RBOT 250: Homework Solutions

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Homework 1. SCARA Robot Mobility Analysis:
Using the mobility condition, determine and explain why the SCARA robot of Fig 2.5 has the number degrees of freedom that you find.

Solution 1. We have the following general equation from the mobility condition:

$$\mathfrak{M} = 6(n - g - 1) + \sum_{i=1}^{g} f_i. \tag{1}$$

where n is the number of links, g is the total number of joints and f_i are the respective relative degrees of freedoms of the individual joints. The SCARA manipulator has three joints with an RRP configuration $i.e.$, $g = 3$. These connect four links (including the end-effector) so that $n = 4$. Each of the joints have $f_i = 1$. Plugging these into (1), we find that

$$\mathfrak{M} = 6(4 - 3 - 1) + \sum_{i=1}^{3} f_i = 3 \tag{2}$$

Hence, the SCARA manipulator is a 3DOF robot.

Homework 2. Parallel Robot Mechanism Analysis
With the *Grübler-Kutzbach's mobility condition* that we have learned, analyze the mobility criteria of the mechanism of Fig. 1. Hint: This mechanism is made up of two chains: chains $A_3\,B_3\,B_1\,A_1$ and $A_2\,B_2\,B_4\,A_4$, and there is a fixed distance between the U-joints, $A_2\,A_3$ as well as A_1, A_4.

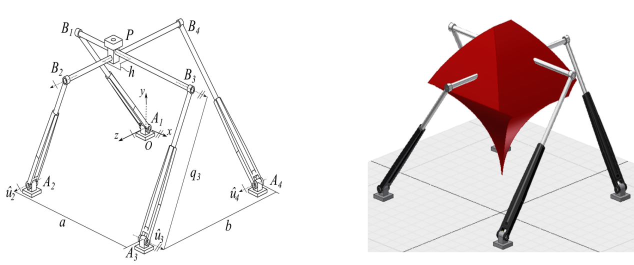

Fig. 1. *Left*: A Parallel Planar Robot Mechanism. *Right:* Workspace of the mechanism.

Solution 2. This robot has four RPRP-type kinematic chains. Three of these chains have their prismatic pair closest to the base; they serve as active joints. The fourth kinematic chain is completely passive. Points B_1 and B_2 are connected by a rod, which is perpendicular to the rod that connects points B_2 and B_4. Both rods are connected to the moving platform by prismatic joints, which are separated from each other by a vertical offset h. Point $P = (P_x, P_y, P_z)$ is the interconnecting point for all the chains on the mobile platform and the top rods. Notice that the rotational axes of the revolute joints $i.e.$, $\hat{\boldsymbol{u}}_i$ are parallel to the x-axes of joints 1 and 3; also, they are parallel to the z-axis for joints 2 and 4.

There are many ways of solving this mobility problem. However, note that this mechanism has multiple closed loops and care must be taken when using the formulas we have introduced in our notes. Here, I will be using Gogu's method as proposed in [1]. It decomposes the mechanism into the different closed loops in order to properly analyze the mobility constraint. The mobility of the mechanism is

$$\mathfrak{M} = \sum_{i=1}^{g} f_i - r \tag{3}$$

with r being the number of joint parameters that lose independence after closing the loops of the mechanism. We define r as

$$r = \sum_{j=1}^{k} SH_j - SFr_l \tag{4}$$

where k is the number of closed-loops in the mechanism, SH_j is the connectivity of the j'th closed loop H_j when separated from the mechanism; SF is the overall connectivity of the mechanism and r_l is the total number of parameters that lose independence in the closed loops. We may find the variables as follows

$$SH_j = \dim(RH_j), \quad SF = \dim(RF), \quad r_l = \sum_{j=1}^{k} r_l^{H_j} \tag{5}$$

where RH_j is the velocity vector associated with the interest point P in a closed loop H_j; RF is the resultant velocity vector formed from the intersection of RH_j $i.e.$ $RF = RH_1 \cap RH_2 \ldots \cap RH_k$ and $r_l^{H_j}$ is the number of parameters that lose independence in the loop H_j. We define the variable $r_h^{H_j}$ as

$$r_l^{H_j} = SG_1^{H_j} + SG_2^{H_j} - SF_{H_j} \tag{6}$$

with $SG_i^{H_j}, i = 1, 2$ being the connectivity of $G_i^{H_j}$ in H_j, whereas SF_{H_j} is the loop's connectivity. We define

$$SG_i^{H_j} = \dim\left(RG_{1,2}^{H_j}\right), \quad SF_{H_j} = \dim\left(RF_{H_j}\right) \tag{7}$$

where $RG_i^{H_j}$ are the velocity vectors for $G_i^{H_j}$ and RF_{H_j} is the resultant velocity vector formed by the intersection of the $RG_i^{H_j}$s.

For this mechanism, we have $g = 14$ and $k = 2$; furthermore, we have $RH_1 = \{v_x, v_y, v_z, \omega_x\}$ and $RH_2 = \{v_x, v_y, v_z\}$ so that the loops' connectivity are $SH_1 = 4$ and $SH_2 = 4$. Hence $RF = RH_1 \cap RH_2 = \{v_x, v_y, v_z\}$ and the mechanism's connectivity is thus $SF = \dim(RF) = 3$.

If we disconnect the limbs, the velocity vector of $RG_1^{H_1} = RG_2^{H_1} = \{v_x, v_z, \omega_x\}$ and $RG_1^{H_2} = RG_2^{H_2} = \{v_x, v_y, \omega_z\}$, so that the connectivity of each limb is $SG_{1,2}^{H_1} = SG_{1,2}^{H_2} = 3$. Thus,

$$\begin{aligned} RF_{H_1} &= RG_1^{H_1} \cap RG_2^{H_1} = \{v_x\, v_z\, \omega_x\}, \\ RF_{H_2} &= RG_1^{H_2} \cap RG_2^{H_2} = \{v_x\, v_z\, \omega_z\} \end{aligned} \tag{8}$$

so that $SG_{H_1} = SF_{H_2} = 3$. Moreover, we see that $r_l^{H_1} = r_l^{H_2} = 3$ and $r_l = 6$. Thus, we have

$$r = SH_1 + SH_2 - SF + r_l = 11. \tag{9}$$

Since p = 14 and $f_i = 1$, we have

$$\mathfrak{M} = 14 - 11 = 3. \tag{10}$$

Hence, the mechanism has only 3 degrees of freedom. It is a planar parallel manipulator.

Homework 3. What is the geometric meaning of equation 3.1.6 on a twist axis to you. Define *pure rotation* and a *pure translation* in terms of equation 3.1.6[1].

Solution 3. Twists and pitches of twists.

1) The pitch of the twist is the ratio of the magnitude of a point on the axis of the twist to the magnitude of the angular velocity about the axis of the twist. By this, we see that the pitch of the twist is the unit velocity traveled along the axis of the twist.
2) A pure rotation occurs when the numerator of equation 3.1.6 is zero $i.e.$ when a point travels only in the angular velocity direction of the twist axis; this is called a zero pitch twist.
3) A pure translation occurs when the right hand side of 3.1.6 is infinite. This corresponds to an infinite pitch twist $i.e.$ a point on the twist axis only travels along the linear velocity direction of the twist axis.

Homework 4. A unit screw, twist or wrench is one where the magnitude of the screw, twist or wrench is 1.

1) What is the geometric meaning of a unit screw to you?
2) Consult the identified reference materials and explain what a reciprocal screw is in no more than five sentences.

Solution 4. Here is a geometric meaning of the screw:

1) Imagine a nut fitted upon a mechanical screw. As we tighten the nut around the threads of the screw, there exists a rectilinear distance by which the screw travels into the nut. This rectilinear distance is called the *pitch* of the screw. Thus, the pitch is a linear magnitude. When a nut is rotated about the threads of a screw, the rectilinear distance by which the nut moves when rotated through a particular angle is the product of the pitch and the the circular measure of the angle. *A screw then may be geometrically seen as a straight line with which a definite linear magnitude, $i.e.$ the pitch, travels in space.*
Often with twist and wrenches, we wish to associate a magnitude other than 1 so that there are ∞^6 different twists and wrenches if we consider their magnitudes as well. With this interpretation we can think of a unit screw as defining only an axis and a pitch. Then we can think of a given magnitude as defining a twist (if its units are rotation/time) or a wrench (if its units are force) acting along the screw. In both cases the pitch is expressed in length units. These six-spaces of (infinitesial) twists or wrenches can be considered to be vector spaces in that they are closed under vector addition and scalar multiplication.
2) **Reciprocal Screws:** Suppose a body is free to twist about a screw x and that body is in equilibrium, while being acted upon by a wrench on another screw χ; we can extend this logic and infer that a body that is free to rotate about the screw χ will be in equilibrium, while being acted upon by a wrench on a screw x. By the principle of virtual velocities, if the body is in equilibrium the work done in a small displacement against external forces must be zero with the virtual coefficient vanishing. *A pair of screws are reciprocal when their virtual coefficient is zero.*

[1]Figure out how they correspond to zero pitch and infinite pitch twists.

Homework 5. Given a matrix $\hat{m} \in so(3)$, suppose that the following relation holds,

$$\hat{m}^2 = mm^T - \|m\|^2 I \tag{11}$$
$$\hat{m}^3 = -\|m\|^2 \hat{m} \tag{12}$$

with the fact that higher powers of $\hat{m}$ can be recursively found. Utilizing this lemma along with $m = \omega\theta$, and $\|\omega\| = 1$, show that

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta). \tag{13}$$

Solution 5. First recall that we can expand $e^{\hat{\omega}\theta}$ using Taylor series so that

$$\begin{aligned} e^{\hat{\omega}\theta} = I + \theta\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + \frac{\theta^4}{4!}\hat{\omega}^4 + \frac{\theta^5}{5!}\hat{\omega}^5 + \\ \frac{\theta^6}{6!}\hat{\omega}^6 + \frac{\theta^7}{7!}\hat{\omega}^7 + \cdots \end{aligned} \tag{14}$$

Rewriting, we find that

$$\begin{aligned} e^{\hat{\omega}\theta} = I + \hat{\omega}\left(\theta + \frac{\theta^3}{3!}\hat{\omega}^2 + \frac{\theta^5}{5!}\hat{\omega}^4 + \frac{\theta^7}{7!}\hat{\omega}^6 + \cdots\right) + \\ \hat{\omega}^2\left(\frac{\theta^2}{2!} + \frac{\theta^4}{4!}\hat{\omega}^2 + + \frac{\theta^6}{6!}\hat{\omega}^4 + \cdots\right). \end{aligned} \tag{15}$$

Since we have from the lemma that

$$\hat{\omega}^2 = \omega\omega^T - \|\omega\|^2 I \tag{16a}$$
$$\hat{\omega}^3 = -\|\omega\|^2 \hat{\omega} \tag{16b}$$

we may write (15) as

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \hat{\omega}\left(\theta + \frac{\theta^3}{3!}\left(\omega\omega^T - \|\omega\|^2 I\right) + \frac{\theta^5}{5!}\left(\omega\omega^T - \|\omega\|^2 I\right)^2 + \cdots\right) \\ &\quad + \hat{\omega}^2\left(\frac{\theta^2}{2!} + \frac{\theta^4}{4!}\left(\omega\omega^T - \|\omega\|^2 I\right) + \frac{\theta^6}{6!}\left(\omega\omega^T - \|\omega\|^2 I\right)^2 + \cdots\right) \\ &= I + \hat{\omega}\left(\theta - \frac{\theta^3}{3!}\hat{\omega}^2 + \frac{\theta^5}{5!}\hat{\omega}^4 - \frac{\theta^7}{7!}\hat{\omega}^6 + \cdots\right) \\ &\quad + \hat{\omega}^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\hat{\omega}^2 + + \frac{\theta^6}{6!}\hat{\omega}^4 + \cdots\right) \end{aligned} \tag{17}$$

Recall from trigonometric identities that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{18a}$$

Therefore, (17) becomes

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2\left(1 - \cos\theta\right) \tag{19}$$

where we have used the identities $\hat{\omega}^2 = -1$ and $\hat{\omega}^4 = 1$ etc.

Homework 6. Verify that $R_{ij} = R_{ji}^{-1} \equiv R_{ji}^T$. Furthermore, verify that the determinant of the rotation matrix is ± 1 *i.e.* $detR = \pm 1$.

Solution 6. We have

$$R_{ij} = \begin{bmatrix} \boldsymbol{x}_i \cdot \boldsymbol{x}_j & \boldsymbol{y}_i \cdot \boldsymbol{x}_j & \boldsymbol{z}_i \cdot \boldsymbol{x}_j \\ \boldsymbol{x}_i \cdot \boldsymbol{y}_j & \boldsymbol{y}_i \cdot \boldsymbol{y}_j & \boldsymbol{z}_i \cdot \boldsymbol{y}_j \\ \boldsymbol{x}_i \cdot \boldsymbol{z}_j & \boldsymbol{y}_i \cdot \boldsymbol{z}_j & \boldsymbol{z}_i \cdot \boldsymbol{z}_j \end{bmatrix}, \tag{20}$$

$$R_{ji} = \begin{bmatrix} \boldsymbol{x}_j \cdot \boldsymbol{x}_i & \boldsymbol{y}_j \cdot \boldsymbol{x}_i & \boldsymbol{z}_j \cdot \boldsymbol{x}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{y}_i & \boldsymbol{y}_j \cdot \boldsymbol{y}_i & \boldsymbol{z}_j \cdot \boldsymbol{y}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{z}_i & \boldsymbol{y}_j \cdot \boldsymbol{z}_i & \boldsymbol{z}_j \cdot \boldsymbol{z}_i \end{bmatrix}, \tag{21}$$

and that

$$R_{ji}^T = \begin{bmatrix} \boldsymbol{x}_j \cdot \boldsymbol{x}_i & \boldsymbol{x}_j \cdot \boldsymbol{y}_i & \boldsymbol{x}_j \cdot \boldsymbol{z}_i \\ \boldsymbol{y}_j \cdot \boldsymbol{x}_i & \boldsymbol{y}_j \cdot \boldsymbol{y}_i & \boldsymbol{y}_j \cdot \boldsymbol{z}_i \\ \boldsymbol{z}_j \cdot \boldsymbol{x}_i & \boldsymbol{z}_j \cdot \boldsymbol{y}_i & \boldsymbol{z}_j \cdot \boldsymbol{z}_i \end{bmatrix}. \tag{22}$$

Inspecting the rows of R_{ij} and R_{ij}^T, the dot products between the respective vector elements are simply the cosine of the angles between them *i.e.* $p_j \cdot p_i = cos\theta$, where θ is the angle between the vectors p_j and p_i. Evaluating, it follows that $R_{ij}^T = R_{ij}$. Similarly, we know that a matrix X is invertible if there exists a matrix B such that

$$AB = BA = I_n \tag{23}$$

where I_n is an $n \times n$ matrix. In this case, we have that

$$R_{ji}R_{ji}^T = R_{ji}^T R_{ji} = I_3. \tag{24}$$

It follows that,

$$R_{ji} = \frac{1}{R_{ji}^T} \tag{25}$$

or

$$R_{ji}^T = R_{ji}^{-1}. \tag{26}$$

From (20) and (21), it follows that see that

$$R_{ji}^T = R_{ji}^{-1} = R_{ij}. \tag{27}$$

Homework 7. Compose the rotation matrix in three dimensions where all axes of the inertial frame are rotated by an angle β around each of the x_0, y_0 and z_0 axes respectively using the foregoing logic. In addition, for each transformation, verify that (1) $R_{e,0} = I$[2] where e is the axes about which we are rotating and β is the angle of rotation, (2) the composition of rotations about the angles β and α in a successive manner implies that $R_{z,\beta}, R_{z,\alpha} = R_{z,\beta+\alpha}$, and (3) $\left(R_{z,\beta}\right)^{-1} = R_{z,-\beta}$. Bonus points will be awarded for cool 3D visualizations.

Solution 7. In three dimensions, a rotation angle of β around the principal axes of the moving frame x_0, y_0, z_0, gives the rotation matrix

$$\begin{bmatrix} \boldsymbol{x}_1 \cdot \boldsymbol{x}_0 & \boldsymbol{y}_1 \cdot \boldsymbol{x}_0 & \boldsymbol{z}_1 \cdot \boldsymbol{x}_0 \\ \boldsymbol{x}_1 \cdot \boldsymbol{y}_0 & \boldsymbol{y}_1 \cdot \boldsymbol{y}_0 & \boldsymbol{z}_1 \cdot \boldsymbol{y}_0 \\ \boldsymbol{x}_1 \cdot \boldsymbol{z}_0 & \boldsymbol{y}_1 \cdot \boldsymbol{z}_0 & \boldsymbol{z}_1 \cdot \boldsymbol{z}_0 \end{bmatrix} \tag{28}$$

A rotation about x by β gives,

$$R_{x_0,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}, \tag{29}$$

a rotation about y_0 by β gives,

$$R_{y_0,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{30}$$

and a rotation about z_0 by β gives,

$$R_{z_0,\beta} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{31}$$

Substituting 0 for β in (29), (30), and (31), we have

$$R_{x_0,\beta} = R_{\beta,y_0} = R_{\beta,z_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

The composition of rotations about the angles β and α in a successive manner implies that $R_{z,\beta}R_{z,\alpha} = R_{z,\beta+\alpha}$
Proof:

$$R_{\beta,z_0}R_{\alpha,z_0} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$R_{\beta,z_0}R_{\alpha,z_0} = \begin{bmatrix} c_\alpha c_\beta - s_\alpha s_\beta & c_\alpha s_\beta - c_\beta s_\alpha & 0 \\ c_\alpha s_\beta + s_\alpha c_\beta & c_\alpha c_\beta - s_\alpha s_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{34}$$

or

$$R_{\beta,z_0}R_{\alpha,z_0} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 0 \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

Now, we find that

$$R_{z,\beta+\alpha} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 0 \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

Since (35) = (36), the supposition is confirmed.
Prove that $\left(R_{z,\beta}\right)^{-1} = R_{z,-\beta}$: We have from (31) that

$$R_{z,-\beta} = \begin{bmatrix} \cos(-\beta) & -\sin(-\beta) & 0 \\ \sin(-\beta) & \cos(-\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

[2]In your notes, this was written as $R_{e,\beta}$. You will not be penalized if you could not arrive at the right solution because of this mistake.

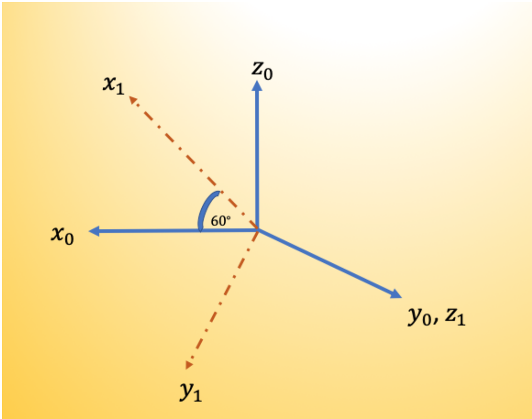

Fig. 2. Relative orientation between two frames.

Now, $(R_{z,\beta})^{-1} = (R_{z,\beta})^T$ so that

$$R^T_{z,\beta} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The equivalence of the two preceding equations prove our case. (QED)

Homework 8. For the two frames shown in Fig. 2, determine the rotation matrix between them. In addition, explain the difference between rotating about a *current frame* and rotating about a *fixed frame*[3]. In particular, when is it necessary to carry out a *pre-multiplication* and when is it necessary to carry out a *post-multiplication* when transforming points or vectors about coordinate frames?

Solution 8. From the given figure, observe that the x_0 axis was rotated counterclockwise by $60°$ from the horizontal. To find the rotation matrix between the two frames, we can project the unit vectors x_1, y_1, z_1 onto x_0, y_0, z_0 to generate the coordinates of x_1, y_1, z_1 in the x_0, y_0, z_0 frame. We therefore have

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad y_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{39}$$

Homework 9. For the robot manipulator we are using in this class, suppose that you have the following point in the base frame of the robot, $q_o = [-2, 3, 1]$. Furthermore, suppose that the joint angles for all six joints are respectively $\{-90, 60, 30, 45, 90, 125\}$, transform the point q_0 in the base frame to a coordinate frame on the sixth joint.

Solution 9. Since we changed the robot we are using, this homework is rendered moot and you will not be graded for it.

Consider the composition of rotations of Fig. 3 where we first rotate by an angle θ about the x axis and then rotate about an angle ψ about the z axis. The rotation matrix can be composed as

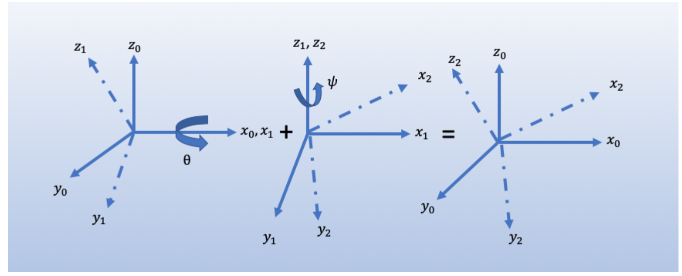

Fig. 3. Illustration of composition of rotations about a **current axis**.

$$R = R_{x,\theta} R_{z,\psi} \quad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \cdot \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$R = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ c_\theta s_\psi & c_\theta c_\psi & -s_\theta \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \tag{41}$$

Notice how the order of multiplication is carried out, owing to the axis about which we are making the transformation.

Homework 10. Carry out the transformation above in reverse order. What do you notice?

Solution 10. Reversing the order of transformation reveals that

$$R_{z,\psi} R_{x,\theta} = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \tag{42}$$

$$R = R_{z,\psi} R_{x,\theta} = \begin{pmatrix} c_\psi & -c_\theta s_\psi & s_\psi s_\theta \\ s_\psi & c_\psi c_\theta & -c_\psi s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \tag{43}$$

$i.e.\ R_{x,\theta} R_{z,\psi} = (R_{x,-\theta} R_{z,-\psi})^T$

which goes to show that reversing the order of rotations is tantamount to reversing the order of angle rotations and then taking the transpose of the ensuing result.

References

[1] G. Gogu, "Maximally regular t3-type translational parallel robots," *Structural Synthesis of Parallel Robots: Part 2: Translational Topologies with Two and Three Degrees of Freedom*, pp. 687–748, 2009. 1

[3]See sections 2.4.1 and 2.4.2 of Spong's book.