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SUCCESSIVE DIFFERENTIATION AND ERROR APPROXIMATIONS

If $y = f(x)$ is a differentiable function, then the first derivative of y, $\frac{dy}{dx} = f'(x)$. Again, if $f'(x)$ is a derivable function, then its second derivative is given by $\frac{d^2y}{dx^2} = f''(x)$. And so on. In general, the nth derivative of $y = f(x)$ is denoted by $\frac{d^ny}{dx^n} = f^n(x)$.

This process of finding the derivatives of the same function again and again is called **Successive Differentiation.** Sometimes we use the notations $y_1, y_2, \ldots$ or y', y'' ... or Dy, D^2y, ... to denote the successive derivetes of y; where $D = \frac{d}{dx}$ is called the differential operator.

1.1 *n*th Derivatives of Some Standard Functions

We now derive the formulae for computing the nth order derivative of some simple functions, where n is a positive integer.

1. *n*th Derivative of $(ax + b)^m$

Let $\quad y = (ax+b)^m$

Then, $\quad y_1 = ma(ax+b)^{m-1}$

$$y_2 = m(m-1)\, a^2(ax+b)^{m-2}$$

$$y_3 = m(m-1)\,(m-2)\, a^3(ax+b)^{m-3}$$

In general, $\quad y_n = m(m-1)\ldots(m-n+1)\, a^n(ax+b)^{m-n}$

Remarks:

1. We can re-write the above formula as:

$$y_n = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$$

If m is a positive integer such that $m \geq n$.

For $n = m$, we get

$$y_m = m!\, a^m$$

Also, $\quad y_n = 0\ \forall m < n.$

2. For $\quad m = -1,$

$$y = (ax+b)^{-1}$$

Then, $y_1 = -(ax+b)^{-2}$

$y_2 = (-1)(-2)(-3)\, a(ax+b)^{-3} = \lfloor 2 \; a(ax+b)^{-3}$

In general, $y_n = (-1)^n\, n!\, a^n (ax+b)^{-(n+1)}$

$$= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

2. *n*th Derivative of log $(ax + b)$

Let $y = \log(ax+b)$

Then, $y_1 = \dfrac{a}{ax+b} = a(ax+b)^{-1}$

$y_2 = (-1)\, a^2(ax+b)^{-2}$

$y_3 = (-1)\,(-2)\, a^3(ax+b)^{-3} = 2!\, a^3(ax+b)^{-3}$

$y_4 = (-1)\,(-2)\,(-3)\, a^4(ax+b)^{-4} = -\, 3!\, a^4\, (ax+b)^{-4}$

In general, $y_n = \dfrac{(-1)^n (n-1)! a^n}{(ax+b)^n}$

3. *n*th Derivative of a^{mx}

Let $y = a^{mx}$

Then, $y_1 = m\, a^{mx} \log a$

$y_2 = m^2\, a^{mx}(\log a)^2$

$\vdots$

In general, $y_n = m^n\, a^{mx}(\log a)^n$

Remark: Taking $a = e$, we get

$y = e^{mx}$

and $y_n = m^n\, e^{mx}$ (as $\log e = 1$)

4. *n*th Derivative of sin$(ax + b)$ and cos$(ax + b)$

Let $y = \sin(ax+b)$

Then, $y_1 = a\cos(ax+b)$

$$= a\sin\left(ax+b+\frac{\pi}{2}\right) \qquad \left[\text{as } \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta\right]$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(\frac{\pi}{2}+ax+b+\frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

In general

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly,

for $y = \cos(ax+b)$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right) \qquad \left[as \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right]$$

5. nth Derivative of $e^{ax} \cos (bx + c)$ and $e^{ax} \sin(bx + c)$

Let $\quad y = e^{ax} \cos(bx + c)$

Then, $\quad y_1 = e^{ax} a\cos(bx + c) - e^{ax}.b\sin(bx + c)$

Put $\quad a = r \cos\theta$ and $b = r \sin\theta$

$\therefore \quad r^2 = a^2 + b^2$ and $\tan\theta = \dfrac{b}{a}$

Thus, $\quad y_1 = e^{ax}[r\cos\theta\cos(bx + c) - r\sin\theta\sin(bx + c)]$

$= r\, e^{ax} \cos(bx + c + \theta)$

[Using cos (A + B = cos A cos B – sin A sin B]

Now $\quad y_2 = r\, a\, e^{ax} \cos(bx + c + \theta) - r\, be^{ax} \sin(bx + c + \theta)$

$= r\, e^{ax}[r \cos\theta \cos(bx + c + \theta) - r \sin \theta \sin(bx + c + \theta)]$

[Using $a = r$ cos, $b = r \sin \theta$]

$= r^2 e^{ax} \cos(bx + c + 2\theta)$

Similarly, $\quad y_3 = r^3 e^{ax} \cos(bx + c + 3\theta)$ and soon.

In general, $\quad y_n = r^n e^{ax} \cos(bx + c + n\theta)$

where $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

Similarly, for $\quad y = e^{ax} \sin (bx + c)$

$y_n = r^n e^{ax} \sin(bx + c + n\theta)$

where $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

Example 1. If $\quad y = \left(\dfrac{1}{x}\right)^x$, show that $y_2(1) = 0$.

Solution. $\quad \log y = x\log\dfrac{1}{x} = -x\log x$

Differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = -\left(x.\frac{1}{x} + \log x\right) = -(1 + \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{1}{x}\right)^x (1 + \log x) \qquad \text{...(1)}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -\left[\left(\frac{1}{x}\right)^x .\frac{1}{x} + (1 + \log x)\frac{d}{dx}\left(\frac{1}{x}\right)^x\right]$$

$$= -\left[\left(\frac{1}{x}\right)^x \cdot \frac{1}{x} - (1+\log x)\left(\frac{1}{x}\right)^x (1+\log x)\right] \text{ Using (1)}$$

$$= \left(\frac{1}{x}\right)^x (1+\log x)^2 - \left(\frac{1}{x}\right)^x \cdot \frac{1}{x}$$

$\therefore$

$$y_2(1) = 1(1 + \log 1)^2 - 1 = 1 - 1 = 0.$$

Example 2. If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, show that

$$p + \frac{d^2p}{d\theta^2} = \frac{d^2b^2}{p^3}$$

Solution. To show $p^4 + p^3 \dfrac{d^2p}{d\theta^2} = a^2b^2$

Differentiating, w.r.t. 'θ', we get

$$\Rightarrow \quad 2p.\frac{dp}{d\theta} = -2a^2 \cos\theta \sin\theta + 2b^2 \sin\theta \cos\theta$$

$$\Rightarrow \quad p\frac{dp}{d\theta} = (b^2 - a^2)\cos\theta\sin\theta \quad ...(1)$$

Again differentiating w.r.t 'θ' we get.

$$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)\,[\cos^2\theta - \sin^2\theta]$$

Example.3. If $x = a\left(a\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, find $\dfrac{d^2y}{dx^2}$

Solution.

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan t/2}\sec^2\frac{t}{2}.\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1-\sin^2 t}{\sin t}\right] = a\frac{\cos^2 t}{\sin t}$$

Also,

$$\frac{dy}{dt} = a\cos t$$

Thus,

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = a\cos t.\frac{\sin t}{a\cos^2 t} = \tan t$$

Differentiating again w.r.t, we get

$$\frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx}$$

$$= \sec^2 t\frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t} = \frac{1}{a}\sin t\sec^4 t$$

Example 4. If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, show that

$$p+\frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Solution. To show $p^4 + p^3\frac{d^2p}{d\theta^2} = a^2b^2$ differentiating w.r.t 'w.r.t. 'θ', we get

$$2p\frac{dp}{d\theta} = -2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta$$

$$\Rightarrow \quad p\frac{dp}{d\theta} = (b^2-a^2)\cos\theta\sin\theta \quad ...(1)$$

Again diff. w.r.t. 'θ' we get.

$$\Rightarrow \quad p\frac{d^2p}{d\theta^2}+\frac{(b^2-a^2)^2\sin^2\theta\cos^2\theta}{p^2} = (b^2-a^2)(\cos^2\theta-\sin^2\theta)$$

$$\Rightarrow \quad p^3\frac{d^2p}{d\theta^2} = (b^2-a^2)(\cos^2\theta-\sin^2\theta)p^2-(b^2-a^2)^2\sin^2\theta\cos^2\theta$$

$$\Rightarrow \quad p^3\frac{d^2p}{d\theta^2} = (b^2-a^2)(\cos^2\theta-\sin^2\theta)(a^2\cos^2\theta+b^2\sin^2\theta)$$

$$-(b^2-a^2)\sin^2\theta\cos^2\theta$$

$$= b^2a^2\cos^4\theta+b^4\cos^2\theta\sin^2\theta-b^2a^2\sin^2\theta\cos^2\theta$$

$$-b^4\sin^4\theta-a^4\cos^4\theta-a^2b^2\cos^2\theta\sin^2\theta$$

$$+a^4\sin^2\theta\cos^2\theta+a^2b^2\sin^4\theta-b^4\sin^2\theta\cos^2\theta$$

$$-a^4\sin^2\theta\cos^2\theta+2a^2b^2\sin^2\theta\cos^2\theta$$

$$= a^2b^2(\cos^4\theta+\sin^4\theta)-a^4\cos^4\theta-b^4\sin^4\theta$$

$$\Rightarrow \quad p^4+p^3\frac{d^2p}{d\theta^2} = (a^2\cos^2\theta+b^2\sin^2\theta)^2+a^2b^2(\cos^4\theta+\sin^4\theta)$$

$$-a^4\cos^4\theta-b^4\sin^4\theta$$

$$= a^4\cos^4\theta+b^4\sin^4\theta+2a^2b^2\cos^2\theta\sin^2\theta$$

$$+a^2b^2(\cos^4\theta+\sin^4\theta)-a^4\cos^4\theta-b^4\sin^4\theta$$

$$= 2a^2b^2\sin^2\theta\cos^2\theta+a^2b^2(\cos^4\theta+\sin^4\theta)$$

$$= a^2b^2[\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta]$$
$$= a^2 b^2 (\sin^2\theta + \cos^2\theta)^2$$
$$= a^2 b^2$$

Example 5. Find nth derivatives of the following:

(*i*) $\frac{1}{x^2-4}$ (*ii*) $\frac{x}{x^2-3x+2}$ (*ii*) $\log(1-x^2)$

Solution. (*i*) Let $y = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right]$

$$\therefore \quad y_n = \frac{1}{4}\left[\frac{d^n}{dx^n}\frac{1}{(x-2)} - \frac{d^n}{dx^n}\frac{1}{(x+2)}\right]$$

$$= \frac{1}{4}\left[\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}\right]$$

(*ii*) Let
$$y = \frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\Rightarrow \quad x = A(x-2) + B(x-1)$$

For $x = 1$
$$A = -1$$
$$x = 2$$
$$\Rightarrow \quad B = 2$$

$$\therefore \quad y = \frac{-1}{x-1} + \frac{2}{x-2}$$

$$\Rightarrow \quad y_n = (-1)\frac{d^n}{dx^n}\frac{1}{(x-1)} + 2\frac{d^n}{dx^n}\frac{1}{(x-2)}$$

$$= \frac{(-1)(-1)^n n!}{(x-1)^{n+1}} + \frac{2(-1)^n n!}{(x-2)^{n+1}}$$

(*iii*) Let
$$y = \log(1-x^2)$$

$$\therefore \quad y_1 = \frac{-2x}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$

$$\Rightarrow \quad A(1-x) + B(1+x) = -2x$$
$$x = -1 \Rightarrow A = 1$$
$$x = 1 \Rightarrow B = -1$$

$$\therefore \quad y = \frac{1}{1+x} - \frac{1}{1-x}$$

Differentiating $(n-1)$ times, we get

$$y_n = \frac{d^{n-1}}{x^{n-1}}\left(\frac{1}{1+x}\right) \frac{-d^{n-1}}{dx^{n-1}}\left(\frac{1}{1-x}\right)$$

$$= \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} - \frac{(-1)^{n-1}(n-1)!}{(1-x)^n}$$

$$= (-1)^{n-1}(n-1)!\left[\frac{1}{(1+x)^n} - \frac{1}{(1-x)^n}\right]$$

Q.6. Find nth derivative of the following :
(*i*) $\sin^3 x$ (*ii*) $\sin^4 x \cos 2x$ (*iii*) $\sin^4 x$

(*i*) Let $y = \sin^3 x$

As $\sin 3x = 3\sin x - 4\sin^3 x$

$$\therefore \quad y = \frac{3\sin x - \sin 3x}{4}$$

$$\therefore \quad y_n = \frac{3}{4}\frac{d^n}{dx^n}\sin x - \frac{1}{4}\frac{d^n}{dx^n}\sin 3x$$

$$= \frac{3}{4}\sin\left(\frac{n\pi}{2}+x\right) - \frac{1}{4}.3^n\sin\left(\frac{n\pi}{2}+3x\right)$$

(*ii*) Let $y = \sin 4x \cos 2x = \frac{1}{2}[\sin 6x + \sin 2x]$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{d^n}{dx^n}\sin 6x + \frac{d^n}{dx^n}\sin 2x\right]$$

$$= \frac{1}{2}\left[6^n\sin\left(\frac{n\pi}{2}+6x\right) + 2^n\sin\left(\frac{n\pi}{2}+2x\right)\right]$$

(*iii*) Let $y = \sin^4 x$

$$= (\sin^2 x)^2 = \frac{(1-\cos 2x)^2}{4}$$

$$= \frac{1 - 2\cos 2x + \cos^2 2x}{4}$$

$$= \frac{1 - 2\cos 2x + (1+\cos 4x)/2}{4}$$

$$= \frac{1}{4} - \frac{\cos 2x}{2} + \frac{1}{8} + \frac{\cos 4x}{8} = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\therefore \quad y_n = 0 - \frac{1}{2}.2^n\cos\left(\frac{n\pi}{2}+2x\right) + \frac{1}{8}4^n\cos\left(\frac{n\pi}{2}+4x\right)$$

$$= \frac{-1}{2}.2^n\cos\left(\frac{n\pi}{2}+2x\right) + \frac{1}{8}4^n\cos\left(\frac{n\pi}{2}+4x\right)$$

Example 7. Prove that the nth derivative of $\frac{x^3}{x^2-1}$ at $x = 0$ is zero if n is even and $-(n)!$ if n is odd $(n > 1)$.

Solution. Let
$$y = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$
$$= x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

Thus,
$$y_n = 0 + \frac{1}{2}(-1)^n n!\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

$\therefore$
$$y_n(0) = \frac{1}{2}(-1)^n n!\left[\frac{1}{(-1)^{n+1}} + 1\right]$$

Case (i) $\quad n$ is even $\Rightarrow y_n(0) = \frac{n!}{2}(-1)^{n}[-1+1] = 0$

Case (ii) $\quad n$ is odd $\Rightarrow y_n(0) = \frac{n!}{2}(-1)^n[1+1] = -(n!)$

Example 8. Find nth derivative of the following:

(i) $e^{2x}\cos^2 x$. (ii) $e^x \sin x \sin 2x$

Solution. (i) Let
$$y = e^{2x}\cos^2 x$$
$$= e^{2x}\left(\frac{1+\cos 2x}{2}\right)$$
$$= \frac{1}{2}e^{2x} + \frac{1}{2}e^{2x}\cos 2x$$

$\therefore$
$$y_n = \frac{1}{2}\frac{d^n}{dx^n}e^{2x} + \frac{1}{2}\frac{d^n}{dx^n}e^{2x}\cos 2x$$
$$= \frac{2^n}{2}e^{2x} + \frac{1}{2}e^{2x}(2^2+2^2)^{n/2}\cos\left[2x + n\tan^{-1}\frac{2}{2}\right]$$
$$= \frac{2^n}{2}e^{2x} + \frac{1}{2}e^{2x}8^{n/2}\cos(2x + n\tan^{-1})$$
$$= \frac{e^{2x}}{2}[2^n + 8^{n/2}\cos(2x + n\tan^{-1})]$$

(ii) Let
$$y = e^x \sin x \sin 2x$$
$$= \frac{e^x}{2}[\cos x - \cos 3x]$$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{d^n}{dx^n}e^x\cos x - \frac{d^n}{dx^n}e^x\cos 3x\right]$$

$$= \frac{1}{2}\left[e^x(1^2+1^2)^{n/2}\cos\left(x+n\tan^{-1}\frac{1}{1}\right)\right.$$

$$\left. - e^x(1^2+3^2)^{n/2}\cos\left(3x+n\tan^{-1}\frac{3}{1}\right)\right]$$

$$= \frac{e^x}{2}[2^{n/2}\cos(x+n\tan^{-1}1) - 10^{n/2}\cos(3x+n\tan^{-1}3)]$$

Example 9. If $I_n = \frac{d^n}{dx^n}(x^n\log x)$ prove that $I_n = n\, I_{n-1} + (n-1)!$

Deduce that $I_n = n!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$

Sol.

$$I_n = \frac{d^n}{dx^n}(x^n\log x) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n\log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[x^n\frac{1}{x} + nx^{n-1}\log x\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}[x^{n-1} + nx^{n-1}\log x]$$

$$= \frac{d^{n-1}}{dx^{n-1}}x^{n-1} + n\frac{d^{n-1}}{dx^{n-1}}[x^{n-1}\log x]$$

$$\therefore \quad I_n = (n-1)! + n\, I_{n-1} \qquad (1)$$

Deduction (1) gives

$$I_{n-1} = (n-2)! + (n-1)\, I_{n-2}$$
$$I_{n-2} = (n-3)! + (n-2) I_{n-3}$$
$$\vdots$$
$$I_3 = 2! + 3\, I_2$$
$$I_2 = 1! + 2\, I_1$$

Now,

$$I_1 = \frac{d}{dx}(x\log x) = 1 + \log x$$

Substituting the above values successively in (1), we get

$$I_n = (n-1)! + n\, I_{n-1}$$

$$\Rightarrow \quad \frac{I_n}{n!} = \frac{1}{n} + \frac{I_{n-1}}{(n-1)!}$$

Now,

$$\frac{I_3}{3!} = \frac{1}{3} + \frac{I_2}{2!}$$

$$\frac{I_2}{2!} = \frac{1}{2} + I_1$$

Adding these we get

$$\frac{I_n}{n!} = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + I_1$$

$$= 1 + \log x + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

$\therefore$
$$I_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right]$$

Example 10. Find nth derivative of $\dfrac{1}{x^2 + a^2}$

Solution. Let
$$y = \frac{1}{x^2 + a^2} = \frac{1}{(x + ia)(x - ia)}$$

Differentiating n times w.r.t. 'x', we get

$$y_n = \frac{1}{2ai}(-1)^n n!\left[\frac{1}{(x - ia)^{n+1}} - \frac{1}{(x + ia)^{n+1}}\right] \quad \dots(1)$$

Let
$$x = r\cos\theta,\ a = r\sin\theta,\ r = \sqrt{x^2 + a^2}\ \&\ \theta = \tan^{-1}\frac{a}{x}$$

$\therefore$
$$x + ia = r(\cos\theta + i\sin\theta)$$

$\therefore$
$$(x + ia)^{n+1} = r^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta]$$ (By De-Moivre's Theorem)

and
$$(x - ia)^{n+1} = r^{n+1}\{\cos(n+1)\theta - i\sin(n+1)\theta\}^{-1}$$

$\therefore$ (1) $\Rightarrow$
$$y_n = \frac{1}{2ai}(-1)^n n!\left[\frac{1}{r^{n+1}}\{\cos(n+1)\theta - i\sin(n+1)\theta\}^{-1} - \frac{1}{r^{n+1}}\{\cos(n+1)\theta + i\sin(n+1)\theta\}^{-1}\right]$$

$$= \frac{1}{2ai}(-1)^n n!\left[\frac{1}{r^{n+1}}\{\cos(n+1)\theta + i\sin(n+1)\theta\} - \frac{1}{r^{n+1}}\{\cos(n+1)\theta - i\sin(n+1)\theta\}\right]$$

$$= \frac{1}{2ai}(-1)^n . \frac{n!}{r^{n+1}} 2i\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{a(x^2 + a^2)^{(n+1)/2}}\sin(n+1)\theta$$

where
$$\theta = \tan^{-1}\frac{a}{x}$$

Example 10. If $y = \tan^{-1} x$, show that

$$y_n = (-1)^{n-1}(n-1)!\sin n\left(\frac{\pi}{2} - y\right)\sin^n\left(\frac{\pi}{2} - y\right)$$

Solution.

$$y = \tan^{-1} x$$

Differentiating w.r.t. x, we get

$$y_1 = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)}$$

$$= \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

Differentiating $(n-1)$ times w.r.t. 'x', we get

$$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

Taking $x = r\cos\theta,\ 1 = r\sin\theta$, we get

$$x + i = r(\cos\theta + i\sin\theta)\ r = \sqrt{x^2+1},\ \theta = \tan^{-1}\frac{1}{x}$$

$$\therefore \quad (x+i)^n = r^n[\cos n\theta + i\sin n\theta]$$

$$(x-i)^n = r^n[\cos n\theta - i\sin n\theta]$$

$$\therefore \quad y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n}\{\cos n\theta - i\sin n\theta\} - \frac{1}{r^n}\{\cos n\theta + i\sin\theta\}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\frac{1}{r^n}[-2i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}}\sin n\theta,\ \theta = \tan^{-1}\frac{1}{x} \qquad \text{...(2)}$$

As $\quad y = \tan^{-1} x,\ \theta = \tan^{-1}\frac{1}{x}$

$$\frac{1}{x} = \tan\theta \Rightarrow x = \cot\theta$$

$$\therefore \quad y = \tan^{-1}(\cot\theta) = \tan^{-1}\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} - \theta$$

$$\therefore \quad \theta = \frac{\pi}{2} - y$$

Also $\quad (x^2+1)^{n/2} = (\cot^2\theta + 1)^{n/2}$

$$= (\operatorname{cosec}^2\theta)^{n/2} = \operatorname{cosec}^n\theta$$

$$= \operatorname{cosec}^n\left(\frac{\pi}{2} - y\right)$$

$\therefore$ from (2) we get $\quad y_n = \dfrac{(-1)^n(n-1)!}{\operatorname{cosec}^n\left(\dfrac{\pi}{2}-y\right)}\sin n\left(\dfrac{\pi}{2}-y\right)$

$$= (-1)^n(n-1)!\sin n\left(\frac{\pi}{2}-y\right)\sin^n\left(\frac{\pi}{2}-y\right)$$

1.2 LEIBNITZ'S THEOREM

Theorem

If $y = u\,v$, where u and v are functions of x possessing nth order derivatives, then

$$y_n = {}^nc_0u_nv + {}^nc_1u_{n-1}v_1 + {}^nc_2u_{n-2}v_2 + \ldots + {}^nc_nuv_n$$

Where u_k, v_k, $k = 1, 2, \ldots, n$ denote the kth order derivatives of u and v, respectively.

Remark: To use Leibnitz's Theorem effectively it is very important that u and v are chosen properly. In general, u is taken to be the function whose derivative vanishes after few successive differentiation and v is taken as another function. For eg, in $x^3 \sin x$, we take $u = x^3$ and $v = \sin x$ as fourth derivative of u is zero, whereas v on differentiation does not vanish.

Example 11. If $y = (x^2-1)^n$, prove that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

Solution. $\quad y = (x^2-1)^n$

$$\Rightarrow \quad y_1 = n(x^2-1)^{n-1}.2x. = 2nx\frac{(x^2-1)^n}{(x^2-1)} = \frac{2nxy}{(x^2-1)}$$

$$\Rightarrow \quad y_1(x^2-1) = 2\,nxy$$

$$\Rightarrow \quad y_2(x^2-1) + 2x.\,y_1 = 2n(y + xy_1) \qquad \text{[on differentiation]}$$

$$\Rightarrow \quad y_2(x^2-1) + 2(1-n)xy_1 - 2ny = 0$$

Differentiating 'n' times w.r.t. 'x', we get

$$y_{n+2}(x^2-1) + n.y_{n+1}.2x + \frac{n(n-1)}{2}y_n2 + 2(1-n)\{y_{n+1}x + ny_n\} - 2ny_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xn.y_{n+1} + n(n-1)y_n + 2xy_{n+1} - 2nxy_{n+1} + 2n(1-n)y_n - 2ny_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} + n^2y_n - ny_n + 2ny_n - 2n^2y_n - 2ny_n = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$$

Example 12. If $\quad y = \left[x+\sqrt{1+x^2}\right]^m$ show that

$$(x^2+1)\,y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Hence find $y_n(0)$.

Solution: $y = \left[x + \sqrt{1+x^2}\right]^m$

Differentiating we get

$$y_1 = m\left[x + \sqrt{1+x^2}\right]^{m-1}\left[1 + \frac{2x}{2\sqrt{1+x^2}}\right]$$

$$= m\left[x + \sqrt{1+x^2}\right]^{m-1}\left\{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right\}$$

$$= \frac{my}{\sqrt{1+x^2}} \qquad ...(2)$$

$\Rightarrow$ $y_1^2(1 + x^2) = m^2y^2$

Differentiating, we get

$$2y_1 y_2 (1 + x^2) + y_1^2\, 2x = 2m^2 yy_1$$

$$\Rightarrow \quad y_2(1 + x^2) + xy_1 - m^2y = 0 \qquad ...(3)$$

By Leibnitz theorem,

$$y_{n+2}(1 + x^2) + {}^nc_1\, y_{n+1} \cdot 2x + {}^nc_2\, y_n .2 + y_{n+1}\, x + {}^nc_1\, y_n - m^2y_n = 0$$

$$\Rightarrow \quad y_{n+2}(1+x^2) + 2nxy_{n+1} + n(n-1)y + xy_{n+1} + ny_n - m^2y_n = 0.$$

$$\Rightarrow \quad (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0 \qquad ...(4)$$

Putting $x = 0$, we get

$$y_{n+2}(0) + (n^2 - m^2)y_n(0) = 0$$

Putting $x = 0$ in (1), (2) & (3), we get

$$y(0) = 1,\ y_1(0) = m,\ y_2(0) = m^2$$

Putting $n = 1, 2, 3, 4, ...$ in (4), we get

$$y_3(0) = (m^2 - 1^2)\, y_1(0) = m(m^2 - 1^2)$$
$$y_4(0) = (m^2 - 2^2)\, y_2(0) = m^2(m^2 - 2^2)$$
$$y_5(0) = (m^2 - 3^2)\, y_3(0) = m(m^2 - 1^2)(m^2 - 3^2)$$
$$y_6(0) = (m^2 - 4^2)\, y_4(0) = m^2(m^2 - 2^2)(m^2 - 4^2)$$

In general,

$$y_n(0) = m(m^2 - 1^2)(m^2 - 3^2) .... [m^2 - (2n - 1)^2], \text{ is odd}$$
$$m^2(m^2 - 2^2)(m^2 - 4^2)....[m^2 - (2n - 2)^2],\ n \text{ is even.}$$

Example 13. $y = e^{m\sin^{-1}x}$, show that

$$(1- x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$$

Solution. $y_1 = e^{m\sin^{-1}x}\dfrac{m}{\sqrt{1-x^2}} = \dfrac{my}{\sqrt{1-x^2}}$

$\Rightarrow$ $y_1^2(1 - x^2) = m^2y^2$

Differentiating, we get

$$2y_1\, y_2(1-x^2) - 2xy_1^{\,2} = 2m^2yy_1$$

$$\Rightarrow \quad y_2(1-x^2) - xy_1 = m^2y$$

Differentiating n times w.r.t x, we get

$$y_{n+2}(1-x^2) + {}^nc_1y_{n+1}(-2x) + {}^nc_3y_n(-2) - \{y_{n+1}\,x + {}^nc_1\, y_n\} = m^2y_n.$$

As $\quad {}^nc_1 = n,\ {}^nc_2 = \dfrac{n(n-1)}{2!}$ therefore

$$y_{n+2}(1-x^2) - 2nxy_{n+1} - \frac{2n(n-1)}{2}y_n - xy_{n+1} - ny_n - m^2y_n = 0.$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - x^2y_n + xy_n - xy_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Example 15. If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that, $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$

Solution. $\quad y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow y^2(1-x^2) = (\sin^{-1}x)^2$

$$\Rightarrow \quad 2yy_1(1-x^2) + y^2(1-2x) = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow y^2(1-x^2) = (\sin^{-1}x)^2 \text{ (On Differentiation)}$$

$$\Rightarrow \quad y_1(1-x^2) - xy = 1$$

$$\Rightarrow y_2(1-x^2) + y_1(-2x) - (xy_1 + y) = 0 \quad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_2(1-x^2) - 3xy_1 - y = 0 \quad \text{(On Differentiation)}$$

Differentiating 'n' times w.r.t. 'x', we get

$$y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2) - y_n - (3xy_{n+1} + ny_n) = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+3)\,xy_{n+1} - (n^2+2n+1)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2\,y_n = 0$$

Example 16. If $y^{1/m} + y^{-1/m} = 2x$, Prove that:

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Solution. Let $\quad w = y^{1/m}$

$$\Rightarrow \quad w + \frac{1}{w} = 2x \Rightarrow w^2 - 2xw + 1 = 0$$

$$\therefore \quad w = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

Case (*i*) $\quad w = x + \sqrt{x^2-1}$

$$\Rightarrow \quad y^{1/m} = x + \sqrt{x^2-1} \Rightarrow y = \left(x + \sqrt{x^2-1}\right)^m$$

Done earlier

Case (*ii*)

$$w = x-\sqrt{x^2-1}$$

$$y = \left(x-\sqrt{x^2-1}\right)^m$$

$$\Rightarrow \quad y_1 = m\left[x-\sqrt{x^2-1}\right]^{m-1}\left\{1-\frac{1}{2\sqrt{x^2-1}}.2x\right\}$$

$$= m\left[x-\sqrt{x^2-1}\right]^{m-1}\left\{\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right\}$$

$$= \frac{-my}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad y_1^2(x^2-1) = m^2y^2$$

$$\Rightarrow \quad 2y_1\, y_2\,(x^2-1)+2x.\,y_1^2 = 2m^2\, yy_1 \qquad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_2\,(x^2-1)+xy_1 = m^2y$$

Differentiating '*n*' times, we get

$$y_{n+2}(x^2-1)+ny_{n+1}.2x+n(n-1).y_n+xy_{n+1}+y_n = m^2y_n$$

$$\Rightarrow (x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n = 0$$

Example 17. If $y = \log\left[x+\sqrt{1+x^2}\right]$, find $y_n(0)$

Solution.

$$y = \log\left[x+\sqrt{1+x^2}\right] \qquad \text{...(1)}$$

$$\Rightarrow \quad y_1 = \frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{1}{\sqrt{1+x^2}}.x\right] \qquad \text{(On Differentiation)}$$

$$= \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{\sqrt{1+x^2}} \qquad \text{...(2)}$$

$$\Rightarrow \quad (1+x^2)\,y_1^2 = 1$$

$$\Rightarrow \quad 2y_1\,y_2\,(1+x^2)+y_1^2.\,2x = 0 \qquad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_2\,(1+x^2)+xy_1 = 0 \qquad \text{...(3)}$$

Differentiating '*n*' times, we get

$$y_{n+2}(1+x^2)+ny_{n+1}.2x+\frac{n(n-1)}{2}y_n.2+xy_{n+1}+ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2}+2nx\,y_{n+1}+n^2y_n-ny_n+xy_{n+1}+ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2}+(2n+1)x\,y_{n+1}+n^2\,y_n = 0 \qquad \text{...(4)}$$

Now (1) $\Rightarrow$ $\quad y(0) = 0$

(2) $\Rightarrow$ $\quad y_1(0) = 1$

(3) $\Rightarrow$ $\quad y_2(0) = 0$

(4) $\Rightarrow$ $\quad y_{n+2}(0) = -n^2 y_n(0)$

Putting $n = 1, 2, 3, 4, 5, \ldots$, we get

$$y_1(0) = 1$$
$$y_2(0) = 0$$
$$y_3(0) = -1^2 y_1(0) = -1^2$$
$$y_4(0) = 0$$
$$y_5(0) = -3^2\, y_3(0) = 1^2.3^2$$

Thus $\quad y_n(0) = (-1)^n\, 1^2.\, 3^2\, .\, 5^2 \ldots. (2n-1)^2$, if n is odd.

$= 0$, if n is even.

Example 18. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log}{x}\right) = \dfrac{(-1)^n n!}{x^{n+1}}\left(\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots \dfrac{1}{n}\right)$

Solution: Let $\quad u = \dfrac{1}{x},\ v = \log x$

Then,
$$u_1 = \frac{-1}{x^2}, v_1 = \frac{1}{x}$$
$$u_2 = \frac{2}{x^3}, v_2 = \frac{-1}{x^2}$$
$$u_3 = \frac{-2.3}{x^4}\ v_3 = \frac{2}{x^3}$$
$$u_n = (-1)^n \frac{n!}{x^{n+1}}, v_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

Now by Leibnitz theorem,

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \ldots + u v_n$$
$$= (-1)^n \frac{n!}{x^{n+1}}.\log x + n(-1)^{n-1}\frac{n-1!}{x^n}.\frac{1}{x} + \frac{n(n-1)}{2!}$$
$$\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\left(\frac{-1}{x^2}\right) + \ldots + \frac{1}{x}(-1)^{n-1}\frac{(n-1)}{x^n}$$

$$\therefore \quad \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 \frac{-1}{2}\frac{-1}{3} - \ldots. \frac{-1}{n}\right]$$

Example 19. If $y = \cos(m \sin^{-1} x)$, show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$

Solution. $\quad y = \cos(m \sin^{-1} x)$

$$y_1 = -\sin(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}} = \frac{-m\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1^2(1-x^2) = m^2(1-y^2) = m^2 - m^2y^2$$

Differentiating we get

$$\Rightarrow \quad 2y_1y_2(1-x^2) + y_1^2(-2x) = -2m^2yy_1$$

$$\Rightarrow \quad y_2(1-x^2) - xy_1 = -m^2y$$

$$\Rightarrow \quad y_2(1-x^2) - xy_1 + m^2y = 0$$

By Leibnitz theorem, differentiating n times we get

$$y_{n+2}(1-x^2) + {}^nc_1y_{n+1}(-2x) + {}^nc_2y_n(-2) - \{y_{n+1}x + {}^nc_1y_n\} + m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2y_n + ny_n - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0$$

Example 20. If $y = e^{\frac{x^2}{2}\cos x}$, show that

$$y_{2n+2}(0) - 4n\,y_{2n}(0) + 2n(2n-1)y_{2n-2}(0) = 0$$

Solution:

$$y_1 = e^{x^2/2}(-\sin x) + e^{x^2/2}.x.\cos x \quad \text{(On Differentiatingly)}$$

$$= -e^{x^2/2}\sin x + xy \quad \text{...(1)}$$

$$\Rightarrow \quad y_2 = -e^{x^2/2}\cos x - e^{x^2/2}.x\sin x + xy_1 + y_1 \text{ (On Differenting } y\text{)}$$

$$= -y + x(y_1 - e^{x^2/2}\sin x) + y_1x + y$$

$$= -y + x(y_1 - xy) + y_1x + y \text{ (by (1))}$$

$$\Rightarrow \quad y_2 - y_1x - xy_1 + x^2y = 0$$

$$\Rightarrow \quad y_2 - 2y_1x + x^2y = 0$$

'2n' times and using Leibnitz theorem, we get

Differentiating $y_{2n+2} - 2[y_{2n+1}.x + 2n.y_{2n}] + [x^2y_{2n} + 2ny_{2n-1}.2x + \frac{2n(2n-1)}{2}y_{2n-2}.2] = 0$

Putting $x = 0$ we get

$$y_{2n+2}(0) - 2[2n\,y_{2n}(0)] + 2n(2n-1)y_{2n-2}(0) = 0$$

$$\Rightarrow y_{2n+2}(0) - 4n\,y_{2n}(0) + 2n(2n-1)y_{2n-2}(0) = 0$$

Example. 21. Find nth derivative of $\tan^{-1}(x/a)$.

Solution. $y = \tan^{-1}x/a$

$$y_1 = \frac{1}{a}\frac{1}{\left(\frac{x^2}{a^2}+1\right)} = \frac{a}{a^2+x^2} \quad \text{(On Differentiation)}$$

$$= \frac{a}{(x+ia)(x-ia)} = \frac{1}{2i}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$$\Rightarrow \quad y_n = \frac{1}{2i}(-1)^{n-1} n-1!\left[\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right]$$

(On Differentiating y_1, $(n-1)$ times)

$$= \frac{1}{2ia^n}(-1)^{n-1}(n-1)!\left[\frac{1}{(x/a-i)^n} - \frac{1}{(x/a+i)^n}\right]$$

Let $$\theta = \tan^{-1}\frac{a}{x} \Rightarrow \frac{a}{x} = \tan\theta \Rightarrow \frac{x}{a} = \cot\theta$$

Now $$\frac{x}{a}+i = r(\cos\theta + i\sin\theta), r = \sqrt{\frac{x^2}{a^2}+1}$$

$$\Rightarrow \quad \left(\frac{x}{a}+i\right)^n = r^n [\cos n\theta + i \sin n\theta]$$

Similarly, $$\left(\frac{x}{a}-i\right)^n = r^n [\cos n\theta - i \sin n\theta]$$

Thus, $$y_n = \frac{1}{2ia^n}(-1)^{n-1}(n-1)!\left[\frac{1}{r^n}\{\cos n\theta + i\sin n\theta\} - \frac{1}{r^n}\{\cos n\theta - i\sin n\theta\}\right]$$

$$= \frac{a^{-n}}{2i}(-1)^{n-1}\frac{(n-1)!}{r^n}2i\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!a^n}{a^n(x^2+a^2)^{n/2}}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)}{(x^2+a^2)^{n/2}}\sin n\theta$$

$$\frac{x}{a} = \cot\theta \Rightarrow \theta = \cot^{-1}\frac{x}{a}$$

Now $$(x^2+a^2)^{n/2} = a^n\left(\frac{x^2}{a^2}+1\right)^{n/2}$$

$$= a^n(\cot^2\theta+1)^{n/2} = a^n(\operatorname{cosec}^2\theta)^{n/2}$$

$$= a^n \operatorname{cosec}^n\theta$$

$$\therefore \quad y_n = (-1)^{n-1}(n-1)!a^{-n}\sin^n\theta\sin n\theta$$

where $\theta = \cot^{-1}\frac{x}{a}$

Example 22. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2 + 1)y_n = 0$

Solution.
$$y = a\cos(\log x) + b\sin(\log x)$$

$$\Rightarrow \quad y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x).\frac{1}{x} \quad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_1 x = b\cos(\log x) - a\sin(\log x)$$

$$\Rightarrow \quad y_2 x + y_1 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x).\frac{1}{x} \quad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_2 x^2 + y_1 x = -y$$

Differentiating 'n' times w.r.t. 'x', we get

$$y_{n+2}x^2 + n.y_{n+1}.2x + \frac{n(n-1)}{2}y_n.2 + y_{n+1}.x + n.y_n = -y_n$$

$$\Rightarrow x^2 y_{n+2} + 2nx.y_{n+1} + n^2 y_n - ny_n + x.y_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)y_n = 0$$

1.3 Maclaurin's Series

Let $f(x)$ be differentiable and all its derivatives exist. If $f(x)$ can be expanded in an infinite convergent series of positive integral powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots \frac{x^n}{n!}f^n(0) + \ldots$$

which is known as Maclaurin's Series. ...(i)

Proof: Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

Then, $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

$f''(x) = 2a_2 + 3.2a_3 x + \ldots$

and so on.

In general,

$f^n(x) = n(n-1)\ldots 3.2.1\, a_n$ + term containing + ve power of x

At $x = 0$, $f(0) = a_0$; and

$f^n(0) = n(n-1)\ldots 3.2.1\, a_n = n!\, a_n$.

Thus, $f'(0) = a_1$

$$f''(0) = 2!a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(0) = 3!a_3 \Rightarrow a_3 = \frac{f'''(0)}{3!}$$

and so on.

Substituting the above values in equation (1), we get where $\theta = \cot^{-1}\frac{x}{a}$

Example 22. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0$

Solution.
$$y = a\cos(\log x) + b\sin(\log x)$$

$$\Rightarrow \quad y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x).\frac{1}{x} \quad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_1\,x = b\cos(\log x) - a\sin(\log x)$$

$$\Rightarrow \quad y_2x + y_1 = -b\sin(\log x)\frac{1}{x} - a\cos(\log x).\frac{1}{x} \quad \text{(On Differentiation)}$$

$$\Rightarrow \quad y_2\,x^2 + y_1x = -y$$

Differentiating 'n' times w.r.t. 'x', we get

$$y_{n+2}x^2 + n.y_{n+1}.2x + \frac{n(n-1)}{2}y_n.2 + y_{n+1}.x + n.y_n = -y_n$$

$$\Rightarrow x^2y_{n+2} + 2nx.y_{n+1} + n^2y_n - ny_n + x.y_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+1)y_n = 0$$

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which is known as Maclaurin's Series. ...(*i*)

Proof: Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.$

Then, $f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots$

$f''(x) = 2a_2 + 3.2a_3x + \ldots..$

and so on.

In general,

$f^n(x) = n(n-1)\ldots.3.2.1\;a_n$ + term containing + *ve* power of x

At $x = 0, f(0) = a_0$; and

$f^n(0) = n(n-1)\ldots.3.2.1\;a_n = n!\,a_n.$

Thus, $f'(0) = a_1$

$$f''(0) = 2!a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(0) = 3!a_3 \Rightarrow a_3 = \frac{f'''(0)}{3!}$$

and so on.

Substituting the above values in equation (1), we get