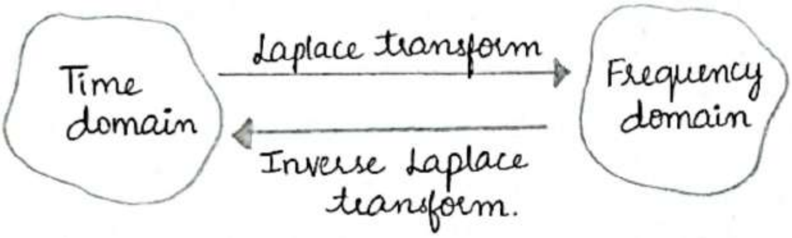
LAPLACE TRANSFORMS

A Laplace transform is an extremely diverse function that can transform a real function of time t to one in the complex plane s, referred to as the frequency domain.

One thing to note is that the Laplace transform is a complex transform of a complex variable, while the Fourier transform is a complex transform of a real variable.



Laplace transforms is one of the methods used in solving initial and boundary value problems involving homogenous and non-homogenous equations without actually finding the general solution of the differential equation by various known methods.

The primary use of this transform is to change an ordinary differential equation in a real domain into an algebraic equation in the complex domain, making the equation much easier to solve. The subsequent solution that is found by solving the algebraic equation is then taken and inverted by use of inverse Laplace transform, acquiring a solution for the original differential equation, or ODE.

The applications of Laplace transform are numerous, ranging from heating, ventilation, and air conditioning systems modeling to modeling radioactive decay in nuclear physics. Along with these applications, some of its more well-known uses are in electrical circuits and in analog signal processing. It is also used in harmonic oscillators and systems such as HVAC and many other types of systems that deal with sinusoids and exponentials.

The subject was first introduced by Oliver Heaviside of England around 1800 AD and later on by Bromwich and Carson during 1916-17. French mathematician Pierre de Laplace (1749 - 1827) used such transforms much earlier in 1799, while developing the theory of probability. Hence the transform is named after him.

Pierre-Simon, marquis de Laplace was a prominent French mathematical physicist and astronomer of the 19th century, who made crucial contributions in the arena of planetary motion by applying Sir Isaac Newton's theory of gravitation to the entire solar system. His work regarding the theory of probability and statistics is considered pioneering and has influenced a whole new generation of mathematicians. Laplace heavily contributed in the development of differential equations, difference equations, probability and statistics. His work was important to the development of engineering, mathematics, statistics, physics, astronomy and philosophy.

* **Definition:**

Let $f(t)$ be a real valued function defined for $0 \le t < \infty$. Then the Laplace transform of $f(t)$ denoted by $\mathcal{L}\{f(t)\}$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad . \rightarrow (1)$$

provided the integral exists. s is a parameter real or complex number, is called the transform parameter.

Note that the value of integral on the right-hand side of (1) depends on s. Thus $\mathcal{L}\{f(t)\}$ is a function of s. This function is denoted by $F(s)$ or $\bar{f}(s)$.

Thus, $\mathcal{L}\{f(t)\} = F(s) = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

Note : (i) The symbol $\mathcal{L}$ is called the Laplace transform operator.
(ii) If $\mathcal{L}\{f(t)\} = F(s) = \bar{f}(s)$, then $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\bar{f}(s)\}$ and is called the inverse Laplace transform.
i.e., $f(t)$ is called the inverse Laplace transform of $F(s)$.

(iii) In a practical situation, the variable t represents time and s represents frequency. Hence the Laplace transform converts time domain into the frequency domain.

* (Or) Formal definition (Just for reference)

The Laplace transform of a function, $f(t)$, $t \geq 0$ with t being in the time domain, is normally denoted by the following equation,

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

This function transforms the equation from being in the time domain to being in the complex domain where s is a complex variable representing frequency denoted by the equation,

$$s = \sigma + i\omega$$

In the case of the equation denoting s, σ and ω are both real numbers with i being the complex portion. This means we are putting the differential equation into a completely different domain, as previously mentioned with σ and $i\omega$ being our individual coordinates respectively. This domain will be referred to as the frequency domain. Note that this transform is invertible. The equation to do so is as follows,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{\eta \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

It is worth noting that γ is a real number in this integral, which is also known by the name The Bromwich Integral. The importance of this inverse transform cannot be understated, as it is what allows us to convert the equation back into the real domain to get the solution for the original equation.

* Suppose $f(t)$ is defined as follows :

$$f(t) = \begin{cases} f_1(t), & 0<t<a \\ f_2(t), & a<t<b \\ f_3(t), & t>b \end{cases}$$

Note that $f(t)$ is piecewise continuous.

$$\therefore \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} f_1(t)\,dt + \int_a^b e^{-st} f_2(t)\,dt + \int_b^\infty e^{-st} f_3(t)\,dt$$

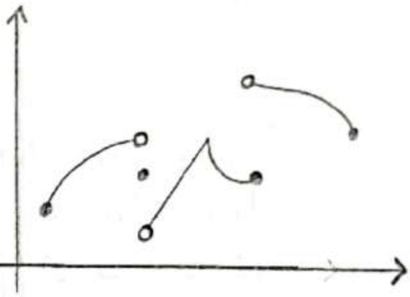
* Piecewise continuous function :

If an interval $[a,b]$ can be partitioned by a finite number of points $a_0 = t_0 < t_1 < t_2 < \dots < t_n = b$ such that

(i) f is continuous on each sub interval (t_i, t_{i+1}).

(ii) $\left| \lim_{t \to t_i^+} f(t) \right| < \infty \quad \forall i = 0,1,2,3,\dots,n-1.$

(iii) $\left| \lim_{t \to t_{i+1}^-} f(t) \right| < \infty \quad \forall i = 1,2,3,\dots,n,$



then the function is piecewise continuous.

Eg: 1. The function defined by $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3-t, & 1<t\le 2 \\ t+1, & 2<t\le 3 \end{cases}$ is piecewise continuous in $[0,3]$.

2. The function defined by $f(t) = \begin{cases} t^2+1, & 0\le t\le 1 \\ \frac{1}{2-t}, & 1<t\le 2 \\ 4, & 2<t\le 3 \end{cases}$ is not piecewise continuous in $[0,3]$.

* Exponential order :

A function $f(t)$ is said to be of exponential order α, if there exists constants M and α such that

$$|f(t)| \le Me^{\alpha t} \text{ for sufficiently large } t.$$

Eg: Any polynomial is of exponential order.

$\because \; e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!}$

$\Rightarrow t^n \le \frac{n!}{a^n} e^{at}$

But, $f(t) = e^{t^2}$ is not of exponential order.

* Existence and Uniqueness of Laplace Transform.

- Sufficient conditions for existence of Laplace Transform :

The Laplace transform of a function $f(t)$ exists when the following conditions are satisfied

(i) $f(t)$ is piecewise continuous. i.e., $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α. i.e., there exists M, α such that $|f(t)| \le Me^{\alpha t}$, for all $t \ge 0$. In other words,

$$\lim_{t\to\infty} \{e^{-\alpha t} f(t)\} = \text{finite quantity}.$$

Eg : 1. $\mathcal{L}\{\tan t\}$ does not exist since $\tan t$ is not piece wise continuous.

2. $\mathcal{L}\{e^{t^2}\}$ does not exist since e^{t^2} is not of any exponential order.

- Uniqueness of Laplace transform :

Let $f(t)$ and $g(t)$ be continuous functions such that $F(S) = G(S)$, then $f(t) = g(t)$ for all t.

* Region of Convergence (ROC) :

Laplace transform $F(S)$ of a function $f(t)$ converges provided that the limit $\lim_{R\to\infty} \int_0^R f(t) e^{-St} dt$ exists.

The Laplace transform converges absolutely if the integral $\int_0^{\infty} |f(t)e^{-St}| dt$ exists. The set of values of S for which F(S) exists is known as the region of convergence.

* Basic properties of Laplace transform

1. Linearity property

For any two functions f(t) and g(t) and any two constants a and b,

$$\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$$

Proof :

Using the definition of the Laplace transform, we get

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^{\infty} e^{-St}\{af(t) + bg(t)\}dt$$

$$= a\int_0^{\infty} e^{-St} f(t)\, dt + b\int_0^{\infty} e^{-St} g(t)\, dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Similarly $\mathcal{L}\{af(t) - bg(t)\} = a\mathcal{L}\{f(t)\} - b\mathcal{L}\{g(t)\}$

In particular, for $a = b = 1$, $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$ and
for $a = -b = 1$, $\mathcal{L}\{f(t) - g(t)\} = \mathcal{L}\{f(t)\} - \mathcal{L}\{g(t)\}$

The linearity of the Laplace transform follows from the (its) definition as an integral and the fact that integration is a linear operation.

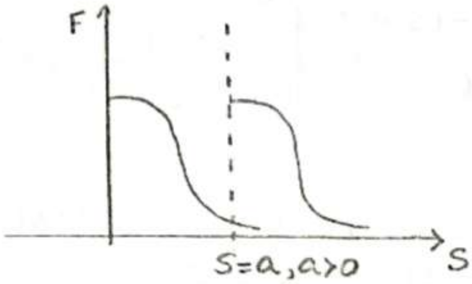
2. S-domain shift / First shifting property

If $\mathcal{L}\{f(t)\} = F(S)$, then $\mathcal{L}\{e^{at} f(t)\} = F(S-a)$, where a is any real constant.

Proof : By definition, $\mathcal{L}\{e^{at} f(t)\} = \int_0^{\infty} e^{-St}\{e^{at} f(t)\}dt = \int_0^{\infty} e^{-(S-a)t} f(t)\, dt$

$$= F(S-a)$$

Multiplication by an exponential in time introduces a shift in frequency s to the Laplace transform of $f(t)$. i.e., Laplace transform of $e^{at}f(t)$ can be written down directly by changing s to s-a in $F(s)$.

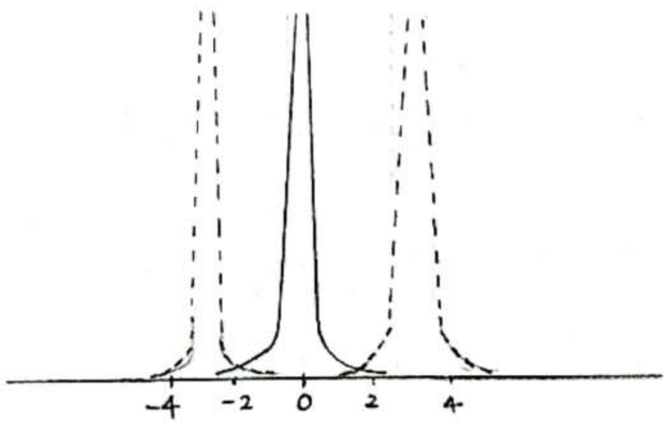


Example:

If $f(t) = t$, then $F(s) = \frac{1}{s^2}$

If $f(t) = e^{3t}.t$ then $F(s) = \frac{1}{(s-3)^2}$ (Shifting towards right)

If $f(t) = e^{-3t}.t$ then $F(s) = \frac{1}{(s+3)^2}$ (Shifting towards left)



* Laplace transform of some standard functions

1. $\mathcal{L}(a) = a/s$, where 'a' is a constant.

$$\mathcal{L}(a) = \int_0^\infty e^{-st}.a\,dt = a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}(0-1) = \frac{a}{s}, \text{ where } s>0.$$

In particular, if $a=1$, then $\mathcal{L}\{1\} = \frac{1}{s}$.

2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ where $s>a$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[-\frac{e^{-(s-a)t}}{(s-a)}\right]_0^\infty = \frac{1}{(s-a)}, \text{ where } s>a.$$

3. $\mathcal{L}\{t^n\} = \overset{(i)}{\frac{\Gamma(n+1)}{s^{n+1}}} = \overset{(ii)}{\frac{n!}{s^{n+1}}}$, where n is a positive $\overset{(i)}{\text{real number}}$ or a positive integer. $^{(ii)}$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n\, dt$$

Put $st = x \Rightarrow dt = \frac{dx}{s}$, x varies from 0 to ∞.

Now $\mathcal{L}\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx = \frac{\Gamma(n+1)}{s^{n+1}}$,

where n is a positive real number.

Note : If n is a positive integer, $\Gamma(n+1) = n!$. $\therefore \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

(or)

The above result can also be established without involvement of gamma functions.

$$\mathcal{L}\{t^n\} = \int_0^\infty \underset{II}{e^{-st}}\, \underset{I}{t^n}\, dt = \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (n t^{n-1})\, dt \quad \text{(by parts)}$$

$$= 0 + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1}\, dt$$

$$\therefore \mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$$

Similarly, $\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s} \mathcal{L}\{t^{n-2}\}$; $\mathcal{L}\{t^{n-2}\} = \frac{n-2}{s} \mathcal{L}\{t^{n-3}\}$ etc.

Using all these results, we have

$$\mathcal{L}\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} \mathcal{L}\{t^0\}$$

$$= \frac{n!}{s^n} \mathcal{L}\{1\} = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$= \frac{n!}{s^{n+1}}, \text{ where } n \text{ is a positive integer.}$$

4. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$ where $s>a$ and 'a' is a constant.

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}+e^{-at}\} = \frac{1}{2}\left[\mathcal{L}\{e^{at}\}+\mathcal{L}\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right] = \frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right) = \frac{s}{s^2-a^2}$$

5. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$, where $s>a$ and 'a' is a constant.

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathcal{L}\{e^{at}\}-\mathcal{L}\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right] = \frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$$

6. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ where $s>0$ and 'a' is a constant.

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt.$$

Using $\int e^{at}\cos bt\, dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$, we have

$$\mathcal{L}\{\cos at\} = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty = \frac{1}{s^2+a^2}\left[0 - e^0(-s\cos 0 + a\sin 0)\right]$$

$$= s/_{s^2+a^2} \text{ where } s>0.$$

7. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ where $s>0$.

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\, dt.$$

Using $\int e^{at}\sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$, we have

$$\mathcal{L}\{\sin at\} = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty = -\frac{1}{s^2+a^2}(0-a)$$

$$= \frac{a}{s^2+a^2}, \text{ where } s>0$$

For ready reference, the results obtained above are tabulated.

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1.	a	$\frac{a}{s}$, $s>0$
2.	e^{at}	$\frac{1}{s-a}$, $s>a$
3.	t^n	$\frac{n!}{s^{n+1}}$, $s>0$, n is a +ve integer.
4.	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, $s>0$, n is a +ve real.
5.	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>a$
6.	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>a$
7.	$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
8.	$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
	$e^{at}f(t)$	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
1.	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$, $s>0$, n is a +ve integer
2.	$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
3.	$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
4.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
5.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

→ Find the Laplace transforms of the following

1. $\sin 3t \sin 4t$

Soln WKT $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\therefore \mathcal{L}\{\sin 3t \sin 4t\} = \mathcal{L}\{\frac{1}{2}(\cos t - \cos 7t)\}$

$= \frac{1}{2}[\mathcal{L}\{\cos t\} - \mathcal{L}\{\cos 7t\}]$

$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right] = \frac{24s}{(s^2+1)(s^2+49)}$

2. $\cos^2 4t$

Soln WKT $\cos 2\theta = 2\cos^2\theta - 1.$

$\therefore \mathcal{L}\{\cos^2 4t\} = \mathcal{L}\left\{\frac{1}{2}(1+\cos 8t)\right\} = \frac{1}{2}\left[\mathcal{L}(1) + \mathcal{L}(\cos 8t)\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$

3. $\sin^3 2t$

Soln WKT $\sin 3x = 3\sin x - 4\sin^3 x.$

$\therefore \mathcal{L}\{\sin^3 2t\} = \mathcal{L}\left\{\frac{1}{4}(3\sin 2t - \sin 6t)\right\} = \frac{1}{2^2}\left[3\mathcal{L}\{\sin 2t\} - \mathcal{L}\{\sin 6t\}\right]$

$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+6^2}\right] = \frac{3}{2} \times \frac{32}{(s^2+4)(s^2+36)}$

$= \frac{48}{(s^2+4)(s^2+36)}$

4. $\cos t \cos 2t \cos 3t$

Soln WKT $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$; $\cos(-x) = \cos x.$

$\cos 2t \cos 3t = \frac{1}{2}(\cos 5t + \cos t)$

$\cos t \cos 2t \cos 3t = \frac{1}{2}[\cos t \cos 5t + \cos^2 t] = \frac{1}{2}\left[\left\{\frac{1}{2}(\cos 6t + \cos 4t)\right\} + \left\{\frac{1+\cos 2t}{2}\right\}\right]$

$= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$

$\therefore \mathcal{L}\{\cos t \cos 2t \cos 3t\} = \frac{1}{4}\left[\mathcal{L}\{\cos 6t\} + \mathcal{L}\{\cos 4t\} + \mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}\right]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$

5. $\cosh^2 2t$

Sol: WKT $\cosh 2t = 2\cosh^2 t - 1$

$\therefore \mathcal{L}\{\cosh^2 2t\} = \mathcal{L}\{\frac{1}{2}(1+\cosh 4t)\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$

6. Evaluate (i) $\mathcal{L}(\sqrt{t})$ (ii) $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)$ (iii) $\mathcal{L}\left(t^{-3/2}\right)$

Sol: Note : $\Gamma(n+1) = n\Gamma n$, $\Gamma\frac{1}{2} = \sqrt{\pi}$, $\Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}$, $\Gamma -\frac{1}{2} = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = -2\sqrt{\pi}$

$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

(i) For $n = \frac{1}{2}$, $\mathcal{L}\{\sqrt{t}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

(ii) For $n = -\frac{1}{2}$, $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

(iii) For $n = -\frac{3}{2}$, $\mathcal{L}\left\{t^{-3/2}\right\} = \frac{\Gamma(-\frac{3}{2}+1)}{s^{-\frac{3}{2}+1}} = \frac{\Gamma(-\frac{1}{2})}{s^{-\frac{1}{2}}} = -2\sqrt{\pi s}$

7. Evaluate (i) $\mathcal{L}(t^2)$ (ii) $\mathcal{L}(t^3)$

Sol: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

(i) For $n = 2$ $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

(ii) For $n = 3$ $\mathcal{L}\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

8. $e^{-5t} + 5e^{+2t}$

Soln $\mathcal{L}\{e^{-5t}+5e^{+2t}\} = \mathcal{L}\{e^{-5t}\} + 5\mathcal{L}\{e^{+2t}\} = \frac{1}{s+5} + \frac{5}{s-2}$

* Exercise

Find the Laplace transform of the following functions.

(i) $\sin(3t+4)$

Soln WKT $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$$\therefore \mathcal{L}\{\sin(3t+4)\} = \mathcal{L}\{\sin 3t \cos 4 + \cos 3t \sin 4\}$$

$$= \cos 4\, \mathcal{L}\{\sin 3t\} + \sin 4\, \mathcal{L}\{\cos 3t\}$$

$$= \frac{3\cos 4}{s^2+9} + \sin 4 \cdot \frac{s}{s^2+9}$$

$$= \frac{3\cos 4 + s\sin 4}{s^2+9}$$

(ii) $\cos 2t \sin 3t$

Soln WKT $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$\therefore \mathcal{L}\{\cos 2t \sin 3t\} = \frac{1}{2}\mathcal{L}\{\sin 5t + \sin t\}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

(iii) $\sin t \sin 2t \sin 3t$

Soln WKT $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$$\therefore \sin 2t \sin 3t = -\frac{1}{2}\left[\cos 5t - \cos t\right]$$

$$\sin t \sin 2t \sin 3t = \frac{-1}{2}\left[\cos 5t \sin t - \cos t \sin t\right]$$

$$= \frac{-1}{2}\left[\frac{1}{2}(\sin 6t - \sin 4t) - \frac{1}{2}(\sin 2t)\right]$$

$$= \frac{-1}{4}\left[\sin 6t - \sin 4t - \sin 2t\right]$$

$$\therefore \mathcal{L}\{\sin t \sin 2t \sin 3t\} = \frac{1}{4}\mathcal{L}\{\sin 2t + \sin 4t - \sin 6t\} = \frac{1}{4}\left(\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2+36}\right)$$

(iv) $\cos^3 t$

Soln WKT $\cos 3x = 4\cos^3 x - 3\cos x$

$$\therefore L\{\cos^3 t\} = L\left\{\frac{1}{4}(\cos 3t + 3\cos t)\right\}$$

$$= \frac{1}{4}\left(\frac{s}{s^2+9} + \frac{3s}{s^2+1}\right) = \frac{s(s^2+7)}{(s^2+9)(s^2+1)}$$

(v) $(\sin t - \cos t)^2$

Soln $(\sin t - \cos t)^2 = \sin^2 t + \cos^2 t - 2\sin t\cos t$

$$= 1 - \sin 2t$$

$$\therefore L\{(\sin t - \cos t)^2\} = L\{1 - \sin 2t\}$$

$$= \frac{1}{s} - \frac{2}{s^2+4}$$

→* Find the laplace transforms of the following functions.

1. $e^{-3t}(2\cos 5t - 3\sin 5t)$

Soln consider $f(t) = 2\cos 5t - 3\sin 5t$.

$$L\{f(t)\} = L\{2\cos 5t - 3\sin 5t\}$$

$$= \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25}$$

WKT $L\{e^{at}f(t)\} = F(s-a)$ if $L\{f(t)\} = F(s)$.

$$\therefore L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

2. $\cosh at \sin at$

Soln $\cosh at \sin at = \frac{e^{at}+e^{-at}}{2} \cdot \sin at$

$$L\{\cosh at \sin at\} = \frac{1}{2}\left[L\{e^{at}\sin at\} + L\{e^{-at}\sin at\}\right]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right] = \frac{a(s^2+2a^2)}{[(s-a)^2+a^2][(s+a)^2+a^2]}$$

3. $\cosh t \sin^3 2t$

Sol: $\mathcal{L}\{\cosh t \sin^3 2t\} = \mathcal{L}\left\{\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{3\sin 2t - \sin 6t}{4}\right)\right\}$

$$= \frac{1}{8}\left[3\mathcal{L}(e^t \sin 2t) - \mathcal{L}(e^t \sin 6t) + 3\mathcal{L}(e^{-t}\sin 2t) - \mathcal{L}(e^{-t}\sin 6t)\right]$$

$$= \frac{1}{8}\left[\frac{3\cdot 2}{(s-1)^2+4} - \frac{6}{(s-1)^2+36} + \frac{3\cdot 2}{(s+1)^2+4} - \frac{6}{(s+1)^2+36}\right]$$

$$= \frac{3}{4}\left[\frac{1}{(s-1)^2+4} - \frac{1}{(s-1)^2+36} + \frac{1}{(s+1)^2+4} - \frac{1}{(s+1)^2+36}\right]$$

4. $e^{-4t} t^{-5/2}$

Sol: $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -5/2$, $\mathcal{L}\{t^{-5/2}\} = \frac{\Gamma(-3/2)}{s^{-3/2}} = \frac{4\sqrt{\pi}}{3s^{-3/2}}$

$\therefore \mathcal{L}\{e^{-4t} t^{-5/2}\} = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$

Note: $n\Gamma n = \Gamma n+1$

$-\frac{1}{2}\Gamma -\frac{1}{2} = \Gamma -\frac{1}{2}+1 \Rightarrow \Gamma -\frac{1}{2} = -2\Gamma \frac{1}{2} = -2\sqrt{\pi}$

$-\frac{3}{2}\Gamma -\frac{3}{2} = \Gamma -\frac{3}{2}+1 \Rightarrow \Gamma -\frac{3}{2} = -\frac{2}{3}\cdot\Gamma -\frac{1}{2} = -\frac{2}{3}(-2\sqrt{\pi}) = \frac{4}{3}\pi$

5. a^{kt}

Sol: $a^{kt} = e^{\log a^{kt}} = e^{kt\log a}$

$= e^{(k\log a)t}$

$\therefore \mathcal{L}\{a^{kt}\} = \mathcal{L}\{e^{(k\log a)t}\}$

$= \frac{1}{s - k\log a}$

6. $e^{-2t}\cos^2 2t$.

Soln $L\{e^{-2t}\cos^2 2t\} = L\left\{e^{-2t}\left(\frac{1+\cos 4t}{2}\right)\right\}$

$$= \frac{1}{2}\left[L\{e^{-2t}\} + L\{e^{-2t}\cos 4t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{S+2} + \frac{S+2}{(S+2)^2+16}\right] = \frac{(S+2)^2+16+(S+2)^2}{2(S+2)\left[(S+2)^2+16\right]}$$

$$= \frac{(S+2)^2+8}{(S+2)\left[(S+2)^2+16\right]}$$

7. $e^{2t}\sin 3t\cos 2t$

Soln $L\{e^{2t}\sin 3t\cos 2t\} = L\left\{\frac{1}{2}.e^{2t}(\sin 5t + \sin t)\right\}$

$$= \frac{1}{2}\left[L\{e^{2t}\sin 5t\} + L\{e^{2t}\sin t\}\right]$$

$$= \frac{1}{2}\left[\frac{5}{(S-2)^2+25} + \frac{1}{(S-2)^2+1}\right]$$

$$= \frac{3(S-2)^2+15}{\left[(S-2)^2+25\right]\left[(S-2)^2+1\right]}$$

8. $(e^t+1)^2\cos t$

Soln $L\{(e^t+1)^2\cos t\} = L\{(e^{2t}+1+2e^t)\cos t\}$

$$= L\{e^{2t}\cos t\} + L\{\cos t\} + 2L\{e^t\cos t\}$$

$$= \frac{S-2}{(S-2)^2+1} + \frac{S}{S^2+1} + \frac{2(S-1)}{(S-1)^2+1}$$

* Note :

1. If $L\{f(t)\} = F(S)$, (i) $L\{(\cosh at)f(t)\} = \frac{1}{2}\left[F(S-a)+F(S+a)\right]$

(ii) $L\{(\sinh at)f(t)\} = \frac{1}{2}\left[F(S-a)-F(S+a)\right]$

2. $e^{it} = \cos t + i\sin t$

(i) $L\{t\sin at\} = \frac{2aS}{(S^2+a^2)^2}$; (ii) $L\{t\cos at\} = \frac{S^2-a^2}{(S^2+a^2)^2}$

* Properties of Laplace transforms continued

3. Change of scale property : If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof : By definition $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt$$

Put $at = u$ so that $dt = \frac{1}{a}du$; $t=0 \Rightarrow u=0$, $t=\infty \Rightarrow u=\infty$.

$$\therefore L\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-\frac{su}{a}} f(u)\,du = \frac{1}{a}F\left(\frac{s}{a}\right)$$

4. Second shifting or translation property.

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$, then $L\{g(t)\} = e^{-as}F(s)$

Proof :

By definition, $L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$

$$= \int_0^a e^{-st}.0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = u$ so that $dt = du$.

When $t = a$, $u = 0$ and when $t = \infty$, $u = \infty$.

$$L\{g(t)\} = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du.$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-as}F(s)$$

Example : Find $\mathcal{L}\{\cos 5t\}$ using change of scale property.

Soln: We have $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

Applying change of scale property, we get $\mathcal{L}\{\cos 5t\} = \frac{1}{5} F\left(\frac{s}{5}\right)$

$$\therefore \mathcal{L}\{\cos 5t\} = \frac{1}{5} \frac{s/5}{(s/5)^2+1} = \frac{s}{s^2+25}$$

* Find the Laplace transform of the following functions

1. $f(t) = \begin{cases} t, & 0<t<3 \\ 4, & t>3 \end{cases}$

Soln
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st}\, t\,dt + \int_3^\infty e^{-st}.4\,dt$$

$$= \left[t.\frac{e^{-st}}{-s}\right]_0^3 - \int_0^3 1.\frac{e^{-st}}{-s}\,dt + \left[4\,\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= -3\frac{e^{-3s}}{s} - \left[\frac{e^{-st}}{s^2}\right]_0^3 + 4\frac{e^{-3s}}{s}$$

$$= \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} = \frac{e^{-3s}}{s} + \frac{1}{s^2}(1-e^{-3s})$$

2. $f(t) = \begin{cases} \sin 2t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

Soln
$$\mathcal{L}\{f(t)\} = \int_0^\pi e^{-st} f(t)\,dt + \int_\pi^\infty e^{-st} f(t)\,dt$$

Let
$$I = \int_0^\pi \underset{\text{II}}{e^{-st}}\, \underset{\text{I}}{\sin 2t}\,dt + 0.$$

$$= \left[\sin 2t\,\frac{e^{-st}}{-s}\right]_0^\pi - \int_0^\pi 2\cos 2t\,\frac{e^{-st}}{-s}\,dt$$

$$I = \frac{2}{s}\left[\cos 2t\,\frac{e^{-st}}{-s}\Big|_0^\pi - \int_0^\pi -2\sin 2t.\,\frac{e^{-st}}{-s}\,dt\right]$$

$$I = \frac{2}{s^2}\left[-e^{-s\pi}+1\right] - \frac{4}{s^2} I \Rightarrow I\left(1+\frac{4}{s^2}\right) = \frac{2}{s^2}(1-e^{-\pi s})$$

$$\therefore I = \frac{2}{s^2+4}(1-e^{-\pi s})$$

3. $f(t) = \begin{cases} e^t, & 0<t<5 \\ 3, & t>5 \end{cases}$

Sol: $\mathcal{L}\{f(t)\} = \int_0^5 e^{-st} f(t)\,dt + \int_5^\infty e^{-st} f(t)\,dt$

$$= \int_0^5 e^{-st} e^t\,dt + \int_5^\infty e^{-st}.3\,dt = \int_0^5 e^{(1-s)t}dt + 3\int_0^5 e^{-st}dt$$

$$= \left[\frac{e^{(1-s)t}}{(1-s)}\right]_0^5 + \left[3\frac{e^{-st}}{-s}\right]_5^\infty$$

$$= \frac{e^{5(1-s)}-1}{1-s} + \frac{3}{s}e^{-5s}$$

* Laplace transform of derivatives

1. Differentiation in the time domain:

If $f(t)$ is a continuous function and $\mathcal{L}\{f(t)\} = F(s)$, then

(i) $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ (ii) $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

Proof: (i) By definition, $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t) e^{-st}(-s)\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= sF(s) - f(0)$$

(ii) $\mathcal{L}\{f''(t)\} = \mathcal{L}\{f'(t)\}' = s\mathcal{L}\{f'(t)\} - f'(0)$

$$= s[sF(s) - f(0)] - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Extension : $\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots - f^{n-1}(0)$

provided $f(t)$ and all its derivatives upto order $(n-1)$ are of exponential order and that $f^n(t)$ is continuous.

2. Differentiation in the s-domain.

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\{F(s)\} = -F'(s)$

Proof: By definition,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds}\{e^{-st} f(t)\}\,dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\,dt = -\int_0^\infty e^{-st}\{t f(t)\}\,dt$$

$$= -\mathcal{L}\{t f(t)\} = (-1)^1 \mathcal{L}\{t f(t)\}$$

$$\therefore \mathcal{L}\{t f(t)\} = (-1)^1 F'(s)$$

Similarly, $\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s) = (-1)^2 F''(s)$

Extension: In general $\mathcal{L}\{t^n f(t)\} = (-1)^n F^n(s) = (-1)^n \frac{d^n}{ds^n} F(s)$,

where n is a positive integer.

Proof:

$$F^n(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d^n}{ds^n}\{e^{-st} f(t)\}\,dt$$

$$= \int_0^\infty (-t)^n e^{-st} f(t)\,dt = (-1)^n \int_0^\infty e^{-st}\{t^n f(t)\}\,dt$$

$$= (-1)^n \mathcal{L}\{t^n f(t)\}$$

$$\therefore \mathcal{L}\{t^n f(t)\} = (-1)^n F^n(s)$$

Also, $\mathcal{L}^{-1}\left[F^{(n)}(s)\right] = (-1)^n t^n f(t)$.

Note: Differentiation in s-domain corresponds to multiplication by t in the time domain.

* Examples:

1. By using the Laplace transform of sin(at), find the Laplace transform of cosat.

Soln Let $f(t) = \sin at$. Then, $\mathcal{L}\{f(t)\} = \frac{a}{s^2+a^2} = F(s)$

$$f'(t) = a\cos at.$$

Taking Laplace transforms, $\mathcal{L}\{f'(t)\} = \mathcal{L}\{a\cos at\} = a\,\mathcal{L}\{\cos at\}$.

$$\Rightarrow \mathcal{L}\{\cos at\} = \frac{1}{a}\mathcal{L}\{a\cos at\} = \frac{1}{a}\mathcal{L}\{f'(t)\}$$

$$= \frac{1}{a}\left[sF(s) - f(0)\right]$$

$$= \frac{1}{a}\left[\frac{sa}{s^2+a^2} - 0\right] = \frac{s}{s^2+a^2}$$

2. Given $\mathcal{L}\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, show that $\mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Soln Let $f(t) = 2\sqrt{\frac{t}{\pi}}$, given $\mathcal{L}\{f(t)\} = \frac{1}{s^{3/2}} = F(s)$

Note that $f'(t) = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$

$$\therefore \mathcal{L}\{f'(t)\} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = sF(s) - f(0)$$

$$= s.\frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$$

3. Find $\mathcal{L}(t^2 \sin 3t)$

Soln $\mathcal{L}(\sin 3t) = \frac{3}{s^2+9} = F(s)$

$$\mathcal{L}(t^2 f(t)) = (-1)^2 F''(s) = F''(s)$$

$$F'(s) = \frac{-3.2s}{(s^2+9)^2} = \frac{-6s}{(s^2+9)^2}$$

$$F''(s) = -6\left[\frac{(s^2+9)^2.1 - 2s(s^2+9).2s}{(s^2+9)^4}\right] = -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$\mathcal{L}(t^2 f(t)) = F''(s) = \frac{18(s^2-3)}{(s^2+9)^3}$$

4. Find $\mathcal{L}[t e^{-t} \sin 4t]$

Sol: $\mathcal{L}[\sin 4t] = \frac{4}{s^2+16} = F(s)$

$\mathcal{L}[t \sin 4t] = -F'(s) = \frac{+8s}{(s^2+16)^2}$

$\therefore \mathcal{L}[te^{-t}\sin 4t] = \frac{8(s+1)}{[(s+1)^2+16]^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$ (s-shift property)

5. Prove that $\int_0^\infty e^{-3t}.t.\sin t\,dt = \frac{3}{50}$

Sol: $\int_0^\infty e^{-st}.t \sin t\,dt = \mathcal{L}[t \sin t] = -\frac{d}{ds}\mathcal{L}(\sin t)$

$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{+2s}{(s^2+1)^2}$

Put $s = 3$, $\int_0^\infty e^{-3t}.t \sin t\,dt = \frac{6}{100} = \frac{3}{50}$

6. Find $\mathcal{L}[t \sin 3t \cos 2t]$

Sol: $\mathcal{L}[\sin 3t \cos 2t] = \frac{1}{2}\mathcal{L}[\sin 5t + \sin t] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] = F(s)$

$\mathcal{L}[t \sin 3t \cos 2t] = -F'(s) = \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$

7. Find $\mathcal{L}[t^2 e^{-t} \cos t]$

Sol: $\mathcal{L}[\cos t] = \frac{s}{s^2+1} = F(s)$

$\mathcal{L}[t^2 \cos t] = F''(s)$

$F'(s) = \frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$

$F''(s) = \frac{(s^2+1)^2(-2s) - (1-s^2).2(s^2+1).2s}{(s^2+1)^4} = \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}$

$= \frac{2s(s^2-3)}{(s^2+1)^3}$

$$L[t^2\cos t] = \frac{2s(s^2-3)}{(s^2+1)^3}$$

$$L\left[t^2 e^{-t}\cos t\right] = \frac{2(s+1)\left[(s+1)^2-3\right]}{\left[(s+1)^2+1\right]^3} = \frac{2(s+1)(s^2+2s-2)}{(s^2+2s+2)^3}$$

8. Evaluate the following integrals using Laplace transforms :

(i) $\int_0^\infty t e^{-2t} \sin 3t \, dt$

Sol

$$\int_0^\infty e^{-st} t \sin 3t \, dt = L\{t \sin 3t\}$$

$$L\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$$

$$L\{t \sin 3t\} = -F'(s) = \frac{6s}{(s^2+9)^2}$$

Put $s = 2$ to obtain $\int_0^\infty e^{-2t} . t \sin 3t = \frac{6(2)}{(4+9)^2} = \frac{12}{169}$

(ii) $\int_0^\infty e^{3t} t^3 \sin t \, dt$

Sol

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = L\{t^3 \sin t\}$$

$$L\{\sin t\} = \frac{1}{s^2+1} = F(s)$$

$$L\{t^3 \sin t\} = (-1)^3 F'''(s) = -F'''(s)$$

$$F'(s) = \frac{-2s}{(s^2+1)^2}$$

$$F''(s) = \frac{(s^2+1)^2(-2) - (-2s).2(s^2+1).(2s)}{(s^2+1)^4} = \frac{-2s^2-2+8s^2}{(s^2+1)^3} = \frac{6s^2-2}{(s^2+1)^3}$$

$$F'''(s) = \frac{(s^2+1)^3(12s) - (6s^2-2).3(s^2+1)^2.2s}{(s^2+1)^6} = \frac{12s^3+12s-36s^3+12s}{(s^2+1)^4}$$

$$= \frac{-24s^3+24s}{(s^2+1)^4}$$

$$\therefore L\{t^3 \sin t\} = (24s^3 - 24s)/(s^2+1)^4$$

Put $s=-3$, $\int_0^\infty e^{3t} t^3 \sin t \, dt = \dfrac{24(-3)^3-24(-3)}{[(-3)^2+1]^4} = \dfrac{-576}{10^4}$

(iii) $\int_0^\infty e^{3t} t^3 \cos t \, dt$

Sol: $\int_0^\infty e^{-st} t^3 \cos t \, dt = \mathcal{L}\{t^3 \cos t\}$

$\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1} = F(s)$

$\mathcal{L}\{t^3 \cos t\} = -F'''(s)$

$F'(s) = \dfrac{(s^2+1)(1)-s(2s)}{(s^2+1)^2} = \dfrac{1-s^2}{(s^2+1)^2}$

$F''(s) = \dfrac{(s^2+1)^2(-2s)-(1-s^2).2(s^2+1)2s}{(s^2+1)^4}$

$= \dfrac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} = \dfrac{2s^3-6s}{(s^2+1)^3}$

$F'''(s) = \dfrac{(s^2+1)^3(6s^2-6)-(2s^3-6s).3(s^2+1)^2.2s}{(s^2+1)^6}$

$= \dfrac{6s^4-6s^2+6s^2-6-12s^4+36s^2}{(s^2+1)^4}$

$= \dfrac{-6s^4+36s^2-6}{(s^2+1)^4}$

$\therefore \mathcal{L}\{t^3 \cos t\} = \dfrac{6s^4-36s^2+6}{(s^2+1)^4}$

Put $s=-3$,

$\int_0^\infty e^{3t} t^3 \cos t \, dt = \dfrac{6(-3)^4-36(-3)^2+6}{[(-3)^2+1]^4}$

$= \dfrac{168}{10^4}$

* Laplace transform of integrals.

1. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$. (Provided the integral exists).

Proof: By definition of Laplace transform, we have

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Integrating both sides w.r.t s from s to ∞, we get

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left\{ \int_0^\infty e^{-st} f(t)\,dt \right\} ds$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\,ds \right] dt$$

$$= \int_0^\infty f(t) \cdot \left[\frac{e^{-st}}{-t} \right]_s^\infty dt = \int_0^\infty f(t) \frac{e^{-st}}{t}\,dt$$

$$= \int_0^\infty e^{-st} \left\{ \frac{f(t)}{t} \right\} dt = \mathcal{L}\left\{ \frac{f(t)}{t} \right\}$$

Thus, $\mathcal{L}\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(s)\,ds$

2. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{ \int_0^t f(t)\,dt \right\} = \frac{F(s)}{s}$

Proof: Let $\phi(t) = \int_0^t f(t)\,dt$. Then, $\phi'(t) = f(t)$ and $\phi(0) = 0$.

Now, $$\mathcal{L}[\phi(t)] = \int_0^\infty \underset{\mathrm{II}}{e^{-st}}\, \underset{\mathrm{I}}{\phi(t)}\,dt$$

$$= \left[\phi(t) \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \phi'(t) \cdot \frac{e^{-st}}{-s}\,dt.$$

$$= (0 - 0) + \frac{1}{s} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s} F(s)$$

Examples

1. Find $\mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\}$

Soln $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s)$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}s\right]_s^\infty$$

$$= \pi/2 - \tan^{-1}s = \cot^{-1}s$$

$$\therefore \mathcal{L}\left\{\frac{e^{-t}\sin t}{t}\right\} = \cot^{-1}(s+1) \qquad \because \text{ s-shifting property}$$

2. Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$ and hence evaluate $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$

Soln $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s)$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^\infty$$

$$= \pi/2 - \tan^{-1}s = \cot^{-1}s$$

Now, let $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s = F(s)$

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \mathcal{L}\left\{a\frac{\sin at}{at}\right\} = a\,\mathcal{L}\left\{\frac{\sin at}{at}\right\} = a\,\mathcal{L}\{f(at)\}$$

$$= a\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right] \quad \text{(Scaling)}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

3. Find $\mathcal{L}\left[\frac{1-e^{-2t}}{t}\right]$

$\lim_{s\to\infty} \ln\left(\frac{s}{s+2}\right) = 0$
$\because \lim_{s\to\infty} \ln\left(\frac{s}{s(1+\frac{2}{s})}\right) = \ln 1 = 0$

Soln $\mathcal{L}[1-e^{-2t}] = \frac{1}{s} - \frac{1}{s+2}$

$$\therefore \mathcal{L}\left[\frac{1-e^{-2t}}{t}\right] = \int_s^\infty\left(\frac{1}{s} - \frac{1}{s+2}\right)ds = \left[\ln(s) - \ln(s+2)\right]_s^\infty = \ln\left(\frac{s+2}{s}\right)$$

4. Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

Soln $L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = F(s)$

$$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\ln(s^2+a^2) - \ln(s^2+b^2)\right]_s^\infty \qquad \because \int \frac{f'(x)}{f(x)} = \ln|f(x)|$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \ln\left(\frac{s^2+a^2}{s^2+b^2}\right) - \ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \qquad \left[\because = \ln\left(\lim_{s\to\infty}\frac{s^2+a^2}{s^2+b^2}\right) = \ln\left(\lim_{s\to\infty}\frac{2s}{2s}\right) = \ln(1) = 0\right]$$

$$= \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

5. Find $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

Soln $L[e^{-at} - e^{-bt}] = \frac{1}{(s+a)} - \frac{1}{(s+b)}$

$$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \lim_{s\to\infty}\log\left(\frac{s+a}{s+b}\right) - \log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{s\to\infty}\log\left[\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right] + \log\left(\frac{s+b}{s+a}\right)$$

$$= \log 1 + \log\left(\frac{s+b}{s+a}\right) = \log\left(\frac{s+b}{s+a}\right)$$

6. Find $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\}$

Sol: $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$$\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \ln \frac{s}{\sqrt{s^2+4}} - \ln \frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \ln \frac{\cancel{s}}{\cancel{s}\sqrt{1+\frac{4}{s^2}}} + \ln\left(\frac{\sqrt{s^2+4}}{s}\right)\right.$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{s^2+4}}{s}\right)$$

7. Evaluate $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt$

Sol: $= \int_0^\infty e^{-3t}\left(\frac{1-e^{-3t}}{t}\right) dt = \mathcal{L}\left\{\frac{1-e^{-3t}}{t}\right\}$ where $s=3$

$$\int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}(1-e^{-3t}) = \frac{1}{s} - \frac{1}{s+3}$$

$$\mathcal{L}\left\{\frac{1-e^{-3t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+3}\right) ds = \left[\ln s - \ln(s+3)\right]_s^\infty$$

$$= \lim_{s\to\infty} \ln\left(\frac{s}{s+3}\right) - \ln\left(\frac{s}{s+3}\right) = 0 - \ln\left(\frac{s}{s+3}\right)$$

$$= \ln\left(\frac{s+3}{s}\right)$$

Put $s = 3$ to obtain $\int_0^\infty e^{-3t}\left(\frac{1-e^{-3t}}{t}\right) dt = \ln\left(\frac{6}{3}\right) = \ln 2.$

$$\therefore \int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt = \ln 2$$

8. Evaluate $\int_0^\infty e^{-t} \frac{\sin\sqrt{3}t}{t}\, dt$

Sol

$\int_0^\infty e^{-st} \frac{\sin\sqrt{3}t}{t}\, dt = L\left\{\frac{\sin\sqrt{3}t}{t}\right\}$

$L\{\sin\sqrt{3}t\} = \frac{\sqrt{3}}{s^2+(\sqrt{3})^2}$

$L\left\{\frac{\sin\sqrt{3}t}{t}\right\} = \int_s^\infty \frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\, ds = \cancel{\sqrt{3}}\cdot\left[\frac{1}{\cancel{\sqrt{3}}}\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{\sqrt{3}}\right)$

$= \pi/2 - \tan^{-1}\left(\frac{s}{\sqrt{3}}\right) = \cot^{-1}\left(\frac{s}{\sqrt{3}}\right)$

Put $s=1$ to obtain $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}\, dt = \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/3$

9. Find $L\left[\int_0^t \frac{\cos at - \cos bt}{t}\, dt\right]$

Sol

$L\{f(t)\} = L\left[\frac{\cos at - \cos bt}{t}\right]$

$L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds = \frac{1}{2}\left[\ln(s^2+a^2) - \ln(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

WKT, $L\left[\int_0^t f(t)\, dt\right] = \frac{1}{s} L\{f(t)\}$

$\therefore L\left[\int_0^t \frac{\cos at - \cos bt}{t}\, dt\right] = \frac{1}{2s}\ln\left(\frac{s^2+b^2}{s^2+a^2}\right)$

10. Find $L\{\int_0^t te^{-t}\sin 4t\, dt\}$

Sol $L\{f(t)\} = L\{te^{-t}\sin 4t\}$

$L\{\sin 4t\} = \frac{4}{s^2+16}$

$$\mathcal{L}\{t\sin 4t\} = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{-4\cdot(-2s)}{(s^2+16)^2} = \frac{8s}{(s^2+16)^2}$$

$$\mathcal{L}\{e^{-t}\,t\sin 4t\} = \frac{8(s+1)}{[(s+1)^2+16]^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$$

$$\therefore \mathcal{L}\left\{\int_0^t e^{-t}\,t\sin 4t\,dt\right\} = \frac{8(s+1)}{s(s^2+2s+17)^2}$$

11. Find $\mathcal{L}\left\{\int_0^t e^t \frac{\sin t}{t}\,dt\right\}$

Soln $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}(s)\right]_s^\infty$$

$$= \pi/2 - \tan^{-1}(s) = \cot^{-1}(s)$$

$$\mathcal{L}\left\{e^t\frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$$

$$\therefore \mathcal{L}\left\{\int_0^t e^t \frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1}(s-1)$$

12. $\mathcal{L}\left\{\int_0^t t^2\sin at\,dt\right\}$

Soln $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

$$\mathcal{L}\{t^2\sin at\} = \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\} = \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} = \frac{(s^2+a^2)^2(-2a)+(2as)\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{-2as^2-2a^3+8as^2}{(s^2+a^2)^3} = \frac{6as^2-2a^3}{(s^2+a^2)^3}$$

$$\therefore \mathcal{L}\left\{\int_0^t t^2\sin at\,dt\right\} = \frac{2a(3s^2-a^2)}{s(s^2+a^2)^3}$$

13. Find $\mathcal{L}\left\{\int_0^t e^t \cosh t \, dt\right\}$

Sol: $\mathcal{L}\{\cosh t\} = \mathcal{L}\left\{\frac{e^t+e^{-t}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^t+e^{-t}\}$

$$= \frac{1}{2}\left[\frac{1}{S-1}+\frac{1}{S+1}\right] = \frac{1}{2}\cdot\frac{2S}{S^2-1} = \frac{S}{S^2-1}$$

$$\mathcal{L}\{e^t\cosh t\} = \frac{S-1}{(S-1)^2-1} = \frac{S-1}{S^2-2S}$$

$$\therefore \mathcal{L}\left\{\int_0^t e^t\cosh t\, dt\right\} = \frac{S-1}{S^2(S-2)}$$

14. Find $\mathcal{L}\left\{\int_0^t (e^{-t}\sin 4t + t\cos 2t)\,dt\right\}$

Sol: $\mathcal{L}\{\sin 4t\} = \frac{4}{S^2+16}$

$$\mathcal{L}\{e^{-t}\sin 4t\} = \frac{4}{(S+1)^2+16} = \frac{4}{S^2+2S+17}$$

$$\mathcal{L}\{\cos 2t\} = \frac{S}{S^2+4}$$

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{dS}\left(\frac{S}{S^2+4}\right) = -\frac{(S^2+4)(1)-S(2S)}{(S^2+4)^2}$$

$$= \frac{S^2-4}{(S^2+4)^2}$$

$$\therefore \mathcal{L}\{e^{-t}\sin 4t + t\cos 2t\} = \frac{4}{S^2+2S+17} + \frac{S^2-4}{(S^2+4)^2}$$

Hence $\mathcal{L}\left\{\int_0^t (e^{-t}\sin 4t + t\cos 2t)\,dt\right\} = \frac{4}{S(S^2+2S+17)} + \frac{S^2-4}{S(S^2+4)^2}$

* **Periodic function**

A function $f(t)$ is said to be a periodic function of period $T>0$ if $f(t+nT) = f(t)$, where $n = 1, 2, 3, \ldots$ The graph of the periodic function repeats itself in equal intervals.

For example, $\sin t$ and $\cos t$ are periodic functions of period 2π because $\sin(t+2n\pi) = \sin t$, $\cos(t+2n\pi) = \cos t$.

* **Laplace transform of periodic functions**

If $f(t)$ is a periodic function of period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Proof:

By the definition, we have $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-su} f(u)\,du$

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-su} f(u)\,du + \int_T^{2T} e^{-su} f(u)\,du + \ldots + \int_{nT}^{(n+1)T} e^{-su} f(u)\,du + \ldots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u)\,du \quad \rightarrow (1)$$

Now, put $u = t+nT$ $\therefore du = dt$

If $u = nT$, then $t = 0$. If $u = (n+1)T$, then $t = T$.

Further $f(u) = f(t+nT) = f(t)$ by the periodic property.

Using these, (1) becomes $\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \int_0^T e^{-s(t+nT)} f(t)\,dt$

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} e^{-snT} \int_{t=0}^T e^{-st} f(t)\,dt \quad \rightarrow (2)$$

But, $\sum_{n=0}^{\infty} e^{-snT} = \sum_{n=0}^{\infty} (e^{-sT})^n = 1 + e^{-sT} + (e^{-sT})^2 + \ldots = \frac{1}{1-e^{-sT}} \rightarrow (3)$

(Geometric Series) (Sum to infinity).

Using (3) in (2), we obtain $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt.$

1. For the periodic function $f(t)$ of period 4 defined by

$$f(t) = \begin{cases} 3t, & 0<t<2 \\ 6, & 2<t<4 \end{cases}, \text{ find } L\{f(t)\}.$$

Soln Here, period of $f(t) = T = 4$.

By definition, $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 \underset{I}{3t}\, \underset{II}{e^{-st}}\, dt + \int_2^4 6e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[3\left\{ \left(t\, \frac{e^{-st}}{-s}\right)_0^2 - \int_0^2 \frac{e^{-st}}{-s} dt \right\} + 6\left(\frac{e^{-st}}{-s}\right)_2^4 \right]$$

$$= \frac{1}{1-e^{-4s}} \left[-6\,\cancel{\frac{e^{-2s}}{s}} - 3\left(\frac{e^{-st}}{(-s)^2}\right)_0^2 - 6\frac{e^{-4s}}{s} + \cancel{\frac{6e^{-2s}}{s}} \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{-3e^{-2s}}{s^2} + \frac{1.3}{s^2} - \frac{6e^{-4s}}{s} \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{3\left(1-e^{-2s} - 2se^{-4s}\right)}{s^2} \right]$$

2. A periodic function of period $\frac{2\pi}{\omega}$ is defined by

$$f(t) = \begin{cases} E \sin \omega t, & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t \le \frac{2\pi}{\omega} \end{cases}$$

, where E and ω are positive constants.

Show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$.

Soln Here, $T = \frac{2\pi}{\omega}$

By definition, $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} E.e^{-st} \sin \omega t\, dt \longrightarrow (1)$$

Let $I = \int_0^{\pi/\omega} e^{-st} \sin\omega t \, dt$

$$= \left[\sin\omega t \ \frac{e^{-st}}{-s}\right]_0^{\pi/\omega} - \int_0^{\pi/\omega} \omega\cos\omega t \cdot \frac{e^{-st}}{-s} \, dt$$

$$I = \frac{\omega}{s}\left[\left(\cos\omega t \cdot \frac{e^{-st}}{-s}\right)_0^{\pi/\omega} - \int_0^{\pi/\omega} -\omega\sin\omega t \cdot \frac{e^{-st}}{-s} dt\right]$$

$$I = \frac{\omega}{s}\left[\frac{e^{-\pi s/\omega}}{s} + \frac{1}{s}\right] - \frac{\omega^2}{s^2} I$$

$$I\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2}\left(1 + e^{-\pi s/\omega}\right)$$

$$I = \frac{\omega}{\omega^2 + s^2}\left(1 + e^{-\pi s/\omega}\right)$$

Substituting in (1), we get

$$L\{f(t)\} = \frac{E}{1 - e^{-2\pi s/\omega}} \cdot \frac{\omega}{(\omega^2 + s^2)}\left(1 + e^{-\pi s/\omega}\right)$$

$$= \frac{E}{(1 + e^{-\pi s/\omega})(1 - e^{-\pi s/\omega})} \cdot \frac{\omega}{(\omega^2 + s^2)}\left(1 + e^{-\pi s/\omega}\right)$$

$$= \frac{E\omega}{(1 - e^{-\pi s/\omega})(\omega^2 + s^2)}$$

3. A periodic function $f(t)$ of period $2a$, $a > 0$ is defined by

$$f(t) = \begin{cases} E, & 0 < t \le a \\ -E, & a < t \le 2a \end{cases}.$$

Show that $L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$

Soln Here, $T = 2a$.

By definition, $L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) \, dt.$

$$= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) \, dt$$

$$= \frac{1}{1 - e^{-2as}}\left[\int_0^a E e^{-st} dt + \int_a^{2a} -E e^{-st} dt\right]$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}}\left[\left(E\frac{e^{-st}}{-s}\right)_0^a - \left(E\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{e^{-as}}{-s} + \frac{1}{s} + \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$

$$= \frac{E\left(1-2e^{-as}+e^{-2as}\right)}{s(1+e^{-as})(1-e^{-as})} = \frac{E\left(1-e^{-as}\right)^2}{s(1+e^{-as})(1-e^{-as})}$$

$$= \frac{E(1-e^{-as})}{s(1+e^{-as})} \quad \times \text{ly} \div \text{by } e^{as/2}$$

$$= \frac{E\left(e^{as/2}-e^{-as/2}\right)}{s\left(e^{as/2}+e^{-as/2}\right)} = \frac{E\, 2\sinh\left(\frac{as}{2}\right)}{s\, 2\cosh\left(\frac{as}{2}\right)}$$

$$= \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

4. $f(t) = t^2$, $0<t<2$ and $f(t+2) = f(t)$ for $t>2$. Find $\mathcal{L}\{f(t)\}$

Soln Here, $T = 2$.

By definition, $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-2s}}\int_0^2 \underset{II}{e^{-st}}\, \underset{I}{t^2}\, dt$$

Applying Bernoulli's rule of integration by parts,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - 2t\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right]$$

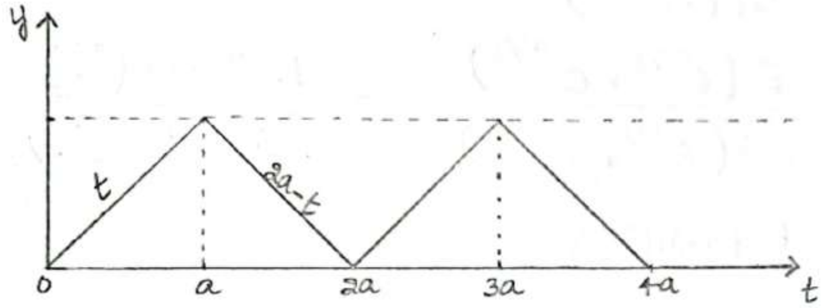
$$= \frac{2}{s^3(1-e^{-2s})}\left(1 - e^{-2s} - 2s^2e^{-2s} - 2se^{-2s}\right)$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[1 - e^{-2s}(1+2s+2s^2)\right]$$

5. Find $\mathcal{L}\{f(t)\}$ of periodic function, given $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t & a < t \le 2a \end{cases}$,

where $f(2a+t) = f(t)$. Also sketch the graph of $f(t)$ as a periodic function.

Sol. Let $f(t) = y$. Now $y = t$ is a straight line passing through origin making an angle 45° with the t-axis. $y = 2a - t$ or $y + t = 2a$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$. The graph is as follows :



This is known as triangular wave function.

Here, $T = 2a$.

By definition $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st} dt + \int_a^{2a} (2a-t) e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\left\{ t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{(-s)^2} \right\}_0^a + \left\{ (2a-t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{(-s)^2} \right\}_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\cancel{-\frac{a e^{-as}}{s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{a e^{-as}}{s}} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{(1 - 2e^{-as} + e^{-2as})}{s^2 (1 - e^{-2as})} = \frac{(1-e^{-as})^2}{s^2 (1-e^{-as})(1+e^{-as})}$$

$$= \frac{(1 - e^{-as})}{s^2 (1 + e^{-as})} \quad \times\ \xi_1 \div\ e^{as/2}$$

$$= \frac{(e^{as/2} - e^{-as/2})}{s^2 (e^{as/2} + e^{-as/2})} = \frac{2 \sinh(as/2)}{s^2 . 2\cosh(as/2)} = \frac{1}{s^2} \tanh(as/2)$$

6. Show that the Laplace transform of the periodic function defined by $f(t) = \frac{Kt}{T}$; $0<t<T$; $f(t+T) = f(t)$ is $\frac{-Ke^{-ST}}{S(1-e^{-ST})} + \frac{K}{S^2T}$

Soln By definition, $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ST}} \int_0^T f(t)\, e^{-st}\, dt$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-ST}} \int_0^T \frac{Kt}{T} e^{-St} dt$$

$$= \frac{K}{T(1-e^{-ST})} \left[t \frac{e^{-St}}{-S} - (1)\frac{e^{-St}}{S^2} \right]_0^T$$

$$= \frac{K}{T(1-e^{-ST})} \left[T\frac{e^{-ST}}{-S} - \frac{e^{-ST}}{S^2} + \frac{1}{S^2} \right]$$

$$= \frac{-Ke^{-ST}}{S(1-e^{-ST})} + \frac{K(1-e^{-ST})}{TS^2(1-e^{-ST})}$$

$$= \frac{-Ke^{-ST}}{S(1-e^{-ST})} + \frac{K}{S^2T}$$

* Heaviside function (or unit step function)

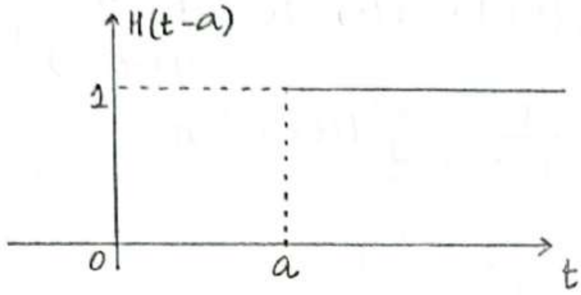
In many Engineering applications, we deal with the important discontinuous function $H(t-a)$ or $U(t-a)$ defined as follows :

$$H(t-a) \text{ or } U(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t>a \end{cases}$$, where 'a' is a non-negative constant.

This function is known as the Heaviside function or the unit step function named after the British electrical engineer Oliver Heaviside (1850 - 1925).

He was an English self-taught electrical engineer, physicist and mathematician who adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations (equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated Vector analysis.

The graph of the function is shown below :



From the graph of $H(t-a)$, the value of this function suddenly jumps (steps up) from the value zero to the value 1 as $t \to a$ from the left and retains the value 1 for all $t > a$. This is why $H(t-a)$ is called the unit step function.

In particular when $a = 0$, we get

$$H(t) \text{ or } u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

* Laplace transform of unit step function :

$$L\{H(t-a)\} = L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Proof: $L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} \cdot 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

Note : If $a = 0$, $L\{u(t)\} = \frac{1}{s}$

* Properties associated with the unit step function.

1. Heaviside shift property (or theorem) / t-shifting property.

$$L[f(t-a)\, u(t-a)] = e^{-as} F(s) \quad \text{where } L\{f(t)\} = F(s).$$

Proof: We have $L\{f(t-a) u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt$

$$= \int_0^a e^{-st} f(t-a).0\, dt + \int_a^\infty e^{-st} f(t-a).1\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt$$

Put $t-a = v$

$\therefore dt = dv$. If $t=a$, $v=0$ and if $t=\infty$, $v=\infty$.

Hence, $\mathcal{L}\{f(t-a)u(t-a)\} = \int_{v=0}^{\infty} e^{-s(a+v)} f(v)\, dv$

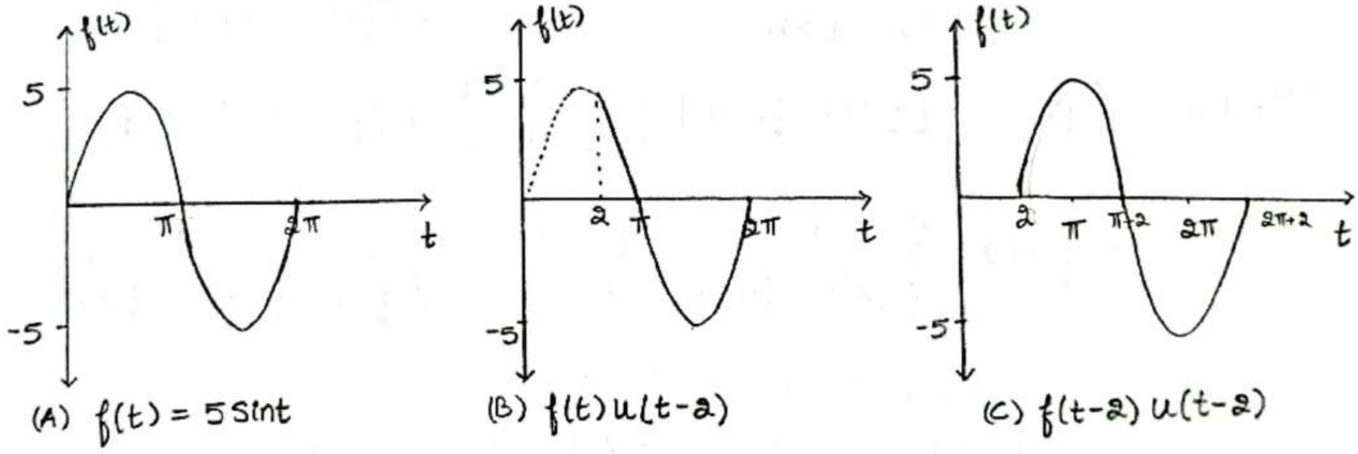
$$= e^{-as}\int_0^\infty e^{-sv} f(v)\, dv$$

$$= e^{-as} F(s)$$

Also $\mathcal{L}^{-1}\left[e^{-as}\mathcal{L}\{f(t)\}\right] = f(t-a)\,u(t-a)$ (or) $f(t-a)H(t-a)$

The unit step function is a common tool used in engineering to model scenarios where systems (such as mechanical or electrical driving forces) are either in an 'off' or 'on' state. The functions $f(t)$ multiplied with $u(t-a)$ represent signals that exhibit such discrete transitions, producing various effects in the system's response.

The simple basic idea is illustrated in the following figures.



(A) $f(t) = 5\sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

(i) In fig (A), function $f(t) = 5\sin t$.

(ii) In fig (B), it is switched off between $t=0$ and $t=2$ ($u(t-2)=0$ when $t<2$) and switched on beginning at $t=2$.

(iii) It is shifted to the right by 2 units.

The following two results will be useful in working problems connected with unit step function to find their Laplace transforms.

2. If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$, then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$

Sol: By the definition of $u(t-a)$, we have

$$[f_2(t) - f_1(t)]\,u(t-a) = \begin{cases} f_2(t) - f_1(t) & \text{for } t > a \\ 0 & \text{for } t \le a \end{cases}$$

Adding $f_1(t)$ both sides, this becomes

$$f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) = \begin{cases} f_1(t) + \{f_2(t) - f_1(t)\} & \text{for } t > a \\ f_1(t) + 0 & \text{for } t \le a \end{cases}$$

$$= \begin{cases} f_2(t) & \text{for } t > a \\ f_1(t) & \text{for } t \le a \end{cases} = f(t).$$

$$\therefore f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$$

3. If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$, then $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$

Sol: $$RHS = f_1(t) + [f_2(t) - f_1(t)]\begin{cases} 0, & t \le a \\ 1, & t > a \end{cases} + [f_3(t) - f_2(t)]\begin{cases} 0, & t \le b \\ 1, & t > b \end{cases}$$

$$= f_1(t) + \begin{cases} 0, & t \le a \\ f_2(t) - f_1(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases} + \begin{cases} 0, & t \le b \\ f_3(t) - f_2(t), & t > b \end{cases}$$

$$= \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a, t \le b \\ f_3(t), & t > b \end{cases} = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases} = f(t) = LHS.$$

* Working procedure :

- Type 1 : To find $L[F(t)u(t-a)]$ where $F(t)$ is a polynomial in t.

  Step 1 : Let $F(t) = f(t-a) \Rightarrow F(t+a) = f(t)$.

  Step 2 : Replace t by t+a to obtain $f(t)$

  Step 3 : Find $L[f(t)] = F(s)$.

  Step 4 : $L[F(t)u(t-a)] = e^{-as}F(s)$ by property (1).

- Type 2 : Given $f(t)$ as a discontinuous function to find $L[f(t)]$ by expressing $f(t)$ in terms of unit step function.

  Step 1 : Express $f(t)$ in terms of unit step function by directly making use of results 2 or 3 as the case may be.

  Step 2 : Find $L[f(t)]$ as in Type-I.

* Examples :

1. Find $L\{[e^{t-2} + \sin(t-2)]H(t-2)\}$

Sol: Let $f(t-2) = e^{t-2} + \sin(t-2)$.

Then, $f(t) = e^t + \sin t$

$$L\{f(t)\} = \frac{1}{s-1} + \frac{1}{s^2+1} = F(s)$$

By Heaviside shift theorem, $L[f(t-2)H(t-2)] = e^{-2s}F(s)$

$$\therefore L\left[(e^{t-2} + \sin(t-2))H(t-2)\right] = e^{-2s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$$

2. Find $L\{(3t^2+2t+3)u(t-1)\}$

Sol:

Let $f(t-1) = 3t^2+2t+3$

$$f(t) = 3(t+1)^2 + 2(t+1) + 3 = 3t^2+6t+3+2t+2+3$$

$$= 3t^2+8t+8$$

$$\therefore L\{f(t)\} = \frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s} \qquad \left(\because L\{t^n\} = \frac{n!}{s^{n+1}}\right)$$

$$\therefore L\{(3t^2+2t+3)u(t-1)\} = e^{-s}L\{f(t)\}$$

$$= e^{-s}\left[\frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}\right]$$

3. Find $\mathcal{L}\{e^{-t}H(t-2)\}$

Soln

Let $f(t-2) = e^{-t}$. Then, $f(t) = e^{-(t+2)} = e^{-2}e^{-t}$

Thus, $\mathcal{L}\{f(t)\} = \dfrac{e^{-2}}{s+1}$

$$\therefore \mathcal{L}\{e^{-t}H(t-2)\} = e^{-2s}\mathcal{L}\{f(t)\} = \frac{e^{-2(s+1)}}{s+1}$$

4. Express the following functions in terms of unit step function and hence find their Laplace transform.

(i) $f(t) = \begin{cases} t^2, & 1<t\leq 2 \\ 4t, & t>2 \end{cases}$

Soln WKT $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$

$$\therefore f(t) = t^2 + (4t-t^2)\,u(t-2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{(4t-t^2)u(t-2)\} \longrightarrow (1)$$

Consider $\mathcal{L}\{(4t-t^2)u(t-2)\}$

Let $F(t-2) = 4t-t^2$, Then $F(t) = 4(t+2) - (t+2)^2 = 4t+8-t^2-4-4t$

$$= -t^2+4$$

$$\mathcal{L}\{F(t)\} = \frac{-2}{s^3} + \frac{4}{s}$$

$$\therefore \mathcal{L}\{(4t-t^2)u(t-2)\} = \mathcal{L}\{F(t-2)u(t-2)\} = e^{-2s}\mathcal{L}\{F(t)\}$$

$$= e^{-2s}\left[\frac{-2}{s^3} + \frac{4}{s}\right]$$

Substituting in (1), we have

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} + e^{-2s}\left[-\frac{2}{s^3} + \frac{4}{s}\right]$$

(ii) $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

Soln WKT $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$

$$f(t) = \cos t + [\sin t - \cos t]\,u(t-\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + \mathcal{L}[\{\sin t - \cos t\}\,u(t-\pi)]$$

Consider $\mathcal{L}\{(\sin t - \cos t)\, u(t-\pi)\}$

Let $F(t-\pi) = \sin t - \cos t$. Then $F(t) = \sin(t+\pi) - \cos(t+\pi)$
$= \cos t - \sin t$

$$\mathcal{L}\{F(t)\} = \frac{s}{s^2+1} - \frac{1}{s^2+1} = \frac{s-1}{s^2+1}$$

$$\therefore \mathcal{L}\{(\sin t - \cos t)\, u(t-\pi)\} = e^{-\pi s}.\mathcal{L}\{F(t)\} = e^{-\pi s}\left(\frac{s-1}{s^2+1}\right)$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s-1}{s^2+1}\right)$$

(iii) $f(t) = \begin{cases} 2t & 0<t\le\pi \\ 1 & t>\pi \end{cases}$

Soln WKT, $f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a)$.

$\therefore f(t) = 2t + (1-2t)\, u(t-\pi)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{2t\} + \mathcal{L}[(1-2t)u(t-\pi)] \rightarrow (1)$

$\mathcal{L}\{2t\} = \frac{2}{s^2}$

Consider $\mathcal{L}[(1-2t)\,u(t-\pi)]$

Let $F(t-\pi) = 1-2t$. Then $F(t) = 1-2(t+\pi)$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{1-2\pi-2t\} = \frac{(1-2\pi)}{s} - \frac{2}{s^2}$$

$$\mathcal{L}\{F(t-\pi).u(t-\pi)\} = e^{-\pi s}.\mathcal{L}\{F(t)\} = e^{-\pi s}\left[\frac{(1-2\pi)}{s} - \frac{2}{s^2}\right]$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{2}{s^2} + e^{-\pi s}\left[\frac{(1-2\pi)}{s} - \frac{2}{s^2}\right]$$

5. Find $\mathcal{L}[t^2 H(t-3)]$

Soln Let $F(t-3) = t^2$. Then $F(t) = (t+3)^2 = t^2+6t+9$.

$$\mathcal{L}\{F(t)\} = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}$$

$$\therefore \mathcal{L}\{F(t-3)H(t-3)\} = e^{-3s}\mathcal{L}\{F(t)\} = \frac{e^{-3s}}{s^3}(9s^2+6s+2)$$

6. Find $\mathcal{L}[(e^{-t}\cos 2t)H(t-\pi)]$.

Soln Let $F(t-\pi) = e^{-t}\cos 2t$. Then $F(t) = e^{-(t+\pi)}\cos 2(t+\pi) = e^{-(t+\pi)}\cos(2\pi+2t)$

$$= e^{-\pi}e^{-t}\cos 2t$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{F(t)\} = e^{-(\pi)}\cdot\frac{s+1}{(s+1)^2+4} = e^{-\pi}\frac{(s+1)}{s^2+2s+5}$$

$$\mathcal{L}\{F(t-\pi)H(t-\pi)\} = e^{-\pi s}\mathcal{L}\{F(t)\}$$

$$= e^{-\pi(1+s)}\frac{(s+1)}{s^2+2s+5}$$

7. Find $\mathcal{L}[\sin^3 t\ H(t-2\pi)]$

Soln Let $F(t-2\pi) = \sin^3 t$. Then $F(t) = \sin^3(t+2\pi) = \sin^3 t$

$$\mathcal{L}\{F(t)\} = \frac{1}{4}\mathcal{L}\{3\sin t - \sin 3t\} = \frac{1}{4}\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$= \frac{6}{(s^2+1)(s^2+9)}$$

$$\mathcal{L}\{F(t-2\pi)H(t-2\pi)\} = e^{-2\pi s}\cdot\mathcal{L}\{F(t)\} = \frac{6e^{-2\pi s}}{(s^2+1)(s^2+9)}$$

8. Express the following functions in terms of unit step function and hence find their Laplace transform.

(i) $f(t) = \begin{cases} t^2, & 0<t<2 \\ 8t, & t>2 \end{cases}$

Soln WKT , $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

$$= t^2 + (8t - t^2)u(t-2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{(8t-t^2)u(t-2)\}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

Consider $\mathcal{L}\{(8t-t^2)u(t-2)\}$

Let $F(t-2) = 8t - t^2$. Then $F(t) = 8(t+2) - (t+2)^2$

$$= 8t + 16 - t^2 - 4 - 4t$$

$$= -t^2 + 4t + 12$$

$$\mathcal{L}\{F(t)\} = -\frac{2}{s^3} + \frac{4}{s^2} + \frac{12}{s}$$

$$\mathcal{L}\{F(t-2)u(t-2)\} = e^{-2s}\mathcal{L}\{F(t)\} = e^{-2s}\left[\frac{-2}{s^3} + \frac{4}{s^2} + \frac{12}{s}\right].$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{e^{-2s}}{s^3}(12s^2+4s-2)$$

(ii) $f(t) = \begin{cases} \pi - t, & 0<t\leq\pi \\ \sin t, & t>\pi \end{cases}$

Soln $\mathcal{L}\{f(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}[f_2(t) - f_1(t)]u(t-a)$

$$\therefore f(t) = (\pi - t) + [\sin t - \pi + t]u(t-\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}(\pi - t) + \mathcal{L}[(\sin t - \pi + t)u(t-\pi)]$$

$$\mathcal{L}(\pi - t) = \frac{\pi}{s} - \frac{1}{s^2}$$

Consider $\mathcal{L}[(\sin t - \pi + t)u(t-\pi)]$

Let $F(t-\pi) = \sin t - \pi + t$. Then $F(t) = \sin(t+\pi) - \pi + (t+\pi)$
$= t - \sin t.$

$$\mathcal{L}\{F(t)\} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\mathcal{L}\{F(t-\pi)u(t-\pi)\} = e^{-\pi s}\mathcal{L}\{F(t)\} = e^{-\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{\pi}{s} - \frac{1}{s^2} + e^{-\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

(iii) $f(t) = \begin{cases} \cos t, & 0<t<\pi/2 \\ \sin t, & t>\pi/2 \end{cases}$

Soln $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

$$f(t) = \cos t + [\sin t - \cos t]u(t-\pi/2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\{(\sin t - \cos t)u(t-\pi/2)\}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

Consider $\mathcal{L}\{(\sin t - \cos t)u(t-\pi/2)\}$

Let $F(t-\pi/2) = \sin t - \cos t$. Then $F(t) = \sin(t+\pi/2) - \cos(t+\pi/2)$
$= \cos t + \sin t$.

$$\therefore \mathcal{L}\{F(t)\} = \frac{s}{s^2+1} + \frac{1}{s^2+1} = \frac{s+1}{s^2+1}$$

$$\mathcal{L}\{F(t-\pi/2)u(t-\pi/2)\} = e^{-\pi/2\, s}\ \frac{s+1}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi/2\, s}\cdot\frac{s+1}{s^2+1}$$

9. Express the following functions in terms of unit step function and hence find their Laplace transform.

(i) $f(t) = \begin{cases} \sin t, & 0 < t \le \pi \\ \sin 2t, & \pi < t \le 2\pi \\ \sin 3t, & t > 2\pi. \end{cases}$

Soln. WKT $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$$\therefore f(t) = \sin t + [\sin 2t - \sin t]u(t-\pi) + [\sin 3t - \sin 2t]u(t-2\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{(\sin 2t - \sin t)u(t-\pi)\} + \mathcal{L}\{(\sin 3t - \sin 2t)u(t-2\pi)\}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

Consider,
$\mathcal{L}\{(\sin 2t - \sin t)u(t-\pi)\}$

Let $F(t-\pi) = \sin 2t - \sin t$. Then $F(t) = \sin(2\pi+2t) - \sin(\pi+t)$
$= \sin 2t + \sin t$

$$\therefore \mathcal{L}\{F(t)\} = \frac{2}{s^2+4} + \frac{1}{s^2+1}$$

$$\therefore \mathcal{L}\{F(t-\pi)u(t-\pi)\} = e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right)$$

Now, consider $\mathcal{L}\{(\sin 3t - \sin 2t)u(t-2\pi)\}$

Let $G(t-2\pi) = \sin 3t - \sin 2t$. Then $G(t) = \sin 3(t+2\pi) - \sin(2t+4\pi)$
$= \sin 3t - \sin 2t$

$$\therefore \mathcal{L}\{G(t)\} = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

$$\mathcal{L}\{G(t-2\pi)\,u(t-2\pi)\} = e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$$

(ii) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ t, & 1 \le t \le 2 \\ t^2, & t > 2 \end{cases}$

Soln WKT $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$

$$f(t) = 1 + (t-1)\,u(t-1) + (t^2 - t)\,u(t-2)$$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\,u(t-1)\} + \mathcal{L}\{(t^2-t)\,u(t-2)\}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

Consider $\mathcal{L}\{(t-1)\,u(t-1)\}$

Let $F(t-1) = t-1$. Then $F(t) = t+1-1 = t$.

$$\mathcal{L}\{F(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\{F(t-1)\,u(t-1)\} = e^{-s}\cdot\mathcal{L}\{F(t)\} = \frac{e^{-s}}{s^2}$$

Consider $\mathcal{L}\{(t^2-t)\,u(t-2)\}$

Let $G(t-2) = t^2 - t$. Then $G(t) = (t+2)^2 - (t+2) = t^2+4+\overset{4}{2}t-t-2$

$$= t^2+3t+2.$$

$$\mathcal{L}\{G(t)\} = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$$

$$\therefore \mathcal{L}\{G(t-2)\,u(t-2)\} = e^{-2s}\,\mathcal{L}\{G(t)\}$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^3}(2s^2+3s+2)$$

(iii) $f(t) = \begin{cases} t^2, & 0<t\le 2 \\ 4, & 2<t\le 4 \\ 0, & t>4 \end{cases}$

Sol. $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$\therefore f(t) = t^2 + (4-t^2)u(t-2) + (0-4)u(t-4)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{(4-t^2)u(t-2)\} - 4\mathcal{L}\{u(t-4)\}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$

Consider $\mathcal{L}\{(4-t^2)u(t-2)\}$

Let $F(t-2) = 4-t^2$. Then $F(t) = 4-(t+2)^2 = 4-t^2-4-4t$
$= -t^2-4t$

$\mathcal{L}\{F(t)\} = \frac{-2}{s^3} - \frac{4}{s^2}$

$\mathcal{L}\{F(t-2)u(t-2)\} = -e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}\right)$

Now, consider $\mathcal{L}\{u(t-4)\}$

Let $G(t-4) = 1$. Then $G(t) = 1$.

$\mathcal{L}\{G(t)\} = 1/s$.

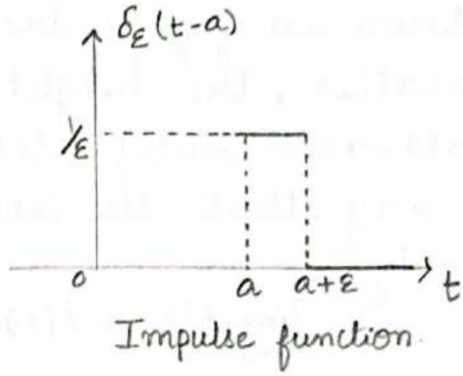
$\mathcal{L}\{G(t-4)u(t-4)\} = e^{-4s}\cdot\frac{1}{s}$

$\therefore \mathcal{L}\{f(t)\} = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}\right) - \frac{4}{s}\cdot e^{-4s}$

* Unit Impulse function / Dirac Delta function

Phenomena of an impulsive nature, such as the action of forces or voltages over short interval of time, arise in various applications, for instance, if a mechanical system is hit by a hammer blow, an airplane makes hard landing, a ship is hit by a single high wave, hit a tennis ball by a racket and so on. These problems are modeled by 'Dirac's delta function' $\delta(t)$ and efficiently solved by Laplace transform. This function is named after British Physicist P.A.M Dirac (1902-1984).

The displaced (delayed) delta or unit impulse function $\delta(t-a)$ represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.



Impulse function

Dirac delta function $\delta(t-a)$ is defined as $\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$

and $\delta_\varepsilon(t-a) = \begin{cases} 1/\varepsilon, & a \le t \le a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$

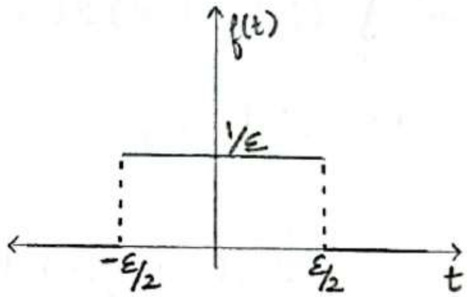
where 'a' is a non-negative constant.

From the graph, we observe that the value of the function $\delta_\varepsilon(t-a)$ suddenly increases from 0 to $1/\varepsilon$ as $t \to a$ from the left and suddenly decreases back to 0 for $t \geqslant a+\varepsilon$. Thus when ε is infinitesimally small, $\delta_\varepsilon(t-a)$ suddenly jumps from 0 to an infinitely large value as $t \to a$ and jumps back to 0 immediately thereafter. For this reason, the function $\delta(t-a)$ which is the limiting case of $\delta_\varepsilon(t-a)$ as $\varepsilon \to 0$ is called the impulse function.

* NOTE : (Alternate explanation).

Consider the function $f(t)$ given by

$$f(t) = \begin{cases} 1/\varepsilon, & -\varepsilon/2 < t < \varepsilon/2 \\ 0, & \text{otherwise} \end{cases}$$



The width of this function is ε and its amplitude is $1/\varepsilon$.

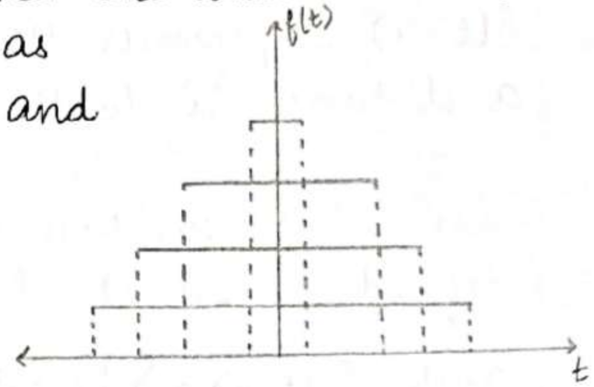
The area enclosed by the function $f(t)$ and t-axis is given by

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{-\varepsilon/2} f(t)\,dt + \int_{-\varepsilon/2}^{\varepsilon/2} f(t)\,dt + \int_{\varepsilon/2}^{\infty} f(t)\,dt$$

$$= 0 + \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\,dt + 0 = \frac{1}{\varepsilon}[t]_{\varepsilon/2}^{\varepsilon/2}$$

$$= \frac{1}{\varepsilon} \cdot \varepsilon = 1$$

Hence, the area of this function is one unit. That is, as $\varepsilon \to 0$, the width of the strip as shown in figure becomes smaller and smaller, the height of the strip increases indefinitely in such a way that the area remains unity.



$$\lim_{\varepsilon \to 0} f(t) = \delta(t).$$

$$\delta(t) = 0 \ , t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \ , t = 0.$$

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases} \quad \text{such that} \int_0^{\infty} \delta(t-a)\,dt = 1 \quad (a \geq 0)$$

* Some properties of Dirac delta function

1. $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$
2. $\int_{0}^{\infty} f(t)\,\delta(t)\,dt = f(0)$
3. $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$
4. $\int_{0}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$.

<u>Proof</u>: Using $\delta_\varepsilon(t-a) = \begin{cases} 1/\varepsilon, & a \leq t \leq a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$

$$\int_0^{\infty} \delta_\varepsilon(t-a)\,f(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon} f(t)\,dt = \frac{1}{\varepsilon}\left[\phi(a+\varepsilon) - \phi(a)\right]$$

where $\phi(t) = \int f(t)\,dt \quad \rightarrow$ (i)

By the Lagrange's mean value theorem, we have

$$\phi(a+\varepsilon) - \phi(a) = \varepsilon\,\phi'(a+\theta\varepsilon) \quad 0<\theta<1$$
$$= f(a+\theta\varepsilon) \quad \text{(using (i))}$$

$$\therefore \int_0^{\infty} \delta_\varepsilon(t-a)\,f(t)\,dt = f(a+\theta\varepsilon) \quad \rightarrow \text{(ii)}$$

Taking the limits as $\varepsilon \to 0$ on both sides of (ii) and using the relation $\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)$, we get $\int_0^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$.

* Laplace transform of Unit impulse function.

1. $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$

WKT $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

Taking $f(t) = e^{-st}$, we get

$$\int_0^\infty e^{-st}\,\delta(t-a) = e^{-sa}$$

i.e., $\mathcal{L}\{\delta(t-a)\} = e^{-sa}$

For $a = 0$,

$$\mathcal{L}\{\delta(t)\} = 1.$$

2. $\mathcal{L}\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

* Examples :

1. Find $\mathcal{L}\{(t-1)^2\,\delta(t-a)\}$

Soln Here, $f(t) = (t-1)^2$

$\therefore f(a) = (a-1)^2 = a^2-2a+1.$

Thus, $\mathcal{L}\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

$\Rightarrow \mathcal{L}\{(t-1)^2\,\delta(t-a)\} = e^{-as}(a^2-2a+1)$

2. Find $\mathcal{L}\{\sin t\;\delta(t-\pi/2)\}$

Soln Here, $f(t) = \sin t$, $a = \pi/2$

$\therefore f(a) = f(\pi/2) = \sin(\pi/2) = 1.$

$\therefore \mathcal{L}\{\sin t\;\delta(t-\pi/2)\} = e^{-\frac{\pi}{2}s}$

3. Evaluate $\int_0^\infty t^m (\log t)^n\,\delta(t-3)\,dt$

Soln $f(t) = t^m(\log t)^n$; $a = 3$; $f(a) = 3^m(\log 3)^n$

$$\therefore \int_0^\infty t^m(\log t)^n\,\delta(t-3)\,dt = 3^m(\log 3)^n$$

4. Find the Laplace transform of the following functions.

(i) $t^2 \delta(t-3)$

Soln $f(t) = t^2 \quad ; \quad a = 3$

$\therefore f(a) = a^2 = 9$

$L\{t^2 \delta(t-3)\} = 9e^{-3s}$

(ii) $e^{-t}\cos 2t \, \delta(t-\pi)$

Soln $f(t) = e^{-t}\cos 2t \quad ; a = \pi$

$f(a) = e^{-\pi}\cos 2\pi = e^{-\pi}$

$\therefore L\{e^{-t}\cos 2t \, \delta(t-\pi)\} = e^{-\pi s} . e^{-\pi} = e^{-\pi(1+s)}$

(iii) $\dfrac{e^{-t} + \log t}{t} \delta(t-3)$

Soln $f(t) = \dfrac{e^{-t} + \log t}{t} \quad ; a = 3$

$f(a) = \dfrac{e^{-3} + \log 3}{3}$

$L\left\{ \dfrac{e^{-t} + \log t}{t} \delta(t-3) \right\} = e^{-3s}\left(\dfrac{e^{-3} + \log 3}{3} \right)$