Continuous Finite-Time Stabilization of the Translational and Rotational Double Integrator

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Introduction

- Feedback linearization often generates closed-loop Lipschitzian dynamics
- Convergence in such systems is often exponential, carrying the burden of infinite settling time
- Undesirable in time-critical applications such as minimum-energy control, conservation of momentum et cetera
- It is therefore imperative to design continuous finite-time stabilizing controllers
- Classical optimal control provides examples of systems that converge to the equilibrium in finite time such as the double integrator.

The case for continuous finite-time stabilizing (FTS) controllers

- Optimal synthesis methods such as the bang-bang time-optimal feedback control of the double integrator has discontinuous dynamics which leads to chattering or excitation of HF dynamics in flexible structures applications [Fuller et. al., '66]
- The double integrator is open-loop controllable such that its states can be driven to the origin in finite time
- The minimum-energy control is one such method which drives the state of the system $\ddot{x} = u$ from an initial condition $x(0) = x_0$, $\dot{x}(0) = y_0$ to the origin in a given time t_f [Athans, '66]
- But such open-loop methods invite insensitivity to uncertainties and have poor disturbance rejection

Non-Lipschitzian Dynamics of a Continuous FTS Feedback Controller

Overview of Problem

- The design of FTS continuous time-invariant feedback controllers involve non-Lipschitzian closed-loop dynamics
- Such controllers will exhibit non-unique solutions in backward time, i.e., better robustness and good disturbance rejection
- Such non-unique (revert time) solutions would violate uniqueness conditions for Lipschitz differential equations

Statement of Problem

- Consider a rigid body rotating under the action of a mechanical torque about a fixed axis
- Its equations of motion resemble those of a double integrator. States differ by $2n\pi$ (where $n = 0, 1, 2, \ldots$) in angular modes which correspond to the same physical configuration of the body.
- State space for this system is $S^1 \times \mathbb{R}$ rather than $\mathbb{R}^2$ [Andronov et. al]
- Developing stabilizing controls for the double integrator on $\mathbb{R}^2$ (translational double integrator) will lead to unwinding since the configuration space is actually $\mathbb{R}$
- This makes an interesting problem when designing feedback controllers for the rotational double integrator with anti-wind-up compensation
- Discontinuous feedback controllers are practically infeasible due to the chattering they introduce because of plant uncertainties
- They could also excite high-frequency dynamics when used in controlling lightly damped structures [Baruh et. al.]

Finite-Time Stabilization: A Definition

For the System of differential equations,

$$\dot{y}(t) = f(y(t) \quad (1)$$

where $f : \mathcal{D} \mapsto \mathbb{R}^n$ is continuous on an open neighborhood $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin and $f(0) = 0$. A continuously differentiable function $y : I \to \mathcal{D}$ is said to be a solution of 1 on the interval $I \subset \mathbb{R}$ if y satisfies (1) for all $t \in I$

- We assume (1) possesses unique solutions in forward time except possibly at the origin for all initial conditions
- Uniqueness in forward time and the continuity of f ensure that solutions are continuous functions of initial conditions even when f is no longer Lipschitz continuous [Hartman et. al., '82, Th. 2.1, p. 94]

Definition

The origin is finite-time stable if there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a settling time function $T : \mathcal{N} \setminus 0 \mapsto (0, \infty)$, such that we have the following:

1. Finite-time convergence: For every $x \in \mathcal{N} \setminus \{0\}$, $\rho_t(x)$ is defined for $t \in \left[0, T(x)\right)$, $\rho_t(x) \in \mathcal{N} \setminus \{0\}$, for $t \in \left[0, T(x)\right)$, and $lim_{t \mapsto T(x)} \rho_t(x) = 0$
2. Lyapunov stability: For every open set $\mathcal{U}_\epsilon$ such that $0 \in \mathcal{U}_\epsilon \subseteq \mathcal{N}$, there exists an open set $\mathcal{U}_\delta$ such that $0 \in \mathcal{U}_\delta \subseteq \mathcal{N}$ and for every $x \in U_\delta \setminus \{0\}$, $\rho_t(x) \in \mathcal{U}\bullet$ for $t \in \left[0, T(x)\right)$.

When $\mathcal{D} = \mathcal{N} = \mathbb{R}^n$, we have global finite-time convergence.

Theorem: For a continuously differentiable function $V : \mathcal{D} \mapsto \mathbb{R}$, such that $k > 0$, $\alpha \in (0, 1)$, where α and $k \in \mathbb{R}$ if there exists a neighborhood of the origin $\mathcal{U} \subset \mathcal{D}$ such that V is positive definite, $\dot{V}$ is negative definite and $\dot{V} + kV^\alpha$ is negative semi-definite on $\mathcal{U}$, where $\dot{V}(x) = \dfrac{\partial V}{\partial x}(x) f(x)$. Then, the origin of (1) is finite-time stable. Also, the settling time, $T(x)$ is defined as $T(x) = \dfrac{1}{k(1-\alpha)} V(x)^{1-\alpha}$

Continuous Finite-Time Stabilizing Controllers

We want to find a continuous feedback law, $u = \psi(x, y)$ such that the double integrator defined as, $\dot{x} = y, \quad \dot{y} = u$ is finite-time stabilized.

Proposition 1

The origin of the double integrator is globally finite-time stable [Bhat et. al., '98, §III] under the feedback control law u where

$$\psi(x, y) = -\text{sign}\,(y)|y|^{\alpha} - \text{sign}\,(\phi_\alpha\,(x, y))\left|(\phi_\alpha\,(x, y))\right|^{\dfrac{\alpha}{2-\alpha}} \tag{2}$$

where $\phi_\alpha(x, y) \triangleq x + \dfrac{1}{2-\alpha}\text{sign}(y)|y|^{\dfrac{\alpha}{2-\alpha}}$

See Appendix A. for proof.

Remarks

The vector field obtained by using the feedback control law u is locally Lipschitz everywhere except the x-axis (denoted Γ), and the zero-level set $\mathcal{S} = \{(x, y) : \phi_\alpha(x, y) = 0\}$ of the function ϕ_α. The closed-loop vector field f_α is transversal to Γ at every point in $\Gamma \setminus \{0, 0\}$

- Every initial condition in $\Gamma \setminus \{0, 0\}$ has a unique solution in forward time
- The set $\mathcal{S}$ is positively invariant for the closed-loop system
- On the set $\mathcal{S}$ the closed-loop system is

$$\dot{x} = -\text{sign}\,(x)\left[(2-\alpha)|x|\right]^{\frac{1}{2-\alpha}} \tag{3}$$

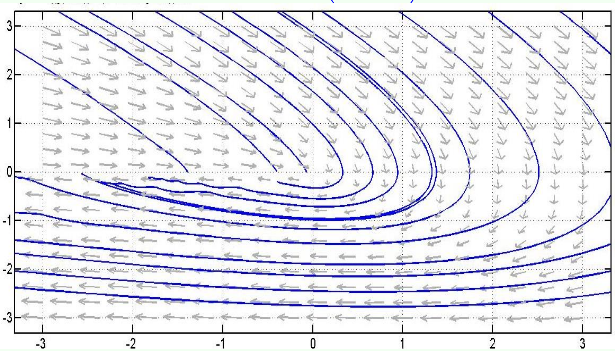
$$\dot{y} = -\text{sign}\,(y)|y|^{\alpha} \tag{4}$$

The resulting closed loop system (4) is locally Lipschitz everywhere except the origin and therefore possesses unique solutions in forward time for initial conditions in $\mathcal{S} \setminus \{0, 0\}$.

Example 1: Implementation of the proposed controller

By choosing $\alpha = \frac{2}{3}$ in (2), we have the phase portrait shown in Figure 6 for the resulting feedback law

$$\psi(x, y) = -y^{\frac{2}{3}} - \left(x + \frac{3}{4} y^{\frac{4}{3}}\right)^{\frac{1}{2}} \tag{5}$$



We see that all trajectories converge to the set $\mathcal{S} = \{(x, y) : x + \frac{3}{4} y^{\frac{4}{3}} = 0\}$ in finite-time. The term $-y^{\frac{2}{3}}$ in (5) makes the set $\mathcal{S}$ positively invariant while the other term $-\left(x + \frac{3}{4} y^{\frac{4}{3}}\right)^{\frac{1}{2}}$ drives the states to $\mathcal{S}$ in finite-time. Therefore, (2) represents an example of a terminal sliding mode control without using discontinuous or high gain feedback.

Bounded Continuous Finite-Time Controllers

In the previous section, the designed feedback controller is unbounded, meaning the controller will lead to the so-called ”unwinding” phenomenon.

- In a spacecraft application, for example, such unwinding can lead to the mismanagement of fuel and and momentum-consuming devices
- In order to finite-time stabilize the controller, we saturate the elements of the controller given in (2)

It follows that for a positive number ε,

$$\begin{aligned} sat_{\varepsilon}(y) &= y, \qquad |y| < \varepsilon \\ &= \varepsilon \,\mathrm{sign}(y), \qquad |y| \geq \varepsilon \end{aligned} \tag{6}$$

$$\text{such that} \quad \left|\mathrm{sat}_{\varepsilon}(y)\right| \leq \varepsilon \quad \text{for all} \quad y \in \mathbb{R} \tag{7}$$

The relation in (7) ensures that the controller does not operate in the nonlinear region by bounding its operating range within the defined perimeter of operation (6)

Proposition 2

The origin of the double integrator system is a globally stable equilibrium under the bounded feedback control law $u = \psi(x, y)$ *with*

$$\psi_{sat}(x, y) = -\mathrm{sat}_{\varepsilon}\left\{\mathrm{sign}(y)|y|^{\alpha}\right\} - \left\{\mathrm{sat}_{\varepsilon}\,\mathrm{sign}\left(\phi_{\alpha}(x, y)\right)\Big|\left(\phi_{\alpha}(x, y)\right)\Big|^{\frac{\alpha}{2-\alpha}}\right\} \tag{8}$$

for every $\alpha \in (0, 1)$ and $\varepsilon > 0$, where $\phi_{\alpha}(x, y) \triangleq x + \frac{1}{2-\alpha}\mathrm{sign(y)}|y|^{\frac{\alpha}{2-\alpha}}$

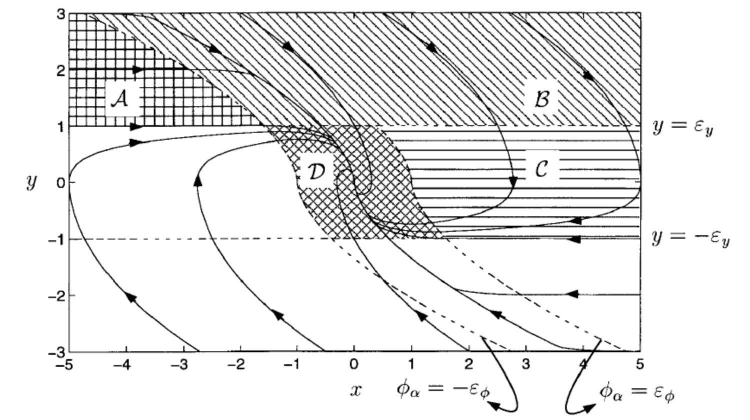
- See Proof in [Bhat et. al., ’98, §IV, p. 680].

Example 2: Bounded continuous finite time controller

For the double integrator under the feedback control law

$$\psi_{sat}(x,) = -sat_1(y^{\frac{1}{3}}) - sat_1\{(x + \frac{3}{5}y^{\frac{5}{3}})\}^{\frac{1}{5}} \tag{9}$$

obtained from (2) with $\alpha = \frac{1}{3}$ and $\varepsilon = 1$. Again, all trajectories converge to the set $\mathcal{S} = \{(x, y) : x + \frac{3}{5}y^{\frac{5}{3}} = 0\}$. But in some phase plane regions, $\psi_{sat}(x,) = 0$.



The Rotational Double Integrator

Let us denote the motion of a rigid body rotating about a fixed axis with unit moment of inertia as

$$\ddot{\theta}(t) = u(t) \tag{10}$$

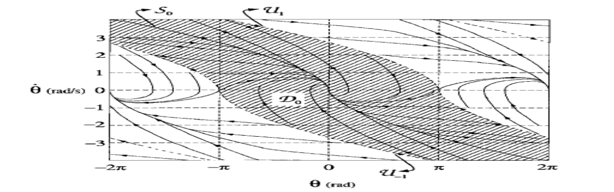
where θ is the angular displacement from some reference and u is the applied control. We can rewrite the equation as a first-order equation with $\dot{x} = \theta$ and $\dot{y} = u$.

- Suppose we require the angular position to be finite-time stable, then the feedback law given in (2) can only finite-time stabilize the origin such that if applied to the rotational double integrator, it leads to the *unwinding* phenomenon.
- Feedback controllers designed for the translational double integrator does not suffice for the rotational double integrator

  How to avoid the *unwinding* phenomenon?
- Modify (2) such that it is periodic in x with period 2π, i.e.,

$$\psi_{rot}(x, y) = -\text{sign}\,(y)|y|^{\alpha} - \text{sign}\,(sin(\phi_{\alpha}(x, y)))\,\left|sin(\phi_{\alpha}(x, y))\right|^{\frac{\alpha}{2-\alpha}} \tag{11}$$

  ϕ_{α} is same as defined in Proposition 1. For $\alpha = \frac{1}{3}$ with $u = \psi_{rot}$ the resulting phase portrait is shown below



Remarks from Phase Portrait of Rotational Double Integrator

- The closed loop system has equilibrium points at $s_n = (2n\pi, 0)$, $u_n = ((2n+1)\pi, 0)$, $n = \cdots, -1, 0, 1, \cdots$
- The equilibrium points s_n are locally finite-time stable in forward time, while the points u_n are finite-time saddles
- The domain of attraction of the equilibrium point s is $\mathcal{D}_n = \{(x, y) : (2n-1)\pi < \phi_\alpha(x, y) < (2n+1)\pi\}$.
- The shaded region of the plot shows a portion of $\mathcal{D}_0$. The sets $\mathcal{U}_{n-1}$ and $\mathcal{U}_n$ represent the stable manifolds of the equilibrium points u_{n-1} and u_n respectively where $\mathcal{U}_n = \{(x, y) : \phi_\alpha(x, y) = (2n+1)\pi\}$ and $n = \cdots, -1, 0, 1, \cdots,$
- All trajectories starting in the set $\mathcal{D}_n$ converge to the set $\mathcal{S}_n = \{(x, y) : \phi_\alpha(x, y) = 2n\pi\}$ in finite, forward time and to the set $\mathcal{U}_{n-1} \cup \mathcal{U}_n$ in finite, reverse time
- The sets $\mathcal{S}_n$ are positively invariant while the sets $\mathcal{U}_n$ are negatively invariant

- The solutions in the figure have no uniqueness to initial conditions lying in any of the sets $\mathcal{U}_n, n = \cdots, -1, 0, 1, \cdots$.
- All solutions initialized in u_n are equivalent to the rigid body resting in an unstable configuration and then starting to move spontaneously clockwise or counterclockwise
- Departure from the unstable equilibrium is a unique feature to non-Lipschitzian systems as Lipschitzian systems do not possess solutions that depart from equilibrium
- The desired final configuration is however not globally finite-time stable due to the presence of the unstable equilibrium configuration at $\theta = \pi$. These are saddle points $u_n,\ n = \cdots, -1, 0, 1, \cdots$
- This is a basic drawback to every continuous feedback controller that stabilizes the rotational double integrator without generating the unwinding effect.
- The desired final configuration in the phase plane corresponds to multiple equilibria in the phase plane meaning every controller that stabilizes the desired configuration stabilizes each equilibria
- But stability, continuous dependence on initial conditions and solutions' uniqueness imply that the domain of attraction of any two equilibrium points in the plane are non-empty, open and disjoint.

- We cannot write $\mathbb{R}^2$ as the union of a collection of disjoint sets.
- Thus, there are initial conditions in the plane that do not converge to the equilibria of the desired final configuration.
- With respect to (11), these initial conditions are the stable manifolds of the unstable configuration.
- The designed controller is practically globally stable as its non-Lipschitzian property increases the sensitivity of the unstable configuration to perturbations

Appendix A: Proof of Proposition 1

If we denote $\phi_\alpha(x, y)$ by ϕ_α and fix $\alpha \in (0, 1)$, we could choose the $\mathcal{C}^2$ Lyapunov function candidate,

$$V(x, y) = \frac{2-\alpha}{3-\alpha}|\phi_\alpha|^{\frac{3-\alpha}{2-\alpha}} + sy\phi_\alpha + \frac{r}{3-\alpha}|y|^{3-\alpha} \tag{12}$$

where r and s are positive numbers. Along the closed loop trajectories,

$$\dot{V}(x, y) = s\phi_\alpha \dot{y} + r|y|^{2-\alpha}\dot{y} + sy\dot{\phi_\alpha} + |\phi_\alpha|^{\frac{1}{2-\alpha}}\dot{\phi_\alpha} \quad = s\phi_\alpha\left[-\text{sign}(y)|y|^\alpha - \text{sign}(\phi_\alpha)|\phi_\alpha|^{\frac{\alpha}{2-\alpha}}\right] + r|y|^{2-\alpha} \tag{13}$$

$$+|\phi_\alpha|^{\frac{1}{2-\alpha}}\dot{\phi_\alpha}$$

But,

$$\dot{\phi_\alpha} = \dot{x} + \dot{y}\,\text{sign}(y)\,|y|^{1-\alpha} \tag{14}$$

from (2), it therefore follows that,

Continuation of Proof of Proposition 1

$$\dot{\phi}_\alpha = -\text{sign}\,(y)\,\text{sign}\,(\phi_\alpha)|y|^{1-\alpha}|\phi_\alpha|^{\frac{\alpha}{2-\alpha}} \tag{15}$$

Putting (14) into (13), and noting that

$$sy\dot{\phi}_\alpha = -s\,\text{sign}\,(y\phi_\alpha)|y|^{1-\alpha}|\phi_\alpha|^{\frac{1+\alpha}{2-\alpha}} \tag{16}$$

$$\text{and } \dot{\phi}_\alpha|\phi_\alpha|^{\frac{1}{2-\alpha}} = -\text{sign}\,(y)\,\text{sign}\,(\phi_\alpha)\,|y|^{1-\alpha}|\phi_\alpha|^{\frac{1+\alpha}{2-\alpha}} \tag{17}$$

we find that,

$$\dot{V}\,(x,y) = -ry^2 - s|\phi_\alpha|^{\frac{2}{2-\alpha}} - |y|^{1-\alpha}|\phi_\alpha|^{\frac{1+\alpha}{2-\alpha}} - s\phi_\alpha\,\text{sign}\,(y)|y|^\alpha - (r+s)\text{sign}(y\phi_\alpha)|y|^{2-\alpha}|\phi_\alpha|^{\frac{\alpha}{2-\alpha}} \tag{18}$$

Remarks

The obtained Lyapunov derivative in the foregoing is continuous everywhere since $\alpha \in (0,1)$. For $k > 0$ and $(x,y) \in \mathbb{R}^2$ the following holds,

- Introduce $x = k^{2-\alpha}$, $y = ky$ such that

Remarks

$$\begin{aligned}\phi_\alpha(k^{2-\alpha}x, ky) &= k^{2-\alpha}x - \frac{1}{2-\alpha}\text{sign}(ky)|ky|^{2-\alpha} \\ &= k^{2-\alpha}\phi_\alpha(x,y))\end{aligned} \tag{19}$$

and

$$V(k^{2-\alpha}x, ky) = \frac{2-\alpha}{3-\alpha}\left|\phi_\alpha(k^{2-\alpha}x, ky)\right|^{\frac{3-\alpha}{2-\alpha}} + sky\phi_\alpha(k^{2-\alpha}x, ky) + \frac{r}{3-\alpha}|ky|^{3-\alpha} \tag{20}$$

such that

$$V(k^{2-\alpha}x, ky) = k^{3-\alpha}V(x,y) \tag{21}$$

Following a similar logic as in (20), we find that

$$\dot{V}(k^{2-\alpha}x, ky) = k^2\dot{V}(x,y) \tag{22}$$

The results of the previous section imply that for $r > 1$ and $s < 1$, both V and $\dot{V}$ are positive on the set $\mathcal{O} = \{(x,y) : max_{(x,y)\neq(0,0)}|\phi_\alpha|^{\frac{1}{2-\alpha}}, |y| = 1\}$ which is a closed curve encircling the origin.

For every $(x,y) \in \mathbb{R}^2 \setminus \{0,0\}$ there exists $k > 0$ such that $k^{2-\alpha}x, ky \in \mathcal{O}$, the homogeneity properties of (20) and (22) imply a positive definite V and negative semi-definite $\dot{V}$.

From (20), V is radially unbounded so that the set $\mathcal{V} = \{(x,y) : V(x,y) = 1\}$ is compact. Therefore $\dot{V}$ achieves its maximum on the compact set $\mathcal{V}$. If we define $c = -max_{\{(x,y)\in\mathcal{V}\}}\dot{V}(x,y)$, then $\dot{V}(x,y) \leq -c\{V(x,y)\}^{\frac{2}{3-\alpha}}$ for all $(x,y) \in \mathbb{R}^2$ [Bhat et. al. '96]

The homogeneity of (20) and (22) ensures $\dot{V}(x,y) \leq -cV(x,y)^{\frac{2}{3-\alpha}}$ for all $(x,y) \in \mathbb{R}^2$ [Bhat et. al. '96]. Since $\alpha \in (0,1)$ is $\equiv \frac{2}{3-\alpha} \in (0,1)$, we can conclude finite time stability

References

A. A. Andronov, A. A. Vitt, and S. E. Khaikin. *Theory of Oscillators*. Oxford, UK: Pergamon, 1966.

M. Athans and P.L. Falb. *Optimal Control: An Introduction to the Theory and Its Applications*. McGraw-Hill, New York, 1966.

H. Baruh and S.S.K. Tadikonda. *Gibbs phenomenon in structural control*. *J. Guidance, Contr., and Dynamics*. vol., no. 1, pp. 51-58, 1991.

S.P. Bhat and D.S. Bernstein. *Continuous Finite-Time Stabilization of the Translational and Rotational Double Integrators. IEEE Transactions on Automatic Control*. vol. 43., no. 5, 1998

S.P. Bhat and D.S. Bernstein. *Continuous, Bounded, Finite-Time Stabilization of the Translational and Rotational Double Integrators. in Proc. Amer, Contr. Conf.,* Seattle, WA, June 1996, pp. 1831 - 1832

A. Fuller. *Optimization of some nonlinear control systems by means of Bellman's equation and dimensional analysis*. *Int. J. Contr.*, vol. 3, no. 4, pp. 359–394, 1966.

P. Hartman. *Ordinary Differential Equations*. 2nd edition, Boston, MA: Birkhauser, 1982.