Screw Theory: Old and New

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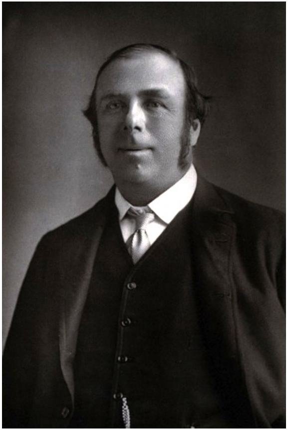
Origins—Sir Robert Stawell Ball FRS



- Irish Astronomer
- Born Dublin 1 July 1840 – Died Cambridge 25 November 1913
- Gave huge number of public lectures on Astronomy and other subject
- Avid golfer
- Screw theory

Definitions

Definition

A **screw** is a geometrical object consisting of a line in space together with a parameter called the **pitch** of the screw.

Two derived definitions,

Definition

A **twist** is a screw together with an amplitude.

Definition

A **wrench** is a screw together with an intensity.

Interpretation

- A wrench is supposed to represent a force or system of forces. Pitch zero forces correspond to pure forces along a line. Pure torque represented as infinite pitch wrenches.
- A twist represents an infinitesimal rigid-body displacement, essentially a generalised velocity. Zero pitch twist are infinitesimal pure rotations, infinite pitch twists correspond to infinitesimal pure translations.

First problem: "infinite pitch"?—corresponds to lines at infinity.

So in definition, line in space means line in projective space $\mathbb{P}^3$, not line in $\mathbb{R}^3$.

Early Work

- Much of Ball's book concerned with the problem:
  How does a rigid-body begin to move when acted on by an impulsive wrench?
- Klein and Study looked at screws as an extension of Line geometry in particular thought about Screw systems.
- According to Klein the pinnacle of screw theory was Kirchhoff's analysis of a rigid-body moving in a fluid. (Also studied by Thomson and Tait).

The Death of Screw Theory

After World War I, interest in Screw theory declined. Several possible reasons for this.

- Ball died in 1913, he had no students.
- Other British/Irish mathematicians who might have carried these ideas forward died young,
  - Clifford, a contemporary and friend of Ball's died 1879 aged 33.
  - Charles Jasper Joly (1864-1906) studied under Helmholz and Königs in Berlin, was a successor of Ball's as Royal Astronomer of Ireland.
  - Arthur Buchheim (1859–1888), studied under Klein at Leipzig. Taught at Manchester grammar school, died aged 29.
- Relativity became popular, and Euclidean geometry less so. Ball joked, "The Theory of Screws is now all done; it is quite obsolete; it is all going over into non-Euclidean space."

Kept alive in Soviet Union, Kotelnikov and others.

Rediscovered by Mechanical Engineers

In the 1960s two mechanical engineers in Australia rediscovered Balls work. Ball's theory of screws was just what they needed to study mechanisms.

- Kenneth Henderson Hunt (1920–2002) was born in the UK, worked at Monash University. "Kinematic Geometry of Mechanisms", first published in 1978. Applied screw theory to the problem of designing constant velocity joints.
- Jack Raymond Phillips (1923–2009) University of Sydney. Studied agricultural machinery, (trailed disc ploughs) and the mechanics of the lobster's claw. Two volume work "Freedom in Machinery: Introducing Screw Theory".

Modern Screw Theory

- Hervé and others realised that the Lie group of rigid-body displacements $SE(3)$, was crucial in the theory of Robots and Mechanisms.
- Perhaps Chris Gibson first identified twists with elements of the Lie algebra to $SE(3)$.
- Roger Brockett, introduced Product of Exponentials formula into robotics. Use of Lie algebra implicit here.
- Screw systems are then linear subspaces of the Lie algebra.
- Gibson and Hunt classified screw systems upto rigid transformation.

Rigid Body Displacements

Lie Group $SE(3)$ of proper rigid body displacements can be represented by 4×4 matrices,

$$M = \begin{pmatrix} R & \vec{t} \\ 0 & 1 \end{pmatrix}$$

where R is a 3×3 rotation matrix and $\vec{t}$ is a translation vector.

Group is the semi-direct product of the rotations with the translations, $SE(3) = SO(3) \ltimes \mathbb{R}^3$.

Twists—Lie algebra Elements

Many ways to describe the Lie algebra to a Lie group. Infinitesimal displacements, velocities, tangent vectors at the identity, left-invariant vector fields.

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Consider a path in the group parameterised by t, $M(t)$, this is a sequence of displacements—a rigid-body motion. Assume that $M(0) = I$ the identity in the group, then the Lie algebra element given by the path at I is,

$$S = \frac{d}{dt} M(0)\, M^{-1}(0)$$

i.e. the derivative is evaluated at $t = 0$. As a 4×4 matrix this is,

$$S = \begin{pmatrix} \Omega & \vec{v} \\ 0 & 0 \end{pmatrix}$$

Twists—Continued

Explicitly Ω is a 3×3 antisymmetric matrix,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

Can write this as a vector,

$$\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Can also write twist (Lie algebra element) as a 6-vector, in partitioned form,

$$\mathbf{s} = \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix}$$

Pitch

The pitch of the twist is given by the quotient

$$h = \frac{\vec{\omega} \cdot \vec{v}}{\vec{\omega} \cdot \vec{\omega}}$$

When $h = 0$ these 6 components give the Plücker coordinates of a line with direction $\vec{\omega}$ and moment $\vec{v}$.

When $\vec{\omega} = \vec{0}$, pitch said to be infinite.

Wrenches

Wrenches are elements of the dual space to the twists. So a wrench is a 6-vector with partitioned form,

$$\mathcal{W} = \begin{pmatrix} \vec{\tau} \\ \vec{F} \end{pmatrix}$$

such that the product,

$$\mathcal{W}^T \mathbf{s} = \vec{\tau} \cdot \vec{\omega} + \vec{F} \cdot \vec{v}$$

is a scalar, that is it must be invariant with respect to a rigid change of coordinates.

If $\vec{\omega}$ and $\vec{v}$ are the angular and linear velocities of the body and if $\vec{F}$ and $\vec{\tau}$ are the force and torque acting on the body then, $\mathcal{W}^T \mathbf{s}$ gives the power—a scalar.

Screws

- Notice in this modern view, twists are the fundamental concept.
- Wrenches are elements of the dual to the Lie algebra—linear functional on the twists
- Screws are elements of the projective space formed from the Lie algebra. (and not very interesting or important)

Many other classical definitions of screws, keep being rediscovered,

- vector fields with helicoidal symmetry
- linear line complexes

Velocities of Points

Suppose $\vec{p}_0$ is a point on the moving body. Its subsequent positions will be given by,

$$\begin{pmatrix} \vec{p}(t) \\ 1 \end{pmatrix} = M(t) \begin{pmatrix} \vec{p}_0 \\ 1 \end{pmatrix}$$

where $M(t)$ a rigid-body motion as above.

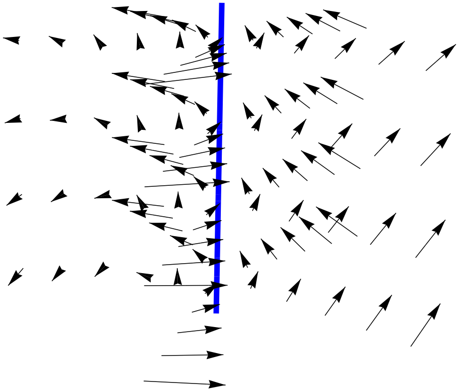
Velocity of point $\vec{p}(t)$ on the moving body given by,

$$\begin{pmatrix} \dot{\vec{p}}(t) \\ 0 \end{pmatrix} = \dot{M}(t) \begin{pmatrix} \vec{p}_0 \\ 1 \end{pmatrix} = \dot{M} M^{-1} M \begin{pmatrix} \vec{p}_0 \\ 1 \end{pmatrix} = S \begin{pmatrix} \vec{p}(t) \\ 1 \end{pmatrix}$$

As 3-vectors this is,

$$\dot{\vec{p}} = \Omega \vec{p} + \vec{v} = \vec{\omega} \times \vec{p} + \vec{v}$$

Helicoidal Vector Field



The blue line is the screw axis. Velocities are invariant with respect to rotations about the screw axis and translations along it.

Near the axis velocities are nearly parallel with the blue line, further away velocity vectors are almost perpendicular to the axis.

Dynamics of a Rigid Body

As an application, look at how simple robot dynamics is in this formalism. For a single rigid body the equation of motion is a single wrench equation, it can be derived quite simply from the equation,

$$\frac{d}{dt}\big(N\mathbf{s}\big) = \mathcal{W}$$

where $\mathcal{W}$ is the applied wrench—force and torque and $N\mathbf{s}$ is the momentum 6-vector.

Momentum and Inertia

The 6×6 inertia matrix N, is given in partitioned form as,

$$N = \begin{pmatrix} \mathbb{I} & mC \\ mC^T & I_3 \end{pmatrix}$$

where $\mathbb{I}$ is the usual 3×3 inertia matrix of the body, I_3 is the 3×3 identity matrix, m is the body's mass and C is the position of the body's centre of mass written as an antisymmetric 3×3 matrix,

$$C = \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix}$$

Equation of Motion

Performing the differentiation of the momentum 6-vector give the equation of motion of a single rigid body

$$N\dot{\mathbf{s}} + \{\mathbf{s},\, N\mathbf{s}\} = \mathcal{W}$$

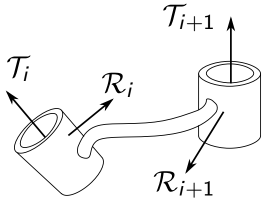
The bracket operation here is the co-adjoint representation of the Lie algebra on its dual, it is given by,

$$\{\mathbf{s},\, \mathcal{M}\} = \left\{ \begin{pmatrix} \vec{\omega} \\ \vec{v} \end{pmatrix}, \begin{pmatrix} \vec{J} \\ \vec{l} \end{pmatrix} \right\} = \begin{pmatrix} \vec{\omega} \times \vec{J} + \vec{v} \times \vec{l} \\ \vec{\omega} \times \vec{l} \end{pmatrix}$$

Note, this is the equation in the global frame of reference, the inertia will change as the body moves.

Equations of Motion for a Single Link

Consider the ith link in a serial robot. Ignore gravity.



As before the equation of motion, in the global fixed frame, just,

$$N_i \dot{\mathbf{s}}_i + \{\mathbf{s}_i,\, N_i \mathbf{s}_i\} = \mathcal{W}_i$$

where the subscript i refers to the ith link.

Here $\mathbf{s}_i$ is twist about ith joint.

Can say more about the applied wrench — composed of the joint torques and the reaction wrenches.

$$N_i \dot{\mathbf{s}}_i + \{\mathbf{s}_i,\, N_i \mathbf{s}_i\} = \mathcal{T}_i + \mathcal{R}_i - \mathcal{T}_{i+1} - \mathcal{R}_{i+1}$$

The Last Link

The equation for the final link, usually link 6, will be a little different,

$$N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\} = \mathcal{T}_6 + \mathcal{R}_6$$

We might want to include a wrench due to the weight of the payload, but since we are ignoring gravity let's not.

The Other Links

writing all the equations for all the links gives,

$$
\begin{aligned}
N_1\dot{\mathbf{s}}_1 + \left\{\mathbf{s}_1,\, N_i\mathbf{s}_1\right\} &= \mathcal{T}_1 + \mathcal{R}_1 - \mathcal{T}_2 - \mathcal{R}_2 \\
N_2\dot{\mathbf{s}}_2 + \left\{\mathbf{s}_2,\, N_2\mathbf{s}_2\right\} &= \mathcal{T}_2 + \mathcal{R}_2 - \mathcal{T}_3 - \mathcal{R}_3 \\
N_3\dot{\mathbf{s}}_3 + \left\{\mathbf{s}_3,\, N_3\mathbf{s}_3\right\} &= \mathcal{T}_3 + \mathcal{R}_3 - \mathcal{T}_4 - \mathcal{R}_4 \\
N_4\dot{\mathbf{s}}_4 + \left\{\mathbf{s}_4,\, N_4\mathbf{s}_4\right\} &= \mathcal{T}_4 + \mathcal{R}_4 - \mathcal{T}_5 - \mathcal{R}_5 \\
N_5\dot{\mathbf{s}}_5 + \left\{\mathbf{s}_5,\, N_5\mathbf{s}_5\right\} &= \mathcal{T}_5 + \mathcal{R}_5 - \mathcal{T}_6 - \mathcal{R}_6 \\
N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\} &= \mathcal{T}_6 + \mathcal{R}_6
\end{aligned}
$$

Can add cumulatively from the bottom to eliminate torques and reactions from higher links,

$$
\sum_{i=j}^{6}\left(N_6\dot{\mathbf{s}}_6 + \left\{\mathbf{s}_6,\, N_6\mathbf{s}_6\right\}\right) = \mathcal{T}_j + \mathcal{R}_j, \qquad j = 1, 2, \ldots 6
$$

Equations of Motion for a Serial Robot

Now the reaction wrench at any joint cannot do work on the joint twist,

$$\mathcal{R}_i^T \mathbf{s}_i = 0, \qquad i = 1, 2, \dots 6$$

Moreover, the wrench for the joint torque and joint twist have the same axis so,

$$\mathcal{T}_i^T \mathbf{s}_i = \mathbf{s}_i^T \mathcal{T}_i = \tau_i, \qquad i = 1, 2, \dots 6$$

where τ_i is the torque of the ith motor.

Thus pairing with the appropriate joint twist eliminates the reaction wrenches and produces 6 scalar equations,

$$\sum_{i=j}^{6} \mathbf{s}_j^T \Big(N_i \dot{\mathbf{s}}_i + \big\{ \mathbf{s}_i,\, N_i \mathbf{s}_i \big\} \Big) = \tau_j, \qquad j = 1, 2, \dots 6$$

Conclusions—Other Applications

- Errors and Numerical Methods. Robot Calibration.
- Stiffness and Compliance.
- Hybrid control, pseudo-inverses, SVD, manipulability ellipsoids and other sins.