	-	-	-	0.2 ± 0.02	0.17 ± 0.03
1.9	-	-	-	-	-	-	-	-	-	0.1 ± 0.01

Table A12: Average computational convergence rate (mean±STD) for qPpNWAWS on FF100MEOP.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	45.4645	75.5157	**78.5411**	68.5563	64.3221	60.4422	60.0741	56.2867	51.9187	48.8151
	SR	0.1061	0.1166	**0.1172**	0.1143	0.1129	0.1116	0.1114	0.1102	0.1087	0.1076
1.1	CW	-	70.6336	73.1394	62.1984	61.933	64.6889	59.7303	54.9652	52.7812	53.578
	SR	-	0.1153	0.1158	0.1124	0.1121	0.1128	0.1113	0.1097	0.109	0.1092
1.2	CW	-	-	66.0843	57.5627	58.7638	60.6815	57.1283	53.0324	54.1895	56.1977
	SR	-	-	0.1138	0.1109	0.1111	0.1115	0.1104	0.109	0.1094	0.1101
1.3	CW	-	-	-	50.2791	54.8675	55.6076	53.038	51.177	51.75	54.4712
	SR	-	-	-	0.1084	0.1098	0.1099	0.1091	0.1084	0.1086	0.1095
1.4	CW	-	-	-	-	51.8308	52.7809	50.6453	50.9802	51.4462	54.142
	SR	-	-	-	-	0.1087	0.109	0.1082	0.1083	0.1084	0.1093
1.5	CW	-	-	-	-	-	52.0359	51.4891	51.3103	51.1554	54.3556
	SR	-	-	-	-	-	0.1087	0.1085	0.1084	0.1083	0.1094
1.6	CW	-	-	-	-	-	-	52.727	52.3873	52.002	53.9062
	SR	-	-	-	-	-	-	0.1089	0.1088	0.1086	0.1092
1.7	CW	-	-	-	-	-	-	-	54.0681	54.2055	54.4498
	SR	-	-	-	-	-	-	-	0.1094	0.1093	0.1093
1.8	CW	-	-	-	-	-	-	-	-	56.2806	55.3808
	SR	-	-	-	-	-	-	-	-	0.11	0.1096
1.9	CW	-	-	-	-	-	-	-	-	-	55.1256
	SR	-	-	-	-	-	-	-	-	-	0.1095

Table A13: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on FTSE100. The CW and SR for the original setting $(q, p) = (1, 2)$ are 50.1765 and 0.1081, respectively.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	1.3759	7.6786	**8.1831**	8.0839	7.0116	6.1615	5.5029	5.1351	4.8972	4.6473
	SR	0.0393	0.0873	**0.0892**	0.0892	0.0854	0.0818	0.0783	0.0761	0.0746	0.073
1.1	CW	-	6.4397	7.2775	7.0786	6.1042	5.3003	4.9398	4.7081	4.5468	4.3697
	SR	-	0.0823	0.0858	0.0852	0.0813	0.0772	0.0751	0.0735	0.0724	0.0712
1.2	CW	-	-	6.7903	6.4068	5.2845	4.7138	4.545	4.458	4.2981	4.0487
	SR	-	-	0.0839	0.0824	0.0771	0.0738	0.0726	0.0719	0.0708	0.069
1.3	CW	-	-	-	6.1334	4.8934	4.2499	4.2272	4.2161	4.0197	3.8176
	SR	-	-	-	0.0813	0.0749	0.0708	0.0705	0.0703	0.0689	0.0674
1.4	CW	-	-	-	-	4.9208	4.368	4.1212	3.9263	3.7898	3.6361
	SR	-	-	-	-	0.0752	0.0716	0.0698	0.0683	0.0673	0.066
1.5	CW	-	-	-	-	-	4.3312	3.8162	3.5874	3.4547	3.3215
	SR	-	-	-	-	-	0.0714	0.0676	0.0658	0.0646	0.0635
1.6	CW	-	-	-	-	-	-	3.5415	3.2637	3.1451	3.0999
	SR	-	-	-	-	-	-	0.0655	0.0631	0.062	0.0615
1.7	CW	-	-	-	-	-	-	-	3.0056	2.8606	2.8844
	SR	-	-	-	-	-	-	-	0.0607	0.0593	0.0595
1.8	CW	-	-	-	-	-	-	-	-	2.6534	2.6911
	SR	-	-	-	-	-	-	-	-	0.0572	0.0576
1.9	CW	-	-	-	-	-	-	-	-	-	2.5943
	SR	-	-	-	-	-	-	-	-	-	0.0566

Table A14: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on NASDAQ100. The CW and SR for the original setting $(q, p) = (1, 2)$ are 4.4003 and 0.0714, respectively.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	**78.2733**	66.1289	65.4248	62.4756	60.487	60.581	59.5244	58.6327	58.2801	57.9964
	SR	**0.1684**	0.161	0.1606	0.1584	0.1572	0.1575	0.1571	0.1564	0.1561	0.1558
1.1	CW	-	71.5448	70.1542	68.908	63.4955	64.5905	62.6362	61.8175	61.2322	61.8582
	SR	-	0.1636	0.1625	0.1617	0.1589	0.1592	0.1587	0.1582	0.1578	0.158
1.2	CW	-	-	71.6484	67.7225	69.1062	65.4196	64.0382	63.7063	63.1599	63.7408
	SR	-	-	0.163	0.161	0.1618	0.1599	0.1593	0.1592	0.1589	0.1591
1.3	CW	-	-	-	71.903	77.1214	66.4467	65.345	65.4356	64.8402	65.3721
	SR	-	-	-	0.1635	0.1653	0.1609	0.1603	0.1602	0.1598	0.16
1.4	CW	-	-	-	-	68.3831	69.2298	66.8632	67.1484	66.3637	66.7174
	SR	-	-	-	-	0.1616	0.1623	0.161	0.161	0.1605	0.1607
1.5	CW	-	-	-	-	-	70.7036	69.0821	68.5513	67.7569	68.025
	SR	-	-	-	-	-	0.1628	0.162	0.1617	0.1612	0.1613
1.6	CW	-	-	-	-	-	-	71.661	70.1756	69.0988	69.2098
	SR	-	-	-	-	-	-	0.1632	0.1624	0.1618	0.1618
1.7	CW	-	-	-	-	-	-	-	71.4712	70.1408	69.7693
	SR	-	-	-	-	-	-	-	0.163	0.1623	0.1621
1.8	CW	-	-	-	-	-	-	-	-	70.967	70.3666
	SR	-	-	-	-	-	-	-	-	0.1626	0.1623
1.9	CW	-	-	-	-	-	-	-	-	-	71.0373
	SR	-	-	-	-	-	-	-	-	-	0.1626

Table A15: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on FF100. The CW and SR for the original setting $(q, p) = (1, 2)$ are 60.2157 and 0.1569, respectively.

q \ p		1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.0	CW	**117.2168**	99.7265	97.2394	95.0089	95.7351	95.961	96.2792	95.7062	95.7199	95.7733
	SR	**0.1799**	0.1737	0.1727	0.1719	0.1723	0.1724	0.1726	0.1725	0.1725	0.1726
1.1	CW	-	99.9262	98.5504	96.8576	96.7715	96.9106	97.2259	97.5471	97.8084	98.3201
	SR	-	0.1736	0.1731	0.1725	0.1726	0.1727	0.1729	0.1731	0.1732	0.1735
1.2	CW	-	-	100.3027	98.9845	98.8171	98.9833	99.4045	99.7495	100.1719	100.8471
	SR	-	-	0.1737	0.1733	0.1734	0.1735	0.1737	0.1739	0.1741	0.1744
1.3	CW	-	-	-	102.0236	101.7188	101.856	102.1336	102.4828	102.8048	103.4895
	SR	-	-	-	0.1744	0.1744	0.1745	0.1747	0.1749	0.175	0.1753
1.4	CW	-	-	-	-	104.8142	105.0038	105.1909	105.4119	105.5615	106.1576
	SR	-	-	-	-	0.1755	0.1756	0.1758	0.1759	0.176	0.1762
1.5	CW	-	-	-	-	-	108.0347	108.0789	108.1193	108.2833	108.7254
	SR	-	-	-	-	-	0.1766	0.1767	0.1768	0.1769	0.1771
1.6	CW	-	-	-	-	-	-	110.7385	110.7341	110.5933	111.0338
	SR	-	-	-	-	-	-	0.1776	0.1777	0.1777	0.1779
1.7	CW	-	-	-	-	-	-	-	112.885	112.6995	113.0243
	SR	-	-	-	-	-	-	-	0.1784	0.1783	0.1785
1.8	CW	-	-	-	-	-	-	-	-	114.5054	114.8594
	SR	-	-	-	-	-	-	-	-	0.1789	0.1791
1.9	CW	-	-	-	-	-	-	-	-	-	116.612
	SR	-	-	-	-	-	-	-	-	-	0.1796

Table A16: Cumulative wealth (CW) and Sharpe Ratio (SR) of qPpNWAWS on FF100MEOP. The CW and SR for the original setting $(q, p) = (1, 2)$ are 95.7632 and 0.1726, respectively.