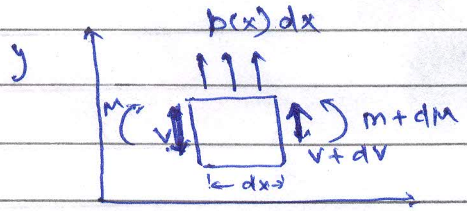
Vibration in Beams (Euler Equation)

Consider the forces and moments acting on an element of the beam with load intensity (load per unit length) as shown in the figure



$\Sigma F_y = 0: \; -V + p(x)dx + (V+dV) = 0$

$$\frac{dV}{dx} = -p(x)$$

$\Sigma M = 0: \; M + dM - M + V\,dx - p(x) \times \frac{(dx)^2}{2} = 0$

in the limiting case

$$\frac{dM}{dx} = -V$$

or

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = p(x)$$

From flexural equation $M = EI \dfrac{d^2y}{dx^2}$

$$\therefore \; \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) = p(x)$$

For a beam vibrating about its static equilibrium position under its own weight $p(x)$ is due to inertial load (caused by its mass and acceleration). Therefore,

$$p(x) = -\frac{1}{l}\left(\rho v \frac{\partial^2 y}{\partial t^2}\right)$$

$p(x)$ is load per unit length

let $\dfrac{\rho v}{l} = m$ .... mass per unit length

$$p(x) = -m \frac{\partial^2 y}{\partial t^2}$$

or

$$EI \frac{\partial^4 y}{\partial x^4} = -m \frac{\partial^2 y}{\partial t^2}$$

Assuming a solution

$$y(x) = Y(x) \cdot T(x)$$

$$EI \frac{\partial^4 Y}{\partial x^4}(T(x)) = -m\, Y(x) \frac{\partial^2 T}{\partial t^2}$$

or substituting $\frac{EI}{m} = \alpha^2$

$$\alpha^2 \frac{Y''''}{Y} = -\frac{\ddot{T}}{T}$$

Since left hand side is a function of x only and the right hand side is the function of t, each must be equal to a constant.

let $$\alpha^2 \frac{Y''''}{Y} = -\frac{\ddot{T}}{T} = \omega^2$$

This leads to two ordinary differential equations.

OD-1 : $$Y'''' - \left(\frac{\omega^2}{\alpha^2}\right) Y = 0$$

OD-2 : $$\ddot{T} + \omega^2 T = 0$$

Solution to OD-2 is

$$T(t) = A \cos \omega t + B \sin \omega t$$

Let $\frac{\omega^2}{\alpha^2} = \beta^4$, therefore, $\left[\omega = \beta^2 \sqrt{\frac{EI}{m}}\right]$

$$y'''' - \beta^4 y = 0$$

Assuming the solution of the form $y = e^{\lambda x}$

$$\lambda = \pm \beta \quad \text{and} \quad \lambda = \pm i\beta$$

General solution can be written in the form

$$Y(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

where $\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

and $e^{\pm \beta x} = \cos \beta x \pm i \sin \beta x$

also $e^{\pm \beta x} = \cosh \beta x \pm \sinh \beta x$

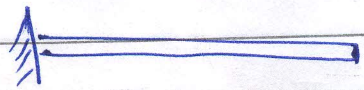
The constants C_1, C_2, C_3 and C_4 depend upon the boundary conditions.

Therefore,

$$y(x) = \left[C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x\right] \left(A \cos \omega t + B \sin \omega t\right)$$

Oscillatory part of the solution is given by the term $A \cos \omega t + B \sin \omega t$, where ω is angular frequency.

Consider a cantilever beam for its natural frequency.



BC: 1 : $y = 0$ @ $x = 0$: $C_1 + C_3 = 0$

or $C_1 = -C_3$

BC: 2 : $\frac{dy}{dx} = 0$ @ $x = 0$: $C_2 + C_4 = 0$

or $C_2 = -C_4$

BC: 3 :

$M = 0$ @ $x = l$ (using $C_1 = -C_3$ & $C_2 = -C_4$)

$\uparrow$

$d^2y/dx^2 = 0$

$$C_1 (\cos \beta l + \cosh \beta l) + C_2 (\sin \beta l + \sinh \beta l) = 0$$

BC: 4 :

$V = 0$ @ $x = l$ (using $C_1 = -C_3$ & $C_2 = -C_4$)

$\uparrow$

$\frac{d^3y}{dx^3} = 0$

$$C_1 (\sin \beta l - \sinh \beta l) - C_2 (\cos \beta l + \cosh \beta l) = 0$$

The only possible solution for C_1 and C_2 from BC 3 & 4 is its determinant is zero. OR

$$\begin{vmatrix} \cos \beta l + \cosh \beta l & \sin \beta l + \sinh \beta l \\ \sin \beta l - \sinh \beta l & -\cos \beta l - \cosh \beta l \end{vmatrix} = 0$$

multiplying and simplifying gives.

$$\cos \beta l \cosh \beta l + 1 = 0$$

Solutions for βl are

$$1.8751, \quad 4.6941, \quad 7.8548, \quad 10.995$$

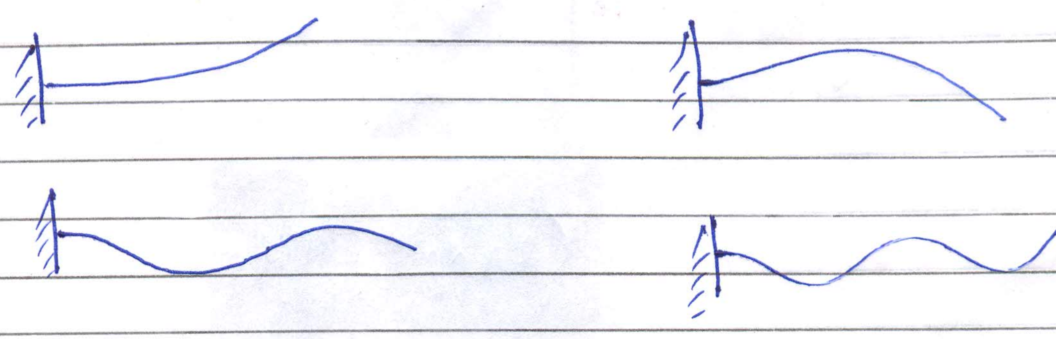
Therefore natural frequency for first, second ... modes:

$$\omega_1 = \left(\frac{1.8751}{l}\right)^2 \sqrt{\frac{EI}{m}}$$

$$\omega_2 = \left(\frac{4.6941}{l}\right)^2 \sqrt{\frac{EI}{m}}$$

$$\vdots$$

mode shapes.



For free-free BCs:

$$\beta l = 4.7300, \; 7.8532, \; 10.9956$$