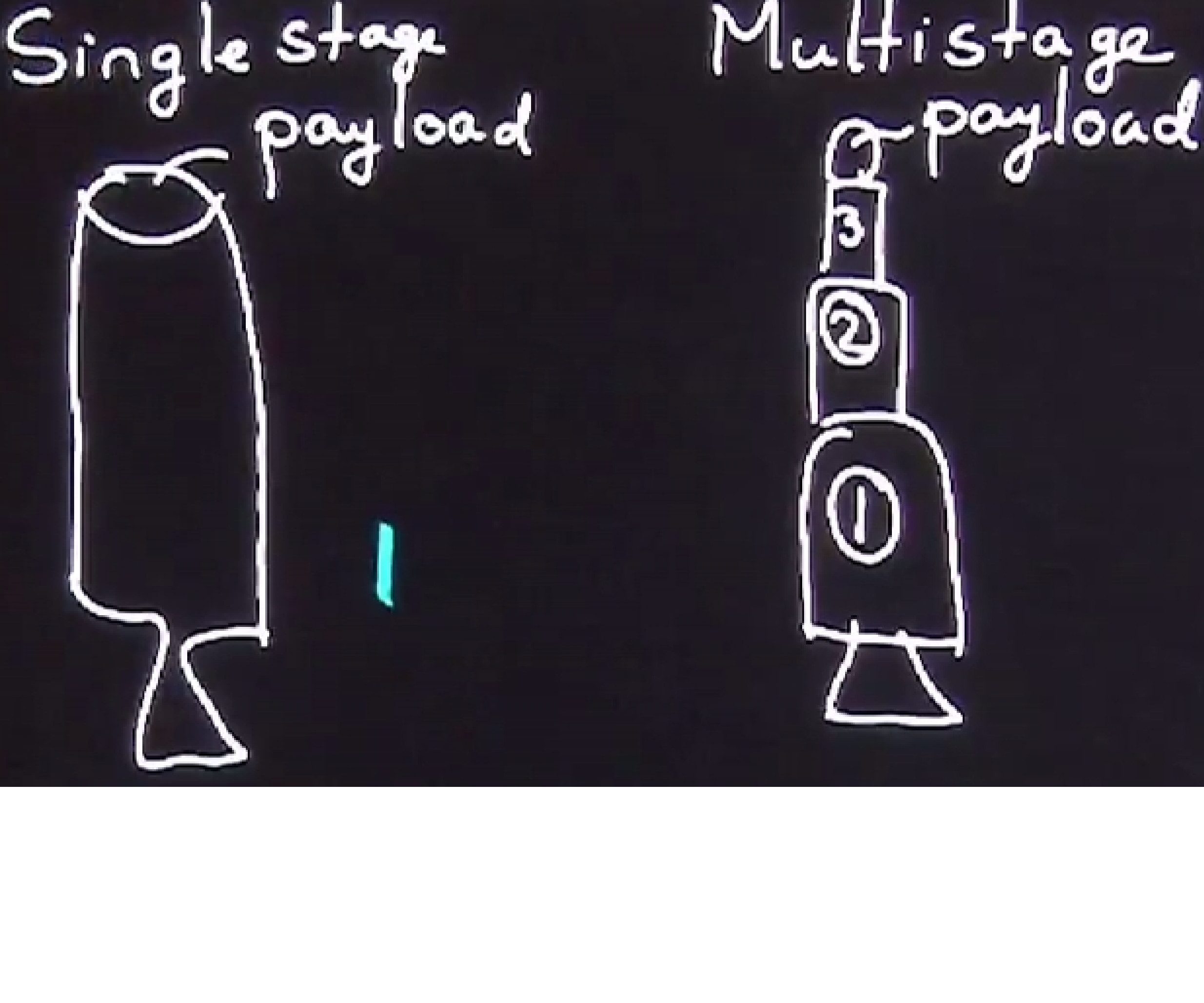
1) Velocity increment, Δu

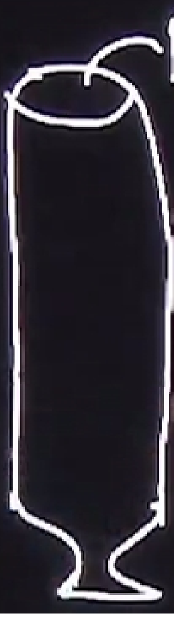
2) Range/ height, h

$$\Delta u = u_{eq} \ln\left(\frac{M_0}{M_b}\right)$$
$$= g_e I_{sp} \ln(R) \qquad R = \frac{M_0}{M_b}$$

Multistaging



Single stage rocket



M_0 – Total mass of rocket before firing

M_ℓ – Mass of payload

M_p – Mass of propellant

M_s – Structural mass (fuel/O_2 tanks, pumps, navigation systems...)

$$\text{Mass ratio, } R = \frac{M_0}{M_b} = \frac{M_0}{M_\ell + M_s}$$

$$M_b = M_\ell + M_s$$

$$M_0 = M_\ell + M_s + M_p \qquad = \frac{M_0}{M_0 - M_p}$$

$$\text{Payload ratio, } \lambda = \frac{M_\ell}{M_s + M_p}$$

λ large is desirable

$$\lambda = \frac{M_\ell}{M_0 - M_\ell}$$

Structural coefficient, ϵ

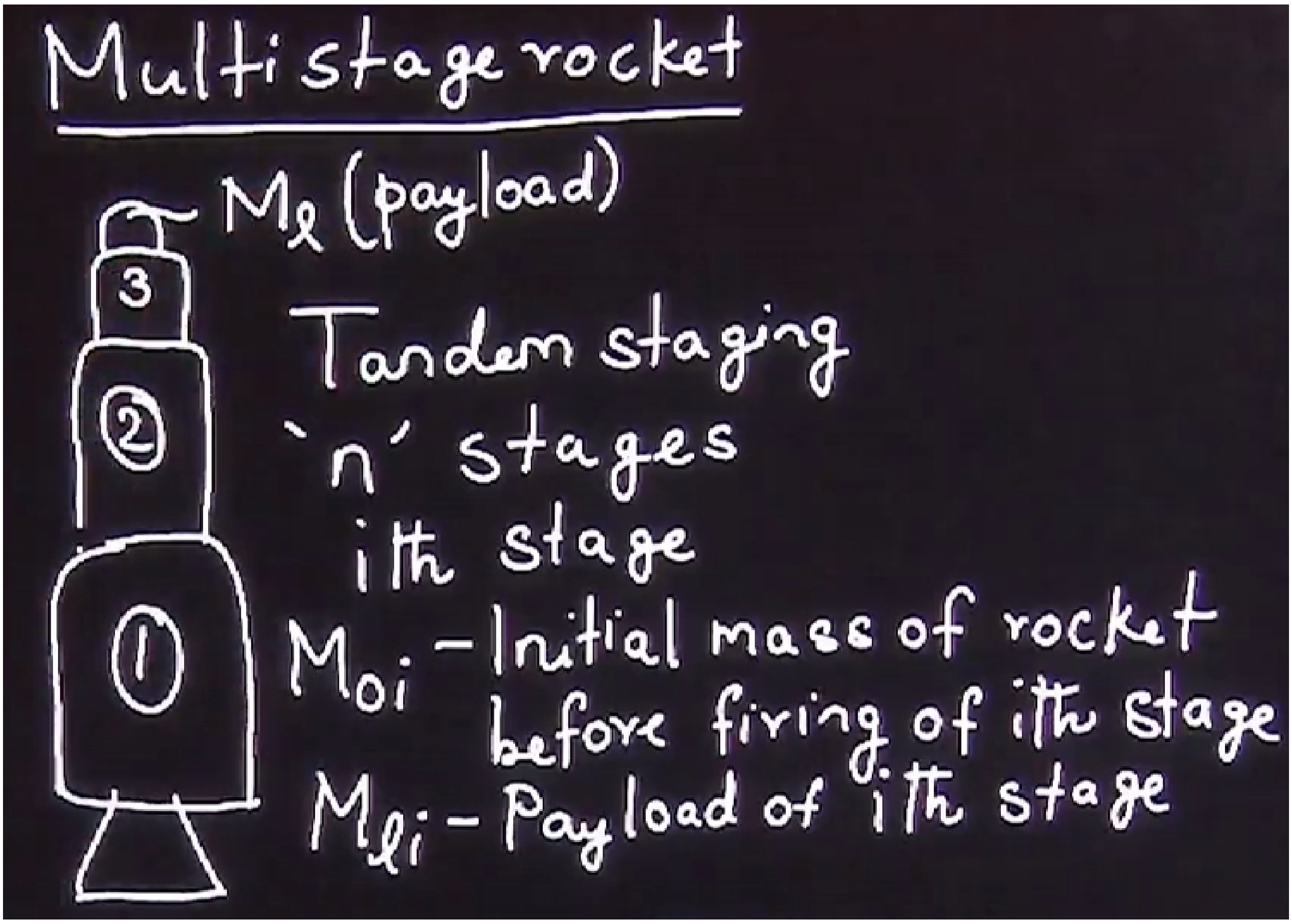
$$\epsilon = \frac{M_s}{M_s + M_p} = \frac{1}{1 + (M_p/M_s)}$$

$\downarrow$

small values are desirable

$$\epsilon = \frac{M_s}{M_o - M_\ell}, \quad \boxed{\frac{1+\lambda}{\epsilon+\lambda} = R}$$

HW


Multi stage rocket
M_l (payload)
3
2
1
Tandem staging
'n' stages
ith stage
M_{0i} - Initial mass of rocket before firing of ith stage
M_{li} - Payload of ith stage

M_{pi} – Mass of propellant of the ith stage

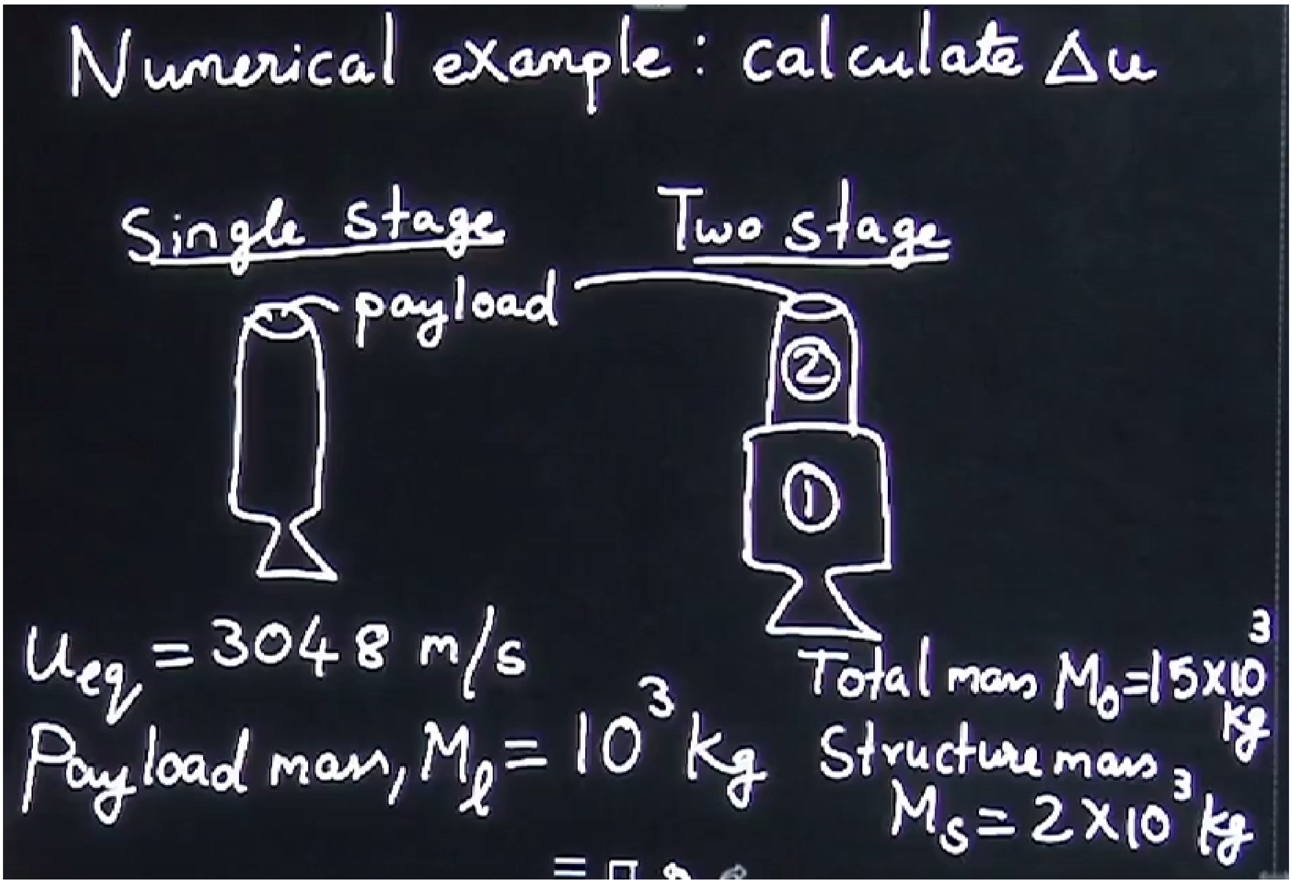
M_{si} – Structural mass of the ith stage

M_{bi} – Mass at the end of burnout of ith stage.

Mass ratio of ith stage, $R_i = \dfrac{M_{oi}}{M_{bi}}$

Payload ratio, $\lambda_i = \dfrac{M_{o(i+1)}}{M_{o(i)} - M_{o(i+1)}}$

$$\boxed{\frac{1+\lambda_i}{\epsilon_i+\lambda_i} = R_i}$$


Numerical example: calculate Δu
Single stage
Two stage
payload
2
1
$u_{eq} = 3048$ m/s
Payload mass, $M_l = 10^3$ kg
Total mass $M_0 = 15 \times 10^3$ kg
Structure mass $M_S = 2 \times 10^3$ kg

Multistaging benefits in rockets. Chap. 10. example problem in Hill & Peterson

Equivalent velocity

Pg. 1

$u_{eq} = 3048$ m/s

Total initial mass, $M_0 = 15000$ kg

Payload, $M_l = 1000$ kg.

structural mass, $M_s = 2000$ kg

Single stage: rocket.

Mass ratio, $R = \frac{M_0}{M_l + M_s} = \frac{15000}{1000 + 2000} = 5.$

$\Delta u_{single} = u_{eq} \ln R = 3048 \ln 5 = \boxed{4906 \text{ m/s} = \Delta u}$

structural coefficient $\epsilon = \frac{M_s}{M_0 - M_l} = \frac{2000}{15000 - 1000} = 0.143$

Payload ratio $\lambda = \frac{M_l}{M_0 - M_l} = \frac{1000}{15000 - 1000} = 0.0714$

Two stage rocket:

$M_{01} = 15000$ kg., $M_l = 1000$ kg.

Assuming same $\lambda = \lambda_1 = \lambda_2$.

$\lambda_1 = \frac{M_{02}}{M_{01} - M_{02}}$ $\quad \lambda_2 = \frac{M_l}{M_{02} - M_l}$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda = \frac{M_{02}}{M_{01} - M_{02}} = \frac{M_l}{M_{02} - M_l}$

$$\frac{M_{02}}{15000 - M_{02}} = \frac{1000}{M_{02} - 1000}$$

$M_{02}^2 - \cancel{1000 M_{02}} = 15 \times 10^6 - \cancel{1000 M_{02}}$

$\Rightarrow \boxed{M_{02} = 3873 \text{ kg}}$

$$\lambda_1 = \lambda_2 = \lambda = \frac{M_l}{M_{02} - M_l} = \frac{1000}{3873-1000} = \boxed{0.348 = \lambda}$$

structural coefficients

$$\epsilon_1 = \frac{M_{S1}}{M_{01} - M_{02}} \qquad \epsilon_2 = \frac{M_{S2}}{M_{02} - M_l}$$

Assuming $\epsilon_1 = \epsilon_2 = \epsilon = \frac{M_{S1}}{15000 - 3873} = \frac{M_{S2}}{3873-1000}$

$$\Rightarrow 2873 \, M_{S1} = 11127 \, M_{S2} \quad - (A)$$

Also the total structural mass,

$$M_{S1} + M_{S2} = 2000 \quad - (B)$$

solving (A) & (B) $M_{S1} = 1589.57$ kg $M_{S2} = 410.43$ kg.

$$\epsilon = 1/7 = 0.143$$

~~Ve~~ Mass ratio, $R_i = \frac{1+\lambda_i}{\epsilon_i + \lambda_i}$

since all ϵ_i & λ_i's are same, R_i's are also same.

$$R_1 = R_2 = R = \frac{1+0.348}{0.143+0.348} = 2.745$$

velocity increment,

$$\Delta u_i = u_{eq} \ln R_i$$

$$\Rightarrow \Delta u_1 = \Delta u_2 = u_{eq} \ln R = 3048 \ln 2.745$$

$$= 3077 \text{ m/s}$$

Total velocity increment, $\Delta u_{multi} = \Delta u_1 + \Delta u_2$.

$$\Delta u_{multi} = 6154 \text{ m/s}.$$

* since $\Delta u_{multi} > \Delta u_{single}$ for the same mass of propellant & structure, multistaging is preferred.