Another Short Video by Frederi Viens

Watch this video between Dr. Katherine's two intro videos.

EXPLICIT RESOLUTION FOR LINEAR BAYESIAN ANALYSIS.

⊛ A simple example: Re-imagine Quadri's setup, but more quantitative.

- let X = true nitrogen level in our field
- let Y = value of nitrogen test.
- ⚠ Experimentalists use expensive methods to determine true X.

  ☺ Academics & extension agents design a much less expensive test to estimate X.
- Assume linear relationship betw. X & Y:

$$Y = \beta X + \sigma \underbrace{\varepsilon}_{\text{standard normal}} \quad \text{"}N(0,1)\text{"}$$

For this quick example, we assume:

→ σ is known, X & Y are known.

→ probab. density of $z = \sigma\varepsilon$ is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{z^2}{\sigma^2}}$

⊛ $Y = \beta X + \sigma \varepsilon$

$\varepsilon \sim N(0,1)$.

- We want to estimate β
- <u>In frequentist statistics</u> the "LS" estimator for β is $\hat{\beta} = (X^T X)^{-1} X^T Y$.
We want to ignore this since its uncertainty is typically underestimated.
- In Bayesian statistics we set a PRIOR DENSITY ON β: $\beta \sim N(0, s^2)$

  Here $\frac{1}{s^2}$ = PRIOR PRECISION ON β.

  Hence $f_{\text{Prior}}(\beta) = \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{1}{2}\frac{\beta^2}{s^2}}$.
- $Y = \beta X + \sigma \varepsilon$ is the LIKELIHOOD EQ.

Therefore $f_{LIK}(Y \mid \beta, X, \sigma) \sim N(\beta X, \sigma^2)$

For the purpose of this presentation, we assume now that X & Y are scalars.

- <u>BAYES' THEOREM FOR DENSITIES:</u>

$$f_{POST}(\beta \mid X, Y, \sigma) \propto f_{PRI}(\beta) f_{LK}(Y \mid \beta, X, \sigma)$$

proportionality sign

We can ignore any multipl. constants not involving β.

⊗ LET'S DO THE MATH TO DISCOVER TRUE NATURE of $f_{post}(\beta)$

- $f_{POST}(\beta | X, Y, \sigma)$

$$\propto e^{-\frac{1}{2}\frac{\beta^2}{s^2}} e^{-\frac{1}{2}(Y-\beta X)^2 \frac{1}{\sigma^2}}$$

$$= e^{-\frac{1}{2}\left(\beta^2 \frac{1}{s^2} + \beta^2 \frac{X^2}{\sigma^2} - 2XY\beta + \frac{Y^2}{\sigma^2}\right)}$$

(factor this out and ignore since no β there)

$$\propto e^{-\frac{1}{2}\left(\left(\frac{1}{s^2}+\frac{X^2}{\sigma^2}\right)\beta^2 - 2XY\beta\right)}$$

- NOW COMPLETE THE SQUARE in β!

$$= e^{-\frac{1}{2}\left(\frac{1}{s^2}+\frac{X^2}{\sigma^2}\right)\left(\beta^2 - \frac{2XY}{\frac{1}{s^2}+\frac{X^2}{\sigma^2}}\beta\right)}$$

$$\propto e^{-\frac{1}{2}\left(\frac{1}{s^2}+\frac{X^2}{\sigma^2}\right)\left(\beta - \frac{XY}{\frac{1}{s^2}+\frac{X^2}{\sigma^2}}\right)^2}$$

- As a function of β, we RECOGNIZE the NORMAL DENSITY with

$$\beta\text{'s POSTERIOR MEAN} = \frac{XY}{\frac{1}{s^2}+\frac{X^2}{\sigma^2}}$$

$$\beta\text{'s POSTERIOR VAR} = \frac{1}{\frac{1}{s^2}+\frac{X^2}{\sigma^2}}$$

☺.

- Note: POST PRECISION $= \frac{1}{s^2} + \frac{X^2}{\sigma^2}$: larger when σ is smaller. It all makes sense! Always larger than PRIOR PRECISION!!

ONE MORE THING:

- A similar calculation works out for posterior density of σ^2 assuming that β is known and assuming that prior on σ^2 is "inverse-gamma" distributed.
- In that case, posterior dens. of σ^2 is also inverse-gamma, with typically a higher precision that the prior on σ^2.

SO WE CAN DO THE MATH EXPLICITLY

→ If we know σ^2 & we estimate β.

→ If we know β & we estimate σ^2.

⚠ By iterating this procedure when σ & β are both unknown, we obtain the so-called GIBBS SAMPLER to find the joint posterior density of (β, σ^2). YAY!!