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# Stellar Structure Report

Lakeisha M. Ramos-Rosado[1]

[1] *William H. Miller III Department of Physics and Astronomy, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, USA*

## ABSTRACT

This class project consisted of the stellar modeling of a Zero-Age Main Sequence (ZAMS) star that was carried out using stellar structure equations, basic physical principles and assumptions that will be discussed later on. We then compare our model to Modules for Experiments in Stellar Astrophysics (MESA) to test our model's accuracy.

## 1. INTRODUCTION

It is of crucial knowledge to underand how stars behave and all the processes that lead them to their current state. Mathematical models are the most accessible ways to understand the internal structure and evolutions of stars.

In this project, we aim to create a ZAMS stellar model using stellar structure equations. We also aim to compare our model with MESA, an advance sofware that does exactly what we're aiming to achieve.

## 2. METHODS

### 2.1. *Assumptions*

To be able to make a simple ZAMS model we had to make the following assumptions in our model.

- Spherical symmetry, that is assuming a star that is isolated, non-rating, therefore doesn't contain strong magnetic fields which makes it spherically symmetric.
- Isolation, we assume that distances between stars are big. This allow us to ignore the gravitational field and radiation of other stars when considering stellar structure. Stars won't collide with each other.
- Initial composition is uniform, assumed star forms from a molecular cloud with composition of hydrogen, helium and metals and that this composition is constant throughout the star.
- Ideal gas can be used everywhere.
- The star is fully ionized.

Corresponding author: Lakeisha M. Ramos-Rosado
lramosr1@jhu.edu

### 2.2. *Stellar Structure Equations*

The fundamental stellar structure equations are four coupled differential equations that have the following parameters: luminosity (L), pressure (P), distance (r) and temperature (T). These can be written in a typical form and in Lagrangian form. The latter is what we used which uses the mass enclosed as an independent variable. These can be expressed as:

$$\begin{aligned} \frac{d\mathcal{L}}{d\mathcal{M}_r} &= \varepsilon, \\ \frac{dP}{d\mathcal{M}_r} &= -\frac{G\mathcal{M}_r}{4\pi r^4}, \\ \frac{dr}{d\mathcal{M}_r} &= \frac{1}{4\pi r^2 \rho}, \\ \frac{dT}{d\mathcal{M}_r} &= -\frac{G\mathcal{M}_r T}{4\pi r^4 P}\nabla. \end{aligned} \tag{1}$$

where $\varepsilon$ is defined as the total energy generation rate, $G$ is the gravitational constant, $\nabla = \frac{\partial \ln T}{\partial \ln P}$, and $\rho$ is the density for that particular mass.

After establishing our assumptions and our four, coupled differential equations to solve, we started the stellar structure modeling which would be outlined in a few steps.

### 2.3. *Star Selection*

We selected a star of 3.5 solar masses and a composition of X = 0.35, Y = 0.55 and Z = 0.1.

### 2.4. *Step 1: Interpolation Function for Opacities*

We use opacity tables from the Lawrence Livermore National Laboratory's OPAL database (Iglesias & Rogers 1996) which provides opacity values for different stellar compositions from using log R values (from -8 to 1) and log T (from 3.75 to 8.7). Using these tables we created an interpolation function to calculate

an opacity value. To do this, we use scipy's linear interpolation (scipy.interpolate.griddata) to facilitate our interpolation. One crucial equation to achieve the interpolated values is:

$$R = \frac{\rho}{T_6^3} \tag{2}$$

where $T_6 = T/10^6$.

### 2.5. *Step 2: Energy Functions*

#### 2.5.1. *Energy Generation Rate*

We know that stars fuse elements in their core which then generates energy that is liberated at the surface. In our case, for simplicity, we are assuming that fusion is the main source for energy generation. We assume that pp-chains and the CNO cycle are the drivers of fusion reactions. For that case, let's consider:

$$\frac{d\mathcal{L}}{d\mathcal{M}_r} = \varepsilon = \varepsilon_{pp} + \varepsilon_{CNO}. \tag{3}$$

where the energy generation rates from each reaction is given by the following.

For the pp-chain:

$$\varepsilon_{pp} = 2.57 \times 10^4 \psi f_{11} g_{11} \rho X^2 T_9^{-2/3} e^{-3.381/T_9^{1/3}}, \tag{4}$$

$$g_{11} \equiv 1 + 3.82T_9 + 1.51T_9^2 + 0.144T_9^3 - 0.0114T_9^4. \tag{5}$$

$$f_{11} = e^{E_D/kT}, \tag{6}$$

$f_{11}$ is determined from assuming weak screening, and $\frac{E_D}{kT} = 5.92 \times 10^{-3} \left(\frac{\rho}{kT}\right)^{1/2}$ from the pp-chain and we also assume $\zeta \approx 1$. Yielding to:

$$f_{11} = e^{5.92\times10^{-3} Z_1 Z_2 \left(\frac{\zeta\rho}{T_7^3}\right)^{1/2}}. \tag{7}$$

We're also assuming that $Z_1$, $Z_2$ and $\psi$ are 1.

For the CNO cycle:

$$\varepsilon_{CNO} = 8.24\times10^4 \psi g_{14,1} X_{CNO} X \rho \times T_9^{-2/3} e^{(-15.231 T_9^{-1/3}) - \frac{T_9}{0.8}^2}. \tag{8}$$

$$g_{14,1} = 1 - 2.00T_9 + 3.41T_9^2 - 2.43T_9^3, \tag{9}$$

where $X_{CNO} = X_C + X_N + X_O \approx .7Z$ using approximately solar composition of C, N, O, and metals ($Z$). We test our functions by replicating Figure 18.8 from the book Stellar Structure and Evolution and we're able to succeed as seen in Figure 1.

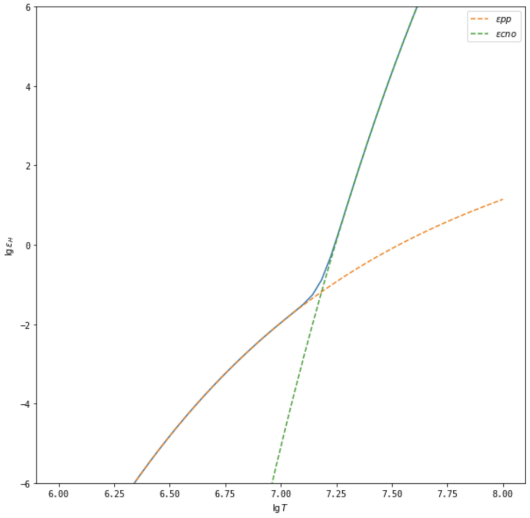


**Figure 1.** Adapted from Figure 18.8 from Stellar Structure and Evolution Kippenhahn et al. (2012)

#### 2.5.2. *Energy Transport*

Let's consider now how energy is transported, $\frac{dT}{d\mathcal{M}_r}$. Particularly focusing on $\nabla$ because it allow us to differentiate between radiative from convective energy transport.

- Radiation, with local thermodynamic equilibrium (LTE) and convection neglected:

$$\nabla = \nabla_{rad} = \left(\frac{d\ln T}{d\ln P}\right)_{rad} = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{\mathcal{L}_r}{G\mathcal{M}_r}. \tag{10}$$

$$\ln T_r = \ln T_c - \left(\frac{\pi}{6}\right)^{1/3} G\frac{\nabla_c \rho_c^{4/3}}{P_c}\mathcal{M}_r^{2/3}. \tag{11}$$

- Convection, we assume that the transport is adiabatic:

$$\nabla = \nabla_{ad} = \left(\frac{d\ln T}{d\ln P}\right)_{ad} = \frac{\gamma - 1}{\gamma}. \tag{12}$$

We also assume complete ionization, $\gamma = \frac{5}{3}$, making $\nabla_{ad} = 0.4$.

$$T_r^4 = T_c^4 - \frac{1}{2ac}\left(\frac{3}{4\pi}\right)^{2/3} \kappa_c \varepsilon_c \rho_c^{4/3} \mathcal{M}_r^{2/3}. \tag{13}$$

### 2.6. *Step 3: Equation of State Function*

Now we create a function to define the equation of state which includes both ideal gas pressure and radiation pressure:

$$P = P_{\text{ideal}} + P_{\text{radiation}} = \frac{\rho k T}{\mu} + \frac{1}{3} a T^3 \tag{14}$$

here, $k$ and $a$ are constants, while $\mu$ is dependent on the mass fraction of hydrogen $X$: $\mu = \frac{4}{3+5X}$. This equation can be solved for the density as a function of pressure, temperature, and composition. We are assuming full ionization.

### 2.7. *Step 4: Derivatives Function*

We created a function to take the independent variable m and the four dependent variables of luminosity, pressure, radius and temperature and return the derivatives like equation 1

### 2.8. *Step 4: Shooting Function*

We use the shooting method which is a robust numerical technique to solve boundary value problems. For our case, our boundaries are at the surface and at the core. So we want to integrate numerically outwards from the core and inwards from the surface. We want then to repeatedly integrate while changing the boundary conditions until both parts meet. We find the differences at the point they meet and then use a minimizer to converge our solutions within some threshold.

### 2.9. *Boundary Conditions*

We define functions for our boundary conditions for which we use our shooting function. For which we use a combination of the book Stellar Structure and Evolution (Kippenhahn et al. 2012) and Stellar Interiors (Hansen et al. 2004).

#### 2.9.1. *Core Boundary*

In the core we cannot select the exact center because our equations will be non-existent so we select an arbitrary point away from the core, say $\mathcal{M} = 1 \times 10^{-12} \mathcal{M}_*$.

Similarly we use the following corrections for central pressure and radius. Our equations for the core are:

$$\begin{aligned} &\mathcal{L} = \varepsilon \mathcal{M}, \\ &P = P_c - \tfrac{3G}{8\pi} \left(\tfrac{4\pi}{3}\rho_c\right)^{4/3} \mathcal{M}^{1/3}, \\ &r = \left(\tfrac{3}{4\pi\rho_c}\right)^{1/3} \mathcal{M}^{1/3}, \\ &T = T_c \end{aligned} \tag{15}$$

For our initial guesses, we provide initial pressure and temperature which we use the constant density model given by:

$$\begin{aligned} P_c &= \tfrac{3}{8\pi} \tfrac{G\mathcal{M}^2}{\mathcal{R}^4}, \\ T_c &= \tfrac{1}{2} \tfrac{G\mathcal{M}}{R} \tfrac{\mu}{N_A k}. \end{aligned} \tag{16}$$

#### 2.9.2. *Surface Boundary*

For the surface we use the homology relations discussed in class to assume radius and luminosity by:

$$\begin{aligned} \tfrac{\mathcal{L}}{\mathcal{L}_\odot} &\approx \left(\tfrac{\mathcal{M}}{\mathcal{M}_\odot}\right)^{3.5}, \\ \tfrac{\mathcal{R}}{\mathcal{R}_\odot} &\approx \left(\tfrac{\mathcal{M}}{\mathcal{M}_\odot}\right)^{0.75}. \end{aligned} \tag{17}$$

We have the following equations for our surface boundary.

$$\begin{aligned} &\mathcal{L} = \mathcal{L}, \\ &P = \tfrac{G\mathcal{M}}{R^2} \tfrac{2}{3\kappa}, \\ &r = R, \\ &T = T_{\text{eff}} = \left(\tfrac{\mathcal{L}}{4\pi R^2 \sigma}\right)^{1/4} \end{aligned} \tag{18}$$

### 2.10. *Step 5:Iterating Process*

We integrate our equations by using Scipy's solver, optimize.solveivp which integrates differential equations for our convenience. We use Runge-Kutta (RK45) which the integrator has available.

### 2.11. *Step 6: Minimizer*

We finally use Scipy's least squares minimizer which acts similarly to Newton-Raphson (NR) method.

## 3. RESULTS

Our code can be found in Github. The results are outlined below:

The results of our solution and MESA's solution are outlined below (Paxton et al. 2011).

| Model | $\mathcal{L}/\mathcal{L}_\odot$ | log $P$ | $\mathcal{R}/\mathcal{R}_\odot$ | log $T$ |
|---|---|---|---|---|
| Solution | 401.0589 | 16.5835 | 2.5588 | 7.4042 |
| MESA Solution | 245.6874 | 16.3614 | 3.7608 | 7.3157 |
| Percent Errors (%) | 63.23 | 66.76 | 31.96 | 22.62 |

**Table 1.** Table of values for our solution, MESA's solution and percent errors between MESA and the solution solution. Here we use: $\mathcal{L}_\odot = 3.839 \times 10^{33}$ erg/s, $\mathcal{R}_\odot = 6.957 \times 10^{10}$ cm

Unfortunately our values are not within MESA's expected values and are way off. This could be due to the poor physics included and assumptions made. It could also be a possibility that our star was not favorable for the assumptions made here.

In Figure 2 we can see a plot of our star's structure. We see that in the boundary between inward and outward is where my results are having a huge step which definitely ruined our comparison with MESA. However,

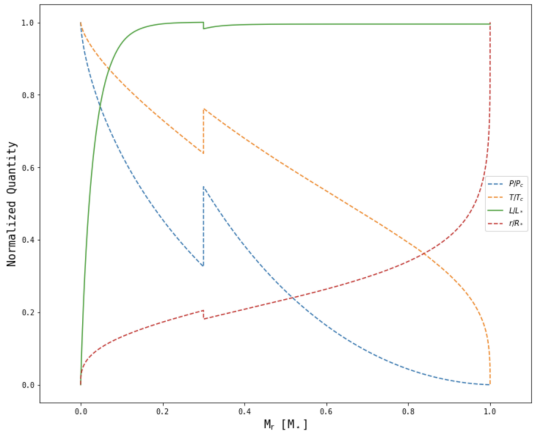


**Figure 2.** Our ZAMS's star's interior structure using luminosity, pressure, radius, and temperature. The values are normalized for plotting conveniences.

we see that the behavior seems to be consistent with a typical ZAMS star.

## REFERENCES

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