Modeling of soft robots using products of exponentials

Stanislao Grazioso[1], Giuseppe Di Gironimo[1], Bruno Siciliano[2]

[1] IDEAS Lab, Department of Industrial Engineering
[2] PRISMA Lab, Department of Electrical Engineering and Information Technology

Università degli Studi di Napoli Federico II

Tutorial on *Screw Theory for Robotics: a Practical Approach for Modern Robot Mechanics*
2018 IEEE/RSJ International Conference on Intelligent Robots and Systems, Madrid, Spain

5 October 2018

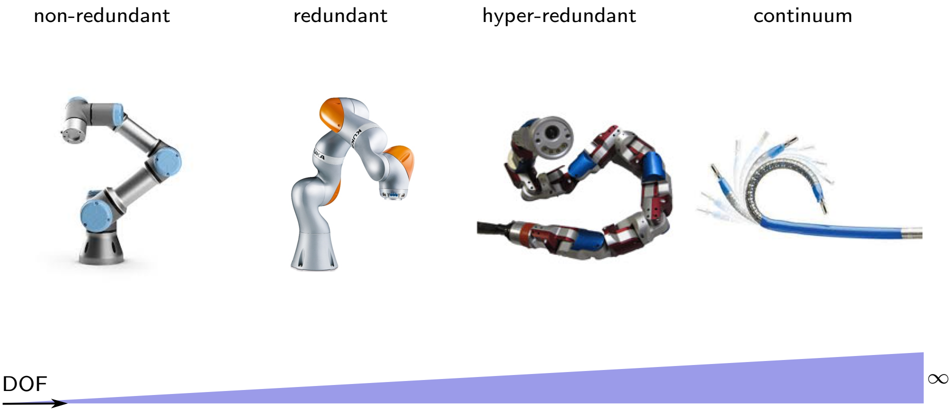
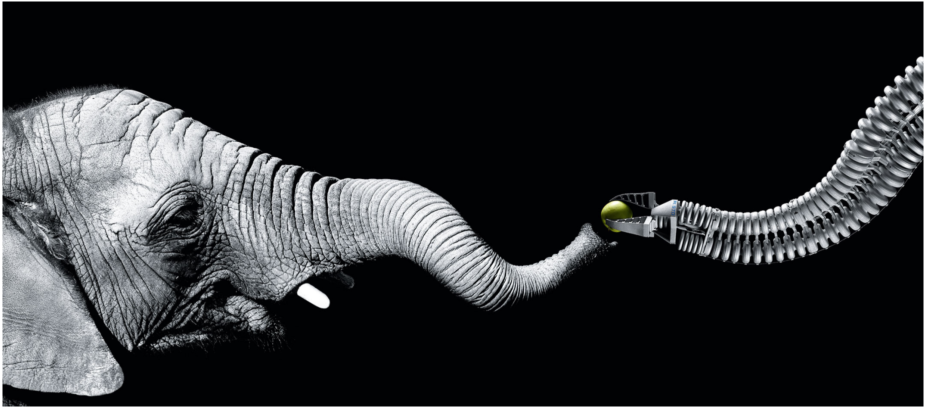


Figure: Classification of robots based on degrees of freedom.

Introduction (cont'd)

Soft robots

Continuously deformable **bioinspired** robots with an **elastic structure**.



Introduction (cont'd)

How to describe the three-dimensional shape of soft robots?

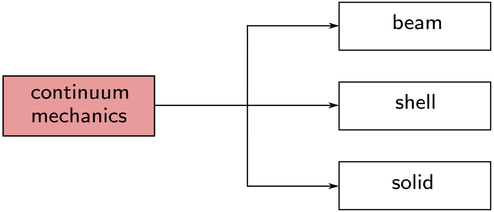
$\Downarrow$

Using theories from **continuum mechanics**

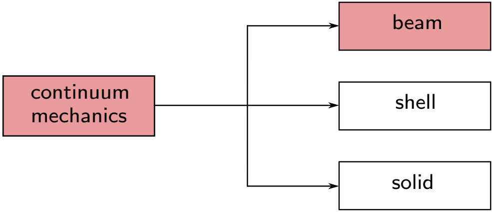
Continuum mechanics for soft robotics

continuum mechanics

Continuum mechanics for soft robotics (cont'd)

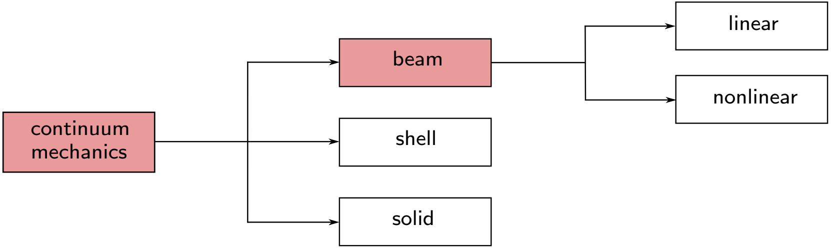


Continuum mechanics for soft robotics (cont'd)



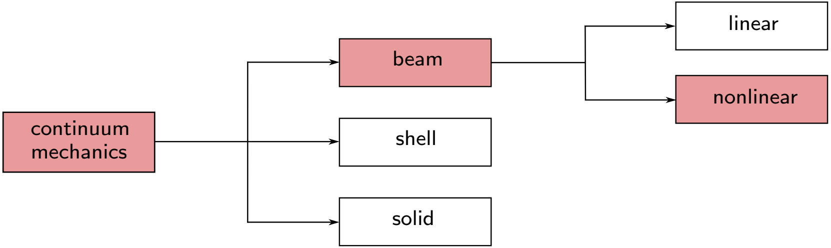
soft robotic arm:
one dimension is dominant
over the two others

Continuum mechanics for soft robotics (cont'd)



soft robotic arm:
one dimension is dominant
over the two others

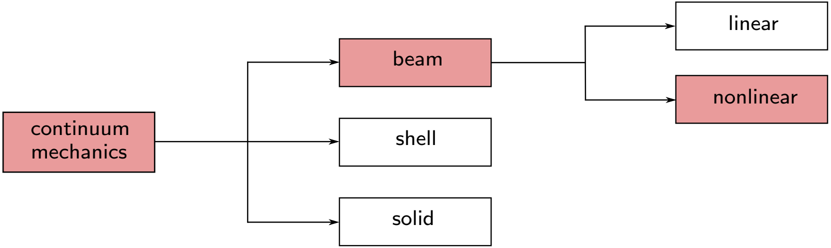
Continuum mechanics for soft robotics (cont'd)



soft robotic arm:
one dimension is dominant over the two others

- large motions
- no approximation on kinematic variables

Continuum mechanics for soft robotics (cont'd)



soft robotic arm:
one dimension is dominant over the two others

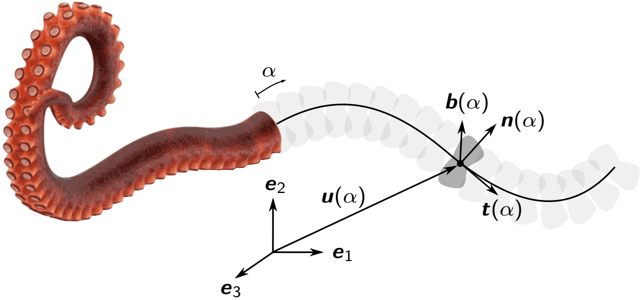
- large motions
- no approximation on kinematic variables

↓

COSSERAT THEORY

Modeling assumption

A soft robotic arm is the continuous assembly of **two–dimensional cross sections** moving upon a **three-dimensional curve** according to infinite **rigid–body transformations** defined by distributed laws of **internal deformations**



Continuum formulation

Position field

$$\alpha \in \mathbb{R} \mapsto \boldsymbol{H}(\alpha) = \mathcal{H}(\boldsymbol{R}(\alpha), \boldsymbol{u}(\alpha)) \in SE(3)$$

- $\alpha \in \mathbb{R}$, material abscissa along the arm
- $\boldsymbol{u}(\alpha) \in \mathbb{R}^3$, position vector of the cross-section
- $\boldsymbol{R}(\alpha) = [\boldsymbol{t}(\alpha) \;\; \boldsymbol{n}(\alpha) \;\; \boldsymbol{b}(\alpha)] \in SO(3)$, rotation matrix of the cross-section

with

$SO(3)$ special Orthogonal group: the Lie group of the rotation matrices

$SE(3)$ special Euclidean group: the Lie group of the homogeneous matrices

Continuum formulation (cont'd)

Deformation field

The homogeneous matrix $\boldsymbol{H}$ evolves along the material abscissa α according to the differential kinematic relationship

$$\boldsymbol{H}'(\alpha) = \boldsymbol{H}(\alpha)\widetilde{\boldsymbol{f}}(\alpha)$$

where $\widetilde{\boldsymbol{f}}$ is a left invariant vector field.

$\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{f}_\omega} & \boldsymbol{f}_u \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathfrak{se}(3)$ deformation twist

$\boldsymbol{f}_u \in \mathbb{R}^3$ vector of linear deformations

$\widetilde{\boldsymbol{f}_\omega} = \begin{bmatrix} 0 & -f_{\omega,3} & f_{\omega,2} \\ f_{\omega,3} & 0 & -f_{\omega,1} \\ -f_{\omega,2} & f_{\omega,1} & 0 \end{bmatrix} \in \mathfrak{so}(3)$ skew-symmetric matrix of angular deformations

Deformations as Lie algebra elements

- $\widetilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3)$
- $\widetilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3)$

with

$\mathfrak{so}(3)$ Lie algebra associated to the Lie group SO(3)

$\mathfrak{se}(3)$ Lie algebra associated to the Lie group SE(3)

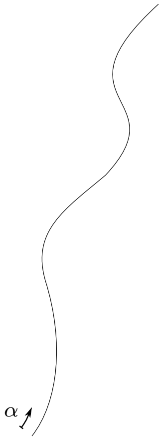
$$\Downarrow$$

$\widetilde{\mathbf{f}}(\alpha) \in \mathfrak{se}(3)$ deformation twist

$\mathbf{f}(\alpha) \in \mathbb{R}^6$ deformation vector (axial, shear, bending, torsion)

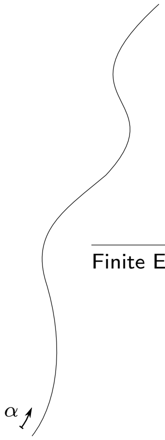
How to obtain a finite–dimensional system?

The discretization strategy



continuum arm: continuous field $\boldsymbol{f}(\alpha)$

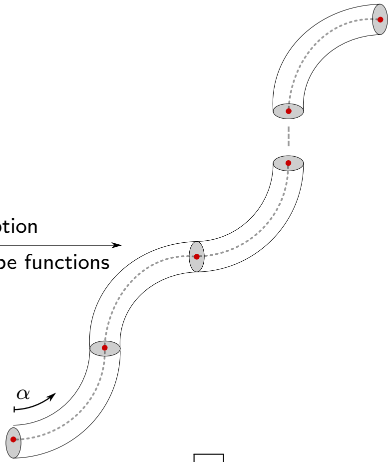
The discretization strategy (cont'd)



Piecewise constant deformation assumption

$\longrightarrow$

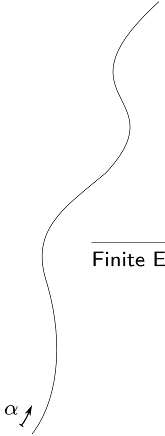
Finite Element Method (FEM) using helical shape functions



continuum arm: continuous field $\boldsymbol{f}(\alpha)$

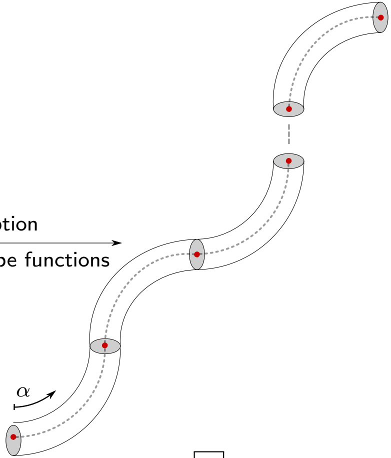
discrete arm: n deformation vectors $\boxed{\boldsymbol{f}}$ (each one is a 6-D vector)

The discretization strategy (cont'd)



Piecewise constant deformation assumption

Finite Element Method (FEM) using helical shape functions

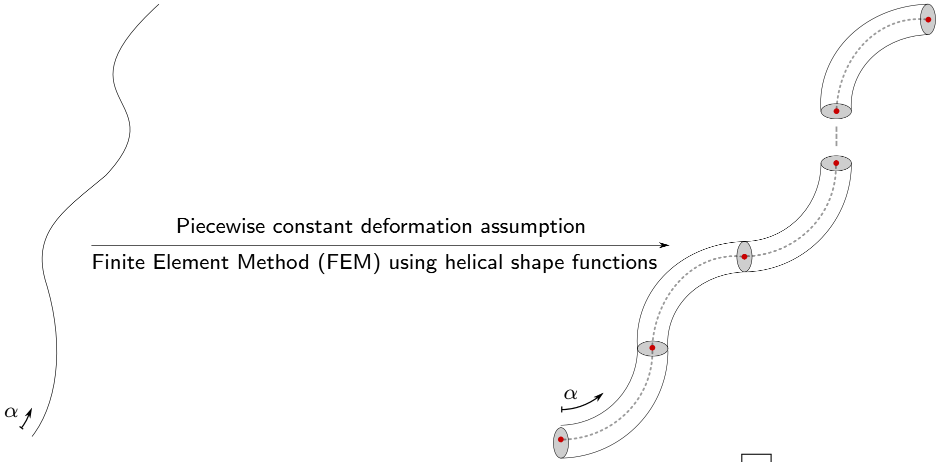


continuum arm: continuous field $\boldsymbol{f}(\alpha)$

discrete arm: n deformation vectors $\boxed{\boldsymbol{f}}$ (each one is a 6-D vector)

$\boxed{\boldsymbol{q}}$ traditional rigid robotics

The discretization strategy (cont'd)



continuum arm: continuous field $\boldsymbol{f}(\alpha)$

discrete arm: n deformation vectors $\boxed{\boldsymbol{f}}$ (each one is a 6-D vector)

$\boxed{\boldsymbol{q}}$ traditional rigid robotics

THE FE DEFORMATION SPACE FORMULATION

The discretization strategy (cont'd)

Summary

- **Continuum mechanics**: Cosserat rod theory
- **Finite element method**: from infinite to finite–dimensional system (∞ to $6n$)
- **Soft robotics**: deformation-based formulation

Kinematics

$$\mathbf{H}'(\alpha) = \mathbf{H}(\alpha)\widetilde{\boldsymbol{f}}(\alpha)$$

- $\alpha \in [0, L_n] = [0, L_1), (L_1, L_2), \ldots, (L_{n-1}, L_n]$
  L_n: total length of the arm
- $\boldsymbol{f}_i$: constant deformation vector of each discrete element of the arm

$\Downarrow$ . . . since $\boldsymbol{f}$ does not depend on α

Product of exponentials (PoE)

$$\boldsymbol{H}(\alpha) = \boldsymbol{H}_0 \prod_{i=1}^{n} \exp_{SE(3)}\left((\min(L_i, \alpha_i) - L_{i-1})\widetilde{\boldsymbol{f}}_i\right) \tag{1}$$

- $\boldsymbol{H}_0$ in SE(3): configuration of the arm at $\alpha = 0$

Kinematics (cont'd)

The exponential map on $SE(3)$

$$\exp_{SE(3)}(\cdot) : \mathbb{R}^6 \to SE(3), \qquad \boldsymbol{f} \mapsto \exp_{SE(3)}(\boldsymbol{f}) \quad (*)$$

$$\exp_{SE(3)}(\boldsymbol{f}) = \begin{bmatrix} \exp_{SO(3)}(\boldsymbol{f}_\omega) & \boldsymbol{T}^T_{SO(3)}(\boldsymbol{f}_\omega)\boldsymbol{f}_u \\ \boldsymbol{0}_{1\times 3} & 1 \end{bmatrix}$$

- $\exp_{SO(3)}(\boldsymbol{f}_\omega) = \boldsymbol{I}_{3\times 3} + \alpha(\boldsymbol{f}_\omega)\widetilde{\boldsymbol{f}}_\omega + \frac{\beta(\boldsymbol{f}_\omega)}{2}\widetilde{\boldsymbol{f}}^2_\omega$: Rodrigues' formula
- $\boldsymbol{T}_{SO(3)}(\boldsymbol{f}_\omega) = \boldsymbol{I}_{3\times 3} - \frac{\beta(\boldsymbol{f}_\omega)}{2}\widetilde{\boldsymbol{f}}_\omega + \frac{1-\alpha(\boldsymbol{f}_\omega)}{\|\boldsymbol{f}_\omega\|^2}\widetilde{\boldsymbol{f}}^2_\omega$: Tangent operator

$$\alpha(\boldsymbol{f}_\omega) = \frac{\sin(\|\boldsymbol{f}_\omega\|)}{\|\boldsymbol{f}_\omega\|} \qquad \beta(\boldsymbol{f}_\omega) = 2\frac{1-\cos(\|\boldsymbol{f}_\omega\|)}{\|\boldsymbol{f}_\omega\|^2}$$

(*) The formal definition of exponential map uses the Lie algebra $\mathfrak{se}(3)$ instead of $\mathbb{R}^6$.
However, due to the isomorphism between Lie algebra $\mathfrak{se}(3)$ and $\mathbb{R}^6$, Hence, with a slight abuse of notation, we use $\mathbb{R}^6$ instead of $\mathfrak{se}(3)$

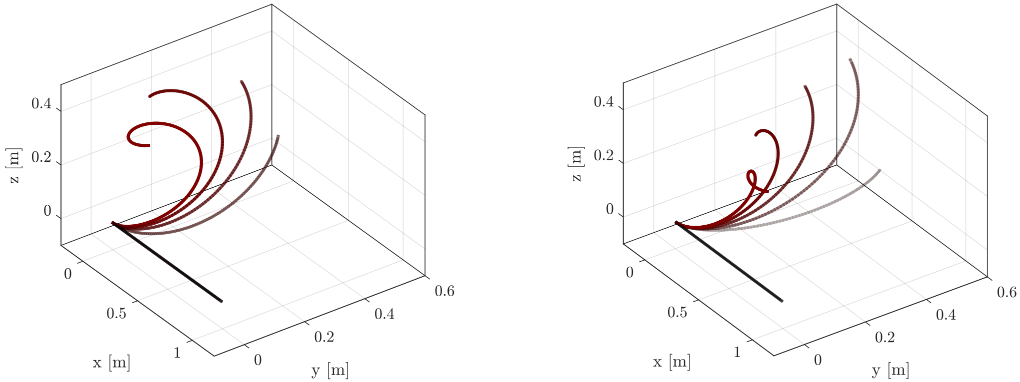
Example

- Computing the shape of a soft arm with one element ($n = 1$) with internal deformations $\boldsymbol{f}_u = \mathbf{0}_{3\times 1}$; $\quad \boldsymbol{f}_\omega = [\tau \; 0 \; \kappa]^T$ (no axial and shear deformations; bending about z; torsion about x)

$\Downarrow \qquad$ PoE ($n = 1$): $\boldsymbol{H}(\alpha) = \boldsymbol{H}_0 \exp_{SE(3)}(\alpha \boldsymbol{f})$

$$
\boldsymbol{H}(\alpha) = \begin{bmatrix} 1-(1-\cos(\alpha\kappa_g))\frac{\kappa^2}{\kappa_g^2} & -\sin(\alpha\kappa_g)\frac{\kappa}{\kappa_g} & (1-\cos(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^2} & \alpha+(\sin(\alpha\kappa_g)-\alpha\kappa_g)\frac{\kappa^2}{\kappa_g^3} \\ \sin(\alpha\kappa_g)\frac{\kappa}{\kappa_g} & \cos(\alpha\kappa_g) & -\sin(\alpha\kappa_g)\frac{\tau}{\kappa_g} & (1-\cos(\alpha\kappa_g))\frac{\kappa}{\kappa_g^2} \\ (1-\cos(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^2} & \sin(\alpha\kappa_g)\frac{\tau}{\kappa_g} & 1-(1-\cos(\alpha\kappa_g))\frac{\tau^2}{\kappa_g^2} & (\alpha\kappa_g-\sin(\alpha\kappa_g))\frac{\kappa\tau}{\kappa_g^3} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- $\kappa_g = \sqrt{\kappa^2 + \tau^2}$: Gaussian curvature of the arm



(a) $\tau = 3\,\mathrm{m}^{-1}$; $\kappa = 0 : \pi/2 : 2\pi\ \mathrm{m}^{-1}$.

(b) $\kappa = 3\,\mathrm{m}^{-1}$; $\tau = 0 : \pi/2 : 2\pi\ \mathrm{m}^{-1}$

Figure: Whole arm screw motion of a manipulator with constant curvature and torsion. $L = 1\,\mathrm{m}$.

Differential kinematics

Continuum formulation

The velocity field and compatibility equations are given by

$$\dot{\boldsymbol{H}}(\alpha) = \boldsymbol{H}(\alpha)\widetilde{\boldsymbol{\eta}}(\alpha)$$
$$\boldsymbol{\eta}'(\alpha) - \dot{\boldsymbol{f}}(\alpha) = \widehat{\boldsymbol{\eta}}(\alpha)\boldsymbol{f}(\alpha)$$

$\overset{(1)}{\longrightarrow}$

Deformation space formulation

$$\boldsymbol{\eta}(\alpha) = \mathbb{J}(\alpha, \boldsymbol{f})\dot{\boldsymbol{f}} \tag{2}$$

$\dot{(\cdot)}$	derivative *wrt* time
$\widetilde{\boldsymbol{\eta}} \in \mathfrak{se}(3)$	velocity twist
$\widehat{(\cdot)} : \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$	matrix operator

$\boldsymbol{\eta} \in \mathbb{R}^{6n}$	total velocity vector
$\boldsymbol{f} \in \mathbb{R}^{6n}$	total deformation vector
$\mathbb{J} \in \mathbb{R}^{6n\times 6n}$	soft geometric Jacobian

Statics

Principle of virtual work

The manipulator is in static equilibrium iff the virtual work done by the internal forces balances the virtual work done by the external forces, i.e.

$$\delta(\mathcal{V}_{int}) = \delta(\mathcal{V}_{ext})$$

$$\begin{matrix} (1) & (2) \\ \downarrow & \downarrow \end{matrix}$$

Static model

$$\boldsymbol{\sigma} = \int_L \mathbb{J}^T(\alpha, \boldsymbol{f})\boldsymbol{g}_{ext}\, \mathrm{d}\alpha \tag{3}$$

$\boldsymbol{\sigma} = \mathbb{K}\boldsymbol{f}$	internal force vector (linear elastic material model)
$\mathbb{K}$	stiffness matrix
$\mathbb{F} = \int_L \mathbb{J}^T(\alpha, \boldsymbol{f})\boldsymbol{g}_{ext}\, \mathrm{d}\alpha$	external force vector

Dynamics

Hamilton's principle

The *action integral* over the time interval $[t_i, t_f]$ is stationary provided that the initial and final configurations are fixed, i.e.

$$\delta \left(\int_{t_i}^{t_f} (\mathcal{K} - \mathcal{V}_{int} + \mathcal{V}_{ext}) \, \mathrm{d}t \right) = 0$$

where $\mathcal{K}$ is the kinetic energy.

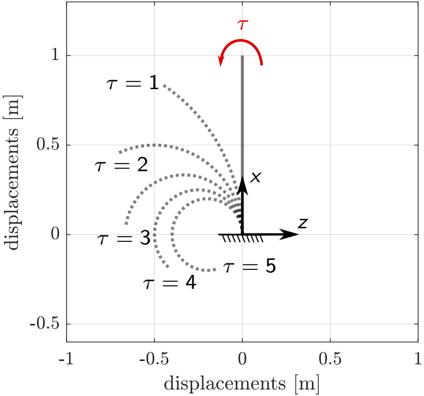
$$\begin{array}{ccc} (1) & (2) & (3) \\ \downarrow & \downarrow & \downarrow \end{array}$$

Dynamic model

$$\mathbb{M}(\alpha, \boldsymbol{f})\ddot{\boldsymbol{f}} + \left(\mathbb{C}_1\left(\alpha, \boldsymbol{f}, \dot{\boldsymbol{f}}\right) - \mathbb{C}_2(\alpha, \boldsymbol{f}, \dot{\boldsymbol{f}})\right)\dot{\boldsymbol{f}} - \mathbb{K}\boldsymbol{f} = \mathbb{F}$$

$\mathbb{M}, \mathbb{C}_1, \mathbb{C}_2$ mass and velocity matrices

- Computing the shape of a soft arm with one element ($n = 1$) subject to an external generalized force $\boldsymbol{g}_{ext,u} = \mathbf{0}_{3\times 1}$; $\boldsymbol{g}_{ext,u} = [0\ \tau\ 0]^T$



$$\overset{(3)}{\downarrow} \downarrow \boldsymbol{\sigma}(L) = \mathbf{K}(L)\boldsymbol{f}(L) = \boldsymbol{g}_{ext}(L)$$

$$\boldsymbol{f}_u = \mathbf{0}_{3\times 1};\quad \boldsymbol{f}_\omega = [0\ \tau/EI_y\ 0]^T$$

$$\overset{(1)}{\downarrow}$$

$$\mathbf{H}(\alpha) = \begin{bmatrix} \cos(\alpha\kappa_y) & 0 & \sin(\alpha\kappa_y) & \frac{1}{\kappa_y}\sin(\alpha\kappa_y) \\ 0 & 1 & 0 & 0 \\ -\sin(\alpha\kappa_y) & 0 & \cos(\alpha\kappa_y) & \frac{1}{\kappa_y}(1-\cos(\alpha\kappa_y)) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\kappa_y = \tau/EI_y$; EI_y: flexural rigidity

Conclusions

- Soft robotic arm modeled as a Cosserat rod
- Finite element method involving helical shape functions for arm discretization
- Geometric formalisms of screw–theory
- Kinematics described using product of exponentials
- The deformation space formulation for soft robot kinematics, statics and dynamics: equivalence with mechanics of rigid robots.

Conclusions

- Soft robotic arm modeled as a Cosserat rod
- Finite element method involving helical shape functions for arm discretization
- Geometric formalisms of screw–theory
- Kinematics described using product of exponentials
- The deformation space formulation for soft robot kinematics, statics and dynamics: equivalence with mechanics of rigid robots.

Conclusions

- Soft robotic arm modeled as a Cosserat rod
- Finite element method involving helical shape functions for arm discretization
- Geometric formalisms of screw–theory
- Kinematics described using product of exponentials
- The deformation space formulation for soft robot kinematics, statics and dynamics: equivalence with mechanics of rigid robots.

Conclusions

- Soft robotic arm modeled as a Cosserat rod
- Finite element method involving helical shape functions for arm discretization
- Geometric formalisms of screw–theory
- Kinematics described using product of exponentials
- The deformation space formulation for soft robot kinematics, statics and dynamics: equivalence with mechanics of rigid robots.

Conclusions

- Soft robotic arm modeled as a Cosserat rod
- Finite element method involving helical shape functions for arm discretization
- Geometric formalisms of screw–theory
- Kinematics described using product of exponentials
- The deformation space formulation for soft robot kinematics, statics and dynamics: equivalence with mechanics of rigid robots.

Modeling of soft robots using product of exponentials