Proceeding of the 11th World Congress on Intelligent Control and Automation
Shenyang, China, June 29 - July 4 2014

Two-Wheeled Self-balancing Robot Dynamic Model and Controller Design

Yufeng Zhuang
Automation School
Beijing University of Posts and Telecommunications, Beijing, China
zhuangyf @bupt.edu.cn

Zeyan Hu and Yi Yao
Automation School
Beijing University of Posts and Telecommunications, Beijing, China
hzy890621@bupt.edu.cn

Abstract - In this paper, we focus on the two-wheeled self-balancing robot dynamic model and controller design. Establishing the dynamic model, through Lagrange equation of the generalized coordinates of the robot system. According to the partial feedback linearization principle, making the two free degree of the left and right wheel be linear. The pitch angle between the vertical shaft and the centroid direction of the under-actuated robot is as the internal dynamic consideration. We design the nonlinear control algorithm of the robot balance movement and use the Matlab simulation to achieve the balance control of the two-wheeled self-balancing robot.

Index Terms - Two-wheeled self-balancing. Partial feedback linearization. Under-actuated. Nonlinear control. Lagrange equation

I. Introduction

As everyone knows, the wheeled robot has distinctly superior in the mobile robots. This kind of robots can keep high moving speed in the flat environment with a certain stability. The drive and control of the robots are easy relative to other mobile robots. In recent years, it has become one of the important direction in robot research[1].

The two-wheeled robots which can have a high complexity as the control object, have been a typical example of the application of control theory, because of their simple structures and low cost. The different control methods of the two-wheeled self-balancing robots will be the important embodiment of the control ability of the system with unstable essence.

The existing two-wheeled self-balancing robots are generally modeled as the structure model of the inverted pendulum. There are two kinds of robots, fixed centroid and moving centroid, according to the robots have weight unit or not. This paper is for the fixed one. A. Salerno and others from the McGill University proposed a concept of two - wheeled robot named “Quasimoro”. The centroid is placed below the wheel. They used LQR method to achieve the robust control of the performances and the dynamic uncertainties[2-3]. R. C. Ooi used the structure of the inverted pendulum , the fixed centroid method and the pole placement method to achieve the balance of the two-wheeled robot[4].The Ubot from MIT University used the fixed centroid and the inverted pendulum method similarly. It equipped with a camera and can obstacle avoidance[5]. Xudong Duan and Henghua Wei from University of Science and Technology of China designed a two-wheeled inverted pendulum system[6]. They established the dynamic model on the slope according to the classical Newtonian mechanics principle and used the robust control algorithm to ensure the dynamic and steady state performance.

In this paper, the centroid of the two-wheeled self-balancing robot is on the axis of the two wheels. We established the dynamic model, through Lagrange equation of the generalized coordinates of the robot system. According to the partial feedback linearization principle, we design the nonlinear control algorithm of the robot balance movement. Through the simulation, the method is reasonable and feasible, which can quickly and smoothly realize the robot balance control.

II. The Dynamic Model of the Two-Wheeled Self-Balancing Robot

In view of the above background, we need to simplify the physical structure, in order to make the dynamic model of the two-wheeled self-balancing robot easily. This chapter mainly shows the ideal assumptions, kinematics analysis and dynamics model of the robot and so on.

A. The ideal assumptions

The assumptions of the two-wheeled self-balancing robot are as follows. The part that connects the two wheels can be regarded as a particle and the mass concentrate on the center of mass. The left and right wheels can be regarded as circles and the centroid on the center of the circle.

We assume that the mass of left wheel is m and it equals the right wheel. The radius of the wheel is r . The distance between the two wheels is $2l$. The intermediate part of the robot exclusive of the wheels is as a cylinder connected on the central axis of the two wheels, the mass is m_1 and the radius is r_1 . Its centroid is on the central axis and the distance between it and the axis of the two wheels is h . After a disturbance, the robot will deviate from the vertical direction and the pitch angle is β .The robot will turn around the central axis and the angle is θ . In order to keep balance, need to add torque M_1 on the left wheel and torque M_2 on the right wheel. The angle of the wheels are α_1 and α_2 .

In addition, the assumptions also include follows.

(1) The body of the robot and the wheels are rigid.

(2) The geometric dimensions and mass of left and right wheels are equal.

(3) The wheels always keep contact with ground. They are pure rolling and do not slip during the movement. The

instantaneous speed of the landing places of the wheels is zero.

(4) Neglect the gear clearance and sensor noise in actual use.

(5) Neglect the inductance in the armature winding and the friction of the motor. Neglect the no-load torque and assume that the motor output torque is the electromagnetic torque.

(6) Neglect the system internal energy loss.

(7) Neglect the moment of couple of the frictional resistance and only consider the friction and friction torque.

B. The Kinematic analysis

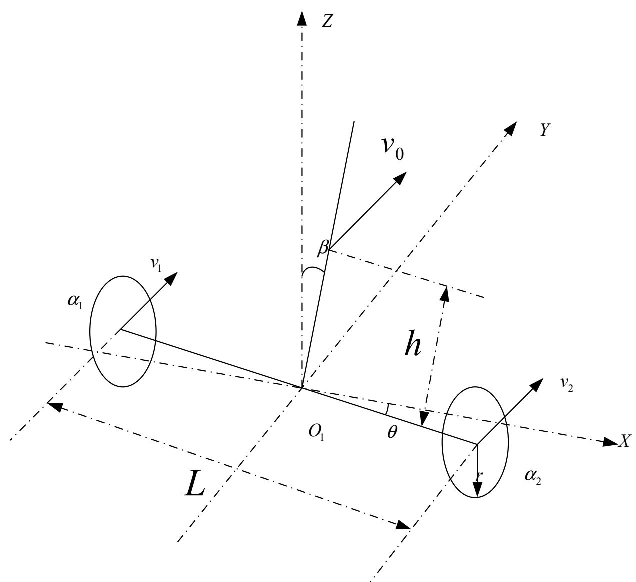


Fig. 1 the kinematic analysis chart of the two-wheeled self-balancing robot.

We assume that the centroid speed of the left wheel is v_1. The movement of the left wheel is that the centroid turn around the central axis at the rate of $r\dot{\alpha}_1$. The moment of the inertia is J_1.It rotates around its absolute instantaneous center of velocity O_1.

Similarly, we assume that the centroid speed of the right wheel is v_2. The movement of the right wheel is that the centroid turn around the central axis at the rate of $r\dot{\alpha}_2$. The moment of the inertia is J_2. It rotates around its absolute instantaneous center of velocity O_1. So, we can get the following equations.

$$v_1 = r\dot{\alpha}_1 \tag{1}$$

$$v_2 = r\dot{\alpha}_2 \tag{2}$$

$$v_0 = \frac{v_1 + v_2}{2} \tag{3}$$

$$\dot{\theta} = \frac{|v_1 - v_2|}{L} \tag{4}$$

Among the above equation $L = 2l$.

The kinetic energy of the left wheel is

$$T_1 = \frac{1}{2}mv_1^2 + \frac{1}{2}J_1(\frac{v_1}{r})^2 . \tag{5}$$

The kinetic energy of the right wheel is

$$T_2 = \frac{1}{2}mv_2^2 + \frac{1}{2}J_2(\frac{v_2}{r})^2 . \tag{6}$$

In addition, $J_1 = J_2 = mr^2$.

Assume the centroid speed of the cylinder is v_3 [7]. The movement of the body is the rotation around the absolute instantaneous center of velocity O_1 at the rate of v_0 and the rotation around the central axis at the rate of $h\dot{\beta}$. The moment of the inertia is J_3. The angle of v_0 and $h\dot{\beta}$ is β.Then from the vector correlation we can get the following equation.

$$v_3^2 = v_0^2 + (h\dot{\beta})^2 + 2v_0h\dot{\beta}\cos\beta \tag{7}$$

The kinetic energy of the intermediate part is

$$T_3 = \frac{1}{2}m_1{v_3}^2 + \frac{1}{2}J_3\dot{\beta}^2 . \tag{8}$$

We can know $J_3 = \frac{4}{3}m_1h^2$.

Above all, the total kinetic energy of the two wheeled robot is

$$T = T_1 + T_2 + T_3 . \tag{9}$$

After the simplification, we can get

$$T = mr^2(\dot{\alpha}_1^2 + \dot{\alpha}_2^2) + \frac{m_1}{8}r^2(\dot{\alpha}_1 + \dot{\alpha}_2)^2 + \frac{1}{2}m_1rh(\dot{\alpha}_1 + \dot{\alpha}_2)\dot{\beta}\cos\beta + \frac{7}{6}m_1h^2\dot{\beta}^2 . \tag{10}$$

Take the ground as a reference surface. The total potential energy of the robot is

$$V = 2mgr + m_1g(h\cos\beta + r) . \tag{11}$$

C. The dynamics equation of the two- wheeled robot

Then we synthesize the total kinetic energy and potential energy, and get the Lagrange operator of the system[8].

$$L = mr^2(\dot{\alpha}_1^2 + \dot{\alpha}_2^2) + \frac{m_1}{8}r^2(\dot{\alpha}_1 + \dot{\alpha}_2)^2 + \frac{1}{2}m_1rh(\dot{\alpha}_1 + \dot{\alpha}_2)\dot{\beta}\cos\beta + \frac{7}{6}m_1h^2\dot{\beta}^2 - 2mgr - m_1g(h\cos\beta + r) \tag{12}$$

The second Lagrange equation is

$$Q_i = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i}(i = 1,2,3...,k) . \tag{13}$$

In the above equation, $L = T - V_e$, q_i is the generalized coordinate and Q_i is the corresponding generalized force.

In this system, we choose the angle of the left wheel α_1 ,the angle of the right wheel α_2 and the pitch angle β as the generalized coordinates. The left and right motors output torque M_1 and M_2 are as the generalized forces.

Take the Lagrange operator into the Lagrange equation and arrange the equation.

$$J(q)\ddot{q}+f(q,\dot{q})\dot{q}+G(q)=\tau \tag{14}$$

Among this equation, some unknown parts are as follows. $q^T=[\alpha_1,\alpha_2,\beta]$, $\tau^T=[M_1,M_2,0]$,

$$\boldsymbol{J}(\boldsymbol{q})=\begin{bmatrix} 2mr^2+\frac{m_1}{4}r^2 & \frac{m_1}{4}r^2 & \frac{m_1}{2}rh\cos\beta \\ \frac{m_1}{4}r^2 & 2mr^2+\frac{m_1}{4}r^2 & \frac{m_1}{2}rh\cos\beta \\ \frac{m_1}{2}rh\cos\beta & \frac{m_1}{2}rh\cos\beta & \frac{7}{3}m_1h^2 \end{bmatrix},$$

$$f(q,\dot{q})=\begin{bmatrix} 0 & 0 & -\frac{1}{2}m_1rh\dot{\beta}\sin\beta \\ 0 & 0 & -\frac{1}{2}m_1rh\dot{\beta}\sin\beta \\ 0 & 0 & 0 \end{bmatrix},$$

and $G(q)=\begin{bmatrix} 0 \\ 0 \\ -m_1gh\sin\beta \end{bmatrix}$.

III. THE DESIGN OF BALANCE CONTROLLER FOR THE TWO-WHEELED ROBOT

According to the dynamic model, we establish the balance controller for the two wheeled robot. The model is from the generalized coordinates of the robot system through the Lagrange equation. The dynamic model contains drive system and under-actuated system. According to the partial feedback linearization principle, we make the two drive degrees of freedom of the left and right wheels be linear. The pitch angle is as the internal dynamic of the system. Consequently, the dynamic model could be rewritten into two subsystems[9].

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} \ddot{\Theta}_1 \\ \ddot{\Theta}_2 \end{bmatrix}+\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}=\begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{15}$$

Θ_1 is the drive part and $\Theta_1=[\alpha_1,\alpha_2]^{\mathrm{T}}$. Θ_2 is the under-actuated part and $\Theta_2=\beta$.

$$M_{11}=\begin{bmatrix} 2mr^2+\frac{m_1}{4}r^2 & \frac{m_1}{4}r^2 \\ \frac{m_1}{4}r^2 & 2mr^2+\frac{m_1}{4}r^2 \end{bmatrix} \tag{16}$$

$$M_{12}=\begin{bmatrix} \frac{m_1}{2}rh\cos\beta \\ \frac{m_1}{2}rh\cos\beta \end{bmatrix} \tag{17}$$

$$M_{21}=\begin{bmatrix} \frac{m_1}{2}rh\cos\beta & \frac{m_1}{2}rh\cos\beta \end{bmatrix} \tag{18}$$

$$M_{22}=\frac{7}{3}m_1h^2 \tag{19}$$

$$F_1=\begin{bmatrix} -\frac{1}{2}m_1rh\dot{\beta}^2\sin\beta \\ -\frac{1}{2}m_1rh\dot{\beta}^2\sin\beta \end{bmatrix} \tag{20}$$

$$F_2=-m_1gh\sin\beta \tag{21}$$

$$\tau^T=[M_1,M_2] \tag{22}$$

$M_{21}=\begin{bmatrix} \frac{m_1}{2}rh\cos\beta & \frac{m_1}{2}rh\cos\beta \end{bmatrix}\neq 0$, we could get the equation below.

$$\ddot{\Theta}_1=-M_{21}^{-1}(M_{22}\ddot{\Theta}_2+F_2) \tag{23}$$

Take the equation (23) into the equation (15), we could get

$$\begin{aligned}&(-M_{11}M_{21}^{-1}M_{22}+M_{12})\ddot{\Theta}_2- \\ &M_{11}M_{21}^{-1}F_2+F_1=\tau\end{aligned}. \tag{24}$$

Then apply the partial feedback linearization. Make Θ_2 be linear and $\ddot{\Theta}_2=\upsilon$.

The equation (24) would be

$$\begin{aligned}&\tau=(-M_{11}M_{21}^{-1}M_{22}+M_{12})\upsilon \\ &-M_{11}M_{21}^{-1}F_2+F_1\end{aligned}. \tag{25}$$

The system model could be rearranged.

$$\begin{cases} \ddot{\Theta}_1=-M_{21}^{-1}(M_{22}\ddot{\Theta}_2+F_2) \\ \ddot{\Theta}_2=\upsilon \\ y=[\Theta_1\ \ \Theta_2]^T \end{cases} \tag{26}$$

Take the $\Theta_1=[\alpha_1,\alpha_2]^{\mathrm{T}}$, $\Theta_2=\beta$, equation (18), equation (19) and equation (21) into the equation (26).

$$\ddot{\Theta}_1=\begin{bmatrix} \frac{g}{r}\tan\beta-\frac{7h}{3r\cos\beta}\ddot{\beta} \\ \frac{g}{r}\tan\beta-\frac{7h}{3r\cos\beta}\ddot{\beta} \end{bmatrix} \tag{27}$$

So, we could get in the system $\alpha_1=\alpha_2=\alpha$. Take the equation (16-22) into the equation (25), we could get $M_1=M_2=M$.

The equation (26) could be simplified as the follow equation.

$$\begin{cases} \ddot{\alpha}=\frac{g}{r}\tan\beta-\frac{7h}{3r\cos\beta}\ddot{\beta} \\ \ddot{\beta}=\upsilon \\ y=[\alpha\quad \beta]^T \end{cases} \tag{28}$$

Assume the expected value of the system is

$$y^d=[\ \alpha^d\ \ \dot{\alpha}^d\ \ \beta^d\ \ \dot{\beta}^d\ ]^T. \tag{29}$$

The ideal target of the system is the static equilibrium position, $[\alpha^d\ \ \dot{\alpha}^d\ \ \beta^d\ \ \dot{\beta}^d]^T=[0\ 0\ 0\ 0]^T$.

Then select the state variables as : $x_1=\alpha-\alpha^d$, $x_2=\dot{\alpha}-\dot{\alpha}^d$, $x_3=\beta-\beta^d$, $x_4=\dot{\beta}-\dot{\beta}^d$.

The controlling quantity is

$$\upsilon = \beta^d - k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4 . \tag{30}$$

k_1 , k_2 , k_3 and k_4 are the parameters can be adjusted.

The whole system can be written as

$$\begin{cases} \dot{x} = f(x) + g(x)\upsilon \\ y = Cx \end{cases} . \tag{31}$$

The variables of the equation (31) are

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, f(x) = \begin{bmatrix} x_2 \\ \frac{g}{r}\tan x_3 \\ x_4 \\ -k_1x_1 - k_2x_2 - k_3x_3 - k_4x_4 \end{bmatrix},$$

$$\mathrm{g(x)} = \begin{bmatrix} 0 \\ -\frac{7h}{3r\cos x_3} \\ 0 \\ 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \ \cos x_3 \neq 0$$

IV. THE STABILITY ANALYSIS OF THE TWO-WHEELED ROBOT SYSTEM

In order to make the robot stable at the equilibrium point, the closed-loop system should have the stability. The proofs are as follows.

The Jacobian matrix of the equation $f(x)$, when the robot in the static equilibrium position is showed as the equation (32)

$$\left.\frac{\partial f(x)}{\partial(x)}\right|_{x=0,} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 \end{bmatrix} \tag{32}$$

$w = g/r$. The characteristic equation is

$$D(s) = s^4 + k_4s^3 + k_3s^2 + wk_2s + wk_1 . \tag{33}$$

According to the Hurwitz-Routh criterion[10], Routh decision table as follows.

$$\begin{array}{c|ccc} s_4 & 1 & k_3 & wk_1 \\ s_3 & k_4 & wk_2 & 0 \\ s_2 & b_3 & wk_1 & 0 \\ s_1 & c_3 & 0 & 0 \\ s_0 & wk_1 & 0 & 0 \end{array} \tag{34}$$

$b_3 = \frac{k_3k_4 - wk_2}{k_4}$, $c_3 = \frac{wk_2(k_3k_4 - wk_2) - wk_1k_4}{k_3k_4 - wk_2}$. On the basis of the Routh criterion, conditions for the stability of the system is

$$\begin{cases} k_1, k_2, k_3, k_4 > 0 \\ k_3k_4 > wk_2 \\ k_2(k_3k_4 - wk_2) > k_1k_4 \end{cases} . \tag{35}$$

we select the appropriate k_1 , k_2 , k_3 , k_4 . The characteristic equation is critical stable and the output of the system is stable.

V. THE SIMULATION ANALYSIS OF THE BALANCE CONTROLLER

Take the specific parameters in the TABLE 1. $g = 9.8\text{m/s}^2$. We used the Matlab Simulilnk to build the nonlinear systems showed by the equation (26). The system block diagram is shown as follows.

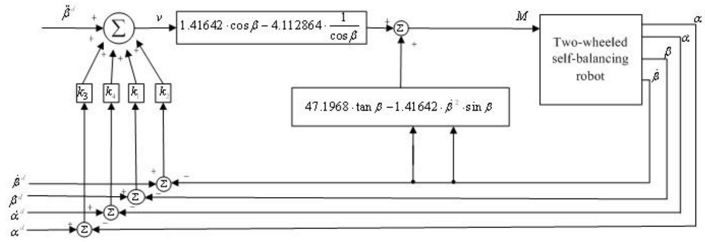


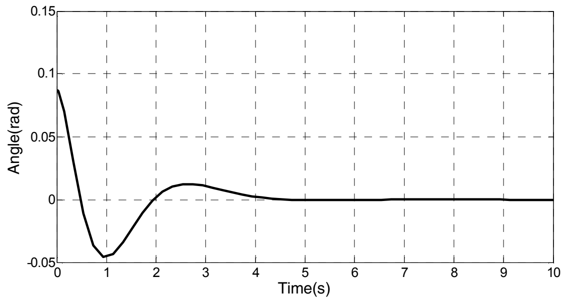
Fig. 2 The system block diagram of the two-wheeled self-balancing robot.

Set the simulation time is 10s, the step is 0.005s. In order to adjust the system parameters quickly, we set the shaft with a 5 degree angle in the initial state, that can reduce the simulation time and show the simulation results. The initial state is $(\alpha, \dot{\alpha}, \beta, \dot{\beta}) = (0, 0, 0.0873, 0)$. With the set initial state, we choose the control parameters k_1=0.2 , k_2=0.32 , k_3=23 and k_4=10 for the equilibrium.

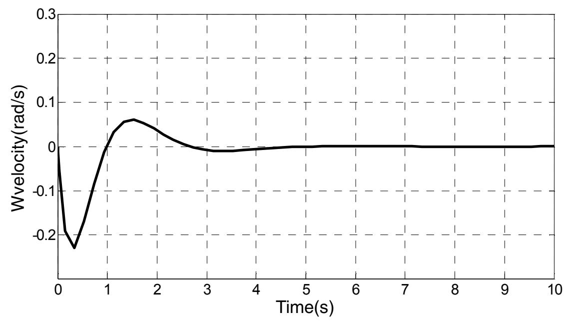
TABLE I
THE PARAMETERS OF THE TWO-WHEELED SELF-BALANCING ROBOT

Name	physical significance	values
m	the mass of the wheel	4.73 kg
m_1	The mass of the intermediate cylinder	77.4 kg
r	The radius of the wheel	0.1 m
h	the distance between the centroid and the axis of the wheels	0.366 m

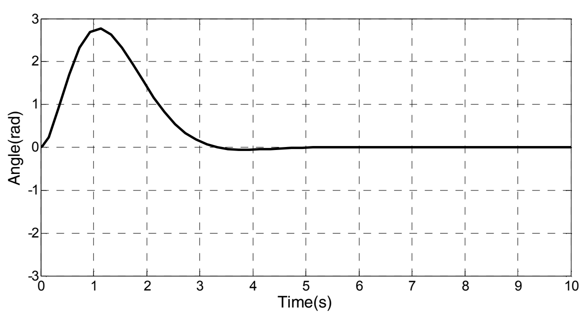
The Figure below is the simulation parameters curves of the two-wheeled self-balancing robot motion control which from the set angle to the equilibrium position. From the figure, we can see that after about 4S adjustment of the feedback controller, the angle α and the pitch angle β converged to 0.At the same time, the corresponding angular velocity changed in a certain range, and eventually converged to 0. In addition, the adjust torques of the two wheels eventually stabilized for 0.



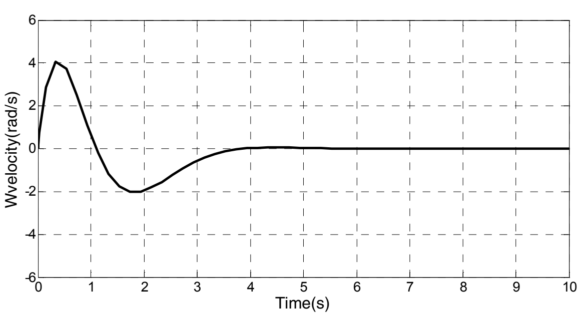
a. the pitch angle β



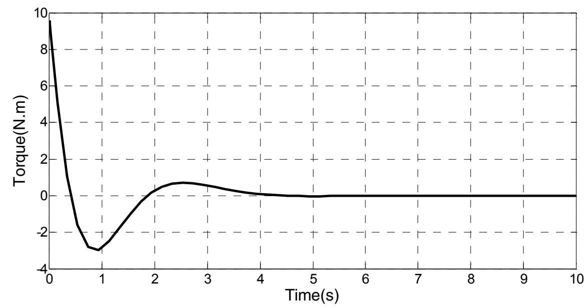
b. the angular velocity of the pitch angle $\dot{\beta}$



c. The angle of the wheels α



d. the angular velocity $\dot{\alpha}$



e. the torque M

Fig. 5 The simulation parameters curves.

IV. CONCLUSION

Through the calculation and simulation analysis above, we can see that the angle β and α become stable in a short time. The angular velocity of the β , α and the torque M is in accordance with the physical significance .These also prove the dynamics model is correct. Therefore we have achieved the linearization of the two-wheeled self-balancing robot nonlinear system. The designed controller is reasonable and reliable. It can stabilize the robot faster and more smoothly and has laid a good foundation for the motion control of the robot prototype .

REFERENCES

[1] F. Jifen, et al, "Controller Design of a Self-banlancing Two Wheeled Mobile Robot," *Development & Innovation of Machinery & Electrical Products*,vol.17, no.6, pp.75-77, 2004.

[2] A. Salerno and J. Angeles, "Design and Implementation of a Quasiholonomic Mobile Robot," in Proceedings of 2007 IEEE International Conference on Robotics and Automation, Roma, , pp.2977-2983, 2007.

[3] A. Salerno and J. Angeles, "A New Family of Two-Wheeled Mobile Robots: Modeling and Controllability," in Proceedings of IEEE Transactions on Robotics and Automation, vol. 23, no. 1, pp.169-173, 2007.

[4] R. C. Ooi, "Balancing a Two-Wheeled Autonomous Robot," The Dissertation for Master Degree in the University of Western Australia, pp. 10-15, 2003.

[5] P. Deegan, et al, "Mobile Manipulators for Assisted Living in Residential Settings," *Autonomous Robots*, vol. 24, no. 2, pp.179-192, 2008.

[6] D. Xudong, W. Henghua and C. Xing , "Dual-wheel Cart-inverted Pendulum System Based on DSP," *Control Engineering of China*, vol. 11, no. 6, pp. 521-524, 2004.

[7] F. liyi, L. Haihong and Z. Fuhai, "Design and Kinematic Analysis of Anti- terrorism Robot With Wheel- tracked Variable Structure," Machinery & Electronics, no. 10, pp, 66-69, 2011.

[8] L. Tijun, "On Selecting Generalized Coordinates in Analytic Mechanics," *Journal of Changchun Normal University(Natural Science)*, no.4, pp.28-29, 2008.

[9] D. Fuguo and L Yurong, "Consistency analysis of addition chains for several fast algorithms of modular exponentiation," *Computer Engineering and Applications*, vol. 46, no. 36, 2010.

[10]R. C.Dorf and R. H.Bishop, Modern control system (Eighth Edition), Higher Education Press (HEP), pp. 250-257, 2001.06.