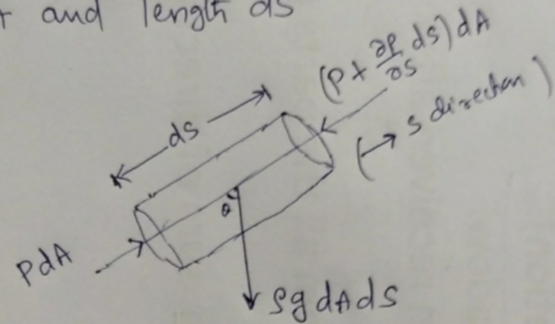
Bernoullis equation from Euler's equation

Consider a streamline in which flow is taking place in S-direction as shown in fig. Consider a cylindrical element of cross-section dA and length ds



The forces acting on the cylindrical element are

1) pressure force in the direction of flow, PdA
2) pressure force opposite to the direction of flow
$$\left(P + \frac{\partial p}{\partial s} ds\right) dA$$
3) weight of element, ie $\rho g\, dA\, ds$

$$\text{weight density} = \frac{\text{weight of element}}{\text{Volume of fluid element}}$$

weight of element = weight density × Volume of fluid element

weight of element = $\rho g\,(dA\,ds)$

The resultant force on the fluid element in the direction of S must be equal to the mass of fluid x accellaration in the direction S

$$\text{Resultant force} = PdA - \left(P + \frac{\partial P}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$

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$$(\rho dA ds) a_s = PdA - \left(P + \frac{\partial P}{\partial s} ds\right) dA - \rho g dA ds \cos\theta \longrightarrow (1)$$

where a_s is the accelaration in the direction of S

Now $a_s = \frac{dv}{dt}$, where V is the function of S and t

$$a_s = \frac{\partial v}{\partial s} \frac{ds}{dt} + \frac{\partial v}{\partial t} \cdot \frac{dt}{dt}$$

$$a_s = \frac{\partial v}{\partial s} V + \frac{\partial v}{\partial t}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\boxed{a_s = \frac{V \partial v}{\partial s}}$$

Substitute the value of a_s in equation 1 and simplifying the equation, we get

$$\cancel{PdA} - \cancel{PdA} - \frac{\partial P}{\partial s} ds\, dA - \rho g\, dA\, ds \cos\theta = \rho\, dA\, ds \times \frac{V \partial V}{\partial s}$$

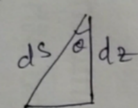
$$-\frac{\partial p}{\partial s} ds\, dA - \rho g\, dA\, ds \cos\theta = \rho\, dA\, ds \times \frac{V \partial V}{\partial s}$$

Divide the above equation by $\rho\, dA\, ds$
then

$$-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{V \partial V}{\partial s}$$

$$\frac{\partial p}{\rho \partial s} + g\cos\theta + \frac{V \partial V}{\partial s} = 0$$

But from fig



$\cos\theta = \frac{dz}{ds}$

Now we have $\cos\theta = \frac{dz}{ds}$

$$\frac{\partial P}{\rho \partial s} + g\frac{dz}{ds} + \frac{V \partial V}{\partial s} = 0$$

$$\boxed{\frac{\partial P}{\rho} + g\, dz + V \partial V = 0}$$

↳ Euler's equation of motion

Bernoullis equation

Bernoulli's equation is obtained by integrating the Euler's equation of motion.

$$\int \frac{dp}{\rho} + \int g\, dz + \int v\, dv = \text{constant}$$

If the flow is incompressible, ρ is constant

$$\boxed{\frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}}$$

or

$$\boxed{\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}}$$

↳ Bernoulli's equation

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head

$\frac{v^2}{2g}$ = kinetic energy per unit weight or kinetic head

z = potential energy per unit weight or potential head.

Assumptions

The following assumptions made in derivation of Bernoullis equation

(1) The fluid is ideal, (viscosity is zero)

(2) The flow is steady

(3) The flow is incompressible

(4) The flow is irrotational.