Comprehensive Linear Algebra for Computer Science

1 Introduction to Linear Algebra in Computing

Linear algebra forms the mathematical foundation for many computer science applications:

- 3D Graphics and Animation
- Machine Learning and AI
- Data Compression
- Search Engines
- Quantum Computing

2 Vectors

2.1 Basic Definitions

A vector $\mathbf{v}$ in $\mathbb{R}^n$ is an ordered list of n numbers:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Practical Application

Real-World Applications:

- **Recommendation Systems:** User preferences as vectors
- **Natural Language Processing:** Word embeddings
- **Physics Engines:** Position, velocity, force vectors

2.2 Vector Operations

2.2.1 Basic Operations

1. **Addition:**

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

2. **Scalar Multiplication:**

$$c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

Practical Application

Vector addition and scaling are used in:

- **Graphics:** Combining multiple transformations
- **Physics:** Adding multiple forces
- **Animation:** Interpolation between keyframes

2.2.2 Advanced Operations

1. **Dot Product:**

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Practical Application

Dot products are essential in:

- **Machine Learning:** Computing similarities
- **Computer Graphics:** Light intensity calculations
- **Game Physics:** Projection calculations

2. **Cross Product** (3D vectors):

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Practical Application

Cross products are used in:

- **3D Graphics:** Normal vector calculations
- **Robotics:** Torque calculations
- **Computer Vision:** Camera positioning

3 Matrices

3.1 Fundamental Concepts

A matrix A is a rectangular array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Practical Application

Matrices are fundamental in:

- **Image Processing:** Images as pixel matrices
- **Neural Networks:** Weight matrices
- **Graph Algorithms:** Adjacency matrices

3.2 Matrix Operations

1. **Addition:** $(A + B)_{ij} = a_{ij} + b_{ij}$
2. **Scalar Multiplication:** $(cA)_{ij} = ca_{ij}$
3. **Matrix Multiplication:** $(AB)_{ij} = \sum_k a_{ik}b_{kj}$

   Practical Application

   Matrix multiplication enables:

   - **3D Transformations:** Combining multiple transforms
   - **Neural Networks:** Layer computations
   - **Computer Graphics:** Scene rendering

4. **Transpose:** $(A^T)_{ij} = a_{ji}$

4 Transformations

4.1 2D Transformations

1. **Translation:**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2. **Rotation:**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Practical Application

Benefits of matrix transformations:

- **Efficiency:** Multiple transforms combined into one matrix
- **Precision:** Accurate geometric calculations
- **Speed:** Hardware-accelerated operations
- **Applications:**
  - Game character movement
  - UI animations
  - Image processing
  - Robotics control

4.2 3D Transformations

1. **3D Rotation Matrices:**
   - Around X-axis:
   $$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
   - Similar matrices for Y and Z rotations

Practical Application

3D transformations enable:

- Virtual reality experiences
- 3D modeling software
- Scientific visualization
- Medical imaging

5 Machine Learning Applications

5.1 Neural Networks

Basic feedforward operation:

$$z = Wx + b$$

Where:

- W is the weight matrix
- x is the input vector
- b is the bias vector

Practical Application

Matrix operations in neural networks enable:

- Parallel processing on GPUs
- Efficient gradient computations
- Batch processing of data
- Applications:
  - Image recognition
  - Natural language processing
  - Speech recognition
  - Autonomous vehicles

5.2 Dimensionality Reduction

Principal Component Analysis (PCA):

$$X = U\Sigma V^T$$

Practical Application

Benefits of dimensionality reduction:

- Reduced storage requirements
- Faster computation
- Noise reduction
- Applications:
  - Image compression
  - Feature extraction
  - Data visualization
  - Pattern recognition

6 Advanced Topics

6.1 Eigenvalues and Eigenvectors

For matrix A:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Practical Application

Applications of eigenvalues:

- Google's PageRank algorithm
- Facial recognition
- Vibration analysis
- Quantum mechanics simulations

6.2 Matrix Decompositions

1. **LU Decomposition:** $A = LU$
2. **QR Decomposition:** $A = QR$
3. **SVD:** $A = U\Sigma V^T$

Practical Application

Matrix decompositions enable:

- Efficient linear system solving
- Stable numerical computations
- Data compression
- Signal processing