

Applications of Partial Differential Equations

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18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem*.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. Solve (by the method of separation of variables) :

$$\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \quad (\text{P.T.U., 2009 S; Bhopal 2008; U.P.T.U., 2005})$$

Solution. Assume the trial solution $z = X(x)Y(y)$ where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \quad \text{where } X' = \frac{dX}{dx}, Y' = \frac{dY}{dy} \text{ etc.}$$

$$\text{Separating the variables, we get } \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} \quad \dots(ii)$$

Since x and y are independent variables, therefore, (ii) can only be true if each side is equal to the same constant, a (say).

$$\therefore \frac{X'' - 2X'}{X} = a, \text{ i.e. } X'' - 2X' - aX = 0 \quad \dots(iii)$$

...(iv)

and $-Y'/Y = a, \text{ i.e., } Y' + aY = 0$

To solve the ordinary linear equation (iii), the auxiliary equation is

$$m^2 - 2m - a = 0, \text{ whence } m = 1 \pm \sqrt{1+a}.$$

\therefore the solution of (iii) is $X = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}$

and the solution of (iv) is $Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (i), we get

$$z = \{c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}\} \cdot c_3 e^{-ay}$$

i.e.,

$$z = \{k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x}\} e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6e^{-3x}$.

(V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$XT' = 2XT'' + XT \quad \text{or} \quad (X' - X)T = 2XT''$$

or

$$\frac{X' - X}{2X} = \frac{T'}{T} = k \quad (\text{say})$$

$$\therefore X' - X - 2kX = 0 \quad \text{or} \quad \frac{X'}{X} = 1 + 2k \quad \dots(i) \quad \text{and} \quad \frac{T'}{T} = k \quad \dots(ii)$$

$$\text{Solving (i), } \log X = (1+2k)x + \log c \quad \text{or} \quad X = ce^{(1+2k)x}$$

$$\text{From (ii), } \log T = kt + \log c' \quad \text{or} \quad T = c'e^{kt}$$

$$\text{Thus } u(x, t) = XT = cc'e^{(1+2k)x}e^{kt} \quad \dots(iii)$$

$$\text{Now } 6e^{-3x} = u(x, 0) = cc'e^{(1+2k)x}$$

$$\therefore cc' = 6 \text{ and } 1+2k = -3 \quad \text{or} \quad k = -2$$

Substituting these values in (iii), we get

$$u = 6e^{-3x}e^{-2t} \quad \text{i.e., } u = 6e^{-(3x+2t)} \text{ which is the required solution.}$$

PROBLEMS 18.1

Solve the following equations by the method of separation of variables :

$$1. \quad py^3 + qx^2 = 0. \quad (\text{V.T.U., 2011 ; S.V.T.U., 2008}) \quad 2. \quad x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0. \quad (\text{V.T.U., 2008})$$

$$3. \quad \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}, \text{ given that } u(0, y) = 8e^{-3y}. \quad (\text{J.N.T.U., 2006})$$

$$4. \quad 4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u, \text{ given } u = 3e^{-y} - e^{-5y} \text{ when } x = 0. \quad (\text{S.V.T.U., 2008})$$

$$5. \quad 3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0, \quad u(x, 0) = 4e^{-x}. \quad (\text{V.T.U., 2008 S})$$

$$6. \quad \text{Find a solution of the equation } \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \text{ in the form } u = f(x)g(y). \text{ Solve the equation subject to the conditions } u = 0 \text{ and } \frac{\partial u}{\partial x} = 1 + e^{-3y}, \text{ when } x = 0 \text{ for all values of } y. \quad (\text{Andhra, 2000})$$

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

$$(i) \text{Wave equation : } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

$$(ii) \text{One dimensional heat flow equation : } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

(iii) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) *Transmission line equations*.

(v) *Vibrating membrane*. Two dimensional wave equation.

(vi) *Laplace's equation in three dimensions*.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB , of the string entirely in one plane.

Taking the end A as the origin, AB as the x -axis and AY perpendicular to it as the y -axis ; so that the motion takes place entirely in the xy -plane. Figure 18.1 shows the string in the position APB at time t . Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x -axis. Clearly the element is moving upwards with the acceleration $\frac{\partial^2 y}{\partial t^2}$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T [\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T \left[\left\{ \frac{\partial y}{\partial x} \right\}_{x+\delta x} - \left\{ \frac{\partial y}{\partial x} \right\}_x \right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \cdot \frac{\partial^2 y}{\partial t^2} = T \left[\left\{ \frac{\partial y}{\partial x} \right\}_{x+\delta x} - \left\{ \frac{\partial y}{\partial x} \right\}_x \right] \quad \text{i.e.,} \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \left[\frac{\left\{ \frac{\partial y}{\partial x} \right\}_{x+\delta x} - \left\{ \frac{\partial y}{\partial x} \right\}_x}{\delta x} \right]$$

$$\text{Taking limits as } Q \rightarrow P \text{ i.e., } dx \rightarrow 0, \text{ we have } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \text{ where } c^2 = \frac{T}{m} \quad \dots(1)$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

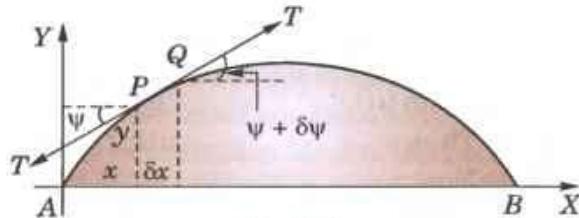


Fig. 18.1

Then $\frac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\frac{\partial^2 y}{\partial x^2} = X'' \cdot T$

Substituting these in (1), we get $XT'' = c^2 X'' T$ i.e., $\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}$... (2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2 X}{dx^2} - kX = 0 \quad \dots(3) \quad \text{and} \quad \frac{d^2 T}{dt^2} - kc^2 T = 0 \quad \dots(4)$$

Solving (3) and (4), we get

(i) When k is positive and $= p^2$, say $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{cpt} + c_4 e^{-cpt}$.

(ii) When k is negative and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(iii) When k is zero. $X = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad \dots(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad \dots(6)$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad \dots(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t . Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), i.e., of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \dots(8)$$

is the only suitable solution of the wave equation.

(Bhopal, 2008)

Example 18.3. A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin(\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = a \sin(\pi x/l) \cos(\pi ct/l). \quad (\text{V.T.U., 2010; S.V.T.U., 2008; Kerala, 2005; U.P.T.U., 2004})$$

Solution. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (i)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad \dots(ii) \quad \text{and} \quad y(l, t) = 0 \quad \dots(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \dots(iv)$

Also $y(x, 0) = a \sin(\pi x/l) \quad \dots(v)$

Now we have to solve (i) subject to the boundary conditions (ii) and (iii) and initial conditions (iv) and (v). Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \dots(vi)$$

By (ii), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt) \quad \dots(vii)$

and $\frac{\partial y}{\partial t} = C_2 \sin px [C_3(-cp \cdot \sin cpt) + C_4(cp \cdot \cos cpt)]$

\therefore By (iv), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \cdot (C_4 cp) = 0$, whence $C_2 C_4 cp = 0$.

If $C_2 = 0$, (vii) will lead to the trivial solution $y(x, t) = 0$,

\therefore the only possibility is that $C_4 = 0$.

Thus (vii) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt \quad \dots(viii)$

∴ By (iii), $y(l, t) = C_2 C_3 \sin pl \cos cpt = 0$ for all t .

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. ∴ $pl = n\pi$, i.e., $p = n\pi/l$, where n is an integer.

Hence (i) reduces to $y(x, t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$.

[These are the solutions of (i) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \dots$ is called its **spectrum**.]

Finally, imposing the last condition (v), we have $y(x, 0) = C_2 C_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$... (ix)

Obs. We have from (ix) $\frac{\partial^2 y}{\partial t^2} = -a \left(\frac{\pi c}{l}\right)^2 \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l} = -\left(\frac{\pi c}{l}\right)^2 y$.

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, i.e., $2l/c$.

Thus we can look upon (ix) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l , wave-velocity c and amplitude $y_0 = a \cos (\pi c t/l)$ which varies harmonically with time t . Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called **nodes**, remain undisturbed during wave motion. Thus (ix) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.

(Rajasthan, 2006; V.T.U., 2003; J.N.T.U., 2002)

Solution. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$... (ii)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\frac{\pi x}{l}\right)$... (iii)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$... (iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

∴ $y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$

Also by (ii), $y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t .

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l} \right)$... (v)

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{cn\pi}{l} \left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l} \right)$$

By (iv), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{cn\pi}{l} \cdot c_4 = 0$, i.e. $c_4 = 0$.

Thus (v) becomes $y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$

Adding all such solutions the general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$$
 ... (vi)

$$\therefore \text{from (iii), } y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{or } y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \dots$$

Comparing both sides, we have

$$b_1 = 3y_0/4, b_2 = 0, b_3 = -y_0/4, b_4 = b_5 = \dots = 0.$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi c t}{l}.$$

Example 18.5. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$... (ii)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$... (iii)

$$\text{and } \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots (\text{iv})$$

The solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\text{By (ii), } y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

For this to be true for all time, $c_1 = 0$.

$$\therefore y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$$

$$\text{Also by (ii)} \quad y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0 \text{ for all } t.$$

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

$$\text{Thus } y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi c t}{l} + c_4 \sin \frac{n\pi c t}{l} \right) \quad \dots (\text{v})$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \left(-c_3 \sin \frac{n\pi c t}{l} + c_4 \cos \frac{n\pi c t}{l} \right)$$

$$\therefore \text{by (iv)} \quad \left(\frac{\partial y}{\partial t} \right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \cdot c_4 = 0$$

$$\text{Thus (v) becomes } y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l}$$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi c t}{l} \quad \dots (\text{vi})$$

$$\text{From (iii), } \mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{l} \int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l} dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l} \left\{ \left[(lx - x^2) \left(-\frac{\cos n\pi x/l}{n\pi/l} \right) \right]_0^l - \int_0^l (l - 2x) \left(-\frac{\cos n\pi x/l}{n\pi/l} \right) dx \right\}$$

$$\begin{aligned}
 &= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l-2x) \frac{\cos n\pi x}{l} dx \right\} = \frac{2\mu}{n\pi} \left\{ (l-2x) \frac{\sin n\pi x/l}{n\pi/l} \Big|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} dx \right\} \\
 &= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx = \frac{4\mu l}{n^2 \pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3 \pi^3} (1 - (-1)^n)
 \end{aligned}$$

Hence from (vi), the desired solution is

$$\begin{aligned}
 y(x, t) &= \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \\
 &= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi}{l} x \cos \frac{(2m-1)\pi ct}{l}.
 \end{aligned}$$

Example 18.6. A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$.

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$... (ii)

Also the initial conditions are $y(x, 0) = 0$... (iii)

$$\text{and } \left(\frac{\partial y}{\partial t} \right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \quad \dots (iv)$$

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$\text{By (ii), } y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

For this to be true for all time $c_1 = 0$.

$$\therefore y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$$

$$\text{Also } y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0 \text{ for all } t.$$

$$\text{This gives } pl = n\pi \quad \text{or} \quad p = \frac{n\pi}{l}, n \text{ being an integer.}$$

$$\text{Thus } y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t \right)$$

$$\text{By (iii), } 0 = c_2 c_3 \sin \frac{n\pi x}{l} \quad \text{for all } x \text{ i.e., } c_2 c_3 = 0$$

$$\therefore y(x, t) = b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \quad \text{where } b_n = c_2 c_4$$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \quad \dots (v)$$

$$\text{Now } \frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$$

$$\text{By (iv), } v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t} \right)_{t=0} = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$$

$$\begin{aligned}
 \text{or } \frac{v_0}{4} \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) &= \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l} \quad [\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta] \\
 &= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + ...
 \end{aligned}$$

Equating coefficients from both sides, we get

$$\begin{aligned} \frac{3v_0}{4} &= \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \dots \\ \therefore b_1 &= \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \dots = 0 \end{aligned}$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi} \left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l} \right).$$

Example 18.7. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t . (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$... (i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$... (ii)

Also the initial conditions are $y(x, 0) = 0$... (iii)

and

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = \lambda x(l-x) \quad \dots (iv)$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi ct}{l} \quad \dots (v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l} \cdot \left(\frac{n\pi c}{l} \right)$$

$$\text{By (iv), } \lambda x(l-x) = \left(\frac{\partial y}{\partial t} \right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

$$\begin{aligned} \therefore \frac{\pi c}{l} n b_n &= \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l} dx \\ &= \frac{2\lambda}{l} \left| (lx - x^2) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l-2x) \left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2) \left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right|_0^l \end{aligned}$$

$$= \frac{4\lambda l^2}{n^3\pi^3} (1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n]$$

$$\text{or } b_n = \frac{4\lambda l^3}{c\pi^4 n^4} [1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m-1)^4} \text{ taking } n = 2m-1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}.$$

Example 18.8. The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest. (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA (= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(i)$$

and the boundary conditions are

$$y(0, t) = 0 \quad \dots(ii)$$

$$y(l, t) = 0 \quad \dots(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \dots(iv)$$

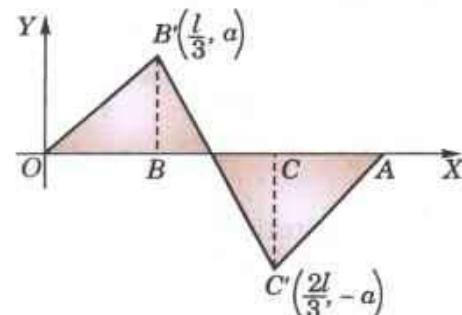


Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)x$;

$$\text{the equation of } B'C' \text{ is } y - a = \frac{-2a}{(l/3)} \left(x - \frac{l}{3}\right), \text{ i.e., } y = \frac{3a}{l}(l - 2x)$$

$$\text{and the equation of } C'A \text{ is } y = \frac{3a}{l}(x - l)$$

Hence the fourth boundary condition is

$$\left. \begin{aligned} y(x, 0) &= \frac{3a}{l}x, 0 \leq x \leq \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \frac{l}{3} \leq x \leq \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \frac{2l}{3} \leq x \leq l \end{aligned} \right\} \quad \dots(v)$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad [\text{Where } b_n = C_2 C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \dots(vi)$$

$$\text{Putting } t = 0, \text{ we have } y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(vii)$$

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

\therefore by (1) of § 10.7,

$$\begin{aligned} b_n &= \frac{2}{l} \left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l} dx + \int_{l/3}^{2l/3} \frac{3a}{l} (l - 2x) \sin \frac{n\pi x}{l} dx + \int_{2l/3}^l \frac{3a}{l} (x - l) \sin \frac{n\pi x}{l} dx \right] \\ &= \frac{6a}{l^2} \left[\left| x \left\{ -\frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - 1 \left\{ -\frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right|_{0}^{l/3} \right. \\ &\quad \left. + \left| (l - 2x) \left\{ -\frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - (-2) \left\{ \frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right|_{l/3}^{2l/3} \right. \\ &\quad \left. + \left| (x - l) \left\{ -\frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - (1) \cdot \left\{ -\frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right|_{2l/3}^l \right] \\ &= \frac{6a}{l^2} \left[\left(-\frac{l^2}{3n\pi} \cos \frac{n\pi}{3} + \frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{3} \right) + \frac{l^2}{3n\pi} \cos \frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2} \sin \frac{2n\pi}{3} + \frac{l^2}{3n\pi} \cos \frac{n\pi}{3} \right. \\ &\quad \left. + \frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{3} - \left(\frac{l^2}{3n\pi} \cos \frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2} \sin \frac{2n\pi}{3} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2} \left(\sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3} \right) \\
 &= \frac{18a}{n^2\pi^2} \sin \frac{n\pi}{3} [1 + (-1)^n] \quad \left[\because \sin \frac{2n\pi}{3} = \sin \left(n\pi - \frac{n\pi}{3} \right) = -(-1)^n \sin \frac{n\pi}{3} \right]
 \end{aligned}$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2} \sin \frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$\begin{aligned}
 y(x, t) &= \sum_{n=2, 4, \dots}^{\infty} \frac{36a}{n^2\pi^2} \sin \frac{n\pi}{3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad [\text{Take } n = 2m] \\
 &= \frac{9a}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \sin \frac{2m\pi x}{l} \cos \frac{2m\pi ct}{l} \quad \dots(vii)
 \end{aligned}$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, i.e. $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m .

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v .

$$\text{Then } \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial}{\partial u} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

$$\text{Similarly, } \frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right)$$

$$\text{Substituting in (1), we get } \frac{\partial^2 y}{\partial u \partial v} = 0 \quad \dots(2)$$

$$\text{Integrating (2) w.r.t. } v, \text{ we get } \frac{\partial y}{\partial u} = f(u) \quad \dots(3)$$

where $f(u)$ is an arbitrary function of u . Now integrating (3) w.r.t. u , we obtain

$$y = \int f(u) du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v . Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.

$$y(x, t) = \phi(x + ct) + \psi(x - ct) \quad \dots(4)$$

This is the *general solution of the wave equation (1)*.

Now to determine ϕ and ψ , suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

$$\text{Differentiating (4) w.r.t. } t, \text{ we get } \frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$$

$$\text{At } t = 0, \quad \phi'(x) = \psi'(x) \quad \dots(5)$$

$$\text{and } y(x, 0) = \phi(x) + \psi(x) = f(x) \quad \dots(6)$$

$$(5) \text{ gives, } \phi(x) = \psi(x) + k$$

$$\therefore (6) \text{ becomes } 2\psi(x) + k = f(x)$$

$$\text{or } \psi(x) = \frac{1}{2} [f(x) - k] \text{ and } \phi(x) = \frac{1}{2} [f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2} [f(x - ct) - k] = f(x + ct) + f(x - ct) \quad \dots(7)$$

which is the *d'Alembert's solution** of the wave equation (1)

(V.T.U., 2011 S)

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$. (V.T.U., 2011)

Solution. By d'Alembert's method, the solution is

$$\begin{aligned} y(x, t) &= \frac{1}{2} [f(x + ct) + f(x - ct)] \\ &= \frac{1}{2} [k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}] \\ &= k[\sin x \cos ct - \sin 2x \cos 2ct] \end{aligned}$$

Also $y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\frac{\partial y(x, 0)}{\partial t} = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibrations of a string of length l , fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\frac{\partial y(x, 0)}{\partial t} = 0$, $0 < x < l$. (Bhopal, 2007 S ; U.P.T.U., 2005)

2. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.

3. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection

$$f(x) = \frac{2k}{l}x \text{ when } 0 < x < \frac{l}{2}, \quad \frac{2k}{l}(l-x) \text{ when } \frac{l}{2} < x < l,$$

and initial velocity zero. (Bhopal, 2006 ; Kerala, M.E., 2005)

4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. (Anna, 2005)

5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by

$$f(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1-x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

If it is released from this position with velocity a , perpendicular to the x -axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by

$$u(x, t) = \frac{4\sqrt{2}}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{\sin((4pi - 3)\pi x) \cos((4pi - 3)\pi ct - \pi/4)}{(4n - 3)^2} - \frac{\sin((4pi - 1)\pi x) \cos((4pi - 1)\pi ct - \pi/4)}{(4n - 1)^2} \right\}$$

6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $0 < x < l/2$

$c(l - x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t . (Anna, 2008)

7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :

(i) $f(x) = a(x - x^2)$ (Kerala, M. Tech., 2005)

(ii) $f(x) = a \sin^2 \pi x$.

*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α (cm^2). Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α . Take one end of the bar as the origin and the direction of flow as the positive x -axis (Fig. 18.3). Let p be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O . If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (ii) p. 466, the quantity of heat in this slab = $s\alpha \delta x \delta u$. Hence the rate of increase of heat in this slab, i.e., $s\alpha \delta x \frac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

$$\text{Now by (A) of p. 466, } R_1 = -k\alpha \left(\frac{\partial u}{\partial x} \right)_x \text{ and } R_2 = -k\alpha \left(\frac{\partial u}{\partial x} \right)_{x+\delta x}$$

the negative sign appearing as a result of (i) on p. 466.

$$\text{Hence } s\alpha \delta x \frac{\partial u}{\partial t} = -k\alpha \left(\frac{\partial u}{\partial x} \right)_x + k\alpha \left(\frac{\partial u}{\partial x} \right)_{x+\delta x} \text{ i.e., } \frac{\partial u}{\partial t} = \frac{k}{sp} \left\{ \frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x} \right\}$$

Writing $k/sp = c^2$, called the *diffusivity* of the substance ($\text{cm}^2/\text{sec.}$), and taking the limit as $\delta x \rightarrow 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

This is the *one-dimensional heat-flow equation*.

(V.T.U., 2011)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \cdot T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT'' = c^2 X''T, \text{ i.e., } X''/X = T'/c^2 T \quad \dots(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2 X}{dx^2} - kX = 0 \quad \dots(3) \quad \text{and} \quad \frac{dT}{dt} - kc^2 T = 0 \quad \dots(4)$$

Solving (3) and (4), we get

(i) When k is positive and $= p^2$, say :

$$X = c_1 e^{px} + c_2 e^{-px}, T = c_3 e^{c^2 p^2 t};$$

(ii) When k is negative and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px, T = c_6 e^{-c^2 p^2 t};$$

(iii) When k is zero :

$$X = c_7 x + c_8, T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t} \quad \dots(5)$$

$$u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2 p^2 t} \quad \dots(6)$$

$$u = (c_7 x + c_8) c_9 \quad \dots(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, i.e., u is to decrease with the increase of time t . Accordingly, the solution given by (6), i.e., of the form

$$u = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t} \quad \dots(8)$$

is the only suitable solution of the heat equation.

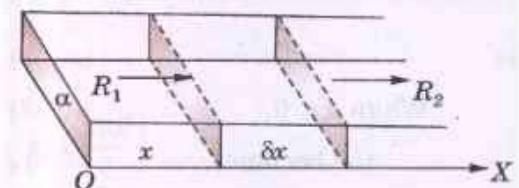


Fig. 18.3

Example 18.10. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ and $u(1, t) = 0$, where $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$... (i)

$$\text{is } u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-p^2 t} \quad \dots(ii)$$

$$\text{When } x = 0, \quad u(0, t) = c_1 e^{-p^2 t} = 0 \quad \text{i.e., } c_1 = 0. \quad \dots(iii)$$

$$\therefore (ii) \text{ becomes } u(x, t) = c_2 \sin p x e^{-p^2 t} \quad \dots(iv)$$

$$\text{When } x = 1, \quad u(1, t) = c_2 \sin p \cdot e^{-p^2 t} = 0 \text{ or } \sin p = 0 \quad \dots(v)$$

$$\text{i.e., } p = n\pi. \quad \dots(vi)$$

$$\therefore (v) \text{ reduces to } u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x \text{ where } b_n = c_2 \quad \dots(vii)$$

$$\text{Thus the general solution of (i) is } u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x \quad \dots(viii)$$

$$\text{When } t = 0, 3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 t} \sin n\pi x$$

$$\text{Comparing both sides, } b_n = 3 \quad \dots(ix)$$

Hence from (ix), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \sin n\pi x. \quad \dots(x)$$

Example 18.11. Solve the differential equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions:

$$(i) u \text{ is not infinite for } t \rightarrow \infty, (ii) \frac{\partial u}{\partial x} = 0 \text{ for } x = 0 \text{ and } x = l,$$

$$(iii) u = lx - x^2 \text{ for } t = 0, \text{ between } x = 0 \text{ and } x = l. \quad (\text{P.T.U., 2007})$$

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad \text{i.e., } X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)} \quad \dots(1)$$

$$\therefore \frac{d^2 X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + k^2 \alpha^2 T = 0 \quad \dots(1)$$

$$\text{Their solutions are } X = c_1 \cos kx + c_2 \sin kx, T = c_3 e^{-k^2 \alpha^2 t} \quad \dots(2)$$

$$\text{If } k^2 \text{ is changed to } -k^2, \text{ the solutions are}$$

$$X = c_4 e^{kx} + c_5 e^{-kx}, T = c_6 e^{k^2 \alpha^2 t} \quad \dots(3)$$

$$\text{If } k^2 = 0, \text{ the solutions are } X = c_7 x + c_8, T = c_9 \quad \dots(4)$$

In (3), $T \rightarrow \infty$ for $t \rightarrow \infty$ therefore, u also $\rightarrow \infty$ i.e., the given condition (i) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (ii) to (4), we get $c_7 = 0$.

$$\therefore u = XT = c_8 c_9 = a_0 \quad (\text{say}) \quad \dots(5)$$

$$\text{From (2), } \frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx) k c_3 e^{-k^2 \alpha^2 t} \quad \dots(6)$$

$$\text{Applying the condition (ii), we get } c_2 = 0 \text{ and } -c_1 \sin kl + c_2 \cos kl = 0 \quad \dots(7)$$

$$\text{i.e., } c_1 = 0 \quad \text{and} \quad kl = n\pi \quad (n \text{ an integer}) \quad \dots(8)$$

$$\therefore u = c_1 \cos kx \cdot c_3 e^{-k^2 \alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2 \pi^2 \alpha^2 t}}{l^2} \quad \dots(9)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \sum a_n \cos(n\pi x/l) e^{-n^2\pi^2\alpha^2 t/l^2} \quad \dots(7)$$

Now using the condition (iii), we get

$$lx - x^2 = a_0 + \sum a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l} \int_0^l (lx - x^2) dx = \frac{1}{l} \left| \frac{lx^2}{2} - \frac{x^3}{3} \right|_0^l = \frac{l^2}{6}$$

and

$$\begin{aligned} a_n &= \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \left| (lx - x^2) \left(\frac{l}{n\pi} \sin \frac{n\pi x}{l} \right) \right. \\ &\quad \left. - (l - 2x) \left(-\frac{l^2}{n^2\pi^2} \cos \frac{n\pi x}{l} \right) + (-2) \left(-\frac{l^3}{n^3\pi^3} \sin \frac{n\pi x}{l} \right) \right|_0^l \\ &= \frac{2}{l} \left\{ 0 - \frac{l^3}{n^2\pi^2} (\cos n\pi + 1) + 0 \right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0. \end{aligned}$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \left(\frac{2m\pi x}{l} \right) e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18.12. (a) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at a distance x from A at time t .
(U.P.T.U., 2005)

(b) Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C .
(Madras, 2000 S)

Solution. (a) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (steady state condition). When u depends only on x , (i) reduces to $\partial^2 u / \partial x^2 = 0$.

Its general solution is $u = ax + b$...(ii)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (ii) gives $b = 0$ and $a = 100/l$.

Thus the initial condition is expressed by $u(x, 0) = \frac{100}{l} x$...(iii)

Also the boundary conditions for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \quad \dots(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \quad \dots(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (i) subject to the initial condition (iii) and the boundary conditions (iv) and (v).

Now the solution of (i) is of the form

$$u(x, t) = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t} \quad \dots(vi)$$

By (iv), $u(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t .

Hence $C_1 = 0$ and (vi) reduces to $u(x, t) = C_2 \sin px \cdot e^{-c^2 p^2 t}$...(vii)

Applying (v), (vii) gives $u(l, t) = C_2 \sin pl \cdot e^{-c^2 p^2 t} = 0$, for all values of t .

This requires $\sin pl = 0$ i.e., $pl = n\pi$ as $C_2 \neq 0$. $\therefore p = n\pi/l$, where n is any integer.

Hence (vii) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2} \quad \dots(viii)$$

$$\text{Putting } t = 0, \quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(ix)$$

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\begin{aligned} \frac{100x}{l} &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l} \int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} dx \\ &= \frac{200}{l^2} \left[x \left\{ -\frac{\cos(n\pi x/l)}{(n\pi/l)} \right\} - (1) \left\{ -\frac{\sin(n\pi x/l)}{(n\pi/l)^2} \right\} \right]_0^l = \frac{200}{l^2} \left(-\frac{l^2}{n\pi} \cos n\pi \right) = \frac{200}{n\pi} (-1)^{n+1} \end{aligned}$$

$$\text{Hence (viii) gives } u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \quad \dots(x)$$

$$u(l, t) = 80 \text{ for all values of } t \quad \dots(xi)$$

In part (a), the boundary values (i.e., the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \quad \dots(xii)$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi); $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t .

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \quad \dots(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \quad \dots(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \quad \dots(xv)$$

$$\begin{aligned} \text{Also } u_t(x, 0) &= u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20 \right) && \text{[by (iii) and (xiii)]} \\ &= \frac{40x}{l} - 20 && \dots(xvi) \end{aligned}$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px) e^{-c^2 p^2 t}$

$$\text{By (xiv), } u_t(0, t) = C_1 e^{-c^2 p^2 t} = 0, \text{ for all values of } t.$$

$$\text{Hence } C_1 = 0 \text{ and } u_t(x, t) = C_2 \sin px \cdot e^{-c^2 p^2 t} \quad \dots(xvii)$$

$$\text{Applying (xv), it gives } u_t(l, t) = C_2 \sin pl e^{-c^2 p^2 t} = 0 \text{ for all values of } t.$$

$$\text{This requires } \sin pl = 0, \text{ i.e. } pl = n\pi \text{ as } C_2 \neq 0, p = n\pi/l, \text{ when } n \text{ is any integer.}$$

$$\text{Hence (xvii) reduces to } u_t(x, t) = b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} \text{ where } b_n = C_2.$$

Adding all such solutions, the most general solution of (xvii) satisfying the boundary conditions (xiv) and (xv) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} \quad \dots (xviii)$$

$$\text{Putting } t = 0, \text{ we have } u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots (xix)$$

In order that the condition (xvi) may be satisfied, (xvi) and (xix) must be same. This requires the expansion of $(40/l)x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l} \int_0^l \left(\frac{40x}{l} - 20 \right) \sin \frac{n\pi x}{l} dx = -\frac{40}{nx} (1 + \cos nx)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

$$\begin{aligned} \text{Hence (xviii) becomes } u_t(x, t) &= \sum_{n=2, 4, \dots}^{\infty} \left(\frac{-80}{n\pi} \right) \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t / l^2} && [\text{Take } n = 2m] \\ &= -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t / l^2} \end{aligned} \quad \dots (xx)$$

Finally combining (xiii) and (xx), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t / l^2}.$$

Example 18.13. The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t .

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$...(i)

In steady state condition, u is independent of time and depends on x only, (i) reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0. \quad \dots (ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u(x, 0) = 30 + \frac{5}{2}x \quad \dots (iii)$$

The boundary conditions are $u(0, t) = 40$, $u(20, t) = 60$

Using (ii), the steady state temperature is

$$u(x, 0) = 40 + \frac{60 - 40}{20} x = 40 + x \quad \dots (iv)$$

To find the temperature u in the intermediate period,

$$u(x, t) = u_s(x) + u_t(x, t)$$

where $u_s(x)$ is the steady state temperature distribution of the form (iv) and $u_t(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t(x, t)$ satisfies one dimensional heat equation

$$\therefore u(x, t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} \quad \dots (v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t} \quad \text{whence } a_n = 0.$$

$$\therefore (v) \text{ reduces to } u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{-p^2 t} \quad \dots(vi)$$

Also $u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20 pe^{-p^2 t}$

or $\sum_{n=1}^{\infty} b_n \sin 20 pe^{-p^2 t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$

Thus (vi) becomes $u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \quad \dots(vii)$

Using (iii), $30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

or $\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

where $b_n = \frac{2}{20} \int_0^{20} \left(\frac{3x}{2} - 10\right) \sin \frac{n\pi x}{20} dx = -\frac{20}{n\pi} (1 + 2 \cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1 + 2 \cos n\pi}{n} \sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ ($= 100$ cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(l, t) = 0 \text{ for all } t \quad \dots(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l$. $\dots(iii)$

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$ $\dots(iv)$

Differentiating partially w.r.t. x , we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px) e^{-c^2 p^2 t} \quad \dots(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 p e^{-c^2 p^2 t} = 0 \quad \text{for all } t$. [By (ii)]

$$\therefore c_2 = 0$$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl e^{-c^2 p^2 t} \text{ for all } t$. [By (ii)]

$$\therefore c_1 p \sin pl = 0 \text{ i.e., } p \text{ being } \neq 0, \text{ either } c_1 = 0 \text{ or } \sin pl = 0.$$

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $pl = n\pi \quad \text{or} \quad p = n\pi/l, \quad n = 0, 1, 2, \dots$

Hence (iv) becomes $u(x, t) = c_1 \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$.

\therefore the most general solution of (i) satisfying the boundary conditions (ii) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} \quad (\text{where } A_n = c_1) \dots (vi)$$

$$\text{Putting } t = 0, u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} = x \quad [\text{by (iii)}]$$

This requires the expansion of x into a half range cosine series in $(0, l)$.

$$\text{Thus } x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \quad \text{where } a_0 = \frac{2}{l} \int_0^l x dx = l$$

$$\text{and } a_n = \frac{2}{l} \int_0^l x \cos \frac{n\pi x}{l} dx = \frac{2l}{n^2 \pi^2} (\cos n\pi - 1)$$

$$= 0, \text{ where } n \text{ is even; } = -4l/n^2 \pi^2, \text{ when } n \text{ is odd.}$$

$$\therefore A_0 = \frac{a_0}{2} = l/2, \text{ and } A_n = a_n = 0 \text{ for } n \text{ even; } = -4l/n^2 \pi^2 \text{ for } n \text{ odd.}$$

Hence (vi) takes the form

$$\begin{aligned} u(x, t) &= \frac{l}{2} + \sum_{n=1, 3, \dots}^{\infty} \frac{4l}{n^2 \pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2} \\ &= \frac{l}{2} - \frac{4l}{\pi^2} \sum_1^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t / l^2} \end{aligned} \dots (vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t .

Obs. The sum of the temperatures at any two points equidistant from the centre is always 100°C , a constant.

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

\therefore Replacing x by $l - x$ in (vii), we get the temperature at P_2 as

$$\begin{aligned} u(l-x, t) &= \frac{l}{2} - \frac{4l}{\pi^2} \sum_1^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{-c^2 (2n-1)^2 \pi^2 t / l^2} \\ &= \frac{l}{2} + \frac{4l}{\pi^2} \sum_1^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t / l^2} \end{aligned} \dots (viii)$$

$$\left\{ \because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos \left[2n\pi - \pi - \frac{(2n-1)\pi x}{l} \right] = -\cos \frac{(2n-1)\pi x}{l} \right.$$

Adding (vii) and (viii), we get $u(x, t) + u(l-x, t) = l = 100^\circ\text{C}$.

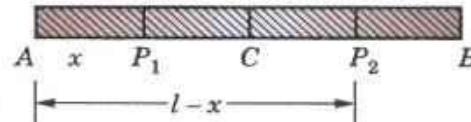


Fig. 18.4

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$\begin{aligned} u(x, 0) &= x, & 0 \leq x \leq 50 \\ &= 100 - x, & 50 \leq x \leq 100. \end{aligned}$$

Find the temperature $u(x, t)$ at any time.

(Bhopal, 2007; S.V.T.U., 2007; Kurukshetra, 2006)

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l , whose ends are kept at temperature 0°C and whose initial temperature in $(^\circ\text{C})$ is given by $ax(l-x)/l^2$. (P.T.U., 2009)
3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A . (Anna, 2008)
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C . Find the temperature distribution in the bar at time t . (P.T.U., 2010)
- Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
- $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad \frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0, \quad u(x, 0) = x. \quad (\text{S.V.T.U., 2008})$$
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.
7. Find the solution of $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that
- $$(i) \theta \text{ is not infinite when } t \rightarrow +\infty; \quad (ii) \left. \begin{array}{l} \frac{\partial \theta}{\partial x} = 0 \quad \text{when } x = 0 \\ \theta = 0, \quad \text{when } x = l \end{array} \right\} \text{for all values of } t; \\ (iii) \theta = \theta_0, \text{ when } t = 0, \text{ for all values of } x \text{ between } 0 \text{ and } l. \quad (\text{S.V.T.U., 2008})$$
8. Find the solution of $\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm³), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z -coordinate and depends only on x, y and time t , then the flow is said to be two-dimensional. In this case, the heat flow is in the XY -plane only and is zero along the normal to the XY -plane.

Consider a rectangular element $ABCD$ of the plane with sides δx and δy . By (A) on p. 466, the amount of heat entering the element in 1 sec. from the side AB

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y} \right)_y$$

and the amount of heat entering the element in 1 second from the side AD = $-k\alpha\delta y \left(\frac{\partial u}{\partial x} \right)_x$

The quantity of heat flowing out through the side CD per sec. = $-k\alpha\delta x \left(\frac{\partial u}{\partial y} \right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second = $-k\alpha\delta y \left(\frac{\partial u}{\partial x} \right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y} \right)_y - k\alpha\delta y \left(\frac{\partial u}{\partial x} \right)_x + k\alpha\delta x \left(\frac{\partial u}{\partial y} \right)_{y+\delta y} + k\alpha\delta y \left(\frac{\partial u}{\partial x} \right)_{x+\delta x}$$

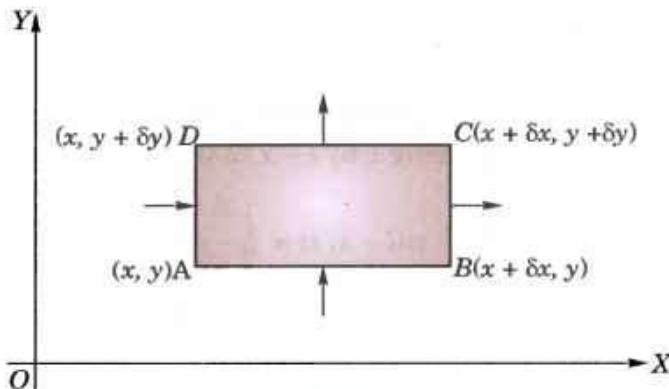


Fig. 18.5

$$\begin{aligned}
 &= k\alpha\delta x \left[\left(\frac{\partial u}{\partial y} \right)_{y+\delta y} - \left(\frac{\partial u}{\partial y} \right)_y \right] + k\alpha\delta y \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right] \\
 &= k\alpha\delta x\delta y \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y} \right)_{y+\delta y} - \left(\frac{\partial u}{\partial y} \right)_y}{\delta y} \right] \quad \dots(1)
 \end{aligned}$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s \frac{\partial u}{\partial t} \quad \dots(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y} \right)_{y+\delta y} - \left(\frac{\partial u}{\partial y} \right)_y}{\delta y} \right] = \rho\delta x\delta y\alpha s \frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \rightarrow 0, \delta y \rightarrow 0$, we get

$$\begin{aligned}
 k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= \rho s \frac{\partial u}{\partial t} \\
 i.e., \quad \frac{\partial \mathbf{u}}{\partial t} &= c^2 \left(\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} \right) \text{ where } c^2 = k/\rho s \text{ is the diffusivity.} \quad \dots(3)
 \end{aligned}$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state*.

Cor. In the *steady state*, u is independent of t , so that $\partial u / \partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} = 0$$

which is the well known **Laplace's equation in two dimensions**.

Obs. When the stream lines are curves in space, i.e., the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

In a *steady state*, it reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

which is the *three dimensional Laplace's equation*.

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} = 0 \quad \dots(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\frac{d^2 X}{dx^2} Y + X \frac{d^2 Y}{dy^2} = 0$

or separating the variables, $\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2}$...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2 X}{dx^2} - kX = 0 \text{ and } \frac{d^2 Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(i) When k is positive and is equal to p^2 , say

$$X = c_1 e^{px} + c_2 e^{-px}, Y = c_3 \cos py + c_4 \sin py$$

(ii) When k is negative, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px, Y = c_7 e^{py} + c_8 e^{-py}$$

(iii) When k is zero ; $X = c_9 x + c_{10}$, $Y = c_{11} y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \dots(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \dots(4)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \dots(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(V.T.U., 2011 S ; Kerala, 2005)

Temperature distribution in long plates

Example 18.15. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π ; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.

(P.T.U., 2005 ; J.N.T.U., 2002 S)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(i)$$

The boundary conditions are $u(0, y) = 0$ for all values of y ...(ii)

$$u(\pi, y) = 0 \text{ for all values of } y \quad \dots(iii)$$

$$u(x, \infty) = 0 \text{ in } 0 < x < \pi \quad \dots(iv)$$

$$u(x, 0) = u_0 \text{ in } 0 < x < \pi \quad \dots(v)$$

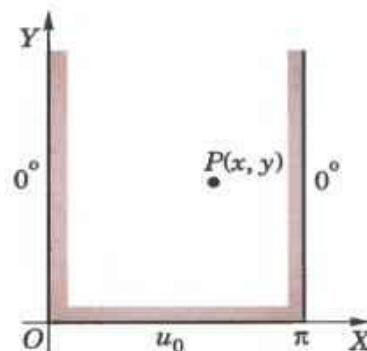


Fig. 18.6

Now the three possible solutions of (i) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \dots(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \dots(vii)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \dots(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (vi) cannot satisfy the condition (ii) for $u \neq 0$ for $x = 0$, for all values of y . The solution (viii) cannot satisfy the condition (iv). Thus the only possible solution is (vii), i.e. of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \dots(ix)$$

By (ii), $u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all y .

Hence $C_1 = 0$ and (ix) reduces to

$$u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py}) \quad \dots(x)$$

By (iii), $u(\pi, y) = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py}) = 0$, for all y .

This requires $\sin p\pi = 0$, i.e. $p\pi = n\pi$ as $C_2 \neq 0$. $\therefore p = n$, an integer.

Also to satisfy the condition (iv), i.e., $u = 0$ as $y \rightarrow \infty$, $C_3 = 0$.

Hence (x) takes the form $u(x, y) = b_n \sin nx \cdot e^{-ny}$, where $b_n = C_2 C_4$.

\therefore the most general solution satisfying (ii), (iii) and (iv) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \cdot e^{-ny} \quad \dots(xi)$$

$$\text{Putting } y = 0, \quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad \dots(xii)$$

In order that the condition (v) may be satisfied, (v) and (xii) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin nx dx = \frac{2u_0}{n\pi} [1 - (-1)^n]$$

i.e.,

$$b_n = 0, \text{ if } n \text{ is even}; = 4u_0/n\pi, \text{ if } n \text{ is odd.}$$

$$\text{Hence (xi) becomes } u(x, y) = \frac{4u_0}{\pi} \left[e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x + \frac{1}{5} e^{-5y} \sin 5x + \dots \right].$$

Temperature distribution in finite plates

Example 18.16. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin n\pi x/l$. (V.T.U., 2011; J.N.T.U., 2006; Kerala M. Tech., 2005, U.P.T.U., 2004)

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(i)$$

$$\text{are } u = (c_1 e^{px} + c_2 e^{-px}) (c_3 \cos py + c_4 \sin py) \quad \dots(ii)$$

$$u = (c_5 \cos px + c_6 \sin px) (c_7 e^{py} + c_8 e^{-py}) \quad \dots(iii)$$

$$u = (c_9 x + c_{10}) (c_{11} y + c_{12}) \quad \dots(iv)$$

We have to solve (i) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \dots(v) \quad u(l, y) = 0 \quad \dots(vi)$$

$$u(x, 0) = 0 \quad \dots(vii) \quad u(x, a) = \sin n\pi x/l \quad \dots(viii)$$

Using (v) and (vi) in (ii), we get

$$c_1 + c_2 = 0, \text{ and } c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (v) and (vi) in (iv). Hence the suitable solution for the present problem is solution (iii). Using (v) in (iii), we have $c_5(c_7 e^{py} + c_8 e^{-py}) = 0$ i.e., $c_5 = 0$

$$\therefore (iii) \text{ becomes } u = c_6 \sin px (c_7 e^{py} + c_8 e^{-py}) \quad \dots(ix)$$

$$\text{Using (vi), we have } c_6 \sin pl (c_7 e^{py} + c_8 e^{-py}) = 0$$

$$\therefore \text{either } c_6 = 0 \text{ or } \sin pl = 0$$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \dots$

$$\therefore (ix) \text{ becomes } u = c_6 \sin(n\pi x/l) (c_7 e^{n\pi y/l} + c_8 e^{-n\pi y/l}) \quad \dots(x)$$

$$\text{Using (vii), we have } 0 = c_6 \sin n\pi x/l \cdot (c_7 + c_8) \text{ i.e., } c_8 = -c_7.$$

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin \frac{n\pi x}{l} (e^{n\pi y/l} - e^{-n\pi y/l}) \text{ where } b_n = c_6 c_7$$

Now using the condition (viii), we have

$$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l} (e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get

$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin \frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)} \sin \frac{n\pi x}{l}.$$

Example 18.17. The function $v(x, y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0, x = a, y = 0, y = b$, it satisfies the conditions $v(0, y) = v(a, y) = v(x, b) = 0$ and $v(x, 0) = x(a - x)$, $0 < x < a$. Show that $v(x, y)$ is given by

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x/a}{(2n+1)^3} \frac{\sinh((2n+1)\pi(b-y)/a)}{\sinh((2n+1)\pi b/a)}$$

(Madras, 2003)

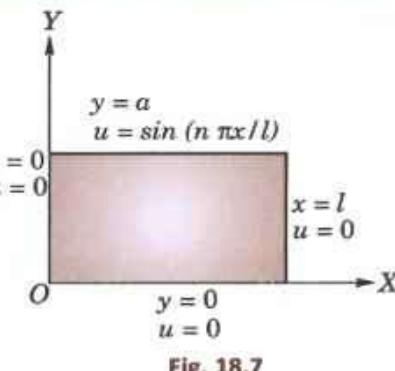


Fig. 18.7

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \dots(i)$$

is of the form

$$v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \dots(ii)$$

The boundary conditions are

$$v(0, y) = 0; \quad v(a, y) = 0 \quad \dots(iii)$$

$$v(x, b) = 0 \quad \dots(iv)$$

$$v(x, 0) = x(a-x), \quad 0 < x < a. \quad \dots(v)$$

Using (iii)

$$v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$$

∴ (ii) becomes

$$v(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \dots(vi)$$

Again using (iii),

$$v(a, y) = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py}) = 0.$$

i.e.,

$$\sin pa = 0, \quad i.e. \quad pa = n\pi \quad \text{or} \quad p = n\pi/a$$

∴ (vi) becomes

$$v(x, y) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}} \right)$$

or

$$v(x, y) = \sin \frac{n\pi x}{a} (Ae^{n\pi y/a} + Be^{-n\pi y/a}) \quad \text{where} \quad A = c_2 c_3, \quad B = c_2 c_4 \quad \dots(vii)$$

Now using (iv),

$$v(x, b) = \sin \frac{n\pi x}{a} \left(Ae^{\frac{n\pi b}{a}} + Be^{-\frac{n\pi b}{a}} \right) = 0$$

i.e.,

$$Ae^{n\pi b/a} + Be^{-n\pi b/a} = 0 \quad \text{or} \quad Ae^{n\pi b/a} - Be^{-n\pi b/a} = -\frac{1}{2} b_n \quad (\text{say})$$

Thus (vii) becomes

$$\begin{aligned} v(x, y) &= \sin \frac{n\pi x}{a} \cdot \frac{1}{2} b_n \left\{ e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a} \right\} \\ &= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \end{aligned}$$

∴ the most general solution of (i) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \quad \dots(viii)$$

Using the condition (v), we have

$$x(a-x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

$$\text{where } b_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a x(a-x) \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a} \left| (ax - x^2) \left(\frac{-\cos n\pi x/a}{n\pi/a} \right) - (a-2x) \left(-\frac{\sin n\pi x/a}{(n\pi/a)^2} \right) + (-2) \left\{ \frac{\cos n\pi x/a}{(n\pi/a)^3} \right\} \right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3 \pi^3} (1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3 \pi^3} \quad \text{when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from (viii), the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1, 3, 5, \dots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0, v(a, y) = 0, v(x, \infty) = 0, v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \quad (\text{J.N.T.U., 2005})$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin(\pi x/8), \quad 0 < x < 8;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C , show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100e^{-\pi y/8} \sin(\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20x \quad \text{for } 0 \leq x \leq 5$$

and

$$u = 20(10 - x) \quad \text{for } 5 \leq x \leq 10$$

and the two long edges $x = 0, x = 10$ as well as the other short edge are kept at 0°C , prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} e^{-(2n-1)\pi y/10}. \quad (\text{Anna, 2009})$$

4. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi, 0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0, u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C . Find the steady state temperature in the plate. (Madras, 2003)

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm, and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y) . Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi} \left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \dots \right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0, x = a, y = 0, y = a$. The edges $x = 0, y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0 . Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \frac{(2n-1)\pi x}{2a} \cos \frac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \frac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b . Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0, x = a, y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \frac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(1)$$

Assume that a solution of (1) is of the form $u = R(r) \cdot \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2 R'' \phi + r R' \phi + R \phi'' = 0$ or $\phi(r^2 R'' + r R') + R \phi'' = 0$.

$$\text{Separating the variables } \frac{r^2 R'' + r R'}{R} = -\frac{\phi''}{\phi} \quad \dots(2)$$

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - kR = 0 \quad \dots(3) \quad \text{and} \quad \frac{d^2 \phi}{d\theta^2} + k\phi = 0 \quad \dots(4)$$

$$\text{Putting } r = e^z, (3) \text{ reduces to } \frac{d^2 R}{dz^2} - kR = 0 \quad \dots(5)$$

Solving (5) and (4), we get

(i) When k is positive and $= p^2$, say :

$$R = c_1 e^{pz} + c_2 e^{-pz} = c_1 r^p + c_2 r^{-p}, \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(ii) When k is negative and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos(p \log r) + c_6 \sin(p \log r), \phi = c_7 e^{p\theta} + c_8 e^{-p\theta}$$

(iii) When k is zero :

$$R = c_9 z + c_{10} = c_9 \log r + c_{10}, \phi = c_{11} \theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \dots(6)$$

$$u = [c_5 \cos(p \log r) + c_6 \sin(p \log r)](c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad \dots(7)$$

$$u = (c_9 \log r + c_{10})(c_{11} \theta + c_{12}) \quad \dots(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (S.V.T.U., 2008)

Example 18.18. The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is $T^\circ\text{C}$. Show that the steady state temperature in the plate is given by

$$u(r, \theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{r}{a}\right)^{2n-1} \sin(2n-1)\theta. \quad \text{(Kerala M. Tech., 2005)}$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(i)$$

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \leq r \leq a \quad \dots(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \leq r \leq a \quad \dots(iii)$$

$$u(a, \theta) = T \quad \dots(iv)$$

and

The three possible solutions of (i) are

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad \dots(v)$$

$$u = [c_5 \cos(p \log r) + c_6 \sin(p \log r)](c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad \dots(vi)$$

$$u = (c_9 \log r + c_{10})(c_{11} \theta + c_{12}) \quad \dots(vii)$$

From (ii) and (iii), $u = 0$ when $r = 0$ i.e., u must be finite at the origin. Thus the solutions (vi) and (vii) are to be rejected. Hence the only suitable solution is (v).

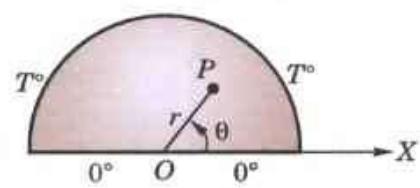


Fig. 18.8

By (ii),

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_3 = 0$$

Hence $c_3 = 0$ and (v) becomes

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta \quad \dots(viii)$$

By (iii),

$$u(r, \pi) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi = 0.$$

As $c_4 \neq 0$, $\sin p\pi = 0$, i.e., $p = n$, where n is any integer.

Hence (viii) reduces to

$$u(r, \theta) = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta \quad \dots(ix)$$

Since $u = 0$, when $r = 0$, $\therefore c_2 = 0$ and (ix) becomes

$$u(r, \theta) = b_n r^n \sin n\theta, \text{ where } b_n = c_1 c_4.$$

\therefore the most general solution of (i) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \dots(x)$$

Putting $r = a$,

$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \quad \dots(xi)$$

In order that (iv) may be satisfied, (iv) and (xi) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where } B_n = \frac{2}{\pi} \int_0^{\pi} T \sin n\theta d\theta = \frac{2T}{n\pi} (1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n} (1 - \cos n\pi)$$

$$i.e., \quad b_n = 0, \text{ if } n \text{ is even}$$

$$= \frac{4T}{n\pi a^n}, \text{ if } n \text{ is odd.}$$

$$\text{Hence (x) gives } u(r, \theta) = \frac{4T}{\pi} \left\{ \frac{(r/a)}{1} \sin \theta + \frac{(r/a)^3}{3} \sin 3\theta + \frac{(r/a)^5}{5} \sin 5\theta + \dots \right\}$$

Example 18.19. The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta \leq \pi/2 \\ 50(\pi - \theta), & \text{when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function $u(r, \theta)$.

(Madras, 2003)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r \frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial r^2} = 0 \quad \dots(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$

$$\text{and } u(a, \theta) = 50\theta \text{ for } 0 \leq \theta \leq \pi/2; u(a, \theta) = 50(\pi - \theta) \text{ for } \pi/2 \leq \theta \leq \pi \quad \dots(ii)$$

As in example 18.18, the most general solution of (i) satisfying the boundary conditions (ii) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \dots(iv)$$

Putting $r = a$,

$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (iii) is satisfied, we have $u(a, \theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$

$$\text{where } b_n a^n = B_n = \frac{2}{\pi} \left\{ \int_0^{\pi/2} 50\theta \sin n\theta d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta) \sin n\theta d\theta \right\} \quad \dots(v)$$

$$\begin{aligned}
 &= \frac{100}{\pi} \left\{ \left| \theta \left(\frac{-\cos n\theta}{\theta} \right) - (1) \left(\frac{-\sin n\theta}{n^2} \right) \right|_0^{\pi/2} + \left| (\pi - \theta) \left(\frac{-\cos n\theta}{n} \right) - (-1) \left(\frac{-\sin n\theta}{n^2} \right) \right|_{\pi/2}^{\pi} \right\} \\
 &= \frac{100}{\pi} \left\{ -\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2} + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2} \right\} = \frac{200}{\pi n^2} \sin n\pi/2.
 \end{aligned}$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (iv) gives

$$\begin{aligned}
 u(r, \theta) &= \sum_{n=1, 3, 5, \dots}^{\infty} \left(\frac{200}{\pi n^2} \sin \frac{n\pi}{2} \right) \frac{1}{a^n} \cdot r^n \sin n\theta \\
 &= \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2} \left(\frac{r}{a} \right)^{2m-1} \sin (2m-1)\theta.
 \end{aligned}$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \dots$; gives $m = 1, 2, 3, \dots$, $\sin n\pi/2 = \sin (2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.]

PROBLEMS 18.5

- A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
- A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \leq \theta \leq \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
- A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
- Determine the steady state temperature at the points on the sector $0 \leq \theta \leq \pi/4$, $0 \leq r \leq a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature $k^\circ\text{C}$ along the curved edges.
- Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C .
- If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and

$$u(10, \theta) = 15 \cos \theta, u(20, \theta) = 30 \sin \theta,$$

find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]

- A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are

$$u(2, \theta) = 10 \sin \theta + 6 \cos \theta, u(4, \theta) = 17 \sin \theta + 15 \cos \theta$$

Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy -plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$$\begin{aligned}
 &= T\delta y \sin \beta - T\delta y \sin \alpha \\
 &= T\delta y (\tan \beta - \tan \alpha) \text{ approximately, since } \alpha \text{ and } \beta \text{ are small}
 \end{aligned}$$

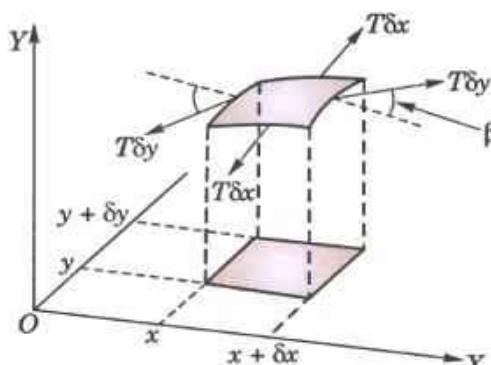


Fig. 18.9

$$= T\delta y \left\{ \left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right\} = T\delta y \frac{\partial^2 u}{\partial x^2} \delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x \frac{\partial^2 u}{\partial y^2} \delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y \frac{\partial^2 u}{\partial t^2} = T \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \delta x\delta y \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad \text{where } c^2 = T/m \quad \dots(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT , we get

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2 X}{dx^2} + k^2 X = 0, \quad \frac{d^2 Y}{dy^2} + l^2 Y = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + (k^2 + l^2) c^2 T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos \sqrt{(k^2 + l^2)} ct + c_6 \sin \sqrt{(k^2 + l^2)} ct] \quad \dots(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0, x = a, y = 0, y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)} ct + c_6 \sin \sqrt{(k^2 + l^2)} ct] \quad \text{i.e., } c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \quad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2 c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (c_5 \cos p_{mn} t + c_6 \sin p_{mn} t)$$

$$\text{where } p_{mn} = \pi c \sqrt{[(m/a)^2 + (n/b)^2]} \quad \dots(3)$$

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2 c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \quad \dots(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, i.e., $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

This is *double Fourier series*. Multiplying both sides by $\sin(m\pi x/a) \sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx = \frac{ab}{4} A_{mn} \quad \dots(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula**.

Rectangular Membranes

Example 18.20. Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$\begin{aligned} A_{mn} &= 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y dy dx \\ &= 4A \int_0^1 \sin \pi x \sin m\pi x dx \left(\int_0^1 \sin 2\pi y \sin n\pi y dy \right) = 0, \quad \text{for } m \neq 1 \\ &= 4A \left(\frac{1}{2} \right) \int_0^1 \sin 2\pi y \sin n\pi y dy, \quad \text{for } m = 1 \quad \left[\because \int_0^1 \sin \pi x \sin \pi x dx = \frac{1}{2} \right] \end{aligned}$$

$$\text{i.e., } A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y dy = 0, \quad \text{for } n \neq 2$$

$$= 2A \left(\frac{1}{2} \right), \quad \text{for } n = 2.$$

$$\therefore A_{12} = A. \text{ Also from (3), } p_{mn} = \pi \sqrt{(m^2 + n^2)}$$

$$\therefore p_{12} = \pi \sqrt{(1^2 + 2^2)} = \sqrt{5}\pi.$$

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos(\sqrt{5}\pi t)$.

Example 18.21. Find the vibration $u(x, y, t)$ of a rectangular membrane ($0 < x < a$, $0 < y < b$) whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.

Solution. Proceeding as in § 18.9 (2), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c \sqrt{[(m/a)^2 + (n/b)^2]}$$

Since the membrane starts from rest $\partial u / \partial t = 0$ when $t = 0$,

$$\therefore \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn} p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \quad \dots(i)$$

$$\text{Then } hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\text{where } A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a - x)(b - y) \cdot \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dy dx$$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a - x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| \left(ax - x^2 \right) \left(\frac{-\cos m\pi x/a}{m\pi/a} \right) - (a - 2x) \left\{ \frac{-\sin m\pi x}{(m\pi/a)^2} \right\} + (-2) \frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| \left(by - y^2 \right) \left(\frac{-\cos n\pi y/b}{n\pi/b} \right) - (b - 2y) \left\{ \frac{-\sin n\pi y/b}{(n\pi/b)^2} \right\} + (-2) \frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (i), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \frac{16ha^2b^2}{m^3n^3\pi^6} (1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c \sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos(c\alpha_m t) \cdot J_0(\alpha_m r) \text{ where } A_m = \frac{2}{J_0^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r) dr,$$

and α_m ($m = 1, 2, \dots$) are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates i.e.,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad \dots(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \quad \dots(ii)$$

For solutions not depending on θ ,

$$\text{initial deflection } u(r, 0) = f(r) \quad \dots(iii)$$

$$\text{and initial velocity } \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \quad \dots(iv)$$

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ satisfying the boundary condition (ii).

Differentiating and substituting (v) in (i), we get

$$\frac{\partial^2 T}{\partial t^2} = \frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

$$\text{This leads to } \frac{\partial^2 T}{\partial t^2} + p^2 T = 0 \text{ where } p = ck \quad \dots(vi)$$

$$\text{and } \frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k^2 R = 0 \quad \dots(vii)$$

Now putting $s = kr$, (vii) transforms to $\frac{d^2 R}{ds^2} + \frac{1}{s} \frac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution

$R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m, m = 1, 2, \dots$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (vii) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \dots$ and the corresponding solutions of (vi) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (i) satisfying (ii) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t) J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m .

To find that solution which also satisfies the initial conditions (iii) and (iv), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t) J_0(\alpha_m r)$$

Putting $t = 0$ and using (iii), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, i.e.,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r) dr$$

Using (iv), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

Let the instantaneous voltage and current at any point P , distant x km from the sending end O , and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \rightarrow 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L \frac{\partial i}{\partial t} \quad \dots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$$\therefore -\delta i = C \frac{\partial v}{\partial t} \delta x + Gv\delta x \text{ from which as before, we get}$$

$$-\frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t} + Gv \quad \dots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L \frac{\partial}{\partial t} \right) i + \frac{\partial v}{\partial x} = 0 \quad \dots(3)$$

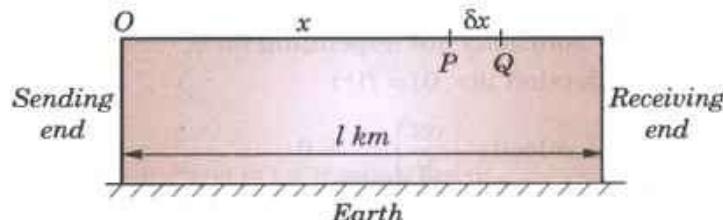


Fig. 18.10

and

$$\frac{\partial i}{\partial x} + \left(C \frac{\partial}{\partial t} + G \right) v = 0, \quad \dots(4)$$

we shall eliminate i and v in turn.

\therefore operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R + L \frac{\partial}{\partial t} \right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2} - \left(R + L \frac{\partial}{\partial t} \right) \left(C \frac{\partial}{\partial t} + G \right) v = 0$$

or

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} + (LG + RC) \frac{\partial v}{\partial t} + RGv \quad \dots(5)$$

Again operating (3) by $\left(C \frac{\partial}{\partial t} + G \right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C \frac{\partial}{\partial t} + G \right) \left(R + L \frac{\partial}{\partial t} \right) i - \frac{\partial^2 i}{\partial x^2} = 0$$

or

$$\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} + (LG + RC) \frac{\partial i}{\partial t} + RGi \quad \dots(6)$$

which is (5) with v replaced by i . The equations (5) and (6) are called the *telephone equations*.

Cor. 1. If $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \dots(7) \qquad \frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t} \quad \dots(8)$$

which are known as the *telegraph equations*.

Rewriting (7) as $\frac{\partial v}{\partial t} = \frac{1}{RC} \frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. If $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \dots(9) \qquad \frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} \quad \dots(10)$$

which are called the *radio equations*.

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2} = k^2 \frac{\partial^2 v}{\partial x^2}$ where $k^2 = \frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$.

[See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. If $L = C = 0$, e.g., in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2} = GRv, \text{ i.e. } (D^2 - GR)v = 0$$

$$\therefore v(x) = A \cosh(\sqrt{GR} \cdot x) + B \sinh(\sqrt{GR} \cdot x) \quad \dots(11)$$

$$\text{Since by (1), } Ri = -\frac{\partial v}{\partial x} = -\sqrt{GR} [A \sinh(\sqrt{GR} \cdot x) + B \cosh(\sqrt{GR} \cdot x)]$$

$$\therefore i(x) = -\sqrt{G/R} [A \sinh(\sqrt{GR} \cdot x) + B \cosh(\sqrt{GR} \cdot x)] \quad \dots(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(G/R)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x) = v_0 \cosh \gamma x - i_0 z_0 \sinh \gamma x \quad \dots(13)$$

and

$$i(x) = i_0 \cosh \gamma x - \frac{v_0}{z_0} \sinh \gamma x. \quad \dots(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. Neglecting R and G , find the e.m.f. $v(x, t)$ in a line of length l , t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$. (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the Radio equation $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$... (i)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, v(l, t) = 0 \quad \dots (ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and

$$v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} \quad \dots (iii)$$

$$\therefore \frac{\partial i}{\partial x} = -c \frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad \dots (iv)$$

Let $v = X(x)T(t)$ be the solution of (i).

$$\therefore (i) \text{ gives } X''T = LCXT''$$

$$\frac{X''}{X} = LC \frac{T''}{T} = -k^2 \quad (\text{say})$$

$$\therefore X'' + k^2 X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx) \left(c_3 \cos \frac{k}{\sqrt{LC}} t + c_4 \sin \frac{k}{\sqrt{LC}} t \right)$$

Using the boundary conditions (ii), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore v = \sin \frac{n\pi x}{l} \left(a_n \cos \frac{n\pi}{l\sqrt{LC}} t + b_n \sin \frac{n\pi}{l\sqrt{LC}} t \right)$$

Using the initial condition (iv), we get $b_n = 0$

$$\therefore v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t$$

Thus the most general solution of (i) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (iii), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$$\therefore a_1 = e_1 \quad \text{and} \quad a_5 = e_5 \quad \text{while all other } a's \text{ are zero.}$$

$$\text{Hence} \quad v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC \frac{T'}{T} = -k^2 \quad (\text{say})$$

$$\therefore X'' + k^2X = 0 \quad \text{and} \quad T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx, \quad T = c_3 e^{-k^2 t/RC}$$

giving

$$v = (c_1 \cos kx + c_2 \sin kx)c_3 e^{-k^2 t/RC} \quad \dots(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$$\therefore 0 = c_1 c_3; \quad 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e.,

$$c_1 c_3 = 0 \quad \text{and} \quad kl = n\pi \quad (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0 x + b_0 + \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 t / RCl^2} \quad \dots(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

$$\text{Then (ii) becomes} \quad v = \frac{E}{l} x + \sum b_n \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 t / RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \sum b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$$\begin{aligned} \therefore b_n &= \frac{2}{l} \int_0^l \left(-\frac{Ex}{l} \right) \sin \left(\frac{n\pi x}{l} \right) dx \\ &= \frac{2}{l} \left[\left(-\frac{Ex}{l} \right) \left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - \left(-\frac{E}{l} \right) \left(-\frac{l^2}{n^2 \pi^2} \sin \frac{n\pi x}{l} \right) \right]_0^l = \frac{2}{l} \left(\frac{El}{n\pi} \cos n\pi \right) = \frac{2E}{n\pi} (-1)^n. \end{aligned}$$

$$\text{Thus} \quad v = \frac{Ex}{l} + \frac{2E}{\pi} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} e^{-n^2 \pi^2 t / RCl^2} \quad \dots(iii)$$

$$\text{Also when } L = 0, \quad \frac{-\partial v}{\partial x} = Ri$$

$$\text{i.e.,} \quad i = -\frac{1}{R} \frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR} \sum_{n=1}^{\infty} (-1)^n \cos \frac{n\pi x}{l} e^{-n^2 \pi^2 t / RCl^2}$$

At the grounded end ($x = 0$), the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR} \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi^2 t / RCl^2}$$

$$\text{When } t = 1 \text{ sec,} \quad i = -\frac{E}{lR} \left(1 - 2e^{\pi^2 / RCl^2} + 2e^{-4\pi^2 / RCl^2} - \dots \right) \quad \dots(iv)$$

$$\text{Since} \quad \frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5 \times 10^{-7})(3000)^2} = 0.548$$

$$\therefore e^{-\pi^2 / RCl^2} = e^{-0.548} = 0.578$$

$$\text{When} \quad t \rightarrow \infty, \quad i \rightarrow -E/lR$$

Hence from (iv), we have

$$\begin{aligned} i &= -\frac{E}{lR} \{1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + \dots\} \\ &= -\frac{E}{lR} \{1 - 1.156 + 0.223 - 0.014 + \dots\} \\ &= i_{\infty}(0.053) = 5.3\% \text{ of } i_{\infty}. \end{aligned}$$

Example 18.25. A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential $v(x, t)$. (Andhra, 2000)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC} \frac{\partial^2 v}{\partial x^2} \quad \dots(i)$$

$$v_s = \text{initial steady voltage satisfying } \frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0) \quad \dots(ii)$$

$$v'_s = \text{steady voltage (after grounding the terminal end) when steady conditions are ultimately reached} = 1300 - 1.3x$$

$$\therefore v(x, t) = v'_s + v_t(x, t) \text{ where } v_t(x, t) \text{ is the transient part}$$

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2 \pi^2 t)/(l^2 RC)} \sin \frac{n\pi x}{l} \quad [\text{By (viii), p. 614}] \quad \dots(iii)$$

where $l = 1000$ kilometers.

Putting $t = 0$, we have from (ii) and (iii)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\text{i.e. } 1.2x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l} \int_0^l 1.2 \sin \frac{n\pi x}{l} dx = \frac{2400}{\pi} \cdot \frac{(-1)^{n+1}}{n}$$

$$\text{Hence } v(x, t) = 1300 - 1.3x + \frac{2400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-(n^2 \pi^2 t)/(l^2 RC)} \sin \frac{n\pi x}{1000}.$$

PROBLEMS 18.7

- Find the current i and voltage e in a line of length l , t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$. Also R and G are negligible.
- Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
- A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l . At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
- Obtain the solution of the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$$

appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

which is the *Laplace's equation in three dimensions*.

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R , then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r} d\xi d\eta d\zeta \quad \dots(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z , we get

$$\nabla^2 u = \iiint_R \rho \nabla^2 (1/r) d\xi d\eta d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the potential theory and its solutions are called the harmonic functions.*

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

$$(1) \text{ Cartesian form of } \nabla^2 u = 0 \text{ is } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

$$\text{Let } u = X(x)Y(y)Z(z) \quad \dots(2)$$

be a solution of (1). Substituting (2) in (1) and dividing by XYZ , we obtain

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad \dots(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2 X}{dx^2} - k^2 X = 0, \quad \frac{d^2 Y}{dy^2} - l^2 Y = 0, \quad \frac{d^2 Z}{dz^2} + (k^2 + l^2) Z = 0$$

Their solutions are $X = c_1 e^{kx} + c_2 e^{-kx}, Y = c_3 e^{ly} + c_4 e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)} z + c_6 \sin \sqrt{(k^2 + l^2)} z$$

Thus a possible solution of (1) is

$$u = (c_1 e^{kx} + c_2 e^{-kx})(c_3 e^{ly} + c_4 e^{-ly}) [c_5 \cos \sqrt{(k^2 + l^2)} z + c_6 \sin \sqrt{(k^2 + l^2)} z].$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) [c_5 e^{\sqrt{(k^2 + l^2)} z} + c_6 e^{-\sqrt{(k^2 + l^2)} z}].$$

$$(2) \text{ Cylindrical form of } \nabla^2 u = 0 \text{ is } \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

Let

$$u = R(\rho) H(\phi) Z(z)$$

[(iv) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ , we get

$$\frac{1}{R} \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} \right) + \frac{1}{\rho^2 H} \frac{d^2 H}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad \dots(2)$$

Assuming that $\frac{d^2 H}{d\phi^2} = -n^2 H$ and $\frac{d^2 Z}{dz^2} = k^2 Z$, ..(3)

(2) reduces to $\frac{1}{R} \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} \right) - \frac{n^2}{\rho^2} + k^2 = 0$

or $\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + (k^2 \rho^2 - n^2) R = 0$.

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1 J_n(k\rho) + c_2 Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi, Z = c_5 e^{kz} + c_6 e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1 J_n(k\rho) + c_2 Y_n(k\rho)][c_3 \cos n\phi + c_4 \sin n\phi][c_5 e^{kz} + c_6 e^{-kz}]$$

(Assam, 1999)

which is known as a *cylindrical harmonic*.

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2 u}{r^2} + \frac{2}{r} \frac{\partial u}{r} + \frac{1}{r^2} \frac{\partial^2 u}{\theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\phi^2} = 0 \quad \dots(1) \quad [(iv) p. 361]$$

Let $u = R(r) G(\theta) H(\phi)$ be a solution of (1).

Then $\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} \right) + \frac{1}{G} \left(\frac{d^2 G}{d\theta^2} + \cot \theta \frac{dG}{d\theta} \right) + \frac{1}{H \sin^2 \theta} \frac{d^2 H}{d\phi^2} = 0$

Putting $\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} \right) = n(n+1) \quad \dots(2) \quad \text{and} \quad \frac{1}{H} \frac{d^2 H}{d\phi^2} = -m^2, \quad \dots(3)$

the above equation takes the form

$$\frac{d^2 G}{d\theta^2} + \cot \theta \frac{dG}{d\theta} + [n(n+1) - m^2 \operatorname{cosec}^2 \theta] G = 0 \quad \dots(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

m times with respect to x and writing $u = d^m y / dx^m$, we get

$$(1-x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \quad \dots(5)$$

Now putting $G = (1-x^2)^{m/2} u$ in (5), we get

$$(1-x^2) \frac{d^2 G}{dx^2} - 2x \frac{dG}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2} \right] G = 0 \quad \dots(6)$$

Now putting $x = \cos \theta$ in (6), it reduces to (4) and its solution is

$$G = c_1 P_n^m(\cos \theta) + c_2 Q_n^m(\cos \theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $R = c_5 r^n + c_6 r^{-n-1}$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [c_1 P_n^m(\cos \theta) + c_2 Q_n^m(\cos \theta)] (c_3 \cos m\phi + c_4 \sin m\phi) \times (c_5 r^n + c_6 r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic*.

Example 18.26. Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2 \theta$.

Solution. Take the origin at the centre of the given sphere S . Since the potential is independent of ϕ on S , so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0 \quad \dots(i)$$

Putting $u(r, \theta) = R(r) G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{\partial \theta^2} + \cot \theta \frac{dG}{d\theta} + n(n+1)G = 0 \quad \dots(ii)$$

and

$$\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} \right) = n(n+1) \quad \dots(iii)$$

Putting $\cot \theta = v$, (ii) takes the form

$$(1-v^2) \frac{d^2 G}{dv^2} - 2v \frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos \theta) \text{ for } n = 0, 1, 2, \dots$$

The solutions of (iii) are $R_n(r) = r^n$, $\bar{R}_n(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos \theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos \theta)/r^{n+1}, \text{ where } n = 0, 1, 2, \dots$$

For points inside S , we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos \theta)$ $\dots(iv)$

On the boundary of S , $u(1, \theta) = f(\theta) \therefore f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos \theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$\begin{aligned} c_n &= \left(n + \frac{1}{2} \right) \int_{-1}^1 f(\theta) P_n(x) dx \text{ where } x = \cos \theta. \\ &= \left(n + \frac{1}{2} \right) \int_{-1}^1 x^2 P_n(x) dx \quad [\because f(\theta) = \cos^2 \theta] \\ &= \left(n + \frac{1}{2} \right) \int_{-1}^1 \left[\frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \right] P_n(x) dx \quad [\because P_2(x) = \frac{1}{2}(3x^2 - 1)] \end{aligned}$$

Using the orthogonality of Legendre polynomials, we get

$$c_n = 0, \text{ except for } n = 0, 2. \text{ Hence}$$

$$c_0 = \frac{1}{2} \cdot \frac{1}{3} \int_{-1}^1 P_0^2(x) dx = \frac{1}{3}, \quad c_2 = \frac{5}{2} \cdot \frac{2}{3} \int_{-1}^1 P_2^2(x) dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3} r^2 P_2(\cos \theta)$ or $u(r, \theta) = \frac{1}{3} + r^2 (\cos^2 \theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr) \{ e^{iz} (A_n \cos n\theta + B_n \sin n\theta) + e^{-iz} (C_n \cos n\theta + D_n \sin n\theta) \}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2 \theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and

$$u(r, \theta) = 2r^{-3}(\cos^2 \theta - 1/3) - r^{-1/3} \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ) , Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos \theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

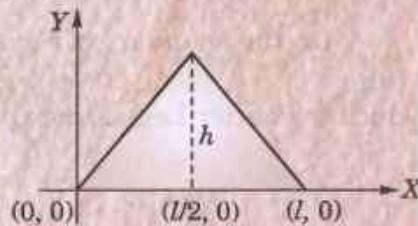
$$(1 - \mu^2) \frac{d^2 P_n}{d\mu^2} - 2\mu \frac{dP_n}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1 - \mu^2} \right\} P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are
2. The partial differential equation representing variable heat flow in three dimensions, is
3. Temperature gradient is defined as
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as
5. The partial differential equation of the transverse vibrations of a string is
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is
7. The equation $u_t = c^2 u_{xx}$ is classified as
8. The two dimensional steady state heat flow equation in polar coordinates is
9. The mathematical function of the initial form of the string given by the following graph is
10. When a vibrating string fastened to two points l apart, has an initial velocity u_0 , its initial conditions are
11. In two dimensional heat flow, the temperature along the normal to the xy -plane is
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u , then the boundary conditions for this problem are
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are
14. D'Alembert's solution of the wave equation is
15. The partial differential equation of 2-dimensional heat flow in
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is (Anna, 2008)
17. The three possible solutions of Laplace equation in polar coordinates are
18. Solution of $\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is
19. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is
20. In the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, α^2 represents
21. The telegraph equations for potential and current are
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form
23. The three possible solutions of Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ are



Complex Numbers and Functions

1. Complex Numbers. 2. Argand's diagram. 3. Geometric representation of $z_1 \pm z_2$; $z_1 z_2$ and z_1/z_2 . 4. De Moivre's theorem. 5. Roots of a complex number. 6. To expand $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$ in powers of $\sin \theta$, $\cos \theta$ and $\tan \theta$ respectively; Addition formulae for any number of angles; To expand $\sin^n \theta$, $\cos^n \theta$ and $\sin^n \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ . 7. Complex function: Definition. 8. Exponential function of a complex variable. 9. Circular functions of a complex variable. 10. Hyperbolic functions. 11. Inverse hyperbolic functions. 12. Real and imaginary parts of circular and hyperbolic functions. 13. Logarithmic functions of a complex variable. 14. Summation of series - 'C + iS' method. 15. Approximations and Limits. 16. Objective Type of Questions.

19.1 COMPLEX NUMBERS

Definition. A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a complex number.

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.

(2) Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

$$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

and

(3) Sum, difference, product and quotient of any two complex numbers is itself a complex number.

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

$$(i) \text{ their sum} = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(ii) \text{ their difference} = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

$$(iii) \text{ their product} = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

$$\text{and (iv) their quotient} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

(4) Every complex number $x + iy$ can always be expressed in the form $r(\cos \theta + i \sin \theta)$.

$$\text{Put } R(x + iy), \text{ i.e., } x = r \cos \theta$$

...(i)

and

$$I(x + iy), \text{ i.e., } y = r \sin \theta$$

...(ii)

Squaring and adding, we get $x^2 + y^2 = r^2$ i.e. $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (ii) by (i), we get $y/x = \tan \theta$ i.e. $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos \theta + i \sin \theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. The number $r = +\sqrt{x^2 + y^2}$ is called the **modulus** of $x + iy$ and is written as $\text{mod}(x + iy)$ or $|x + iy|$.

The angle θ is called the **amplitude** or **argument** of $x + iy$ and is written as $\text{amp}(x + iy)$ or $\arg(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the **principal value of the amplitude**. Unless otherwise specified, we shall take $\text{amp}(z)$ to mean the principal value.

Note. $\cos \theta + i \sin \theta$ is briefly written as $\text{cis } \theta$ (pronounced as 'sis θ ')

(5) If the conjugate of $z = x + iy$ be \bar{z} , then

$$(i) R(z) = \frac{1}{2}(z + \bar{z}), I(z) = \frac{1}{2i}(z - \bar{z}) \quad (ii) |z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$$

$$(iii) z\bar{z} = |z|^2 \quad (iv) \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(v) \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2 \quad (vi) \overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2, \text{ where } \bar{z}_2 \neq 0.$$

Example 19.1. Reduce $1 - \cos \alpha + i \sin \alpha$ to the modulus amplitude form.

Solution. Put $1 - \cos \alpha = r \cos \theta$ and $\sin \alpha = r \sin \theta$

$$\therefore r = (1 - \cos \alpha)^2 + \sin^2 \alpha = 2 - 2 \cos \alpha = 4 \sin^2 \alpha/2$$

i.e.,

$$r = 2 \sin \alpha/2$$

and

$$\begin{aligned} \tan \theta &= \frac{\sin \alpha}{1 - \cos \alpha} = \frac{2 \sin \alpha/2 \cos \alpha/2}{2 \sin^2 \alpha/2} = \cot \alpha/2 \\ &= \tan \left(\frac{\pi}{2} - \frac{\alpha}{2} \right) \quad \therefore \theta = \frac{\pi - \alpha}{2}. \end{aligned}$$

$$\text{Thus } 1 - \cos \alpha + i \sin \alpha = 2 \sin \frac{\alpha}{2} \left[\cos \frac{\pi - \alpha}{2} + i \sin \frac{\pi - \alpha}{2} \right].$$

Example 19.2. Find the complex number z if $\arg(z + 1) = \pi/6$ and $\arg(z - 1) = 2\pi/3$.

(Mumbai, 2009)

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

$$\text{Since } \arg(z + 1) = \pi/6, \quad \therefore \tan^{-1} \left(\frac{y}{x+1} \right) = 30^\circ$$

i.e.,

$$\frac{y}{x+1} = \tan 30^\circ = 1/\sqrt{3}, \text{ or } \sqrt{3}y = x + 1 \quad \dots(i)$$

$$\text{Also since } \arg(z - 1) = 2\pi/3, \quad \therefore \tan^{-1} \left(\frac{y}{x-1} \right) = 120^\circ$$

i.e.,

$$\frac{y}{x-1} = \tan 120^\circ = -\sqrt{3}, \quad \text{or } y = -\sqrt{3}x + \sqrt{3} \quad \text{or } \sqrt{3}y = -3x + 3 \quad \dots(ii)$$

Subtracting (ii) from (i), we get $4x - 2 = 0$ i.e., $x = 1/2$

$$\text{From (i), } \sqrt{3}y = 1/2 + 1, \quad \text{i.e., } y = \sqrt{3}/2$$

$$\text{Hence } z = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Example 19.3. Find the real values of x, y so that $-3 + ix^2y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that

$$z = (x^2 + y) - 4i$$

$$\therefore -3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating

$$x, (y+3)y = -4$$

or

$$y^2 + 3y - 4 = 0 \text{ i.e., } y = 1 \text{ or } -4$$

When $y = 1$,

$$x^2 = -3 - 1 \text{ or } x = +2i \text{ which is not feasible}$$

When $y = -4$,

$$x^2 = 1 \text{ or } x = \pm 1$$

Hence $x = 1$,

$$y = -4 \text{ or } x = -1, y = -4.$$

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX' . Let OA be equal to one unit of measurement (Fig. 19.1).

Take a point L on OX such that $OL = x(OA)$.

Then L on OX represents the positive real number x and $i \cdot ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O .

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'' .

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL'' .

*Hence, if $Y'Y$ be a line perpendicular to the real axis $X'OX$, then all imaginary numbers are represented by points on $Y'Y$, called the **imaginary axis**, the positive ones along Y and negative ones along Y' .**

Obs. Geometric interpretation of i^* . From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'Y$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX' . Let the point L on OX represent the real number x (Fig. 19.2).

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'Y$ (called the *imaginary axis*), the positive ones along Y and negative ones along Y' . Let the point M on Y represent the imaginary number iy .

Complete the rectangle $OLPM$. Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z . The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P , then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. Hereafter we shall often refer to the complex number $z = x + iy$ as

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to $P(x, y)$.

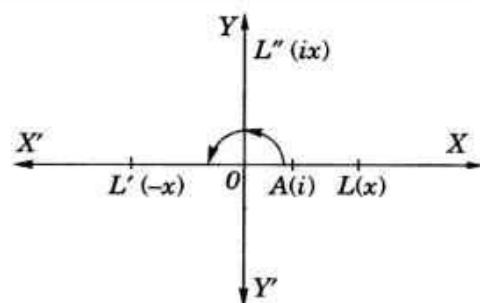


Fig. 19.1

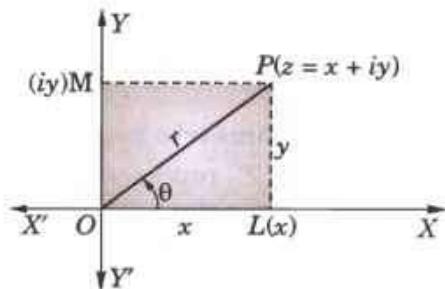


Fig. 19.2

Example 19.4. The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.

* The first mathematician to propose a geometric representation of imaginary number i was Kuhn of Denzig (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoir of Jean Robert Argand, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$$OA = 2 \text{ and } \angle XOA = \tan^{-1} 1/\sqrt{3} = 30^\circ. (\text{Fig. 19.3})$$

Being a regular hexagon, $OB = OC = 2$

$$\angle XOB = 30^\circ + 60^\circ = 90^\circ$$

and

$$\angle XOC = 30^\circ + 120^\circ = 150^\circ$$

$$\therefore \vec{OB} = 2(\cos 90^\circ + i \sin 90^\circ) = 2i$$

$$\vec{OC} = 2(\cos 150^\circ + i \sin 150^\circ) = -\sqrt{3} + i$$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O ,

$$\therefore \vec{OD} = -\vec{OA} = -\sqrt{3} - i$$

$$\vec{OE} = -\vec{OB} = -2i \text{ and } \vec{OF} = -\vec{OC} = \sqrt{3} - i.$$

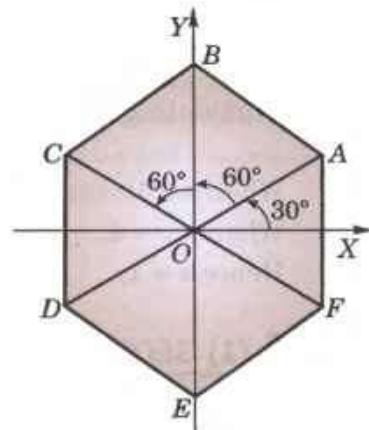


Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2 . Draw P_1L, P_2M and $PN \perp s$ to OX .

Also draw $P_1K \perp PN$.

Since $ON = OL + LN = OL + OM = x_1 + x_2$ [$\because LN = P_1K = OM$]

and $NP = NK + KP = LP_1 + MP_2 = y_1 + y_2$.

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and } \text{amp}(z_1 + z_2) = \angle XOP.$$

Obs. Vectorially, we have $\vec{OP}_1 + \vec{P}_1P = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $-z_2$ is represented by OQ i.e., $z_1 - z_2 = \vec{OQ} = \vec{P}_2P_1$.

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1 .

Obs. By means of the relation $\vec{P}_2P_1 = \vec{OP}_1 - \vec{OP}_2$, any vector \vec{P}_2P_1 may be referred to the origin.

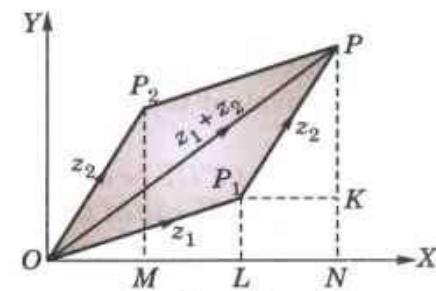


Fig. 19.4

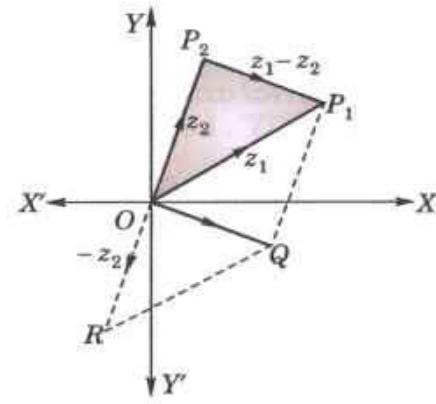


Fig. 19.5

Example 19.5. Find the locus of $P(z)$ when

$$(i) |z - a| = k;$$

$$(ii) \text{amp}(z - a) = \alpha, \text{ where } k \text{ and } \alpha \text{ are constants.}$$

(Gorakhpur, 1999)

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

$$\text{Then } z - a = \vec{OP} - \vec{OA} = \vec{AP}$$

$$(i) |z - a| = k \text{ means that } AP = k.$$

Thus the locus of $P(z)$ is a circle whose centre is $A(a)$ and radius k .

(ii) $\text{amp}(z - a)$, i.e., $\text{amp}(\vec{AP}) = \alpha$, means that AP always makes a constant angle with the X -axis.

Thus the locus of $P(z)$ is a straight line through $A(a)$ making an $\angle\alpha$ with OX .

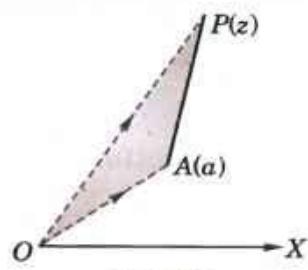


Fig. 19.6

Example 19.6. Determine the region in the z -plane represented by

- (i) $1 < |z + 2i| \leq 3$ (ii) $R(z) > 3$ (iii) $\pi/6 \leq \text{amp}(z) \leq \pi/3$.

Solution. (i) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \leq 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

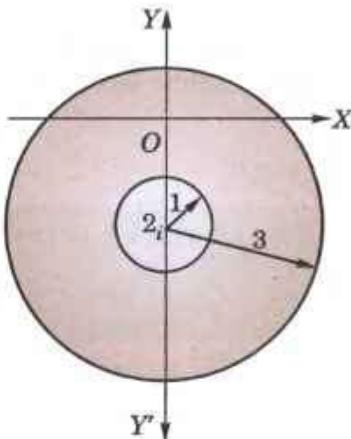


Fig. 19.7

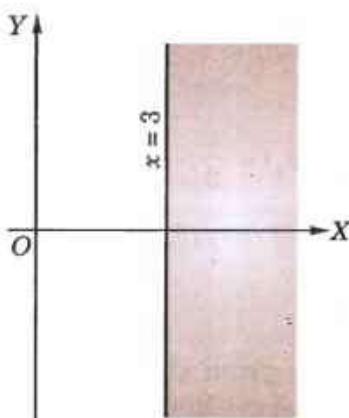


Fig. 19.8

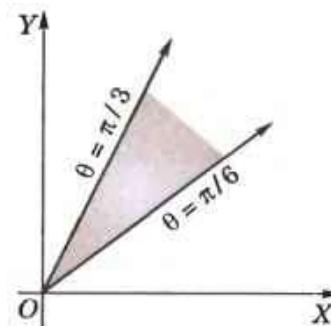


Fig. 19.9

(ii) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(iii) If $z = r(\cos \theta + i \sin \theta)$, then $\text{amp}(z) = \theta$.

$\therefore \pi/6 \leq \text{amp}(z) \leq \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. If z_1, z_2 be any two complex numbers, prove that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$ [i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli].

(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$ [i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10).

Complete the parallelogram OP_1PP_2 , so that

$$|z_1| = OP_1, |z_2| = OP_2 = P_1P,$$

and

$$|z_1 + z_2| = OP.$$

Now from ΔOP_1P , $OP \leq OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \dots(i)$$

Again

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2| \quad [\text{By (i)}]$$

Thus

$$|z_1 - z_2| \geq |z_1| - |z_2| \quad \dots(ii)$$

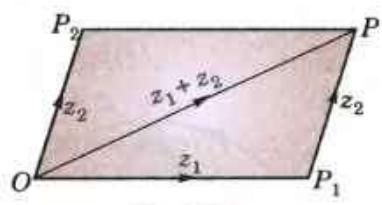


Fig. 19.10

Obs. $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$.

In general, $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$.

Example 19.8. If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of amplitudes of z_1 and z_2 is $\pi/2$.

(Mumbai, 2007)

Solution. Let $z_1 + z_2 = r(\cos \theta + i \sin \theta)$ and $z_1 - z_2 = r(\cos \phi + i \sin \phi)$

Then $2z_1 = r[(\cos \theta + \cos \phi) + i(\sin \theta + \sin \phi)]$

$$= r \left\{ 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} + 2i \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \right\}$$

or $z_1 = r \cos \frac{\theta - \phi}{2} \left(\cos \frac{\theta + \phi}{2} + i \sin \frac{\theta + \phi}{2} \right)$ i.e., $\text{amp}(z_1) = \frac{\theta + \phi}{2}$... (i)

Also

$$2z_2 = r(\cos \theta - \cos \phi) + i(\sin \theta - \sin \phi)$$

$$= 2r \sin \frac{\theta - \phi}{2} \left(-\sin \frac{\theta + \phi}{2} + i \cos \frac{\theta + \phi}{2} \right)$$

or $z_2 = r \sin \frac{\theta - \phi}{2} \left\{ \cos \left(\frac{\pi}{2} + \frac{\theta + \phi}{2} \right) + i \sin \left(\frac{\pi}{2} + \frac{\theta + \phi}{2} \right) \right\}$

i.e., $\text{amp}(z_2) = \frac{\pi}{2} + \frac{\theta + \phi}{2}$... (ii)

Hence [(ii) - (i)], gives $\text{amp}(z_2) - \text{amp}(z_1) = \frac{\pi}{2}$.

Example 19.9. Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$, is $|z - z_1| + |z - z_2| = 2a$.

Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z - z_1|$ and $S'P = |z - z_2|$.

We know that $SP + S'P = AA' (= 2a)$

i.e., $|z - z_1| + |z - z_2| = 2a$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or $|\vec{OS}' - \vec{OS}| = 2ae$ or $|z_2 - z_1| = 2ae$

or $|z_1 - z_2| = 2ae$ whence $e = |z_1 - z_2|/2a$.

(3) Geometric Representation of $z_1 z_2$. Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1(\cos \theta_1 + i \sin \theta_1)$$

and $z_2 = x_2 + iy_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct $\Delta OP_2 P$ on OP_2 directly similar to ΔOAP_1 ,

so that $OP/OP_1 = OP_2/OA$ i.e., $OP = OP_1 \cdot OP_2 = r_1 r_2$

and $\angle AOP = \angle AOP_2 + \angle P_2 OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$

$\therefore P$ represents the number

$$r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

Hence the product of two complex numbers z_1, z_2 is represented by the point P , such that (i) $|z_1 z_2| = |z_1| \cdot |z_2|$.

(ii) $\text{amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2)$.

Cor. The effect of multiplication of any complex number z by $\cos \theta + i \sin \theta$ is to rotate its direction through an angle θ , for the modulus of $\cos \theta + i \sin \theta$ is unity.

(4) Geometric representation of z_1/z_2 .

Let P_1, P_2 represent the complex numbers

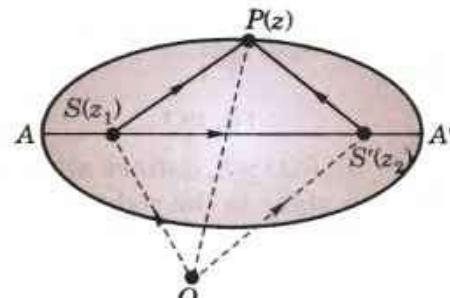


Fig. 19.11

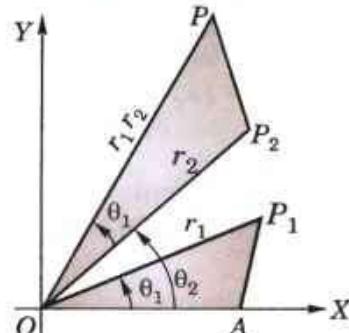


Fig. 19.12

$$\begin{aligned} z_1 &= x_1 + iy_1 = r_1(\cos \theta_1 + i \sin \theta_1) \\ z_2 &= x_2 + iy_2 = r_2(\cos \theta_2 + i \sin \theta_2) \end{aligned}$$

and Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad \text{i.e.,} \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and $\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2$.

$\therefore P$ represents the number

$$(r_1/r_2)[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

Hence the complex number z_1/z_2 is represented by the point P , such that

$$(i) |z_1/z_2| = |z_1|/|z_2|$$

$$(ii) \operatorname{amp}(z_1/z_2) = \operatorname{amp}(z_1) - \operatorname{amp}(z_2).$$

Note. If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then

$$\operatorname{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O , the origin, to P_1 , P_2 , and P_3 . Then from the figure 19.14, we have

$$\vec{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \vec{P_2P_3} = z_3 - z_2$$

$$\therefore \operatorname{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \operatorname{amp}\left[\frac{\vec{P_2P_3}}{\vec{P_2P_1}}\right]$$

$$= \operatorname{amp}(\vec{P_2P_3}) - \operatorname{amp}(\vec{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

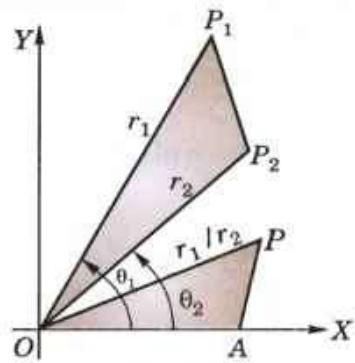


Fig. 19.13

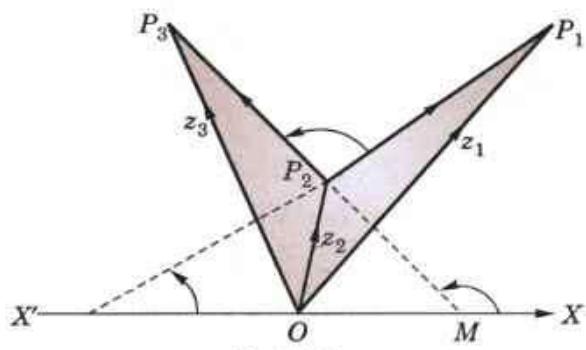


Fig. 19.14

Example 19.10. Find the locus of the point z , when

$$(i) \left| \frac{z-a}{z-b} \right| = k \quad (ii) \operatorname{amp}\left(\frac{z-a}{z-b}\right) = \alpha \text{ where } k \text{ and } \alpha \text{ are constants.}$$

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(i) Since $|z-a| = AP$ and $|z-b| = BP$,

$$\therefore \text{The point } P \text{ moves so that } \left| \frac{z-a}{z-b} \right| = \left| \frac{z-a}{z-b} \right| = \frac{AP}{BP} = k$$

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the Appollonius circle.

When $k = 1$, $BP = AP$ i.e., P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB .

Hence the locus of $P(z)$ is a circle (unless $k = 1$, when the locus is the right bisector of AB).

Obs. For different values of k , the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

$$(ii) \text{ From the figure 19.16, we have } \operatorname{amp}\left(\frac{z-a}{z-b}\right) = \angle APB = \alpha.$$

Hence the locus of $P(z)$ is the arc APB of the circle which passes through the fixed points A and B .

If, however, $P'(z')$ be a point on the lower arc AB of this circle, then

$\operatorname{amp}\left(\frac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

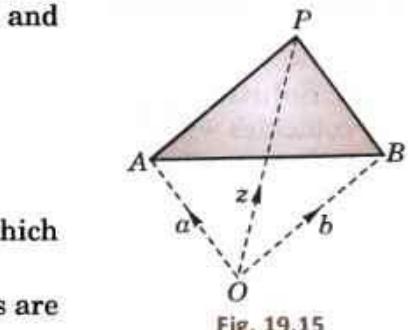


Fig. 19.15

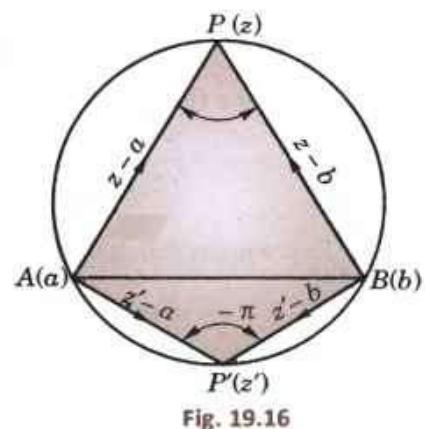


Fig. 19.16

Obs. For different values of α from $-\pi$ to π , the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. If z_1, z_2 be two complex numbers, show that

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2(|z_1|^2 + |z_2|^2).$$

Solution. Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$ so that

$$\begin{aligned}|z_1 + z_2|^2 &= (r_1 \cos \theta_1 + r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 + r_2 \sin \theta_2)^2 \\&= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1) \\|z_1 - z_2|^2 &= (r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2 \\&= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)\end{aligned}$$

and

$$\therefore |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\{|z_1|^2 + |z_2|^2\}.$$

Example 19.12. If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2 , prove that

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1), B(z_2), C(z_3)$ be the vertices of ΔABC such that

$$AB = BC \text{ and } \angle ABC = \pi/2. \text{ (Fig. 19.17)}$$

Then $|z_1 - z_2| = |z_3 - z_2| = r$ (say).

If $\operatorname{amp}(z_1 - z_2) = \theta$ then $\operatorname{amp}(z_3 - z_2) = \pi/2 + \theta$

$\therefore z_1 - z_2 = r(\cos \theta + i \sin \theta)$,

and $z_3 - z_2 = r \left[\cos \left(\frac{\pi}{2} + \theta \right) + i \sin \left(\frac{\pi}{2} + \theta \right) \right] = r(-\sin \theta + i \cos \theta)$

i.e., $z_3 - z_2 = ir(\cos \theta + i \sin \theta) = i(z_1 - z_2)$

or $(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2)$.

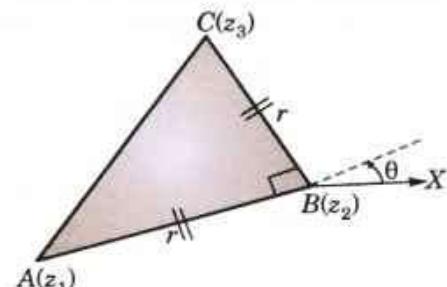


Fig. 19.17

Example 19.13. If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through 60° coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle \theta$, we multiply it by $\cos \theta + i \sin \theta$.

$\therefore \vec{BC} (\cos \pi/3 + i \sin \pi/3) = \vec{BA}$

i.e., $(z_3 - z_2) \left(\frac{1+i\sqrt{3}}{2} \right) = z_1 - z_2$

or $i\sqrt{3}(z_3 - z_2) = 2z_1 - z_2 - z_3$

Squaring, $-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$

or $4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$

whence follows the required condition.

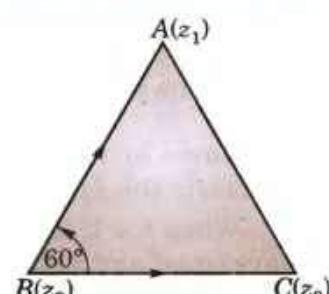


Fig. 19.18

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

(i) $1 + \sin \alpha + i \cos \alpha$ (ii) $\frac{1}{(2+1)^2} - \frac{1}{(2-1)^2}$. (V.T.U., 2011 S)

2. If $\frac{1}{x+iy} + \frac{1}{u+iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y .

3. If x and y are real, solve the equation $\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \frac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (Mumbai, 2008 S)
5. Find what curve $z\bar{z} + (1+i)z + (1-i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9+i$, $4+13i$, $-8+8i$ and $-3-4i$ form a square.
7. If $|z_1| = |z_2|$ and $\text{amp}(z_1) + \text{amp}(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1+i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1+i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1+i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z -plane is represented by
 (i) $2 \leq |z+3| < 4$ (ii) $I(z) > 2$
 (iii) $\pi/3 < \text{amp}(z) < \pi/2$ (iv) $|z+2| + |z-2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. (Mumbai, 2007)
13. What are the loci given by (i) $|z-1| + |z+1| = 3$, (ii) $|z-3| = k|z+1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
 (i) $|z| = |z-2|$. (ii) $|3z-1| = |z-3|$.
15. Find the locus of z :
 (i) when $\frac{z+i}{z+2}$ is real, (ii) when $\frac{z-i}{z-2}$ is purely imaginary. (Osmania, 2003 S)

19.4 DE MOIVRE'S THEOREM*

Statement : If n be (i) an integer, positive or negative $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$;
 (ii) a fraction, positive or negative, one of the values of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$.

Proof. Case I. When n is a positive integer.

By actual multiplication

$$\begin{aligned}\text{cis } \theta_1 \text{ cis } \theta_2 &= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2), \text{ i.e., cis } (\theta_1 + \theta_2)\end{aligned}$$

Similarly $\text{cis } \theta_1 \text{ cis } \theta_2 \text{ cis } \theta_3 = \text{cis } (\theta_1 + \theta_2) \text{ cis } \theta_3 = \text{cis } (\theta_1 + \theta_2 + \theta_3)$

Proceeding in this way,

$$\text{cis } \theta_1 \text{ cis } \theta_2 \text{ cis } \theta_3 \dots \text{ cis } \theta_n = \text{cis } (\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \dots = \theta_n = \theta$, we obtain $(\text{cis } \theta)^n = \text{cis } n\theta$.

Case II. When n is a negative integer.

Let $n = -m$, where m is a + ve integer.

$$\begin{aligned}\therefore (\text{cis } \theta)^n &= (\text{cis } \theta)^{-m} = \frac{1}{(\text{cis } \theta)^m} = \frac{1}{\text{cis } m\theta} \quad (\text{By case I}) \\ &= \frac{\cos m\theta - i \sin m\theta}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}\end{aligned}$$

[Multiplying the num. and denom. by $(\cos m\theta - i \sin m\theta)$]

*One of the remarkable theorems in mathematics; called after the name of its discoverer Abraham De Moivre (1667–1754), a French Mathematician.

$$\begin{aligned}
 &= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta \\
 &= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis } n\theta
 \end{aligned}
 \quad [\because -m = n]$$

Case III. When n is a fraction, positive or negative.

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or - ve

Now $(\text{cis } \theta/q)^q = \text{cis}(q \cdot \theta/q) = \text{cis } \theta$

\therefore Taking q th root of both sides $\text{cis } (\theta/q)$ is one of the q values of $(\text{cis } \theta)^{1/q}$, i.e., one of the values of $(\text{cis } \theta)^{1/q} = \text{cis } \theta/p$

Raise both sides to power p , then one of the values of $(\text{cis } \theta)^{p/q} = (\text{cis } \theta/q)^p = \text{cis}(p/q)\theta$ i.e., one of the values of $(\text{cis } \theta)^n = \text{cis } n\theta$. (By case I and II)

Thus the theorem is completely established for all rational values of n .

- Cor.
1. $\text{cis } \theta_1 \cdot \text{cis } \theta_2 \dots \text{cis } \theta_n = \text{cis}(\theta_1 + \theta_2 + \dots + \theta_n)$
 2. $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta = (\cos \theta + i \sin \theta)^{-n}$
 3. $(\text{cis } m\theta)^n = \text{cis } mn\theta = (\text{cis } n\theta)^m$.

Example 19.14. Simplify $\frac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$

$$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$$

$$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$$

$$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$$

$$\therefore \text{The given expression} = \frac{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}}{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}} = 1.$$

Example 19.15. Prove that

$$(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n(\theta/2) \cdot (\cos n\theta/2).$$

Solution. Put $1 + \cos \theta = r \cos \alpha, \sin \theta = r \sin \alpha$.

$$\therefore r^2 = (1 + \cos \theta)^2 + \sin^2 \theta = 2 + 2 \cos \theta = 4 \cos^2 \theta/2 \quad \text{i.e., } r = 2 \cos \theta/2$$

and $\tan \alpha = \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \sin \theta/2 \cdot \cos \theta/2}{2 \cos^2 \theta/2} = \tan \theta/2 \quad \text{i.e., } \alpha = \theta/2$.

$$\therefore \text{L.H.S.} = [r(\cos \alpha + i \sin \alpha)]^n + [r(\cos \alpha - i \sin \alpha)]^n$$

$$= r^n[(\cos \alpha + i \sin \alpha)^n + (\cos \alpha - i \sin \alpha)^n] = r^n(\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha)$$

$= r^n \cdot 2 \cos n\alpha \quad [\text{Substituting the values of } r \text{ and } \alpha]$

$$= 2^{n+1} \cos^n(\theta/2) \cos(n\theta/2).$$

Example 19.16. If $2 \cos \theta = x + \frac{1}{x}$, prove that

$$(i) 2 \cos r\theta = x^r + 1/x^r, \quad (ii) \frac{x^{2n} + 1}{x^{2n-1} + x} = \frac{\cos n\theta}{\cos(n-1)\theta} \quad (\text{Madras, 2000 S})$$

Solution. Since

$$x + 1/x = 2 \cos \theta \quad \therefore x^2 - 2x \cos \theta + 1 = 0$$

whence

$$x = \frac{2 \cos \theta \pm \sqrt{(4 \cos^2 \theta - 4)}}{2} = \cos \theta \pm i \sin \theta.$$

(i) Taking the + ve sign, $x^r = (\cos \theta + i \sin \theta)^r = \cos r\theta + i \sin r\theta$

(S.V.T.U., 2009)

and

$$x^{-r} = (\cos \theta + i \sin \theta)^{-r} = \cos r\theta - i \sin r\theta$$

Adding $x^r + 1/x^r = 2 \cos r\theta$. Similarly with the - ve sign, the same result follows.

$$\begin{aligned}
 (ii) \quad & \frac{x^{2n} + 1}{x^{2n-1} + x} = \frac{(\cos \theta + i \sin \theta)^{2n} + 1}{(\cos \theta + i \sin \theta)^{2n-1} + \cos \theta + i \sin \theta} \\
 & = \frac{\cos 2n\theta + i \sin 2n\theta + 1}{\cos (2n-1)\theta + i \sin (2n-1)\theta + \cos \theta + i \sin \theta} \\
 & = \frac{(1 + \cos 2n\theta) + i \sin 2n\theta}{(\cos 2n-1\theta + \cos \theta) + i(\sin 2n-1\theta + \sin \theta)} \\
 & = \frac{2 \cos^2 n\theta + 2i \sin n\theta \cos \theta}{2 \cos n\theta \cos n-1\theta + 2i \sin n\theta \cos n-1\theta} \\
 & = \frac{\cos n\theta (2 \cos n\theta + 2i \sin n\theta)}{\cos n-1\theta (2 \cos n\theta + 2i \sin n\theta)} = \frac{\cos n\theta}{\cos n-1\theta}.
 \end{aligned}$$

Example 19.17. If $\sin \alpha + \sin \beta + \sin \gamma = \cos \alpha + \cos \beta + \cos \gamma = 0$,

prove that (i) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$

(iii) $\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 2 \sum \sin 2(\alpha + \beta)$

(iv) $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$.

(Mumbai, 2009)

Solution. Let $a = \text{cis } \alpha$, $b = \text{cis } \beta$ and $c = \text{cis } \gamma$.

Then $a + b + c = (\cos \alpha + i \sin \alpha) + (\cos \beta + i \sin \beta) + (\cos \gamma + i \sin \gamma) = 0$... (1)

$$\begin{aligned}
 (i) \quad & \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = (\cos \alpha + i \sin \alpha)^{-1} + (\cos \beta + i \sin \beta)^{-1} + (\cos \gamma + i \sin \gamma)^{-1} \\
 & = \sum \frac{\cos \alpha - i \sin \alpha}{\cos \alpha + i \sin \alpha} \cdot \frac{1}{\cos \alpha + i \sin \alpha} = \sum (\cos \alpha - i \sin \alpha) \\
 & = (\cos \alpha + \cos \beta + \cos \gamma) - i(\sin \alpha + \sin \beta + \sin \gamma) = 0 \quad (\text{Given})
 \end{aligned}$$

or

$$bc + ca + ab = 0 \quad \dots (2)$$

$$\therefore a^2 + b^2 + c^2 = (a + b + c)^2 - 2(bc + ca + ab) = 0$$

[By (1) & (2) ... (3)]

$$\text{or} \quad (\text{cis } \alpha)^2 + (\text{cis } \beta)^2 + (\text{cis } \gamma)^2 = \text{cis } 2\alpha + \text{cis } 2\beta + \text{cis } 2\gamma = 0$$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

$$(ii) \text{ Since } a + b + c = 0, \quad \therefore a^3 + b^3 + c^3 = 3abc$$

$$\text{or} \quad (\text{cis } \alpha)^3 + (\text{cis } \beta)^3 + (\text{cis } \gamma)^3 = 3 \text{ cis } \alpha \text{ cis } \beta \text{ cis } \gamma$$

$$\text{or} \quad \text{cis } 3\alpha + \text{cis } 3\beta + \text{cis } 3\gamma = 3 \text{ cis } (\alpha + \beta + \gamma)$$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$$

$$(iii) \text{ From (1), } a + b = -c \text{ or } (a + b)^2 = c^2 \text{ or } a^2 + b^2 - c^2 = -2ab$$

$$\text{Again squaring, } a^4 + b^4 + c^4 + 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = 4a^2b^2$$

i.e.,

$$a^4 + b^4 + c^4 = 2(a^2b^2 + b^2c^2 + c^2a^2)$$

$$\text{or} \quad (\text{cis } \alpha)^4 + (\text{cis } \beta)^4 + (\text{cis } \gamma)^4 = 2 \sum (\cos \alpha)^2 (\cos \beta)^2$$

$$\text{or} \quad \text{cis } 4\alpha + \text{cis } 4\beta + \text{cis } 4\gamma = 2 \sum \text{cis } 2\alpha \text{ cis } 2\beta = 2 \sum \text{cis } 2(\alpha + \beta)$$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 2 \sum \sin 2(\alpha + \beta)$$

$$(iv) \text{ From (2), } ab + bc + ca = 0$$

$$\text{or} \quad \text{cis } \alpha \text{ cis } \beta + \text{cis } \beta \text{ cis } \gamma + \text{cis } \gamma \text{ cis } \alpha = 0$$

$$\text{or} \quad \text{cis } (\alpha + \beta) + \text{cis } (\beta + \gamma) + \text{cis } (\gamma + \alpha) = 0$$

Equating imaginary parts from both sides, we get

$$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$$

PROBLEMS 19.2

1. Prove that (i) $\frac{(\cos 5\theta - i \sin 5\theta)^2 (\cos 7\theta + i \sin 7\theta)^3}{(\cos 4\theta - i \sin 4\theta)^3 (\cos \theta + i \sin \theta)^5} = 1$
(ii) $\frac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^5} = \sin(4\alpha + 5\beta) - i \cos(4\alpha + 5\beta)$. (iii) $\left(\frac{\cos \theta + i \sin \theta}{\sin \theta + i \cos \theta} \right)^4 = \cos 8\theta + i \sin 8\theta$.
2. If $p = \text{cis } \theta$ and $q = \text{cis } \phi$, show that
(i) $\frac{p - q}{p + q} = i \tan \frac{\theta - \phi}{2}$ (Mumbai, 2008) (ii) $\frac{(p + q)(pq - 1)}{(p - q)(pq + 1)} = \frac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi}$. (Kurukshetra, 2005)
3. If $a = \text{cis } 2\alpha$, $b = \text{cis } 2\beta$, $c = \text{cis } 2\gamma$ and $d = \text{cis } 2\delta$, prove that
(i) $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$ (ii) $\sqrt{\frac{ab}{cd}} + \sqrt{\frac{cd}{ab}} = 2 \cos(\alpha + \beta - \gamma - \delta)$.
4. If $x_r = \text{cis}(\pi/2^r)$, show that $\lim_{n \rightarrow \infty} x_1 x_2 x_3 \dots x_n = -1$. (S.V.T.U., 2009; Mumbai, 2007)
5. Find the general value of θ which satisfies the equation
 $(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots (\cos n\theta + i \sin n\theta) = 1$.
6. Prove that (i) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{m/2n} \cos\left(\frac{m}{n} \tan^{-1} \frac{b}{a}\right)$.
(ii) $(1+i)^n + (1-i)^n = 2^{n/2+1} \cos n\pi/4$.
7. Simplify $[\cos \alpha - \cos \beta + i(\sin \alpha - \sin \beta)]^n + [\cos \alpha - \cos \beta - i(\sin \alpha - \sin \beta)]^n$
8. Prove that (i) $(1 + \sin \theta + i \cos \theta)^n + (1 + \sin \theta - i \cos \theta)^n = 2^{n+1} \cos^n\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \cos\left(\frac{n\pi}{4} - \frac{n\theta}{2}\right)$.
(ii) $\left[\frac{1 + \sin \alpha + i \cos \alpha}{1 + \sin \alpha - i \cos \alpha} \right]^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i \sin\left(\frac{n\pi}{2} - n\alpha\right)$. (S.V.T.U., 2006)
9. If $2 \cos \theta = x + 1/x$ and $2 \cos \phi = y + 1/y$, show that one of the values of
(i) $x^m y^n + \frac{1}{x^m y^n}$ is $2 \cos(m\theta + n\phi)$. (S.V.T.U., 2007)
(ii) $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ is $2 \cos(m\theta - n\phi)$. (Nagpur, 2009)
10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1} \cos n\pi/3$. (Delhi, 2002)
11. If α, β are the roots of the equation $z^2 \sin^2 \theta - z \sin \theta + 1 = 0$, then prove that
(i) $\alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$ (ii) $\alpha^n \beta^n = \operatorname{cosec}^{2n} \theta$. (Mumbai, 2009)
12. If $x^2 - 2x \cos \theta + 1 = 0$, show that $x^{2n} - 2x^n \cos n\theta + 1 = 0$.
13. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$ and $x + y + z = 0$, then prove that
 $x^{-1} + y^{-1} + z^{-1} = 0$.
14. If $\sin \theta + \sin \phi + \sin \psi = 0 = \cos \theta + \cos \phi + \cos \psi$, prove that
(i) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (Mumbai, 2009)
(ii) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3 \cos(\theta + \phi + \psi)$
(iii) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2 \sum \cos 2(\phi + \psi)$.
15. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, prove that
(i) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3/2$
(ii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$ (Mumbai, 2009; S.V.T.U., 2008)
16. If $\sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$, $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = 0$, prove that $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma = 18 \cos(\alpha + \beta + \gamma)$.

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos \theta + i \sin \theta)^{1/q}$, q being an integer.

Since $\cos \theta = \cos(2n\pi + \theta)$ and $\sin \theta = \sin(2n\pi + \theta)$, where n is any integer.

$\therefore \text{cis } \theta = \text{cis}(2n\pi + \theta)$.

By De Moivre's theorem one of the values of

$$(\operatorname{cis} \theta)^{1/q} = [\operatorname{cis}(2n\pi + \theta)]^{1/q} = \operatorname{cis}(2n\pi + \theta)/q \quad \dots(1)$$

Giving n the values $0, 1, 2, 3, \dots, (q - 1)$ successively, we get the following q values of $(\operatorname{cis} \theta)^{1/q}$:

$$\left. \begin{array}{ll} \operatorname{cis} \theta/q & (\text{for } n=0) \\ \operatorname{cis}(2\pi + \theta)/q & (\text{for } n=1) \\ \operatorname{cis}(4\pi + \theta)/q & (\text{for } n=2) \\ \dots & \dots \\ \operatorname{cis}[2(q-1)\pi + \theta]/q & (\text{for } n=q-1) \end{array} \right\} \quad \dots(2)$$

Putting $n = q$ in (1), we get a value of $(\operatorname{cis} \theta)^{1/q} = \operatorname{cis}(2\pi + \theta)/q = \operatorname{cis} \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\operatorname{cis} \theta)^{1/q}$ to be $\operatorname{cis}(2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\operatorname{cis} \theta)^{1/q}$ for $n = q, q+1, q+2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π .

Hence $(\operatorname{cis} \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. $(\operatorname{cis} \theta)^{p/q}$ where p/q is a rational fraction in its lowest terms, has also q and only q distinct values; which are obtained by putting $n = 0, 1, 2, \dots, q-1$ successively in $\operatorname{cis} p(2n\pi + \theta)/q$.

Note that $(\operatorname{cis} \theta)^{6/15}$ has only 5 distinct values and not 15; because $6/15$ in its lowest terms = $2/5$

∴ In order to find the distinct values of $(\operatorname{cis} \theta)^{p/q}$ always see that p/q is in its lowest terms.

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r(\cos \theta + i \sin \theta)$ and proceed as above.

Example 19.18. Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\operatorname{cis} 0)^{1/3} = (\operatorname{cis} 2n\pi)^{1/3} = \operatorname{cis} 2n\pi/3$$

where $n = 0, 1, 2$.

∴ the three values of x are $\operatorname{cis} 0 = 1$,

$$\operatorname{cis} 2\pi/3 = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + i \frac{\sqrt{3}}{2},$$

$$\text{and} \quad \operatorname{cis} 4\pi/3 = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - i \frac{\sqrt{3}}{2}.$$

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120^\circ$, $\angle AOC = 240^\circ$ (Fig. 19.19).

∴ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120^\circ$ i.e., $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

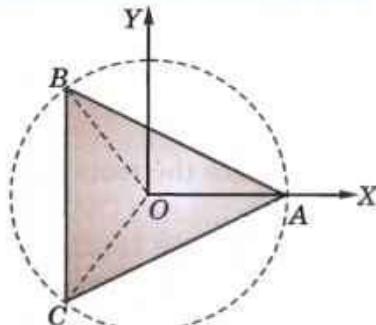


Fig. 19.19

Example. 19.19. Find all the values of $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1.

(Nagpur, 2009)

Solution. Put $1/2 = r \cos \theta$ and $\sqrt{3}/2 = r \sin \theta$ so that $r = 1$ and $\theta = \pi/3$

$$\begin{aligned} \therefore (1/2 + \sqrt{3}i/2)^{3/4} &= [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\operatorname{cis} \pi)^{1/4} \\ &= [\operatorname{cis} (2n+1)\pi]^{1/4} = \operatorname{cis} (2n+1)\pi/4 \text{ where } n = 0, 1, 2, 3. \end{aligned}$$

Hence the required values are $\operatorname{cis} \pi/4$, $\operatorname{cis} 3\pi/4$, $\operatorname{cis} 5\pi/4$ and $\operatorname{cis} 7\pi/4$.

$$\therefore \text{their continued product} = \operatorname{cis} \left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4} \right) = \operatorname{cis} 4\pi = 1.$$

Example 19.20. Use De Moivre's theorem to solve the equation.

(P.T.U., 2005)

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. ' $x^4 - x^3 + x^2 - x + 1$ ' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1 - (-x)^5}{1 - (-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e.,

$$x^5 = -1 = \text{cis } \pi = \text{cis } (2n + 1)\pi$$

$$\therefore x = [\text{cis } (2n + 1)\pi]^{1/5} = \text{cis } (2n + 1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are $\text{cis } \pi/5, \text{cis } 3\pi/5, \text{cis } \pi, \text{cis } 7\pi/5, \text{cis } 9\pi/5$

or

$$\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}, \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}, -1, \cos \frac{7\pi}{5} - i \sin \frac{7\pi}{5}, \cos \frac{9\pi}{5} - i \sin \frac{9\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos \pi/5 \pm i \sin \pi/5, \cos 3\pi/5 \pm i \sin 3\pi/5.$$

Example 19.21. Show that the roots of the equation $(x - 1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i \cot r\pi/n)$, where r has the values $1, 2, 3, \dots, n - 1$.

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1 \quad \text{or} \quad 1 - \frac{1}{x} = (1)^{1/n}$

$$\text{or} \quad \frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis } \frac{2r\pi}{n}, r = 0, 1, 2, \dots (n-1).$$

[$\because 1 = \text{cis } 2\pi r$]

$$\text{or} \quad = \left(1 - \cos \frac{2r\pi}{n}\right) - i \sin \frac{2r\pi}{n} = 2 \sin^2 \frac{r\pi}{n} - 2i \sin \frac{r\pi}{n} \cos \frac{r\pi}{n}$$

$$\therefore x = \frac{1}{2 \sin \frac{r\pi}{n}} \cdot \frac{1}{\left(\sin \frac{r\pi}{n} - i \cos \frac{r\pi}{n}\right)} = \frac{\sin \frac{r\pi}{n} + i \cos \frac{r\pi}{n}}{2 \sin \frac{r\pi}{n}}$$

$$= \frac{1}{2} \left(1 + i \cot \frac{r\pi}{n}\right), r = 1, 2, \dots (n-1). \quad [\because \cot 0 \rightarrow \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i \cot r\pi/n)$ where $r = 1, 2, 3, \dots (n-1)$.

Example 19.22. Find the 7th roots of unity and prove that the sum of their n th powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.

(Mumbai, 2008; Kurukshetra, 2005)

Solution. We have $(1)^{1/7} = (\cos 2r\pi + i \sin 2r\pi)^{1/7} = \text{cis } \frac{2r\pi}{7} = \left(\text{cis } \frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

\therefore sum S of the n th powers of these roots $= 1 + \rho^n + \rho^{2n} + \dots + \rho^{6n}$... (i)

$$= \frac{1 - \rho^{7n}}{1 - \rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis } 2\pi)^n = 1$.

i.e., $1 - \rho^{7n} = 0$ and $1 - \rho^n \neq 0$, as n is not a multiple of 7.

Thus $S = 0$.

When n is a multiple of 7 = $7p$ (say)

$$\text{From (i), } S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \dots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

Example 19.23. Find the equation whose roots are $2 \cos \pi/7, 2 \cos 3\pi/7, 2 \cos 5\pi/7$.

Solution. Let $y = \cos \theta + i \sin \theta$, where $\theta = \pi/7, 3\pi/7, \dots, 13\pi/7$.

Then $y^7 = (\cos \theta + i \sin \theta)^7 = \cos 7\theta + i \sin 7\theta = -1$ or $y^7 + 1 = 0$
 or $(y + 1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$... (i)

Its roots are $y = \text{cis } \theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (i) by y^3 , $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $x^3 - x^2 - 2x + 1 = 0$... (ii)

where $x = y + 1/y = 2 \cos \theta$.

Now since $\cos 13\pi/7 = \cos \pi/7, \cos 11\pi/7 = \cos 3\pi/7, \cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (ii) are $2 \cos \frac{\pi}{7}, 2 \cos \frac{3\pi}{7}, 2 \cos \frac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of

$$(i) (1+i)^{1/4} \quad (ii) (-1+i)^{2/5} \\ (iii) (-1+i\sqrt{3})^{3/2} \quad (iv) (1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$$

2. If w is a complex cube root of unity, prove that $1+w+w^2=0$.

3. Find all the values of $(-1)^{1/6}$.

4. Mark by points on the Argand diagram, all the values of $(1+i\sqrt{3})^{1/5}$ and verify that they form a pentagon.

5. Use De Moivre's theorem to solve the following equations :

$$(i) x^5 + 1 = 0 \quad (ii) x^7 + x^4 + x^3 + 1 = 0 \\ (iii) x^9 + x^5 - x^4 - 1 = 0 \quad (iv) (x-1)^5 + x^5 = 0$$

6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.

7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.

8. Show that the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i \cot r\pi/7$, $r = 1, 2, 3$. (Mumbai, 2008)

9. Prove that the n th roots of unity form a geometric progression. (Mumbai, 2007)

Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.

10. Find the equation whose roots are $2 \cos 2\pi/7, 2 \cos 4\pi/7, 2 \cos 6\pi/7$.

19.6 (1) TO EXPAND $\sin n\theta, \cos n\theta$ AND $\tan n\theta$ IN POWERS OF $\sin \theta, \cos \theta$ AND $\tan \theta$ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$ (By De Moivre's theorem)
 $= \cos^n \theta + {}^nC_1 \cos^{n-1} \theta (i \sin \theta) + {}^nC_2 \cos^{n-2} \theta (i \sin \theta)^2 + {}^nC_3 \cos^{n-3} \theta (i \sin \theta)^3 + \dots$ (By Binomial theorem)
 $= (\cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + \dots) + i ({}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + \dots)$

Equating real and imaginary parts from both sides, we get

$$\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots \quad \dots(1)$$

$$\sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + {}^nC_5 \cos^{n-5} \theta \sin^5 \theta - \dots \quad \dots(2)$$

Replacing every $\sin^2 \theta$ by $1 - \cos^2 \theta$ in (1) and every $\cos^2 \theta$ by $1 - \sin^2 \theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + {}^nC_5 \cos^{n-5} \theta \sin^5 \theta - \dots}{\cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots}$$

and dividing numerator and denominator by $\cos^n \theta$, we get

$$\tan n\theta = \frac{{}^nC_1 \tan \theta - {}^nC_3 \tan^3 \theta + {}^nC_5 \tan^5 \theta - \dots}{1 - {}^nC_2 \tan^2 \theta + {}^nC_4 \tan^4 \theta - \dots}$$

Example 19.24. Express $\cos 6\theta$ in terms of $\cos \theta$.

(Madras, 2002)

Solution. We know that $\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + {}^nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots$

$$\begin{aligned}\text{Put } n = 6, \text{ then } \cos 6\theta &= \cos^6 \theta - {}^6C_2 \cos^4 \theta \sin^2 \theta + {}^6C_4 \cos^2 \theta \sin^4 \theta - {}^6C_6 \sin^6 \theta \\ &= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3 \\ &= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.\end{aligned}$$

(2) Addition formulae for any number of angles

$$\begin{aligned}\text{We have, } \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n) \\ &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots (\cos \theta_n + i \sin \theta_n)\end{aligned}$$

Now $\cos \theta_1 + i \sin \theta_1 = \cos \theta_1 (1 + i \tan \theta_1)$, $\cos \theta_2 + i \sin \theta_2 = \cos \theta_2 (1 + i \tan \theta_2)$ and so on.

$$\begin{aligned}\therefore \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n) \\ &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n (1 + i \tan \theta_1)(1 + i \tan \theta_2) \dots (1 + i \tan \theta_n) \\ &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n [1 + i(\tan \theta_1 + \tan \theta_2 + \dots + \tan \theta_n) \\ &\quad + i^2(\tan \theta_1 \tan \theta_2 + \tan \theta_2 \tan \theta_3 + \dots) + i^3(\tan \theta_1 \tan \theta_2 \tan \theta_3 + \dots) + \dots + \dots] \\ &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n (1 + is_1 - s_2 - is_3 + s_4 + \dots)\end{aligned}$$

where $s_1 = \tan \theta_1 + \tan \theta_2 + \dots + \tan \theta_n$, $s_2 = \sum \tan \theta_1 \tan \theta_2$, $s_3 = \sum \tan \theta_1 \tan \theta_2 \tan \theta_3$ etc.

Equating real and imaginary parts, we have

$$\begin{aligned}\cos(\theta_1 + \theta_2 + \dots + \theta_n) &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n (1 - s_2 + s_4 - \dots) \\ \sin(\theta_1 + \theta_2 + \dots + \theta_n) &= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n (s_1 - s_3 + s_5 - \dots)\end{aligned}$$

and by division, we get $\tan(\theta_1 + \theta_2 + \dots + \theta_n) = \frac{s_1 - s_3 + s_5 - \dots}{1 - s_2 + s_4 - s_6 + \dots}$.

Example 19.25. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi/2$, show that $xy + yz + zx = 1$.

(P.T.U., 2003)

Solution. Let $\tan^{-1} x = \alpha$, $\tan^{-1} y = \beta$, $\tan^{-1} z = \gamma$ so that $x = \tan \alpha$, $y = \tan \beta$, $z = \tan \gamma$

$$\begin{aligned}\text{We know that } \tan(\alpha + \beta + \gamma) &= \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha} \\ \therefore \tan \pi/2 &= \frac{x + y + z - xyz}{1 - xy - yz - zx} \quad \text{or} \quad 1 - xy - yz - zx = 0\end{aligned}$$

Hence $xy + yz + zx = 1$.

Example 19.26. If $\theta_1, \theta_2, \theta_3$ be three values of θ which satisfy the equation $\tan 2\theta = \lambda \tan(\theta + \alpha)$ and such that no two of them differ by a multiple of π , show that $\theta_1 + \theta_2 + \theta_3 + \alpha$ is a multiple of π .

Solution. Given equation can be written as $\frac{2t}{1-t^2} = \lambda \frac{t + \tan \alpha}{1 - t \cdot \tan \alpha}$ where $t = \tan \theta$

or $\lambda t^3 + (\lambda - 2) \tan \alpha \cdot t^2 + (2 - \lambda) t - \lambda \tan \alpha = 0$

$\therefore \tan \theta_1, \tan \theta_2, \tan \theta_3$, being its roots, we have

$$s_1 = \sum \tan \theta_i = -\frac{\lambda - 2}{\lambda} \tan \alpha \quad [\text{By } \S 1.3]$$

$$s_2 = \sum \tan \theta_i \tan \theta_j = \frac{2 - \lambda}{\lambda} \quad \text{and} \quad s_3 = \tan \alpha$$

$$\begin{aligned}\therefore \tan(\theta_1 + \theta_2 + \theta_3) &= \frac{s_1 - s_3}{1 - s_2} = \frac{(-1 + 2/\lambda) \tan \alpha - \tan \alpha}{1 - (2/\lambda - 1)} \\ &= -\tan \alpha = \tan(n\pi - \alpha)\end{aligned}$$

Thus $\theta_1 + \theta_2 + \theta_3 = n\pi - \alpha$, whence follows the result.

(3) To expand $\sin^m \theta, \cos^n \theta$ or $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ

If $z = \cos \theta + i \sin \theta$ then $1/z = \cos \theta - i \sin \theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i \sin p\theta$ and $1/z^p = \cos p\theta - i \sin p\theta$

$$\therefore z + 1/z = 2 \cos \theta, z - 1/z = 2i \sin \theta; z^p + 1/z^p = 2 \cos p\theta, z^p - 1/z^p = 2i \sin p\theta$$

These results are used to expand the powers of $\sin \theta$ or $\cos \theta$ or their products in a series of sines or cosines of multiples of θ .

Example 19.27. Expand $\cos^8 \theta$ in a series of cosines of multiples of θ .

Solution. Let $z = \cos \theta + i \sin \theta$, so that $z + 1/z = 2 \cos \theta$ and $z^p + 1/z^p = 2 \cos p\theta$.

$$\therefore (2 \cos \theta)^8 = (z + 1/z)^8$$

$$\begin{aligned} &= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8} \\ &= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4 \\ &= (2 \cos 8\theta) + 8(2 \cos 6\theta) + 28(2 \cos 4\theta) + 56(2 \cos 2\theta) + 70. \end{aligned}$$

$$\text{Hence } \cos^8 \theta = \frac{1}{128} [\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 70].$$

Example 19.28. Expand $\sin^7 \theta \cos^3 \theta$ in a series of sines of multiples of θ .

Solution. Let $z = \cos \theta + i \sin \theta$

so that $z + 1/z = 2 \cos \theta$, $z - 1/z = 2i \sin \theta$ and $z^p - 1/z^p = 2i \sin p\theta$.

$$\begin{aligned} \therefore (2i \sin \theta)^7 (2 \cos \theta)^3 &= (z - 1/z)^7 (z + 1/z)^3 \\ &= (z - 1/z)^4 [(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4 (z^2 - 1/z^2)^3 \\ &= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \right) \left(z^6 - 3z^4 + \frac{3}{z^2} - \frac{1}{z^6} \right) \\ &= \left(z^{10} - \frac{1}{z^{10}} \right) - 4 \left(z^8 - \frac{1}{z^8} \right) + 3 \left(z^6 - \frac{1}{z^6} \right) + 8 \left(z^4 - \frac{1}{z^4} \right) - 14 \left(z^2 - \frac{1}{z^2} \right) \\ &= 2i \sin 10\theta - 4(2i \sin 8\theta) + 3(2i \sin 6\theta) + 8(2i \sin 4\theta) - 14(2i \sin 2\theta) \end{aligned}$$

$$\text{Since } i^7 = -i,$$

$$\therefore \sin^7 \theta \cos^3 \theta = -\frac{1}{2^9} [\sin 10\theta - 4 \sin 8\theta + 3 \sin 6\theta + 8 \sin 4\theta - 14 \sin 2\theta].$$

Obs. The expansion of $\sin^m \theta \cos^n \theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express $\sin 6\theta / \sin \theta$ as a polynomial in $\cos \theta$?

Prove that (2–5):

2. $\sin 7\theta / \sin \theta = 7 - 56 \sin^2 \theta + 112 \sin^4 \theta - 64 \sin^6 \theta$.

3. $\frac{1 + \cos 7\theta}{1 + \cos \theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2 \cos \theta$. (Madras, 2002)

4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2 \cos \theta$. 5. $\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan \theta$.

6. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, show that $x + y + z = xyz$.

7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, prove that
 $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = n\pi$ radians except in one particular case.

Prove that (8–12):

8. $\cos^7 \theta = \frac{1}{16} (\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta)$. (Madras, 2003 S)

9. $\cos^6 \theta - \sin^6 \theta = \frac{1}{16} (\cos 6\theta + 15 \cos 2\theta)$. (Mumbai, 2007)

10. $\sin^8 \theta = 2^{-7} (\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 2\theta + 35)$.

11. $32 \sin^4 \theta \cos^2 \theta = \cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2$.

12. $\sin^5 \theta \cos^2 \theta = \frac{1}{64} (\sin 7\theta - 3 \sin 5\theta + \sin 3\theta + 5 \sin \theta)$. (Madras, 2003)

13. Expand $\cos^5 \theta \sin^7 \theta$ in a series of sines of multiples of θ ?
 (Madras, 2002)
14. If $\cos^5 \theta = A \cos \theta + B \cos 3\theta + C \cos 5\theta$, find $\sin^5 \theta$ in terms of A, B, C .
15. If $\sin^4 \theta \cos^3 \theta = A_1 \cos \theta + A_3 \cos 3\theta + A_5 \cos 5\theta + A_7 \cos 7\theta$, prove that

$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. If for each value of the complex variable $z (= x + iy)$ in a given region R , we have one or more values of $w (= u + iv)$, then w is said to be a **complex function** of z and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y .

If to each value of z , there corresponds one and only one value of w , then w is said to be a *single-valued function* of z otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z . The former is defined at all points of the z -plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) **Definition.** When x is real, we are already familiar with the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \infty.$$

Similarly, we define the exponential function of the complex variable $z = x + iy$, as

$$e^z \text{ or } \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \infty \quad \dots(i)$$

(2) **Properties :**

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get

$$\begin{aligned} e^{iy} &= 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \infty \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) = \cos y + i \sin y \end{aligned}$$

Thus $e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$

Also $x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, [$\because e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z$].

III. e^z is not zero for any value of z .

Since $e^z = e^x \cdot e^{iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore r = e^x > 0, y = \theta, |e^{iy}| = 1,$

Thus $|e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0$.

IV. $e^{\bar{z}} = \overline{e^z}$

Since $e^{\bar{z}} = e^{\bar{x}-iy} = e^x \cdot e^{-iy} = e^x (\cos y - i \sin y)$

$$= \overline{e^x (\cos y + i \sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) **Definitions:**

Since $e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$.

\therefore the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to define the circular functions of the complex variable z by the equations :

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}$$

with $\operatorname{cosec} z$, $\sec z$ and $\cot z$ as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π . [$a \sin(z + 2n\pi) = \sin z$, $\tan(z + n\pi) = \tan z$ etc.]

II. Even and odd functions : $\cos z$, $\sec z$ are even functions while $\sin z$, $\operatorname{cosec} z$ are odd functions. [∴ $\cos z$

$$= \frac{e^{-iz} + e^{iz}}{2} = \cos z, \text{ and } \sin(-z) = \frac{e^{-iz} - e^{iz}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = -\sin z]$$

III. Zeros of $\sin z$ are given by $z = \pm 2n\pi$ and zeros of $\cos z$ are given by $z = \pm \frac{1}{2}(2n+1)\pi$, $n = 0, 1, 2, \dots$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1$, $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

$$\text{By definition } \cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \frac{e^{iz} - e^{-iz}}{2i} = e^{iz} \quad \text{where } z = x + iy.$$

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos \theta + i \sin \theta$, where θ is real or complex. This is called the Euler's theorem.*

Cor. De Moivre's theorem for complex numbers

Whether θ is real or complex, we have

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. Prove that (i) $[\sin(\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha e^{-in\theta}$

$$(ii) \sin(\alpha - n\theta) + e^{-i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha.$$

Solution. (i) L.H.S. = $[\sin \alpha \cos \theta + \cos \alpha \sin \theta - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$$

$$= \sin^n \alpha (\cos \theta - i \sin \theta)^n = \sin^n \alpha (e^{-i\theta})^n = \sin^n \alpha e^{-in\theta}$$

(ii) L.H.S. = $\sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta$

$$= \sin \alpha \cos n\theta - i \sin \alpha \sin n\theta$$

$$= \sin \alpha (\cos n\theta - i \sin n\theta) = \sin \alpha \cdot e^{-in\theta}.$$

Example 19.30. Given $\frac{1}{\rho} = \frac{1}{L\rho i} + C\rho i + \frac{1}{R}$, where L , ρ , R are real, express ρ in the form $Ae^{i\theta}$ giving the values of A and θ .

$$\text{Solution. } \frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or

$$\begin{aligned} \rho &= L\rho \frac{Ri}{(R - L\rho^2 CR) + iLR} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho} \\ &= \frac{L^2 \rho^2 R + iL\rho R (R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos \theta + i \sin \theta), \text{ say} \end{aligned}$$

*See footnote p. 205.

Equating real and imaginary parts, we have

$$A \cos \theta = \frac{L^2 \rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad \dots(i)$$

$$A \sin \theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad \dots(ii)$$

Squaring and adding (i) and (ii),

$$A^2 = \frac{(L^2 \rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2}} \quad \dots(iii)$$

Dividing (ii) by (i),

$$\tan \theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1} \left\{ \frac{R(1 - LC\rho^2)}{L\rho} \right\} \quad \dots(iv)$$

Hence $P = A(\cos \theta + i \sin \theta) = Ae^{i\theta}$

where A and θ are given by (iii) and (iv).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: If x be real or complex,

(i) $\frac{e^x - e^{-x}}{2}$ is defined as **hyperbolic sine of x** and is written as **sinh x** .

(ii) $\frac{e^x + e^{-x}}{2}$ is defined as **hyperbolic cosine of x** and is written as **cosh x** .

Thus $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. *Periodic functions*: $\sinh z$ and $\cosh z$ are periodic functions having imaginary period $2\pi i$.

[$\because \sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. *Even and odd functions*: $\cosh z$ is an even function while $\sinh z$ is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ , $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$$\begin{aligned} \therefore \text{ Putting } \theta = ix, \text{ we have } \sin ix &= \frac{e^{-x} - e^x}{2i} = -\frac{e^x - e^{-x}}{2i} \\ &= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh x \end{aligned} \quad [\because e^{i\theta} = e^{i \cdot ix} = e^{-x}]$$

and, therefore,

$$\cos ix = \frac{e^{-x} + e^x}{2} = \cosh x \quad \dots(i)$$

$$\sin ix = i \sinh x \quad \dots(ii)$$

$$\cos ix = \cosh x \quad \dots(iii)$$

$$\tan ix = i \tanh x \quad \dots(iv)$$

$$\sinh ix = i \sin x \quad \dots(v)$$

$$\cosh ix = \cos x \quad \dots(vi)$$

$$\tanh ix = i \tan x \quad \dots(vii)$$

$$\operatorname{cosech} ix = i \operatorname{cosec} x \quad \dots(viii)$$

$$\operatorname{sech} ix = i \operatorname{sec} x \quad \dots(ix)$$

V. Formulae of hyperbolic functions

(a) Fundamental formulae

$$(1) \cosh^2 x - \sinh^2 x = 1 \quad (2) \operatorname{sech}^2 x + \tanh^2 x = 1 \quad (3) \coth^2 x - \operatorname{cosech}^2 x = 1.$$

(b) Addition formulae

$$(4) \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \quad (5) \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$(6) \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

(c) Functions of 2x.

$$(7) \sinh 2x = 2 \sinh x \cosh x$$

$$(8) \cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$(9) \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

(d) Functions of 3x

$$(10) \sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

$$(11) \cosh 3x = 4 \cosh^3 x - 3 \cosh x \quad (12) \tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

$$(e) (13) \sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (14) \sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$$

$$(15) \cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2} \quad (16) \cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}.$$

Proofs. (1) Since, for all values of θ , we have $\cos^2 \theta + \sin^2 \theta = 1$.

∴ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

$$\text{Otherwise : } \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 = \frac{1}{4} [e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1.$$

Similarly we can establish the formulae (2) and (3).

$$(4) \sinh(x + y) = (1/i) \sin i(x + y) = -i[\sin ix \cos iy + \cos ix \sin iy] \\ = -i[i \sinh x \cosh y + \cosh x i \sinh y] = \sinh x \cosh y + \cosh x \sinh y.$$

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x + y)$$

Similarly we can establish the formulae (5) and (6).

$$(12) \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

$$\text{Putting } A = ix, \tan 3ix = \frac{3 \tan ix - \tan^3 ix}{1 - 3 \tan^2 ix} \quad \text{or} \quad i \tanh 3x = \frac{3(i \tanh x) - (i \tanh x)^3}{1 - 3(i \tanh x)^2}$$

$$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

$$(16) \cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$\text{Putting } C = ix, \text{ and } D = iy, \cos ix - \cos iy = -2 \sin i \frac{x+y}{2} \sin i \frac{x-y}{2}$$

$$\text{or} \quad \cosh x - \cosh y = -2 \left(i \sinh \frac{x+y}{2} \right) \left(i \sinh \frac{x-y}{2} \right) = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If $\sinh u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) To show that (i) $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$

(Mumbai, 2009)

$$(ii) \cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}], \quad (iii) \tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}.$$

$$(i) \text{ Let } \sinh^{-1} z = u, \text{ then } z = \sinh u = \frac{1}{2}(e^u - e^{-u})$$

$$\text{or } 2z = e^u - e^{-u} \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u , we have

$$e^u = \frac{2z \pm \sqrt{(4z^2 + 4)}}{2} = z \pm \sqrt{(z^2 + 1)}$$

∴ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2 + 1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2 + 1)}]$$

Similarly we can establish (ii)

(iii) Let $\tanh^{-1} z = u$, then $z = \tanh u$

$$\text{i.e., } z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\frac{1+z}{1-z} = e^u/e^{-u} = e^{2u}$

$$\text{or } 2u = \log \left(\frac{1+z}{1-z} \right) \text{ whence follows the result.} \quad (\text{P.T.U., 2005})$$

Example 19.31. If $u = \log \tan (\pi/4 + \theta/2)$, prove that

(i) $\tanh u/2 = \tan \theta/2$

(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)

(ii) $\theta = -i \log \tan \left(\frac{\pi}{4} + \frac{iu}{2} \right)$.

(Kurukshetra, 2006)

Solution. We have $e^u = \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right)$ or $\frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \theta/2}{1 - \tan \theta/2}$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad \text{i.e.,} \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \quad \dots(i)$$

$$\text{or } \frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1} \left(\tan \frac{iu}{2} \right) = \frac{1}{2} \log \frac{1 + \tan iu/2}{1 - \tan iu/2}$$

$$\text{or } \theta = \frac{1}{i} \log \tan \left(\frac{\pi}{4} + \frac{iu}{2} \right) = -i \log \tan \left(\frac{\pi}{4} + \frac{iu}{2} \right). \quad \dots(ii)$$

Example 19.32. Show that $\tanh^{-1} (\cos \theta) = \cosh^{-1} (\operatorname{cosec} \theta)$.

(Kurukshetra, 2005)

Solution. Let $\tanh^{-1} (\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

$$\text{or} \quad \tanh^2 \phi = \cos^2 \theta \quad \text{or} \quad 1 - \operatorname{sech}^2 \phi = \cos^2 \theta$$

$$\text{or} \quad \operatorname{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta \quad \text{or} \quad \operatorname{sech} \phi = \sin \theta$$

$$\text{or} \quad \cosh \phi = \operatorname{cosec} \theta \quad \text{or} \quad \phi = \cosh^{-1} (\operatorname{cosec} \theta).$$

Example 19.33. Find $\tanh x$, if $5 \sinh x - \cosh x = 5$.

(Mumbai, 2004)

Solution. We have $5(\sinh x - 1) = \cosh x$

$$\text{or } 25(\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$$

$$\text{or } 24 \sinh^2 x - 50 \sinh x + 24 = 0 \quad \text{or} \quad 12 \sinh^2 x - 25 \sinh x + 12 = 0$$

$$\text{or } (3 \sinh x - 4)(4 \sinh x - 3) = 0 \quad \text{whence } \sinh x = 4/3 \quad \text{or} \quad 3/4.$$

$$\therefore \cosh x = \sqrt{1 + \sinh^2 x} = 5/3 \quad \text{or} \quad -5/4 \quad [\because \cosh x = 5/4 \text{ doesn't satisfy (i)}]$$

$$\text{Hence } \tanh x = \frac{4}{5} \quad \text{or} \quad -\frac{3}{5}.$$

PROBLEMS 19.5

1. Separate into real and imaginary parts

$$(i) \exp(z^2) \text{ where } z = x + iy \quad (ii) \exp(5 + i\pi/2) \quad (iii) \exp(5 + 3i)^2.$$

2. From the definitions of $\sin z$ and $\cos z$, prove that

$$(i) \cos 2z = 2 \cos^2 z - 1 \quad (ii) \frac{\sin 2z}{1 - \cos 2z} = \cot z \quad (iii) \sin 3z = 3 \sin z - 4 \sin^3 z.$$

3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha [\sin(\alpha - n\theta) + e^{-in\alpha} \sin n\theta]$

4. If $z = e^{i\theta}$, show that $\frac{z^2 - 1}{z^2 + 1} = i \tan \theta$.

5. Eliminate z from $p \operatorname{cosech} z + q \operatorname{sech} z + r = 0$, $p' \operatorname{cosech} z + q' \operatorname{sech} z + r' = 0$.

6. If $y = \log \tan x$, show that $\sinh ny = \frac{1}{2} (\tan^n x - \cot^n x)$.

7. If $\tan y = \tan \alpha \tanh \beta$ and $\tan z = \cot \alpha \tanh \beta$, prove that $\tan(y+z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.

8. Prove that

$$(i) \cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2 \sinh \alpha \sinh \beta$$

$$(ii) \sinh(\alpha + \beta) \cosh(\alpha - \beta) = \frac{1}{2} (\sinh 2\alpha + \sinh 2\beta).$$

9. Prove that (i) $(\cosh \theta \pm \sinh \theta)^n = \cosh n\theta \pm \sinh n\theta$; (ii) $\left(\frac{1 + \tanh \theta}{1 - \tanh \theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.

10. Express $\cosh^7 \theta$ in terms of hyperbolic cosines of multiples of θ .

11. If $\sin \theta = \tanh x$, prove that $\tan \theta = \sinh x$.

12. If $\tan x/2 = \tanh u/2$, prove that

$$(i) \tan x = \sinh u \text{ and } \cos x \cosh u = 1; \quad (ii) u = \log_e \tan(\pi/4 + x/2).$$

13. If $\cosh x = \sec \theta$, prove that

$$(i) \tanh^2 x/2 = \tan^2 \theta/2; \quad (ii) x = \log_e \tan(\pi/4 + \theta/2).$$

14. Show that $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$.

15. Prove that

$$(i) \sinh^{-1} x = \cosh^{-1} \sqrt{1+x^2} = \tanh^{-1} \frac{x}{\sqrt{1-x^2}} = \frac{1}{2} \operatorname{cosech}^{-1} \frac{1}{2x\sqrt{1+x^2}}$$

$$(ii) \tanh^{-1} x = \sinh^{-1} \frac{x}{\sqrt{1-x^2}}.$$

16. Show that

$$(i) \sinh^{-1}(\tan \theta) = \log \tan(\pi/4 + \theta/2) \quad (ii) \operatorname{sech}^{-1}(\sin \theta) = \log \cot \theta/2.$$

17. Solve the equation $7 \cosh x + 8 \sinh x = 1$ for real values of x .

(Mumbai, 2008)

18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) To separate the real and imaginary parts of

(i) $\sin(x+iy)$; (ii) $\cos(x+iy)$; (iii) $\tan(x+iy)$; (iv) $\cot(x+iy)$; (v) $\sec(x+iy)$; (vi) $\operatorname{cosec}(x+iy)$.

Proofs. (i) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

(iii) Let $\alpha + i\beta = \tan(x+iy)$ then $\alpha - i\beta = \tan(x-iy)$

Adding, $2\alpha = \tan(x+iy) + \tan(x-iy)$

$$\text{i.e., } \alpha = \frac{\sin(x+iy) + \sin(x-iy)}{2 \cos(x+iy) \cos(x-iy)} = \frac{\sin 2x}{\cos 2x + \cos 2y} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x+iy) - \tan(x-iy)$

$$\text{i.e., } i\beta = \frac{\sin 2iy}{2 \cos(x+iy) \cos(x-iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore \beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\text{Similarly, } \cot(x+iy) = \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}.$$

(v) Let $\alpha + i\beta = \sec(x+iy)$ then $\alpha - i\beta = \sec(x-iy)$

Adding, $2\alpha = \sec(x+iy) + \sec(x-iy)$

$$\text{i.e., } \alpha = \frac{\cos(x-iy) + \cos(x+iy)}{2 \cos(x+iy) \cos(x-iy)} = \frac{2 \cos x \cos iy}{\cos 2x + \cos 2y} = \frac{2 \cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x+iy) - \sec(x-iy)$

$$\text{i.e., } i\beta = \frac{\cos(x-iy) - \cos(x+iy)}{2 \cos(x+iy) \cos(x-iy)} = \frac{2 \sin x \sin iy}{\cos 2x + \cos 2y} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\therefore \beta = \frac{2 \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\text{Similarly, } \operatorname{cosec}(x+iy) = 2 \frac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}.$$

(2) To separate the real and imaginary parts of

(i) $\sinh(x+iy)$; (ii) $\cosh(x+iy)$; (iii) $\tanh(x+iy)$.

Proofs. (i) $\sinh(x+iy) = (1/i) \sin i(x+iy) = (1/i) \sin(ix-y)$

$$= (1/i) [\sin ix \cos y - \cos ix \sin y] = (1/i) [i \sinh x \cos y - \cosh x \sin y]$$

$$= \sinh x \cos y + i \cosh x \sin y$$

Similarly, $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$.

(iii) If $\alpha + i\beta = \tanh(x+iy) = (1/i) \tan(ix-y)$

then $\alpha - i\beta = \tanh(x-iy) = (1/i) \tan(ix+y)$

Adding, $2\alpha = (1/i) [\tan(ix-y) + \tan(ix+y)]$

$$\alpha = \frac{\sin(ix-y+ix+y)}{i \cdot 2 \cos(ix-y) \cos(ix+y)} = \frac{(1/i) \sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cosh 2y}.$$

Subtracting, $2i\beta = (1/i) [\tan(ix-y) - \tan(ix+y)]$

$$\text{i.e., } i\beta = - \frac{\sin[(ix+y)-(ix-y)]}{i \cdot 2 \cos(ix+y) \cos(ix-y)}$$

$$\therefore \beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cosh 2y}.$$

Example 19.34. If $\cosh(u+iv) = x+iy$, prove that

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad (\text{P.T.U., 2009 S}) \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

(Madras, 2000)

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$
 $= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v$.
 \therefore equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e., $\frac{x}{\cosh u} = \cos v$ and $\frac{y}{\sinh u} = \sin v$

Squaring and adding, we get the first result.

Again $\frac{x}{\cos v} = \cosh u$ and $\frac{v}{\sin v} = \sinh u$.

\therefore squaring and subtracting, we get the second result.

Example 19.35. If $\tan(\theta + i\phi) = e^{i\alpha}$, show that

$$\theta = (n + 1/2)\pi/2 \text{ and } \phi = \frac{1}{2} \log \tan(\pi/4 + \alpha/2). \quad (\text{S.V.T.U., 2007; Rohtak, 2005})$$

Solution. Since $\tan(\theta + i\phi) = \cos \alpha + i \sin \alpha \therefore \tan(\theta - i\phi) = \cos \alpha - i \sin \alpha$
 $\therefore \tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi) \tan(\theta - i\phi)} = \frac{2 \cos \alpha}{1 - (\cos^2 \alpha + \sin^2 \alpha)} = \frac{2 \cos \alpha}{0} \rightarrow \infty$$

i.e., $2\theta = n\pi + \pi/2 \text{ or } \theta = (n + 1/2)\pi/2$

Also $\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi) \tan(\theta - i\phi)}$

or $i \tanh 2\phi = \frac{2i \sin \alpha}{1 + (\cos^2 \alpha + \sin^2 \alpha)} = i \sin \alpha \text{ or } \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin \alpha}{1}$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin \alpha}{1 - \sin \alpha} = \frac{\cos^2 \alpha/2 + \sin^2 \alpha/2 + 2 \sin \alpha/2 \cos \alpha/2}{\cos^2 \alpha/2 + \sin^2 \alpha/2 - 2 \sin \alpha/2 \cos \alpha/2}$$

or $e^{4\phi} = \frac{(\cos \alpha/2 + \sin \alpha/2)^2}{(\cos \alpha/2 - \sin \alpha/2)^2} = \left(\frac{1 + \tan \alpha/2}{1 - \tan \alpha/2} \right)^2$

or $e^{2\phi} = \frac{1 + \tan \alpha/2}{1 - \tan \alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$. Hence $\phi = \frac{1}{2} \log \tan(\pi/4 + \alpha/2)$.

Example 19.36. Separate $\tan^{-1}(x + iy)$ into real and imaginary parts.

(S.V.T.U., 2009)

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding, $2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)* = \tan^{-1} \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$

$\therefore \alpha = \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2}$

Subtracting, $2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1} \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$

$$= \tan^{-1} i \frac{2y}{1 + x^2 + y^2} = i \tanh^{-1} \frac{2y}{1 + x^2 + y^2} \quad [\because \tan^{-1} iz = i \tanh^{-1} z]$$

$\therefore \beta = \frac{1}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}$.

Example 19.37. Separate $\sin^{-1}(\cos \theta + i \sin \theta)$ into real and imaginary parts, where θ is a positive acute angle.

* $\tan^{-1} A \pm \tan^{-1} B = \tan^{-1} \frac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos \theta + i \sin \theta) = x + iy$

Then $\cos \theta + i \sin \theta = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

$\therefore \cos \theta = \sin x \cosh y \quad \dots(i)$ and $\sin \theta = \cos x \sinh y \quad \dots(ii)$

Squaring and adding, we have

$$\begin{aligned} 1 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \sinh^2 y (\sin^2 x + \cos^2 x) \end{aligned}$$

or $1 - \sin^2 x = \sinh^2 y, \text{ i.e., } \cos^2 x = \sinh^2 y.$

Hence from (ii), we have $\sin^2 \theta = \cos^4 x, \text{ i.e., } \cos^2 x = \sin \theta$ because θ being a positive acute angle, $\sin \theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = \sqrt{\sin \theta} \text{ or } x = \cos^{-1} \sqrt{\sin \theta}$$

The relation (ii), then, gives $\sinh y = \sqrt{\sin \theta}$ so that $y = \log [\sqrt{\sin \theta} + \sqrt{1 + \sin \theta}]$.

PROBLEMS 19.6

1. If $\sin(A + iB) = x + iy$, prove that

$$(i) \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1 \quad (ii) \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1. \quad (P.T.U., 2010)$$

$$2. \text{ If } \cos(\alpha + i\beta) = r(\cos \theta + i \sin \theta), \text{ prove that } (i) e^{2\beta} = \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}. \quad (Kurukshetra, 2005; Madras, 2003)$$

$$(ii) \beta = \frac{1}{2} \log \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}. \quad (V.T.U., 2006)$$

3. If $\cos(\theta + i\phi) = \cos \alpha + i \sin \alpha$, prove that

$$(i) \sin^2 \theta = \pm \sin \alpha \quad (Madras, 2003) \quad (ii) \cos 2\theta + \cosh 2\phi = 2.$$

4. If $\tan(A + iB) = x + iy$, prove that

$$(i) x^2 + y^2 + 2x \cot 2A = 1. \quad (ii) x^2 + y^2 - 2y \coth 2B + 1 = 0. \quad (iii) x \sinh 2B = y \sin 2A.$$

$$5. \text{ If } \tan(\theta + i\phi) = \tan \alpha + i \sec \alpha, \text{ prove that } e^{2\phi} = \pm \cot \alpha/2 \text{ and } 2\theta = \left(n + \frac{1}{2}\right)\pi + \alpha. \quad (Nagpur, 2009; S.V.T.U., 2008)$$

$$6. \text{ If } \tan(x + iy) = \sin(u + iv), \text{ prove that } \frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tan v}. \quad (S.V.T.U., 2006)$$

$$7. \text{ If } \operatorname{cosec}(\pi/4 + ix) = u + iv, \text{ prove that } (u^2 + v^2) = 2(u^2 - v^2). \quad (Mumbai, 2009)$$

$$8. \text{ If } x = 2 \cos \alpha \cosh \beta, y = 2 \sin \alpha \sinh \beta, \text{ prove that } \sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \frac{4x}{x^2 + y^2}.$$

$$9. \text{ If } a + ib = \tanh(v + i\pi/4), \text{ prove that } a^2 + b^2 = 1.$$

$$10. \text{ Reduce } \tan^{-1}(\cos \theta + i \sin \theta) \text{ to the form } a + ib. \quad (Mumbai, 2009)$$

$$\text{Hence show that } \tan^{-1}(e^{i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2} \log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right).$$

11. Separate $\cos^{-1}(\cos \theta + i \sin \theta)$ into real and imaginary parts, where θ is a positive acute angle.

12. If $\sin^{-1}(u + iv) = \alpha + i\beta$, prove that $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$

13. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that

$$(i) x^2 \sec^2 \alpha - y^2 \operatorname{cosec}^2 \alpha = 1, \quad (ii) x^2 \operatorname{sech}^2 \beta + y^2 \operatorname{cosech}^2 \beta = 1.$$

$$14. \text{ Prove that } (i) \sin^{-1}(ix) = 2n\pi + i \log(\sqrt{1 + x^2} + x) \quad (ii) \sin^{-1}(\operatorname{cosec} \theta) = \pi/2 + i \log \cot \theta/2.$$

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) **Definition.** If $z (= x + iy)$ and $w (= u + iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$ (i)

$$\text{Also } e^{w+2in\pi} = e^w \cdot e^{2in\pi} = z \quad [\because e^{2in\pi} = 1]$$

$$\therefore \log z = w + 2in\pi \quad \dots(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $\text{Log } z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $\text{Log } z$.

Thus from (i) and (ii), $\text{Log}(x + iy) = 2in\pi + \log(x + iy)$.

Obs. 1. If $y = 0$, then $\text{Log } x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.

e.g.

$$\log_e(-2) = \log_e 2(-1) = \log_e 2 + \log_e(-1) = \log_e 2 + i\pi \quad [\because -1 = \cos \pi + i \sin \pi = e^{i\pi}] \\ = 0.6931 + i(3.1416).$$

(2) Real and imaginary parts of $\text{Log}(x + iy)$.

$$\text{Log}(x + iy) = 2in\pi + \log(x + iy)$$

$$= 2in\pi + \log[r(\cos \theta + i \sin \theta)] \quad \left\{ \begin{array}{l} \text{Put } x = r \cos \theta, y = r \sin \theta \text{ so that} \\ r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}(y/x) \end{array} \right. \\ = 2in\pi + \log(re^{i\theta}) \\ = 2in\pi + \log r + i\theta = \log \sqrt{x^2 + y^2} + i\{2n\pi + \tan^{-1}(y/x)\}$$

(3) Real and imaginary parts of $(\alpha + i\beta)^{x+iy}$

$$(\alpha + i\beta)^{x+iy} = e^{(x+iy)} \text{Log}(\alpha + i\beta) = e^{(x+iy)} [2in\pi + \log(\alpha + i\beta)] \quad \left\{ \begin{array}{l} \text{Put } \alpha = r \cos \theta, \beta = r \sin \theta \text{ so that} \\ r = \sqrt{\alpha^2 + \beta^2} \text{ and } \theta = \tan^{-1}\beta/\alpha \end{array} \right. \\ = e^{(x+iy)[2in\pi + \log r e^{i\theta}]} = e^{(x+iy)[\log r + i(2n\pi + \theta)]} \\ = e^{A+iB} = e^A (\cos B + i \sin B).$$

where $A = x \log r - y(2n\pi + \theta)$ and $B = y \log r + x(2n\pi + \theta)$.

\therefore the required real part $= e^A \cos B$ and the imaginary part $= e^A \sin B$.

Example 19.38. Find the general value of $\log(-i)$.

Solution. $\text{Log}(-i) = 2in\pi + \log[0 + i(-1)]$ $\left\{ \begin{array}{l} \text{Put } 0 = r \cos \theta, -1 = r \sin \theta \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2 \end{array} \right.$

$$= 2in\pi + \log[r(\cos \theta + i \sin \theta)] = 2in\pi + \log(re^{i\theta}) \\ = 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i(-\pi/2) = i\left(2n + \frac{1}{2}\right)\pi.$$

Example 19.39. Prove that (i) $i^i = e^{-(4n+1)\pi/2}$ and $\text{Log } i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(ii) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha} \text{ cis } \alpha$ where $\alpha = \pi/4 \sqrt{2}$.

(Mumbai, 2008)

Solution. (i) By definition, we have

$$i^i = e^{i \text{Log } i} = e^{i(2in\pi + \log i)} = e^{-2n\pi + i \log[\exp(i\pi/2)]} \quad [\because i = \text{cis } \pi/2 = \exp(i\pi/2)] \\ = e^{-2n\pi + i(i\pi/2)} = e^{-(2n + 1/2)\pi}$$

Taking logarithms, we get (ii)

$$(ii) \quad (\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i} \log \sqrt{i}}$$

Now $\sqrt{i} \log \sqrt{i} = \frac{1}{2} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{1/2} \log \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$

$$= \frac{1}{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \log(e^{i\pi/2}) = \frac{1}{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \frac{i\pi}{2}$$

$$= \frac{i\pi}{4} \left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = -\frac{\pi}{4\sqrt{2}} + i \frac{\pi}{4\sqrt{2}}$$

Hence $(\sqrt{i})^{\sqrt{I}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4 \sqrt{2}$
 $= e^{-\alpha} \cdot e^{i\alpha} = e^{-\alpha} (\cos \alpha + i \sin \alpha).$

Example 19.40. If $(a + ib)^p = m^{x+iy}$, prove that one of the values of y/x is
 $2 \tan^{-1}(b/a) / \log(a^2 + b^2)$.

Solution. Taking logarithms, $(a + ib)^p = m^{x+iy}$ gives $p \log(a + ib) = (x + iy) \log m$

or $p \left(\frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \frac{b}{a} \right) = x \log m + iy \log m$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2} \log(a^2 + b^2) = x \log m \quad \dots(i), \quad p \tan^{-1} \frac{b}{a} = y \log m \quad \dots(ii)$$

Division of (ii) by (i) gives

$$y/x = 2 \tan^{-1}(b/a) / \log(a^2 + b^2).$$

Example 19.41. If $i^{A+iB} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi B}$. (S.V.T.U., 2006 S)

Solution. $i^{A+iB} = A + iB$ i.e. $i^{A+iB} = A + iB$

or $A + iB = e^{(A+iB) \log i} = e^{(A+iB) \log(\cos \pi/2 + i \sin \pi/2)}$
 $= \exp[(A+iB) \log(e^{i\pi/2})] = e^{(A+iB)(i\pi/2)}$
 $= e^{-B\pi/2} \cdot e^{i\pi A/2} = e^{-B\pi/2} \left(\cos \frac{\pi A}{2} + i \sin \frac{\pi A}{2} \right)$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2} \cos \frac{\pi A}{2} \quad \dots(i) \quad B = e^{-B\pi/2} \sin \frac{\pi A}{2} \quad \dots(ii)$$

Division of (ii) by (i) gives $B/A = \tan \pi A/2$

Squaring and adding (i) and (ii), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. Prove that $\log \left(\frac{a+ib}{a-ib} \right) = 2i \tan^{-1} \left(\frac{b}{a} \right)$. Hence evaluate $\cos \left[i \log \left(\frac{a+ib}{a-ib} \right) \right]$.

(P.T.U., 2006)

Solution. Putting $a = r \cos \theta$, $b = r \sin \theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\begin{aligned} \log \left(\frac{a+ib}{a-ib} \right) &= \log \frac{r(\cos \theta + i \sin \theta)}{r(\cos \theta - i \sin \theta)} = \log (e^{i\theta} + e^{-i\theta}) \\ &= \log e^{2i\theta} = 2i\theta = 2i \tan^{-1} b/a. \end{aligned}$$

Thus $\cos \left[i \log \left(\frac{a+ib}{a-ib} \right) \right] = \cos [i(2i\theta)] = \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - (b/a)^2}{1 + (b/a)^2} = \frac{a^2 - b^2}{a^2 + b^2}.$

Example 19.43. Separate into real and imaginary parts $\log \sin(x+iy)$.

Solution. $\log \sin(x+iy) = \log(\sin x \cos iy + \cos x \sin iy)$
 $= \log(\sin x \cosh y + i \cos x \sinh y) = \log r(\cos \theta + i \sin \theta),$

where

$$r \cos \theta = \sin x \cosh y \text{ and } r \sin \theta = \cos x \sinh y,$$

so that

$$r = \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)}$$

$$= \sqrt{\frac{1 - \cos 2x}{2} \cdot \frac{1 + \cosh 2y}{2} + \frac{1 + \cos 2x}{2} \cdot \frac{\cosh 2y - 1}{2}} = \sqrt{\frac{1}{2} (\cosh 2y - \cos 2x)}$$

and

$$\theta = \tan^{-1}(\cot x \tanh y).$$

Thus $\log \sin(x+iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2} \log \left[\frac{1}{2} (\cosh 2y - \cos 2x) \right] + i \tan^{-1}(\cot x \tanh y).$$

Example 19.44. Find all the roots of the equation

$$(i) \sin z = \cosh 4$$

$$(ii) \sinh z = i.$$

Solution. (i)

$$\sin z = \cosh 4 = \cos 4i = \sin(\pi/2 - 4i)$$

∴

$$z = n\pi + (-1)^n (\pi/2 - 4i)$$

$\left\{ \begin{array}{l} \text{If } \sin \theta = \sin \alpha \\ \text{then } \theta = n\pi + (-1)^n \alpha \end{array} \right.$

(ii)

$$i = \sinh z = \frac{e^z - e^{-z}}{2}$$

or

$$e^{2z} - 2ie^z - 1 = 0, \quad \text{i.e.} \quad (e^z - i)^2 = 0 \quad \text{i.e.,} \quad e^z = i$$

or

$$z = \operatorname{Log} i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i \left(2n + \frac{1}{2} \right) \pi.$$

PROBLEMS 19.7

1. Find the general value of

$$(i) \log(6+8i) \quad (\text{Rohtak, 2006})$$

$$(ii) \log(-1).$$

(J.N.T.U., 2003)

2. Show that (i) $\log(1+i\tan\alpha) = \log(\sec\alpha) + i\alpha$, where α is an acute angle.

$$(ii) \operatorname{Log}_e \frac{3-i}{3+i} = 2i \left(n\pi - \tan^{-1} \frac{1}{3} \right).$$

3. If $(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n) = A + iB$, prove that

$$(i) (a_1^2 + b_1^2)(a_2^2 + b_2^2) \dots (a_n^2 + b_n^2) = A^2 + B^2$$

$$(ii) \tan^{-1} \frac{b_1}{a_1} + \tan^{-1} \frac{b_2}{a_2} + \dots + \tan^{-1} \frac{b_n}{a_n} = \tan^{-1} \frac{B}{A}.$$

4. Find the modulus and argument of (i) $(1-i)^{1+i}$. (P.T.U., 2010) (ii) $i^{\log(1+i)}$

5. If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

(Kurukshetra, 2005)

6. Prove that $\log \left\{ \frac{\sin(x+iy)}{\sin(x-iy)} \right\} = 2i \tan^{-1}(\cot x \tanh y)$.

(Mumbai, 2007)

7. Prove that $\tan \left[i \log \left(\frac{a-ib}{a+ib} \right) \right] = \frac{2ab}{a^2 - b^2}$.

8. If $\tan \log(x+iy) = a+ib$ where $a^2+b^2 \neq 1$, show that $\tan \log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$.

9. If $\sin^{-1}(x+iy) = \log(A+iB)$, show that $\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.

10. Separate into real and imaginary parts $\log \cos(x+iy)$.

11. Find all the roots of the equation, (i) $\cos z = 2$, (ii) $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – ‘C + iS’ METHOD

This is the most general method and is applied to find the sum of a series of the form

$$a_0 \sin \alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + \dots$$

or

$$a_0 \cos \alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + \dots$$

Procedure. (i) Put the given series = S (or C) according as it is a series of sines (or cosines).

Then write C (or S) = a similar series of cosines (or sines).

e.g., If

$$S = a_0 \sin \alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + \dots$$

then

$$C = a_0 \cos \alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) \dots$$

(ii) Multiply the series of sines by i and add to the series of cosines, so that

$$\begin{aligned} C + iS &= a_0 [\cos \alpha + i \sin \alpha] + a_1 [\cos(\alpha + \beta) + i \sin(\alpha + \beta)] + \dots \\ &= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \dots \end{aligned}$$

(iii) Sum up this last series using any of the following standard series :

(1) **Exponential series i.e.,** $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty = e^x$

(2) **Sine, cosine, sinh or cosh series**

i.e., $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty = \sin x, \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty = \cos x$

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty = \sinh x, \quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \infty = \cosh x$$

(3) **Logarithmic series**

i.e., $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty = \log(1+x), \quad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right) = \log(1-x)$

(4) **Gregory's series**

i.e., $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty = \tan^{-1} x, \quad x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(5) **Binomial series**

i.e., $1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots \infty = (1+x)^n$

$$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1+x)^{-n}$$

$$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1-x)^{-n}$$

(6) **Geometric series**

i.e., $a + ar + ar^2 + \dots \text{ to } n \text{ terms} = a \frac{1-r^n}{1-r}, a + ar + ar^2 + \dots \infty = \frac{a}{1-r}, |r| < 1.$

(iv) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. Sum the series $\sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \dots \infty$.

Solution. Let $S = \sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \dots \infty$

and $C = \cos \alpha + x \cos(\alpha + \beta) + \frac{x^2}{2!} \cos(\alpha + 2\beta) + \dots \infty$

$$\begin{aligned} C + iS &= [\cos \alpha + i \sin \alpha] + x [\cos(\alpha + \beta) + i \sin(\alpha + \beta)] \\ &\quad + \frac{x^2}{2!} [\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)] + \dots \infty \\ &= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!} \cdot e^{i(\alpha+2\beta)} + \dots \infty = e^{i\alpha} \left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2 e^{2i\beta}}{2!} + \dots \infty \right] \\ &= e^{i\alpha} \cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos \beta + i \sin \beta)} = e^x \cos \beta + i (\alpha + x \sin \beta) = e^x \cos \beta e^{i(\alpha + x \sin \beta)} \\ &= e^{x \cos \beta} [\cos(\alpha + x \sin \beta) + i \sin(\alpha + x \sin \beta)] \end{aligned}$$

Equating imaginary parts from both sides, we have $S = e^{x \cos \beta} \sin(\alpha + x \sin \beta)$.

Series depending on logarithmic series

Example 19.46. Sum the series

$$\sin^2 \theta - \frac{1}{2} \sin 2\theta \sin^2 \theta + \frac{1}{3} \sin 3\theta \sin^3 \theta - \frac{1}{4} \sin 4\theta \sin^4 \theta + \dots \infty.$$

(P.T.U., 2010 ; V.T.U., 2006 S)

Solution. Let $S = \sin \theta \cdot \sin \theta - \frac{1}{2} \sin 2\theta \cdot \sin^2 \theta + \frac{1}{3} \sin 3\theta \cdot \sin^3 \theta - \dots \infty$

and $C = \cos \theta \cdot \sin \theta - \frac{1}{2} \cos 2\theta \cdot \sin^2 \theta + \frac{1}{3} \cos 3\theta \cdot \sin^3 \theta - \dots \infty$

$$\therefore C + iS = e^{i\theta} \sin \theta - \frac{e^{2i\theta} \sin^2 \theta}{2} + \frac{e^{3i\theta} \sin^3 \theta}{3} - \dots \infty$$

$$= \log(1 + e^{i\theta} \sin \theta) = \log[1 + (\cos \theta + i \sin \theta) \sin \theta]$$

$$= \log[1 + \cos \theta \sin \theta + i \sin^2 \theta] \quad [\text{Put } 1 + \cos \theta \sin \theta = r \cos \alpha; \sin^2 \theta = r \sin \alpha]$$

$$= \log r (\cos \alpha + i \sin \alpha) = \log r e^{i\alpha} = \log r + i\alpha \quad \dots(i)$$

Equating imaginary parts, we have $S = \alpha = \tan^{-1} \left(\frac{\sin^2 \theta}{1 + \cos \theta \sin \theta} \right)$.

[from (i)]

Series depending on binomial series

Example 19.47. Find the sum to infinity of the series

$$1 - \frac{1}{2} \cos \theta + \frac{1.3}{2.4} \cos 2\theta - \frac{1.3.5}{2.4.6} \cos 3\theta + \dots \quad (-\pi < \theta < \pi). \quad (\text{S.V.T.U., 2009})$$

Solution. Let $C = 1 - \frac{1}{2} \cos \theta + \frac{1.3}{2.4} \cos 2\theta - \frac{1.3.5}{2.4.6} \cos 3\theta + \dots \infty$

and $S = 0 - \frac{1}{2} \sin \theta + \frac{1.3}{2.4} \sin 2\theta - \frac{1.3.5}{2.4.6} \sin 3\theta + \dots \infty$

$$\therefore C + iS = 1 - \frac{1}{2} e^{i\theta} + \frac{1.3}{2.4} e^{2i\theta} - \frac{1.3.5}{2.4.6} e^{3i\theta} - \dots$$

$$= 1 + \left(-\frac{1}{2} \right) e^{i\theta} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1 \right)}{1.2} e^{2i\theta} + \frac{-\frac{1}{2} \left(-\frac{1}{2} - 1 \right) \left(-\frac{1}{2} - 2 \right)}{1.2.3} e^{3i\theta} + \dots$$

$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos \theta + i \sin \theta)^{-1/2} = \left(2 \cos^2 \frac{\theta}{2} + i \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right)^{-1/2}$$

$$= \left(2 \cos \frac{\theta}{2} \right)^{-1/2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)^{-1/2} = \left(2 \cos \frac{\theta}{2} \right)^{-1/2} \left(\cos \frac{\theta}{4} - i \sin \frac{\theta}{4} \right).$$

Equating real parts, we have $C = (2 \cos \theta/2)^{-1/2} \cos \theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos \theta + \sin \theta \cos 2\theta + \frac{\sin^2 \theta}{1.2} \cos 3\theta + \dots \infty. \quad (\text{P.T.U., 2005})$

2. $\sin \alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \dots \infty.$

3. $x \sin \theta - \frac{1}{2} x^2 \sin 2\theta + \frac{1}{3} x^3 \sin 3\theta - \dots \infty. \quad (\text{Kurukshetra, 2005})$

4. $\cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \dots \infty. \quad (\text{S.V.T.U., 2006}) \quad 5. \quad e^\alpha \cos \beta - \frac{e^{3\alpha}}{3} \cos 3\beta + \frac{e^{5\alpha}}{5} \cos 5\beta - \dots \infty.$

6. $c \sin \alpha + \frac{c^3}{3} \sin 3\alpha + \frac{c^5}{5} \sin 5\alpha + \dots \infty.$

7. $1 - \frac{1}{2} \cos 2\theta + \frac{1.3}{2.4} \cos 4\theta - \frac{1.3.5}{2.4.6} \cos 6\theta + \dots \infty. \quad (\text{Kurukshetra, 2006})$

8. $n \sin \alpha + \frac{n(n+1)}{1.2} \sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3} \sin 3\alpha + \dots \infty.$

9. $\sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots \sin(\alpha + (n-1)\beta) \quad (\text{P.T.U., 2009 S})$

10. $\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots \text{to } n \text{ terms.} \quad (\text{Kurukshetra, 2006})$

11. $\sin \alpha \cos \alpha + \sin^2 \alpha \cos 2\alpha + \sin^3 \alpha \cos 3\alpha + \dots \infty.$

12. $1 + x \cos \theta + x^2 \cos 2\theta + \dots + x^{n-1} \cos(n-1)\theta.$

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. If $\frac{\sin \theta}{\theta} = \frac{599}{600}$, find an approximate value of θ in radians.

Solution. Since $\frac{\sin \theta}{\theta} = 1 - \frac{1}{600}$ which is nearly equal to 1. $\therefore \theta$ must be very small.

We know that $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$

$$\therefore \frac{\sin \theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin \theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. Solve approximately $\sin \left(\frac{\pi}{6} + \theta \right) = 0.51$.

Solution. Since 0.51 is nearly equal to 1/2, which is the value of $\sin \pi/6$, so θ must be very small.

$$\begin{aligned} \therefore \sin \left(\frac{\pi}{6} + \theta \right) &= \sin \frac{\pi}{6} \cos \theta + \cos \frac{\pi}{6} \sin \theta = \frac{1}{2} \left(1 - \frac{\theta^2}{2!} + \dots \right) + \frac{\sqrt{3}}{2} \left(\theta - \frac{\theta^3}{3!} + \dots \right) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} \theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta. \end{aligned}$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2} \theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

$$\text{or} \quad \theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\frac{\sin \theta}{\theta} = \frac{5045}{5046}$, show that θ is $1^\circ 58'$ nearly.

2. If $\frac{\sin \theta}{\theta} = \frac{2165}{2166}$, find an approximate value of θ in radians.

(Madras, 2003)

3. If $\cos \theta = \frac{1681}{1682}$, find θ approximately.

4. Solve approximately the equation $\cos \left(\frac{\pi}{3} + \theta \right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is

(a) 7	(b) 5	(c) 13	(d) $\sqrt{2} + 3$.
-------	-------	--------	----------------------
2. The real part of $(\sin x + i \cos x)^5$ is

(a) $-\cos 5x$	(b) $-\sin 5x$	(c) $\sin 5x$	(d) $\cos 5x$.
----------------	----------------	---------------	-----------------

3. The number $(i)^i$ is
 (a) a purely imaginary number (b) an irrational number
 (c) a rational number (d) an integer.
4. The relation $|3 - z| + |3 + z| = 5$ represents
 (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
5. z is a complex number with $|z| = 1$ and $\arg(z) = 3\pi/4$. The value of z is
 (a) $(1+i)/\sqrt{2}$ (b) $(-1+i)/\sqrt{2}$ (c) $(1-i)/\sqrt{2}$ (d) $(-1-i)/\sqrt{2}$.
6. If $f(z) = e^{2z}$, then the imaginary part of $f(z)$ is
 (a) $e^x \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.
7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is
8. Expansion of $\cos 6\theta$ in terms of $\cos \theta$ is
9. If $f(z) = 3\bar{z}$, then the value of $f(z)$ at $z = 2 + 4i$ is
10. If $x = \cos \theta + i \sin \theta$, then $x^n - 1/x^n =$
11. Imaginary part of $(2+i3)/(3-i4)$ is
12. Real part of $\cosh(x+iy)$ is
13. If $\frac{\sin \theta}{\theta} = \frac{2165}{2166}$, then $\theta =$ approximately.
14. If $\tan x/2 = \tanh y/2$, then $\cos x \cosh y =$
15. Imaginary part of $\sin \bar{z}$ is
16. Modulus of $(\sqrt{i})^{\sqrt{i}}$ =
17. If $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\dots)$
18. $\log(-1) =$
19. $(i)^i$ is purely real or imaginary
20. If $\sin \theta = \tanh \phi$, then $\tan \theta =$
21. Imaginary part of $\tan(\theta + i\phi) =$
22. $\cos 5\alpha = (\dots) \cos^5 \alpha + (\dots) \cos^3 \alpha + (\dots) \cos \alpha$.
23. Cube roots of unity form triangle.
24. If $|z_1 + z_2| = |z_1 - z_2|$ then $\text{amp}(z_1) - \text{amp}(z_2)$ is
25. If $-3 + ix^2y$ and $x^2 + y + 4i$ represent conjugate complex numbers then $x =$ and $y =$
26. If $\left| \frac{z-a}{z-b} \right| = k \neq 1$, then the locus of z is
27. $(-i)^{-i}$ is purely real. (True or False)
28. The statements $\text{Re } z > 0$ and $|z - 1| < |z + 1|$ are equivalent. (Mumbai, 2007) (True or False)
29. Hyperbolic functions are periodic. (True or False)
30. n th roots of unity form a G.P. (True or False)
31. $\sin ix = -i \sinh x$. (Mumbai, 2008) (True or False)
32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. (Mumbai, 2008) (True or False)
33. The modulus of the sum of two complex numbers \geq to the sum of their moduli. (True or False)

Calculus of Complex Functions

1. Introduction. 2. Limit and continuity of $f(z)$. 3. Derivative of $f(z)$ —Cauchy-Riemann equations. 4. Analytic functions. 5. Harmonic functions—Orthogonal system. 6. Applications to flow problems. 7. Geometrical representation of $f(z)$. 8. Some standard transformations. 9. Conformal transformation. 10. Special conformal transformations. 11. Schwarz-Christoffel transformation. 12. Integration of complex functions. 13. Cauchy's theorem. 14. Cauchy's integral formula. 15. Morera's theorem, Cauchy's inequality, Liouville's theorem, Poisson's integral formulae. 16. Series of complex terms—Taylor's series—Laurent's series. 17. Zeros and Singularities of an analytic function. 18. Residues. Residue theorem. 19. Calculation of residues—20. Evaluation of real definite integrals. 21. Objective Type of Questions.

20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A function $w = f(z)$ is said to tend to **limit** l as z approaches a point z_0 , if for every real ϵ , we can find a positive real δ such that

$$|f(z) - l| < \epsilon \quad \text{for } |z - z_0| < \delta$$

i.e., for every $z \neq z_0$ in the δ -disc (dotted) of z -plane, $f(z)$ has a value lying in the ϵ -disc of w -plane (Fig. 20.1). In symbols, we write $\lim_{z \rightarrow z_0} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here z approaches z_0 from any direction in the z -plane.

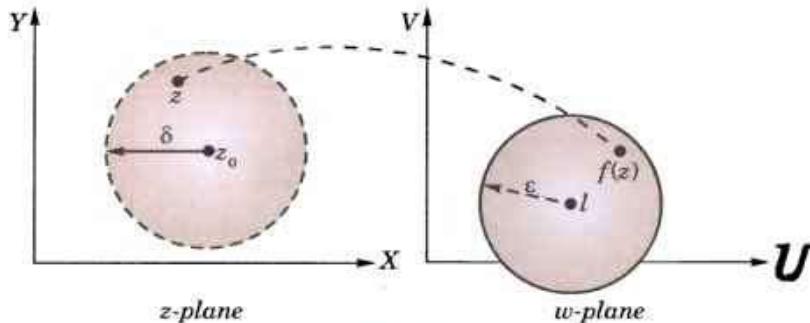


Fig. 20.1

(2) Continuity of $f(z)$. A function $w = f(z)$ is said to be **continuous** at $z = z_0$, if

$$\underset{z \rightarrow z_0}{\text{Lt}} f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z -plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, i.e., at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0) , then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF $f(z)$

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then the derivative of $w = f(z)$ is defined to be

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \rightarrow 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, i.e., δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

(2) Theorem. The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R , are

- (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R ;
- (ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann*** equations or briefly C-R equations.

(a) Condition is necessary.

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$\begin{aligned} f'(z) &= \underset{\delta z \rightarrow 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} \\ &= \underset{\substack{\delta x \rightarrow 0 \\ \delta y \rightarrow 0}}{\text{Lt}} \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y} \end{aligned}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore f'(z) = \underset{\delta x \rightarrow 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \dots(1)$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore f'(z) = \underset{i\delta y \rightarrow 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \dots(2)$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

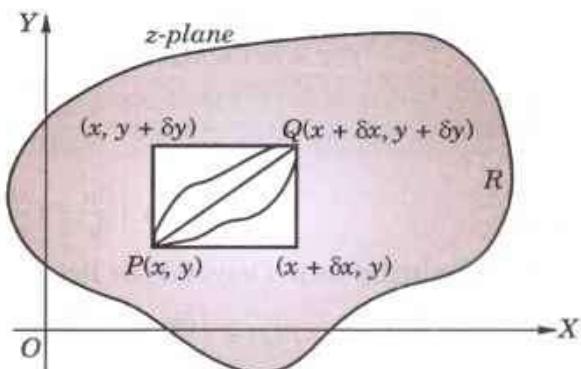


Fig. 20.2

* Named after Cauchy (p. 144) and the German mathematician Bernhard Riemann (1826–1866) who along with Weierstrass (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots(3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied. (V.T.U., 2011 S)

(b) Condition is sufficient. Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the C-R equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$\begin{aligned} f(z + \delta z) &= u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) \\ &= u(x, y) + \left(\frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) + \dots + i \left[v(x, y) + \left(\frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right) + \dots \right] \\ &= f(z) + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y \end{aligned}$$

[Omitting terms beyond the first powers of δx and δy]

$$\text{or } f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y.$$

Now using the C-R equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

$$\begin{aligned} \text{Then } f(z + \delta z) - f(z) &= \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta x + \left[-\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right] \delta y = \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta x + \left[i \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \right] i \delta y \\ &= \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] (\delta x + i \delta y) = \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta z \\ \therefore f'(z) &= \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{aligned}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R , is called an **analytic function** of z in that region. An analytic function is also called a regular function or an holomorphic function.

A function which is analytic everywhere in the complex plane, is known as an **entire function**. As derivative of a polynomial exists at every point, a polynomial of any degree is an entire function.

A point at which an analytic function ceases to possess a derivative is called a **singular point** of the function.

Thus if u and v are real single-valued functions of x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous throughout a region R , then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R . The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions*. The relation between two conjugate functions is given by C-R equation (1).

Example 20.1. If $w = \log z$, find dw/dz and determine where w is non-analytic.

(U.P.T.U., 2005; J.N.T.U., 2005)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}y/x$ [By (2), p. 665]
so that $u = \frac{1}{2}\log(x^2 + y^2)$, $v = \tan^{-1}y/x$.
 $\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \frac{\partial v}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}$.

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at $(0, 0)$. Hence w is analytic everywhere except at $z = 0$.

$$\therefore \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z} (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. **Thus if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.

(U.P.T.U., 2008; Mumbai, 2005 S; Madras 2003; Bhopal, 2002 S)

Solution. If $f(z) = u + iv$ is an analytic function, then

$$|f(z)| = \sqrt{(u^2 + v^2)} \text{ is constant} = c \text{ (say)} \text{ or } u^2 + v^2 = c^2 \quad \dots(i)$$

Differentiating (i) partially w.r.t. x and y , we get

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0; \quad 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$\text{or} \quad u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \quad \dots(ii) \quad u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0 \quad \dots(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$$\therefore (iii) \text{ becomes } -u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x} = 0 \quad \dots(iv)$$

Squaring and adding (ii) and (iv), we obtain

$$u^2 \left(\frac{\partial u}{\partial x} \right)^2 + v^2 \left(\frac{\partial v}{\partial x} \right)^2 + u^2 \left(\frac{\partial v}{\partial x} \right)^2 + v^2 \left(\frac{\partial u}{\partial x} \right)^2 = 0$$

$$\text{or} \quad (u^2 + v^2) \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] = 0 \quad \text{or} \quad \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = 0 \quad [\because u^2 + v^2 = c^2 \neq 0] \quad \dots(v)$$

$$\text{Now} \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\therefore |f'(z)|^2 = \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 = 0 \quad [\text{By (v)}]$$

or $f'(z) = 0$. or $f(z) = \text{constant}$.

Example 20.3. Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof. (A.M.I.E.T.E., 2005 S; Osmania, 2003)

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However

$$f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{|mx^2|} - 0}{x(1+im)}, \text{ when } z \rightarrow 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1+im} \text{ which is not unique.}$$

$\therefore f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. Prove that the function $f(z)$ defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \quad (z \neq 0), \quad f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009; V.T.U., 2001)

Solution.

$$\lim_{z \rightarrow 0} f(z) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \lim_{y \rightarrow 0} \frac{-y^3(1-i)}{y^2} = \lim_{y \rightarrow 0} [-y(1-i)] = 0$$

$$\lim_{z \rightarrow 0} f(z) = \lim_{\substack{y \rightarrow 0 \\ x \rightarrow 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^3(1+i)}{x^2} = \lim_{x \rightarrow 0} [x(1+i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\lim_{z \rightarrow 0} f(z) = f(0)$ when $x \rightarrow 0$ first and then $y \rightarrow 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\begin{aligned} \lim_{z \rightarrow 0} f(z) &= \lim_{\substack{y \rightarrow mx \\ x \rightarrow 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \\ &= \lim_{x \rightarrow 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2} = \lim_{x \rightarrow 0} \frac{x[1+i-m^3(1-i)]}{1+m^2} = 0 \end{aligned}$$

Hence $\lim_{z \rightarrow 0} f(z) = f(0)$, in whatever manner $z \rightarrow 0$. $\therefore f(z)$ is continuous at the origin.

Now $f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i \frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y)$.

Also $u(0, 0) = 0$, and $v(0, 0) = 0$

$\therefore f(0) = 0$

$$\left(\frac{\partial u}{\partial x} \right)_{0,0} = \lim_{x \rightarrow 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y} \right)_{0,0} = \lim_{y \rightarrow 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x} \right)_{0,0} = \lim_{x \rightarrow 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\left(\frac{\partial v}{\partial y} \right)_{0,0} = \lim_{y \rightarrow 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{y}{y} = 1.$$

and

Hence at $(0, 0)$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus the C-R equations are satisfied at the origin.

But $f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \frac{f(z)}{z} = \lim_{z \rightarrow 0} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$.

If $z \rightarrow 0$ along the path $y = mx$, then $f'(0) = \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at $(0, 0)$ i.e., $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the C-R equations thereat.

Example 20.5. Show that polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \quad (\text{U.P.T.U., 2008; V.T.U., 2006})$$

Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$. (Bhopal, 2009; Kurukshetra, 2005)

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y) , then $z = x + iy = re^{i\theta}$.

$$\therefore u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ .

Differentiating it partially w.r.t. r and θ , we have

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

and $\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \dots(i) \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \quad \dots(ii)$$

Differentiating (i) partially w.r.t. r , we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta \partial r} \quad \dots(iii)$$

Differentiating (ii) partially w.r.t. θ , we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r \frac{\partial^2 v}{\partial r \partial \theta} \quad \dots(iv)$$

Thus using (i), (ii) and (iv)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} \right) + \frac{1}{r^2} \left(-r \frac{\partial^2 v}{\partial r \partial \theta} \right) = 0 \quad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta} \right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z -plane, then the Cauchy-Riemann equations are satisfied.

i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots(1) \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad \dots(2)$$

Differentiating (1) with respect to x and (2) with respect to y , we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \dots(3) \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad \dots(4)$$

Adding (3) and (4) and assuming that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \dots(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad \dots(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory**. (Rohtak, 2005)

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad \dots(7) \quad \text{and} \quad v(x, y) = c_2 \quad \dots(8)$$

Differentiating (7), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

or

$$\frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} = \frac{\partial v / \partial y}{\partial v / \partial x} = m_1 \text{ (say)} \quad [\text{By (1) and (2)}]$$

Similarly (8) gives $\frac{dy}{dx} = -\frac{\partial v / \partial x}{\partial v / \partial y} = m_2 \text{ (say)}$

$\therefore m_1 m_2 = -1$, i.e., (7) and (8) form an orthogonal system.

Hence every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system. (U.P.T.U., 2009)

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy -plane, the velocity \mathbf{V} of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \quad \dots(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla \phi(x, y) = \frac{\partial \phi}{\partial x} \mathbf{I} + \frac{\partial \phi}{\partial y} \mathbf{J} \quad \dots(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines*.]

Thus from (1) and (2), $v_x = \frac{\partial \phi}{\partial x}$ and $v_y = \frac{\partial \phi}{\partial y}$... (3)

Also the fluid being incompressible $\operatorname{div} \mathbf{V} = 0$ [by § 8.7 (1)] i.e., $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$.

Substituting the values of v_x and v_y from (3), we get $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ is analytic. ... (4)

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = \frac{\partial \phi / \partial y}{\partial \phi / \partial x} \\ &= v_y/v_x \end{aligned} \quad \begin{matrix} \text{[By C-R equations]} \\ \text{[By (3)]} \end{matrix}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, i.e. the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4),

$$\begin{aligned}\frac{\partial w}{\partial z} &= \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i \frac{\partial \phi}{\partial y} \\ &= v_x - iv_y\end{aligned}$$

[By C-R equations]

[By (3)]

\therefore The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = | dw/dz |$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential**.

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and vice-versa.

Example 20.6. If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 +$

$$\frac{x}{x^2 + y^2}, \text{ determine the function } \phi.$$

(V.T.U., 2011; Mumbai, 2008; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

$\therefore \phi$ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \dots(i) \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \dots(ii)$$

$$\therefore \text{by (i), } \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y} \left[x^2 - y^2 + \frac{x}{x^2 + y^2} \right] = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r.t. x , we get $\phi = -2xy + \frac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y .

$$\therefore (ii) \text{ gives } -2x + \frac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

whence $\eta'(y) = 0$, i.e., $\eta(y) = c$, an arbitrary constant.

Thus

$$\phi = -2xy + \frac{y}{x^2 + y^2} + c$$

Otherwise (Milne-Thomson's method*):

We have

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i \frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2 + y^2)^2} \right] + i \left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2} \right]$$

By Milne-Thomson's method, we express dw/dz in terms of z , on replacing x by z and y by 0.

$$\therefore \frac{dw}{dz} = i \left(2z - \frac{1}{z^2} \right)$$

Integrating w.r.t. z , we get $w = i(z^2 + 1/z) + A$ where A is a complex constant.

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z})$$

$$\therefore f(z) = \phi(x, y) + i\psi(x, y) \quad \dots(1)$$

$$= \phi \left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right] + i\psi \left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i} \right]$$

Now considering this as a formal identity in the two independent variables z, \bar{z} and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad \dots(2)$$

\therefore (2) is the same as (1), if we replace x by z and y by 0.

Thus to express any function in terms of z , replace x by z and y by 0. This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.

Hence $\phi = R \left[i \left(z^2 + \frac{1}{z} \right) + A \right] = -2xy + \frac{y}{x^2 + y^2} + c.$

Example 20.7. Find the analytic function, whose real part is $\sin 2x / (\cosh 2y - \cos 2x)$.

(J.N.T.U., 2005; Anna, 2003)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x / (\cosh 2y - \cos 2x)$

$$\begin{aligned}\therefore f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \\ &= \frac{(\cosh 2y - \cos 2x) 2 \cos 2x - \sin 2x (2 \sin 2x)}{(\cosh 2y - \cos 2x)^2} - i \frac{\sin 2x (-2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2} \\ &= \frac{2 \cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i \frac{2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}\end{aligned}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore f'(z) = \frac{2 \cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2 \sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z , we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. Determine the analytic function $f(z) = u + iv$, if $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$.

(A.M.I.E.T.E., 2005; Osmania, 2003)

Solution. We have $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x) \cosh y + 1 - e^{-y} \sin x}{2(\cos x - \cosh y)^2} \quad \dots(i)$$

and

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y) e^{-y} + (\cos x + \sin x - e^{-y}) \sinh y}{2(\cos x - \cosh y)^2}$$

or

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x) \sinh y + e^{-y} (\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \quad \dots(ii)$$

Subtracting (ii) from (i), we get

$$2 \frac{\partial u}{\partial x} = \frac{(\sin x - \cos x) \cosh y - (\sin x + \cos x) \sinh y + 1 - e^{-y} (\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (i) and (ii), we have

$$-2 \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x) \cosh y + (\sin x + \cos x) \sinh y + 1 + e^{-y} (-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \\ &= \frac{1}{2(1 - \cos z)} = \frac{1}{4 \sin^2 z/2} = \frac{1}{4} \operatorname{cosec}^2 \frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2} \cot \frac{z}{2} + c\end{aligned}$$

Since $f(\pi/2) = 0$,

$$0 = -\frac{1}{2} \cot \pi/4 + c, \quad \text{whence } c = \frac{1}{2}$$

Hence

$$f(z) = \frac{1}{2} \left(1 - \cot \frac{z}{2} \right).$$

Example 20.9. Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.
(Marathwada, 2008)

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad \dots(i)$$

$$-\frac{1}{r} \frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad \dots(ii)$$

$$\therefore (i) \text{ gives,} \quad \frac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta) \quad \text{where } \phi(\theta) \text{ is an arbitrary function.}$$

$$\therefore \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad \dots(iii)$$

From (ii) and (iii), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$$\therefore \phi'(\theta) = 0 \quad \text{or} \quad \phi(\theta) = c$$

Thus $u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v .

Now v will be harmonic if it satisfies the Laplace equation $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} + \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$

From (i), $\frac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (ii), $\frac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\begin{aligned} \therefore \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} &= 2 \cos 2\theta + \frac{1}{r} (2r \cos 2\theta - \cos \theta) + \frac{1}{r^2} (-4r^2 \cos 2\theta + r \cos \theta) \\ &= 4 \cos 2\theta - \frac{1}{r} \cos \theta - 4 \cos 2\theta + \frac{1}{r} \cos \theta = 0 \end{aligned}$$

Hence v is harmonic.

Example 20.10. (a) Find the orthogonal trajectories of the family of curves

$$x^4 + y^4 - 6x^2y^2 = \text{constant}$$

(b) Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.

(Mumbai, 2005; J.N.T.U., 2003)

Solution. (a) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y) = \text{constant}$ will be the required trajectories if $f(z) = u + iv$ is analytic.

$$\text{Now} \quad \frac{\partial u}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial u}{\partial y} = 4y^3 - 12x^2y$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\text{Integrating,} \quad v = 4x^3y - 4xy^3 + c(x)$$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are $v = \text{constant}$ or $x^3y - xy^3 = \text{constant}$.

(b) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,

we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n (\cos n\theta + i \sin n\theta) = r^n \cdot e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. Two concentric circular cylinders of radii r_1, r_2 ($r_1 < r_2$) are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda/\log(r_2/r_1)$ where λ is the dielectric constant of the medium.

Solution. We have $\phi + i\psi = a \log(re^{i\theta}) + c$ where $z = x + iy = re^{i\theta}$

$$\therefore \phi = a \log r + c, \quad \text{and} \quad \psi = a\theta$$

so that

$$\phi_1 = a \log r_1 + c, \quad \phi_2 = a \log r_2 + c$$

$$\text{Thus the potential difference} = \phi_2 - \phi_1 = a(\log r_2 - \log r_1)$$

$$\text{Also the total charge (or flux)} = \int_0^{2\pi} d\psi = \int_0^{2\pi} a d\theta = 2\pi a.$$

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log(r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric = $2\pi\lambda/\log(r_2/r_1)$.

Example 20.12. If $f(z)$ is a regular function of z , prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2. \quad (\text{J.N.T.U., 2006 ; Kottayam, 2005})$$

or

$$\nabla^2 |f(z)|^2 = 4 |f'(z)|^2 \quad (\text{Madras, 2006})$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $|f(z)|^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = 2 \left\{ u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x} \right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x} \right)^2 \right\}$$

$$\text{Similarly, } \frac{\partial^2 \phi}{\partial y^2} = 2 \left\{ u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y} \right)^2 + v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y} \right)^2 \right\}$$

Adding, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 \left\{ u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right\} + 2 \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right\} \quad \dots(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \left(\frac{\partial u}{\partial x} \right)^2 = \left(\frac{\partial v}{\partial y} \right)^2; \left(\frac{\partial u}{\partial y} \right)^2 = \left(-\frac{\partial v}{\partial x} \right)^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

$$\text{Thus (i) takes the form } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4 \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right\}$$

$$\text{Hence } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2 \quad \text{or} \quad \nabla^2 |f(z)|^2 = 4 |f'(z)|^2.$$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0, & z = 0 \end{cases}$ prove that $|f(z) - f(0)|/z \rightarrow 0$ as $z \rightarrow 0$ along any radius vector but not as $z \rightarrow 0$ along the curve $y = ax^3$.

2. Show that (a) $f(z) = xy + iy$ is everywhere continuous but is not analytic. (Osmania, 2003 S)
 (b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (J.N.T.U., 2003)
3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$. (Mumbai, 2007)
4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. (Mumbai, 2008)
5. Show that z^n is analytic. Hence find its derivative. (V.T.U., 2010 S)
6. Determine which of the following functions are analytic :
 (i) $2xy + i(x^2 - y^2)$ (ii) $(x - iy)/(x^2 + y^2)$ (iii) $\cosh z$.
7. (a) Determine p such that the function $f(z) = \frac{1}{2} \log_e(x^2 + y^2) + i \tan^{-1}(px/y)$ be an analytic function. (Mumbai, 2007 ; J.N.T.U., 2003)
 (b) Show that $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. (U.P.T.U., 2010)
8. Show that each of the following functions is not analytic at any point :
 (i) \bar{z} (J.N.T.U., 2003) (ii) $|z|^2$.
9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.
10. Show that $f(z) = \begin{cases} xy^2(x + iy) + (x^2 + y^2), & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not analytic at $z = 0$, although C-R equations are satisfied at the origin. (J.N.T.U., 2003)
11. Verify if $f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$ is analytic or not. (U.P.T.U., 2008)
12. Examine the nature of the function $f(z) = \frac{x^2y^5(x + iy)}{x^4 + y^{10}}, z \neq 0; f(0) = 0$. (Rohtak, 2004)
13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z, z \neq 0; f(0) = 0$, show that the C-R equations are satisfied at $(0, 0)$, but $f(z)$ is not differentiable at $(0, 0)$. (P.T.U., 2010)
14. Determine the analytic function whose real part is
 (i) $x^3 - 3xy^2 + 3x^2 - 3y^2$ (Bhopal, 2009) (ii) $\cos x \cosh y$ (Rohtak, 2004)
 (iii) $y/(x^2 + y^2)$ (iv) $y + e^x \cos y$ (S.V.T.U., 2008 ; V.T.U., 2006)
 (v) $e^{-x}(x \sin y - y \cos y)$ (vi) $e^{2x}(x \cos 2y - y \sin 2y)$ (U.P.T.U., 2008)
 (vii) $x \sin x \cosh y - y \cos x \sinh y$ (V.T.U., 2008 S ; Mumbai, 2005 ; Kottayam, 2005)
 (viii) $e^x[(x^2 - y^2) \cos y - 2xy \sin y]$. (V.T.U., 2010 S ; Rohtak, 2005)
15. Find the regular function whose imaginary part is
 (i) $(x - y)/(x^2 + y^2)$ (ii) $-\sin x \sinh y$ (iii) $e^x \sin y$
 (iv) $e^{-x}(x \sin y - y \cos y)$ (v) $e^{-x}(x \cos y + y \sin y)$ (U.P.T.U., 2009) (vi) $\frac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. (Mumbai, 2006)
16. Find the analytic function $z = u + iv$, if
 (i) $u - v = (x - y)(x^2 + 4xy + y^2)$ (Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005)
 (ii) $u - v = \frac{\cos x + \sin x - e^{-y}}{2 \cos x - e^y - e^{-y}}$ when $f\left(\frac{\pi}{2}\right) = 0$ (Mumbai, 2007)
 (iii) $u + v = \frac{2 \sin 2x}{e^{2y} - e^{-2y} - 2 \cos 2x}$. (P.T.U., 2002)
17. An electrostatic field in the xy -plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.
18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.
19. Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. (U.P.T.U., 2004 S)

20. Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z . (Bhopal, 2007)
21. For $w = \exp(z^2)$, find u and v , and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. (J.N.T.U., 2003)
22. Find the orthogonal trajectories of the family of curves
 (i) $x^3y - xy^3 = c$ (Mumbai, 2007) (ii) $e^x \cos y - xy = c$ (Mumbai, 2008) (iii) $r^2 \cos 2\theta = c$.
23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :
 (i) $\psi = -y/(x^2 + y^2)$ (ii) $\psi = \tan^{-1}(y/x)$.
24. Find the analytic function $f(z) = u + iv$, given
 (i) $u = a(1 + \cos \theta)$ (ii) $v = (r - 1/r) \sin \theta, r \neq 0$.
25. If $f(z)$ is an analytic function of z , show that
- $$\left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = |f'(z)|^2. \quad (\text{U.P.T.U., 2009; V.T.U., 2008 S; P.T.U., 2005})$$
26. If $f(z)$ is an analytic function of z , prove that
 (i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log |f(z)| = 0$ (Madras, 2000 S) (ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |R f(z)|^2 = 2 |f'(z)|^2$
 (iii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}$. (Kerala, 2005)
27. Prove that $\psi = \log |(x-1)^2 + (y-2)^2|$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z .

20.7 GEOMETRICAL REPRESENTATION OF $w = f(z)$

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e.,

$$u + iv = f(x + iy) \quad \dots(1)$$

involves four real variables x, y, u, v . Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z -plane, the point w will move along a corresponding curve C' in the w -plane, since to each point (x, y) , there corresponds a point (u, v) (Fig. 20.3). We then, say that a curve C in the z -plane is mapped into the corresponding curve C' in the w -plane by the function $w = f(z)$ which defines a **mapping or transformation** of the z -plane into the w -plane.

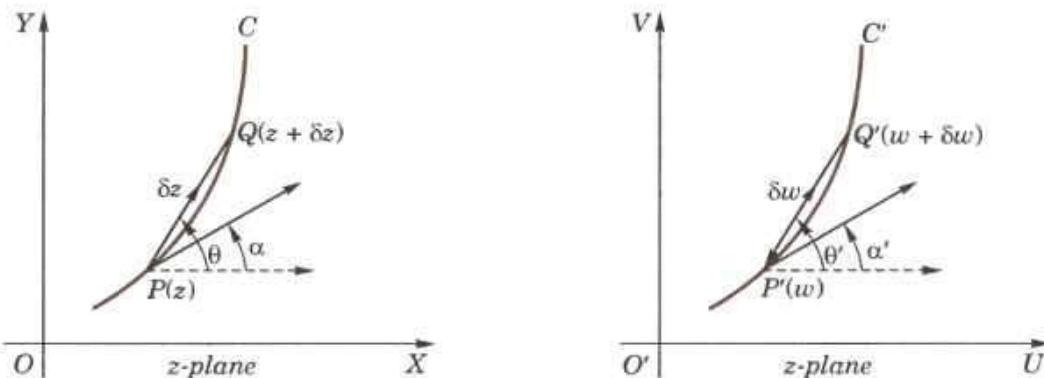


Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, i.e. the point $P(x, y)$ in the z -plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w -plane. Every point in the z -plane is mapped onto w -plane in the same way. Thus if the w -plane is superposed on the z -plane, figure is shifted through a distance given by the vector c . Accordingly, this transformation maps a figure in the z -plane into a figure in the w -plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) **Magnification and rotation.** $w = cz$, where c is a complex constant.

If $c = pe^{i\alpha}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$, then

$$Re^{i\phi} = pe^{i\alpha} \cdot re^{i\theta} = pre^{i(\theta + \alpha)}$$

whence $R = pr$ and $\phi = \theta + \alpha$, i.e. the point $P(r, \theta)$ in the z -plane is mapped onto the point $P'(pr, \theta + \alpha)$ in the w -plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $|c|$ and its rotation through an $\angle\alpha = \text{amp}(c)$. Accordingly any figure in the z -plane is transformed into a geometrically similar figure in the w -plane. In particular, this transformation maps circles into circles.

(3) **Inversion and reflection.** $w = 1/z$.

Here it is convenient to think the w -plane as superposed on z -plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \frac{1}{r} e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, i.e. P_1 is the inverse* of P w.r.t. the unit circle with centre O , then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.

To prove this, we write $z = 1/w$ as $x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$

so that

$$x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}. \quad \dots(1)$$

Now the general equation of any circle in the z -plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(2)$$

which on substituting from (1), becomes $\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + 2g \frac{u}{u^2 + v^2} + 2f \frac{-v}{u^2 + v^2} + c = 0$

or

$$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \quad \dots(3)$$

This is the equation of a circle in the w -plane. If $c = 0$, the circle (2) passes through the origin and its image, i.e., (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the transformation $w = 1/z$ always maps a circle into a circle.

(4) **Bilinear transformation.** The transformation

$$w = \frac{az + b}{cz + d} \quad \dots(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation**.** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, i.e., the transformation is conformal. If $ad - bc = 0$ every point of the z -plane is a critical point.

The inverse mapping of (1) is

$$z = \frac{-dw + b}{cw - a} \quad \dots(2)$$

which is also a bilinear transformation.

* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA \cdot OB = k^2$.

** First studied by Möbius (p. 337). Hence, sometimes called Möbius transformation

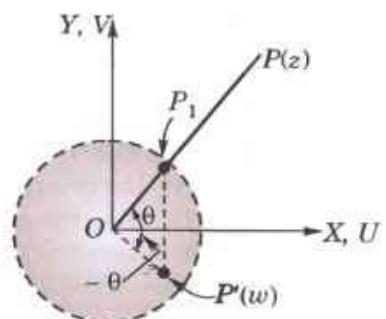


Fig. 20.4

Obs. 1. From (1), we see that each point in the z -plane except $z = -d/c$, corresponds a unique point in the w -plane. Similarly, (2) shows that each point in the w -plane except $w = a/c$, maps into a unique point in the z -plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes*.

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w -plane (i.e., $w = z$), then (1) gives

$$z = \frac{az + b}{cz + d} \quad \text{or} \quad cz^2 + (d - a)z - b = 0$$

The roots of this equation (say : z_1, z_2) are defined as the invariant or fixed points of the bilinear transformation (1). If however, the two roots are equal, the bilinear transformation is said to be *parabolic*.

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation*. For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3 .

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + d/c}$

which is a combination of the transformations

$$w_1 = z + d/c, w_2 = 1/w_1, w_3 = \frac{bc - ad}{c^2} w_2, w = \frac{a}{c} + w_3.$$

By these transformations, we successively pass from z -plane to w_1 -plane, from w_1 -plane to w_2 -plane, from w_2 -plane to w_3 -plane and finally from w_3 -plane to w -plane. Now each of these transformations is one or other of the standard transformations $w = z + c, w = cz, w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio[†] of four points.

Let the points z_1, z_2, z_3, z_4 of the z -plane map onto the points w_1, w_2, w_3, w_4 of the w -plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j + b}{cz_j + d} - \frac{az_k + b}{cz_k + d} = \frac{ad - bc}{(cz_j + d)(cz_k + d)} (z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \rightarrow \infty$, the quotient of those two differences which contain z_1 , is replaced by 1 i.e.,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{z_3 - z_4}{z_3 - z_2}.$$

Example 20.13. Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.

Hence find (a) the image of $|z| < 1$,

(Mumbai, 2006 ; Delhi, 2002)

(b) the invariant points of this transformation.

(U.P.T.U., 2008 ; V.T.U., 2000)

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$. Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\frac{2w}{2i} = \frac{(z+1)(1-i) + (z-1)(1+i)}{(z+1)(1-i) - (z-1)(1+i)}$

[†] **Def.** If t_1, t_2, t_3, t_4 be any four numbers, then $\frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)}$ is said to be their cross-ratio and is denoted (t_1, t_2, t_3, t_4) .

$$\therefore w = \frac{1+iz}{1-iz} \quad \dots(i)$$

which is the required bilinear transformation.

$$(a) \text{ Rewriting (i) as } z = i \frac{1-w}{1+w}$$

$$\therefore \left| \frac{i(1-w)}{1+w} \right| = |z| < 1 \quad \text{or} \quad |i| \cdot |1-w| < |1+w|$$

$$\begin{aligned} \text{or} \quad & |1-u-iv| < |1+u+iv| \quad [\because |i|=1] \\ \text{or} \quad & (1-u)^2 + v^2 < (1+u)^2 + v^2 \text{ which reduces to } u > 0. \end{aligned}$$

Hence the interior of the circle $x^2 + y^2 = 1$ in the z -plane is mapped onto the entire half of the w -plane to the right of the imaginary axis.

(b) To find the invariant points of the transformation, we put $w = z$ in (i).

$$\therefore z = \frac{1+iz}{1-iz} \quad \text{or} \quad iz^2 + (i-1)z + 1 = 0$$

$$\text{or} \quad z = \frac{1-i \pm \sqrt{(i-1)^2 - 4i}}{2i} = -\frac{1}{2}[1+i \pm \sqrt{(6i)}]$$

which are the required invariant points.

Example 20.14. Show that $w = \frac{i-z}{i+z}$ maps the real axis of z -plane into the circle $|w|=1$ and the half plane $y>0$ into the interior of the unit circle $|w|=1$ in the w -plane. (Mumbai, 2007)

Solution. Since $w = (i-z)/(i+z)$,

$$\begin{aligned} \therefore |w|=1 \text{ becomes } (i-z)/(i+z) = 1 & \quad \text{or} \quad |i-z| = |i+z| \\ \text{i.e.,} \quad |i-x-iy| = |i+x+iy| & \quad \text{or} \quad |-x+i(1-y)| = |x+i(1+y)| \\ \therefore \sqrt{x^2 + (1-y)^2} = \sqrt{(x^2 + (1+y)^2)} & \quad \text{or} \quad (1-y)^2 = (1+y)^2 \\ \therefore 4y = 0 \quad \text{or} \quad y = 0 & \quad \text{which is the real axis.} \end{aligned}$$

Hence the real axis of the z -plane is mapped to the circle $|w|=1$.

Now for the interior of the circle $|w|=1$

$$\begin{aligned} |w| < 1 \quad \text{i.e.,} \quad |i-z| < |i+z| & \quad \text{or} \quad (1-y)^2 < (1+y)^2 \\ \therefore -4y < 0 \quad \text{i.e.,} \quad y > 0 & \end{aligned}$$

Hence the half plane $y>0$ is mapped into the interior of the circle $|w|=1$.

PROBLEMS 20.2

- Find the invariant points of the transformation $w = (z-1)/(z+1)$. (Madras, 2003)
- Find the transformation which maps the points $-1, i, 1$ of the z -plane onto $1, i, -1$ of the w -plane respectively. Also find its *invariant points*. (V.T.U., 2011)
- Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. (S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006)
- Determine the bilinear transformation that maps the points $1-2i, 2+i, 2+3i$ respectively into $2+2i, 1+3i, 4$. (J.N.T.U., 2003 ; Coimbatore, 1999)
- Find the bilinear transformation which maps
 - the points $z=1, i, -1$ into the points $w=0, 1, \infty$ (V.T.U., 2008 ; Mumbai, 2007)
 - the points $z=0, 1, i$ into the points $w=1+i, -i, 2-i$ (V.T.U., 2010 S)
 - $R(z)>0$ into interior of unit circle so that $z=\infty, i, 0$ map into $w=-1, -i, 1$.
- Under the transformation $w = \frac{z-1}{z+1}$, show that the map of the straight line $x=y$ is a circle and find its centre and radius. (Marathwada, 2008)

7. Show that the bilinear transformation $w = (2z + 3)/(z - 4)$ maps the circle $x^2 + y^2 - 4x = 0$ into the line $4u + 3 = 0$.
(Mumbai, 2007; J.N.T.U., 2003; Bhopal, 2002)
8. Show that the condition for transformation $w = (az + b)/(cz + d)$ to make the circle $|w| = 1$ correspond to a straight line in the z -plane is $|a| = |c|$.
9. Show that the transformation $w = i(1-z)/(1+z)$ maps the circle $|z| = 1$ into the real axis of the w -plane and the interior of the circle $|z| < 1$ into the upper half of the w -plane.
(Osmania, 2003 S; V.T.U., 2001)
10. If z_0 is the upper half of the z -plane, show that the bilinear transformation $w = e^{i\alpha} \left(\frac{z - z_0}{z + z_0} \right)$ maps the upper half of the z -plane into the interior of the unit circle at the origin in the w -plane.

20.9 (1) CONFORMAL TRANSFORMATION

Suppose two curves C, C_1 in the z -plane intersect at the point P and the corresponding curves C', C'_1 in the w -plane intersect at P' (Fig. 20.5). If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P' in magnitude and sense, then the transformation is said to be **conformal**.

(2) Theorem. The transformation effected by an analytic function $w = f(z)$ is conformal at every point of the z -plane where $f'(z) \neq 0$.

Let $P(z)$ be a point in the region R of the z -plane and $P'(w)$ the corresponding point in the region R' of the w -plane (Fig. 20.3). Suppose z moves on a curve C and w moves on the corresponding curve C' . Let $Q(z + \delta z)$ be a neighbouring point on C and $Q'(w + \delta w)$ be the corresponding point on C' so that $\vec{PQ} = \delta z$ and $\vec{P'Q'} = \delta w$. Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x -axis.

$$\therefore \delta z = r e^{i\theta}$$

Similarly, if the modulus and amplitude of δw be r' and θ' , then $\delta w = r' e^{i\theta'}$.

Hence

$$\frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Now if the tangent at P to the curve C makes an $\angle \alpha$ with the x -axis and the tangent at P' to C' makes an $\angle \alpha'$ with the u -axis, then as $\delta z \rightarrow 0$, $\theta \rightarrow \alpha$ and $\theta' \rightarrow \alpha'$.

$$\therefore f'(z) = \frac{dw}{dz} = \left(\text{Lt } \frac{r'}{r} \right) \cdot e^{i(\alpha' - \alpha)} \quad \dots(1)$$

If ρ is the modulus and ϕ the amplitude of the function $f(z)$ which is supposed to be non-zero, then

$$f'(z) = \rho e^{i\phi} \quad \dots(2)$$

$$\therefore \text{from (1) and (2), we have } \rho = \text{Lt } \frac{r'}{r} \quad \dots(3)$$

$$\text{and } \phi = \alpha' - \alpha. \quad \dots(4)$$

Now let C_1 be another curve through P in the z -plane and C'_1 the corresponding curve through P' in the w -plane. If the tangent at P to C_1 makes an $\angle \beta$ with the x -axis and tangent at P' to C'_1 makes an $\angle \beta'$ with the u -axis, then as in (4),

$$\psi = \beta' - \beta \quad \dots(5)$$

Equating (4) and (5), $\alpha' - \alpha = \beta' - \beta$ or $\beta - \alpha = \beta' - \alpha' = \gamma$ (Fig. 20.5)

Thus the angle between the curves before and after the mapping is preserved in magnitude and direction. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Obs. 1. A point at which $f'(z) = 0$ is called a **critical point** of the transformation.

Obs. 2. The relation (4), i.e., $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an $\angle \phi = \text{amp } f'(z)$ under the given transformation.

Obs. 3. The relation (3) shows that in the transformation, elements of arc passing through P in any direction are changed in the ratio $\rho : 1$, where $\rho = |f'(z)|$, i.e., an infinitesimal length in the z -plane is magnified by the factor $|f'(z)|$. Consequently the infinitesimal areas are magnified by the factor $|f'(z)|^2$ in a conformal transformation.

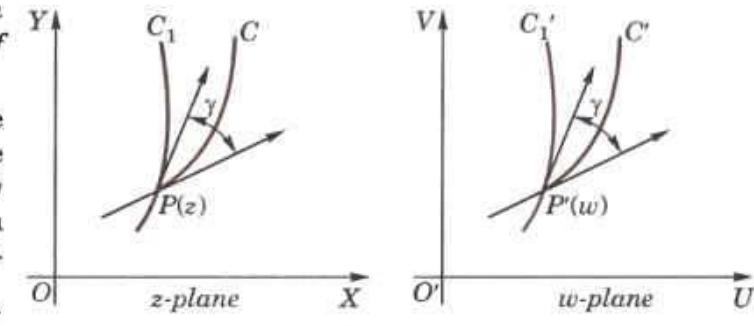


Fig. 20.5

If $w = f(z)$ is analytic then u and v must satisfy C-R equations.

$$\therefore J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Hence in a conformal transformation, infinitesimal areas are magnified by the factor $J\left(\frac{u,v}{x,y}\right)$.

Also the condition of a conformal mapping is $J\left(\frac{u,v}{x,y}\right) \neq 0$.

Obs. 4. The angle preserving property of the conformal transformation has many important physical applications.

For instance, consider the flow of an incompressible fluid in a plane with velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$. We know that ϕ and ψ are real and imaginary parts of some analytic function $w = f(z)$. As $\phi = \text{constant}$ and $\psi = \text{constant}$ represent a system of orthogonal curves; these are transformed by the function $w = f(z)$ into a set of orthogonal lines in the w -plane and vice-versa.

Thus, the conjugate functions ϕ and ψ when subjected to conformal transformation remain conjugate functions, i.e., the solutions of Laplace's equation remain solutions of the Laplace's equation after the transformation. This is the main reason for the great importance of the conformal transformation in applications.

20.10 SPECIAL CONFORMAL TRANSFORMATIONS

(1) Transformation $w = z^2$.

We have $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$.

$$\therefore u = x^2 - y^2 \text{ and } v = 2xy \quad \dots(1)$$

If u is constant (say, a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly, if v is constant (say, b), then $xy = b/2$ which also represents a rectangular hyperbola.

Hence a pair of lines $u = a, v = b$ parallel to the axes in the w -plane, map into a pair of orthogonal rectangular hyperbolae in the z -plane as shown in Fig. 20.7 (p. 455).

Again, if x is constant (say, c), then $y = v/2c$ and $y^2 = c^2 - u$. Elimination of y from these equations gives $v^2 = 4c^2(c^2 - u)$, which represents a parabola. Similarly, if y is a constant (say, d), then elimination of x from the equations (1) gives $v^2 = 4d^2(d^2 + u)$ which is also a parabola.

Hence the pair of lines $x = c$ and $y = d$ parallel to the axes in the z -plane map into orthogonal parabolae in the w -plane as shown in Fig. 20.6.

Also since $\frac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is a critical point of the mapping.

Taking $z = re^{i\theta}$ and $w = Re^{i\phi}$ then in polar form $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$.

This shows that upper half of the z -plane $0 < \theta < \pi$ transforms into the entire w -plane $0 \leq \phi < 2\pi$. The same is true of the lower half. (P.T.U., 2003)

Obs. 1. Taking the axes to represent two walls, a single quadrant could be used to represent fluid flow at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Obs. 2. For the transformation $w = z^n$, n being a positive integer, we have $dw/dz = 0$ at $z = 0$.

Also $Re^{i\phi} = (re^{i\theta})^n = r^n e^{in\theta}$

$\therefore R = r^n$ and $\phi = n\theta$, when $0 < \theta < \pi/n$, correspondingly $0 < \phi < \pi$.

Hence $w = z^n$ gives a conformal mapping of the z -plane everywhere except at the origin and that is fans out a sector of z -plane of central angle π/n to cover the upper half of the w -plane.

(2) Joukowski's transformation* $w = z + 1/z$.

Since $\frac{dw}{dz} = \frac{(z+1)(z-1)}{z^2}$, the mapping is conformal except at the points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ of the w -plane.

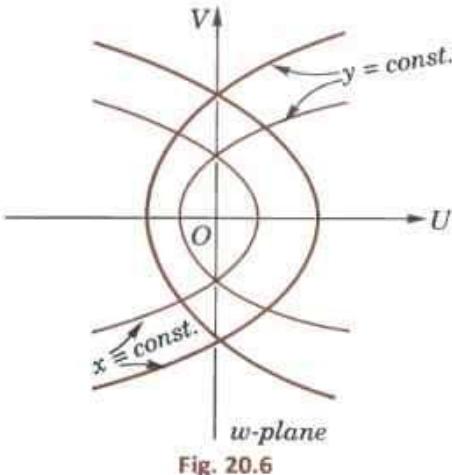


Fig. 20.6

* Named after the Russian mathematician Nikolai Jegorovich Joukowshi (1847-1921).

Changing to polar coordinates,

$$w = u + iv = r(\cos \theta + i \sin \theta) + \frac{1}{r(\cos \theta + i \sin \theta)}$$

$$= r(\cos \theta + i \sin \theta) + \frac{1}{r} (\cos \theta - i \sin \theta)$$

$$\therefore u = (r + 1/r) \cos \theta \text{ and } v = (r - 1/r) \sin \theta$$

$$\text{Elimination of } \theta \text{ gives } \frac{u^2}{(r + 1/r)^2} + \frac{v^2}{(r - 1/r)^2} = 1 \quad \dots(1)$$

$$\text{while the elimination of } r \text{ gives } \frac{u^2}{4 \cos^2 \theta} - \frac{v^2}{4 \sin^2 \theta} = 1 \quad \dots(2)$$

From (1), it follows that the circles $r = \text{constant}$ of z -plane transform into a family of ellipses of the w -plane (Fig. 20.7). These ellipses are confocal for $(r + 1/r)^2 - (r - 1/r)^2 = 4$, i.e., a constant.

In particular, the unit circle ($r = 1$) in the z -plane flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w -plane. Thus the exterior of the unit circle in the z -plane maps into the entire w -plane.

(A.M.I.E.T.E., 2005 S)

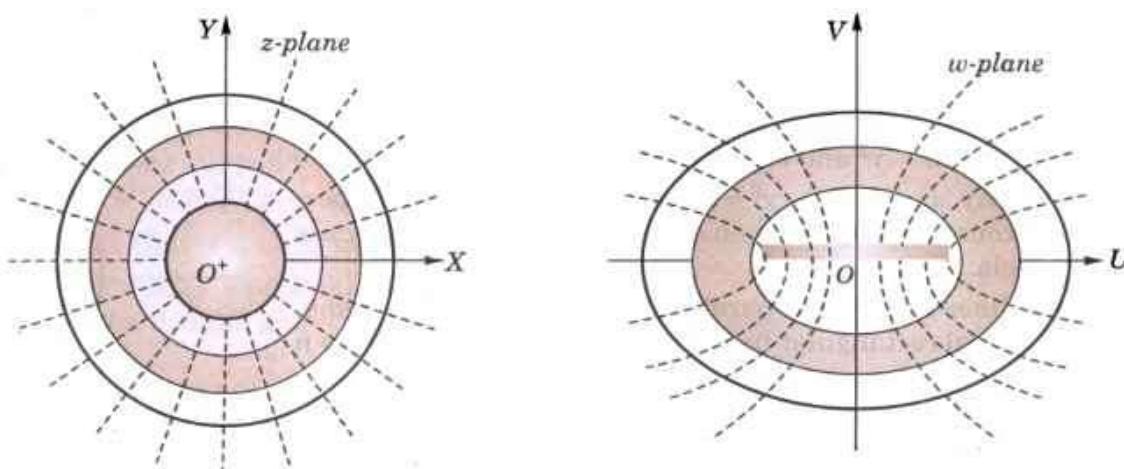


Fig. 20.7

From (2), it is clear that the radial lines $\theta = \text{constant}$ of the z -plane transform into a family of hyperbolae which are also confocal (Fig. 20.7).

Obs. 1. $v = \left(r - \frac{1}{r}\right) \sin \theta = 0$ gives $r = \pm 1$ or $\theta = 0, \pi$, i.e., this streamline consists of the unit circle $r = 1$ and the x -axis ($\theta = 0$ to $\theta = \pi$). For large z , the flow is nearly uniform and parallel to the x -axis. This can be interpreted as a flow around a circular cylinder of unit radius having two **stagnation points*** at $A(z = 1)$ and $B(z = -1)$. (Fig. 20.8)

$\therefore dw/dz = 0$ at $z = \pm 1$

Obs. 2. This transformation is also used to map the exterior of the profile of an aeroplane wing on the exterior of a nearly circular region. These airfoils are known as *Joukowski airfoils*.

(3) Transformation $w = e^z$.

Writing $z = x + iy$ and $w = pe^{i\phi}$, we have $pe^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$

whence $p = e^x \quad \dots(1)$ and $\phi = y \quad \dots(2)$

From (1), it is clear that the lines parallel to y -axis ($x = \text{const.}$) map into circles ($p = \text{const.}$) in the w -plane, their radii being less than or greater than 1 according as x is less than or greater than 0 (Fig. 20.9). (V.T.U., 2011)

Similarly, it follows from (2) that the lines parallel to the x -axis ($y = \text{const.}$) map into the radial lines ($\phi = \text{const.}$) of the w -plane. Thus any horizontal strip of height 2π in the z -plane will cover once the entire w -plane.

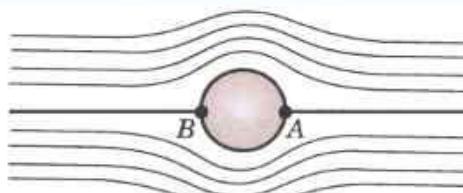


Fig. 20.8

* Stagnation points are those at which the fluid velocity is zero.

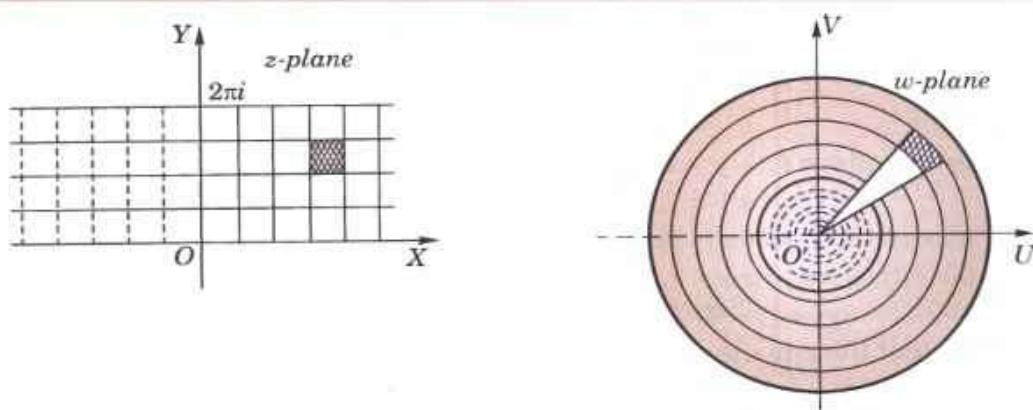


Fig. 20.9

The rectangular region $a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the z -plane (shown shaded) transforms into the region $e^{a_1} \leq \rho \leq e^{a_2}, b_1 \leq \phi \leq b_2$ in the w -plane bounded by circles and rays (shown shaded).

(P.T.U., 2005 ; Kerala, 2005)

Obs. This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a charged circular cylinder etc.

(4) Transformation $w = \cosh z$.

We have

$$u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$$

[By (2) (ii), p. 662]

so that

$$u = \cosh x \cos y \text{ and } v = \sinh x \sin y.$$

Elimination of x from these equations gives

$$\frac{u^2}{\cosh^2 y} - \frac{v^2}{\sinh^2 y} = 1 \quad \dots(1)$$

while elimination of y gives

$$\frac{u^2}{\cos^2 x} + \frac{v^2}{\sinh^2 z} = 1 \quad \dots(2)$$

(1) shows that the lines parallel to x -axis (i.e., $y = \text{const.}$) in the z -plane map into hyperbolae in the w -plane.

(2) shows that the lines parallel to the y -axis (i.e., $x = \text{const.}$) in the z -plane map into ellipse in the w -plane (Fig. 20.10). The rectangular region $a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the z -plane (shown shaded) transforms into the shaded region in the w -plane bounded by the corresponding hyperbolae and ellipses. (Kerala M. Tech., 2005)

Obs. This transformation can be used.

- (i) to obtain the circulation of liquid around an elliptic cylinder;
- (ii) to determine the electrostatic field due to a charged cylinder;
- (iii) to determine the potential between two confocal elliptic (or hyperbolic) cylinders.

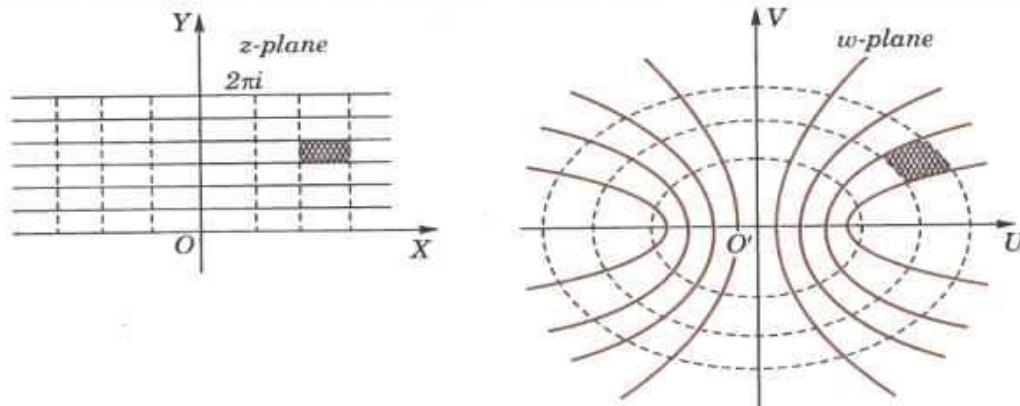


Fig. 20.10

Example 20.15. Show that under the transformation $w = (z - i)/(z + i)$, real axis in the z -plane is mapped into the circle $|w| = 1$. Which portion of the z -plane corresponds to the interior of the circle? (J.N.T.U., 2003)

Solution. We have

$$\begin{aligned}|w| &= \left| \frac{z-i}{z+i} \right| = \frac{|z-i|}{|z+i|} = \frac{|x+i(y-1)|}{|x+i(y+1)|} \\&= \sqrt{x^2 + (y-1)^2}/\sqrt{x^2 + (y+1)^2}\end{aligned}$$

Now the real axis in z -plane i.e., $y = 0$, transforms into

$$|w| = \sqrt{x^2 + 1}/\sqrt{x^2 + 1} = 1.$$

Hence the real axis in the z -plane is mapped into the circle $|w| = 1$.

The interior of the circle, i.e., $|w| < 1$, gives

$$(x^2 + (y-1)^2)/(x^2 + (y+1)^2) < 1 \text{ i.e., } -4y < 0 \text{ or } y > 0.$$

Thus the upper half of the z -plane corresponds to the interior of the circle $|w| = 1$.

PROBLEMS 20.3

- Determine the region of the w -plane into which the following regions are mapped by the transformation $w = z^2$.
 - first quadrant of z -plane
 - region bounded by $x = 1$, $y = 1$, $x + y = 1$
 - the region $1 \leq x \leq 2$ and $1 \leq y \leq 2$
 - circle $|z - 1| = 2$.

(J.N.T.U., 2000)
(Kottayam, 2005 ; V.T.U., 2009 S)
(Osmania, 2003 ; V.T.U., 2000)
- Find the transformation which maps the triangular region $0 \leq \arg z \leq \pi/3$ into the unit circle $w \leq 1$.
- Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin?
- Under the transformation $w = 1/z$, find the image of
 - the circle $|z - 2i| = 2$
 - the straight line $y - x + 1 = 0$
 - the hyperbola $x^2 - y^2 = 1$.

(Bhopal, 2009 ; Kerala M.Tech., 2005)
(P.T.U., 2007)
(Mumbai, 2005 ; J.N.T.U., 2005)
- Show that under the transformation $w = 1/z$, (a) circle $x^2 + y^2 - 6x = 0$ is transformed into a straight line in the w -plane.
(b) the circle $(x - 3)^2 + y^2 = 2$ is transformed into a circle with centre $(3/7, 0)$ and radius $\sqrt{2}/17$.
- Show that the transformation $w = 1/z$ transforms all circles and straight lines into the circles and straight lines in the w -plane. Which circles in the z -plane become straight lines in the w -plane, and which straight lines are transformed into other straight lines?
- Show that the transformation $w = z + 1/z$ converts the straight line $\arg z = \alpha$ ($|\alpha| < \alpha/2$) into a branch of hyperbola of eccentricity $\sec \alpha$.
- Show that the transformation $w = z + (a^2 - b^2)/4z$ transforms the circle of radius $\frac{1}{2}(a+b)$, centre at the origin, in the z -plane into ellipse of semi-axes a, b in the w -plane.
- Show that the transformation $w = z + a^2/z$ transforms circles with origin at the centre in the z -plane into co-axial concentric, confocal ellipses in the w -plane.
- Show that the function $w = A(z + a^2/z)$ may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. Also find the stagnation points.
- Show that by the relation $u + iv = \cos(x + iy)$, the infinite strip bounded by $x = c$, $x = d$, where c and d lie between 0 and $\pi/2$, is mapped into the region between the two branches of the hyperbola lying in $u > 0$.
- Prove that the transformation $w = \sin z$, maps the families of lines $x = \text{constant}$ and $y = \text{constant}$ into two families of confocal central conics.
- Discuss the transformation $w = e^z$, and show that it transforms the region between the real axis and a line parallel to real axis at $y = \pi$, into the upper half of the w -plane.
- Discuss fully the transformation $w = c \cosh z$, where c is a real number. What physical problem can we study with the help of this transformation?

20.11 SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon of the w -plane into the upper half of the z -plane and the boundary of the polygon into the real axis. The formula of this transformation is

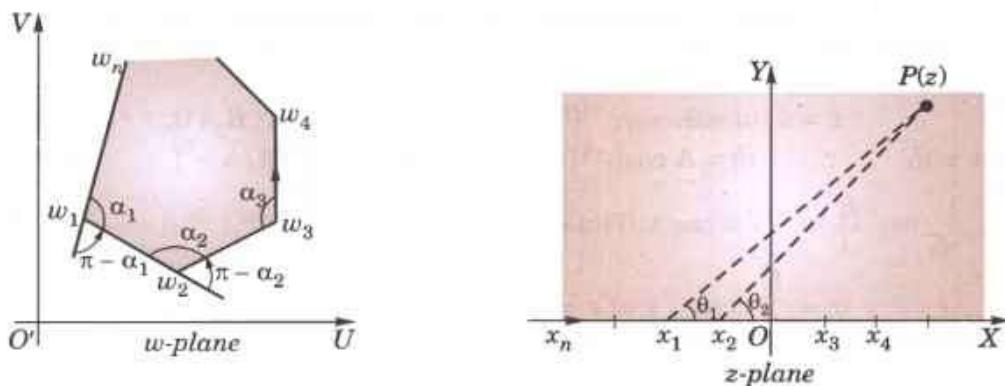


Fig. 20.11

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \quad \dots (1)$$

or

$$w = A \int (z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} dz + B \quad \dots (2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the interior angles of the polygon having vertices w_1, w_2, \dots, w_n which map into the points x_1, x_2, \dots, x_n on the real-axis of the z -plane (Fig. 20.11). Also A and B are complex constants which determines the size and position of the polygon.

Proof. We have from (1),

$$\text{amp} \left(\frac{dw}{dz} \right) = \text{amp} [A] + \left(\frac{\alpha_1}{\pi} - 1 \right) \text{amp} (z - x_1) + \left(\frac{\alpha_2}{\pi} - 1 \right) \text{amp} (z - x_2) \\ \dots + \left(\frac{\alpha_n}{\pi} - 1 \right) \text{amp} (z - x_n) \quad \dots (3)$$

As z moves along the real axis from the left towards x_1 , suppose that w moves along the side $w_n w_1$ of the polygon towards w_1 . As z crosses x_1 from left to right, $\theta_1 = \text{amp} (z - x_1)$ changes from π to 0 while all other terms of (3) remain unaffected. Hence only $\left(\frac{\alpha_1}{\pi} - 1 \right) \text{amp} (z - x_1)$ decreases by $\left(\frac{\alpha_1}{\pi} - 1 \right) \pi = \alpha_1 - \pi$, i.e. increases by $\pi - \alpha_1$ in the anti-clockwise direction. In other words, $\text{amp} (dw/dz)$ increases by $\pi - \alpha_1$. Thus the direction of w_1 turns through the angle $\pi - \alpha_1$ and w now moves along the side $w_1 w_2$ of the polygon.

Similarly when z passes through x_2 , $\theta_1 = \text{amp} (z - x_1)$ and $\theta_2 = \text{amp} (z - x_2)$ change from π to 0 while all other terms remain unchanged. Hence the side $w_1 w_2$ turns through the angle $\pi - \alpha_2$. Proceeding in this way, we see that as z moves along x -axis, w traces the polygon $w_1 w_2 w_3 \dots w_n$ and conversely.

Example 20.16. Find the transformation which maps the semi-infinite strip in the w -plane (Fig. 20.12) into the upper half of the z -plane
(V.T.U., M.E. 2006; Osmania, 2003)

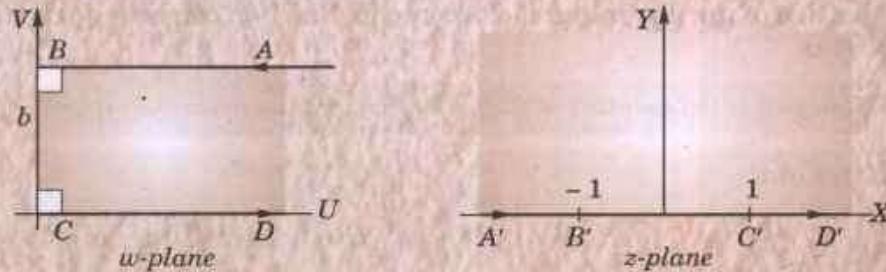


Fig. 20.12

Solution. Consider $ABCD$ as the limiting case of a triangle with two vertices B and C and the third vertex A or D at infinity. Let the vertices B and C map into the points B' (-1) and C' (1) of the z -plane. Since the interior angles at B and C are $\pi/2$, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi} - 1} (z-1)^{\frac{\pi/2}{\pi} - 1} = A/\sqrt{(z^2 - 1)}$$

$$\therefore w = A \int \frac{dz}{\sqrt{(z^2 - 1)}} + B = A \cosh^{-1} z + B$$

When $z = 1, w = 0. \quad \therefore 0 = A \cosh^{-1}(1) + B, i.e., B = 0.$

When $z = -1, w = ib. \quad \therefore ib = A \cosh^{-1}(-1) + 0, i.e., \cosh(ib/A) = -1$

or

$$\cos \frac{b}{A} = -1 = \cos \pi. \text{ Thus } A = \frac{b}{\pi}.$$

Hence $w = \frac{b}{\pi} \cosh^{-1} z \text{ or } z = \cosh \frac{\pi w}{b}.$

PROBLEMS 20.4

- Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0, v = \pi$ and $u = 0$ into the upper half of the z -plane.
- Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w -plane into the upper half of the z -plane.
- Find the mapping function which maps semi-infinite strip in the z -plane $-\pi/2 \leq x \leq \pi/2, y \geq 0$ into half w -plane for which $v \geq 0$, such that the points $(-\pi/2, 0), (\pi/2, 0)$ in the z -plane are mapped into the points $(-1, 0), (1, 0)$ respectively in w -plane.
- Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0, v = \pi$ onto the upper half of the z -plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B . Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \dots, P_i(z_i), \dots, P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^n f(\zeta_i) \delta z_i$ as $n \rightarrow \infty$ in such a way that the length of the chord δz_i approaches zero, is called the **line integral of $f(z)$ taken along the path C** , i.e.,

$$\int_C f(z) dz.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z) dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$



Fig. 20.13

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. The value of the integral is independent of the path of integration when the integrand is analytic.

Example 20.17. Prove that

$$(i) \int_C \frac{dz}{z-a} = 2\pi i. \quad (ii) \int_C (z-a)^n dz = 0 [n, any integer \neq -1]$$

where C is the circle $|z-a| = r$.

(U.P.T.U., 2003)

Solution. The parametric equation of C is $z-a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

$$(i) \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta} d\theta \quad [\because dz = ire^{i\theta} d\theta]$$

$$= i \int_0^{2\pi} d\theta = 2\pi i$$

$$\begin{aligned}
 (ii) \int_C (z-a)^n dz &= \int_0^{2\pi} r^n e^{ni\theta} \cdot i r e^{i\theta} d\theta \\
 &= ir^{n+1} \int_0^{2\pi} e^{(n+1)i\theta} d\theta = \frac{r^{n+1}}{n+1} \left| e^{(n+1)i\theta} \right|_0^{2\pi}, \text{ provided } n \neq -1 \\
 &= \frac{r^{n+1}}{n+1} [e^{2(n+1)\pi i} - 1] = 0 \quad [\because e^{2(n+1)\pi i} = 1]
 \end{aligned}$$

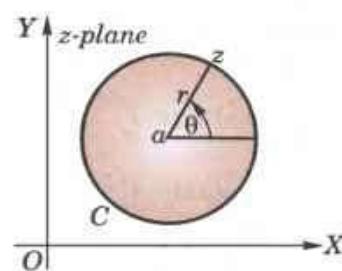


Fig. 20.14

Example 20.18. Evaluate $\int_0^{2+i} (\bar{z})^2 dz$, along (i) the line $y = x/2$, (Bhopal, 2007; U.P.T.U., 2002)

(ii) the real axis to 2 and then vertically to $2+i$. (S.V.T.U., 2009; P.T.U., 2008 S; Mumbai, 2006)

Solution. (i) Along the line OA , $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i) dy$ (Fig. 20.15)

$$\begin{aligned}
 \therefore I &= \int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2-i)^2 y^2 \cdot (2+i) dy \\
 &= 5(2-i) \left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3} (2-i)
 \end{aligned}$$

$$(ii) I = \int_{OB} (\bar{z})^2 dz + \int_{BA} (\bar{z})^2 dz.$$

Now along OB , $z = x$, $\bar{z} = x$, $dz = dx$;

and along BA , $z = 2+iy$, $\bar{z} = 2-iy$, $dz = idy$

$$\begin{aligned}
 \therefore I &= \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \cdot idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [4y + (4-y^2)i] dy \\
 &= \frac{8}{3} + 4 \cdot \frac{1}{2} + \left(4 \cdot 1 - \frac{1}{3} \right) i = \frac{1}{3} (14 + 11i).
 \end{aligned}$$

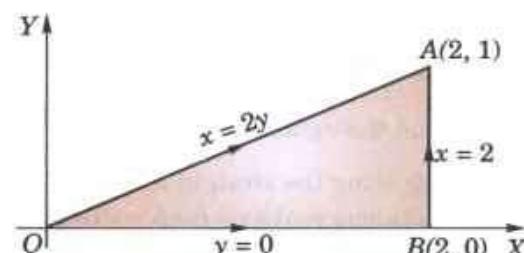


Fig. 20.15

Example 20.19. Evaluate $\int_C (z^2 + 3z + 2) dz$ where C is the arc of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ between the points $(0, 0)$ and $(\pi a, 2a)$. (Rohtak, 2004)

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the z -plane being a polynomial. As such, the line integral of $f(z)$ between O and A is independent of the path (Fig. 20.16). We therefore, take the path from O to L and L to A so that

$$\int_C f(z) dz = \int_{OL} f(z) dz + \int_{LA} f(z) dz \quad \dots(i)$$

$$\therefore \int_{OL} f(z) dz = \int_0^{\pi a} (x^2 + 3x + 2) dx$$

[∴ along OL , $y = 0$, $x = 0$ at O , $x = \pi a$ at L]

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6} (2\pi^2 a^2 + 9\pi a + 12) \quad \dots(ii)$$

and $\int_{LA} f(z) dz = \int_0^{2a} [(\pi a + iy)^2 + 3(\pi a + iy) + 2] idy$

[∴ along LA , $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and y varies from 0 (at L) to $2a$ (at A)]

$$= L \left| \frac{(\pi a + iy)^3}{3i} + 3 \frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3} (\pi + 2i)^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 4ia \quad \dots(iii)$$

∴ substituting from (ii) and (iii) in (i), we get

$$\int_C f(z) dz = \frac{\pi a}{6} (2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3} (\pi + 2i)^3 + \frac{3a^2}{2} (\pi + 2i)^2 + 4ia$$

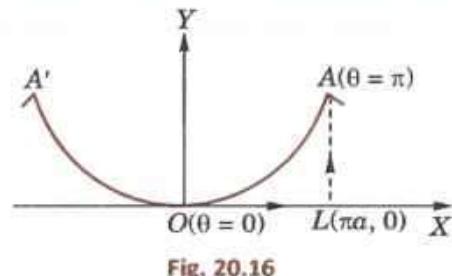


Fig. 20.16

PROBLEMS 20.5

- Evaluate $\int_0^{1+i} (x^2 - iy) dz$ along the paths (a) $y = x$ and (b) $y = x^2$. (U.P.T.U., 2010)
- Evaluate $\int_{1-i}^{2+i} (2x + iy + 1) dz$, along the two paths: (i) $x = t + 1, y = 2t^2 - 1$ (ii) the straight line joining $1 - i$ and $2 + i$. (U.P.T.U., 2006)
- Evaluate $\int_{1-i}^{2+3i} (z^2 + z) dz$ along the line joining the points $(1, -1)$ and $(2, 3)$. (V.T.U., 2004)
- Show that for every path between the limits, $\int_{-2}^{2+i} (2+z)^2 dz = -i/3$. (Delhi, 2002)
- Show that $\oint_C (z+1) dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1+i$ and $z = i$. (Rohtak, 2006)
- Evaluate $\int_C |z| dz$, where C is the contour
 - straight line from $z = -i$ to $z = i$.
 - left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
 - circle given by $|z+1| = 1$ described in the clockwise sense.
- Find the value of $\int_0^{1+i} (x - y + ix^2) dz$
 - along the straight line from $z = 0$ to $z = 1+i$
 - along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1+i$.
(U.P.T.U., 2003)
- Prove that $\int_C dz/z = -\pi i$ or πi , according as C is the semi-circular arc $|z| = 1$ above or below the real axis. (Rohtak, 2005)
- Evaluate $\int_C (z - z^2) dz$, where C is the upper half of the circle $|z| = 1$. What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C , then $\oint_C f(z) dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\oint_C f(z) dz = \oint_C (udx - vdy) = i \oint_C (vdx + udy) \quad \dots(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C .

Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\oint_C f(z) dz = - \iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] dx dy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right] dx dy \quad \dots(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\oint_C f(z) dz = 0$.

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. Hence the line integral of a function $f(z)$ which is analytic in the region D , is independent of the path joining any two points of D .

Obs. 2. Extension of Cauchy's theorem. If $f(z)$ is analytic in the region D between two simple closed curves C and C_1 , then $\oint_C f(z) dz = \oint_{C_1} f(z) dz$.

To prove this, we need to introduce the cross-cut AB . Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, i.e., along AB —along C_1 in clockwise sense and along BA —along C in anti-clockwise sense.

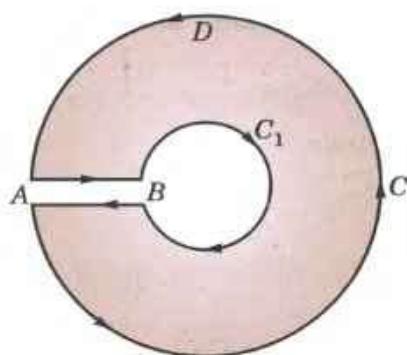


Fig. 20.17(a)

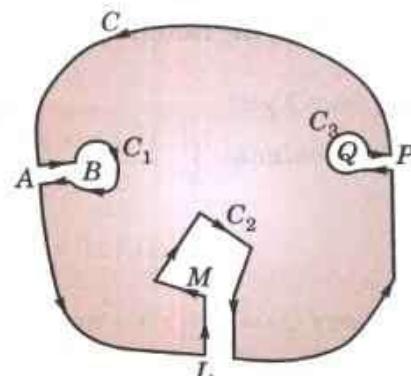


Fig. 20.17(b)

i.e.,

$$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_C f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get

$$\int_C f(z)dz + \int_{C_1} f(z)dz \text{ each integration being taken in the anti-clockwise sense.}$$

If C_1, C_2, C_3, \dots be any number of closed curves within C (Fig. 20.17(b)), then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \dots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve and if a is any point within C , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z=a$. With the point a as centre and radius r , draw a small circle C_1 lying entirely within C .

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1 , we have by Cauchy's theorem,

$$\begin{aligned} \oint_C \frac{f(z)}{z-a} dz &= \oint_{C_1} \frac{f(z)}{z-a} dz && \left\{ \begin{array}{l} \text{For any point on } C_1, \\ z-a=re^{i\theta} \text{ and } dz=ire^{i\theta} d\theta \end{array} \right. \\ &= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} d\theta = i \oint_{C_1} f(a+re^{i\theta}) d\theta \end{aligned} \quad \dots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a , i.e., as $r \rightarrow 0$, the integral (1) will approach to

$$\oint_C f(a)d\theta = if(a) \int_0^{2\pi} d\theta = 2\pi i f(a) \oint_C f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

i.e.,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \quad \dots(2)$$

which is the desired *Cauchy's integral formula*.

(V.T.U., 2011 S)

Cor. Differentiating both sides of (2) w.r.t. a ,

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a} \left[\frac{f(z)}{z-a} \right] dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz \quad \dots(3)$$

Similarly,

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz \quad \dots(4)$$

and in general,

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz. \quad \dots(5)$$

Thus it follows from the results (2) to (5) that if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C . Incidentally, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. Evaluate $\int_C \frac{z^2 - z + 1}{z-1} dz$, where C is the circle

$$(i) |z| = 1, \quad (ii) |z| = \frac{1}{2}. \quad (\text{S.V.T.U., 2007})$$

Solution. (i) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C .

\therefore by Cauchy's integral formula $\frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = f(a) = 1$ i.e., $\int_C \frac{z^2 - z + 1}{z-1} dz = 2\pi i$.

(ii) In this case, $a = 1$ lies outside the circle $C : |z| = 1/2$. So $(z^2 - z + 1)/(z-1)$ is analytic everywhere within C .

\therefore by Cauchy's theorem $\int_C \frac{z^2 - z + 1}{z-1} dz = 0$.

Example 20.21. Evaluate, using Cauchy's integral formula:

$$(i) \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz \text{ where } C \text{ is the circle } |z| = 3 \quad (\text{U.P.T.U., 2010})$$

$$(ii) \oint_C \frac{\cos \pi z}{z^2 - 1} dz \text{ around a rectangle with vertices } 2 \pm i, -2 \pm i$$

$$(iii) \oint_C \frac{e^{iz}}{z^2 + 1} dz \text{ where } C \text{ is the circle } |z| = 3. \quad (\text{U.P.T.U., 2009})$$

Solution. (i) $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\begin{aligned} \therefore \oint_C \frac{f(z) dz}{(z-1)(z-2)} &= \oint_C (\sin \pi z^2 + \cos \pi z^2) \left(\frac{1}{z-2} - \frac{1}{z-1} \right) dz \\ &= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2} dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1} dz \\ &= 2\pi i [\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i [\sin \pi(1)^2 + \cos \pi(1)^2] \end{aligned}$$

[By Cauchy's integral formula]

$$= 2\pi i (0 + 1) - 2\pi i (0 - 1) = 4\pi i$$

(ii) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\begin{aligned} \therefore \oint_C \frac{\cos \pi z}{z^2 - 1} dz &= \frac{1}{2} \oint_C \left(\frac{1}{z-1} - \frac{1}{z+1} \right) \cos \pi z dz \\ &= \frac{1}{2} \oint_C \frac{\cos \pi z}{z-1} dz - \oint_C \frac{\cos \pi z}{z+1} dz \\ &= \frac{1}{2} [2\pi i \cos \pi(1)] - \frac{1}{2} [2\pi i \cos \pi(-1)] = 0. \end{aligned}$$

[By Cauchy's integral formula]

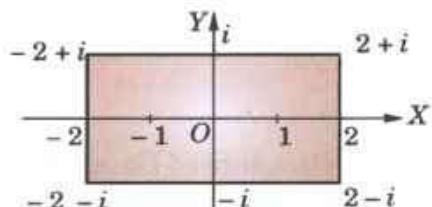


Fig. 20.18

(iii) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ i.e., $z = i$ and $z = -i$ which lie within this circle.

$$\begin{aligned} \oint_C \frac{e^{tz}}{z^2 + 1} dz &= \oint_C \frac{1}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right) e^{tz} dz = \frac{1}{2i} \left\{ \oint_C \frac{e^{tz}}{z-i} dz - \oint_C \frac{e^{tz}}{z+i} dz \right\} \\ &= \frac{1}{2i} \{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \\ &= 2\pi i \left(\frac{e^{it} - e^{-it}}{2i} \right) = 2\pi i \sin t. \end{aligned} \quad [\text{By Cauchy's integral formula}]$$

Example 20.22. Evaluate

$$(i) \oint_C \frac{\sin^2 z}{(z - \pi/6)^3} dz, \text{ where } C \text{ is the circle } |z| = 1 \quad (\text{Rohtak, 2005})$$

$$(ii) \oint_C \frac{e^{2z}}{(z+i)^4} dz, \text{ where } C \text{ is the circle } |z| = 3 \quad (\text{U.P.T.U., 2008})$$

$$(iii) \oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz, \text{ where } C \text{ is } |z| = 4. \quad (\text{U.P.T.U., 2008; J.N.T.U., 2000})$$

Solution. (i) $f(z) = \sin^2 z$ is analytic inside the circle $C: |z| = 1$ and the point $a = \pi/6$ ($= 0.5$ approx.) lies within C .

$$\therefore \text{by Cauchy's integral formula } f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz,$$

$$\begin{aligned} \text{we get } \oint_C \frac{\sin^2 z}{(z - \pi/6)^3} dz &= \pi i \left[\frac{d^2}{dz^2} (\sin^2 z) \right]_{z=\pi/6} \\ &= \pi i (2 \cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i. \end{aligned}$$

(ii) $f(z) = e^{2z}$ is analytic within the circle $C: |z| = 3$. Also $z = -1$ lies inside C .

$$\therefore \text{By Cauchy's integral formula: } f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z) dz}{(z-a)^4}$$

$$\begin{aligned} \text{we get } \int_C \frac{e^{2z}}{(z+1)^4} dz &= \frac{2\pi i}{6} \left| \frac{d^3(e^{2z})}{dz^3} \right|_{z=-1} = \frac{\pi i}{3} [8e^{2z}]_{z=-1} = \frac{8\pi i}{3} e^{-2} \end{aligned}$$

(iii) $\frac{e^z}{(z^2 + \pi^2)^2} = \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}$ is not analytic at $z = \pm \pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

$$\text{Now } \frac{1}{(z+\pi i)^2 (z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\begin{aligned} \therefore \int_C \frac{e^z}{(z^2 + \pi^2)^2} dz &= \frac{7}{2\pi^3 i} \left\{ \int_C \frac{e^z}{z+\pi i} dz - \int_C \frac{e^z}{z-\pi i} dz \right\} - \frac{1}{4\pi^2} \left\{ \int_C \frac{e^z}{(z+\pi i)^2} dz + \int_C \frac{e^z}{(z-\pi i)^2} dz \right\} \\ &= \frac{7}{2\pi^3 i} [2\pi i f(-\pi i) - 2\pi i f(\pi i)] - \frac{1}{4\pi^2} [2\pi i f'(-\pi i) + 2\pi i f'(\pi i)] \\ &= \frac{7}{\pi^2} (e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi} (e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2} \left(\frac{e^{\pi i} - e^{-\pi i}}{2i} \right) - \frac{i}{\pi} \left(\frac{e^{\pi i} + e^{-\pi i}}{2} \right) \\ &= -\frac{14i}{\pi^2} \sin \pi - \frac{i}{\pi} \cos \pi = \frac{i}{\pi}. \end{aligned} \quad [\S 19.9]$$

Example 20.23. Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1+i$, $-1+i$ and $-1-i$.
(U.P.T.U., 2006)

Solution. The boundary of the given triangle consists of three lines AB , BC , CA . (Fig. 29.19).

$$\oint_{ABC} e^{iz} dz = \int_{AB} e^{iz} dz + \int_{BC} e^{iz} dz + \int_{CA} e^{iz} dz$$

Now

$$\int_{AB} e^{iz} dz = \int_1^{-1} e^{i(x+i)} dx \quad \left| \begin{array}{l} \text{: Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx \end{array} \right.$$

$$= \int_1^{-1} e^{ix-1} dx = \left[\frac{e^{ix-1}}{i} \right]_1^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

$$\int_{BC} e^{iz} dz = \int_1^{-1} e^{i(-1+iy)} idy \quad \left| \begin{array}{l} \text{: Along } BC : x = -1 \\ \therefore z = -1 + iy, dz = idy \end{array} \right.$$

$$= i \int_1^{-1} e^{-i-y} dy = i \left[\frac{e^{-i-y}}{-1} \right]_1^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz} dz = \int_{-1}^1 e^{i(1+i)x} (1+i) dx \quad \left| \begin{array}{l} \text{: Along } CA : y = ? \\ \therefore z = (1+i)x, dz = (1+i) dx \end{array} \right.$$

$$= (1+i) \int_{-1}^1 \frac{e^{i(1+i)x} - e^{-i(1+i)x}}{i(1+i)} dx = \frac{e^{i-1} - e^{-i+1}}{i}$$

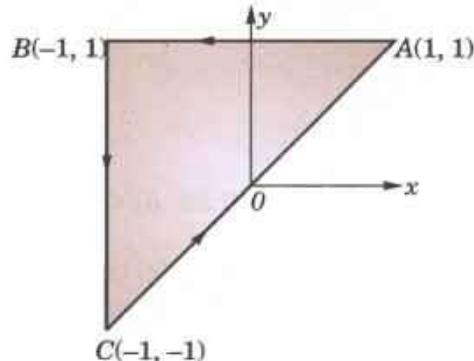


Fig. 29.19

Thus from (i) $\oint_{ABC} e^{iz} dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0$... (ii)

Also since $f(z) = e^{iz}$ is analytic everywhere,

\therefore by Cauchy's theorem $\oint_{ABC} f(z) dz = 0$... (iii)

Hence from (ii) and (iii), \oint_{ABC} Cauchy's theorem is verified.

Example 20.24. If $F(\zeta) = \oint_C \frac{4z^2 + z + 5}{z - \zeta} dz$, where C is the ellipse $(x/2)^2 + (y/3)^2 = 1$, find the value of (a)

$F(3.5)$; (b) $F(i)$, $F''(-1)$ and $F''(-i)$.

(Bhopal, 2009; Marathwada, 2008; Mumbai, 2006)

Solution. (a)

$$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C , therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C .

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0, \text{ i.e., } F(3.5) = 0.$$

(b) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C , therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz$$

i.e.,

$$\oint_C \frac{4z^2 + z + 5}{z - \zeta} dz = 2\pi i(4\zeta^2 + \zeta + 5)$$

i.e.,

$$F(\zeta) = 2\pi i(4\zeta^2 + \zeta + 5)$$

∴

$$F'(\zeta) = 2\pi i(8\zeta + 1) \text{ and } F''(\zeta) = 16\pi i$$

Thus

$$F(i) = 2\pi i(-4 + i + 5) = 2\pi(i - 1)$$

$$F'(-1) = 2\pi i[8(-1) + 1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z) dz = 0$ around every simple closed curve C in D , then $f(z)$ is analytic in D .

Since $\oint_C f(z) dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^z f(z) dz = \phi(z), \text{ (say),}$$

Let $\phi(z) = U + iV$ and $f(z) = u + iv$

$$\text{Then } U + iV = \int_{(x_0, y_0)}^{(x, y)} (u + iv) (dx + idy) = \int_{(x_0, y_0)}^{(x, y)} (udx - vdy) + i \int_{(x_0, y_0)}^{(x, y)} (vdx + udy)$$

$$\therefore U = \int_{(x_0, y_0)}^{(x, y)} (udx - vdy), V = \int_{(x_0, y_0)}^{(x, y)} (vdx + udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u, \frac{\partial U}{\partial y} = -v, \frac{\partial V}{\partial x} = v, \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

∴ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) **Cauchy's inequality†.** If $f(z)$ is analytic within and on the circle C : $|z - a| = r$, then

$$|f^n(a)| \leq \frac{Mn!}{r^n} \quad \dots(I)$$

where M is the maximum value of $|f(z)|$ on C .

From (5) of § 20.14, we get

$$\begin{aligned} |f^n(a)| &= \frac{n!}{2\pi} \left| \oint_C \frac{f(z) dz}{(z - a)^{n+1}} \right| \\ &\leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \oint_C |z| \\ &= \frac{n! M}{2\pi r^{n+1}} \oint_C ds = \frac{Mn!}{2\pi r^{n+1}} 2\pi r = \frac{Mn!}{r^n} \end{aligned} \quad (\text{U.P.T.U., 2005})$$

(3) **Liouville's theorem‡.** If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.

(U.P.T.U., 2008)

* Named after the Italian mathematician, Giacinto Morera (1856–1909) who worked in Turin.

† See footnote p. 144

‡ See footnote p. 573.

Taking $n = 1$ and replacing a by z , (I) gives

$$|f'(z)| \leq M/r$$

As $r \rightarrow \infty$, it gives $f'(z) = 0$ i.e., $f(z)$ is constant for all z .

(4) Poisson's integral formulae. If $f(z)$ is analytic within and on the circle C : $|z| = \rho$ and $z = re^{i\theta}$ is any point within C , then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho \cos(\theta - \phi) + r^2} f(re^{i\phi}) d\phi$$

$$\text{By Cauchy's integral formula, } f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} dw \quad \dots(1)$$

As the inverse of the point x w.r.t. C lies outside C and is given by ρ^2/\bar{z} .

[See footnote p. 685]

∴ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i} \int \frac{f(w)}{w - \rho^2/\bar{z}} dw \quad \dots(2)$$

$$\begin{aligned} \text{Subtracting (2) from (1), } f(z) &= \frac{1}{2\pi i} \int \left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}} \right) f(w) dw \\ &= \frac{1}{2\pi i} \oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)w + z\rho^2} f(w) dw \end{aligned} \quad \dots(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{(r^2 - \rho^2) f(\rho e^{i\phi}) \cdot \rho i e^{i\phi} d\phi}{re^{-i\theta} \cdot \rho^2 e^{2i\phi} - (r^2 + \rho^2) \rho e^{i\phi} + re^{i\theta} \rho^2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi}) d\phi}{\rho^2 + r^2 - r\rho [e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}]} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi}) d\phi}{\rho^2 - 2r\rho \cos(\theta - \phi) + r^2} \end{aligned} \quad \dots(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2) u(\rho, \phi) d\phi}{\rho^2 - 2r\rho \cos(\theta - \phi) + r^2} \quad \dots(5)$$

$$\text{and } v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2) v(\rho, \phi) d\phi}{\rho^2 - 2r\rho \cos(\theta - \phi) + r^2} \quad \dots(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (i) outside C ,
(ii) inside C .

2. Evaluate $\oint_C \frac{dz}{(z - a)^n}$, $n = 2, 3, 4, \dots$, where C is a closed curve containing the point $z = a$.

3. Evaluate (i) $\oint_C \frac{e^z}{z^2 + 1} dz$, where C is the circle $|z| = 1/2$.

(P.T.U., 2010)

- (ii) $\oint_C \frac{e^{3iz}}{(z + \pi)^3} dz$, where C is the circle $|z - \pi| = 3$.

(U.P.T.U., 2007)

* Named after the French mathematician Simeon Denis Poisson (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

$$(i) \oint_C \frac{3z - 5}{z^2 + 2z} dz, \text{ where } C \text{ is } |z| = 1. \quad (\text{P.T.U., 2005 S}) \quad (ii) \oint_C \frac{z^2 + 1}{z(2z + 1)} dz, \text{ where } C \text{ is } |z| = 1.$$

$$(iii) \oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz \text{ where } C \text{ is } |z| = 4. \quad (\text{U.P.T.U., 2008})$$

5. Evaluate (a) $\oint_C \frac{z^3 - 2z + 1}{(z-i)^2} dz$ where C is $|z| = 2$.

$$(b) \oint_C \frac{e^{-z}}{(z-1)(z-2)^2} dz \text{ where } C \text{ is } |z| = 3. \quad (\text{Rohtak, 2003})$$

6. Evaluate, using Cauchy's integral formulae:

$$(i) \oint_C \frac{z}{z^2 - 3z + 2} dz, \text{ where } C \text{ is } |z-2| = \frac{1}{2}. \quad (\text{U.P.T.U., 2009; Hissar, 2007; Madras, 2000})$$

$$(ii) \oint_C \frac{e^z dz}{(z+1)^2}, \text{ where } C \text{ is } |z+1| = 3. \quad (\text{Bhopal, 2009})$$

$$(iii) \oint_C \frac{\log z}{(z-1)^3} dz \text{ where } C \text{ is } |z-1| = \frac{1}{2}. \quad (\text{J.N.T.U., 2003})$$

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2 - z - 2}{z-a} dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2 + 7z + 1}{z - \zeta} dz$, where C is the circle $|z| = 2$ find the values of

$$(i) \phi(3), \quad (ii) \phi'(1-i), \quad (iii) \phi''(1-i). \quad (\text{Mumbai, 2006})$$

9. Evaluate $\oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. (Rohtak, 2006)

10. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1+i, -1+i$; (ii) triangle with vertices $(1, 2), (1, 4), (3, 2)$. (V.T.U., 2003)

20.16 (1) SERIES OF COMPLEX TERMS

Let $(a_1 + ib_1) + (a_2 + ib_2) + \dots + (a_n + ib_n) + \dots$... (1)

be an infinite series of complex terms ; a 's and b 's being real numbers. If the series Σa_n and Σb_n converge to the sums A and B , then series (1) is said to converge to the sum $A + iB$. Also if (1) is a convergent series, then

$$\lim_{n \rightarrow \infty} (a_n + ib_n) = 0.$$

The series (1) is said to be **absolutely convergent** if the series

$$|a_1 + ib_1| + |a_2 + ib_2| + \dots + |a_n + ib_n| + \dots$$

is convergent. Since $|a_n|$ and $|b_n|$ are both $\leq |a_n + ib_n|$, it follows that an absolutely convergent series is convergent.

Next let the series of functions $u_1(z) + u_2(z) + \dots + u_n(z) + \dots$... (2)

converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then the series (2) is said to be **uniformly convergent** in a region R , if corresponding to any positive number ϵ , there exists a positive number N , depending on ϵ , but not on z , such that for every z in R .

$$|S(z) - S_n(z)| < \epsilon \text{ for } n > N. \quad [\text{cf. Def. p. 389}]$$

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \leq M_n$ for all z in R .

Also a uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.

Obs. If a power series $\sum a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.

Since $\sum a_n z_1^n$ converges, therefore, $\lim_{n \rightarrow \infty} a_n z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n . Then

$$\sum a_n z^n = \sum |a_n z_1^n| \cdot |z/z_1|^n < \sum k t^n \text{ where } t = |z/z_1|.$$

But the series $\sum t^n$ converges for $t < 1$. Hence the series $\sum a_n z^n$ converges absolutely for $|z| < |z_1|$, i.e., if a circle with centre at the origin and radius $|z_1|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\sum a_n z^n$ converges, is called the *circle of convergence* and R is called the *radius of convergence*.

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. If $f(z)$ is analytic inside a circle C with centre at a , then for z inside C ,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \quad \dots(i)$$

Proof. Let z be any point inside C . Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1 . We have

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{t-a-(z-a)} = \frac{1}{t-a} \left(1 - \frac{z-a}{t-a}\right)^{-1} \\ &= \frac{1}{t-a} \left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right)^2 + \dots + \left(\frac{z-a}{t-a}\right)^n + \dots\right] \end{aligned} \quad \dots(ii)$$

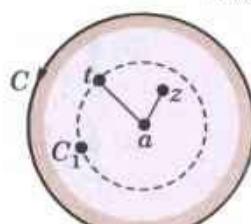


Fig. 20.20

As $|z-a| < |t-a|$, i.e. $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (ii) by $f(t)$, we can integrate over C_1 .

$$\oint_{C_1} \frac{f(t)}{t-z} dz = \oint_{C_1} \frac{f(t)}{t-a} dz + (z-a) \oint_{C_1} \frac{f(t)}{(t-a)^2} dt + \dots + (z-a)^n \cdot \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt + \dots \quad \dots(iii)$$

Since $f(t)$ is analytic on and inside C_1 , therefore, applying the formulae (2) to (5) of p. 697-698 (iii), we get (i) which is known as *Taylor's series*.

Obs. Another remarkable fact is that complex analytic functions can always be represented by power series of the form (i).

(3) Laurent's series†. If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a , then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

$$\text{where } a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt,$$

Γ being any curve in R , encircling C_1 (as in Fig. 20.21).

Proof. Introduce cross-out AB , then $f(z)$ is analytic in the region D bounded by AB , C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D , we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left[\int_{AB} \frac{f(t)}{t-z} dt + \oint_{C_1} \frac{f(t)}{t-z} dt + \int_{BA} \frac{f(t)}{t-z} dt + \oint_C \frac{f(t)}{t-z} dt \right] \\ &= \frac{1}{2\pi i} \left[\oint_C \frac{f(t)}{t-z} dt - \oint_{C_1} \frac{f(t)}{t-z} dt \right] \end{aligned} \quad \dots(i)$$

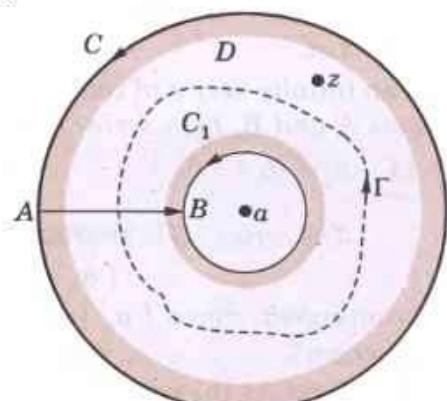


Fig. 20.21

where both C and C_1 are described anti-clockwise in (i) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (i), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{(z-a)^n}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician Pierre Alphonse Laurent (1813–1854) who published this theorem in 1843.

$$= \sum a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \quad \dots(iii)$$

For the second integral in (i), let t lie on C_1 . Then we write

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a} \left(1 - \frac{t-a}{z-a}\right)^{-1} \\ &= -\frac{1}{z-a} \left[1 + \frac{t-a}{z-a} + \left(\frac{t-a}{z-a}\right)^2 + \dots + \left(\frac{t-a}{z-a}\right)^{n-1} + \dots\right] \end{aligned}$$

As $|t-a| < |z-a|$, i.e., $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1 , we get

$$-\frac{1}{2\pi i} \oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i} \oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n} \quad \dots(iv)$$

where

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iv) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}. \quad \dots(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ , we can take the integral giving a_n over Γ . Similarly we can take the integral giving a_{-n} over Γ . Hence (iv) can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series*.

Obs. 1. As $f(z)$ is not given to be analytic inside Γ , $a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}$

However, if $f(z)$ is analytic inside Γ , then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series*.

Obs. 2. To obtain Taylor's or Laurent's series, simply expand $f(z)$ by binomial theorem instead of finding a_n by complex integration which is quite complicated.

Obs. 3. Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + \dots \infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.

Solution. Let the sum of the first n terms of the series be s_n , so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + \dots + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

$\therefore \lim_{n \rightarrow \infty} s_n = z$, i.e., the given series converges for $|z| < 1$.

$$\begin{aligned} |s_n(z)| &= |z(1-z)| + |z^2(1-z)| + \dots + |z^n(1-z)| \\ &= |1-z| \{ |z| + |z|^2 + |z|^3 + \dots + |z|^n \} \end{aligned}$$

$$\text{For } |z| < 1, \quad \lim_{n \rightarrow \infty} |s_n(z)| = |1-z| \frac{|z|}{1-|z|}$$

[G.P.]

Hence the given series converges absolutely.

Example 20.26. Expand $\sin z$ in a Taylor's series about $z = 0$ and determine the region of convergence.
(P.T.U., 2009 S)

Solution. Given $f(z) = \sin z, f'(z) = \cos z, f''(z) = -\sin z, f'''(z) = -\cos z, \dots$
 $\therefore f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0) + \dots$$

i.e., $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \dots$

Hence $\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1}$ where $a_n = \frac{(-1)^{n-1}}{(2n-1)!}$

Since $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n-1)!}{(2n+1)!} \right| = 0$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. Find Taylor's expansion of

$$(i) f(z) = \frac{1}{(z+1)^2} \text{ about the point } z = -i. \quad (\text{V.T.U., 2009 S})$$

$$(ii) f(z) = \frac{2z^3+1}{z^2+z} \text{ about the point } z = i. \quad (\text{P.T.U., 2003})$$

Solution. (i) To expand $f(z)$ about $z = -i$, i.e., in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2} [1+t/(1-i)]^{-2} = \frac{i}{2} \left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \dots \right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n} \right]$$

$$(ii) f(z) = \frac{2z^3+1}{z(z+1)} = 2z-2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \quad \dots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z+1)$ about $z = i$, put $z - i = t$, so that

$$\begin{aligned} \frac{1}{z} &= \frac{1}{(t+i)} = \frac{1}{i} \left(1 + \frac{t}{i} \right)^{-1} && \text{[Expanding by Binomial theorem]} \\ &= \frac{1}{i} \left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \dots \infty \right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \dots \infty \\ &= -i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \end{aligned} \quad \dots(ii)$$

and $\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i} \left(1 + \frac{t}{1+i} \right)^{-1}$ [Expanding by Binomial theorem]

$$\begin{aligned} &= \frac{1}{1+i} \left[1 - \frac{t}{1+i} + \frac{t^2}{(1+i)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \dots \infty \right] \\ &= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \dots \infty \right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}} \end{aligned} \quad \dots(iii)$$

Substituting from (ii) and (iii) in (i), we get

$$\begin{aligned} f(z) &= \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{ \frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}} \right\} (z-i)^n \\ &= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{ \frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}} \right\} (z-i)^n. \end{aligned}$$

Example 20.28. Expand $f(z) = 1/[(z-1)(z-2)]$ in the region:

- (a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $0 < |z-1| < 1$.

(U.P.T.U., 2010; V.T.U., 2010; Bhopal, 2009)

Solution. (a) By partial fractions $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$... (i)

$$= -\frac{1}{2} \left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \quad \dots (ii)$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (ii) gives on expansion

$$\begin{aligned} f(z) &= -\frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right) + (1+z+z^2+z^3+\dots) \\ &= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \dots \text{ which is a Taylor's series.} \end{aligned}$$

(b) For $1 < |z| < 2$, we write (i) as

$$f(z) = -\frac{1}{2} \frac{1}{(1-z/2)} - \frac{1}{z(1-z^{-1})} \quad \dots (iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (iii) gives on expansion

$$\begin{aligned} f(z) &= -\frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right) - \frac{1}{z} (1+z^{-1}+z^{-2}+z^{-3}+\dots) \\ &= \dots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \dots \end{aligned}$$

which is a Laurent's series.

(c) For $|z| > 2$, we write (i) as

$$\begin{aligned} f(z) &= \frac{1}{z(1-2z^{-1})} - \frac{1}{z(1-z^{-1})} \\ &= z^{-1}(1+2z^{-1}+4z^{-2}+8z^{-3}+\dots) - z^{-1}(1+z^{-1}+z^{-2}+z^{-3}+\dots) \\ &= \dots + 7z^{-4} + 3z^{-3} + z^{-2} + \dots \end{aligned}$$

(d) For $0 < |z-1| < 1$, we write (i) as

$$\begin{aligned} f(z) &= \frac{1}{(z-1)-1} - \frac{1}{z-1} \\ &= -(z-1)^{-1} - [1-(z-1)]^{-1} \\ &= -(z-1)^{-1} - [1+(z-1)+(z-1)^2+(z-1)^3+\dots]. \end{aligned}$$

Example 20.29. Find the Laurent's expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < z+1 < 3$.

(S.V.T.U., 2009; Anna, 2003; V.T.U., 2003)

Solution. Writing $z+1 = u$, we have

$$\begin{aligned} f(z) &= \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)} \\ &= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} \quad (\text{splitting into partial fraction}) \\ &= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u} \left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3} \left(1 - \frac{u}{3}\right)^{-1} \end{aligned}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$\begin{aligned} f(z) &= \frac{-3}{u} + \frac{1}{u} \left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty \right) - \frac{2}{3} \left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty \right) \\ &= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty - \frac{2}{3} \left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty \right) \end{aligned}$$

$$\text{Hence } f(z) = -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \dots \infty - \frac{2}{3} \left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \dots \infty \right]$$

which is valid in the region $1 < z+1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. A zero of an analytic function $f(z)$ is that value of z for which $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots \quad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a zero of order m at $z = a$.

When $m = 1$, the zero is said to be simple. In the neighbourhood of zero ($z = a$) of order m ,

$$\begin{aligned} f(z) &= a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots \infty \\ &= (z-a)^m \phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \dots \end{aligned}$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) SINGULARITIES OF AN ANALYTIC FUNCTION

We have already defined a singular point of a function as the point at which the function ceases to be analytic.

(i) **Isolated singularity.** If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood (i.e., there exists a circle with centre a which has no other singularity), then $z = a$ is called an isolated singularity.

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \quad \dots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ i.e. at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all isolated singularities as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the non-isolated singularity of $f(z)$.

(ii) **Removable singularity.** If all the negative powers of $(z-a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a removable singularity.

Thus if $\lim_{z \rightarrow a} f(z)$ exists finitely, then $z = a$ is a removable singularity.

(iii) **Poles.** If all the negative powers of $(z-a)$ in (i) after the n th are missing, then the singularity at $z = a$ is called a pole of order n .

A pole of first order is called a simple pole.

(iv) **Essential singularity.** If the number of negative powers of $(z-a)$ in (1) is infinite, then $z = a$ is called an essential singularity. In this case, $\lim_{z \rightarrow a} f(z)$ does not exist.

Example 20.30. Find the nature and location of singularities of the following functions:

$$(i) \frac{z - \sin z}{z^2}$$

$$(ii) (z+1) \sin \frac{1}{z-2}$$

$$(iii) \frac{1}{\cos z - \sin z}$$

Solution. (i) Here $z = 0$ is a singularity.

$$\text{Also } \frac{z - \sin z}{z^2} = \frac{1}{z^2} \left\{ z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots$$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

$$(ii) (z+1) \sin \frac{1}{z-2} = (t+2+1) \sin \frac{1}{t} \quad \text{where } t = z-2$$

$$\begin{aligned} &= (t+3) \left\{ \frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots \right\} = \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots \right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \dots \right) \\ &= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \dots = 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \dots \end{aligned}$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

(iii) Poles of $f(z) = \frac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, i.e., by $\cos z - \sin z = 0$ or

$\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a simple pole of $f(z)$.

Example 20.31. What type of singularity have the following functions :

$$(i) \frac{1}{1-e^z} \quad (ii) \frac{e^{2z}}{(z-1)^4} \quad (iii) \frac{e^{1/z}}{z^2}. \quad (\text{U.P.T.U., 2009})$$

Solution. (i) Poles of $f(z) = 1/(1-e^z)$ are found by equating to zero $1-e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$$\therefore z = 2n\pi i \quad (n = 0, \pm 1, \pm 2, \dots)$$

Clearly $f(z)$ has a simple pole at $z = 2\pi i$.

$$(ii) \frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4} \cdot e^{2t} \quad \text{where } t = z-1$$

$$\begin{aligned} &= \frac{e^2}{t^4} \left\{ 1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \dots \right\} = e^2 \left\{ \frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \dots \right\} \\ &= e^2 \left\{ \frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \dots \right\} \end{aligned}$$

Since there are finite (4) number of terms containing negative powers of $(z-1)$,

$\therefore z = 1$ is a pole of 4th order.

$$(iii) f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2} \left\{ 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \right\} = z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \dots \infty$$

Since there are infinite number of terms in the negative powers of z , therefore $f(z)$ has an essential singularity at $z = 0$.

PROBLEMS 20.7

- Obtain the expansion of $(z-1)/z^2$ in a Taylor's series in powers of $(z-1)$ and determine the region of convergence.
- Find the first three terms of the Taylor's series expansion of $f(z) = 1/(z^2 + 4)$ about $z = -i$. Also find the region of convergence. (U.P.T.U., 2006)
- Expand in Taylor's series (i) $(z-1)/(z+1)$ about the point $z = 1$. (Andhra, 2000)
- (ii) $\cos z$ about the point $z = \pi/2$. (Marathwada, 2008) (iii) $\frac{1}{z^2 - z - 6}$ about (a) $z = -1$ (b) $z = 1$ (P.T.U., 2009)
- Expand the following functions in Laurent's series :
 - $f(z) = \frac{1}{z-2^{\infty}}$ for $1 < |z+1| < 2$. (Madras, 2006)

- (ii) $f(z) = \frac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (J.N.T.U., 2006)
- (iii) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (V.T.U., 2007)
5. Find the Laurent's expansion of (i) $\frac{e^z}{(z-1)^2}$, about $z=1$. (Rohtak, 2006)
- (ii) $e^{2z}/(z-1)^3$ about the singularity $z=1$.
6. Expand the following functions in Laurent series.
- (i) $(z-1)/z^2$ for $|z-1| > 1$ (ii) $\frac{1-\cos z}{z^3}$, about $z=0$. (Rohtak, 2004)
7. Find the Laurent's series expansion of
- (i) $\frac{z^2-1}{z^2+5z+6}$ about $z=0$ in the region $2 < |z| < 3$ (V.T.U., 2011 S ; Osmania, 2003)
- (ii) $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$
- (iii) $\frac{7z^2-9z-18}{z^3-9z}$ in the region (a) $|z| > 3$ (b) $0 < |z-3| < 3$. (V.T.U., 2010 S)
8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (a) $0 < |z| < 1$; (b) $1 < |z| < \sqrt{2}$; (c) $|z| > 2$.
- Find the nature and location of the singularities of the following functions : (P.T.U., 2005)
9. $\frac{1}{z(2-z)}$ 10. $\sin(1/z)$. (U.P.T.U., 2009) 11. $\tan\left(\frac{1}{z}\right)$. (P.T.U., 2006)
12. $\frac{z^2-1}{(z-1)^3}$. (Osmania, 2003) 13. $\frac{e^z}{(z-1)^4}$. 14. $\frac{\cot \pi z}{(z-a)^n}$. (U.P.T.U., 2008)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the **residue** of $f(z)$ at that point. Thus in the Laurent's series expansion of $f(z)$ around $z=a$ i.e., $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$, the residue of $f(z)$ at $z=a$ is a_{-1} .

$$\therefore \text{Res } f(a) = \frac{1}{2\pi i} \oint_C f(z) dz$$

i.e.,

$$\oint_C f(z) dz = 2\pi i \text{ Res } f(a). \quad \dots(1)$$

(2) Residue Theorem

If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C , then $\oint_C f(z) dz = 2\pi i \times (\text{sum of the residues at the singular points within } C)$.

Let us surround each of the singular points a_1, a_2, \dots, a_n by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles C_1, C_2, \dots, C_n together with C , form a multiply connected region in which $f(z)$ is analytic.

\therefore applying Cauchy's theorem, we have

$$\begin{aligned} \oint_C f(z) dz &= \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \dots + \oint_{C_n} f(z) dz \\ &= 2\pi i [\text{Res } f(a_1) + \text{Res } f(a_2) + \dots + \text{Res } f(a_n)] \text{ which is the desired result.} \end{aligned} \quad [\text{by (1)}]$$

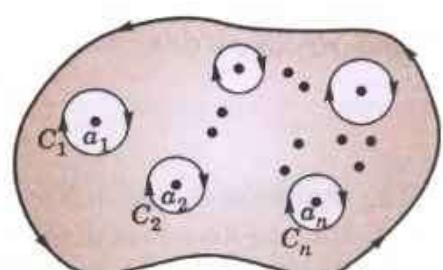


Fig. 20.22

20.19 CALCULATION OF RESIDUES

(1) If $f(z)$ has a simple pole at $z = a$, then

$$\text{Res } f(a) = \lim_{z \rightarrow a} [(z - a)f(z)]. \quad \dots(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z - a) + c_2(z - a)^2 + \dots + c_{-1}(z - a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z - a)f(z) = c_0(z - a) + c_1(z - a)^2 + \dots + c_{-1}.$$

Taking limits as $z \rightarrow a$, we get

$$\lim_{z \rightarrow a} [(z - a)f(z)] = c_{-1} = \text{Res } f(a).$$

(2) Another formula for $\text{Res } f(a)$:

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z - a)F(z)$, $F(a) \neq 0$.

Then

$$\begin{aligned} & \lim_{z \rightarrow a} [(z - a)\phi(z)/\psi(z)] \\ &= \lim_{z \rightarrow a} \frac{(z - a)[\phi(a) + (z - a)\phi'(a) + \dots]}{\psi(a) + (z - a)\psi'(a) + \dots} \\ &= \lim_{z \rightarrow a} \frac{\phi(a) + (z - a)\phi'(a) + \dots}{\psi'(a) + (z - a)\psi''(a) + \dots}, \quad \text{since } \psi(a) = 0 \end{aligned}$$

Thus

$$\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)}.$$

(3) If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [(z - a)^n f(z)] \right\}_{z=a}$$

Here

$$f(z) = c_0 + c_1(z - a) + c_2(z - a)^2 + \dots + c_{-1}(z - a)^{-1} + \dots + c_{-n}(z - a)^{-n}.$$

Multiplying throughout by $(z - a)^n$, we get

$$(z - a)^n f(z) = c_0(z - a)^n + c_1(z - a)^{n+1} + c_2(z - a)^{n+2} + \dots + c_{-1}(z - a)^{n-1} + c_{-2}(z - a)^{n-2} + \dots + c_{-n}.$$

Differentiating both sides w.r.t. z , $n - 1$ times and putting $z = a$, we get

$$\left\{ \frac{d^{n-1}}{dz^{n-1}} [(z - a)^n f(z)] \right\}_{z=a} = (n-1)! c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole ($z = a$) can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|t|$ is quite small.

Example 20.32. Find the sum of the residues of $f(z) = \frac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.

(Rohtak, 2004)

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, \dots$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \text{Res } f(0) = \lim_{z \rightarrow 0} [z \cdot f(z)] = \lim_{z \rightarrow 0} \left(\frac{\sin z}{\cos z} \right) = 0.$$

$$\begin{aligned} \text{Res } f(\pi/2) &= \lim_{z \rightarrow \pi/2} \left[\left(z - \frac{\pi}{2} \right) f(z) \right] = \lim_{z \rightarrow \pi/2} \left\{ \frac{(z - \pi/2) \sin z}{z \cos z} \right\} \\ &= \lim_{z \rightarrow \pi/2} \frac{(z - \pi/2) \cos z + \sin z}{\cos z - z \sin z} \quad \left[\text{Being } \frac{0}{0} \text{ form} \right] \\ &= \frac{1}{-\pi/2} = -\frac{2}{\pi} \end{aligned}$$

and

$$\text{Res } f(-\pi/2) = \lim_{z \rightarrow -\pi/2} \left\{ \frac{(z + \pi/2) \sin z}{z \cos z} \right\} = \lim_{z \rightarrow -\pi/2} \frac{(z + \pi/2) \cos z + \sin z}{\cos z - z \sin z} = \frac{-1}{-\pi/2} = \frac{2}{\pi}$$

$$\text{Hence sum of residues} = 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0.$$

Example 20.33. Determine the poles of the function

$$f(z) = z^2/(z-1)^2(z+2) \text{ and the residue at each pole.} \quad (\text{S.V.T.U., 2008; J.N.T.U., 2005})$$

Hence evaluate $\oint_C f(z) dz$, where C is the circle $|z| = 2.5$.

$$\text{Solution. Since } \lim_{z \rightarrow -2} [(z+2)f(z)] = \lim_{z \rightarrow -2} \frac{z^2}{(z-1)^2} = \frac{4}{9},$$

which is finite and non-zero, the function has a simple pole at $z = -2$ and $\text{Res } f(-2) = 4/9$.

Also since $\lim_{z \rightarrow 1} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \text{Res } f(1) = \frac{1}{1!} \left[\frac{d}{dz} [(z-1)^2 f(z)] \right]_{z=1} = \left[\frac{d}{dz} \left(\frac{z^2}{z+2} \right) \right]_{z=1} = \left[\frac{z^2 + 4z}{(z+2)^2} \right]_{z=1} = \frac{5}{9}.$$

[Otherwise writing $z = 1 + t$,

$$\begin{aligned} f(z) &= \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2} (1+t)^2 (1+t/3)^{-1} = \frac{1}{3t^2} (1+t)^2 \left(1 - \frac{t}{3} + \frac{t^2}{9} - \dots \right) \\ &= \frac{1}{3t^2} \left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \dots \right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \dots \end{aligned} \quad \dots(i)$$

$$\therefore \text{Res } f(1) = \text{coefficient of } \frac{1}{t} \text{ in (i)} = \frac{5}{9}.$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z) dz = 2\pi i [\text{Res } f(-2) + \text{Res } f(1)] = 2\pi i \left[\frac{4}{9} + \frac{5}{9} \right] = 2\pi i.$$

Example 20.34. Find the residue of $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z) dz$

where C is the circle $|z| = 2.5$. (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res } f(1) = \frac{1}{3!} \frac{d^3}{dz^3} \left\{ (z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)} \right\}_{z=1} = \frac{1}{6} \frac{d^3}{dz^3} \left\{ \frac{z^3}{(z-2)(z-3)} \right\}_{z=1}$$

$\therefore z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\begin{aligned} \text{Res } f(1) &= \frac{1}{3!} \frac{d^3}{dz^3} \left\{ (z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)} \right\}_{z=1} = \frac{1}{6} \frac{d^3}{dz^3} \left\{ \frac{z^3}{(z-2)(z-3)} \right\}_{z=1} \\ &= \frac{1}{6} \frac{d^3}{dz^3} \left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3} \right] = \frac{1}{6} \left[-8 \cdot \frac{(-1)^3 3!}{(z-2)^4} + \frac{27 \cdot (-1)^3 3!}{(z-3)^4} \right]_{z=1} \\ &= - \left[-8 + \frac{27}{16} \right] = \frac{101}{16}. \end{aligned}$$

$$\operatorname{Res} f(2) = \lim_{z \rightarrow 2} \left\{ (z-2) \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)} \right\} = \lim_{z \rightarrow 2} \left\{ \frac{z^3}{(z-1)^4(z-3)} \right\} = \frac{8}{(1)^4(-1)} = -8$$

$$\operatorname{Res} f(3) = \lim_{z \rightarrow 3} \left\{ (z-3) \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)} \right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $\oint_C f(z) dz = 2\pi i [\operatorname{Res} f(1) + \operatorname{Res} f(2)]$ [∴ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8 \right) = \frac{-27\pi i}{8}.$$

Example 20.35. Evaluate

$$\oint_C \frac{z-3}{z^2+2z+5} dz, \text{ where } C \text{ is the circle}$$

$$(i) |z| = 1, \quad (ii) |z+1-i| = 2, \quad (iii) |z+1+i| = 2. \quad (\text{J.N.T.U., 2003})$$

Solution. The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by $z^2+2z+5=0$

i.e., by
$$z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i.$$

(i) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C .

Hence by Cauchy's theorem, $\oint_C \frac{z-3}{z^2+2z+5} dz = 0$.

(ii) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z+1-i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\begin{aligned} \therefore \operatorname{Res} f(-1+2i) &= \lim_{z \rightarrow -1+2i} [(z - (-1+2i)) f(z)] = \lim_{z \rightarrow -1+2i} \frac{(z+1-2i)(z-3)}{z^2+2z+5} \\ &= \lim_{z \rightarrow -1+2i} \frac{z-3}{z+1+2i} = \frac{-4+2i}{4i} = i+1/2. \end{aligned}$$

Hence by residue theorem $\oint_C f(z) dz = 2\pi i \operatorname{Res} f(-1+2i) = 2\pi i(i+1/2) = \pi(i+2)$.

(iii) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z+1+i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\begin{aligned} \therefore \operatorname{Res} f(-1-2i) &= \lim_{z \rightarrow -1-2i} \frac{(z+1+2i)(z-3)}{z^2+2z+5} \\ &= \lim_{z \rightarrow -1-2i} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \frac{1}{2} - i \end{aligned}$$

Hence by residue theorem, $\oint_C f(z) dz = 2\pi i \operatorname{Res} f(-1-2i) = 2\pi i(\frac{1}{2} - i) = \pi(2+i)$.

Example 20.36. Evaluate $\oint_C \frac{e^z}{\cos \pi z} dz$, where C is the unit circle $|z| = 1$. (Rohtak, 2006)

Solution. $f(z) = e^z/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \dots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \operatorname{Res} f(1/2) = \lim_{z \rightarrow 1/2} \left[\left(z - \frac{1}{2} \right) f(z) \right] = \lim_{z \rightarrow 1/2} \left\{ \frac{\left(z - \frac{1}{2} \right) e^z}{\cos \pi z} \right\} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \operatorname{Lt}_{z \rightarrow 1/2} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and

$$\begin{aligned} \operatorname{Res} f(-1/2) &= \operatorname{Lt}_{z \rightarrow -1/2} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \\ &= \operatorname{Lt}_{z \rightarrow -1/2} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi} \end{aligned} \quad \left[\begin{matrix} 0 \\ 0 \end{matrix} \text{ form} \right]$$

$$\begin{aligned} \text{Hence } \oint_C \frac{e^z}{\cos \pi z} dz &= 2\pi i \left(\operatorname{Res} f\left(\frac{1}{2}\right) + \operatorname{Res} f\left(-\frac{1}{2}\right) \right) \\ &= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2}. \end{aligned}$$

Example 20.37. Evaluate $\oint_C \tan z dz$ where C is the circle $|z| = 2$.

(V.T.U., 2010 S)

Solution. The poles of $f(z) = \sin z / \cos z$ are given by $\cos z = 0$ i.e. $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$

Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$$\therefore \operatorname{Res} f(\pi/2) = \operatorname{Lt}_{z \rightarrow \pi/2} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \operatorname{Lt}_{z \rightarrow \pi/2} \left(\frac{\sin z}{-\sin z} \right) = -1 \quad [\text{By } \S 20.19 (2)]$$

$$\text{Similarly } \operatorname{Res} f(-\pi/2) = \operatorname{Lt}_{z \rightarrow -\pi/2} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i \{ \operatorname{Res} f(\pi/2) + \operatorname{Res} f(-\pi/2) \} = 2\pi i (-1 - 1) = -4\pi i.$$

Example 20.38. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$, where C is the circle $|z| = 3$.

(V.T.U., 2010; Anna, 2003 S; U.P.T.U., 2002)

$$\text{Solution. } f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\begin{aligned} \therefore \operatorname{Res} f(1) &= \frac{1}{1!} \left[\frac{d}{dz} [(z-1)^2 f(z)] \right]_{z=1} = \left[\frac{d}{dz} \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} \right) \right]_{z=1} \\ &= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2} \right]_{z=1} \\ &= (-1)(-2\pi) - (-1) = 2\pi + 1 \end{aligned}$$

$$\text{Also } \operatorname{Res} f(2) = \operatorname{Lt}_{z \rightarrow 2} [(z-2)f(z)] = \operatorname{Lt}_{z \rightarrow 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i [\operatorname{Res} f(1) + \operatorname{Res} f(2)] = 2\pi i (2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2(z-i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (i) $ze^z/(z-1)^3$ at its pole. (J.N.T.U., 2003)
(ii) $z^2/(z^2+a^2)$ at $z=ai$. (P.T.U., 2009 S)
3. Determine the poles of the following functions and the residue at each pole :
 (i) $\frac{z^2+1}{z^2-2z}$ (ii) $\frac{z^2-2z}{(z+1)^2(z^2+1)}$ (J.N.T.U., 2005) (iii) $\frac{2z+4}{(z+1)(z^2+1)}$ (J.N.T.U., 2006)
4. Find the residues of the following functions at each pole.
 (i) $(1-e^{2z})/z^4$ (ii) $ze^{iz}/(z^2+1)$ (P.T.U., 2010) (iii) $\cot z$.
5. $\oint_C \frac{z^2+4}{(z-2)(z+3)} dz$, where C is (i) $|z+1|=2$ (ii) $|z-2|=2$. (Mumbai, 2006)
6. Evaluate the following integrals :
 (i) $\oint_C \frac{e^{2z} dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z|=3$. (V.T.U., 2009)
 (ii) $\oint_C \frac{4z^2-4z+1}{(z-2)(4+z^2)} dz$, $C: |z|=1$
 (iii) $\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz$, $C: |z|=2$. (U.P.T.U., 2004)
7. Evaluate
 (i) $\int_C \frac{2z+1}{(2z-1)^2} dz$, where C is $|z|=1$ (ii) $\oint_C \frac{z+4}{z^2+2z+5} dz$, where C is $|z+1-i|=2$
 (iii) $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$, where C is the circle $|z|=10$. (U.P.T.U., 2009)
8. Evaluate :
 (i) $\oint_C \frac{z dz}{(z-1)(z-2)^2}$, $C: |z-2|=\frac{1}{2}$. (Madras, 2006)
 (ii) $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)} dz$, $C: |z-2|=2$. (Rohtak, 2005)
 (iii) $\oint_C \frac{dz}{(z^2+4)^2}$, $C: |z-i|=2$. (Hissar, 2007; Anna, 2003 S; Osmania, 2003)
9. Evaluate :
 (i) $\oint_C \frac{e^{-z}}{z^2} dz$, $C: |z|=1$. (ii) $\oint_C z^2 e^{1/z} dz$, $C: |z|=1$,
 (iii) $\oint_C \frac{e^z dz}{z^2+4}$, $C: |z-i|=2$. (V.T.U., 2006) (iv) $\oint_C \frac{e^{2z} dz}{(z+1)^4}$, $C: |z|=2$.
10. Evaluate the following integrals : (i) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3} dz$, $C: |z|=1$
 (ii) $\oint_C \frac{z \sec z}{(1-z)^2} dz$, $C: |z|=3$ (iii) $\oint_C \frac{z \cos z}{(z-\pi/2)^3} dz$, $C: |z-1|=1$. (V.T.U., 2007)
11. Evaluate $\oint_C \frac{dz}{\sinh 2z}$ where C is the circle $|z|=2$. (Marathwada, 2008)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate
 $\oint_C \frac{z}{z^2 \sinh z} dz$, where C is the circle $|z-1|=2$. (J.N.T.U., 2005)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(a) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin \theta, \cos \theta) d\theta$, where the integrand is a rational function of $\sin \theta$ and $\cos \theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$ and $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$, then integral takes the form $\int_C f(z) dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence the integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which are within C .

Example 20.39. Show that

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1 - 2a \cos \theta + a^2} = \frac{2\pi a^2}{1 - a^2}, \quad (a^2 < 1). \quad (\text{Bhopal, 2009; Rohtak, 2003})$$

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos \theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

\therefore the given integral

$$\begin{aligned} I &= \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i} \int_C \frac{(z^4 + 1) dz}{z^2(z - az^2 - a + a^2z)} \\ &= \frac{1}{2i} \int_C \frac{(z^4 + 1) dz}{z^2(z - a)(1 - az)} = \int_C f(z) dz \quad \text{where } C \text{ is the unit circle } |z| = 1. \end{aligned}$$

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$$\therefore \text{Res } f(a) = \lim_{z \rightarrow a} [(z - a)f(z)] = \frac{1}{2i} \lim_{z \rightarrow a} \left[\frac{z^4 + 1}{z^2(1 - az)} \right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

$$\begin{aligned} \text{and } \text{Res } f(0) &= \lim_{z \rightarrow 0} \frac{d}{dz} [z^2 f(z)] = \frac{1}{2i} \lim_{z \rightarrow 0} \frac{d}{dz} \left[\frac{z^4 + 1}{z - az^2 - a + a^2z} \right] \\ &= \frac{1}{2i} \lim_{z \rightarrow 0} \frac{(z - az^2 - a + a^2z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2z)^2} = -\frac{1 + a^2}{2ia^2} \end{aligned}$$

$$\text{Hence } I = 2\pi i [\text{Res } f(a) + \text{Res } f(0)] = 2\pi i \left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2} \right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. By integrating around a unit circle, evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4 \cos \theta} d\theta$.

(S.V.T.U., 2009; U.P.T.U., 2009; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos \theta = \frac{1}{2}(z + 1/z)$

$$\text{and } \cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$$\begin{aligned} \therefore \text{the given integral } I &= \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz} \\ &= -\frac{1}{2i} \int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} dz = -\frac{1}{2i} \int_C \frac{(z^6 + 1) dz}{z^3(2z - 1)(z - 2)} \end{aligned}$$

$$= -\frac{1}{2i} \int_C f(z) dz, \quad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\begin{aligned}\therefore \operatorname{Res} f(1/2) &= \lim_{z \rightarrow 1/2} \frac{(z - 1/2)(z^6 + 1)}{(2z - 1)(z - 2)} = \lim_{z \rightarrow 1/2} \left\{ \frac{z^6 + 1}{2z^3(z - 2)} \right\} = -\frac{65}{24} \\ \operatorname{Res} f(0) &= \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [(z-0)^n f(z)] \right\}_{z=0} \quad \text{where } n = 3 \\ &= \frac{1}{2} \left\{ \frac{d^2}{dz^2} \left(\frac{z^6 + 1}{2z^2 - 5z + 2} \right) \right\}_{z=0} = \frac{d}{dz} \left[\frac{(2z^2 - 5z + 2)6z^5 - (z^6 + 1)(4z - 5)}{2(2z^2 - 5z + 2)^2} \right] \text{ at } z = 0 \\ &= \left\{ \frac{d}{dz} \left[\frac{8z^7 - 25z^6 + 12z^5 - 4z + 5}{2(2z^2 - 5z + 2)^2} \right] \right\}_{z=0} \\ &= \left[\frac{(2z^2 - 5z + 2)^2 (56z^6 - 150z^5 + 60z^4 - 4) - (8z^7 - 25z^6 + 12z^5 - 4z + 5)2(2z^2 - 5z + 2)(4z - 5)}{2(2z^2 - 5z + 2)^4} \right]_{z=0} \\ &= \frac{4(-4) - 5(-20)}{2 \times 16} = \frac{84}{32} = \frac{21}{8}\end{aligned}$$

$$\text{Hence } I = \frac{-1}{2i} (2\pi i [\operatorname{Res} f(1/2) + \operatorname{Res} f(0)]) = -\pi \left[-\frac{65}{24} + \frac{21}{8} \right] = -\pi \left(-\frac{1}{12} \right) = \frac{\pi}{12}.$$

(b) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x) dx$, we consider $\int_C f(z) dz$, where C is

the contour consisting of the semi-circle $C_R : |z| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z) dz + \int_{-R}^R f(x) dx = 2\pi i \sum \operatorname{Res} f(a).$$

Finally making R tend to ∞ , we find the value of $\int_{-\infty}^{\infty} f(x) dx$, provided $\int_{C_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

Example 20.41. Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$. (U.P.T.U., 2008)

Solution. Consider $\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = \int_C f(z) dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = \pm i, z = \pm 2i$ of which $z = i, 2i$ only lie inside C .

\therefore by the Residue theorem,

$$\begin{aligned}\int_C f(z) dz &= 2\pi i [\operatorname{Res} f(i) + \operatorname{Res} f(2i)] \\ &= 2\pi i [\lim_{z \rightarrow i} (z-i)f(z) + \lim_{z \rightarrow 2i} (z-2i)f(z)] \\ &= 2\pi i \left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)} \right] = 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = \frac{\pi}{3}\end{aligned} \quad \dots(ii)$$

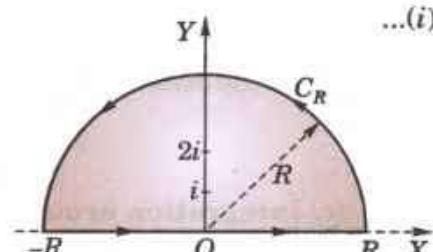


Fig. 20.23

Also $\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz$... (iii)

Now let $R \rightarrow \infty$, so as to show that the second integral in (iii) vanishes. For any point on C_R as $|z| \rightarrow \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z .

Consequently, $\lim_{|z| \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

Hence from (i), (ii) and (iii), we get $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{3}$.

Example 20.42. Evaluate $\int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx$.

(U.P.T.U., 2006; Delhi, 2002)

Solution. Consider $\int_C \frac{e^{iaz}}{z^2 + 1} dz = \int_C f(z) dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ and $z = -i$, of which $z = i$ only lies inside C .

$$\begin{aligned} \therefore \text{by Residue theorem, } \int_C f(z) dz &= 2\pi i \operatorname{Res} f(i) = 2\pi i \lim_{z \rightarrow i} [(z - i) f(z)] \\ &= 2\pi i \lim_{z \rightarrow i} \frac{(z - i) e^{iaz}}{z^2 + 1} = 2\pi i \lim_{z \rightarrow i} \frac{e^{iaz}}{z + i} = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a} \end{aligned} \quad \dots(i)$$

Also $\int_C f(z) dz = \int_{-R}^R f(x) dx + \int_{C_R} f(z) dz$... (ii)

Now $|z| = R$ on C_R and $|z^2 + 1| \geq R^2 - 1$.

Also $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1$ $[\because y > 0]$

$$\therefore \left| \frac{e^{iaz}}{z^2 + 1} \right| = |e^{iaz}| \cdot \frac{1}{|z^2 + 1|} < 1 \cdot \frac{1}{R^2 - 1}$$

Thus $\int_{C_R} f(z) dz = \left| \int_{C_R} \frac{e^{iaz}}{z^2 + 1} dz \right| < \int_{C_R} \frac{1}{R^2 - 1} |dz| < \frac{\pi R}{R^2 - 1}$ which $\rightarrow 0$ as $R \rightarrow \infty$ (iii)

Hence from (i), (ii) and (iii), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x) dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 1} dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + 1} dx = \pi e^{-a}$$

Since $\cos ax/(x^2 + 1)$ is an even function of x , we have

$$2 \int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx = \frac{\pi}{2} e^{-a}.$$

(c) Integration around rectangular contours

Example 20.43. Evaluate $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx$.

Solution. Consider $\int_C \frac{e^{az}}{e^z + 1} dz = \int_C f(z) dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$,

$(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $z = (2n+1)\pi i$, where $n = 0, \pm 1, \pm 2, \pm 3, \dots$

The only pole inside the rectangle is $z = \pi i$.

∴ by Residue theorem,

$$\begin{aligned} \int_C f(z) dz &= 2\pi i \operatorname{Res} f(\pi i) \\ &= 2\pi i \left[e^{az} / \frac{d}{dz} (e^z + 1) \right]_{z=\pi i} \\ &= 2\pi i e^{\pi i a} / e^{\pi i} = -2\pi i e^{\pi i a} \quad [\because e^{\pi i} = -1] \end{aligned} \quad \dots(i)$$

Also $\int_C f(z) dz = \int_{AB} f(z) dz + \int_{BC} f(z) dz + \int_{CD} f(z) dz + \int_{DA} f(z) dz$

$$= \int_0^{2\pi} f(R+iy) idy + \int_R^{-R} f(x+2\pi i) dx + \int_{2\pi}^0 f(-R+iy) idy + \int_{-R}^R f(x) dx$$

[∴ $z = R+iy$ along AB , $z = x+2\pi i$ along BC , $z = -R+iy$ along CD and $z = x$ along DA .]

or $\int_C f(z) dz = i \int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy} + 1} dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i} + 1} dx - i \int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy} + 1} dy + \int_{-R}^R \frac{e^{ax}}{e^x + 1} dx \quad \dots(ii)$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $|e^{R+iy} + 1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

∴ for the integrand of first integral in (ii), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy} + 1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which } \rightarrow 0 \text{ as } R \rightarrow \infty. \quad [\because a > 1]$$

Similarly, for the integrand of the third integral in (ii), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy} + 1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also } \rightarrow 0 \text{ as } R \rightarrow \infty. \quad [\because a < 0]$$

Hence as $R \rightarrow \infty$, since the first and third integrals in (ii) approach zero, we get

$$\int_C f(z) dz = -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = (1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx \quad \dots(iii)$$

Thus from (i) and (iii), we obtain $(1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = -2\pi i e^{\pi a i}$

∴ equating real parts, we get $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1} dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. Show that $\int_0^{\infty} e^{-x^2} \cos 2mx dx = \frac{1}{2} \sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l), B(l), C(l+im), D(-l+im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z) dz + \int_{BC} f(z) dz + \int_{CD} f(z) dz + \int_{DA} f(z) dz = 0 \quad \dots(i)$$

On $AB : z = x$, on $BC : z = l+iy$, on $CD : z = x+im$ and on $DA : z = -l+iy$.

Therefore, (i) becomes

$$\int_{-l}^l e^{-x^2} dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2} dx + \int_m^0 e^{-(l+iy)^2} dy = 0$$

$$\int_{-l}^l e^{-x^2} dx - \int_{-l}^l e^{-x^2 - 2imx + m^2} dx + \int_0^m e^{-l^2 - 2ily + y^2} . idy$$

$$- \int_0^m e^{-l^2 + 2ily + y^2} . idy = 0 \quad \dots(ii)$$

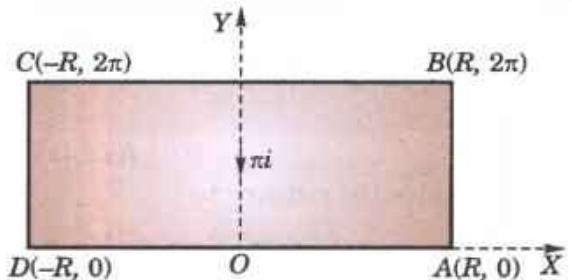


Fig. 20.24

Now let $l \rightarrow \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily}) dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly dy \rightarrow 0$$

[\because As $l \rightarrow \infty$, $e^{-l^2} \rightarrow 0$ and $\sin 2ly$ is finite]

Hence (ii) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

[See p. 289]

or

$$\int_0^{\infty} e^{-x^2} \cos 2mx dx = \frac{1}{2} \sqrt{\pi} e^{-m^2}$$

(d) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (i.e., by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. Evaluate $\int_0^{\infty} \frac{\sin mx}{x} dx$, when $m > 0$.

(U.P.T.U., 2007)

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

- (i) the real axis from r to R ,
- (ii) the upper half of the circle C_R : $|z| = R$,
- (iii) the real axis $-R$ to $-r$,
- (iv) the upper half of the circle C_r : $|z| = r$ (Fig. 20.26).

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x) dx + \int_{C_R} f(z) dz + \int_{-R}^{-r} f(x) dx + \int_{C_r} f(z) dz = 0 \quad \dots(i)$$

$$\text{Now } \int_{C_R} f(z) dz = \int_0^\pi \frac{e^{imR(\cos \theta + i \sin \theta)}}{Re^{i\theta}} \cdot Rie^{i\theta} d\theta \\ = i \int_0^\pi e^{imR(\cos \theta + i \sin \theta)} d\theta$$

[$\because z = Re^{i\theta}$]

Since $|e^{imR(\cos \theta + i \sin \theta)}| = |e^{-mR \sin \theta} + imR \cos \theta| = e^{-mR \sin \theta}$

$$\therefore \left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi e^{-mR \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-mR \sin \theta} d\theta \\ = 2 \int_0^{\pi/2} e^{-2mR \theta/\pi} d\theta \quad [\because \text{for } 0 \leq \theta \leq \pi/2, \sin \theta/\theta \geq 2/\pi] \\ = \frac{\pi}{mR} (1 - e^{-mR}) \text{ which } \rightarrow 0 \text{ as } R \rightarrow \infty,$$

Also $\int_{C_r} f(z) dz = i \int_{\pi}^0 e^{imr(\cos \theta + i \sin \theta)} d\theta \rightarrow i \int_{\pi}^0 d\theta \text{ i.e., } -i\pi \text{ as } r \rightarrow 0$.

Hence as $r \rightarrow 0$ and $R \rightarrow \infty$, we get from (i) $\int_0^{\infty} f(x) dx + 0 + \int_{-\infty}^0 f(x) dx - i\pi = 0$

or

$$\int_{-\infty}^{\infty} f(x) dx = i\pi \text{ i.e., } \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \quad \dots(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

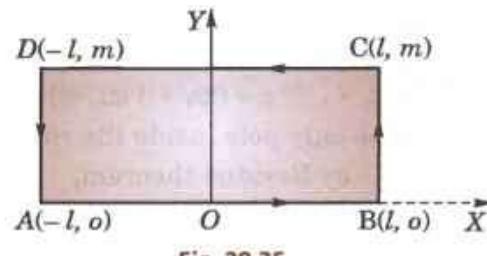


Fig. 20.25

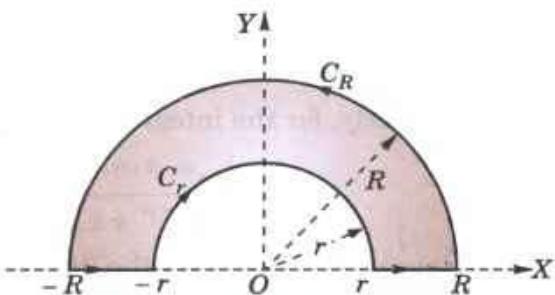


Fig. 20.26

Obs. Equating real parts from both sides of (ii), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. Show that $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x -axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$$\therefore \text{Res } f(-1) = \lim_{z \rightarrow -1} (1+z) \cdot \frac{z^{p-1}}{1+z} = \lim_{z \rightarrow -1} z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$$

$$\text{Thus } \int_{AB} f(z) dz + \int_{\gamma} f(z) dz + \int_{FG} f(z) dz + \int_{\alpha} f(z) dz = 2\pi i e^{i\pi(p-1)} \quad \dots(i)$$

On AB : $z = x$ and on FG : $z = xe^{2\pi i}$

$$\begin{aligned} \therefore \int_{AB} f(z) dz + \int_{FG} f(z) dz &= \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx e^{2\pi i} \\ &= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}] dx \end{aligned}$$

On the circle γ : $z = Re^{i\theta}$. So

$$\int_{\gamma} f(z) dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta} i d\theta$$

For large R , the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ i.e.

R^{p-1} which tends to zero as $R \rightarrow \infty$. $(\because p < 1)$

Hence $\int_{\gamma} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$

On the circle α : $z = ae^{i\theta}$. So

$$\int_{\alpha} f(z) dz = \int_{2\pi}^0 \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta} id\theta$$

For small a , the integrand is of the order a^p which tends to zero as $a \rightarrow 0$. ($\because p > 0$)

Thus on taking limits as $a \rightarrow 0$ and $R \rightarrow \infty$, (i) gives

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}] dx = 2\pi i e^{i\pi(p-1)}$$

$$\text{or } \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{2\pi i e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i e^{ip\pi} (-1)}{1 - e^{2ip\pi} (1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. Prove that $\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)}$. (Osmania, 2003)

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A , part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

e^{-z^2} has no singularity within C .

$$\therefore \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0 \quad \dots(i)$$

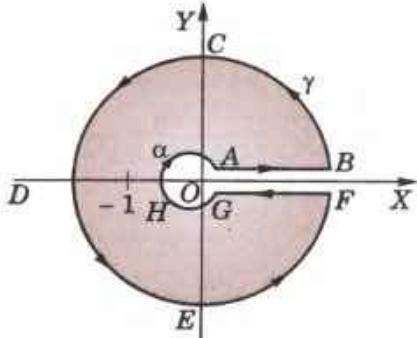


Fig. 20.27

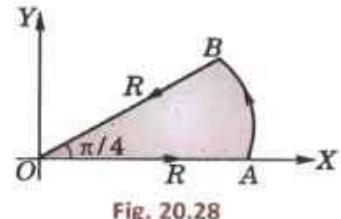


Fig. 20.28

On $OA : z = x$, $\therefore \int_{OA} e^{-z^2} dz = \int_0^R e^{-x^2} dx \rightarrow \sqrt{\pi/2}$ as $R \rightarrow \infty$

[See p. 289]

On $AB : z = Re^{i\theta}$,

$$\therefore \int_{AB} e^{-z^2} dz = \int_0^{\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} \cdot Re^{i\theta} \cdot id\theta \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

[\because integrand $\rightarrow 0$ as $R \rightarrow \infty$]

On $BO : z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$$\therefore \int_{BO} e^{-z^2} dz = \int_R^0 e^{-ir^2} \cdot e^{i\pi/4} dr = - \int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx \\ \rightarrow - \int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \quad \text{when } R \rightarrow \infty$$

Substituting these in (i), we get

$$\frac{1}{2} \sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2) \left(\frac{1+i}{\sqrt{2}} \right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2} \sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)}$.

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin \theta + p^2} = \frac{2\pi}{1 - p^2}$ ($0 < p < 1$). (Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005)
2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r \cos \theta + r^2} = \frac{\pi}{1 - r^2}$. (J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003)
3. $\int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \frac{2\pi}{\sqrt{a^2 - 1}}$ ($a > 1$). (P.T.U., 2010)
4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4 \cos \theta} d\theta = \frac{\pi}{6}$. (U.P.T.U., 2010)
5. $\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{a + b \cos \theta} = \frac{2\pi}{b^2} [a - \sqrt{(a^2 - b^2)}]$, ($0 < b < a$). (J.N.T.U., 2003)
6. $\int_0^{2\pi} \frac{ad\theta}{a^2 + \sin^2 \theta} = \frac{2\pi}{\sqrt{1+a^2}}$, ($a > 0$). (S.V.T.U., 2009)
7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3 \cos \theta)^2} = \frac{5\pi}{32}$.
8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a+b}$ ($a, b > 0$). (P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003)
9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. (A.M.I.E.T.E., 2003 ; Delhi, 2002)
10. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$. (J.N.T.U., 2006)
11. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. (Madras, 2006 ; Kerala, 2005)
12. $\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}$. (Kerala, 2005)
13. $\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}$. (Rohtak, 2006)
14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \operatorname{sech} \frac{m\pi}{2}$. (P.T.U., 2005)
15. $\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \pi/2$. (P.T.U., 2005)
16. $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. (Kerala, 2005)
17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.
18. $\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right)$ ($a > b > 0$).
19. $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions:

1. The only function that is analytic from the following is
 (i) $f(z) = \sin z$ (ii) $f(z) = \bar{z}$ (iii) $f(z) = \operatorname{Im}(z)$ (iv) $R(iz)$.
2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
 (i) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (ii) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (iii) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$.
 3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
 (a) 1 (b) 2 (c) 3 (d) 0.
4. The analytic function which maps the angular region $0 \leq \theta \leq \pi/4$ onto the upper half plane is
 (i) z^2 (ii) $4z$ (iii) z^4 (iv) 20 .
5. An angular domain in the complex plane is defined by $0 < \operatorname{arg}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
 (i) $w = z^4$ (ii) $w = iz^4$ (iii) $w = -z^4$ (iv) $w = -iz^4$.
6. The mapping $w = z^2 - 2z - 3$ is
 (i) conformal within $|z| = 1$ (ii) not conformal at $z = 1$
 (iii) not conformal at $z = -1$ and $z = 3$ (iv) conformal everywhere.
7. If $z = re^{i\theta}$, then the image of $\theta = \text{constant}$ under the mapping $w(z) = Re^{i\phi} = iz^3$ is
 (i) $\phi = 30^\circ$ (ii) $\phi = 3\theta + \pi/2$ (iii) $\phi = 3\theta - \pi/2$ (iv) $\phi = \theta^3$.
8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
 (i) 2, 2 (ii) 2, -2 (iii) -2, -2 (iv) $-4/5, 5$.
9. The value of $\int_C (4x^3 dx + 3y^2 z^2 dy + 2y^3 zdz)$ where C is any path joining A (-1, 1, 0) to B (1, 2, 1) is
 (i) 0 (ii) 1 (iii) 8 (iv) -8.
10. The value of $\int_C \frac{3z^2 + 7z + 1}{z + 1} dz$ where C is $|z| = 1/2$ is
 (i) $2\pi i$ (ii) 0 (iii) πi (iv) $\pi i/2$.
11. The value of $\int_C \frac{3z + 4}{z(2z + 1)} dz$ where C is the circle $|z| = 1$ is
 (i) $2\pi i$ (ii) $3\pi i$ (iii) 4 (iv) -4.
12. The residue of a function can be found if the pole is an isolated singularity :
 (i) True (ii) False (iii) Partially false (iv) none of these.
13. The value of $\int_C \frac{z dz}{\sin z}$ where $C : |z| = 4$ is
 (i) $2\pi i$ (ii) 0 (iii) $-2\pi i$ (iv) $4\pi i$.
14. The value of $\int_C \tanh z dz$, where $C : |z| = 3$, is
 (i) 0 (ii) πi (iii) $2\pi i$ (iv) $4\pi i$.
15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is (U.P.T.U., 2009)
16. Harmonic conjugate of $x^3 - 3xy^2$ is
 17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if
 18. The value of $\int_0^{1+i} z^2 dz$ along the line $x = y$ is 19. Residue of $\frac{\cos z}{z}$ at $z = 0$ is
 20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is
 21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is
 22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are $z =$ 23. $w = \log z$ is analytic everywhere except at $z =$
 24. If $f(z) = -\frac{1}{z-1} - 2[1 + (z-1) + (z-1)^2 + \dots]$, then the residue of $f(z)$ at $z = 1$ is
 25. If $|z| < 1$ then Taylor's series expansion of $\log(1+z)$ about $z = 0$ is

26. The value of $\int_C \frac{4z^2 + z + 5}{z - 4} dz$ where C is $9x^2 + 4y^2 = 36$, is
27. The value of $\int z^4 e^{1/z} dz$, where C is $|z| = 1$, is
 (i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.
28. If $f(z)$ has a pole of order three at $z = a$ $\text{Res}[f(z)] = \dots$
29. The value of $\int_C \frac{e^z dz}{(z - 3)^2}$, C being $|z| = 2$, is
30. The CR equations for $f(z) = u(x, y) + iv(x, y)$ to be analytic are
31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z) dz = \dots$
32. The harmonic conjugate of $e^x \cos y$ is
33. The value of $\oint_C \cos z dz$ where C is the circle $|z| = 1$, is
34. The singularity of $f(z) = z/(z - 2)^3$ is
35. The function $f(z) = \bar{z}$ is analytic at
36. C-R equations for a function to be analytic, in polar form, are
37. If C is the circle $|z - a| = r$, $\int_C (z - a)^n dz$ [n , any integer $\neq -1$] =
38. A simply connected region is that
39. A holomorphic function is that
40. The poles of the function $f(z) = \frac{z^2}{(z - 1)^2(z + 2)}$ are at $z = \dots$
41. The cross-ratio of four points z_1, z_2, z_3, z_4 is
42. The value of $\int_C |z| dz$, where C is the contour represented by the straight line from $z = -i$ to $z = i$, is
43. Taylor's series expansion of $\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$ in the region $|z| < 1$, is
44. The invariant points of the transformation $w = (1+z)/(1-z)$ are $z = \dots$
45. The residue at $z = 0$ of $\frac{1+e^z}{z \cos z + \sin z}$ is
46. The transformation $w = Cz$ consists of
47. The residue of $f(z)$ at a pole is
48. The value of $\int_C \frac{1}{z-1} dz$, C being $|z| = 2$, is
49. If C is $|z| = 1/2$, $\int_C \frac{z^2 - z + 1}{z-1} dz = \dots$
50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are
51. Taylor series expansion of $\frac{1}{z-2}$ in $|z| < 1$ is
52. $\lim_{z \rightarrow \infty} \frac{iz^3 + iz - 1}{(2z+3)(z-1)^2} = \dots$ (P.T.U., 2007)
53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z = \dots$
54. Cauchy's integral theorem states that
55. The critical points of the transformation $w = z + 1/z$ are
56. $\int_C \frac{dz}{2z-3}$, where $|z| = 1$, is
57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are
58. Residue of $\tan z$ at $z = \pi/2$ is
59. Singularity of $e^{z^{-1}}$ at $z = 0$ is of the type
60. $\text{Res}(e^{1/z})_{z=0} = \dots$
61. Taylor's series expansion of $\sin z$ about $z = \pi/4$ is
62. Image of $|z| = 2$ under $w = z + 3 + 2i$ is
63. The poles of $\cot z$ are
64. If a is simple pole, then $\text{Res}[\phi(z)/\psi(z)]_{z=a} = \dots$
65. Bilinear transformation always transforms circles into
66. If $f(z)$ and $\bar{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (Mumbai, 2006)
67. The function $u(x, y) = 2xy + 3xy^2 - 2y^3$ is a harmonic functions. (True or False) (P.T.U., 2009 S)
68. The function $e^x \cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, if $z=a$ is a point within C . (True or False)
70. The transformation affected by an analytic function $w=f(z)$ is conformal at every point of the z -plane where $f'(z) \neq 0$. (True or False)
71. The function \bar{z} is not analytic at any point. (True or False)
72. Under the transformation $w=1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w -plane. (True or False)
73. If $w=f(z)$ is analytic, then $\frac{dw}{dz} = -i \frac{\partial w}{\partial y}$. (True or False)
74. An analytic function with constant imaginary part is constant. (True or False)
75. If $u+iv$ is analytic, then $v-iu$ is also analytic. (True or False)
76. $f(z) = I_m(z)$ is not analytic. (True or False)
77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)
78. $z=0$ is not a critical point of the mapping $w=z^2$. (True or False)
79. $f(z) = \operatorname{Re}(z^2)$ is analytic. (True or False)
80. An analytic function with constant modulus is constant. (True or False)
81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)
82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)

Laplace Transforms

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21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

*Let $f(t)$ be a function of t defined for all positive values of t . Then the **Laplace transforms** of $f(t)$, denoted by $L\{f(t)\}$ is defined by*

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad \dots(1)$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ i.e., $L\{f(t)\} = \bar{f}(s)$, which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L , which transforms $f(t)$ into $\bar{f}(s)$, is called the **Laplace transformation operator**.

*Pierre de Laplace (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ i.e., $\int_0^\infty e^{-st} f(t) dt$ exists for $s > a$, if

(i) $f(t)$ is continuous

(iii) $\lim_{t \rightarrow \infty} \{e^{-at} f(t)\}$ is finite.

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

$$(1) L(1) = \frac{1}{s} \quad (s > 0)$$

$$(2) L(t^n) = \frac{n!}{s^{n+1}}, \text{ when } n = 0, 1, 2, 3, \dots \quad \left[\text{Otherwise } \frac{\Gamma(n+1)}{s^{n+1}} \right]$$

$$(3) L(e^{at}) = \frac{1}{s-a} \quad (s > a)$$

$$(4) L(\sin at) = \frac{a}{s^2 + a^2} \quad (s > 0)$$

$$(5) L(\cos at) = \frac{s}{s^2 + a^2} \quad (s > 0)$$

$$(6) L(\sinh at) = \frac{a}{s^2 - a^2} \quad (s > |a|)$$

$$(7) L(\cosh at) = \frac{s}{s^2 - a^2} \quad (s > |a|)$$

Proofs. (1) $L(1) = \int_0^\infty e^{-st} \cdot 1 dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$

$$(2) L(t^n) = \int_0^\infty e^{-st} \cdot t^n dt = \int_0^\infty e^{-st} \cdot \left(\frac{p}{s} \right)^n \frac{dp}{s}, \text{ on putting } st = p \\ = \frac{1}{s^{n+1}} \int_0^\infty e^{-p} \cdot p^n dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

$$\text{In particular } L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}; L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

In n be a positive integer, $\Gamma(n+1) = n!$ [(v) p. 302],

therefore, $L(t^n) = n!/s^{n+1}.$

$$(3) L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

$$(4) L(\sin at) = \int_0^\infty e^{-st} \sin at dt = \left| \frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2 + a^2}$$

Similarly, the reader should prove (5) himself.

$$(6) L(\sinh at) = \int_0^\infty e^{-st} \sinh at dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt \\ = \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt \right] = \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{a}{s^2 - a^2} \text{ for } s > |a|.$$

Similarly, the reader should prove (7) himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. If a, b, c be any constants and f, g, h any functions of t , then

$$L[af(t) + bg(t) - ch(t)] = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st} [af(t) + bg(t) - ch(t)] dt$$

$$= a \int_0^\infty e^{-st} f(t) dt + b \int_0^\infty e^{-st} g(t) dt - c \int_0^\infty e^{-st} h(t) dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L , it is called a *linear operator*.

II. First shifting property. If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at} f(t)\} = \bar{f}(s-a).$$

$$\begin{aligned} \text{By definition, } L\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt \\ &= \int_0^\infty e^{-rt} f(t) dt, \text{ where } r = s-a = \bar{f}(r) = \bar{f}(s-a). \end{aligned}$$

Thus, if we know the transform $\bar{f}(s)$ of $f(t)$, we can write the transform of $e^{at} f(t)$ simply replacing s by $s-a$ to get $\bar{f}(s-a)$.

Application of this property leads us to the following useful results :

$$(1) L(e^{at}) = \frac{1}{s-a}$$

$$\left[\because L(1) = \frac{1}{s} \right]$$

$$(2) L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$\left[\because L(t^n) = \frac{n!}{s^{n+1}} \right]$$

$$(3) L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$\left[\because L(\sin bt) = \frac{b}{s^2 + b^2} \right]$$

$$(4) L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

$$\left[\because L(\cos bt) = \frac{s}{s^2 + b^2} \right]$$

$$(5) L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$$\left[\because L(\sinh bt) = \frac{b}{s^2 - b^2} \right]$$

$$(6) L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

$$\left[\because L(\cosh bt) = \frac{s}{s^2 - b^2} \right]$$

where in each case $s > a$.

Example 21.1. Find the Laplace transforms of

$$(i) \sin 2t \sin 3t$$

$$(ii) \cos^2 2t$$

$$(iii) \sin^3 2t.$$

Solution. (i) Since $\sin 2t \sin 3t = \frac{1}{2} [\cos t - \cos 5t]$

$$\therefore L(\sin 2t \sin 3t) = \frac{1}{2} [L(\cos t) - L(\cos 5t)] = \frac{1}{2} \left[\frac{s}{s^2 + 1^2} - \frac{s}{s^2 + 5^2} \right] = \frac{12s}{(s^2 + 1)(s^2 + 25)}$$

$$(ii) \text{Since } \cos^2 2t = \frac{1}{2} (1 + \cos 4t)$$

$$\therefore L(\cos^2 2t) = \frac{1}{2} [L(1) + L(\cos 4t)] = \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 + 16} \right)$$

(iii) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$$\begin{aligned}\therefore L(\sin^3 2t) &= \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t) \\ &= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}.\end{aligned}$$

Example 21.2. Find the Laplace transform of

(i) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$.

(ii) $e^{2t} \cos^2 t$ (V.T.U., 2006)

(iii) $\sqrt{t} e^{3t}$. (P.T.U., 2009)

Solution. (i) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s+3}{(s+3)^2 + 5^2} - 3 \cdot \frac{5}{(s+3)^2 + 5^2} = \frac{2s-9}{s^2 + 6s + 34}.$$

(ii) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 4} \right\}$

∴ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2} \left\{ \frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4} \right\}.$$

(iii) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) \cdot \Gamma\pi}{s^{3/2}}$

∴ by shifting property, we obtain $L(e^{3t} \sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s-3)^{3/2}}$.

Example 21.3. If $L f(t) = \bar{f}(s)$, show that

$$L[(\sinh at)f(t)] = \frac{1}{2} [\bar{f}(s-a) - \bar{f}(s+a)]$$

$$L[(\cosh at)f(t)] = \frac{1}{2} [\bar{f}(s-a) + \bar{f}(s+a)]$$

Hence evaluate (i) $\sinh 2t \sin 3t$ (ii) $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at)f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at})f(t)\right] = \frac{1}{2}[L\{e^{at}f(t)\} - L\{e^{-at}f(t)\}]$
 $= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$, by shifting property.

Similarly, $L[(\cosh at)f(t)] = \frac{1}{2}[L\{e^{at}f(t)\} + L\{e^{-at}f(t)\}]$
 $= \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$, by shifting property.

(i) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2} \left\{ \frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2} \right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(ii) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2} \left\{ \frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2} \right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. Show that

$$(i) L(t \sin at) = \frac{2as}{(s^2 + a^2)^2} \quad (Bhopal, 2001) \quad (ii) L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Solution. Since $L(t) = 1/s^2$. $\therefore L(te^{iat}) = \frac{1}{(s - ia)^2} = \frac{(s + ia)^2}{[(s - ia)(s + ia)]^2}$

or $L[t(\cos at + i \sin at)] = \frac{(s^2 - a^2)^2 + i(2as)}{(s^2 + a^2)^2}$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. Find the Laplace transform of $f(t)$ defined as

$$(i) f(t) = t/\tau, \text{ when } 0 < t < \tau \\ = 1, \text{ when } t > \tau. \quad (Kerala, 2005)$$

$$(ii) f(t) = \begin{cases} 1, & 0 < t \leq 1 \\ t, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases} \quad (J.N.T.U., 2006; W.B.T.U., 2005)$$

$$\text{Solution. (i)} \quad Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} \cdot 1 dt = \frac{1}{\tau} \left[\left| t \cdot \frac{e^{-st}}{-s} \right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s} dt \right] + \left| \frac{e^{-st}}{-s} \right|_\tau^\infty \\ = \frac{1}{\tau} \left[\frac{\tau e^{-s\tau} - 0}{-s} - \left| \frac{e^{-st}}{s^2} \right|_0^\tau \right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau} - 1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1 - e^{-s\tau}}{\tau s^2}.$$

$$\text{(ii)} \quad L\{f(t)\} = \int_0^1 e^{-st} \cdot 1 dt + \int_1^2 e^{-st} \cdot t dt + \int_2^\infty e^{-st} \cdot (0) dt \\ = \left| \frac{e^{-st}}{-s} \right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right|_1^2 = \frac{1 - e^{-s}}{s} + \left\{ \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} \right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} \right) \right\} \\ = \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. Find the Laplace transform of (i) $(\sqrt{t} - 1/\sqrt{t})^3$. (Kurukshetra, 2005)

$$(ii) \frac{\cos \sqrt{t}}{\sqrt{t}} \quad (Mumbai, 2009)$$

Solution. (i) Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$$\therefore L(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2}) \\ = \frac{\Gamma(3/2 + 1)}{s^{3/2+1}} - 3 \frac{\Gamma(1/2 + 1)}{s^{1/2+1}} + 3 \frac{\Gamma(-1/2 + 1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2 + 1)}{s^{-3/2+1}} \\ = \frac{\frac{3}{2} \Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3 \frac{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3 \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} \\ = \frac{3}{4} \frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2} \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \right] \\ = \frac{\sqrt{\pi}}{4} \left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}} \right).$$

(ii) We know that $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \infty$

$$\therefore \cos \sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots$$

and

$$\frac{\cos \sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \dots$$

and

$$\begin{aligned} L\left(\frac{\cos \sqrt{t}}{\sqrt{t}}\right) &= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!} \frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!} \frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!} \frac{\Gamma(7/2)}{s^{7/2}} + \dots \\ &= \frac{\Gamma(1/2)}{\sqrt{s}} - \frac{1}{2} \cdot \frac{1/2 \Gamma(1/2)}{s^{3/2}} + \frac{1}{4!} \frac{3/2 \cdot 1/2 \cdot \Gamma(1/2)}{s^{5/2}} - \frac{1}{6!} \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)}{s^{7/2}} + \dots \\ &= \sqrt{\left(\frac{\pi}{2}\right)} \left[1 - \frac{1}{(4s)} + \frac{1}{2!} \frac{1}{(4s)^2} - \frac{1}{3!} \frac{1}{(4s)^3} \dots \right] = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4s}. \end{aligned}$$

Example 21.7. Find the Laplace transform of the function

(i) $f(t) = |t-1| + |t+1|, t \geq 0$

(S.V.T.U., 2009)

(ii) $f(t) = [t]$, where $[]$ stands for the greatest integer function.

(P.T.U., 2010)

Solution. (i) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ 2t, & t \geq 1 \end{cases}$$

$$\begin{aligned} \therefore L[f(t)] &= \int_0^1 e^{-st}(2) dt + \int_1^\infty e^{-st}(2t) dt = 2 \left[\left| \frac{e^{-st}}{-s} \right|_0^1 + 2 \left| \frac{t \cdot e^{-st}}{-s} \right|_1^\infty - \left| \frac{e^{-st}}{(-s)^2} \right|_1^\infty \right] \\ &= 2 \left(\frac{e^{-s}}{-s} + \frac{1}{s} \right) + 2 \left(\frac{0 - e^{-s}}{-s} - \frac{0 - e^{-s}}{s^2} \right) = \frac{2}{s} \left(1 + \frac{e^{-s}}{s} \right) \end{aligned}$$

(ii) Given function is equivalent to

$$[t] = 0 \text{ in } (0, 1) + 1 \text{ in } (1, 2) + 2 \text{ in } (2, 3) + 3 \text{ in } (3, 4) + \dots$$

$$\begin{aligned} \therefore L[f(t)] &= \int_0^\infty e^{-st}[f(t)] dt = \int_0^\infty e^{-st}[t] dt \\ &= \int_0^1 e^{-st}(0) dt + \int_1^2 e^{-st}(1) dt + \int_2^3 e^{-st}(2) dt + \int_3^4 e^{-st}(3) dt + \dots \infty \\ &= 0 + \left| \frac{e^{-st}}{-s} \right|_1^2 + 2 \left| \frac{e^{-st}}{-s} \right|_2^3 + 3 \left| \frac{e^{-st}}{-s} \right|_3^4 + \dots \infty \\ &= -\frac{1}{s} [(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \dots \infty] \\ &= \frac{1}{s} (e^{-s} + e^{-2s} + e^{-3s} + \dots \infty) = \frac{1}{s} \left(\frac{e^{-s}}{1 - e^{-s}} \right) = \frac{1}{s(e^s - 1)}. \end{aligned}$$

III. Change of scale property. If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty e^{-st} f(at) dt = \int_0^\infty e^{-su/a} f(u) \cdot du/a \\ &= \frac{1}{a} \int_0^\infty e^{-su/a} f(u) du = \frac{1}{a} \bar{f}(s/a). \end{aligned}$$

Put $at = u$
 $dt = du/a$

Example 21.8. Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a} \tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a} \tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t$. (J.N.T.U., 2003)
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3 \cosh 5t - 4 \sinh 5t$. (Nagarjuna, 2006)
4. $\cos(at+b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. (Kottayam, 2005)
7. $\sin \sqrt{t}$.
8. $\sin^5 t$. (Mumbai, 2007)
9. $\cos^3 2t$.
10. $e^{-at} \sinh bt$
11. $e^{2t} (3t^5 - \cos 4t)$. (P.T.U., 2007)
12. $e^{-3t} \sin 5t \sin 3t$. (V.T.U., 2006)
13. $e^{-t} \sin^2 t$. (Mumbai, 2009)
14. $e^{2t} \sin^4 t$. (Mumbai, 2007)
15. $\cosh at \sin at$. (Delhi, 2002)
16. $\sinh 3t \cos^2 t$. (Madras, 2000)
17. $t^2 e^{2t}$. (V.T.U., 2008 S)
18. $(1 + te^{-t})^3$
19. $t \sqrt{1 + \sin t}$. (Mumbai, 2007)
20. $f(t) = \begin{cases} 4, & 0 \leq t < 1 \\ 3, & t > 1 \end{cases}$. (U.P.T.U., 2009)
21. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$. (Madras, 2000 S)
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0, & x < \pi/3 \end{cases}$. (Rajasthan, 2006)
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t - 1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$. (Mumbai, 2007)
25. If $L[f(t)] = \frac{1}{s(s^2 + 1)}$, find $L[e^{-t} f(2t)]$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T , i.e., $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$

We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^T e^{-st} f(t) dt + \int_T^{2T} e^{-st} f(t) dt + \int_{2T}^{3T} e^{-st} f(t) dt + \dots$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$\begin{aligned} L\{f(t)\} &= \int_0^T e^{-st} f(t) dt + \int_0^T e^{-s(u+T)} f(u+T) du + \int_0^T e^{-s(u+2T)} f(u+2T) du + \dots \\ &= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-su} f(u) du + e^{-2sT} \int_0^T e^{-su} f(u) du + \dots \\ &\quad [\because f(u) = f(u+T) = f(u+2T) \text{ etc.}] \\ &= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-st} f(t) dt + e^{-2sT} \int_0^T e^{-st} f(t) dt + \dots \\ &= (1 + e^{-sT} + e^{-2sT} + \dots \infty) \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt. \end{aligned}$$

(V.T.U., 2008 ; Mumbai, 2006)

Example 21.9. Find the Laplace transform of the function

$$\begin{aligned} f(t) &= \sin \omega t, \quad 0 < t < \pi/\omega \\ &= 0, \quad \pi/\omega < t < 2\pi/\omega. \end{aligned}$$

(Kurukshetra, 2005 ; Madras, 2003)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\begin{aligned} \therefore L\{f(t)\} &= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 dt \right] \\ &= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st} (-s \sin \omega t - \omega \cos \omega t)}{s^2 + \omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2 + \omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2 + \omega^2)}. \end{aligned}$$

Example 21.10. Draw the graph of the periodic function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform.

(U.P.T.U., 2003)

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

$$\begin{aligned} \therefore L\{f(t)\} &= \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^\pi e^{-st} t dt + \int_\pi^{2\pi} e^{-st} (\pi - t) dt \right\} \\ &= \frac{1}{1-e^{-2\pi s}} \left\{ \left| t \left(\frac{e^{-st}}{-s} \right) - 1 \cdot \left(\frac{e^{-st}}{s^2} \right) \right|_0^\pi + \left| (\pi - t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{s^2} \right) \right|_\pi^{2\pi} \right\} \\ &= \frac{1}{1-e^{-2\pi s}} \left\{ -\frac{\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\} \\ &= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{\pi}{s} (e^{-2\pi s} - e^{-\pi s}) + \frac{1}{s^2} (1 + e^{-2\pi s} - 2e^{-\pi s}) \right\}. \end{aligned}$$

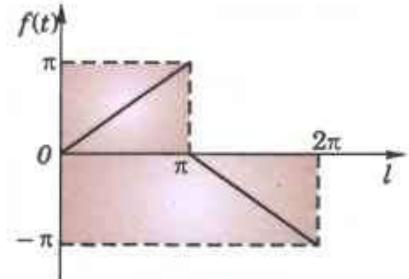


Fig. 21.1

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$

[§ 16.7 (1), p. 553]

$$\begin{aligned} \therefore L\{J_0(x)\} &= \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \frac{6!}{s^7} + \dots \\ &= \frac{1}{s} \left\{ 1 - \frac{1}{2} \left(\frac{1}{s^2} \right) + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{s^4} \right) - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{s^6} \right) + \dots \right\} \\ &= \frac{1}{s} \left(1 + \frac{1}{s^2} \right)^{-1/2} = \frac{1}{\sqrt{s^2 + 1}} \end{aligned} \quad \dots(1)$$

Also since $J_0'(x) = -J_1(x)$.

[Problem 4(i), p. 557]

$$\therefore L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - 1] = 1 - \frac{s}{\sqrt{s^2 + 1}} \quad \dots(2)$$

(2) Transform of Error function

We know that $\text{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt$ (§ 7.18, p. 312)

$$\begin{aligned}
 &= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots \right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5 \cdot 2!} - \frac{x^{7/2}}{7 \cdot 3!} + \dots \right) \\
 \therefore L\{erf(\sqrt{x})\} &= \frac{2}{\sqrt{\pi}} \left\{ \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5 \cdot 2! s^{7/2}} - \frac{\Gamma(9/2)}{7 \cdot 3! s^{9/2}} + \dots \right\} \\
 &= \frac{1}{s^{3/2}} - \frac{1}{2} \frac{1}{s^{5/2}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{s^{7/2}} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{s^{9/2}} + \dots \\
 &= \frac{1}{s^{3/2}} \left\{ 1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{s^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{s^3} + \dots \right\} \\
 &= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s} \right]^{-1/2} = \frac{1}{s \sqrt{(s+1)}}. \tag{Mumbai, 2009} \quad \dots(3)
 \end{aligned}$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$\begin{aligned}
 L[L_n(t)] &= \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt \\
 &= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) \right|_0^\infty + \int_0^\infty e^{-(s-1)t} (s-1) \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt \\
 &= (s-1) \int_0^\infty e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt. \tag{Integrating by parts} \\
 &= (s-1)^n \int_0^\infty e^{-(s-1)t} \cdot e^{-t} \cdot t^n dt = (s-1)^n \int_0^\infty e^{-st} \cdot t^n dt \\
 &= (s-1)^n L(t^n) = (s-1)^n \cdot \frac{n!}{s^{n+1}}
 \end{aligned}$$

Hence $L[L_n(x)] = \frac{n!(s-1)^n}{s^n + 1}$ ($s > 1$).

Example 21.11. Evaluate (i) $L[e^{-at} J_0(at)]$ (ii) $L(erf 2\sqrt{t})$. (Mumbai, 2006)

Solution. (i) We know that $L[J_0(at)] = \frac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L[e^{-at} J_0(at)] = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(ii) We know that $L(erf \sqrt{t}) = \frac{1}{s(s+1)}$

$$\therefore L(erf 2\sqrt{t}) = L[erf \sqrt{(4t)}] = \frac{1}{4} \cdot \frac{1}{\frac{s}{4} \sqrt{\left(\frac{s}{4} + 1\right)}} = \frac{2}{s \sqrt{(s+4)}}.$$

PROBLEMS 21.2

- Find the Laplace transform of the saw-toothed wave of period T , given $f(t) = t/T$ for $0 < t < T$. (V.T.U., 2007)
- Find the Laplace transform of the full-wave rectifier

$$f(t) = E \sin wt, 0 < t < \pi/w, \text{ having period } \pi/w.$$

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$\begin{aligned} f(t) &= k, && \text{when } 0 < t < a \\ &= -k, && \text{when } a < t < 2a. \end{aligned}$$

(V.T.U., 2011)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$\begin{aligned} f(t) &= t, && 0 < t < a \\ &= 2a - t, && a < t < 2a. \end{aligned}$$

(Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002)

Find the Laplace transform of the following functions :

5. $J_0(ax)$.

6. $e^{-at} J_0(bt)$.

7. $e^{2t} \operatorname{erf}(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) If $f'(t)$ be continuous and $L\{f(t)\} = f(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt && [\text{Integrate by parts}] \\ &= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s)e^{-st} \cdot f(t) dt. \end{aligned}$$

Now assuming $f(t)$ to be such that $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s*.

Thus, $L\{f'(t)\} = f(0) + s \int_0^\infty e^{-st} f(t) dt$

whence follows the desired result.

(2) If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0).$$

Using the general rule of integration by parts (Footnote p. 398).

$$\begin{aligned} L\{f^n(t)\} &= \int_0^\infty e^{-st} f^n(t) dt \\ &= \left| e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \dots \right. \\ &\quad \left. + (-1)^{n-1} (-s)^{n-1} e^{-st} \cdot f(t) \right|_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t) dt \\ &= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t) dt \end{aligned}$$

Assuming that $\lim_{t \rightarrow \infty} e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \dots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u) du\right\} = \frac{1}{s} \bar{f}(s)$.

Let $\phi(t) = \int_0^t f(u) du$, then $\phi'(t) = f(t)$ and $\phi(0) = 0$

$$\therefore L\{\phi'(t)\} = s\bar{f}(s) - \phi(0)$$

[By § 21.7 (1)]

or

$$\bar{f}(s) = \frac{1}{s} L\{\phi'(t)\} \quad i.e., \quad L\left(\int_0^t f(u) du\right) = \frac{1}{s} \bar{f}(s).$$

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [\bar{f}(s)], \text{ where } n = 1, 2, 3 \dots$$

We have $\int_0^\infty e^{-st} f(t) dt = \bar{f}(s)$.

Differentiating both sides with respect to s , $\frac{d}{ds} \left\{ \int_0^\infty e^{-st} f(t) dt \right\} = \frac{d}{ds} \{\bar{f}(s)\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t) dt = \frac{d}{ds} \{\bar{f}(s)\}$$

or $\int_0^\infty \{-te^{-st} f(t)\} dt = \frac{d}{ds} [\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st} [tf(t)] dt = -\frac{d}{ds} [\bar{f}(s)]$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st} [t^m f(t)] dt = (-1)^m \frac{d^m}{ds^m} [\bar{f}(s)]$$

Then $\frac{d}{ds} \left[\int_0^\infty e^{-st} t^m f(t) dt \right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)]$

or By Leibnitz's rule, $\int_0^\infty (-te^{-st}) \cdot t^m f(t) dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)]$

or $\int_0^\infty e^{-st} [t^{m+1} f(t)] dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} [\bar{f}(s)].$

This shows that, if the theorem is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n .

(U.P.T.U., 2005)

Example 21.12. Find the Laplace transforms of

(i) $t \cos at$ (Raipur, 2005)

(ii) $t^2 \sin at$

(S.V.T.U., 2007)

(iii) $t^3 e^{-3t}$ (Kottayam, 2005)

(iv) $te^{-t} \sin 3t$

(Kurukshestra, 2005)

Solution. (i) Since $L(\cos at) = s/(s^2 + a^2)$

$$\begin{aligned} L(t \cos at) &= -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) = -\frac{s^2 + a^2 - s \cdot 2s}{(s^2 + a^2)^2} \\ &= \frac{s^2 - a^2}{(s^2 + a^2)^2} \end{aligned}$$

[cf. Example 21.4]

(ii) Since $\sin at = \frac{a}{s^2 + a^2}$,

$$\therefore L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2} \left(\frac{a}{s^2 + a^2} \right) = \frac{d}{ds} \left(\frac{-2as}{(s^2 + a^2)^2} \right) = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

(iii) Since $L(e^{-3t}) = 1/(s + 3)$,

$$\therefore L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3} \left(\frac{1}{s+3} \right) = -\frac{(-1)^3 \cdot 3!}{(s+3)^{3+1}} = 6/(s+3)^4.$$

(iv) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, therefore $L(t \sin 3t) = -\frac{d}{ds} \left(\frac{3}{s^2 + 3^2} \right) = \frac{6s}{(s^2 + 9)^2}$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t} t \sin 3t) = \frac{6(s+1)}{[(s+1)^2 + 9]^2} = \frac{6(s+1)}{(s^2 + 2s + 10)^2}.$$

Example 21.13. Evaluate (i) $L\{t J_0(at)\}$ (ii) $L\{t J_1(t)\}$ (iii) $L\{t \operatorname{erf} 2\sqrt{t}\}$.

Solution. (i) Since $L\{J_0(at)\} = \frac{1}{\sqrt{(s^2 + a^2)}}$

$$\therefore L\{t J_0(at)\} = -\frac{d}{ds} [L\{J_0(at)\}] = -\frac{d}{ds} \frac{1}{\sqrt{(s^2 + a^2)}} = \frac{s}{(s^2 + a^2)^{3/2}}$$

$$(ii) \text{ Since } L\{J_1(t)\} = 1 - \frac{s}{\sqrt{(s^2 + 1)}}$$

$$\therefore L\{t J_1(t)\} = -\frac{d}{ds} [L\{J_1(t)\}] = -\frac{d}{ds} \left\{ 1 - \frac{s}{\sqrt{(s^2 + 1)}} \right\} = \frac{1}{(s^2 + 1)^{3/2}}$$

$$(iii) \text{ Since } L\{\operatorname{erf} \sqrt{t}\} = \frac{1}{s\sqrt{(s+1)}}$$

$$\therefore L\{\operatorname{erf} 2\sqrt{t}\} = L\{\operatorname{erf} \sqrt{4t}\} = \frac{1}{4} \cdot \frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

$$\text{Thus } L\{t \operatorname{erf} 2\sqrt{t}\} = -\frac{d}{ds} \left\{ \frac{2}{s\sqrt{(s+4)}} \right\} = -\frac{d}{ds} \left\{ \frac{2}{\sqrt{(s^3 + 4s^2)}} \right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s) \mathbf{d}s$ provided the integral exists.

$$\text{We have } \bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

Integrating both sides with respect to s from s to ∞ .

$$\begin{aligned} \int_s^\infty \bar{f}(s) ds &= \int_s^\infty \left[\int_0^\infty e^{-st} f(t) dt \right] ds = \int_0^\infty \int_s^\infty f(t) e^{-st} ds dt \\ &\quad [\text{Changing the order of integration}] \\ &= \int_0^\infty f(t) \left[\int_s^\infty e^{-st} ds \right] dt \\ &= \int_0^\infty f(t) \left| \frac{e^{-st}}{-t} \right|_{s}^{\infty} dt = \int_0^\infty e^{-st} \cdot \frac{f(t)}{t} dt = L\left\{\frac{1}{t} f(t)\right\}. \end{aligned}$$

Example 21.14. Find the Laplace transform of (i) $(1 - e^t)/t$

(Madras, 2000)

$$(ii) \frac{\cos at - \cos bt}{t} + t \sin at.$$

(V.T.U., 2010)

Solution. (i) Since $L(1 - e^t) = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1}$

$$\begin{aligned} \therefore L\left(\frac{1-e^t}{t}\right) &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1} \right) ds = \left| \log s - \log(s-1) \right|_s^\infty \\ &= \left| \log\left(\frac{s}{s-1}\right) \right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right) \end{aligned}$$

$$(ii) \text{ Since } L(\cos at - \cos bt) = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \text{ and } L(\sin at) = \frac{a}{s^2 + a^2}$$

$$\begin{aligned}
 \therefore L\left(\frac{\cos at - \cos bt}{t}\right) + L(t \sin at) &= \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}\right) ds - \frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) \\
 &= \left| \frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right|_s^\infty - a \frac{-2s}{(s^2 + a^2)^2} \\
 &= \frac{1}{2} \operatorname{Lt}_{s \rightarrow \infty} \log \frac{s^2 + a^2}{s^2 + b^2} - \frac{1}{2} \log \frac{s^2 + a^2}{s^2 + b^2} + \frac{2as}{(s^2 + a^2)^2} \\
 &= \frac{1}{2} \log \left(\frac{1+0}{1+0} \right) - \frac{1}{2} \log \left(\frac{s^2 + a^2}{s^2 + b^2} \right) + \frac{2as}{(s^2 + a^2)^2} = \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right)^{1/2} + \frac{2as}{(s^2 + a^2)^2} \\
 &\quad [\because \log 1 = 0]
 \end{aligned}$$

Example 21.15. Evaluate (i) $L\left\{e^{-t} \int_0^t \frac{\sin t}{t} dt\right\}$

(Madras, 2006)

(ii) $L\left\{t \int_0^t \frac{e^{-t} \sin t}{t} dt\right\}$ (P.T.U., 2005) (iii) $L\left\{\int_0^t \int_0^t \int_0^t (t \sin t) dt dt dt\right\}$.

(Mumbai, 2006)

Solution. (i) We know that $L(\sin t) = \frac{1}{s^2 + 1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2 + 1} ds = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore L\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s} \cot^{-1}s$$

Thus by shifting property, $L\left\{e^{-t} \left(\int_0^t \frac{\sin t}{t} dt\right)\right\} = \frac{1}{s+1} \cot^{-1}(s+1)$.

(ii) Since $L\left(\frac{\sin t}{t}\right) = \cot^{-1}s$

$$\therefore L\left(e^{-t} \cdot \frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and

$$L\left\{\int_0^t e^{-t} \frac{\sin t}{t} dt\right\} = \frac{1}{s} \cot^{-1}(s+1)$$

Hence $L\left\{t \cdot \int_0^t e^{-t} \frac{\sin t}{t} dt\right\} = -\frac{d}{ds} \left\{\frac{\cot^{-1}(s+1)}{s}\right\}$

$$\begin{aligned}
 &= -\frac{s \cdot \left[\frac{-1}{1+(s+1)^2} \right] - \cot^{-1}(s+1)}{s^2} = \frac{s + (s^2 + 2s + 2) \cot^{-1}(s+1)}{s^2(s^2 + 2s + 2)}
 \end{aligned}$$

(iii) Since $L(\sin t) = \frac{1}{s^2 + 1}$

$$\therefore L(t \sin t) = -\frac{d}{ds} \frac{1}{(s^2 + 1)} = \frac{2s}{(s^2 + 1)^2}$$

$$\text{Thus } L\left\{\int_0^t \int_0^t \int_0^t (t \sin t) dt dt dt\right\} = \frac{1}{s^3} L(t \sin t) = \frac{1}{s^3} \cdot \frac{2s}{(s^2 + 1)^2} = \frac{2}{s^2(s^2 + 1)^2}$$

21.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example 21.16. Evaluate (i) $\int_0^\infty te^{-3t} \sin t dt$ (V.T.U., 2007)

$$(ii) \int_0^\infty \frac{\sin mt}{t} dt$$

$$(iii) \int_0^\infty e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt$$

(Mumbai, 2009)

$$(iv) L \left\{ \int_0^t \frac{e^{-s\tau} \sin \tau}{\tau} d\tau \right\}.$$

Solution. (i) $\int_0^\infty te^{-3t} \sin t dt = \int_0^\infty e^{-st} (t \sin t) dt \quad \text{where } s = 3$

= $L(t \sin t)$, by definition.

$$= (-1) \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right) = \frac{2s}{(s^2 + 1)^2} = \frac{2 \times 3}{(3^2 + 1)^2} = \frac{3}{50}.$$

(ii) Since

$$L(\sin mt) = m/(s^2 + m^2) = f(s), \text{ say.}$$

$$\therefore \text{ Using } \S 21.10, L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s) ds = \int_0^\infty \frac{m ds}{s^2 + m^2} = \left| \tan^{-1} \frac{s}{m} \right|_s^\infty$$

or by Def.,

$$\int_0^\infty e^{-st} \frac{\sin mt}{t} dt = \frac{\pi}{2} - \tan^{-1} \frac{s}{m}$$

Now

$$\lim_{s \rightarrow 0} \tan^{-1}(s/m) = 0 \text{ if } m > 0 \quad \text{or} \quad \pi \text{ if } m < 0.$$

Thus taking limits as $s \rightarrow 0$, we get

$$\int_0^\infty \frac{\sin mt}{t} dt = \frac{\pi}{2} \text{ if } m > 0 \quad \text{or} \quad -\pi/2 \text{ if } m < 0$$

(iii) We know that $L(\cos at) = \frac{s}{s^2 + a^2}$ and $L(\cos bt) = \frac{s}{s^2 + b^2}$

$$\begin{aligned} \therefore L \left(\frac{\cos at - \cos bt}{t} \right) &= \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds \\ &= \frac{1}{2} \left\{ \log \left(\frac{s^2 + a^2}{s^2 + b^2} \right) \right\}_s^\infty = \frac{1}{2} \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right) \end{aligned}$$

This implies that $\int_0^\infty e^{-st} \left(\frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2} \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right)$

Taking $s = 1$, we get $\int_0^\infty \left(e^{-t} \frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2} \log \left(\frac{1+b^2}{1+a^2} \right)$

(iv) Since $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$.

$\therefore L \left\{ e^t \left(\frac{\sin t}{t} \right) \right\} = \cot^{-1}(s-1)$, by shifting property ($\S 21.4 II$).

Thus $L \left[\int_0^t \left\{ e^t \left(\frac{\sin t}{t} \right) \right\} dt \right] = \frac{1}{s} \cot^{-1}(s-1)$, by $\S 21.8$.

PROBLEMS 21.3

1. Find $L \left(\int_0^t e^{-s} \cos t dt \right)$.
 2. Given $L [2\sqrt{t/\pi}] = 1/s^{3/2}$, show that $L [1/\sqrt{\pi t}] = 1/\sqrt{s}$. (U.P.T.U., 2005; Madras, 2003)

3. Given $L [\sin(\sqrt{t})] = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4}s$, prove that $L \left[\frac{\cos(\sqrt{t})}{\sqrt{t}} \right] = \sqrt{\left(\frac{\pi}{s}\right)} e^{-1/4}s$. (Mumbai, 2009)

Find the Laplace transforms of the following functions:

4. $t \sin^2 t$ (Nagarjuna, 2008)
 5. $\sin 2t - 2t \cos 2t$. (Anna, 2003)
 6. $t^2 \cos at$.
 7. $t \sinh at$.
 8. $te^{2t} \sin 3t$. (Madras, 2003)
 9. $te^{-2t} \sin 4t$. (V.T.U., 2008)
 10. $t^2 e^{-3t} \sin 2t$. (Madras, 2000 S)
 11. $(e^{-at} - e^{-bt})/t$. (Anna, 2005 S)
 12. $(\sin t)/t$. (P.T.U., 2010)
 13. $\frac{(\sin t \sin 5t)}{t}$. (Mumbai, 2008)
 14. $(e^{at} - \cos bt)/t$. (U.P.T.U., 2003)
 15. $(e^{-t} \sin t)/t$. (V.T.U., 2009 S)
 16. $(1 - \cos 3t)/t$. (V.T.U., 2006)
 17. $(1 - \cos t)/t^2$. (Hazaribag, 2008)
 18. $2^t + \frac{\cos 2t - \cos 3t}{t} + t \sin t$. (V.T.U., 2004)

19. Evaluate (i) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$ (Mumbai, 2008; P.T.U., 2006)

- (ii) $\int_0^\infty \frac{e^{-\sqrt{2}t} \sinh t \sin t}{t} dt$ (Mumbai, 2005) (iii) $\int_0^\infty te^{-2t} \sin 3t dt$ (V.T.U., 2008)
 (iv) $\int_0^\infty te^{-t} \sin^4 t dt$.

20. Prove that (i) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a}$. (S.V.T.U., 2009; Mumbai, 2007; J.N.T.U., 2006)

- (ii) $\int_0^\infty \frac{e^{-2t} \sinh t}{t} dt = \frac{1}{2} \log 3$ (Mumbai, 2008) (iii) $\int_0^\infty \frac{e^{-t} \sin t}{t} dt = \frac{\pi}{4}$. (V.T.U., 2009 S)
 (iv) $\int_0^\infty \frac{e^{-t} \sin^2 t}{t} dt = \frac{1}{4} \log 5$. (Kurukshetra, 2006)

21. Evaluate (i) $L \left(\int_0^t \frac{\sin t}{t} dt \right)$ (J.N.T.U., 2005)

- (ii) $L \left(\int_0^t e^{-t} \cos t dt \right)$ (iii) $L \int_0^t \frac{e^t \sin t}{t} dt$. (P.T.U., 2009 S; S.V.T.U., 2009; Bhopal, 2008)

22. Show that (i) $L [t J_0(at)] = \frac{s}{(s^2 + a^2)^{3/2}}$ (ii) $\int_0^\infty te^{-3t} J_0(4t) dt = 3/125$.

21.12 INVERSE TRANSFORMS — METHOD OF PARTIAL FRACTIONS

Having found the Laplace transforms of a few functions, let us now determine the inverse transforms of given functions of s . We have seen that $L[f(t)]$ in each case, is a rational algebraic function. Hence to find the inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms :

$$(1) L^{-1} \left[\frac{1}{s} \right] = 1.$$

$$(2) L^{-1} \left[\frac{1}{s-a} \right] = e^{at}.$$

$$(3) L^{-1} \left[\frac{1}{s^n} \right] = \frac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, \dots$$

$$(4) L^{-1} \left[\frac{1}{(s-a)^n} \right] = \frac{e^{at} t^{n-1}}{(n-1)!}.$$

$$(5) L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{1}{a} \sin at.$$

$$(6) L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at.$$

$$(7) L^{-1} \left(\frac{1}{s^2 - a^2} \right) = \frac{1}{a} \sinh at.$$

$$(8) L^{-1} \left(\frac{s}{s^2 - a^2} \right) = \cosh at.$$

$$(9) L^{-1} \left[\frac{1}{(s-a)^2 + b^2} \right] = \frac{1}{b} e^{at} \sin bt.$$

$$(10) L^{-1} \left[\frac{s-a}{(s-a)^2 + b^2} \right] = e^{at} \cos bt.$$

$$(11) L^{-1} \left(\frac{s}{(s^2 + a^2)^2} \right) = \frac{1}{2a} t \sin at.$$

$$(12) L^{-1} \left[\frac{1}{(s^2 + a^2)^2} \right] = \frac{1}{2a^3} (\sin at - at \cos at).$$

The reader is strongly advised to commit these results to memory. The results (1) to (10) follow at once from their corresponding results in § 21.3 and 21.4. As illustrations, we shall prove (11) and (12). Example 21.4 gives

$$L(t \sin at) = \frac{2as}{(s^2 + a^2)^2} \text{ and } L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$\therefore t \sin at = 2a L^{-1} \left[\frac{s}{(s^2 + a^2)^2} \right], \text{ whence follows (11).}$$

$$\begin{aligned} \text{Also } t \cos at &= L^{-1} \left[\frac{s^2 - a^2}{(s^2 + a^2)^2} \right] = L^{-1} \left[\frac{(s^2 + a^2) - 2a^2}{(s^2 + a^2)^2} \right] \\ &= L^{-1} \left[\frac{1}{s^2 + a^2} \right] - 2a^2 L^{-1} \left[\frac{1}{(s^2 + a^2)^2} \right] \\ &= \frac{1}{a} \sin at - 2a^2 L^{-1} \left[\frac{1}{(s^2 + a^2)^2} \right] \text{ whence follows (12).} \end{aligned}$$

Obs. Go through the note on the 'partial fractions' given in para 10 of 'useful information' in Appendix I.

Example 21.17. Find the inverse transforms of

$$(i) \frac{s^2 - 3s + 4}{s^3}$$

$$(ii) \frac{s+2}{s^2 - 4s + 13}$$

(V.T.U., 2008)

$$\text{Solution. (i)} L^{-1} \left(\frac{s^2 - 3s + 4}{s^3} \right) = L^{-1} \left(\frac{1}{s} \right) - 3L^{-1} \left(\frac{1}{s^2} \right) + 4L^{-1} \left(\frac{1}{s^3} \right) = 1 - 3t + 4 \cdot t^2/2! = 1 - 3t + 2t^2.$$

$$\begin{aligned} \text{(ii)} \quad L^{-1} \left(\frac{s+2}{s^2 - 4s + 13} \right) &= L^{-1} \left[\frac{s+2}{(s-2)^2 + 9} \right] = L^{-1} \left[\frac{s-2+4}{(s-2)^2 + 3^2} \right] \\ &= L^{-1} \left[\frac{s-2}{(s-2)^2 + 3^2} \right] + 4L^{-1} \left[\frac{1}{(s-2)^2 + 3^2} \right] = e^{2t} \cos 3t + \frac{4}{3} e^{2t} \sin 3t. \end{aligned}$$

Example 21.18. Find the inverse transforms of

$$(i) \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$$

(V.T.U., 2007; U.P.T.U., 2004)

$$(ii) \frac{4s+5}{(s-1)^2(s+2)}$$

(Kurukshetra, 2005)

Solution. (i) Here the denominator = $(s - 1)(s - 2)(s - 3)$.

So let $\frac{2s^2 - 6s + 5}{(s - 1)(s - 2)(s - 3)} = \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s - 3}$

Then $A = [2 \cdot 1^2 - 6 \cdot 1 + 5]/(1 - 2)(1 - 3) = \frac{1}{2}$

$B = [2 \cdot 2^2 - 6 \cdot 2 + 5]/(2 - 1)(2 - 3) = -1$

and $C = [2 \cdot 3^2 - 6 \cdot 3 + 5]/(3 - 1)(3 - 2) = \frac{5}{2}$.

$$\therefore L^{-1}\left(\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}.$$

(ii) Let $\frac{4s + 5}{(s - 1)^2(s + 2)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{4(-2) + 5}{(-2 - 1)^2(s + 2)}$

Multiplying both sides by $(s - 1)^2(s + 2)$, $4s + 5 = A(s - 1)(s + 2) + B(s + 2) - \frac{1}{3}(s - 1)^2$

Putting $s = 1$, $9 = 3B$, $\therefore B = 3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \frac{1}{3}, \quad \therefore A = \frac{1}{3}.$$

$$\therefore L^{-1}\left[\frac{4s + 5}{(s - 1)^2(s + 2)}\right] = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}.$$

Example 21.19. Find the inverse transforms of

(i) $\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)}$

(Rohitak, 2009; U.P.T.U., 2005)

(ii) $\frac{s}{s^4 + 4a^4}$.

(Mumbai, 2008)

Solution. (i) Let $\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)} = \frac{5(1) + 3}{(s - 1)(1^2 + 2 \cdot 1 + 5)} + \frac{As + B}{s^2 + 2s + 5}$

Multiplying both sides by $(s - 1)(s^2 + 2s + 5)$,

$$5s + 3 = 1 \cdot (s^2 + 2s + 5) + (As + B)(s - 1).$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A, \quad \therefore A = -1.$$

Putting $s = 0$, $3 = 5 - B$, $\therefore B = 2$.

$$\therefore L^{-1}\left[\frac{5s + 3}{(s - 1)(s^2 + 2s + 5)}\right] = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right] = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t} \cos 2t + \frac{3}{2}e^{-t} \sin 2t.$$

(ii) Since $s^4 + 4a^4 = (s^2 + 2a^2)^2 - (2as)^2 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2)$

\therefore Let $\frac{s}{s^4 + 4a^4} = \frac{As + B}{s^2 + 2as + 2a^2} + \frac{Cs + D}{s^2 - 2as + 2a^2}$

Multiplying both sides by $s^4 + 4a^4$,

$$s = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2)$$

- Equating coefficients of s^3 , $0 = A + C$... (i)
 Equating coefficients of s^2 , $0 = -2aA + B + 2aC + D$... (ii)
 Equating coefficients of s , $1 = 2a^2A - 2aB + 2a^2C + 2aD$... (iii)
 Putting $s = 0$, $0 = 2a^2B + 2a^2D$... (iv)
 From (iv), $B + D = 0$... (v)

\therefore (ii) becomes $-A + C = 0$, and by (i), we get $A = C = 0$.

Then (iii) reduces to $D - B = 1/2a$ and by (v), $B = -1/4a$, $D = 1/4a$.

$$\begin{aligned} \therefore L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) &= -\frac{1}{4a} L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a} L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right) \\ &= -\frac{1}{4a} L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right] + \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right] \\ &= -\frac{1}{4a} \cdot \frac{1}{a} e^{-at} \sin at + \frac{1}{4a} \cdot \frac{1}{a} e^{at} \sin at = \frac{1}{2a^2} \sin at \left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2} \sin at \sinh at. \end{aligned}$$

PROBLEMS 21.4

Find the inverse Laplace transforms of :

1. $\frac{2s - 5}{4s^2 + 25} + \frac{4s - 18}{9 - s^2}$.
(S.V.T.U., 2008)
2. $\frac{1}{s^2 - 5s + 6}$.
3. $\frac{s}{(2s - 1)(3s - 1)}$.
(V.T.U., 2010)
4. $\frac{3s}{s^2 + 2s - 8}$.
5. $\frac{3s + 2}{s^2 - s - 2}$.
(V.T.U., 2010 S)
6. $\frac{1}{s(s^2 - 1)}$.
(Nagarjuna, 2008)
7. $\frac{1 - 7s}{(s - 3)(s - 1)(s + 2)}$.
(B.P.T.U., 2005 S)
8. $\frac{s^2 - 10s + 13}{(s - 7)(s^2 - 5s + 6)}$.
9. $\frac{2p^2 - 6p + 5}{p^3 - 6p^2 + 11p - 6}$.
(U.P.T.U., 2004)
10. $\frac{s}{(s^2 - 1)^2}$.
(Kurukshetra, 2005)
11. $\frac{1 + 2s}{(s + 2)^2(s - 1)^2}$.
12. $\frac{s}{(s - 3)(s^2 + 4)}$.
13. $\frac{s}{(s + 1)^2(s^2 + 1)}$.
14. $\frac{s^3}{s^4 - a^4}$.
(Kurukshetra, 2005)
15. $\frac{1}{s^3 - a^3}$.
16. $\frac{s^2 + 6}{(s^2 + 1)(s^2 + 4)}$.
17. $\frac{2s - 3}{s^2 + 4s + 13}$.
18. $\frac{s^2 + s}{(s^2 + 1)(s^2 + 2s + 2)}$.
19. $\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$.
(Mumbai, 2008)
20. $\frac{s}{s^4 + s^2 + 1}$.
(Raipur, 2005)
21. $\frac{a(s^2 - 2a^2)}{s^4 + 4a^4}$.
(Mumbai, 2009)

21.13 OTHER METHODS OF FINDING INVERSE TRANSFORMS

We have seen that the most effective method of finding the inverse transforms is by means of partial fractions. However, various other methods are available which depend on the following *important inversion formulae*.

I. Shifting property for inverse Laplace transforms.

If $L^{-1}[\bar{f}(s)] = f(t)$, then

$$L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)].$$

II. If $L^{-1}[\bar{f}(s)] = f(t)$ and $f(0) = 0$, then

$$L^{-1}[s \bar{f}(s)] = \frac{d}{dt}\{f(t)\}$$

In general, $L^{-1}[s^n \bar{f}(s)] = \frac{d^n}{dt^n}\{f(t)\}$ provided $f(0) = f'(0) = \dots = f^{n-1}(0) = 0$.

The above formulae at once follow from the results of § 21.7 (Transforms of derivatives).

III. If $L^{-1}[\bar{f}(s)] = f(t)$, then

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t) dt$$

This result follows from § 21.8 (Transforms of integrals)

Also $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t) dt\right\} dt$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t) dt\right) dt\right\} dt \text{ and so on.}$$

IV. If $L^{-1}[\bar{f}(s)] = f(t)$, then

$$tf(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L[tf(t)] = -\frac{d}{ds}[\bar{f}(s)]$ (§ 21.9)

V. The formula of § 21.10, i.e.,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s) ds$$

is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transform of $\int_s^\infty \bar{f}(s) ds$ can be conveniently calculated.

Example 21.20. Find the inverse Laplace transforms of the following:

$$(i) \frac{s^2}{(s-2)^3} \quad (ii) \frac{s+3}{s^2-4s+13} \quad (iii) \frac{(s+2)^2}{(s^2+4s+8)^2} \quad (\text{Mumbai, 2005})$$

Solution. (i) Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}t + 2e^{2t}t^2.$$

[using shifting property]

$$(ii) \frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3} L^{-1}\left\{\frac{3}{(s-2^2)+3^2}\right\}$$

$$= e^{2t} \cos 3t + \frac{5}{3} e^{2t} \sin 3t.$$

[Using shifting property]

$$\begin{aligned}
(iii) L^{-1} \frac{(s+2)^2}{(s^2+4s+8)^2} &= L^{-1} \frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1} \frac{(s+2)^2}{[(s+2)^2+4]^2} \\
&= e^{-2t} L^{-1} \left\{ \frac{s^2}{(s^2+4)^2} \right\} = e^{-2t} L^{-1} \left\{ \frac{s^2+4-4}{(s^2+4)^2} \right\} \\
&= e^{-2t} L^{-1} \left\{ \frac{1}{s^2+4} - \frac{4}{(s^2+4)^2} \right\} = \frac{e^{-2t} \sin 2t}{2} - 4e^{-2t} L^{-1} \left\{ \frac{1}{(s^2+4)^2} \right\} \\
&= \frac{e^{-2t} \sin 2t}{2} - 4e^{-2t} \left\{ \frac{1}{4} \left(\frac{\sin 2t}{4} - \frac{t \cos 2t}{2} \right) \right\} \\
&= e^{-2t} \left\{ \frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t \cos 2t}{2} \right\} = e^{-2t} \left\{ \frac{\sin 2t}{4} + \frac{t \cos 2t}{2} \right\}.
\end{aligned}$$

Example 21.21. Find the inverse transform of (i) $1/s(s^2+a^2)$
(ii) $1/s(s+a)^3$.

(P.T.U., 2003)

Solution. (i) Since $L^{-1} \left(\frac{1}{s^2+a^2} \right) = \frac{1}{a} \sin at$,

therefore, by formula III above,

$$\begin{aligned}
L^{-1} \left\{ \frac{1}{s(s^2+a^2)} \right\} &= \int_0^t \frac{1}{a} \sin at \, dt = \frac{1}{a^2} [-\cos at]_0^t = (1 - \cos at)/a^2 \\
(ii) \quad L^{-1} \left\{ \frac{1}{s(s+a)^3} \right\} &= L^{-1} \left\{ \frac{1}{[(s+a)-a](s+a)^3} \right\} = e^{-at} L^{-1} \left\{ \frac{1}{(s-a)s^3} \right\}
\end{aligned}$$

$$\text{Now } L^{-1} \left\{ \frac{1}{s-a} \right\} = e^{at} \quad \therefore \quad L^{-1} \left\{ \frac{1}{(s-a)s} \right\} = \int_0^t e^{at} \, dt = \frac{e^{at}}{a} - \frac{1}{a}, \text{ by III above}$$

$$\therefore L^{-1} \left\{ \frac{1}{(s-a)s^2} \right\} = \frac{1}{a} \int_0^t (e^{at} - 1) \, dt = \frac{1}{a^2} (e^{at} - at - 1)$$

$$L^{-1} \left\{ \frac{1}{(s-a)s^3} \right\} = \frac{1}{a^2} \int_0^t (e^{at} - at - 1) \, dt = \frac{1}{a^3} \left(e^{at} - \frac{a^2}{2} t^2 - at - 1 \right)$$

$$\text{Hence } L^{-1} \left\{ \frac{1}{s(s+a)^3} \right\} = e^{-at} \cdot \frac{1}{a^3} \left(e^{at} - \frac{a^2 t^2}{2} - at - 1 \right) = \frac{1}{a^3} \left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2} t^2 e^{-at} \right).$$

Example 21.22. Find the inverse Laplace transforms of:

$$(i) \frac{s}{(s^2+a^2)^2} \quad (\text{S.V.T.U., 2009}) \quad (ii) \frac{s^2}{(s^2+a^2)^2} \quad (\text{Hazaribag, 2009}) \quad (iii) \frac{1}{(s^2+a^2)^2}.$$

Solution. (i) If $f(t) = L^{-1} \frac{s}{(s^2+a^2)^2}$, then by formula V above,

$$L \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2} \, ds = \frac{1}{2} \int_s^\infty \frac{2s}{(s^2+a^2)^2} \, ds = -\frac{1}{2} \left(\frac{1}{s^2+a^2} \right)_s^\infty = \frac{1}{2} \cdot \frac{1}{s^2+a^2}$$

$$\therefore \frac{f(t)}{t} = \frac{1}{2} L^{-1} \left(\frac{1}{s^2+a^2} \right) = \frac{\sin at}{2a}$$

Hence, $f(t) = \frac{1}{2a} t \sin at$.

Otherwise : Let $f(t) = L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{\sin at}{a}$ so that $\bar{f}(s) = \frac{1}{s^2 + a^2}$

Then by (IV) above, $t f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\} = L^{-1}\left\{-\frac{d}{ds}\left(\frac{1}{s^2 + a^2}\right)\right\}$

or $\frac{t \sin at}{a} = L^{-1}\left\{\frac{2s}{(s^2 + a^2)^2}\right\}$. Hence $L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a} t \sin at$.

(ii) In (i), we have proved that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a} t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$\begin{aligned} L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} &= L^{-1}\left\{s \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \frac{d}{dt}\{f(t)\} \\ &= \frac{d}{dt}\left(\frac{1}{2a} t \sin at\right) = \frac{1}{2a} (\sin at + at \cos at) \end{aligned}$$

(iii) In (i), we have shown that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a} (t \sin at) = f(t), \text{ say}$$

By formula III above, we have

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} &= L^{-1}\left\{\frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \int_0^t f(t) dt = \int_0^t \frac{t \sin at}{2a} dt \\ &= \frac{1}{2a} \left[\left| t \cdot \frac{-\cos at}{a} \right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a} \right) dt \right] \\ &= \frac{1}{2a} \left\{ \frac{-t \cos at}{a} + \frac{\sin at}{a^2} \right\} = \frac{1}{2a^3} (\sin at - at \cos at). \end{aligned}$$

Example 21.23. Find the inverse Laplace transforms of

$$(i) \frac{s+2}{s^2(s+1)(s-2)} \quad (\text{V.T.U., 2003}) \quad (ii) \frac{s+2}{(s^2+4s+5)^2}. \quad (\text{S.V.T.U., 2009; P.T.U., 2005})$$

Solution. (i) $L^{-1}\left\{\frac{s+2}{(s+1)(s-2)}\right\} = \frac{4}{3} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+1}\right) = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t}$

By III above, $L^{-1}\left\{\frac{s+2}{s(s+1)(s-2)}\right\} = \int_0^t L^{-1}\left(\frac{s+2}{(s+1)(s-2)}\right) dt$
 $= \int_0^t \left(\frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} \right) dt = \frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1$

Again by III above, $L^{-1}\frac{s+2}{s^2(s+1)(s-2)} = \int_0^t L^{-1}\left\{\frac{s+2}{s(s+1)(s-2)}\right\} dt$
 $= \int_0^t \left(\frac{2}{3} e^{2t} + \frac{1}{3} e^{-t} - 1 \right) dt = \frac{1}{3} (e^{2t} - e^{-t} - t)$

$$(ii) L^{-1} \left(\frac{1}{s^2 + 4s + 5} \right) = L^{-1} \left\{ \frac{1}{(s+2)^2 + 1} \right\} = e^{-2t} \sin t$$

$$\text{By II above, } L^{-1} \left\{ \frac{d}{ds} \left(\frac{1}{s^2 + 4s + 5} \right) \right\} = (-1)^1 t \cdot e^{-2t} \sin t$$

$$\text{i.e., } L^{-1} \left\{ \frac{-(2s+4)}{(s^2 + 4s + 5)^2} \right\} = -t \cdot e^{-2t} \sin t$$

$$\text{or } L^{-1} \left\{ \frac{s+2}{(s^2 + 4s + 5)^2} \right\} = \frac{1}{2} t \cdot e^{-2t} \sin t.$$

Example 21.24. Find the inverse Laplace transforms of the following :

$$(i) \log \frac{s+1}{s-1} \quad (\text{S.V.T.U., 2009; Bhopal, 2008}) \quad (ii) \log \frac{s^2+1}{s(s+1)} \quad (\text{S.V.T.U., 2009; V.T.U., 2008})$$

$$(iii) \cot^{-1} \left(\frac{s}{2} \right) \quad (iv) \tan^{-1} \left(\frac{2}{s^2} \right). \quad (\text{V.T.U., 2011; Mumbai, 2005 S})$$

Solution. (i) If $f(t) = L^{-1} \log \frac{s+1}{s-1}$, then by IV above,

$$\begin{aligned} t f(t) &= L^{-1} \left\{ -\frac{d}{ds} \log \left(\frac{s+1}{s-1} \right) \right\} = -L^{-1} \left\{ \frac{d}{ds} \log(s+1) \right\} + L^{-1} \left\{ \frac{d}{ds} \log(s-1) \right\} \\ &= -L^{-1} \left(\frac{1}{s+1} \right) + L^{-1} \left(\frac{1}{s-1} \right) = -e^{-t} + e^t = 2 \sinh t \end{aligned}$$

Thus $f(t) = (2 \sinh t)/t$.

(ii) If $f(t) = L^{-1} \log \frac{s^2+1}{s(s+1)}$, then by IV above,

$$\begin{aligned} t f(t) &= L^{-1} \left\{ -\frac{d}{ds} \log \left(\frac{s^2+1}{s(s+1)} \right) \right\} = -L^{-1} \left\{ \frac{d}{ds} \log(s^2+1) \right\} + L^{-1} \left\{ \frac{d}{ds} \log s \right\} \\ &\quad + L^{-1} \left\{ \frac{d}{ds} \log(s+1) \right\} \\ &= -L^{-1} \left(\frac{2s}{s^2+1} \right) + L^{-1} \left(\frac{1}{s} \right) + L^{-1} \left(\frac{1}{s+1} \right) = -2 \cos t + 1 + e^{-t} \end{aligned}$$

Thus $f(t) = \frac{1}{t} (1 + e^{-t} - 2 \cos t)$.

(iii) If $f(t) = L^{-1} \cot^{-1} \left(\frac{s}{2} \right)$, then by IV above,

$$t f(t) = L^{-1} \left\{ -\frac{d}{ds} \cot^{-1} \left(\frac{s}{2} \right) \right\} = L^{-1} \left(\frac{2}{s^2 + 2^2} \right) = \sin 2t$$

Thus $f(t) = (\sin 2t)/t$.

(iv) If $f(t) = L^{-1} \left(\tan^{-1} \frac{2}{s^2} \right)$, then by IV above,

$$t f(t) = L^{-1} \left\{ -\frac{d}{ds} \tan^{-1} \left(\frac{2}{s^2} \right) \right\} = L^{-1} \left\{ \frac{4s}{s^4 + 4} \right\}$$

$$\begin{aligned}
 &= L^{-1} \left\{ \frac{4s}{(s^2 + 2)^2 - (2s)^2} \right\} = L^{-1} \left\{ \frac{4s}{(s^2 + 2 + 2s)(s^2 + 2 - 2s)} \right\} \\
 &= L^{-1} \left\{ \frac{1}{s^2 - 2s + 2} - \frac{1}{s^2 + 2s + 2} \right\} = L^{-1} \left\{ \frac{1}{(s-1)^2 + 1} - \frac{1}{(s+1)^2 + 1} \right\} \\
 &= e^t \sin t - e^{-t} \sin t = 2 \sinh t \sin t.
 \end{aligned}$$

21.14 CONVOLUTION THEOREM

If $L^{-1}\{\bar{f}(s)\} = f(t)$, and $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $L^{-1}\{\bar{f}(s) \bar{g}(s)\} = \int_0^t f(u) g(t-u) du = F * G$

[$F * G$ is called the **convolution** or **faltung** of F and G .]

Let $\phi(t) = \int_0^t f(u) g(t-u) du$

$$L\{\phi(t)\} = \int_0^\infty e^{-st} \left\{ \int_0^t f(u) g(t-u) du \right\} dt = \int_0^\infty \int_0^t e^{-st} f(u) g(t-u) du dt \quad \dots(1)$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ (Fig. 21.2).

On changing the order of integration, we get

$$\begin{aligned}
 L\{\phi(t)\} &= \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u) dt du \\
 &= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-s(t-u)} g(t-u) dt \right\} du \\
 &= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-sv} g(v) dv \right\} du \text{ on putting } t-u=v \\
 &= \int_0^\infty e^{-su} f(u) g(s) du = \int_0^\infty e^{-su} f(u) du \cdot \bar{g}(s) \\
 &= \bar{f}(s) \cdot \bar{g}(s) \text{ whence follows the desired result.}
 \end{aligned}$$

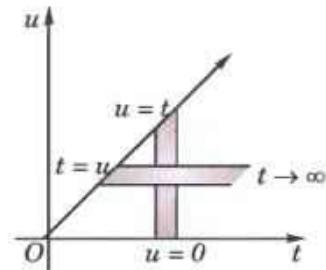


Fig. 21.2

Example 21.25. Apply Convolution theorem to evaluate

$$(i) L^{-1} \frac{s}{(s^2 + a^2)^2}, \quad (\text{V.T.U., 2010})$$

$$(ii) L^{-1} \frac{s^2}{(s^2 + a^2)(s^2 + b^2)}. \quad (\text{V.T.U., 2011 S ; Bhopal, 2008 ; Mumbai, 2007})$$

Solution. (i) Since $f(t) = L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \cos at$ and $g(t) = L^{-1} \left(\frac{s}{s^2 + a^2} \right) = \frac{1}{a} \sin at$

∴ by Convolution theorem, we get

$$\begin{aligned}
 L^{-1} \left[\frac{s}{s^2 + a^2} \cdot \frac{1}{s^2 + a^2} \right] &= \int_0^t \cos au \frac{\sin a(t-u)}{a} du & \because f(u) = \cos au \\
 &= \frac{1}{2a} \int_0^t [\sin at - \sin(2au - at)] dt = \frac{1}{2a} \left| u \sin at + \frac{1}{2a} \cos(2au - at) \right|_0^t = \frac{1}{2a} t \sin at
 \end{aligned}$$

Hence $L^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{1}{2a} t \sin at$.

$$\begin{aligned}
 f(u) &= \cos au \\
 g(t-u) &= \frac{1}{a} \sin a(t-u)
 \end{aligned}$$

(ii) Since $f(t) = L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\frac{s}{s^2 + b^2}\right) = \cos bt$,

\therefore by Convolution theorem, we get

$$\begin{aligned} L^{-1}\left(\frac{s}{s^2 + a^2} \cdot \frac{s}{s^2 + b^2}\right) &= \int_0^t \cos au \cos b(t-u) du \quad [\because f(u) = \cos au, g(t-u) = \cos b(t-u)] \\ &= \frac{1}{2} \int_0^t \{\cos [(a-b)u + bt] + \cos [(a+b)u - bt]\} du \\ &= \frac{1}{2} \left| \frac{\sin[(a-b)u + bt]}{a-b} + \frac{\sin[(a+b)u - bt]}{a+b} \right|_0^t \\ &= \frac{1}{2} \left\{ \frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b} \right\} = \frac{a \sin at - b \sin bt}{a^2 - b^2}. \end{aligned}$$

Example 21.26. Evaluate (i) $L^{-1} \frac{1}{(s^2 + 1)(s^2 + 9)}$

(Mumbai, 2005 S)

(ii) $L^{-1} \frac{s}{(s^2 + 1)(s^2 + 4)(s^2 + 9)}$.

(Madras, 2006)

Solution. (i) Since $L^{-1}\left(\frac{1}{s^2 + 1}\right) = \sin t$, $L^{-1}\left(\frac{1}{s^2 + 9}\right) = \frac{\sin 3t}{3}$

\therefore by Convolution theorem, we get

$$\begin{aligned} L^{-1}\left(\frac{1}{s^2 + 1} \cdot \frac{1}{s^2 + 9}\right) &= \int_0^t \sin u \cdot \frac{\sin 3(t-u)}{3} du \\ &= \frac{1}{6} \int_0^t [\cos(4u - 3t) - \cos(3t - 2u)] du = \frac{1}{6} \left| \frac{\sin(4u - 3t)}{4} - \frac{\sin(3t - 2u)}{-2} \right|_0^t \\ &= \frac{1}{6} \left\{ \frac{1}{4} (\sin t + \sin 3t) + \frac{1}{2} (\sin t - \sin 3t) \right\} = \frac{1}{8} (\sin t - \frac{1}{3} \sin 3t) \end{aligned}$$

(ii) Since $L^{-1}\left(\frac{s}{s^2 + 4}\right) = \cos 2t$ and $L^{-1}\left(\frac{1}{(s^2 + 1)(s^2 + 9)}\right) = \frac{1}{8} \left[\sin t - \frac{1}{3} \sin 3t \right]$

[By (i)]

\therefore by Convolution theorem, we get

$$\begin{aligned} L^{-1} \frac{s}{(s^2 + 1)(s^2 + 4)(s^2 + 9)} &= L^{-1} \left\{ \frac{1}{(s^2 + 1)(s^2 + 9)} \cdot \frac{s}{s^2 + 4} \right\} \\ &= \int_0^t \frac{1}{8} (\sin u - \frac{1}{3} \sin 3u) \cdot \cos 2(t-u) du \\ &= \frac{1}{8} \int_0^t [\sin u \cos 2(t-u) - \frac{1}{3} \sin 3u \cos 2(t-u)] du \\ &= \frac{1}{8} \int_0^t \left[\frac{1}{2} \{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{6} \{\sin(u+2t) - \sin(5u-2t)\} \right] du \\ &= \frac{1}{16} \left[\left| \frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3} \right|_0^t \right] - \frac{1}{48} \left[\left| -\cos(u+2t) + \frac{\cos(5u-2t)}{5} \right|_0^t \right] \\ &= \frac{1}{12} \cos t - \frac{1}{10} \cos 2t + \frac{1}{60} \cos 3t. \end{aligned}$$

PROBLEMS 21.5

Find the inverse transforms of :

1. $\frac{1}{s^2(s+5)}$. (Madras, 2003 S)

2. $\frac{1}{s(s+2)^3}$.

3. $\frac{s}{a^2 s^2 + b^2}$. (Madras, 2000 S)

4. $\frac{1}{s^2(s^2+a^2)}$.

5. $\frac{1}{s^3(s^2+1)}$.

6. $\frac{s+2}{(s^2+4s+8)^2}$. (Mumbai, 2006)

7. $\frac{2as}{(s^2+a^2)^2}$.

8. $\frac{s^2}{(s+a)^3}$.

9. $\log\left(\frac{1+s}{s}\right)$.

10. $\log\left(\frac{s+a}{s+b}\right)$. (Anna, 2003; U.P.T.U., 2003)

12. $\frac{1}{2} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$. (Mumbai, 2008; V.T.U., 2008)

14. $\log\frac{s^2+1}{(s-1)^2}$. (Madras, 2000 S)

15. $\tan^{-1}\left(\frac{2}{s}\right)$

(Mumbai, 2007; P.T.U., 2005)

16. $\cot^{-1}(s)$. (V.T.U., 2005)

17. $s \log \frac{s-1}{s+1}$

(Madras, 1999)

Using Convolution theorem, evaluate :

18. $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$.

19. $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

20. $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$.

21. $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$.

22. $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$

(Mumbai, 2009)

23. $L^{-1}\left\{\frac{s}{(s+2)(s^2+9)}\right\}$.

(V.T.U., 2008 S)

24. $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

(V.T.U., 2007; U.P.T.U., 2005)

25. $\frac{1}{(s^2+4s+13)^2}$.

(Mumbai, 2008)

26. Show that (i) $L^{-1}\left(\frac{1}{s} \sin \frac{1}{s}\right) = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \dots$

(ii) $L^{-1}\left(\frac{1}{s} \cos \frac{1}{s}\right) = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \dots$

21.15 (1) APPLICATION TO DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

(2) Working procedure to solve a linear differential equation with constant coefficients by transform method :

1. Take the Laplace transform of both sides of the differential equation using the formula of § 21.7, and the given initial conditions.

2. Transpose the terms with minus signs to the right.

3. Divide by the coefficient of \bar{y} , getting \bar{y} as a known function of s .

4. Resolve this function of s into partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Example 21.27. Solve by the method of transforms, the equation

$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$

(V.T.U., 2011 S)

Solution. Taking the Laplace transform of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] + 2[s^2 \bar{y} - sy(0) - y'(0)] - [s \bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

$$\text{On inversion, we get } y = L^{-1} \left(\frac{1}{(s-1)} - 3L^{-1} \left(\frac{1}{(s+2)} \right) + 2L^{-1} \left(\frac{1}{s+2} \right) \right)$$

$$\text{or } y = e^t - 3e^{-t} + 2e^{-2t} \text{ which is the desired result.}$$

Example 21.28. Use transform method to solve

$$\frac{d^2x}{dt^2} - 2 \frac{dx}{dt} + x = e^t \text{ with } x = 2, \frac{dx}{dt} = -1 \text{ at } t = 0.$$

(Anna, 2005 S)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2 \bar{x} - sx(0) - x'(0)] - 2[s \bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3} \text{ on breaking into partial fractions.}$$

$$\begin{aligned} \text{On inversion, we obtain } x &= 2L^{-1} \left(\frac{1}{s-1} \right) - 3L^{-1} \left(\frac{1}{(s-1)^2} \right) + L^{-1} \left(\frac{1}{(s-1)^3} \right) \\ &= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \frac{1}{2}t^2e^t. \end{aligned}$$

Example 21.29. Solve $(D^2 + n^2)x = a \sin(nt + \alpha)$, $x = Dx = 0$ at $t = 0$.

Solution. Taking the Laplace transforms of both sides, we get

$$[s^2 \bar{x} - sx(0) - x'(0)] + n^2 \bar{x} = aL\{\sin nt \cdot \cos \alpha + \cos nt \cdot \sin \alpha\}$$

On using the given conditions,

$$(s^2 + n^2)\bar{x} = a \cos \alpha \cdot \frac{n}{s^2 + n^2} + a \sin \alpha \cdot \frac{s}{s^2 + n^2}$$

$$\therefore \bar{x} = an \cos \alpha \cdot \frac{1}{(s^2 + n^2)^2} + a \sin \alpha \cdot \frac{s}{(s^2 + n^2)^2}$$

On inversion, we obtain

$$\begin{aligned} x &= an \cos \alpha \cdot \frac{1}{2n^3} (\sin nt - nt \cos nt) + a \sin \alpha \cdot \frac{t}{2n} \sin nt \\ &= a [\sin nt \cos \alpha - nt \cos(nt + \alpha)]/2n^2. \end{aligned}$$

[By (11) and (12) p. 741]

Example 21.30. Solve $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

(S.V.T.U., 2009)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 \bar{y} - sy(0) - y'(0)] + 3[s \bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using the given conditions, it reduces to

$$\begin{aligned}\bar{y} &= \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6} \\ &= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\end{aligned}$$

$$\begin{aligned}\text{On inversion, we obtain } y &= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{(s-1)^2}\right) - L^{-1}\left(\frac{1}{(s-1)^3}\right) + 2L^{-1}\left(\frac{1}{(s-1)^6}\right) \\ &= e^t \left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right).\end{aligned}$$

Example 21.31. Solve $\frac{d^2x}{dt^2} + 9x = \cos 2t$, if $x(0) = 1$, $x(\pi/2) = -1$. (Bhopal, 2008; U.P.T.U., 2006)

Solution. Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2 \bar{x} - s x(0) - x'(0)] + 9 \bar{x} = \frac{s}{s^2 + 4}$$

$$(s^2 + 9) \bar{x} = s + a + \frac{s}{s^2 + 4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or

$$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5} \cdot \frac{s}{s^2+4} + \frac{4}{5} \cdot \frac{s}{s^2+9}.$$

$$\text{On inversion, we get } x = \frac{a}{3} \sin 3t + \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t$$

$$\text{When } t = \pi/2, -1 = -\frac{a}{3} - \frac{1}{5} \quad \text{or} \quad \frac{a}{3} = \frac{4}{5}$$

$$\left[\because x\left(\frac{\pi}{2}\right) = -1 \right]$$

$$\text{Hence the solution is } x = \frac{1}{5} (\cos 2t + 4 \sin 3t + 4 \cos 3t).$$

Obs. Laplace transform method can also be used for solving ordinary differential equations with variable coefficients of the form $t^m y^{(n)}(t)$ because $L[t^m y^{(n)}(t)] = (-1)^m \frac{d^m}{ds^m} [L y^{(n)}(t)]$.

Example 21.32. Solve $ty'' + 2y' + ty = \cos t$ given that $y(0) = 1$.

(S.V.T.U., 2009)

Solution. Taking Laplace transform of both sides of the equation and noting that

$$L[t f(t)] = -\frac{d}{ds} [L[f(t)]], \text{ we get}$$

$$-\frac{d}{ds} [s^2 \bar{y} - sy(0) - y'(0)] + 2[s \bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2 + 1}$$

or

$$-\left(s^2 \frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2 + 1}$$

or

$$(s^2 + 1) \frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2 + 1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2 + 1} - \frac{s}{(s^2 + 1)^2}$$

On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t f(t)$, we get

$$-ty = -\sin t - \frac{1}{2}t \sin t$$

[See § 21.12 (11)]

or

$$y = \frac{1}{2} \left(1 + \frac{2}{t} \right) \sin t$$

which is the desired solution.

Example 21.33. Solve $x \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$, $y(0) = 2$, $y'(0) = 0$.

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

$$\text{or } -\frac{d}{ds}[s^2 \bar{y} - sy(0) - y'(0)] + [s \bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2 + 1) \frac{d\bar{y}}{ds} + s \bar{y} = 0$$

$$\text{Separating the variables, } \int \frac{d\bar{y}}{\bar{y}} + \int \frac{s ds}{s^2 + 1} = c$$

$$\text{or } \log \bar{y} + \frac{1}{2} \log(s^2 + 1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{s^2 + 1}}$$

$$\text{Inversion gives } y = c' J_0(x)$$

$$\text{To find } c', \text{ we have } y(0) = c' J_0(0), \text{ i.e., } c' = 2$$

$$\text{Hence } y = 2J_0(x).$$

Example 21.34. An alternating e.m.f. $E \sin \omega t$ is applied to an inductance L and a capacitance C in series.

Show by transform method, that the current in the circuit is $\frac{E\omega}{(p^2 - \omega^2)L} (\cos \omega t - \cos pt)$, where $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L \frac{di}{dt} + \frac{q}{C} = E \sin \omega t \quad [\because R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s \bar{i}(s) - i(0)] + \frac{1}{C} L(q) = E \cdot \frac{\omega}{s^2 + \omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore L s \bar{i}(s) + \frac{1}{C} L(q) = \frac{E\omega}{s^2 + \omega^2} \quad \dots(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = s L(q) - q(0)$$

$$L(q) = \bar{i}(s)/s$$

[\because $q(0) = 0$]

$$\therefore (i) \text{ becomes } L s \bar{i}(s) + \frac{1}{C} [\bar{i}(s)/s] = \frac{E\omega}{s^2 + \omega^2}$$

$$\text{or } \left(Ls + \frac{1}{Cs} \right) \bar{i}(s) = \frac{E\omega}{s + \omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2 + 1/LC)(s^2 + \omega^2)}$$

$$\text{or } \bar{i}(s) = \frac{E\omega}{L} \cdot \frac{s}{(s^2 + p^2)(s^2 + \omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2 - \omega^2)} \left\{ \frac{s}{s^2 + \omega^2} - \frac{s}{s^2 + p^2} \right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1} \left\{ \frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)} \right\}$$

or

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$. (V.T.U., 2008 S; Kurukshetra, 2005)
2. $(D^2 - 1)x = a \cosh t$, $x(0) = x'(0) = 0$.
3. $y'' + y = t$, $y(0) = 1$, $y'(0) = 0$. (Mumbai, 2009)
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. (V.T.U., 2010)
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3$, $y(0) = 5$. (Mumbai, 2008)
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2$, $y'(0) = 0$. (S.V.T.U., 2008)
7. $(D^2 + \omega^2)y = \cos \omega t$, $t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\frac{d^2y}{dt^2} + 2 \frac{dy}{dt} - 3y = \sin t$, $y = \frac{dy}{dt} = 0$ when $t = 0$. (Kurukshetra, 2005; Madras, 2003)
9. $\frac{d^4y}{dt^4} - k^4y = 0$, where $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$.
10. $y'''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
11. $\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. (P.T.U., 2010)
12. $y'' + 2y' + 5y = 5(t - 2)$, $y(0) = 0$, $y'(0) = 0$. (P.T.U., 2005 S)
13. $\frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^{2t}$, where $y = 1$, $\frac{dy}{dt} = 0$, $\frac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t$, $x = Dx = 0$ at $t = 0$. (Raipur, 2005; U.P.T.U., 2005)
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1$, $y'(0) = 2$. (P.T.U., 2002)
17. $y'' + 2ty' - y = t$, when $y(0) = 0$, $y'(0) = 1$. (U.P.T.U., 2003)
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3$, $y'(0) = 0$.
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R . Show (by the transform method) that the current at time t is $\frac{E}{R - aL} (e^{-at} - e^{-Rt/L})$. (V.T.U., 2000)
20. Work out example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k , when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

Hint : The equation of motion is $\frac{d^2x}{dt^2} + \frac{k}{m} x = \frac{F_0}{m} \sin at$ and $x = \frac{dx}{dt} = 0$ when $t = 0$.

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. Solve the simultaneous equations $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$ being given $x = y = 0$ when $t = 0$. [Ex. 13.38]

Solution. Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2 \quad i.e., \quad (s+5)\bar{x} - 2\bar{y} = 1/s^2$$

and

$$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad i.e., \quad 2\bar{x} + (s+1)\bar{y} = 0$$

$$\dots(i) \quad [\because x(0) = 0]$$

$$\dots(ii) \quad [\because y(0) = 0]$$

Solving (i) and (ii) for \bar{x} , we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} + \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of \bar{x} in (ii), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}, \quad y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

Example 21.36. The coordinates (x, y) of a particle moving along a plane curve at any time t , are given by $dy/dt + 2x = \sin 2t$, $dx/dt - 2y = \cos 2t$, ($t > 0$). If at $t = 0$, $x = 1$ and $y = 0$, show by transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$. (U.P.T.U., 2003)

Solution. Taking the Laplace transforms of the given equations and noting that $y(0) = 0$, $x(0) = 1$,

we get $[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2 + 2^2} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2 + 4}$... (i)

and $[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2 + 2^2} \quad \text{or} \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2 + 4} + 1$... (ii)

Multiplying (i) by s and (ii) by 2 and subtracting, we get

$$(s^2 + 4)\bar{y} = -2 \quad \text{or} \quad \bar{y} = -2/(s^2 + 4)$$

On inversion, $y = -2L^{-1}\left[\frac{1}{s^2 + 4}\right] = -\sin 2t$

From the given first equation,

$$2x = \sin 2t - dy/dt = \sin 2t - \frac{d}{dt}(-\sin 2t)$$

or $2x = \sin 2t + 2 \cos 2t \quad \text{or} \quad 4x^2 = (\sin 2t + 2 \cos 2t)^2$... (iii)

Also $4xy = (\sin 2t + 2 \cos 2t)(-2 \sin 2t) = -2(\sin^2 2t + 2 \sin 2t \cos 2t)$... (iv)

and $5y^2 = 5 \sin^2 2t$ (v)

Adding (iii), (iv), and (v), we obtain

$$\begin{aligned} 4x^2 + 4xy + 5y^2 &= \sin^2 2t + 4 \sin 2t \cos 2t + 4 \cos^2 2t - 2 \sin^2 2t \\ &\quad - 4 \sin 2t \cos 2t + 5 \sin^2 2t = 4 \sin^2 2t + 4 \cos^2 2t = 4. \end{aligned}$$

Example 21.37. The small oscillations of a certain system with two degrees of freedom are given by the equations : $D^2x + 3x - 2y = 0$, $D^2y + 3x + 5y = 0$ where $D = d/dt$. If $x = 0$, $y = 0$, $x = 3$, $y = 2$ when $t = 0$, find x and y when $t = 1/2$. [Example 13.41]

Solution. Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0 \quad i.e., \quad (s^2 + 3)\bar{x} - 2\bar{y} = 3 \quad \dots(i)$$

and $[s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0 \quad i.e., \quad (s^2 - 3)\bar{x} + (s^2 + 5)\bar{y} = 5 \quad \dots(ii)$

Solving (i) and (ii) for \bar{x} and \bar{y} , we get

$$\begin{aligned} \bar{x} &= \begin{vmatrix} 3 & -2 \\ 5 & s^2 + 5 \end{vmatrix} + \begin{vmatrix} s^2 + 3 & -2 \\ s^2 - 3 & s^2 + 5 \end{vmatrix} = \frac{3s^2 + 25}{(s^2 + 1)(s^2 + 9)} \\ &= \frac{11}{4} \cdot \frac{1}{s^2 + 1} + \frac{1}{4} \cdot \frac{1}{s^2 + 9} \end{aligned}$$

and

$$\bar{y} = \begin{vmatrix} s^2 + 3 & 3 \\ s^2 - 3 & 5 \end{vmatrix} + \begin{vmatrix} s^2 + 3 & -2 \\ s^2 - 3 & s^2 + 5 \end{vmatrix} = \frac{2s^2 + 24}{(s^2 + 1)(s^2 + 9)} = \frac{11}{4} \cdot \frac{1}{s^2 + 1} + \frac{3}{4} \cdot \frac{1}{s^2 + 9}.$$

On inversion, we get $x = \frac{11}{4} \sin t + \frac{1}{12} \sin 3t$; $y = \frac{11}{4} \sin t - \frac{1}{4} \sin 3t$

which are the same as the solution in (vii) on p. 499.

Obs. The student should compare the earlier solutions of the above examples with those given now and appreciate the superiority of the transform method over others.

PROBLEMS 21.7

Solve the following simultaneous equations (by using Laplace transforms):

1. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0) = 1$, $y(0) = 0$. (U.P.T.U., 2006; Delhi, 2002)
2. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$. (Kerala, 2005; U.P.T.U., 2004)
3. $\frac{d^2x}{dt^2} - x = y$, $\frac{d^2y}{dt^2} + y = -x$, given that at $t = 0$; $x = 2$, $y = -1$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. (P.T.U., 2009 S)
4. $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$; given $x = 0$, $y = 0$ when $t = 0$. (Madras, 2003 S)
5. $(D - 2)x - (D + 1)y = 6e^{3t}$; $(2D - 3)x + (D - 3)y = 6e^{3t}$ given $x = 3$, $y = 0$ when $t = 0$.
6. The currents i_1 and i_2 in mesh are given by the differential equations; $di_1/dt - \omega i_2 = a \cos pt$, $di_2/dt + \omega i_1 = a \sin pt$.

Find the currents i_1 and i_2 by Laplace transform, if $i_1 = i_2 = 0$ at $t = 0$.

21.17 (1) UNIT STEP FUNCTION

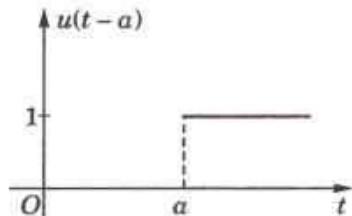
At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the *unit step function* (or *Heaviside's unit function**).

Def. The unit step function $u(t - a)$ is defined as follows:

$$u(t - a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

where, a is always positive (Fig. 21.3). It is also denoted as $H(t - a)$.

Fig. 21.3



(2) Transform of unit function.

$$L\{u(t - a)\} = \int_0^\infty e^{-st} u(t - a) dt = \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} \cdot 1 dt = 0 + \left[\frac{e^{-st}}{-s} \right]_a^\infty$$

Thus $L\{u(t - a)\} = e^{-as}/s$.

$$\text{The product } f(t) u(t - a) = \begin{cases} 0 & \text{for } t < a \\ f(t) & \text{for } t \geq a. \end{cases}$$

The function $f(t - a) \cdot u(t - a)$ represents the graph of $f(t)$ shifted through a distance a to the right and is of special importance.

Second shifting property. If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{f(t - a) \cdot u(t - a)\} = e^{-as} \bar{f}(s)$$

$$L\{f(t - a) \cdot u(t - a)\} = \int_0^\infty e^{-st} f(t - a) u(t - a) dt$$

*Named after the British Electrical Engineer Oliver Heaviside (1850–1925).

$$\begin{aligned}
 &= \int_0^a e^{-st} f(t-a) u(0) dt + \int_a^\infty e^{-st} f(t-a) dt \\
 &= \int_0^\infty e^{-s(u+a)} f(u) du = e^{-sa} \int_0^\infty e^{-su} f(u) du = e^{-as} \bar{f}(s).
 \end{aligned}
 \quad [\text{Put } t-a=u]$$

Example 21.38. Express the following function (Fig. 21.4) in terms of unit step function and find its Laplace transform. (U.P.T.U., 2002)

Solution. We have $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

$$\begin{aligned}
 \text{or } f(t) &= (t-1)[u(t-1)-u(t-2)] + u(t-2) \\
 &= (t-1)u(t-1) - (t-2)u(t-2)
 \end{aligned}$$

By second shifting property,

$$L[f(t-a)u(t-a)] = e^{-as} L[f(t)].$$

$$\text{Also } L[f(t)] = L(t) = 1/s^2.$$

$$\therefore L[(t-1)u(t-1)]$$

$$= e^{-s} \cdot \frac{1}{s^2} \text{ and } L[(t-2)u(t-2)] = e^{-2s} \cdot \frac{1}{s^2}$$

$$\text{Hence } L[f(t)] = L[(t-1)u(t-1) - (t-2)u(t-2)] = \frac{e^{-s} - e^{-2s}}{s^2}.$$

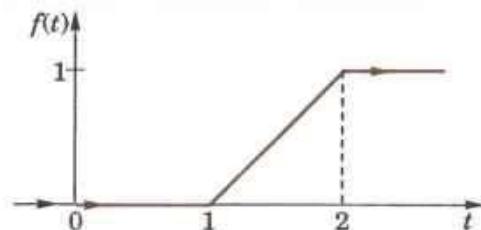


Fig. 21.4

Example 21.39. Using unit step function, find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ \sin 2t, & \pi \leq t < 2\pi \\ \sin 3t, & t \geq 2\pi \end{cases} \quad (\text{V.T.U., 2004})$$

$$\begin{aligned}
 \text{Solution. } f(t) &= \sin t [u(t-0) - u(t-\pi)] + \sin 2t [u(t-\pi) - u(t-2\pi)] + \sin 3t \cdot u(t-2\pi) \\
 &= \sin t + (\sin 2t - \sin t) u(t-\pi) + (\sin 3t - \sin 2t) u(t-2\pi)
 \end{aligned}$$

$$\text{Since } L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s) \text{ and } L(\sin at) = \frac{a}{s^2 + a^2},$$

$$\begin{aligned}
 L[f(t)] &= L(\sin t) + L[(\sin 2t - \sin t) \cdot u(t-\pi)] + L[(\sin 3t - \sin 2t) \cdot u(t-2\pi)] \\
 &= \frac{1}{s^2 + 1} + e^{-\pi s} \left(\frac{2}{s^2 + 4} - \frac{1}{s^2 + 1} \right) + e^{-2\pi s} \left(\frac{3}{s^2 + 9} - \frac{2}{s^2 + 4} \right).
 \end{aligned}$$

Example 21.40. (i) Express the function (Fig. 21.5) in terms of unit step function and find its Laplace transform. (P.T.U., 2005 S)

(ii) Obtain the Laplace transform of $e^{-t}[1 - u(t-2)]$.

Solution. (i) We have $f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$

or

$$\begin{aligned}
 f(t) &= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\
 &= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)
 \end{aligned}$$

$$\text{Since } L[f(t-a)u(t-a)] = e^{-as} \bar{f}(s) \quad \dots(\lambda)$$

$$\therefore L[f(t)] = e^{-s} \cdot \frac{1}{s^2} - 2e^{-2s} \cdot \frac{1}{s^2} + e^{-3s} \cdot \frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2} \quad [\because f(t)=t]$$

$$(ii) L[e^{-t}[1 - u(t-2)]] = L(e^{-t}) - L[e^{-t}u(t-2)] = \frac{1}{s+1} - e^{-2} L[e^{-(t-2)}u(t-2)]$$

Taking $f(t) = e^{-t}$, $\bar{f}(s) = \frac{1}{s+1}$ and using (λ) above,

$$L\{e^{-(t-2)}u(t-2)\} = e^{-2s} \cdot \frac{1}{s+1}$$

Hence $L e^{-t} \{1 - u(t-2)\} = \{1 - e^{-2(s+1)}\}/(s+1)$.

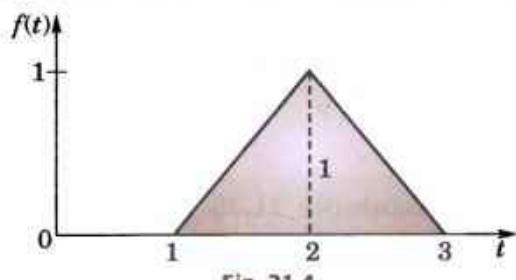


Fig. 21.4

Example 21.41. Using Laplace transform, evaluate $\int_0^\infty e^{-t} (1 + 2t - t^2 + t^3) H(t-1) dt$.

(Mumbai, 2007)

Solution. We have $L\{(1 + 2t - t^2 + t^3) H(t-1)\}$

$$\begin{aligned} &= e^{-s} L[1 + 2(t+1) - (t+1)^2 + (t+1)^3] = e^{-s} L(3 + 3t + 2t^2 + t^3) \\ &= e^{-s} \left(3 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s^2} + 2 \cdot \frac{2!}{s^3} + \frac{3!}{s^4} \right) = e^{-s} \left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4} \right) \end{aligned}$$

By definition, this implies that

$$\int_0^\infty e^{-st} (1 + 2t - t^2 + t^3) H(t-1) dt = e^{-s} \left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4} \right)$$

Taking $s = 1$, we obtain

$$\int_0^\infty e^{-t} (1 + 2t - t^2 + t^3) H(t-1) dt = e^{-1} (3 + 3 + 4 + 6) = 16/e.$$

Example 21.42. Evaluate (i) $L^{-1} \left\{ \frac{e^{-s} - 3e^{-3s}}{s^2} \right\}$

(U.P.T.U., 2002)

(ii) $L^{-1} \left\{ \frac{se^{-as}}{s^2 - w^2} \right\}, a > 0$.

Solution. $L^{-1} \left\{ e^{-s} \cdot \frac{1}{s^2} \right\} = \begin{cases} t-1, & t > 1 \\ 0, & t < 1 \end{cases} = (t-1) u(t-1)$

$$L^{-1} \left\{ e^{-3s} \cdot \frac{1}{s^2} \right\} = \begin{cases} t-3, & t > 3 \\ 0, & t < 3 \end{cases} = (t-3) u(t-3)$$

$$\therefore L^{-1} \left\{ \frac{e^{-s} - 3e^{-3s}}{s^2} \right\} = L^{-1} \left(\frac{e^{-s}}{s^2} \right) - 3L^{-1} \left(\frac{e^{-3s}}{s^2} \right) = (t-1) u(t-1) - 3(t-3) u(t-3)$$

(ii) We know that $L^{-1} \left(\frac{s}{s^2 - w^2} \right) = \cosh wt$

$$\therefore L^{-1} \left(\frac{se^{-as}}{s^2 - w^2} \right) = \begin{cases} \cosh w(t-a), & t > a \\ 0, & t < a \end{cases}$$

= $\cosh w(t-a) u(t-a)$, by second shifting property.

Example 21.43. Find the inverse Laplace transform of :

(i) $\frac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ (V.T.U., 2000) (ii) $\frac{e^{-cs}}{s^2(s+a)}$ ($c > 0$).

(Kurukshestra, 2005)

Solution. (i) Since $L^{-1} \frac{s}{s^2 + \pi^2} = \cos \pi t$, $L^{-1} \left(\frac{\pi}{s^2 + \pi^2} \right) = \sin \pi t$

and

$$L^{-1}[e^{-as} \bar{f}(s)] = f(t-a) \cdot u(t-a) \quad \dots(\lambda)$$

$$\begin{aligned} \therefore L^{-1}\left\{\frac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}\right\} &= L^{-1}\left\{e^{-s/2} \cdot \frac{s}{s^2 + \pi^2}\right\} + L^{-1}\left\{e^{-s} \cdot \frac{\pi}{s^2 + \pi^2}\right\} \\ &= \cos \pi(t - 1/2) \cdot u(t - 1/2) + \sin \pi(t - 1) \cdot u(t - 1) \\ &= \sin \pi t \cdot u(t - 1/2) - \sin \pi t \cdot u(t - 1) = \{u(t - 1/2) - u(t - 1)\} \sin \pi t \end{aligned}$$

$$(ii) L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2} \cdot \frac{1}{s} + \frac{1}{a} \cdot \frac{1}{s^2} + \frac{1}{a^2} \cdot \frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$\begin{aligned} L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} &= -\frac{1}{a^2}\{1 \cdot u(t-c)\} + \frac{1}{a}\{(t-c) \cdot u(t-c)\} + \frac{1}{a^2}\{e^{-a(t-c)} \cdot u(t-c)\} \\ &= \frac{1}{a^2}\{a(t-c) - 1 + e^{-a(t-c)}\} u(t-c). \end{aligned}$$

Example 21.44. A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.

Solution. The constant force F acting from $t = 0$ to $t = T$ can be expressed as

$$F[1 - u(t-T)], \quad 0 < t < T$$

\therefore equation of motion of the particle is

$$m \frac{d^2x}{dt^2} = F[1 - u(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = \frac{F}{m}[1 - u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2 + k^2) \bar{x} = \frac{F}{ms} (1 - e^{-sT}) \quad [\because x = 0, \dot{x} = 0 \text{ at } t = 0]$$

or

$$\begin{aligned} \bar{x} &= \frac{F}{m} \cdot \frac{1 - e^{-sT}}{s(s^2 + k^2)} = \frac{F}{m} (1 - e^{-sT}) \cdot \frac{1}{k^2} \left(\frac{1}{s} - \frac{s}{s^2 + k^2} \right) \\ &= \frac{F}{mk^2} \left\{ (1 - e^{-sT}) \frac{1}{s} - (1 - e^{-sT}) \cdot \frac{s}{s^2 + k^2} \right\} \end{aligned}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2} [(1 - \cos kt) - (1 - \cos k(t-T))] u(t-T)$$

i.e.,

$$x = \frac{F}{mk^2} (1 - \cos kt) \text{ for } 0 < t < T$$

and

$$x = \frac{F}{mk^2} (1 - \cos kt) - (1 - \cos k(t-T)) \text{ for } t > T$$

$$= \frac{F}{mk^2} (\cos k(t-T) - \cos kt) \text{ for } t > T$$

or

$$x = \frac{2F}{mk^2} \sin \frac{kT}{2} \cdot \sin k(t - T/2) \text{ for } t > T$$

Hence the amplitude of subsequent oscillation (i.e., for $t > T$) = $\frac{2F}{mk^2} \sin \frac{kT}{2}$.

Example 21.45. In an electrical circuit with e.m.f. $E(t)$, resistance R and inductance L , the current i builds up at the rate given by

$$L di/dt + Ri = E(t). \quad \dots(i)$$

If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E & \text{for } 0 < t < a \\ 0 & \text{for } t > a \end{cases}$

∴ taking the Laplace transform of both sides, (i) becomes

$$(Ls + R)i = \int_0^\infty e^{-st} E(t) dt = \int_0^a e^{-st} Edt = \frac{E}{s} (1 - e^{-as})$$

or

$$i = \frac{E}{s(Ls + R)} - \frac{Ee^{-as}}{s(Ls + R)}$$

On inversion, we get

$$i = L^{-1} \left\{ \frac{E}{s(Ls + R)} \right\} - L^{-1} \left\{ \frac{Ee^{-as}}{s(Ls + R)} \right\} \quad \dots(ii)$$

$$\text{Now } L^{-1} \left\{ \frac{E}{s(Ls + R)} \right\} = \frac{E}{R} \left\{ L^{-1} \left(\frac{1}{s} \right) - L^{-1} \left(\frac{1}{s + R/L} \right) \right\} = \frac{E}{R} (1 - e^{-Rt/L})$$

and

$$L^{-1} \left\{ \frac{Ee^{-as}}{s(Ls + R)} \right\} = \frac{E}{R} \{1 - e^{-R(t-a)/L}\} u(t-a) \quad [\text{By the second shifting property}]$$

$$\text{Thus (ii) becomes } i = \frac{E}{R} [1 - e^{-Rt/L}] - \frac{E}{R} \{1 - e^{-R(t-a)/L}\} u(t-a)$$

$$\text{Hence } i = \frac{E}{R} \{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

$$\text{and } i = \frac{E}{R} [(1 - e^{-Rt/L}) - (1 - e^{-R(t-a)/L})] = \frac{E}{R} e^{-Rt/L} (e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.

Solution. Taking the origin at the end A, we have

$$EI \frac{d^4 y}{dx^4} = w(x)$$

where $w(x) = w[u(x - l/4) - u(x - 3l/4)]$

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0)]$$

$$= w \left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s} \right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI \bar{y} = w \left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5} \right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$\text{On inversion, we get } EIy = \frac{w}{24} [(x - l/4)^4 u(x - l/4) - (x - 3l/4)^4 u(x - 3l/4)] + \frac{1}{2} c_1 x^2 + \frac{1}{6} c_2 x^3 \quad \dots(i)$$

$$\text{For } x > 3l/4, \quad EIy = \frac{w}{24} [(x - l/4)^2 - (x - 3l/4)^2] + \frac{1}{2} c_1 x^2 + \frac{1}{6} c_2 x^3$$

and

$$EIy' = \frac{w}{6} [(x - l/4)^3 - (x - 3l/4)^3] + c_1 x + \frac{1}{2} c_2 x^2$$

$$\text{Using the conditions } y(l) = 0 \text{ and } y'(l) = 0, \text{ we get } 0 = \frac{w}{24} \left\{ \left(\frac{3l}{4} \right)^4 - \left(\frac{l}{4} \right)^4 \right\} + \frac{1}{2} c_1 l^2 + \frac{1}{6} c_2 l^3$$

and

$$0 = \frac{w}{6} \left\{ \left(\frac{3l}{4} \right)^3 - \left(\frac{l}{4} \right)^3 \right\} + c_1 l + \frac{1}{2} c_2 l^2$$

$$\text{whence } c_1 = 11wl^2/192; c_2 = -wl/4.$$

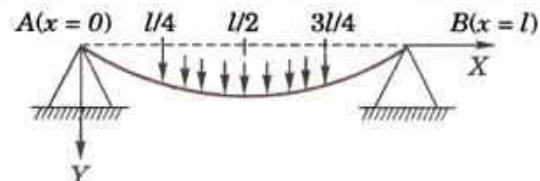


Fig. 21.6

Thus for $l/4 < x < 3l/4$, (i) gives $EIy = \frac{w}{24} \left(x + \frac{1}{4} \right)^4 + \frac{11wl^2}{384} x^2 - \frac{wl}{24} x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

$$\begin{aligned}\delta_\epsilon(t-a) &= 1/\epsilon, \quad a \leq t \leq a + \epsilon \\ &= 0, \quad \text{otherwise}\end{aligned}$$

as $\epsilon \rightarrow 0$. It is clear from Fig. 21.7 that as $\epsilon \rightarrow 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \infty \text{ for } t = a; = 0 \text{ for } t \neq a,$$

such that

$$\int_0^\infty \delta(t-a) dt = 1. \quad (a \geq 0)$$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ϵ per unit length over the portion of the beam between $x = a$ and $x = a + \epsilon$. Thus

$$\begin{aligned}w(x) &= w_0/\epsilon \quad a < x < a + \epsilon, \\ &= 0, \quad \text{otherwise}\end{aligned}$$

i.e.,

$$w(x) = w_0 \delta(x-a).$$

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\begin{aligned}\int_0^\infty f(t) \delta_\epsilon(t-a) dt &= \int_0^{a+\epsilon} f(t) \cdot \frac{1}{\epsilon} dt \\ &= (a + \epsilon - a) f(\eta) \cdot \frac{1}{\epsilon} = f(\eta),\end{aligned} \quad \text{where } a < \eta < a + \epsilon.$$

by Mean value theorem for integrals.

As $\epsilon \rightarrow 0$, we get $\int_0^\infty f(t) \delta(t-a) dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $L\{\delta(t-a)\} = e^{-as}$.

Example 21.47. Evaluate (i) $\int_0^\infty \sin 2t \delta(t-\pi/4) dt$ (ii) $L\left[\frac{1}{t}\delta(t-a)\right]$.

Solution. (i) We know that $\int_0^\infty f(t) \delta(t-a) dt = f(a)$

$$\therefore \int_0^\infty \sin 2t \delta(t-\pi/4) dt = \sin(2 \cdot \pi/4) = 1$$

(ii) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\begin{aligned}\therefore L\left[\frac{1}{t}\delta(t-a)\right] &= \int_s^\infty L[\delta(t-a)] ds = \int_s^\infty e^{-as} ds \\ &= \left[\frac{e^{-as}}{-a} \right]_s^\infty = \frac{1}{a} e^{-as}.\end{aligned}$$

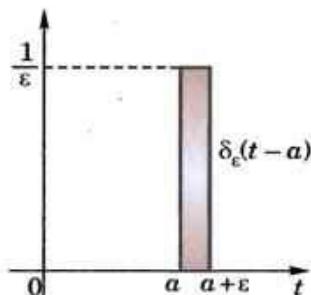


Fig. 21.7

* After the English physicist Paul Dirac (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.

Example 21.48. An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L , R , C in series with zero initial conditions. If i be the current at any subsequent time t , find the limit of i as $t \rightarrow 0$?

Solution. The equation of the circuit governing the current i is

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int_0^t i \, dt = E\delta(t) \quad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L [s \bar{i} - i(0)] + R \bar{i} + \frac{1}{C} \frac{1}{s} \bar{i} = E \quad [\text{Using § 21.7 and 21.8}]$$

or $\left(s^2 + \frac{R}{L}s + \frac{1}{CL} \right) \bar{i} = \frac{E}{L}s \quad \text{or} \quad (s^2 + 2as + a^2 + b^2) \bar{i} = (E/L)s$

where $R/L = 2a$ and $1/CL = a^2 + b^2$

or $\bar{i} = \frac{E}{L} \frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L} \left\{ \frac{s+a}{(s+a)^2+b^2} - a \frac{1}{(s+a)^2+b^2} \right\}$

On inversion, we get

$$i = \frac{E}{L} \left\{ e^{-at} \cos bt - \frac{a}{b} e^{-at} \sin bt \right\}$$

Taking limits as $t \rightarrow 0$, $i \rightarrow E/L$

Although the current $i = 0$ initially, yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact, we have determined the limit of this current which is E/L .

Example 21.49. A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.

Solution. The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI} \delta(x - l/4)$$

Taking the Laplace transform, we have $s^4 \bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{w}{EI} e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI} \frac{e^{-ls/4}}{s^4}.$$

On inversion, it gives $y = c_1 x + c_2 \frac{x^3}{3!} + \frac{w}{EI} \frac{(x-l/4)^3}{3!} u(x-l/4)$

i.e., $y = c_1 x + \frac{1}{6} c_2 x^3, \quad 0 < x < l/4$

and $y = c_1 x + \frac{1}{6} c_2 x^3 + \frac{\omega}{6EI} (x-l/4)^3, \quad l/4 < x < l$

...(i)

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get

$$0 = c_1 l + \frac{1}{6} c_2 l^3 + 9wl^3/128EI \quad \text{and} \quad 0 = c_1 + \frac{1}{2} c_2 l^2 + 9wl^2/32EI$$

whence $c_1 = 9wl^2/256 EI, \quad c_2 = -81w/128EI$.

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

PROBLEMS 21.8

- Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$ and 0 otherwise, in terms of the unit step function and hence find its Laplace transform. (Mumbai, 2005)
- Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms :

(i) $f(t) = 2t$ for $0 < t < \pi$, $f(t) = 1$ for $t > \pi$

(Assam, 1999)

(ii) $f(t) = t^2$ for $0 < t \leq 2$, $f(t) = 0$ for $t > 2$ (iii) $f(t) = \cos(wt + \phi)$ for $0 < t < T$, $f(t) = 0$ for $t > T$.

3. Express the following functions in terms of unit step function and hence find its Laplace transform.

$$(i) f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases} \quad (\text{V.T.U., 2007})$$

$$(ii) f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$$

(Mumbai, 2008 ; V.T.U., 2003 S)

$$(iii) f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$$

(V.T.U., 2011)

4. Evaluate (i) $L\{e^{t-1} u(t-1)\}$ (ii) $L\{(t-1)^2 u(t-1)\}$ (iii) $L\{1 + 2t - 3t^2 + 4t^3\} H(t-2)$ (Mumbai, 2007) (iv) $L\{t^2 u(t-1) + \delta(t-1)\}$.5. Evaluate $\int_0^\infty e^{-t}(1+3t+t^2)u(t-2)dt$.

6. Find the inverse Laplace transforms of:

$$(i) \frac{e^{-8s}}{s^2 + 1}$$

$$(ii) \frac{e^{-2s}}{s^2 + 8s + 25}$$

$$(iii) \frac{e^{-s}}{(s+1)^3} \quad (\text{P.T.U., 2010})$$

$$(iv) \frac{3}{s} - 4 \frac{e^{-s}}{s^2} + 4 \frac{e^{-3s}}{s^2} \quad (\text{P.T.U., 2002 S})$$

7. Solve using Laplace transforms $\frac{d^2y}{dt^2} + 4y = f(t)$ with conditions

$$y(0) = 0, y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 & \text{when } 0 < t < 1 \\ 0 & \text{when } t > 1 \end{cases}$$

(Mumbai, 2007)

8. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0) = 0$

where $u(t) = \begin{cases} 3, & 0 \leq t \leq 4 \\ 2t-5, & t > 4. \end{cases}$

9. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x = l/3$. Find the resulting deflection.

Hint. The differential equation and the boundary conditions are $\frac{d^4y}{dx^4} = \frac{W}{EI} \delta(x - l/3)$ and

$$y(0) = y'(0) = 0, y(l) = y'(l) = 0. \quad \boxed{}$$

10. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load w per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point. (Kurukshetra, 2006)**Hint.** The differential equation and boundary conditions are

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI} \quad (0 < x < l) \text{ where } W(x) = \begin{cases} W_0, & 0 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

and

$$y(0) = y'(0) = 0, y''(0) = y'''(0) = 0.$$

11. An impulse I (kg-sec) is applied to a mass m attached to a spring having a spring constant k . The system is damped with damping constant μ . Derive expressions for displacement and velocity of the mass, assuming initial conditions $x(0) = x'(0) = 0$.

Hint. The equation of motion is $m \frac{d^2x}{dt^2} = I \delta(x) - kx - \mu \frac{dx}{dt}$.

21.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 21.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. Laplace transform of $(t \sin t) = \dots$
2. $L\{\delta(t)\} = \dots$
 - (a) 0
 - (b) e^{-as}
 - (c) ∞
 - (d) 1.
3. If $L[f(t)] = f(s)$, then $L[e^{-at}f(t)]$ is
 - (a) $f(s-a)$
 - (b) $f(s+a)$
 - (c) $f(s)$
 - (d) none of these.
4. $L(e^{2t} \sin t) = \dots$
5. Inverse Laplace transform of $(s+2)^{-2}$ is \dots
6. Inverse Laplace transform of $1/(s^2 + 4s + 13) = \dots$
7. Laplace transform of $f'(t) = \dots$
8. $L^{-1}\left[\frac{s}{(2s+3)^2}\right] = \dots$
9. $L(\cosh^2 2t) = \dots$
10. $L(e^t) = \dots$
11. $L(e^{-t} t^k) = \dots$
12. $\int_0^\infty e^{-2t} \cos 3t dt = \dots$
13. $L[u(t-a)] = \dots$
14. $L^{-1}(\sqrt{t}) = \dots$
15. If $L[F(t)] = f(s)$, then $L\left\{\frac{d^2F(t)}{dt^2}\right\} = \dots$
16. $L(\cos^3 4t) = \dots$
17. $L\left(\frac{\sin at}{t}\right) = \dots$
18. $L^{-1}\left\{\frac{1}{(s+3)^5}\right\} = \dots$
19. $L \cos(2t+3) = \dots$
20. $L^{-1}(1/s^n)$ is possible only when n is
 - (a) zero
 - (b) -ve integer
 - (c) +ve integer
 - (d) negative rational.
21. If $L^{-1}[\phi(s)] = f(t)$, then $L^{-1}[e^{-as}\phi(s)] = \dots$
22. $L[u(t+2)] = \dots$
 - (a) e^{-2s}/s^2
 - (b) e^{2s}
 - (c) $\frac{e^{2s}}{s}$
 - (d) $\frac{e^{-2s}}{s}$(V.T.U., 2011 S)
23. $L^{-1}\left\{\frac{s^2 - 3s + 4}{s^3}\right\} = \dots$ (V.T.U., 2010 S)
24. If $L[f(t)] = \bar{f}(s)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \dots$
25. If $f(t)$ is a periodic function with period T , then $L[f(t)] = \dots$
26. If y satisfies $y'' + 3y' + 2y = e^{-t}$ with $y(0) = y'(0) = 0$, then $L[y(t)] = \dots$
27. $L[e^{3t}(2 \cos 5t + 3 \sin 4t)] = \dots$
28. $L(4^t) = \dots$
29. $L^{-1}\left\{\frac{1}{\sqrt{(s+3)}}\right\} = \dots$
30. Laplace transform of $\sin 2t \delta(t-2)$ is
 - (a) $e^{2s} \sin 4$
 - (b) $e^{-2s} \sin 2$
 - (c) $e^{-4s} \sin 2$
 - (d) $e^{-2s} \sin 4$.(V.T.U., 2009 S)
31. If $L^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \frac{t \sin t}{2}$ then $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$ (P.T.U., 2009)
32. $L^{-1}[e^{-as} F(s)] =$
 - (a) $f(t) u(t)$
 - (b) $f(t-a) u(t)$
 - (c) $f(t-a) u(t-a)$
 - (d) None of these.(V.T.U., 2009 S)
33. $L^{-1}\left\{\frac{1}{(s+a)^2}\right\} =$
 - (a) e^{at}
 - (b) e^{-at}
 - (c) $t e^{-at}$
 - (d) $t e^{at}$
 - (e) $-t$.

34. Laplace transform of $t^4 e^{-at}$ is

- (i) $\frac{4!}{(s+a)^4}$ (ii) $\frac{4!}{(s-a)^5}$ (iii) $\frac{4!}{(s-a)^4}$ (iv) $\frac{5!}{(s-a)^5}$

35. Laplace transform of $te^{at} \sin(at)$, $t > 0$, is

- (i) $\frac{s-a}{(s-a)^2 + a^2}$ (ii) $\frac{a(s-a)}{(s-a)^2 + a^2}$ (iii) $\frac{2a(s-a)}{[(s-a)^2 + a^2]^2}$ (iv) $\frac{(s-a)^2}{(s-a)^2 + a^2}$.

36. $L^{-1} \frac{s^2}{(s^2+4)^2}$ is

- (i) $\frac{1}{4} \sin 2t + t \cos 2t$ (ii) $\frac{1}{4} \sin 2t + \frac{1}{2} \cos 2t$ (iii) $\sin 2t + \frac{t}{2} \cos 2t$ (iv) $\frac{1}{2} \sin 2t + \frac{t}{4} \cos 2t$.

37. $L^{-1} \frac{1}{s(s^2+1)}$ is

- (i) $1 + \sin t$ (ii) $1 - \sin t$ (iii) $1 + \cos t$ (iv) $1 - \cos t$.

38. $L[u(t-a)]$ where $u(t-a)$ is a unit step function, is

- (i) $\frac{e^{-as}}{s}$ (ii) $\frac{e^{as}}{s}$ (iii) $\frac{e^{-as}}{s^2}$ (iv) $\frac{e^{as}}{s^2}$. (V.T.U., 2011)

39. For a periodic function of period 2π , $\int_{a+2\pi}^{b+2\pi} f(x) dx = \dots\dots$ (P.T.U., 2009)

40. $L[\delta(t-a)]$ where $\delta(t-a)$ is a unit impulse function, is

- (i) e^{at} (ii) e^{-at} (iii) e^a (iv) e^{-at}/s (V.T.U., 2010 S)

41. Laplace transform of $\sin^2 3t$ is

- (i) $\frac{3}{s^2+36}$ (ii) $\frac{6}{(s^2+36)}$ (iii) $\frac{18}{s(s^2+36)}$ (iv) $\frac{18}{s^2+36}$ (V.T.U., 2010)

42. $[L(t^2 e^{-3t})] =$

- (i) $\frac{1}{(s+3)^3}$ (ii) $\frac{2}{(s+3)^2}$ (iii) $\frac{3}{(s+3)^3}$ (iv) $\frac{2}{(s+3)^3}$ (V.T.U., 2011)

43. $\frac{d^2}{ds^2} [L f(t)] - L(t^2 f(t)) = 0$. (True or False)

44. Laplace transform of $f(t)$ is defined for +ve and -ve values of t . (True or False)

45. If $L[f(t)] = \phi(s)$, then $L[t f(t)] = \frac{d}{ds} [\phi(s)]$. (True or False)

Fourier Transforms

1. Introduction.
2. Definition.
3. Fourier integrals — Fourier sine and cosine integral – Complex forms of Fourier integral.
4. Fourier transform — Fourier sine and cosine transforms — Finite Fourier sine and cosine transforms.
5. Properties of F-transforms.
6. Convolution theorem for F-transforms.
7. Parseval's identity for F-transforms.
8. Relation between Fourier and Laplace transforms.
9. Fourier transforms of the derivatives of a function—
10. Inverse Laplace transforms by method of residues.
11. Application of transforms to boundary value problems.
12. Objective Type of Questions.

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained e.g., conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x) dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x . The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(i) When $K(s, x) = e^{-sx}$, it leads to the **Laplace transform** of $f(x)$, i.e.,

$$\bar{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx.$$

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(ii) When $K(s, x) = e^{isx}$, we have the **Fourier transform** of $f(x)$, i.e.,

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

(iii) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ i.e.,

$$M(s) = \int_0^\infty f(x) x^{s-1} dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad \dots(1)$$

where

$$a_0 = \frac{1}{c} \int_{-c}^c f(t) dt, \quad a_n = \frac{1}{c} \int_{-c}^c f(t) \cos \frac{n\pi t}{c} dt, \quad \text{and} \quad b_n = \frac{1}{c} \int_{-c}^c f(t) \sin \frac{n\pi t}{c} dt.$$

Substituting the values of a_0 , a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c} \int_{-c}^c f(t) dt + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^c f(t) \cos \frac{n\pi(t-x)}{c} dt \quad \dots(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)| dx$ converges, the first term on the right side of (2) approaches 0 as $c \rightarrow \infty$, since

$$\left| \frac{1}{2c} \int_{-c}^c f(t) dt \right| \leq \frac{1}{2c} \int_{-\infty}^{\infty} |f(t)| dt$$

The second term on the right side of (2) tends to

$$\begin{aligned} & \underset{c \rightarrow \infty}{\text{Lt}} \frac{1}{c} \sum_{n=1}^{\infty} \int_{-c}^{\infty} f(t) \cos \frac{n\pi(t-x)}{c} dt \\ &= \underset{\delta\lambda \rightarrow 0}{\text{Lt}} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda(t-x) dt, \text{ on writing } \pi/c = \delta\lambda \end{aligned}$$

This is of the form $\underset{\delta\lambda \rightarrow 0}{\text{Lt}} \sum_{n=1}^{\infty} F(n\delta\lambda)$, i.e., $\int_0^{\infty} F(\lambda) d\lambda$

$$\text{Thus as } c \rightarrow \infty, (2) \text{ becomes } f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad \dots(3)$$

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos \lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda \quad \dots(4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad \dots(5)$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t dt d\lambda \quad \dots(6)$$

which is known as the *Fourier cosine integral*.

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) Complex form of Fourier integrals. Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad \dots(7)$$

because $\cos \lambda(t-x)$ is an even function of λ . Also since $\sin \lambda(t-x)$ is an odd function of λ , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x) dt d\lambda \quad \dots(8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)} dt d\lambda \quad \dots(9)$$

which is the *complex form of the Fourier integral*.

(4) Fourier integral representation of a function

Using (4), a function $F(x)$ may be represented by a Fourier integral as

$$F(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] d\lambda$$

where $A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos \lambda t dt ; B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin \lambda t dt \quad \dots(10)$

If $f(x)$ is an odd function, then

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin \lambda x d\lambda \text{ where } B(\lambda) = 2 \int_0^\infty f(t) \sin \lambda t dt \quad \dots(11)$$

If $f(x)$ is an even function, then

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x d\lambda \text{ where } A(\lambda) = 2 \int_0^\infty f(t) \cos \lambda t dt \quad \dots(12)$$

Example 22.1. Express $f(x) = 1$ for $0 \leq x \leq \pi$,

$$= 0 \text{ for } x > \pi,$$

as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda. \quad (\text{Kottayam, 2005; J.N.T.U., 2004 S})$$

Solution. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\infty f(t) \sin(\lambda t) dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\infty \sin(\lambda t) dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \left| \frac{-\cos(\lambda t)}{\lambda} \right|_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \leq x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2} \left[\frac{f(\pi - 0) + f(\pi + 0)}{2} \right] = \frac{\pi}{2} \cdot \frac{1+0}{2} = \frac{\pi}{4}.$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} ds \int_{-\infty}^{\infty} f(t)e^{ist} dt,$$

it follows that if

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{ist} dt \quad \dots(1)$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \quad \dots(2)$$

The function $F(s)$, defined by (1), is called the **Fourier transform** of $f(x)$. Also the function $f(x)$, as given by (2), is called the **inverse Fourier transform** of $F(s)$. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x) \sin sx dx \quad \dots(3)$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx ds \quad \dots(4)$$

The function $F_s(s)$, as defined by (3), is known as the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (4) is called the **inverse Fourier sine transform** of $F_s(s)$.

Similarly, it follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x) \cos sx dx \quad \dots(5)$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx ds \quad \dots(6)$$

The function $F_c(s)$ as defined by (5) is known as the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by (6), is called the **inverse Fourier cosine transform** of $F_c(s)$.

(3) Finite Fourier sine and cosine transforms. These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** of $f(x)$, in $0 < x < c$, is defined as

$$F_s(n) = \int_0^c f(x) \sin \frac{n\pi x}{c} dx \quad \dots(7)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** of $F_s(n)$ which is given by

$$f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{c} \quad \dots(8)$$

The **finite Fourier cosine transform** of $f(x)$, in $0 < x < c$, is defined as

$$F_c(n) = \int_0^c f(x) \cos \frac{n\pi x}{c} dx \quad \dots(9)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** of $F_c(n)$ which is given by

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c} \quad \dots(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) Linear property. If $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F[a f(x) + b g(x)] = a F(s) + b G(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x) dx$

$$\begin{aligned}\therefore F[af(x) + bg(x)] &= \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)] dx = a \int_{-\infty}^{\infty} e^{isx} f(x) dx + b \int_{-\infty}^{\infty} e^{isx} g(x) dx \\ &= aF(s) + bG(s)\end{aligned}$$

(2) **Change of scale property.** If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$... (i)

$$\begin{aligned}\therefore F[f(ax)] &= \int_{-\infty}^{\infty} e^{isx} f(ax) dx \quad \left| \begin{array}{l} \text{Put } ax = t \\ \text{so that } dx = dt/a \end{array} \right. \\ &= \int_{-\infty}^{\infty} e^{ist/a} f(t) dt / a = \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt = \frac{1}{a} F\left(\frac{s}{a}\right) \quad [\text{By (i)}]\end{aligned}$$

Cor. If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of $f(x)$ respectively, then

$$F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right).$$

(3) **Shifting property.** If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(x-a)] = e^{isa} F(s)$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$... (i)

$$\begin{aligned}\therefore F[f(x-a)] &= \int_{-\infty}^{\infty} e^{isx} f(x-a) dx \quad \left| \begin{array}{l} \text{Put } x-a = t \\ \text{so that } dx = dt \end{array} \right. \\ &= \int_{-\infty}^{\infty} e^{ist+a} f(t) dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t) dt = e^{isa} F(s) \quad [\text{By (i)}]\end{aligned}$$

(4) **Modulation theorem.** If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)]$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$... (i)

$$\begin{aligned}\therefore F[f(x) \cos ax] &= \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax dx = \int_{-\infty}^{\infty} e^{isx} f(x) \cdot \frac{e^{iax} + e^{-iax}}{2} dx \\ &= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx \right] = \frac{1}{2} [F(s+a) + F(s-a)].\end{aligned}$$

Cor. If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then

$$(i) F_s[f(x) \cos ax] = \frac{1}{2} [F_s(s+a) + F_s(s-a)] \quad (\text{Anna, 2008})$$

$$(ii) F_c[f(x) \sin ax] = \frac{1}{2} [F_c(s+a) - F_c(s-a)]$$

$$(iii) F_s[f(x) \sin ax] = \frac{1}{2} [F_c(s-a) - F_c(s+a)]$$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x} dx$.

(V.T.U., 2010; S.V.T.U., 2009; U.P.T.U., 2008)

Solution. The Fourier transform of $f(x)$, i.e.,

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-1}^1 (1) e^{isx} dx = \left| \frac{e^{isx}}{is} \right|_1^{-1} = \frac{e^{is} - e^{-is}}{is}$$

Thus $F[f(x)] = F(s) = 2 \frac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds, \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin s}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi \quad \therefore \int_0^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. Find the Fourier transform of:

$$f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx$. (V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)

Solution. $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)$, say

$$\begin{aligned} &= \int_{-\infty}^{-1} (0) e^{isx} dx + \int_{-1}^1 (1 - x^2) e^{isx} dx + \int_1^{\infty} (0) e^{isx} dx = \left| (1 - x^2) \frac{e^{isx}}{is} - (2x) \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right|_{-1}^1 \\ &= 2 \left(\frac{e^{is} + e^{-is}}{-s^2} \right) - 2 \left(\frac{e^{is} - e^{-is}}{-is^3} \right) = -\frac{4}{s^3} (s \cos s - \sin s) \end{aligned}$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$\text{or } -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (s \cos s - \sin s) e^{-isx} ds = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (s \cos s - \sin s) e^{-is/2} ds = \frac{3}{4}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \frac{s}{2} - i \sin \frac{s}{2} \right) ds = -\frac{3\pi}{8}$$

$$\text{or } \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \cdot \cos \frac{s}{2} ds = -\frac{3\pi}{8}$$

$$\text{or } \int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cdot \cos \frac{x}{2} dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (a) Find the Fourier transform of e^{-ax^2} , $a < 0$. Hence deduce that $e^{-x^2/2}$ is self reciprocal in respect of Fourier transform. (Madras, 2006 ; Kottayam, 2005)

(b) Find Fourier transform of (i) $e^{-2(x-3)^2}$ (ii) $e^{-x^2} \cos 3x$.

$$\begin{aligned} \text{Solution. (a)} \quad F(e^{-ax^2}) &= \int_{-\infty}^{\infty} e^{-ax^2} \cdot e^{isx} dx = \int_{-\infty}^{\infty} e^{-a(x^2 - isx/a^2)} dx \\ &= \int_{-\infty}^{\infty} e^{-a^2(x-is/2a^2)^2} \cdot e^{-s^2/4a^2} dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2} dt/a \\
 &= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi}
 \end{aligned}
 \quad [\text{Putting } a(x - is/2a^2) = t, dx = dt/a]$$

Hence $F(e^{-a^2x^2}) = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ is self-reciprocal under the Fourier transform.

(b) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$$\therefore \text{ by change of scale property, } F[f(2x)] = \frac{1}{2} F(s/2)$$

$$\text{i.e., } F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

$$\text{By shifting property } F[f(x-3)] = e^{3is} F(3)$$

$$\therefore F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is-s^2)/8} \quad \dots(i)$$

Also by modulation theorem,

$$\begin{aligned}
 F[f(x) \cos 2x] &= \frac{1}{2} [F(s+a) + F(s-a)] \\
 F(e^{-x^2} \cos 3x) &= \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}].
 \end{aligned} \quad \dots(ii)$$

Example 22.5. Find the Fourier cosine transform of e^{-x^2} .

(V.T.U., 2010; Rajasthan, 2006)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx dx = I$ (say)

Differentiating under the integral sign w.r.t. s ,

$$\begin{aligned}
 \frac{dI}{ds} &= - \int_0^{\infty} xe^{-x^2} \sin sx dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx \\
 &= \frac{1}{2} \left\{ \left[\sin sx \cdot e^{-x^2} \right]_0^{\infty} - s \int_0^{\infty} \cos sx \cdot e^{-x^2} dx \right\} \\
 &= -\frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c
 \end{aligned}$$

or

$$\log I = -\frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$$\therefore I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} \cos sx dx = ce^{-s^2/4}$$

$$\text{Putting } s = 0, \quad c = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad \text{i.e. } I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Hence $F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$

Example 22.6. Find the Fourier sine transform of $e^{-|x|}$.

Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx = \frac{\pi e^{-m}}{2}$, $m > 0$. (V.T.U., 2010; S.V.T.U., 2008; Kottayam, 2005)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

\therefore Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx dx = \int_0^\infty e^{-x} \sin sx dx$$

$$= \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin sx dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx ds$$

$$\text{or changing } x \text{ to } m, \quad e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+m^2} dx$$

$$\text{Hence } \int_0^\infty \frac{x \sin mx}{1+m^2} dx = \frac{\pi e^{-m}}{2}.$$

Example 22.7. Find the Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ (J.N.T.U., 2006)

Solution. Fourier cosine transform of $f(x)$ i.e., $F_c\{f(x)\}$

$$\begin{aligned} &= \int_0^\infty f_c(x) \cos sx dx = \int_0^1 x \cos sx dx + \int_1^2 (2-x) \cos sx dx + \int_2^\infty 0 \cdot dx \\ &= \left| x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right|_0^1 + \left| (2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right|_1^2 \\ &= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right) \\ &= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}. \end{aligned}$$

Example 22.8. Find the Fourier sine transform of e^{-ax}/x . (V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

$$\text{i.e. } F_s\{f(x)\} = \int_0^\infty f(x) \sin sx dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s , we get

$$\frac{d}{ds} \{F(s)\} = \int_0^\infty \frac{xe^{-ax} \cos sx}{x} dx = \int_0^\infty e^{-ax} \cos sx dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s , we obtain $F(s) = \int_0^\infty \frac{a}{s^2 + a^2} ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. Find the Fourier cosine transform of $f(x) = 1/(1+x^2)$. (V.T.U., 2011 S ; Anna, 2009)

Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$. (V.T.U., 2009 S)

Solution. $F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I, \text{ say}$... (i)

$$\therefore \frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx \quad \text{... (ii)}$$

$$= - \int_0^\infty \frac{[(1+x^2)-1] \sin sx}{x(1+x^2)} dx = - \int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx$$

or $\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \quad \dots(iii)$

$$\therefore \frac{d^2I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = I$$

or $\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2 - 1)I = 0, \text{ where } D = \frac{d}{ds}$

Its solution is $I = c_1 e^s + c_2 e^{-s} \quad \dots(iv)$

$$\therefore dI/ds = c_1 e^s - c_2 e^{-s} \quad \dots(v)$$

$$\text{When } s = 0, (i) \text{ and } (iv) \text{ give } c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$\text{Also when } s = 0, (iii) \text{ and } (v) \text{ give } c_1 - c_2 = -\pi/2.$$

$$\text{Solving these, } c_1 = 0, c_2 = \pi/2.$$

$$\text{Thus from (i) and (iv), we have } F_c[f(x)] = I = (\pi/2)e^{-s}$$

Now $F_s[\phi(x)] = \int_0^\infty \frac{x \sin sx}{1+x^2} dx = -\frac{dI}{ds}, \text{ from (ii)}$
 $= (\pi/2)e^{-s}, \text{ from (v), with } c_1 = 0, c_2 = \pi/2.$

Example 22.10. Find the Fourier sine and cosine transform of x^{n-1} , $n > 0$.

(Madras, 2006)

Solution. We know that $F_s(x^{n-1}) = \int_0^\infty x^{n-1} \sin sx dx \quad \dots(i)$

and $F_c(x^{n-1}) = \int_0^\infty x^{n-1} \cos sx dx \quad \dots(ii)$

$$\begin{aligned} \therefore F_c(x^{n-1}) + i F_s(x^{n-1}) &= \int_0^\infty (\cos sx + i \sin sx) x^{n-1} dx \\ &= \int_0^\infty e^{isx} x^{n-1} dx = \int_0^\infty e^{-t} \left(-\frac{t}{is}\right)^{n-1} \left(-\frac{dt}{is}\right) \quad [\text{Where } isx = -t] \\ &= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t} t^{n-1} dt = \frac{(i)^{2n}}{(i)^n s^n} \Gamma(n) = \frac{(i)^n}{s^n} \Gamma(n) \\ &= \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right) \Gamma(n)/s^n \end{aligned}$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n} \cos \frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin \frac{n\pi}{2}.$$

Example 22.11. (a) Show that $F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$; $F_c[x f(x)] = \frac{d}{ds}[F_s(s)]$.

(b) Find the Fourier sine and cosine transform of $x e^{-ax}$

(Madras, 2006)

Solution. (a) $\frac{d}{ds}[F_c(s)] = \frac{d}{ds} \left\{ \int_0^\infty f(x) \cos sx dx \right\} = \int_0^\infty f(x) (-x \sin sx) dx$
 $= - \int_0^\infty [x f(x)] \sin sx dx = -F_s[x f(x)] \quad \dots(i)$

$$\begin{aligned} \frac{d}{ds}[F_s(s)] &= \frac{d}{ds} \left\{ \int_0^\infty f(x) \sin sx dx \right\} = \int_0^\infty f(x) (x \cos sx) dx \\ &= \int_0^\infty [x f(x)] \cos sx dx = F_c[x f(x)] \quad \dots(ii) \end{aligned}$$

(b) We have

$$\begin{aligned} F_s(e^{-ax}) &= \int_0^\infty e^{-ax} \sin sx dx = \frac{e^{-ax}}{a^2 + s^2} [-a \sin sx - s \cos sx]_0^\infty \\ &= \frac{s}{a^2 + s^2} \end{aligned} \quad \dots(iii)$$

and

$$\begin{aligned} F_c(e^{-ax}) &= \int_0^\infty e^{-ax} \cos sx dx = \frac{e^{-ax}}{a^2 + s^2} [-a \cos sx + s \sin sx]_0^\infty \\ &= \frac{a}{a^2 + s^2} \end{aligned} \quad \dots(iv)$$

Now

$$F_c(xe^{-ax}) = -\frac{d}{ds} (F_c(e^{-ax})) \quad [\text{by (i)}]$$

$$= -\frac{d}{ds} \left(\frac{a}{a^2 + s^2} \right) = \frac{2as}{(a^2 + s^2)^2} \quad [\text{by (iv)}]$$

$$F_c(xe^{-ax}) = \frac{d}{ds} \{F_s(e^{-ax})\} \quad [\text{by (ii)}]$$

$$= \frac{d}{ds} \left(\frac{s}{a^2 + s^2} \right) = \frac{(a^2 + s^2) - s(2s)}{(a^2 + s^2)^2} = \frac{a^2 - s^2}{(a^2 + s^2)^2}. \quad [\text{by (iii)}]$$

Example 22.12. If the Fourier sine transform of $f(x) = \frac{1 - \cos nx}{n^2 \pi^2}$ ($0 \leq x \leq \pi$), find $f(x)$. (Delhi, 2002)

Solution. We have $f(x) = \text{inverse finite Fourier sine transform of } F_s(n)$

$$\begin{aligned} &= \frac{2}{\pi} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{1 - \cos nn\pi}{n^2 \pi^2} \right\} \sin nx \\ &= \frac{2}{\pi^3} \sum_{n=1}^{\infty} \left\{ \frac{1 - \cos nn\pi}{n^2} \right\} \sin nx. \end{aligned}$$

Example 22.13. Solve the integral equation*

$$\int_0^\infty f(\theta) \cos \alpha \theta d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$. (V.T.U., 2011 S ; Kurukshetra, 2005)

Solution. We have $\int_0^\infty f(\theta) \cos \alpha \theta d\theta = F_c(\alpha)$

$$\therefore F_c(\alpha) = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases} \quad \dots(i)$$

By the inversion formula, we have

$$\begin{aligned} f(\theta) &= \frac{2}{\pi} \int_0^\infty F_c(\alpha) \cos \alpha \theta d\alpha = \frac{2}{\pi} \int_0^1 (1 - \alpha) \cos \alpha \theta d\alpha \quad [\text{Integrating by parts}] \\ &= \frac{2}{\pi} \left[\left| (1 - \alpha) \frac{\sin \alpha \theta}{\theta} \right|_0^1 - \int_0^1 (-1) \frac{\sin \alpha \theta}{\theta} d\alpha \right] = \frac{2}{\pi \theta} \left| -\frac{\cos \alpha \theta}{\theta} \right|_0^1 = \frac{2(1 - \cos \theta)}{\pi \theta^2} \end{aligned}$$

Now

$$F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha \theta d\theta = \int_0^\infty \frac{2(1 - \cos \theta)}{\pi \theta^2} \cos \alpha \theta d\theta \quad \dots(ii)$$

* Refer to Chapter 26.

∴ From (i) and (ii), we have

$$\frac{2}{\pi} \int_0^\infty \frac{1 - \cos \theta}{\theta^2} \cos \alpha \theta d\theta = \begin{cases} 1 - \alpha, & 0 \leq \alpha \leq 1 \\ 0, & \alpha > 1 \end{cases}$$

Now letting $\alpha \rightarrow 0$, we get $\frac{2}{\pi} \int_0^\infty \frac{1 - \cos \theta}{\theta^2} d\theta = 1$ (V.T.U., 2008)

or

$$\int_0^\infty \frac{2 \sin^2 \theta/2}{\theta^2} d\theta = \pi/2 \quad (\text{Put } \theta/2 = t, \text{ so that } d\theta = 2dt)$$

$$\therefore \int_0^\infty \frac{\sin^2 t}{t^2} dt = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as a Fourier integral.

Hence evaluate $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$. (Kottayam, 2005)

2. Find the Fourier integral representation for

$$(i) f(x) = \begin{cases} 1 - x^2, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases} \quad (\text{Mumbai, 2008}) \quad (ii) f(x) = \begin{cases} e^{ax}, & \text{for } x \leq 0, a > 0 \\ e^{-ax}, & \text{for } x \geq 0 a < 0 \end{cases}$$

3. Using the Fourier integral representation, show that

$$(i) \int_0^\infty \frac{\omega \sin x\omega}{1 + \omega^2} d\omega = \frac{\pi}{2} e^{-x} \quad (x > 0) \quad (ii) \int_0^\infty \frac{\cos ax}{1 + \omega^2} d\omega = \frac{\pi}{2} e^{-x} \quad (x \geq 0) \quad (\text{U.P.T.U., 2008})$$

$$(iii) \int_0^\infty \frac{\sin \omega \cos x\omega}{\omega} d\omega = \frac{\pi}{2} \quad \text{when } 0 \leq x < 1. \quad (iv) \int_0^\infty \frac{\sin \pi\alpha \sin \alpha\theta}{1 - \alpha^2} d\alpha = \begin{cases} \frac{1}{2} \pi \sin \theta, & 0 \leq \theta \leq \pi \\ 0, & \theta > \pi \end{cases}$$

4. Find the Fourier transforms of

$$(i) f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \quad (\text{W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003})$$

Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x} dx$ (Mumbai, 2009)

$$(ii) f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases} \quad (\text{S.V.T.U., 2008})$$

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$ (V.T.U., 2007)

Hence deduce that $\int_0^\infty \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$. (Anna, 2009)

6. Given $F(e^{-x^2}) = \sqrt{\pi} e^{-s^2/4}$, find the Fourier transform of

$$(i) e^{-x^2/3} \quad (ii) e^{-4(x-3)^2}$$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 0, & x \geq 2 \end{cases}$ (V.T.U., 2008)

8. Using the Fourier sine transform of e^{-ax} ($a > 0$), show that $\int_0^\infty \frac{x \sin kx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ak}$ ($k > 0$).

Hence obtain the Fourier sine transform of $x/(a^2 + x^2)$. (Rohtak, 2006 ; Madras, 2003 S)

9. Find the Fourier cosine transform of e^{-ax} . (Anna, 2009)

Hence evaluate $\int_0^\infty \frac{\cos \lambda x}{x^2 + \alpha^2} dx$. (V.T.U., 2003 S)

10. If the Fourier sine transform of $f(x)$ is e^{-ax}/s , find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (Mumbai, 2009)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2} .
12. Find the Fourier cosine transform of $e^{-a^2 x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. (Anna, 2009)
13. Find the Fourier sine transform of (i) $\frac{1}{x(x^2 + a^2)}$. (Rohtak, 2006)
(ii) $[e^{-ax}/x]$, $a > 0$ (U.P.T.U., 2008)
14. Obtain Fourier sine transform of
(i) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ (Madras, 2000) (ii) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ (V.T.U., 2006)
15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. (P.T.U., 2006)
16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. (V.T.U., 2011)
17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \leq c \leq \pi$.
18. Show that the inverse finite Fourier sine transform of $F_s(n) = \frac{1}{\pi} \left\{ 1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right\}$ is
 $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$ (V.T.U., 2008)
19. Solve the integral equation $\int_0^\infty f(x) \sin tx dx = \begin{cases} 1, & 0 \leq t < 1, \\ 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ (Kottayam, 2005)
20. Solve the integral equation $\int_0^\infty f(x) \cos ax dx = e^{-a}$. (S.V.T.U., 2009; Rohtak, 2004)

22.6 (1) CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du = h(x).$$

(2) Convolution theorem for Fourier transforms. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, i.e.,

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$$

We have

$$\begin{aligned} F[f(x) * g(x)] &= F \left\{ \int_{-\infty}^{\infty} f(u) g(x-u) du \right\} \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(u) g(x-u) du \right\} e^{isx} dx = \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} g(x-u) \cdot e^{isx} dx \right\} du \\ &\quad [\text{Changing the order of integration}] \\ &= \int_{-\infty}^{\infty} f(u) \left\{ \int_{-\infty}^{\infty} e^{is(x-u)} \cdot g(x-u) d(x-u) \right\} e^{isu} du \\ &= \int_{-\infty}^{\infty} e^{isu} f(u) \left\{ \int_{-\infty}^{\infty} e^{ist} g(t) dt \right\} du \text{ where } x-u=t \\ &= \int_{-\infty}^{\infty} e^{isu} f(u) du \cdot F[g(t)] = \int_{-\infty}^{\infty} e^{isx} f(x) dx \cdot F[g(x)] = F[f(x)] \cdot F[g(x)] \end{aligned}$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

$$(i) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \bar{G}(s) ds = \int_{-\infty}^{\infty} f(x) \bar{g}(x) dx \quad (ii) \frac{1}{2\pi} \int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where bar implies the complex conjugate.

$$\begin{aligned}
 (i) \quad & \int_{-\infty}^{\infty} f(x) \bar{g}(dx) \int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) e^{isx} ds \right\} dx \quad [\text{Using the inversion formula for Fourier transform}] \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) \left\{ \int_{-\infty}^{\infty} f(x) e^{isx} dx \right\} ds \quad [\text{Changing the order of integration}] \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}(s) F(s) ds, \text{ by definition of F-transform.}
 \end{aligned}$$

(ii) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \bar{F}(s) ds = \int_{-\infty}^{\infty} f(x) \bar{f}(x) dx \text{ or } \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

$$\begin{array}{ll}
 (i) \quad \frac{2}{\pi} \int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx & (ii) \quad \frac{2}{\pi} \int_0^{\infty} F_s(s) G_s(s) ds = \int_0^{\infty} f(x) g(x) dx \\
 (iii) \quad \frac{2}{\pi} \int_0^{\infty} |F_c(s)|^2 ds = \int_0^{\infty} |f(x)|^2 dx & (iv) \quad \frac{2}{\pi} \int_0^{\infty} |F_s(s)|^2 ds = \int_0^{\infty} |f(x)|^2 dx.
 \end{array}$$

Example 22.14. Using Parseval's identities, prove that

$$\begin{array}{ll}
 (i) \quad \int_0^{\infty} \frac{dt}{(a^2 + t^2)(b^2 + t^2)} = \frac{\pi}{2ab(a+b)} & (S.V.T.U., 2009 ; I.E.T., 1998) \\
 (ii) \quad \int_0^{\infty} \frac{t^2}{(t^2 + 1)^2} dt = \frac{\pi}{4} & (iii) \quad \int_0^{\infty} \frac{\sin at}{t(a^2 + t^2)} dt = \frac{\pi}{2} \cdot \frac{1 - e^{-a^2}}{a^2}.
 \end{array}$$

Solution. (i) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2 + s^2}$, $G_c(s) = \frac{b}{b^2 + s^2}$

Now using Parseval's identity for Fourier cosine transforms, i.e.,

$$\frac{2}{\pi} \int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx \quad \dots(1)$$

We have $\frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2 + s^2)(b^2 + s^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$

or $\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(a^2 + s^2)(b^2 + s^2)} = \left| \frac{e^{-(a+b)x}}{-(a+b)} \right|_0^{\infty} = \frac{1}{a+b}$

Thus $\int_0^{\infty} \frac{dt}{(a^2 + t^2)(b^2 + t^2)} = \frac{\pi}{2ab(a+b)}$

(ii) Let $f(x) = \frac{x}{x^2 + 1}$ so that $F_s[f(x)] = \frac{\pi}{2} e^{-s}$

Now using Parseval's identity for sine transform, i.e.,

$$\frac{2}{\pi} \int_0^{\infty} [F_s(f(x))]^2 ds = \int_0^{\infty} |f(x)|^2 dx$$

or $\int_0^{\infty} \left(\frac{x}{x^2 + 1} \right)^2 dx = \frac{2}{\pi} \int_0^{\infty} \left(\frac{\pi}{2} e^{-s} \right)^2 ds = \frac{\pi}{2} \left| e^{-2s} / -2 \right|_0^{\infty} = \frac{\pi}{4} (0 - 1) = \frac{\pi}{4}$

Hence $\int_0^{\infty} \frac{t^2}{(t^2 + 1)^2} dt = \frac{\pi}{4}$

(iii) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2 + s^2}$, $G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi} \int_0^\infty \frac{a \sin as}{s(a^2 + s^2)} ds = \int_0^a e^{-ax} \cdot 1 dx = \frac{1 - e^{-a^2}}{a}$

Thus

$$\int_0^\infty \frac{\sin at}{t(a^2 + t^2)} dt = \frac{\pi}{2a^2} (1 - e^{-a^2}).$$

Example 22.15. Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} a - |x|, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$.

Hence show that $\int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$ and $\int_0^\infty \left(\frac{\sin t}{t} \right)^4 dt = \pi/3$. (Anna, 2008)

Solution. Fourier transform of $f(x)$ i.e. $F[f(x)] = \int_{-\infty}^\infty f(x) e^{isx} dx = \int_{-a}^a [a - |x|] e^{isx} dx$

$$\begin{aligned} &= \int_{-a}^a [a - |x|] (\cos sx + i \sin sx) dx \\ &= 2 \int_0^a (a - x) \cos sx dx + 0 && \left[\because [a - |x|] \cos x \text{ is an even function} \right. \\ &= 2 \left| (a - x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right|_0^a = 2 \frac{1 - \cos as}{s^2} = 4 \frac{\sin^2 as/2}{s^2} \end{aligned}$$

(i) By inversion formula,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{-isx} ds = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{4 \sin^2 as/2}{s^2} e^{-isx} ds$$

To evaluate $\int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi} \int_{-\infty}^\infty \frac{\sin^2 s}{s^2} ds = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin s}{s} \right)^2 ds && \left[\because \frac{\sin s}{s} \text{ is an even function} \right]$$

$$\therefore \int_0^\infty \left(\frac{\sin s}{s} \right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{2}. && [\because f(0) = a = 2]$$

(ii) Using Parseval's identity

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^\infty |F(s)|^2 ds &= \int_{-\infty}^\infty |f(x)|^2 dx \\ \frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{4 \sin^2 as/2}{s^2} \right)^2 ds &= \int_{-a}^a |[a - |x|]|^2 dx \\ \frac{16}{\pi} \int_0^\infty \left(\frac{\sin as/2}{s} \right)^4 ds &= 2 \int_0^a (a - x)^2 dx = 2 \left| \frac{(a - x)^3}{-3} \right|_0^a = \frac{2}{3} a^3 \end{aligned}$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi} \int_0^\infty \left(\frac{\sin t}{2t/a} \right)^2 \frac{2}{a} dt = \frac{2}{3} a^3 \quad \text{or} \quad \frac{2a^3}{\pi} \int_0^\infty \left(\frac{\sin t}{t} \right)^4 dt = \frac{2}{3} a^3$$

Hence

$$\int_0^\infty \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}.$$

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)
3. Using Parseval's identity, show that
 - (i) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$. (Hissar, 2007)
 - (ii) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$. (Rohtak, 2003)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$. Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (Anna, 2009)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If $f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}$... (i)
 then $F\{f(t)\} = L\{g(t)\}$.
 We have
$$\begin{aligned} F\{f(t)\} &= \int_{-\infty}^{\infty} e^{ist} f(t) dt = \int_{-\infty}^0 e^{ist} \cdot 0 \cdot dt + \int_0^{\infty} e^{ist} \cdot e^{-xt} g(t) dt \\ &= \int_0^{\infty} e^{(is-x)t} g(t) dt = \int_0^{\infty} e^{-pt} g(t) dt \quad \text{where } p = x - is \end{aligned}$$

Hence the Fourier transform of $f(t)$ [defined by (i)] is the Laplace transform of $g(t)$.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x, t)] = \int_{-\infty}^{\infty} ue^{isx} dx$$

Then the Fourier transform of $\partial^2 u / \partial x^2$, i.e.

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx} dx = \left| e^{isx} \frac{\partial u}{\partial x} - is e^{isx} \cdot u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} ue^{isx} dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm \infty$, then

$$F\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -s^2 \mathbf{F}[\mathbf{u}] \quad \dots (1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F}_s\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s}(\mathbf{u})_{\mathbf{x}=0} - s^2 \mathbf{F}_s[\mathbf{u}] \quad \dots (2)$$

$$\mathbf{F}_c\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - s^2 \mathbf{F}_c[\mathbf{u}] \quad \dots (3)$$

In general, the Fourier transform of the n th derivative of $f(x)$ is given by

$$\mathbf{F} \left[\frac{\mathbf{d}^n \mathbf{f}}{\mathbf{dx}^n} \right] = (-is)^n \mathbf{F}[\mathbf{f}(x)] \quad \dots(4)$$

provided the first $n - 1$ derivatives vanish as $x \rightarrow \pm \infty$.

$$\begin{aligned} \text{For } \mathbf{F}[f^n(x)] &= \int_{-\infty}^{\infty} f^n(x) e^{isx} dx \\ &= \left| e^{isx} f^{n-1} - is e^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \dots \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \cdot e^{isx} dx \end{aligned}$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \quad \dots(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\begin{aligned} \int_{a-ir}^{a+ir} e^{xs} \bar{f}(s) ds &= \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st} dt ds \\ &= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t} dt (-idu) = ie^{ax} \int_{-r}^r e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut} dt du \\ &= ie^{ax} \int_{-r}^r e^{-ixu} \int_{-\infty}^{\infty} \phi(t) e^{iut} dt du \end{aligned} \quad [\text{Put } s = a - iu]$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \rightarrow \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds = ie^{ax} \cdot 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi ie^{ax} \cdot e^{-ax} f(x) \text{ for } x > 0.$$

$$\text{Hence } f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs} \bar{f}(s) ds \quad (x > 0) \quad \dots(2)$$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (i.e., MNL). Then from (2)

$$\frac{1}{2\pi i} \int_{LM} e^{xs} \bar{f}(s) ds = \frac{1}{2\pi i} \int_C e^{xs} \bar{f}(s) ds - \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s) ds$$

The integral over C' tends to zero as $r \rightarrow \infty$ (under certain conditions†). Therefore,

$$\begin{aligned} \mathbf{f}(x) &= \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_C e^{xs} \bar{f}(s) ds \\ &= \text{sum of the residues of } e^{xs} \bar{f}(s) \text{ at the poles of } f(s) \quad \dots(3) \end{aligned}$$

[By § 20.18]

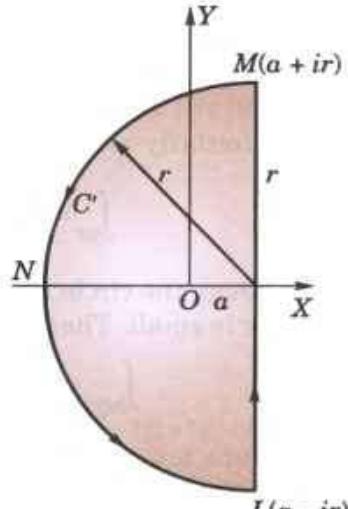


Fig. 22.1

* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C' , then

$$\lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s) ds = 0.$$

(Jordan's Lemma)

Example 22.16. Evaluate $L^{-1} \left\{ \frac{1}{(s-1)(s^2+1)} \right\}$ by the method of residues.

Solution. Since $\left| \frac{1}{(s-1)(s^2+1)} \right| \sim \left| \frac{1}{s^3} \right|$ for $|s| \rightarrow \infty$, therefore,

$$L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \text{sum of Res} \left[\frac{e^{xs}}{(s-1)(s^2+1)} \right] \text{ at the poles } s = 1, \pm i$$

Now $(\text{Res})_{s=1} = \lim_{s \rightarrow 1} \left[\frac{(s-1) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^x}{2}$ [By § 20.19 (1)]

$$(\text{Res})_{s=i} = \lim_{s \rightarrow i} \left[\frac{(s-i) \cdot e^{xs}}{(s-1)(s^2+1)} \right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2} \cdot \frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\frac{1}{2} \cdot \frac{e^{ix}}{1-i}$

$$\therefore L^{-1} \left[\frac{1}{(s-1)(s^2+1)} \right] = \frac{e^x}{2} - \frac{1}{2} \left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i} \right) = \frac{1}{2} (e^x - \sin x - \cos x).$$

Example 22.17. Prove that $L^{-1} \left(\frac{e^{-c\sqrt{s}}}{s} \right) = 1 - \text{erf} \left(\frac{c}{\sqrt{2x}} \right)$.

Solution. By the complex inversion formula,

$$L^{-1} \left(\frac{e^{-c\sqrt{s}}}{s} \right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{xs} \cdot \frac{e^{-c\sqrt{s}}}{s} ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{ \int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL} \right\} \times e^{xs} \frac{e^{-c\sqrt{s}}}{s} ds = 0 \quad \dots(i)$$

If $ON = \rho$ and $OP = \varepsilon$, then along NP , $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\varepsilon} e^{-xR} \frac{e^{-ic\sqrt{R}}}{R} dR$$

Similarly along ST , $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\varepsilon}^{\rho} e^{-xR} \frac{e^{ic\sqrt{R}}}{R} dR$$

Along the circle PQS , $s = \varepsilon e^{i\theta}$. Also e^{xs} and $e^{-c\sqrt{s}}$ are both approximately 1 since ε is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\varepsilon e^{i\theta}} \cdot \varepsilon e^{i\theta} i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|e^{-c\sqrt{s}}/s| < |s|^{-1}$.

But \int_{MN} and \int_{TL} both tend to zero as $r \rightarrow \infty$

Thus (i) takes the form

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs} - e^{-c\sqrt{s}}}{s} ds + \int_{\varepsilon}^{\rho} e^{-xR} \frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R} dR - 2\pi i = 0$$

Taking limits as $\varepsilon \rightarrow 0$ and $\rho \rightarrow \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs} - e^{-c\sqrt{s}}}{s} ds = 2\pi i - 2i \int_0^{\infty} e^{-xR} \frac{\sin c\sqrt{R}}{R} dR$$

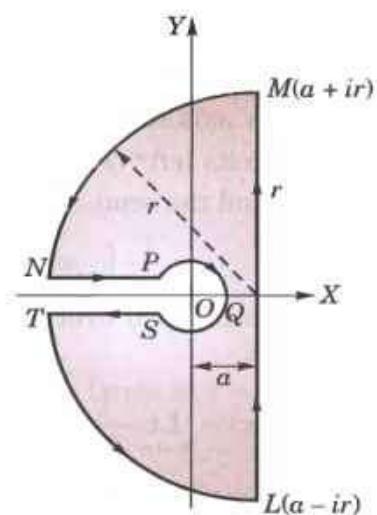


Fig. 22.2

or $\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{xs - c\sqrt{s}}}{s} ds = 1 - \frac{2}{\pi} \int_0^\infty e^{-t^2} \frac{\sin(ct/\sqrt{x})}{t} dt^*$, where $R = t^2/x$
 $= 1 - \frac{2}{\pi} \cdot \frac{\pi}{2} \operatorname{erf}\left(\frac{c}{2\sqrt{x}}\right)$ whence follows the result.

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following:

1. $\frac{1}{(s+1)(s-2)^2}$.

2. $\frac{1}{(s-2)(s^2+1)}$.

3. $\frac{1}{s^2(s^2-a^2)}$.

4. $\frac{1}{(s-1)^2(s^2+1)}$.

5. $\frac{1}{(s^2+1)^2}$.

(V.T.U., 2008 S)

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(i) If in a problem $u(x, t)|_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2 / \partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]|_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u / \partial x^2$.

(ii) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u / \partial x^2$ from the differential equation.

In case $(\partial u / \partial x)|_{x=0}$ and $(\partial u / \partial x)|_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u / \partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

(Osmania, 2003)

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t . We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (x > 0, t > 0) \quad \dots(i)$$

subject to the initial condition $u(x, 0) = f(x)$ $\dots(ii)$

and the boundary condition $u(0, t) = 0$ $\dots(iii)$

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by \bar{u}_s , we have

$$\frac{d\bar{u}_s}{dt} = c^2 [su(0, t) - s^2 \bar{u}_s] \quad [\text{By (2) of § 22.9}]$$

* We know that $\int_0^\infty e^{-t^2} \cos 2mt dt = \frac{1}{2} \sqrt{\pi} e^{-m^2}$ [Example 20.44]

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^\infty e^{-t^2} \left| \frac{\sin 2mt}{2t} \right|_0^{c/2\sqrt{x}} dt = \frac{1}{2} \sqrt{\pi} \int_0^{c/2\sqrt{x}} e^{-m^2} dm$$

or $\int_0^\infty e^{-t^2} \frac{\sin(ct/\sqrt{x})}{t} dt = \frac{\pi}{2} \operatorname{erf}\left(\frac{c}{2\sqrt{x}}\right)$ [By § 7.18(1)]

$$\text{or } \frac{d\bar{u}_s}{dt} + c^2 s^2 \bar{u}_s = 0$$

[By (iii)] ... (iv)

Also the Fourier sine transform of (ii) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$ (v)

Solving (iv) and using (v), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2 s^2 t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(s) e^{-c^2 s^2 t} \sin xs \, ds.$$

Example 22.19. Solve $\partial u / \partial t = 2 \partial^2 u / \partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ ($x > 0$), $u(x, t)$ is bounded where $x > 0$, $t > 0$. (Rohtak, 2006)

Solution. Given $\partial u / \partial t = 2 \partial^2 u / \partial x^2$, $x > 0$, $t > 0$... (i)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ... (ii)

and initial condition $u(x, 0) = e^{-x}$, $x > 0$... (iii)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (i) so that

$$\int_0^\infty \frac{\partial u}{\partial t} \sin px \, dx = 2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin px \, dx$$

$$\text{or } \frac{d}{dt} \int_0^\infty u(x, t) \sin px \, dx = 2 \left[\left| \frac{\partial u}{\partial x} \sin px \right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x} \cdot p \cos px \, dx \right] \quad (\text{Integrating by parts})$$

$$\begin{aligned} \text{or } \frac{d\bar{u}_s}{dt} &= -2p \int_0^\infty \frac{\partial u}{\partial x} \cos px \, dx, \text{ if } \frac{\partial u}{\partial x} \rightarrow \frac{\partial u}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t) \sin px \, dx \\ &= -2p [\int_0^\infty u(x, t) \cos px \, dx - \int_0^\infty u(x, t) - (-p \sin px) \, dx] \quad [\text{Again integrating by parts}] \end{aligned}$$

$$\begin{aligned} &= -2p [0 - u(0, t) + p \int_0^\infty u(x, t) \sin px \, dx] \quad [\because u(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ by (ii)}] \\ &= 2pu(0, t) - 2p^2 \bar{u}_s \end{aligned}$$

$$\text{or } \frac{d\bar{u}_s}{dt} = -2p^2 \bar{u}_s \quad [\text{By (ii)}]$$

$$\text{Integrating } \int \frac{d\bar{u}_s}{\bar{u}_s} - \log c = -2p^2 \int dt \quad \text{or} \quad \log \bar{u}_s - \log c = -2p^2 t$$

$$\therefore \bar{u}_s(p, t) = ce^{-2p^2 t} \quad \dots (iv)$$

Taking Fourier sine transform of both sides of (iii), we get

$$\int_0^\infty u(x, 0) \sin px \, dx = \int_0^\infty e^{-x} \sin px \, dx$$

$$\text{or } \bar{u}_s(p, 0) = \left| \frac{e^{-x}}{1 + p^2} (-\sin px - p \cos px) \right|_0^\infty = \frac{p}{1 + p^2} \quad \dots (v)$$

Putting $t = 0$ in (iv) and using (v), we obtain $p/(1 + p^2) = c$

$$\text{Thus (iv) becomes } \bar{u}_s(p, t) = \frac{p}{1 + p^2} e^{-2p^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{pe^{-2p^2 t}}{1 + p^2} \sin px \, dp.$$

Example 22.20. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, ($x > 0$, $t > 0$) subject to the conditions

$$(i) u = 0, \text{ when } x = 0, t > 0 \quad (ii) u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \leq 1, \text{ when } t = 0 \end{cases} \quad (iii) u(x, t) \text{ is bounded. (U.P.T.U., 2003 S)}$$

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx dx$$

$$\frac{\partial}{\partial t} \int_0^\infty u \sin sx dx = -s^2 \bar{u}(s) + s u(0) \quad [\because u = 0, \text{ when } x = 0]$$

or $\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u}$ or $\frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0$ or $(D^2 + s^2) \bar{u} = 0$ i.e., $D = \pm is$

\therefore Its solution is $\bar{u}(s, t) = e^{-s^2 t}$... (1)

Since

$$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx dx$$

$$\therefore \bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx dx = \int_0^1 1 \cdot \sin sx dx \quad [\text{By (ii)}]$$

$$= \frac{1 - \cos s}{s} \quad \dots (2)$$

From (1) and (2),

$$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives

$$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t} ds$$

which is the desired solution.

Example 22.21. Using finite Fourier transform, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 2x$ where $0 < x < 4$, $t > 0$.

(Rajasthan, 2006)

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x dx$$

$$\bullet \frac{d}{dt} (\bar{u}_s) = F_s \left(\frac{\partial^2 u}{\partial x^2} \right)$$

$$= -\frac{n^2 \pi^2}{16} \bar{u}_s + \frac{n\pi}{4} [u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2 \pi^2}{16} \bar{u}_s \quad [\because u(0, t) = 0, u(4, t) = 0.]$$

or

$$\frac{d \bar{u}_s}{dt} = -\frac{n^2 \pi^2}{16} \bar{u}_s$$

Integrating both sides,

$$\log \bar{u}_s = -\frac{n^2 \pi^2}{16} t + c$$

$$\bar{u}_s(x, 0) = \alpha e^{-\frac{n^2 \pi^2 t}{16}} \quad \dots (i)$$

Putting $t = 0$,

$$\alpha = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} dx \quad [\because u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx = -\frac{32}{n\pi} \cos n\pi$$

or

Thus (i) gives, $\bar{u}_s(x, 0) = -\frac{32}{n\pi} \cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi} (-1)^n e^{-n^2\pi^2 t/16}$

Now taking inverse Fourier sine transform, we get

$$\begin{aligned} u(x, 0) &= \frac{2}{4} \sum_{n=1}^{\infty} \frac{32}{n\pi} (-1)^{n+1} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right) \\ &= 16 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t/16} \sin\left(\frac{n\pi x}{4}\right). \end{aligned}$$

Example 22.22. If the initial temperature of an infinite bar is given by

$$\theta(x) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a, \end{cases}$$

determine the temperature at any point x and at any instant t .

(S.V.T.U., 2008 ; Rohtak, 2004)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad (t > 0) \quad \dots(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases} \theta_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \quad \dots(ii)$

Taking Fourier transform of (i) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2 \bar{\theta} \quad [\text{by (1) of § 22.9}] \quad \dots(iii)$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty} \theta(x, 0) e^{isx} dx = \int_{-a}^a \theta_0 e^{isx} dx = \theta_0 \frac{e^{isa} - e^{-isa}}{is} = 2\theta_0 \frac{\sin as}{s} \quad \dots(iv)$$

Solving (iii) and using (iv), we get $\bar{\theta} = \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t}$

Hence taking its inverse Fourier transform, we get

$$\begin{aligned} \theta(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\theta_0 \sin as}{s} e^{-c^2 s^2 t} e^{-isx} ds = \frac{\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} (\cos xs - i \sin xs) ds \\ &= \frac{2\theta_0}{\pi} \int_0^{\infty} \frac{\sin as}{s} e^{-c^2 s^2 t} \cos xs ds \quad \left. \begin{array}{l} \text{The second integral vanishes as} \\ \text{its integrand is an odd function} \end{array} \right\} \\ &= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-c^2 s^2 t} \frac{\sin(a+x)s + \sin(a-x)s}{s} ds \\ &= \frac{\theta_0}{\pi} \int_0^{\infty} e^{-v^2} \left\{ \sin \frac{(a+x)v}{c\sqrt{t}} + \sin \frac{(a-x)v}{c\sqrt{t}} \right\} \frac{dv}{v} \quad \text{where } v^2 = c^2 s^2 t \\ &= \frac{\theta_0}{\pi} \left\{ \operatorname{erf} \frac{(a+x)}{2c\sqrt{t}} + \operatorname{erf} \frac{(a-x)}{2c\sqrt{t}} \right\}. \end{aligned}$$

[See footnote on p. 783]

Example 22.23. A bar of length a is at zero temperature. At $t = 0$, the end $x = a$ is suddenly raised to temperature u_0 and the end $x = 0$ is insulated. Find the temperature at any point x of the bar at any time $t > 0$, assuming that the surface of the bar is insulated.

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < a, t > 0) \quad \dots(i)$$

subject to the conditions

$$u(x, 0) = 0 \quad \dots(ii); \quad u_x(0, t) = 0 \quad \dots(iii) \quad \text{and} \quad u(a, t) = u_0 \quad (\text{Rohtak, 2005}) \quad \dots(iv)$$

The Laplace transform of (i), if $L[u(x, t)] = \bar{u}(x, s)$, is

$$s\bar{u} - u(x, 0) = c^2 \frac{d^2 \bar{u}}{dx^2}$$

Using (ii), we get $\frac{d^2 \bar{u}}{dx^2} - \frac{s}{c^2} \bar{u} = 0$... (v)

Similarly the Laplace transform of (iii) and (iv) are

$$\bar{u}_x(0, s) = 0 \quad \dots(vi); \quad \bar{u}(a, s) = \frac{u_0}{s} \quad \dots(vii)$$

Solving (v), we have $\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$

Using (vi), we find $C_1 = C_2$ so that

$$\bar{u} = C_1 (e^{\sqrt{sx}/c} + e^{-\sqrt{sx}/c}) = 2C_1 \cosh(\sqrt{sx}/c)$$

Now using (vii), we have $\bar{u} = \frac{u_0 \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)}$

By the inversion formula (3) § 22.10, we get

$$u(x, t) = \text{sum of the residues of } \left(\frac{e^{st} \cdot u_0 \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right) \text{ at all the poles which occur at } s = 0$$

and $\cosh(\sqrt{sa}/c) = 0$ i.e., at $s = 0$, $\sqrt{sa}/c = \left(n - \frac{1}{2}\right)\pi i$, $n = 0, \pm 1, \pm 2, \dots$

or at $s = 0, s (= s_n) = -\frac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, \dots$

Now $(\text{Res})_{s=0} = \underset{s \rightarrow 0}{\text{Lt}} \left\{ s \cdot \frac{u_0 e^{st} \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right\} = u_0$

$$\begin{aligned} (\text{Res})_{s=s_n} &= u_0 \underset{s \rightarrow s_n}{\text{Lt}} \left\{ (s - s_n) \cdot \frac{u_0 e^{st} \cosh(\sqrt{sx}/c)}{s \cosh(\sqrt{sa}/c)} \right\} \\ &= u_0 \underset{s \rightarrow s_n}{\text{Lt}} \left\{ \frac{s - s_n}{\cosh(\sqrt{sa}/c)} \right\} \cdot \underset{s \rightarrow s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{sx}/c)}{s} \right\} \quad \left[\begin{matrix} 0 & \text{form} \\ 0 & \end{matrix} \right] \\ &= u_0 \underset{s \rightarrow s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{sa}/c) \cdot (a/2\sqrt{s}/c)} \cdot \underset{s \rightarrow s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{sx}/c)}{s} \right\} \\ &= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2 \pi^2 c^2 t / 4a^2} \cos \frac{(2n-1)\pi x}{2a} \end{aligned}$$

Thus we get $u(x, t) = u_0 + \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 c^2 t / 4a^2} \cos \frac{(2n-1)\pi x}{2a}$.

Vibrations of a string

Example 22.24. An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string. (Rohtak, 2000)

Solution. The equation for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(i)$$

and the initial conditions are

$$(\frac{\partial y}{\partial t})_{t=0} = 0; y(x, 0) = f(x) \quad \dots(ii)$$

Multiplying (i) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞ , we get

$$\frac{\partial^2 Y}{\partial t^2} = c^2 (-s^2 Y) \quad \text{provided } y \text{ and } \frac{\partial y}{\partial x} \rightarrow 0 \text{ as } x \rightarrow \infty$$

\therefore a solution of $d^2Y/dt^2 + c^2 s^2 Y = 0$ is $Y = A_1 \cos cst + A_2 \sin cst$... (iii)

Also Fourier transforms of (ii) are

$$\frac{\partial y}{\partial t} = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (iii), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus

$$Y = F(s) \cos cst$$

Now taking inverse Fourier transforms, we get

$$\begin{aligned} y(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cst \cdot e^{-isx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{icsx} + e^{-icsx}}{2} \cdot e^{-isx} dx \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} [F(s)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}] ds \\ &= \frac{1}{2} [f(x-ct) + f(x+ct)] \quad [\because f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{-isx} ds] \end{aligned}$$

Example 22.25. An infinitely long string having one end at $x = 0$, is initially at rest along the x -axis. The end $x = 0$ is given a transverse displacement $f(t)$, $t > 0$. Find the displacement of any point of the string at any time.

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t . Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (x > 0, t > 0) \quad \dots(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (i), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2 \bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c} \right]^2 \bar{y} \quad \dots(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$$\bar{y}(0, s) = \bar{f}(s) \quad \text{at } x = 0 \quad \dots(iii) \quad \text{and} \quad \bar{y}(x, s) \text{ is bounded.} \quad \dots(iv)$$

Solving (ii), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (iv), we must have $C_1 = 0$

Using the condition (iii), we get $C_2 = \bar{f}(s)$.

$$\therefore \bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{(t-x/c)s} \bar{f}(s) ds = f(t - x/c).$$

Example 22.26. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l-x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$. (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ $(x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$
and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2 \bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad \dots(i)$$

where $\bar{y}(0, s) = 0, \bar{y}(l, s) = 0 \quad \dots(ii)$

$$\therefore (i) \text{ reduces to } \frac{\partial^2 \bar{y}}{\partial x^2} - \left(\frac{s}{c}\right)^2 \bar{y} = -\frac{\mu s x(l - x)}{c^2}$$

$$\text{Its solution is } \bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \frac{\mu x(l - x)}{s} - \frac{2c^2 \mu}{s^3}$$

Applying the conditions (ii), we get

$$c_1 = 2c^2 \mu / s^2 \quad \text{and} \quad c_2 = \frac{2c^2 \mu}{s^3} \left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)} \right] - \frac{2c^2 \mu}{s^3} \tanh(s/2c)$$

$$\text{Thus } \bar{y}(x, s) = \frac{2c^2 \mu}{s^3} \left[\frac{\cosh(s(2x - l)/2c)}{\cosh(sl/2c)} \right] + \frac{\mu x(l - x)}{s} - \frac{2c^2 \mu}{s^3}$$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) = \text{sum of the residues of}$

$$2c^2 \mu \left[e^{st} \frac{\cosh(s(2x - l)/2c)}{s^3 \cosh(sl/2c)} \right] \text{ at all the poles} + \mu x(l - x) - c^2 \mu t^2$$

Proceeding exactly as in Example 22.23, we have,

$$\begin{aligned} & \text{sum of the residues of } 2c^2 \mu \left[\frac{e^{st} \cosh(s(2x - l)/2c)}{s^3 \cosh(sl/2c)} \right] \text{ at all the poles} \\ &= c^2 \mu \left[t^2 + \left(\frac{2x - l}{2c} \right)^2 - \left(\frac{l}{2c} \right)^2 \right] \\ &\quad - \frac{32c^2 \mu}{\pi^3} \left(\frac{l}{2c} \right)^2 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{(2n-1)^3} \cos \left\{ \frac{(2n-1)\pi(2x-l)}{2l} \right\} \cos \left\{ \frac{(2n-1)\pi ct}{l} \right\} \right] \\ &= c^2 \mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi ct}{l} \right] \end{aligned}$$

$$\text{Hence } y(x, t) = \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi ct}{l} \right].$$

Transmission lines

Example 22.27. A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t . If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \quad \frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \quad \text{i.e.,} \quad \frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \dots(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t .

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0$. $\dots(ii)$

Laplace transforms of (i), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \quad \dots(iii)$$

Laplace transforms of the conditions in (ii), are

$$\bar{v}(0, s) = \frac{v_0}{s} \quad \text{at } x = 0 \quad \dots(iv)$$

and

$$\bar{v}(x, s) \text{ remains finite as } x \rightarrow \infty \quad \dots(v)$$

\therefore the solution of (iii) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}x} + C_2 e^{-\sqrt{RCs}x}$$

To satisfy condition (v), we must have $C_1 = 0$.

Using the condition (iv), we get $C_2 = v_0/s$

$$\text{Thus } \bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}x}$$

Using the inversion formula, we obtain

$$\begin{aligned} v(x, t) &= v_0 L^{-1} \left\{ \frac{e^{-\sqrt{RC}x \cdot \sqrt{s}}}{s} \right\} = v_0 \operatorname{erfc} \left(x \frac{\sqrt{RC}}{2\sqrt{t}} \right) \\ &= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RCx^2/4u)} du \end{aligned} \quad [\text{By Ex. 22.17}]$$

\therefore since $i = -\frac{1}{R} \frac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{C}{R}} t^{-3/2} e^{(-RCx^2/4t)}.$$

Example 22.28. A transmission line of length l has negligible inductance and leakance. A constant voltage v_0 is applied at the sending end ($x = 0$) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t} \quad \text{and} \quad \frac{\partial v}{\partial x} + Ri = 0 \quad \dots(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \quad \dots(ii)$$

The initial condition is $v = 0$ at $t = 0$ ($x > 0$)

Laplace transforms of (i) and (ii) are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \quad \dots(iii)$$

and

$$\bar{v} = v_0/s \text{ at } x = 0, \quad \frac{\partial \bar{v}}{\partial x} = 0 \text{ at } x = l \quad \dots(iv)$$

\therefore the solution of (iii) is

$$\bar{v} = c_1 \cosh \sqrt{(RCs)x} + c_2 \sinh \sqrt{(RCs)x}$$

Applying conditions (iv), it gives

$$v_0/s = c_1, \quad 0 = c_1 \sinh \sqrt{(RCs)l} + c_2 \cosh \sqrt{(RCs)l}$$

$$\therefore \bar{v} = \frac{v_0}{s} \left[\cosh \sqrt{(RCs)x} - \frac{\sinh \sqrt{(RCs)l}}{\cosh \sqrt{(RCs)l}} \sinh \sqrt{(RCs)x} \right]$$

$$= \frac{v_0}{s} \frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}}$$

where $p = \sqrt{(RC)}l$ and $q = (l-x)/l$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\bar{v}) \text{ at all poles of } e^{st}\bar{v}. \quad \dots(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

$$\text{Now } \text{Res}(e^{st}\bar{v})_{s=0} = \underset{s \rightarrow 0}{\text{Lt}} \frac{se^{st} v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} = v_0$$

$$\begin{aligned} \text{and } \text{Res}(e^{st}\bar{v})_{s=-k^2} &= \underset{s \rightarrow -k^2}{\text{Lt}} \frac{(s+k^2)e^{st} v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} \\ &= \underset{s \rightarrow -k^2}{\text{Lt}} \frac{v_0 \cdot e^{st} \cosh pq\sqrt{s} + (s+k^2)(...)}{\cosh p\sqrt{s} + s \sinh p\sqrt{s} \cdot \frac{1}{2} ps^{-1/2}} \\ &= \frac{v_0 e^{-k^2 t} \cosh(ipqk) + 0}{0 + 1/2(ipk) \sinh(ipk)} = \frac{2v_0 e^{-k^2 t} \cos(pqk)}{-pk \sin pk} \end{aligned}$$

Adding up all the residues, (iv) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-[(2n-1)^2 \pi^2 t / 4RCI^2]} \cos [(2n-1) \pi(l-x)/2l]$$

$$\begin{aligned} [\because pk &= (2n-1)\pi/2, -\sin pk = (-1)^n, pqk = \frac{1}{2}(2n-1)\pi(l-x)/l, \\ k^2 &= (2n-1)^2 \pi^2 / 4RCI^2] \end{aligned}$$

$$\text{Also } i = -\frac{1}{R} \frac{\partial v}{\partial x}. \quad [\text{By (i)}]$$

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\frac{\partial y}{\partial t} = 3 \frac{\partial^2 y}{\partial x^2}$
where $y(\pi/2, t) = 0$, $(\partial y / \partial x)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$. (U.P.T.U., 2005)

2. Using suitable transforms, solve the differential equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$, $0 \leq x \leq \pi$, $t \geq 0$,
where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.

3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t . (Rohtak, 2006)

4. Use the complex form of the Fourier transform to show that

$$V = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \bar{f}(u) e^{i[-(x-u)^2/4t]} du$$

is the solution of the boundary value problem

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, -\infty < x < \infty, t > 0; V = f(x) \text{ when } t = 0. \quad (\text{U.P.T.U., 2008})$$

5. A semi-infinite solid ($x > 0$) is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t , is given by $\theta(x, t) = \theta_0 \operatorname{erfc}(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0 , while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x -axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by

$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$

where c is the wave velocity.

8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that

$$y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)].$$

9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage v is applied at the sending end ($x = 0$) which is given by

$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by

$$v(x, t) = v_0 \operatorname{erfc} \left[\frac{x}{2} \sqrt{RCt} \right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

- Fourier cosine transform of $f(t)$ is
- Fourier sine transform of $1/x$ is
- Convolution theorem for Fourier transforms states that
- If Fourier transform of $f(x)$ is $\bar{F}(s)$, then the inversion formula is
- $F[x^n f(x)] =$
- If $F[f'(x)] = F(s)$, then $F[f(x-a)] =$
- Fourier sine integral representation of a function $f(x)$ is given by
- If $F_c[f(ax)] = k F_c(s/a)$, then $k =$
- Fourier transform of second derivative of $u(x, t)$ is
- If $f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is
- Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is
- If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is
- Inverse finite Fourier sine transform of $F_s(p) = \frac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \dots$ and $0 < x < \pi$ is
- If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is
- Fourier cosine transform of e^{-x} is
- $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
- $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$. (True or False)
- Fourier transform is a linear operation. (True or False)
- $F_s[x f(x)] = - \frac{d}{ds} F_c(s)$. (True or False)
- Kernel of Fourier transform is e^{ax} . (True or False)
- Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

Z-Transforms

1. Introduction. 2. Definition. 3. Some standard Z-transforms. 4. Linearity property. 5. Damping rule. 6. Some standard results. 7. Shifting u_n to the right and to the left. 8. Multiplication by n . 9. Two Basic theorems. 10. Some useful Z-transforms. 11. Some useful inverse Z-transforms. 12. Convolution theorems. 13. Convergence of Z-transforms. 14. Two-sided Z-transform. 15. Evaluation of inverse Z-transforms. 16. Application to Difference equations. 17. Objective Type of Questions.

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, i.e. $n = 0, \pm 1, \pm 2, \dots$. The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \dots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \quad \dots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z .

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

$$(1) Z(a^n) = \frac{z}{z-a} \qquad (2) Z(n^p) = -z \frac{d}{dz} Z(n^{p-1}), p \text{ being a +ve integer.}$$

Proof. (1) By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \dots = \frac{1}{1 - (a/z)} = \frac{z}{z - a} \quad (\text{Kottayam, 2005})$$

$$(2) Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} \cdot n \cdot z^{-(n+1)} \quad \dots(i)$$

Changing p to $p - 1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} \cdot z^{-n}$

Differentiating it w.r.t. z ,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} \cdot (-n) z^{-(n+1)} \quad \dots(ii)$$

Substituting (ii) in (i), we obtain $Z(n^p) = -z \frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula*.

In particular, we have the following formulae :

$$(3) Z(1) = \frac{z}{z-1} \quad [\text{Taking } a = 1 \text{ in (1)}] \quad (4) Z(n) = \frac{z}{(z-1)^2} \quad [\text{Taking } p = 1 \text{ in (2)}]$$

$$(5) Z(n^2) = \frac{z^2 + z}{(z-1)^3} \quad (\text{V.T.U., 2006}) \quad (6) Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

$$(7) Z(n^4) = \frac{z^4 + 11z^3 + 11z^2 + z}{(z-1)^5}.$$

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

$$\begin{aligned} \text{Proof. By definition, } Z(au_n + bv_n - cw_n) &= \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n} \\ &= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} - c \sum_{n=0}^{\infty} w_n z^{-n} \\ &= aZ(u_n) + bZ(v_n) - cZ(w_n). \end{aligned}$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

$$\text{Proof. By definition, } Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n \cdot z^{-n} = \sum_{n=0}^{\infty} u_n \cdot (az)^{-n} = U(az). \quad (\text{Madras, 2006})$$

Cor. $Z(a^n u_n) = U(z/a)$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n , hence the name *damping rule*.

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

$$(1) Z(na^n) = \frac{az}{(z-a)^2} \quad (2) Z(n^2 a^n) = \frac{az^2 + a^2 z}{(z-a)^3}$$

$$(3) Z(\cos n\theta) = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$(4) Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$(5) Z(a^n \cos n\theta) = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$$

$$(6) Z(a^n \sin n\theta) = \frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}.$$

Proofs. (1) We know that $Z(n) = \frac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z-1)^2} = \frac{az}{(z-a)^2}.$$

(Madras, 2000 S)

(2) We know that $Z(n^2) = \frac{z^2 + z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2 a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z-1)^3} = \frac{a(z^2 + az)}{(z-a)^3}.$$

(3) and (4) We know that $Z(1) = \frac{z}{z-1}$. Applying damping rule, we have

$$\begin{aligned} Z(e^{-in\theta}) &= Z(e^{-i\theta})^n \cdot 1 = \frac{ze^{i\theta}}{ze^{i\theta}-1} = \frac{z}{z-e^{-i\theta}} = \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})} \\ &= \frac{z(z-\cos \theta)-iz \sin \theta}{z^2-z(e^{i\theta}+e^{-i\theta})+1} = \frac{z(z-\cos \theta)-iz \sin \theta}{z^2-2z \cos \theta+1} \end{aligned}$$

Equating real and imaginary parts, we get (3) and (4).

(V.T.U., 2010 S; Anna, 2009)

(5) We know that $Z(\cos n\theta) = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$. By damping rule, we have

$$Z(a^n \cos n\theta) = \frac{a^{-1}z(a^{-1}z - \cos \theta)}{(a^{-1}z)^2 - 2(a^{-1}z) \cos \theta + 1} = \frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$$

(V.T.U., 2006)

Similarly using (4) above, we get (6).

Example 23.1. Find the Z-transform of the following :

$$(i) 3n - 4 \sin n\pi/4 + 5a$$

$$(ii) (n+1)^2$$

(V.T.U., 2010)

$$(iii) \sin(3n+5).$$

(V.T.U., 2009 S ; Kottayam, 2005)

Solution. (i) $Z(3n - 4 \sin \frac{n\pi}{4} + 5a) = 3Z(n) - 4Z\left(\sin \frac{n\pi}{4}\right) + 5a Z(1)$

[By Linearity property]

$$= 3 \cdot \frac{z}{(z-1)^2} - 4 \cdot \frac{z \sin n\pi/4}{z^2 - 2z \cos \pi/4 + 1} + 5a \cdot \frac{z}{z-1} \quad [\text{Using formulae for } Z(1), Z(n), Z(\sin n\theta)]$$

$$= \frac{(3-5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

$$(ii) \quad Z(n+1)^2 = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1)$$

$$= \frac{z^2 + z}{(z-1)^3} + 2 \frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

$$(iii) \quad Z[\sin(3n+5)] = Z(\sin 3n \cos 5 + \cos 3n \sin 5)$$

= $\cos 5 \cdot Z(\sin 3n) + \sin 5 \cdot Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$)

$$= \cos 5 \cdot \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} + \sin 5 \cdot \frac{z(z - \cos 3)}{z^2 - 2z \cos 3 + 1} = z \cdot \frac{(z \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1}.$$

Example 23.2. Find the Z-transforms of the following

$$(i) e^{an}$$

$$(ii) ne^{an}$$

$$(iii) n^2e^{an}.$$

Solution. (i) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n} u_n) = U(kz)$,

$$\therefore Z(e^{an}) = Z(k^{-n} \cdot 1) = U(kz) = \frac{kz}{kz - 1} \quad \left[\because U(z) = Z(1) = \frac{z}{z - 1} \right]$$

$$= \frac{z}{z - 1/k} = \frac{z}{z - e^a}$$

(ii) Let

$$u_n = n, e^{an} = (e^{-a})^{-n} = k^{-n} \text{ where } k = e^{-a}$$

By damping rule, $Z(e^{an} \cdot n) = Z(k^{-n} \cdot n) = U(kz)$ where $U(z) = Z(n) = \frac{z}{(z - 1)^2}$

$$\frac{kz}{(kz - 1)^2} = \frac{z}{k(z - 1/k)^2} = \frac{e^a z}{(z - e^a)^2}$$

(iii) Let

$$u_n = n^2, e^{an} = (e^{-a})^{-n} = k^{-n} \text{ where } k = e^{-a}$$

By damping rule,

$$\begin{aligned} Z(e^{an} \cdot n^2) &= Z(k^{-n} \cdot n^2) = U(kz) \text{ where } U(z) = Z(n^2) = \frac{z^2 + z}{(z - 1)^3} \\ &= \frac{(kz)^2 + kz}{(kz - 1)^3} = \frac{z(z + 1/k)}{(z - 1/(k))^3} = \frac{ze^a(z + e^a)}{(z - e^a)^3}. \end{aligned}$$

Example 23.3. Find the Z-transform of (i) $\cosh n\theta$. (V.T.U., 2011) (ii) $a^n \cosh n\theta$.

Solution. (i) $Z(\cosh n\theta) = Z\left(\frac{e^{n\theta} + e^{-n\theta}}{2}\right)$

$$= \frac{1}{2} \left[Z\{(e^{-\theta})^{-n} \cdot 1\} + Z\{(e^\theta)^{-n} \cdot 1\} \right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$\begin{aligned} Z(\cosh n\theta) &= \frac{1}{2} \left[\frac{ze^{-\theta}}{ze^{-\theta} - 1} + \frac{ze^\theta}{ze^\theta - 1} \right] \quad \left[\because z(1) = \frac{z}{z - 1} \right] \\ &= \frac{1}{2} \left\{ \frac{2z^2 - z(e^\theta + e^{-\theta})}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right\} = \frac{z^2 - z \cosh \theta}{z^2 - 2z \cosh \theta + 1} \end{aligned}$$

(ii) $Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n} \cdot \cosh n\theta]$ [Apply damping rule using (i)]

$$= \frac{(a^{-1}z)^2 - (a^{-1}z) \cosh \theta}{(a^{-1}z)^2 - 2(a^{-1}z) \cosh \theta + 1} = \frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}.$$

Example 23.4. Find the Z-transforms of

$$(i) e^t \sin 2t$$

(Madras, 2008)

$$(ii) c^k \cos k\alpha. (k \geq 0)$$

(U.P.T.U., 2004 S)

Solution. (i) We know that $Z(\sin 2t) = \frac{z \sin 2}{z^2 - 2z \cos 2 + 1}$... (A)

$$\begin{aligned} \therefore Z(e^t \sin 2t) &= Z[(e^{-1})^{-t} \cdot \sin 2t] \quad [\text{Apply damping rule, using (A)}] \\ &= \frac{(e^{-1}z) \sin 2}{(e^{-1}z)^2 - 2(e^{-1}z) \cos 2 + 1} = \frac{ez \sin 2}{z^2 - 2ez \cos 2 + e^2}. \end{aligned}$$

(ii) We know that $Z(\cos k\alpha) = \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}$... (B)

$$\begin{aligned} \therefore Z(c^k \cos k\alpha) &= Z[(c^{-1})^{-k} \cdot \cos k\alpha] \quad [\text{Apply damping rule, using (B)}] \\ &= \frac{(c^{-1}z)[c^{-1}z - \cos \alpha]}{(c^{-1}z)^2 - 2(c^{-1}z) \cos \alpha + 1} = \frac{z(z - c \cos \alpha)}{z^2 - 2cz \cos \alpha + c^2}. \end{aligned}$$

Example 23.5. Find the Z-transforms of

$$(i) \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) \quad (\text{V.T.U., 2011 S}) \quad (ii) \cosh\left(\frac{n\pi}{2} + \theta\right). \quad (\text{U.P.T.U., 2008})$$

Solution. (i) $Z\left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right] = Z\left(\cos\frac{n\pi}{2} \cos\frac{\pi}{4} - \sin\frac{n\pi}{2} \sin\frac{\pi}{4}\right)$
 $= \cos\frac{\pi}{4} \cdot Z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{4} \cdot Z\left(\sin\frac{n\pi}{2}\right) \quad [\text{Using formulae for } Z(\sin n\alpha) \text{ and } Z(\cos n\alpha)]$
 $= \frac{1}{\sqrt{2}} \left\{ \frac{z(z - \cos\pi/2)}{z^2 - 2z \cos\pi/2 + 1} - \frac{z \sin\pi/2}{z^2 - 2z \cos\pi/2 + 1} \right\} = \frac{1}{\sqrt{2}} \left(\frac{z^2}{z^2 + 1} - \frac{z}{z^2 + 1} \right) = \frac{z(z - 1)}{\sqrt{2}(z^2 + 1)}$

$$(ii) \quad Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] = Z\left[\frac{e^{n\pi/2+\theta} + e^{-(n\pi/2+\theta)}}{2}\right] = \frac{1}{2} [e^\theta Z(e^{n\pi/2}) + e^{-\theta} Z(e^{-n\pi/2})]$$

Since, $Z(a^n) = \frac{z}{z-a}$, $\therefore Z(e^{n\pi/2}) = Z(e^{\pi/2})^n = \frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2}) = \frac{z}{z-e^{-\pi/2}}$

Thus $Z\left[\cosh\left(\frac{n\pi}{2} + \theta\right)\right] = \frac{1}{2} \left\{ e^\theta \cdot \frac{z}{z-e^{\pi/2}} + e^{-\theta} \cdot \frac{z}{z-e^{-\pi/2}} \right\}$
 $= \frac{z}{2} \left\{ \frac{(e^\theta + e^{-\theta}) - [e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)}]}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1} \right\} = \frac{z^2 \cosh\theta - z \cosh\left(\frac{\pi}{2}-\theta\right)}{z^2 - 2z \cosh\left(\frac{\pi}{2}\right) + 1}.$

Example 23.6. Find the Z-transform of

$$(i) {}^n C_p \quad (0 \leq p \leq n) \quad (ii) {}^{n+p} C_p.$$

Solution. (i) $Z({}^n C_p) = \sum_{p=0}^n \left({}^n C_p z^{-p}\right) = 1 + {}^n C_1 z^{-1} + {}^n C_2 z^{-2} + \dots + {}^n C_n z^{-n} = (1 + z^{-1})^n$

$$\begin{aligned} (ii) \quad Z({}^{n+p} C_p) &= \sum_{p=0}^n {}^{n+p} C_p z^{-p} \\ &= 1 + {}^{n+1} C_1 z^{-1} + {}^{n+2} C_2 z^{-2} + {}^{n+3} C_3 z^{-3} + \dots \infty \\ &= 1 + (n+1)z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots \infty \\ &= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!} (-z^{-1})^2 \\ &\quad + \frac{(-n-1)(-n-2)(-n-3)}{3!} (-z^{-1})^3 + \dots \infty \\ &= (1 - z^{-1})^{-n-1}. \end{aligned}$$

Example 23.7. Find the Z-transform of

$$(i) \text{unit impulse sequence } \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases} \quad (ii) \text{unit step sequence } u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

Solution. (i) $Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1 + 0 + 0 + \dots = 1$

$$(ii) Z[u(n)] = \sum_{n=0}^{\infty} u(n) z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

23.7 (1) SHIFTING U_n TO THE RIGHT

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ $(k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \dots = z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$$

$$\begin{aligned} \text{Proof. } Z(u_{n+k}) &= \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)} \\ &= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right] \end{aligned}$$

$$\text{Hence } Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$$

(J.N.T.U., 2002)

In particular, we have the following standard results :

$$(1) Z(u_{n+1}) = z[U(z) - u_0]; (2) Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

$$(3) Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}].$$

Example 23.8. Show that $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ and $Z[1/(n+2)!]$.

(Madras, 2006)

Solution. We have $Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots = e^{1/z}$.

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z \left[Z\left(\frac{1}{n!}\right) - 1 \right] = z(e^{1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z \frac{dU(z)}{dz}$

$$\begin{aligned} \text{Proof. } Z(nu_n) &= \sum_{n=0}^{\infty} n \cdot u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} (z^{-n}). \\ &= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n}) = -z \frac{d}{dz} \left(\sum_{n=0}^{\infty} u_n z^{-n} \right) = -z \frac{d}{dz} U(z). \end{aligned}$$

Obs. We have, $Z(n^2 u_n) = \left(-z \frac{d}{dz}\right)^2 u(z)$

(Madras, 2006)

In general, $Z(n^m u_n) = \left(-z \frac{d}{dz}\right)^m u(z)$.

Example 23.9. Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.

Solution. (i) We know that $Z(nu_n) = -z \frac{dU(z)}{dz}$ and $Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$

$$\begin{aligned} Z(n \sin n\theta) &= -z \frac{d}{dz} [Z(\sin n\theta)] = -z \frac{d}{dz} \left(\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \right) \\ &= -z \frac{\sin \theta - z^2 \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} = \frac{z(z^2 - 1) \sin \theta}{(z^2 - 2z \cos \theta + 1)^2} \end{aligned}$$

(ii) We know that $Z(e^{n\theta}) = \frac{z}{z - e^\theta}$

$$\begin{aligned} Z(n^2 e^{n\theta}) &= \left(-z \frac{d}{dz} \right)^2 (Z(e^{n\theta})) = \left(-z \frac{d}{dz} \right) \left[-z \frac{d}{dz} \left(\frac{z}{z - e^\theta} \right) \right] \\ &= \left(-z \frac{d}{dz} \right) \left\{ -z \frac{(z - e^\theta)(1) - z(1)}{(z - e^\theta)^2} \right\} = -z \frac{d}{dz} \left\{ \frac{ze^\theta}{(z - e^\theta)^2} \right\} \\ &= -ze^\theta \left\{ \frac{(z - e^\theta)^2(1) - z[2(z - e^\theta)]}{(z - e^\theta)^4} \right\} = -ze^\theta \frac{z - e^\theta - 2z}{(z - e^\theta)^3} = \frac{z(z + e^\theta)e^\theta}{(z - e^\theta)^3}. \end{aligned}$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \rightarrow \infty$ without requiring complete knowledge of u_n . We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \rightarrow \infty$ and vice-versa. The precise relationship is given by the following *initial and final value theorems*:

(1) Initial value theorem. If $Z(u_n) = U(z)$, then $u_0 = \lim_{z \rightarrow \infty} U(z)$

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

Taking limits as $z \rightarrow \infty$, we get $\lim_{z \rightarrow \infty} [U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \lim_{z \rightarrow \infty} \{z[U(z) - u_0]\}; u_2 = \lim_{z \rightarrow \infty} \{z^2[U(z) - u_0 - u_1 z^{-1}]\} \text{ and so on.}$$

(2) Final value theorem. If $Z(u_n) = U(z)$, then

$$\lim_{n \rightarrow \infty} (u_n) = \lim_{z \rightarrow 1} (z-1) U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

or $Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

or $z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

or $U(z)(z-1) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

Taking limits of both sides as $z \rightarrow 1$, we get

$$\lim_{z \rightarrow 1} [(z-1) U(z)] - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + \dots + (u_{n+1} - u_n)]$$

$$= \text{Lt}_{n \rightarrow \infty} [u_{n+1} - u_0] = u_\infty - u_0$$

Hence

$$u_\infty = \text{Lt}_{z \rightarrow 1} [(z-1) U(z)].$$

(Anna, 2005 S)

Example 23.10. If $U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 and u_3 .

Solution. Writing $U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1-z^{-1})^4}$

By initial value theorem, $u_0 = \text{Lt}_{z \rightarrow \infty} U(z) = 0$

Similarly, $u_1 = \text{Lt}_{z \rightarrow \infty} \{z [U(z) - u_0]\} = 0$

Now $u_2 = \text{Lt}_{z \rightarrow \infty} \{z^2 [U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$

and

$$u_3 = \text{Lt}_{z \rightarrow \infty} z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] = \text{Lt}_{z \rightarrow \infty} z^3 [U(z) - 0 - 0 - 2z^{-2}]$$

$$= \text{Lt}_{z \rightarrow \infty} z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2} \right] = \text{Lt}_{z \rightarrow \infty} z^3 \left\{ \frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4} \right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :

$$(i) \frac{a^n}{n!} \quad (n \geq 0) \quad (\text{S.V.T.U., 2009}) \quad (ii) \frac{1}{(n+1)!} \quad (\text{iii}) (\cos \theta + i \sin \theta)^n.$$

2. Using the linearity property, find the Z-transforms of the following functions :

$$(i) 2n + 5 \sin n\pi/4 - 3a^n \quad (ii) \frac{1}{2}(n-1)(n+2) \quad (\text{S.V.T.U., 2007})$$

$$(iii) (n+1)(n+2) \quad (\text{Anna, 2008}) \quad (iv) (2n-1)^2 \quad (\text{V.T.U., 2011 S})$$

$$3. \text{ Show that } (i) Z(\sinh n\theta) = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1} \quad (\text{V.T.U., 2011}) \quad (ii) Z(a^n \sinh n\theta) = \frac{az \sinh \theta}{z^2 - 2az \cosh \theta + a^2}.$$

$$4. \text{ Show that } (i) Z(e^{-an} \cos n\theta) = \frac{ze^a(ze^a - \cos \theta)}{z^2 e^{2a} - 2ze^a \cos \theta + 1}; \quad (ii) Z(e^{-an} \sin n\theta) = \frac{ze^a \sin \theta}{z^2 e^{2a} - 2ze^a \cos \theta + 1}$$

Also evaluate $Z(e^{3n} \sin 2n)$. (S.V.T.U., 2007)

$$5. \text{ Using } Z(n^2) = \frac{z^2 + z}{(z-1)^3}, \text{ show that } Z(n+1)^2 = \frac{z^3 + z^2}{(z-1)^3}.$$

$$6. \text{ Find the Z-transforms of } (i) \sin(n+1)\theta, (ii) \cos\left(\frac{k\pi}{8} + \alpha\right). \quad (\text{Marathwada, 2008})$$

$$7. \text{ Find the Z-transform of } \cos n\theta \text{ and hence find } Z(n \cos n\theta). \quad (\text{Anna, 2009})$$

$$8. \text{ Find the Z-transform of } \cos(n\pi/2) \text{ and } a^n \cos(n\pi/2). \quad (\text{Anna, 2008 S})$$

9. Find the Z-transforms of the following

$$(i) e^{-an} \quad (ii) e^{-2n} \quad (\text{V.T.U., 2010 S}) \quad (iii) e^{-an} n^2.$$

$$10. \text{ Show that } (i) Z[\delta(n+1)] = 1/z \quad (ii) (1/2)^n u(n) = \frac{2z}{2z-1}.$$

11. Show that $Z(n+p C_p) = (1 - 1/z)^{-(p+1)}$. Using the damping rule, deduce that

$$Z(n+p C_p a^n) = (1 + a/z)^{-(p+1)}.$$

$$12. \text{ If } Z(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}, \text{ find the Z-transform of } u_{n+2}. \quad (\text{S.V.T.U., 2009})$$

$$13. \text{ If } U(z) = \frac{2z^2 + 3z + 12}{(z-1)^4}, \text{ find the value of } u_2 \text{ and } u_3.$$

14. Given that $Z(u_n) = \frac{2z^2 + 3z + 4}{(z - 3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.
15. Show that (i) $Z\left(\frac{1}{n}\right) = z \log \frac{z}{z-1}$. (Madras, 2003 S) (ii) $Z\left\{\frac{1}{n(n+1)}\right\}$. (Anna, 2005 S)
16. Using $Z(n) = \frac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \frac{(z^3 + z) \cos \theta - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	Sequence u_n ($n \geq 0$)	Z-transform $U(z) = Z(u_n)$
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2 + z)/(z-1)^3$
5.	n^p	$-z d/dz [Z(n^{p-1})]$, p +ve integer.
6.	$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2 + a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$
12.	$\cos n\theta$	$\frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$
13.	$a^n \sin n\theta$	$\frac{az \sin \theta}{z^2 - 2az \cos \theta + a^2}$
14.	$a^n \cos n\theta$	$\frac{z(z - a \cos \theta)}{z^2 - 2az \cos \theta + a^2}$
15.	$\sinh n\theta$	$\frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$
16.	$\cosh n\theta$	$\frac{z(z - \cosh \theta)}{z^2 - 2z \cosh \theta + 1}$
17.	$a^n \sinh n\theta$	$\frac{az \sinh \theta}{z^2 - 2az \cosh \theta + a^2}$
18.	$a^n \cosh n\theta$	$\frac{z(z - a \cosh \theta)}{z^2 - 2az \cosh \theta + a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z) - u_0]$
	u_{n+2}	$z^2[U(z) - u_0 - u_1 z^{-1}]$
21.	u_{n+3}	$z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$
22.	u_{n-k}	$z^{-k} U(z)$
23.	nu_n	$-zd/dz[U(z)]$
24.	u_0	$\text{Lt}_{z \rightarrow \infty} U(z)$
	$\text{Lt}_{n \rightarrow \infty} (u_n)$	$\text{Lt}_{z \rightarrow 1} [(z-1) U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	Inverse Z-transform $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$, then

$$Z^{-1}[U(z) \cdot V(z)] = \sum_{m=0}^n u_m \cdot v_{n-m} = u_n * v_n$$

where the symbol $*$ denotes the convolution operation.

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$$\begin{aligned} \therefore U(z) V(z) &= (u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_n z^{-n} + \dots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \dots + v_n z^{-n} + \dots \infty) \\ &= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0) \end{aligned}$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. Use convolution theorem to evaluate $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$$\begin{aligned} \therefore Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} &= Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n \\ &= \sum_{m=0}^n a^m \cdot b^{n-m} = b^n \sum_{m=0}^n \left(\frac{a}{b}\right)^m \text{ which is a G.P.} \\ &= b^n \cdot \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}. \end{aligned}$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad \dots(1)$$

where u_n represents a number in the sequence for $n = \text{an integer}$. The region of the z -plane in which (1) converges absolutely is known as the region of convergence (ROC) of $U(z)$.

We have so far discussed one-sided Z-transform only for which $n \geq 0$. Here the sequence is always right-sided and the ROC is always outside a prescribed circle say $|z| > |a|$ [Fig. 23.2 (i)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (ii)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \quad \dots(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the annular region $|b| < |z| < |c|$ [Fig. 23.2 (iii)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z . The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.

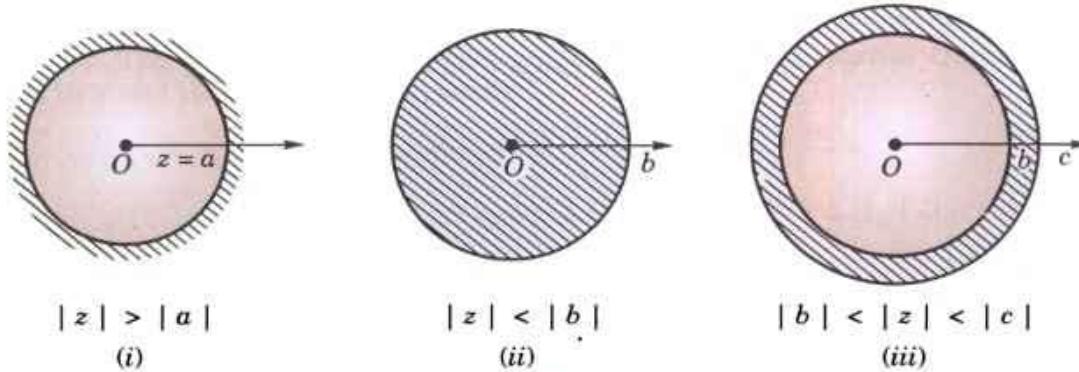


Fig. 23.1

Example 23.12. Find the Z-transform and region of convergence of

$$(a) u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases} \quad (b) u(n) = {}^n c_k, n \geq k.$$

Solution. By definition $Z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) Z^{-n} = \sum_{n=-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$\begin{aligned} Z[u(n)] &= \sum_{m=1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n} \\ &= \left\{ \frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \dots \right\} + \left\{ 1 + \frac{2}{z} + \frac{2^2}{z^2} + \dots \right\} \\ &= \frac{z}{4} \left\{ 1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \dots \right\} + \left\{ 1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \dots \right\} \quad \dots(i) \\ &= \frac{z}{4} \cdot \frac{1}{1 - (z/4)} + \frac{1}{1 - (2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)} \end{aligned}$$

Now the two series in (i) being G.P. will be convergent if $|z/4| < 1$ and $|2/z| < 1$ i.e., if $|z| < 4$ and $2 < |z|$. i.e. $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

$$(b) \text{ By definition, } Z[u(n)] = \sum_{n=-\infty}^{\infty} {}^n C_k z^{-n} = \sum_{n=k}^{\infty} {}^n C_k 2^n z^{-n}$$

To find the sum of this series, we replace n by $k+r$

$$\begin{aligned} \therefore Z[u(n)] &= \sum_{r=0}^{\infty} {}^{k+r} C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r} C_r z^{-r} \\ &= z^{-k} [1 + {}^{k+1} C_1 z^{-1} + {}^{k+1} C_2 z^{-2} + \dots] \\ &= z^{-k} (1 - 1/z)^{-(k+1)} \end{aligned}$$

This series is convergence for $|1/z| < 1$ i.e., for $|z| > 1$.

Hence ROC is $|z| > 1$.

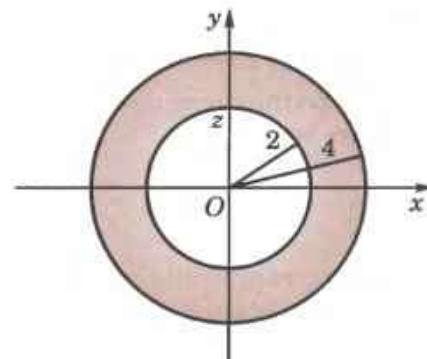


Fig. 23.2

$$[\because {}^k C_r = {}^k C_{k-r}]$$

Example 23.13. Find the Z-transform and the radius of convergence of

$$(a) f(n) = 2^n, n < 0$$

$$(b) f(n) = 5^n/n!, n \geq 0.$$

(Mumbai, 2009)

Solution. (a) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$\begin{aligned} Z[f(n)] &= \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where } m = -n \\ &= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \infty = \frac{z}{2} \{1 + (z/2) + (z/2)^2 + \dots \infty\} \\ &= \frac{z}{2} \cdot \frac{1}{1 - (z/2)} = \frac{z}{2 - z} \end{aligned}$$

This series being a G.P. is convergent if $|z/2| < 1$ i.e., $|z| < 2$.

Hence ROC is $|z| < 2$.

$$\begin{aligned} (b) \text{ By definition, } Z[u(n)] &= \sum_{n=0}^{\infty} \frac{5^n}{n!} z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!} \left(\frac{5}{z}\right)^2 + \frac{1}{3!} \left(\frac{5}{z}\right)^3 + \dots \infty \\ &= e^{5/z} \end{aligned}$$

The above series is convergent for all values of z .

Hence ROC is the entire z -plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0.$
2. $u(n) = 2^n, n < 0.$
3. $u(n) = 4^n, \text{ for } n < 0 \text{ and } = 3^n \text{ for } n \geq 0.$
4. $u(n) = n 5^n, n \geq 0.$
5. $u(n) = 2^n/n, n > 1.$
6. $u(n) = 3^n/n!, n \geq 0.$
7. $u(n) = e^{an}, n \geq 0.$

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. Power series method. This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. Find the inverse Z-transform of $\log(z/z + 1)$ by power series method.

Solution. Putting $z = 1/t$, $U(z) = \log\left(\frac{1/y}{1/y + 1}\right) = -\log(1 + y) = -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \dots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \dots$$

Thus $u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}$.

Example 23.15. Find the inverse Z-transform of $z/(z + 1)^2$ by division method.

Solution. $U(z) = \frac{z}{z^2 + 2z + 1} = z^{-1} - \frac{2 + z^{-1}}{z^2 + 2z + 1}$, by actual division

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1} + 2z^{-2}}{z^2 + 2z + 1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2} + 3z^{-3}}{z^2 + 2z + 1}$$

Continuing this process of division, we get an infinite series i.e.,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} nz^{-n}$$

Thus $u_n = (-1)^{n-1} n$.

II. Partial fractions method. This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. Find the inverse Z-transforms of

$$(i) \frac{2z^2 + 3z}{(z+2)(z-4)} \quad (\text{V.T.U., 2008 S.; S.V.T.U., 2007}) \quad (ii) \frac{z^3 - 20z}{(z-2)^3(z-4)} \quad (\text{V.T.U., 2011})$$

Solution. (i) We write $U(z) = \frac{2z^2 + 3z}{(z+2)(z-4)}$ as $\frac{U(z)}{z} = \frac{2z + 3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$ where $A = 1/6$ and $B = 11/6$

$$\therefore U(z) = \frac{1}{6} \frac{z}{z+2} + \frac{11}{6} \frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \quad [\text{Using § 23.10 (9)}]$$

(ii) We write $U(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)}$

$$\frac{U(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{A + Bz + Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

as

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6, B = 0, C = 1/2$.

Thus $U(z) = \frac{1}{2} \cdot \frac{12z + z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2} \frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4}$

$$= \frac{1}{2} \left\{ \frac{z}{z-2} + 2 \frac{2z^2 + 4z}{(z-2)^3} \right\} - \frac{z}{z-4}$$

On inversion, we get

$$\begin{aligned} u_n &= \frac{1}{2}(2^n + 2 \cdot n^2 2^n) - 4^n \\ &= 2^{n-1} + n^2 2^n - 4^n. \end{aligned}$$

[Using § 23.10 (9) & (11)]

Example 23.17. Find the inverse Z-transform of

$$2(z^2 - 5z + 6.5) / [(z-2)(z-3)^2], \text{ for } 2 < |z| < 3.$$

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}$$

where $A = B = C = 1$

$$\therefore U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2} \left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3} \left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9} \left(1 - \frac{z}{3}\right)^{-2}$$

so that $2/z < 1$ and $z/3 < 1$

$$= \frac{1}{z} \left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots\right) - \frac{1}{3} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots\right) + \frac{1}{9} \left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \dots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \dots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \dots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1) \left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}$, $n \geq 1$ and $u_n = -(n+2)3^{n-2}$, $n \leq 0$.

III. Inversion integral method. The inverse Z-transform of $U(z)$ is given by the formula

$$u_n = \frac{1}{2\pi i} \int_C U(z) z^{n-1} dz$$

= sum of residues of $U(z) z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

The following examples will illustrate the application of this formula :

Example 23.18. Using the inversion integral method, find the inverse Z-transform of

$$\frac{z}{(z-1)(z-2)}$$

(V.T.U., 2010 S)

Solution. Let $U(z) = \frac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i} \int_C U(z) z^{n-1} dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z) z^{n-1}$ at $z = 1$ and $z = 2$.

Now

$$\text{Res} [U(z) z^{n-1}]_{z=1} = \lim_{z \rightarrow 1} \left\{ (z-1) \cdot \frac{z^n}{(z-1)(z-2)} \right\} = -1$$

and

$$\text{Res} [U(z) z^{n-1}]_{z=2} = \lim_{z \rightarrow 2} \left\{ (z-2) \cdot \frac{z^n}{(z-1)(z-2)} \right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1$, $n = 0, 1, 2, \dots$

Example 23.19. Find the inverse Z-transform of $2z / [(z - 1)(z^2 + 1)]$.

(Madras, 2000 S)

Solution. Let $U(z) = \frac{2z}{(z - 1)(z + i)(z - i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i} \int_C U(z) \cdot z^{n-1} dz, \text{ where } C \text{ is a circle large enough to enclose the poles of } U(z).$$

= sum of residues of $U(z) \cdot z^{n-1}$ at $z = 1, z = \pm i$.

$$\begin{aligned} \text{Now } \operatorname{Res} [U(z) z^{n-1}]_{z=1} &= \lim_{z \rightarrow 1} \left\{ (z-1) \frac{2z^n}{(z-1)(z^2+1)} \right\} = 1 \\ \operatorname{Res} [U(z) z^{n-1}]_{z=i} &= \lim_{z \rightarrow i} \left\{ (z-i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{-(i)^n}{1+i} \\ \operatorname{Res} [U(z) z^{n-1}]_{z=-i} &= \lim_{z \rightarrow -i} \left\{ (z+i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{(-i)^n}{i-1} \end{aligned}$$

Hence $u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}$.

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\frac{z^2}{(z-1)(z-3)}$.
2. $\left(\frac{z}{z-a}\right)^2$ (Madras, 2003)
3. $\left(\frac{z}{z-1}\right)^3$.
4. Show that (a) $\frac{1}{n!} * \frac{1}{n!} = \frac{2^n}{n!}$ (b) $Z^{-1}\left(\frac{z^2}{(z+a)(z+b)}\right) = \frac{(-1)}{b-a} (b^{n+1} - a^{n+1})$. (Anna, 2009)

Find the inverse Z-transforms of the following :

5. $\frac{4z}{z-a}$, $|z| > |a|$. (Kottayam, 2005)
6. $\frac{5z}{(2-z)(3z-1)}$. (Madras, 1999)
7. $\frac{z}{(z-1)^2}$.
8. $\frac{18z^2}{(2z-1)(4z+1)}$. (S.V.T.U., 2009)
9. $\frac{8z-z^3}{(4-z)^3}$.
10. $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. (Anna, 2005 S)
11. $\frac{4z^2-2z}{z^3-5z^2+8z-4}$. (V.T.U., 2011 S)
12. $\frac{z^3+3z}{(z-1)^2(z^2+1)}$. (Anna, 2009)
13. $\frac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}\{1/[(z-2)(z-3)]\}$ for (i) $|z| < 2$; (ii) $2 < |z| < 3$; (iii) $|z| > 3$. (Marathwada, 2008)
15. Evaluate $Z^{-1}\{(z-5)^{-3}\}$ for $|z| > 5$. (Mumbai, 2009)

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\frac{z+3}{(z+1)(z-2)}$.
17. $\frac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\frac{1}{z(z-1)(z+0.5)}$. (S.V.T.U., 2008)
19. $\frac{z^2+z}{(z-1)(z^2+1)}$. (Madras, 2003)
20. $\frac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) Working procedure to solve a linear difference equation with constant coefficients by Z-transforms :

1. Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.

2. Transpose all terms without $U(z)$ to the right.

3. Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z .

4. Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.

Example 23.20. Using the Z-transform, solve

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0, u_1 = 1. \quad (\text{U.P.T.U., 2003})$$

Solution. If $Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $Z(2^n) = z/(z-2)$

∴ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8} \frac{1}{z+1} + \frac{1}{24} \frac{1}{z-3} - \frac{5}{12} \frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8} \frac{z}{z+1} + \frac{1}{24} \frac{z}{z-3} - \frac{5}{12} \frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8} Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24} Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12} Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8} (-1)^n + \frac{1}{24} 3^n - \frac{5}{12} (-3)^n.$$

Example 23.21. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.

(V.T.U., 2011; Anna, 2009; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z[(Y(z) - y_0)]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $y_0 = 0$, and $y_1 = 0$, we have $Y(z)(z^2 + 6z + 9) = z/(z-2)$

or $\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25} \left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2} \right]$, on splitting into partial fractions.

or $Y(z) = \frac{1}{25} \left[\frac{z}{z-2} - \frac{z}{z+3} - \frac{5z}{(z+3)^2} \right]$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25} \left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3} Z^{-1}\left(-\frac{3z}{(z+3)^2}\right) \right]$$

$$= \frac{1}{25} \left\{ 2^n - (-3)^n + \frac{5}{3} n(-3)^n \right\}$$

$$\left[\because Z^{-1}\left(\frac{az}{(z-a)^2}\right) = na^n \right]$$

Example 23.22. Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \dots$ by Z-transform method. (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6) Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or $\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \quad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that

$$Y(z) = \frac{1}{2} \frac{z}{z-1} - 2 \frac{z}{z-2} + \frac{3}{2} \frac{z}{z-3}$$

On inversion, we obtain $y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}Y(z) = 1 + \frac{1}{3}z^{-1}$$

or $Y(z) = \left(1 + \frac{1}{3}z^{-1}\right) / \left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right) / \left(z + \frac{1}{4}\right)$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\begin{aligned} \therefore \text{Res}[Y(z)z^{n-1}]_{z=-1/4} &= \lim_{z \rightarrow -1/4} \left\{ \left(z + \frac{1}{4}\right) \cdot \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right) \right\} \\ &= \lim_{z \rightarrow -1/4} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right) \left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} \cdot \left(-\frac{1}{4}\right)^{n-1} \end{aligned}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12} \left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

(S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \cdot \frac{z}{(z-1)^2} + 5 \cdot \frac{z}{z-1}$$

or $U(z)(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$

or $U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0 \frac{z^2 - 2z}{(z-1)^2} + u_1 \frac{z}{(z-1)^2}$

On inversion, we obtain

$$u_n = Z^{-1} \left\{ \frac{5z^2 - 2z}{(z-1)^4} \right\} + u_0 Z^{-1} \left\{ \frac{z^2 - 2z}{(z-1)^2} \right\} + u_1 Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} \quad \dots(i)$$

Noting that $Z(1) = \frac{z}{z-1}$, $Z(n) = \frac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2 + z}{(z-1)^3}, Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

$$\text{We write } \frac{5z^2 - 2z}{(z-1)^4} \equiv A \frac{z^3 + 4z^2 + z}{(z-1)^4} + B \frac{z^2 + z}{(z-1)^3} + C \frac{z}{(z-1)^2} + D \frac{z}{z-1}$$

Equating coefficients of like powers of z , we find

$$A = \frac{1}{2}, B = 1, C = -\frac{3}{2}, D = 0$$

$$\therefore Z^{-1} \left\{ \frac{5z^2 - 2z}{(z-1)^4} \right\} = \frac{1}{2} n^3 + n^2 - \frac{3}{2} n = \frac{1}{2} n(n-1)(n+3)$$

$$\text{Also } Z^{-1} \left\{ \frac{z^2 - 2z}{(z-1)^2} \right\} = Z^{-1} \left\{ \frac{z}{z-1} \right\} - Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} = 1 - n$$

$$\text{and } Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} = n.$$

Substituting these values in (i) above, we get

$$\begin{aligned} u_n &= \frac{1}{2} n(n-1)(n+3) + u_0(1-n) + u_1 n \\ &= \frac{1}{2} n(n-1)(n+3) + c_0 + c_1 n. \end{aligned}$$

where $c_0 = u_0$, $c_1 = u_1 - u_0$

Example 23.25. Using residue method, solve $y_k + \frac{1}{9}y_{k-2} = \frac{1}{3^k} \cos \frac{k\pi}{2}$, $k \geq 0$.

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z \left\{ y_k + \frac{1}{9} y_{k-2} \right\} = Z \left\{ \frac{1}{3^k} \cos \frac{k\pi}{2} \right\}$$

$$\text{or } Y(z) + \frac{1}{9} z^{-2} Y(z) = \frac{z^2}{z^2 + 1/9} \quad \text{or} \quad \left(1 + \frac{1}{9} z^{-2} \right) Y(z) = \frac{z^2}{z^2 + 1/9}$$

$$\text{or } Y(z) = \frac{z^2}{\left(1 + \frac{1}{9} z^{-2} \right) \left(z^2 + \frac{1}{9} \right)} = \frac{z^4}{\left(z^2 + \frac{1}{9} \right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$$\begin{aligned} \therefore \text{Residue at } (z = i/3) &= \left[\frac{d}{dz} \left\{ \left(\frac{z-i}{3} \right)^2 \frac{z^{k-1} z^4}{(z^2 + 1/9)^2} \right\} \right] \\ &= \left[\frac{d}{dz} \left\{ \frac{z^{k+3}}{(z+i/3)^2} \right\} \right]_{z=i/3} = \left[\frac{(z+i/3)^2 (k+3)z^{k+2} - z^{k+3} \cdot 2(z+i/3)}{(z+i/3)^4} \right]_{z=i/3} \\ &= \left[\frac{(z+i/3)(k+3)z^{k+2} - 2z^{k+3}}{(z+i/3)^3} \right]_{z=i/3} = \left(\frac{3}{2i} \right)^3 \left[(2k+6) \left(\frac{i}{3} \right)^{k+3} - 2 \left(\frac{i}{3} \right)^{k+3} \right] \end{aligned}$$

$$= \frac{1}{8} (2k+4) \left(\frac{i}{3}\right)^k = \frac{1}{4} (k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^k = \frac{1}{4} (k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}\right) \quad \dots(i)$$

Changing i to $-i$ in (i), we have

$$\text{Residue at } (z = -i/3) = \frac{1}{4}(k+2) \left(\frac{1}{3}\right)^k \left(\cos \frac{k\pi}{2} - i \sin \frac{k\pi}{2}\right) \quad \dots(ii)$$

$$\text{Adding (i) and (ii), we obtain } y_k = \frac{1}{2}(k+2) \left(\frac{1}{3}\right)^k \cos \frac{k\pi}{2}.$$

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8) :

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. (Kottayam, 2005)
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. (V.T.U., 2008 S)
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$. (U.P.T.U., 2008)
4. $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$. (Madras, 2003 S)
5. $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4, y(1) = 0$ and $y(2) = 8$. (Anna, 2005 S)
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. (Anna, 2009)
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$. 9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0$.
10. $u_{k+2} + u_k = 5(2^k)$ given that $u_0 = 1, u_1 = 0$. (Marathwada, 2008)
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. (Madras, 2006)
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2, y_1 = 1$. 13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos \frac{k\pi}{2}, \quad (k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n; y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) = \dots$
2. If u_n is defined for $n = 0, 1, 2, \dots$ only, then $Z(u_n) = \dots$
3. Z-transform of $n = \dots$ (Anna, 2009)
4. $Z(na^n) = \dots$
5. $Z(\sin n\theta) = \dots$
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) = \dots$
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) = \dots$
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} = \dots$
11. Initial value theorem on Z-transform states that
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\lim_{n \rightarrow \infty} (u_n) = \lim_{z \rightarrow \infty} (z-1)u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}$, $k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}$, $k \geq 0 = e^{az}$. (True or False)
16. Z-transform of $\{^n C_r\}$, $(0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z-1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)

Empirical Laws and Curve-fitting

1. Introduction. 2. Graphical method. 3. Laws reducible to the linear law. 4. Principle of Least squares. 5. Method of Least squares. 6. Fitting of other curves. 7. Method of Group averages. 8. Fitting a parabola. 9. Method of Moments. 10. Objective Type of Questions.

24.1 INTRODUCTION

In many branches of applied mathematics, it is required to express a given data, obtained from observations, in the form of a *Law* connecting the two variables involved. Such a *Law* inferred by some scheme is known as *Empirical Law*. For example, it may be desired to obtain the law connecting the length and the temperature of a metal bar. At various temperatures, the length of the bar is measured. Then, by one of the methods explained below, a law is obtained that represents the relationship existing between temperature and length for the observed values. This relation can then be used to predict the length at an arbitrary temperature.

(2) Scatter diagram. To find a relationship between the set of paired observations x and y (say), we plot their corresponding values on the graph taking one of the variables along the x -axis and other along the y -axis i.e. (x_1, y_1) , (x_2, y_2) , (x_n, y_n) . The resulting diagram showing a collection of dots is called a *scatter diagram*. A smooth curve that approximates the above set of points is known as the *approximating curve*.

(3) Curve fitting. Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of '*best fit*' which may be most suitable for predicting the unknown values. The process of finding such an equation of '*best fit*' is known as *curve-fitting*.

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains n arbitrary constants for we can solve n simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than n arbitrary constants, then we can have recourse to any of the four methods : *Graphical method*, *Method of Least squares*, *Method of Group averages* and *Method of Moments*. The graphical method fails to give the values of the unknowns uniquely and accurately while the other methods do. *The method of Least squares is, probably, the best to fit a unique curve to a given data*. It is widely used in applications and can be easily implemented on a computer.

24.2 GRAPHICAL METHOD

When the curve representing the given data is a **linear law** $y = mx + c$; we proceed as follows :

(i) Plot the given points on the graph paper taking a suitable scale.

(ii) Draw the straight line of best fit such that the points are evenly distributed about the line.

(iii) Taking two suitable points (x_1, y_1) and (x_2, y_2) on the line, calculate m , the slope of the line and c , its intercept on y -axis.

When the points representing the observed values do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

24.3 LAWS REDUCIBLE TO THE LINEAR LAW

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions :

(1) When the law is $y = mx^n + c$.

Taking $x^n = X$ and $y = Y$ the above law becomes $Y = mX + c$

(2) When the law is $y = ax^n$.

Taking logarithms of both sides, it becomes $\log_{10} y = \log_{10} a + n \log_{10} x$

Putting $\log_{10} x = X$ and $\log_{10} y = Y$, it reduces to the form $Y = nX + c$, where $c = \log_{10} a$.

(3) When the law is $y = ax^n + b \log x$.

Writing it as $\frac{y}{\log x} = a \frac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$,

the given law becomes, $Y = aX + b$.

(4) When the law is $y = ae^{bx}$

Taking logarithms, it becomes $\log_{10} y = (b \log_{10} e) x + \log_{10} a$

Putting $x = X$ and $\log_{10} y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10} e$ and $c = \log_{10} a$.

(5) When the law is $xy = ax + by$.

Dividing by x , we have $y = b \frac{x}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.

Example 24.1. R is the resistance to maintain a train at speed V ; find a law of the type $R = a + bV^2$ connecting R and V , using the following data :

V (miles/hour) :	10	20	30	40	50
R (lb/ton) :	8	10	15	21	30

Solution. Given law is $R = a + bV^2$... (i)

Taking $V^2 = x$ and $R = y$, (i) becomes

$$y = a + bx \quad \dots (ii)$$

which is a linear law.

Table for the values of x and y is as follows :

x	100	400	900	1600	2500
y	8	10	15	21	30

Plot these points. Draw the straight line of best fit through these points (Fig. 24.1)

Slope of this line ($= b$)

$$= \frac{MN}{LM} = \frac{21 - 15}{1600 - 900} = \frac{6}{700} = 0.0085 \text{ nearly.}$$

Since $L(900, 15)$ lies on (ii),

$$\therefore 15 = a + 0.0085 \times 900,$$

whence $a = 15 - 7.65 = 7.35$ nearly.

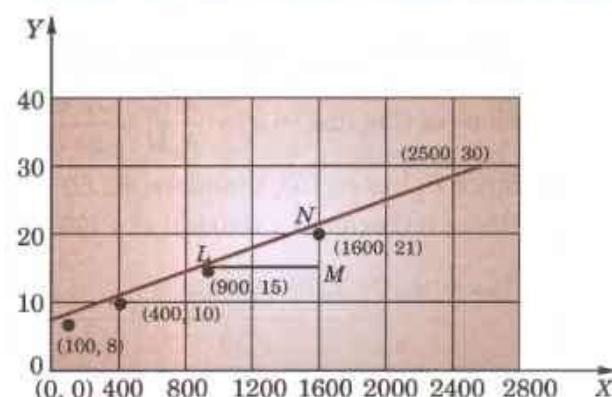


Fig. 24.1

Example 24.2. The following values of x and y are supposed to follow the law $y = ax^2 + b \log_{10} x$. Find graphically the most probable values of the constants a and b .

x	2.85	3.88	4.66	5.69	6.65	7.77	8.67
y	16.7	26.4	35.1	47.5	60.6	77.5	93.4

Solution. Given law is $y = ax^2 + b \log_{10} x$

i.e. $\frac{y}{\log_{10} x} = a \frac{x^2}{\log_{10} x} + b$... (i)

Taking $x^2/\log_{10} x = X$ and $y/\log_{10} x = Y$

(i) becomes $Y = aX + b$... (ii)

This is a linear law. Table for the values of X and Y is as follows :

$X = x^2/\log_{10} x$	17.93	25.56	32.49	42.87	53.75	67.80	80.83
$Y = y/\log_{10} x$	35.59	44.83	52.50	62.90	73.65	87.04	99.56
Points	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Plot these points and draw the straight line of best fit through these points (Fig. 24.2).

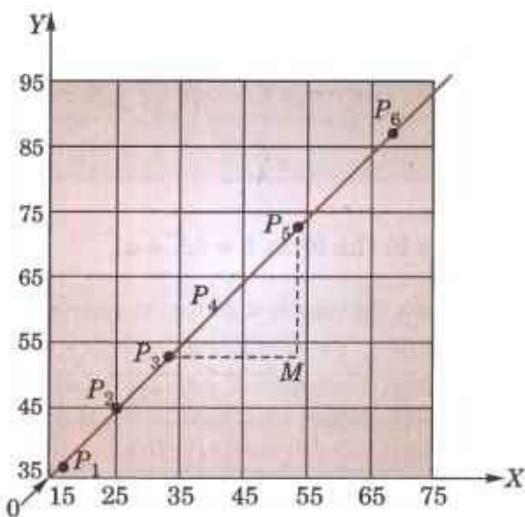


Fig. 24.2

Slope of this line ($= a$) = $\frac{MP_5}{P_3M} = \frac{73.65 - 52.50}{53.75 - 32.49} = \frac{21.15}{21.26} = 0.99$

Since P_3 lies on (ii), therefore, $52.50 = 0.99 \times 32.49 + b$ whence $b = 20.2$

Hence (i) becomes $y = (0.99)x^2 + (20.2)\log_{10} x$.

Example 24.3. The values of x and y obtained in an experiment are as follows :

x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39.1	50.3	67.2	85.6	125.0

The probable law is $y = ae^{bx}$. Test graphically the accuracy of this law and if the law holds good, find the best values of the constants.

Solution. Given law is $y = ae^{bx}$... (i)

Taking logarithms to base 10, we have $\log_{10} y = \log_{10} a + (b \log_{10} e) x$

Putting $x = X$ and $\log_{10} y = Y$, it becomes $y = (b \log_{10} e) X + \log_{10} a$... (ii)

Table for the values of X and Y is as under :

$X = x$	2.30	3.10	4.00	4.92	5.91	7.20
$Y = \log_{10} y$	1.52	1.59	1.70	1.83	1.93	2.1
Points	P_1	P_2	P_3	P_4	P_5	P_6

Scale : 1 small division along x -axis = 0.1

10 small divisions along y -axis = 0.1.

Plot these points and draw the line of best fit. As these points are lying almost along a straight line, the given law is nearly accurate (Fig. 24.3).

Now slope of this line ($= b \log_{10} e$)

$$= \frac{MN}{NM} = 0.12$$

whence $b = \frac{0.12}{\log_{10} e} = 0.12 \times 2.303 = 0.276$

Since the point L (4, 1.71) lies on (ii), therefore, $1.71 = 0.12 \times 4 + \log_{10} a$ whence $a = 17$ nearly.

Hence the curve of best fit is $y = 17 e^{0.276x}$.

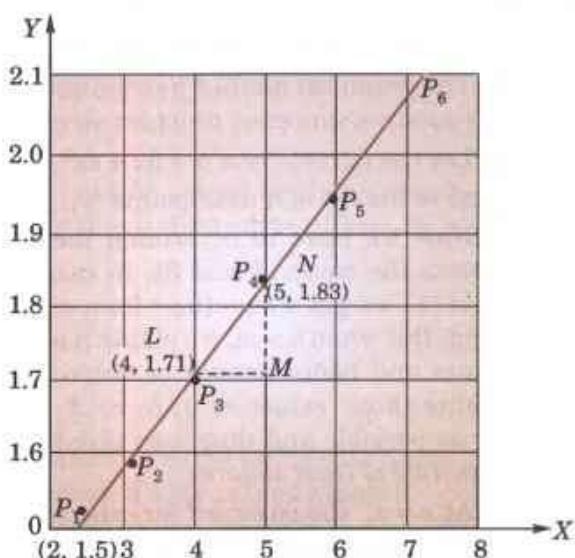


Fig. 24.3

PROBLEMS 24.1

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p and w , using the following data:

w (lb) :	50	70	100	120
p (lb) :	12	15	21	25

Compute p , when $w = 150$ lb.

2. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made:

Voltage	V ...	62	70	78	84	92
Resistance	R ...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best value of a and b .

3. Verify if the values of x and y , related as shown in the following table, obey the law $y = a + b \sqrt{x}$. If so, find graphically the values of a and b .

x :	500	1,000	2,000	4,000	6,000
y :	0.20	0.33	0.38	0.45	0.51

4. The following values of T and t follow the law $T = at^n$. Test if this is so and find the best values of a and n .

$T = 1.0$	1.5	2.0	2.5
$t = 25$	56.2	100	1.56

5. Find the best value of a and b if $y = ax + b \log_{10} x$ is the curve which represents most closely the observed values given below:

x :	2	3	4	5	6
y :	3.74	5.99	7.47	8.92	9.86

6. Fit the curve $y = ae^{bx}$ to the following data:

x :	0	2	4
y :	5.1	10	31.1

(Coimbatore, 1997)

7. The following are the results of an experiment on friction of bearings. The speed being constant, corresponding values of the coefficient of friction and the temperature are shown in the table:

t :	120	110	100	90	80	70	60
μ :	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t .

24.4 PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious drawback of being unable to give a unique curve of fit. *The principle of least squares, however, provides an elegant procedure for fitting a unique curve to a given data.*

Let the curve, $y = a + bx + cx^2 + \dots + kx^{m-1}$... (1)

be fitted to the set of n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

Now we have to determine the constants a, b, c, \dots, k such that it represents the curve of best fit. In case $n = m$, on substituting the values (x_i, y_i) in (1), we get n equations from which a unique set of n constants can be found. But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants. So we try to determine those values of a, b, c, \dots, k which satisfy all the equations as nearly as possible and thus may give the best fit. In such cases, we apply the *principle of least squares*.

At $x = x_i$, the *observed (or experimental) value* of the ordinate is $y_i = P_i L_i$ and the corresponding value on the fitting curve (1) is $a + bx_i + cx_i^2 + \dots + kx_i^{m-1} = M_i L_i$ ($= \eta_i$, say) which is the *expected (or calculated) value* (Fig. 24.4). The difference of the observed and the expected values i.e. $y_i - \eta_i (= e_i)$ is called the *error (or residual)* at $x = x_i$. Clearly some of the errors e_1, e_2, \dots, e_n will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum i.e. $E = e_1^2 + e_2^2 + \dots + e_n^2$.

The curve of best fit is that for which e 's are as small as possible i.e., E , the sum of the squares of the errors is a minimum. This is known as the *principle of least squares* and was suggested by Legendre* in 1806.

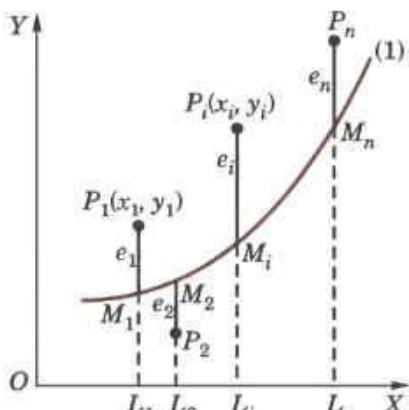


Fig. 24.4

Obs. The principle of least squares does not help us to determine the form of the appropriate curve which can fit a given data. It only determines the best possible values of the constants in the equation when the form of the curve is known before hand. The selection of the curve is a matter of experience and practical considerations.

24.5 (1) METHOD OF LEAST SQUARES

For clarity, suppose it is required to fit the curve

$$y = a + bx + cx^2 \quad \dots(1)$$

to a given set of observations $(x_1, y_1), (x_2, y_2), \dots, (x_5, y_5)$. For any x_i , the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

\therefore the sum of the squares of these errors is

$$\begin{aligned} E &= e_1^2 + e_2^2 + \dots + e_5^2 \\ &= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + \dots + [y_5 - (a + bx_5 + cx_5^2)]^2 \quad [\text{See } \S 5.12 (3)] \end{aligned}$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = 2[y_1 - (a + bx_1 + cx_1^2)] - 2[y_2 - (a + bx_2 + cx_2^2)] - \dots - 2[y_5 - (a + bx_5 + cx_5^2)] \quad \dots(2)$$

$$\begin{aligned} \frac{\partial E}{\partial b} &= 0 = -2x_1[y_1 - (a + bx_1 + cx_1^2)] - 2x_2[y_2 - (a + bx_2 + cx_2^2)] \\ &\quad - \dots - 2x_5[y_5 - (a + bx_5 + cx_5^2)] \dots(3) \end{aligned}$$

$$\begin{aligned} \frac{\partial E}{\partial c} &= 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] \\ &\quad - \dots - 2x_5^2[y_5 - (a + bx_5 + cx_5^2)] \dots(4) \end{aligned}$$

Equation (2) simplifies to

$$\begin{aligned} y_1 + y_2 + \dots + y_5 &= 5a + b(x_1 + x_2 + \dots + x_5) + c(x_1^2 + x_2^2 + \dots + x_5^2) \\ \text{i.e., } \Sigma y_i &= 5a + b \Sigma x_i + c \Sigma x_i^2 \quad \dots(5) \end{aligned}$$

* See footnote on p. 311.

Equation (3) becomes

$$x_1y_1 + x_2y_2 + \dots + x_5y_5 = a(x_1 + x_2 + \dots + x_5) + b(x_1^2 + x_2^2 + \dots + x_5^2) + c(x_1^3 + x_2^3 + \dots + x_5^3) \\ i.e., \quad \Sigma x_i y_i = a \Sigma x_i + b \Sigma x_i^2 + c \Sigma x_i^3 \quad \dots(6)$$

$$\text{Similarly (4) simplifies to } \Sigma x_i^2 y_i = a \Sigma x_i^2 + b \Sigma x_i^3 + c \Sigma x_i^4 \quad \dots(7)$$

The equations (5), (6) and (7) are known as *Normal equations* and can be solved as simultaneous equations in a , b , c . The values of these constants when substituted in (1) give the desired curve of best fit.

(2) Working procedure

(a) To fit the straight line $y = a + bx$

(i) Substitute the observed set of n values in this equation.

(ii) Form normal equations for each constant

$$i.e., \quad \Sigma y = na + b \Sigma x, \quad \Sigma xy = a \Sigma x + b \Sigma x^2$$

[The normal equation for the unknown a is obtained by multiplying the equations by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (i.e., x) and adding.]

(iii) Solve these normal equations as simultaneous equations for a and b .

(iv) Substitute the values of a and b in $y = a + bx$, which is the required line of best fit.

(b) To fit the parabola : $y = a + bx + cx^2$

(i) Form the normal equations $\Sigma y = na + b \Sigma x + c \Sigma x^2$

$$\Sigma xy = a \Sigma x + b \Sigma x^2 + c \Sigma x^3$$

$$\text{and} \quad \Sigma x^2 y = a \Sigma x^2 + b \Sigma x^3 + c \Sigma x^4$$

[The normal equation for c has been obtained by multiplying the equations by the coefficient of c (i.e., x^2) and adding.]

(ii) Solve these as simultaneous equations for a , b , c .

(iii) Substitute the values of a , b , c in $y = a + bx + cx^2$, which is the required parabola of best fit.

(c) In general, the curve $y = a + bx + cx^2 + \dots + kx^{m-1}$ can be fitted to a given data by writing m normal equations.

Example 24.4. If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W , using the following data :

$P = 12$	15	21	25
$W = 50$	70	100	120

where P and W are taken in kg-wt. Compute P when $W = 150$ kg. wt.

(U.P.T.U., 2007; V.T.U., 2002)

Solution. The corresponding normal equations are

$$\left. \begin{aligned} \Sigma P &= 4c + m \Sigma W \\ \Sigma WP &= c \Sigma W + m \Sigma W^2 \end{aligned} \right\} \quad \dots(i)$$

The values of ΣW etc. are calculated by means of the following table :

W	P	W^2	WP
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
Total = 340	73	31800	6750

\therefore The equations (i) becomes $73 = 4c + 340m$ and $6750 = 340c + 31800m$

$$i.e., \quad 2c + 170m = 365 \quad \dots(ii)$$

$$\text{and} \quad 34c + 3180m = 6750 \quad \dots(iii)$$

Multiplying (ii) by 17 and subtracting from (iii), we get

$$m = 0.1879 \quad \therefore \text{from (ii), } c = 2.2785$$

Hence the line of best fit is

$$P = 2.2759 + 0.1879 W$$

When $W = 150 \text{ kg.}$, $P = 2.2785 + 0.1879 \times 150 = 30.4635 \text{ kg.}$

Obs. The calculations get simplified when the central values of x is zero. It is therefore, advisable to make the central value zero, if it be not so. This is illustrated by the next example.

Example 24.5. Fit a second degree parabola to the following data :

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U., 2006)

Solution. Let $u = x - 2$ and $v = y$ so that the parabola of fit $y = a + bu + cu^2$ becomes

$$v = A + Bu + Cu^2 \quad \dots(i)$$

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 12.9 = 5A + 10C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 11.3 = 10B$$

$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad 33.5 = 10A + 34C$$

Solving these as simultaneous equations, we get

$$A = 1.48, \quad B = 1.13, \quad C = 0.55.$$

$\therefore (i)$ becomes, $v = 1.48 + 1.13u + 0.55u^2$

or $y = 1.48 + 1.13(x - 2) + 0.55(x - 2)^2$

Hence $y = 1.42 - 1.07x + 0.55x^2$.

Example 24.6. Fit a second degree parabola to the following data :

$x = 1.0$	1.5	2.0	2.5	3.0	3.5	4.0
$y = 1.1$	1.3	1.6	2.0	2.7	3.4	4.1

(V.T.U., 2009 ; Bhopal, 2008)

Solution. We shift the origin to $(2.5, 0)$ and take 0.5 as the new unit. This amounts to changing the variable x to X , by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. The values of ΣX etc., are calculated as below :

x	X	y	Xy	X^2	$X^2 y$	X^3	X^4
1.0	-3	1.1	-3.3	9	9.9	-27	81
1.5	-2	1.3	-2.6	4	5.2	-8	16
2.0	-1	1.6	-1.6	1	1.6	-1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total	0	16.2	14.3	28	69.9	0	196

The normal equations are

$$7a + 28c = 16.2; \quad 28b = 14.3; \quad 28a + 196c = 69.9$$

Solving these as simultaneous equations, we get

$$a = 2.07, b = 0.511, c = 0.061$$

$$\therefore y = 2.07 + 0.511X + 0.061X^2$$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511(2x - 5) + 0.061(2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$. This is the required parabola of best fit.

Example 24.7. Fit a second degree parabola to the following data :

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(U.P.T.U., 2009)

Solution. Taking $u = x - 1993$ and $v = y - 357$, the equation $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \quad \dots(i)$$

x	$u = x - 1993$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1989	-4	352	-5	20	16	-80	-64	256
1990	-3	360	-1	3	9	-9	-27	81
1991	-2	357	0	0	4	0	-8	16
1992	-1	358	1	-1	1	1	-1	1
1993	0	360	3	0	0	0	0	0
1994	1	361	4	4	1	4	1	1
1995	2	361	4	8	4	16	8	16
1996	3	360	3	9	9	27	27	81
1997	4	359	2	8	16	32	64	256
Total	$\Sigma u = 0$		$\Sigma v = 11$	$\Sigma uv = 51$	$\Sigma u^2 = 60$	$\Sigma u^2v = -9$	$\Sigma u^3 = 0$	$\Sigma u^4 = 708$

The normal equations are

$$\Sigma v = 9A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 11 = 9A + 60C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 51 = 60B \quad \text{or} \quad B = \frac{17}{20}$$

$$\Sigma u^2v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \frac{694}{231}$, $B = \frac{17}{20}$, $C = -\frac{247}{924}$

$$\therefore (i) \text{ becomes } v = \frac{694}{231} + \frac{17}{20}u - \frac{247}{924}u^2$$

$$\text{or } y - 357 = \frac{694}{231} + \frac{17}{20}(x - 1993) - \frac{247}{924}(x - 1993)^2$$

$$\text{or } y = \frac{694}{231} - \frac{32861}{20} - \frac{247}{924}(1993)^2 + \frac{17}{20}x + \frac{247 \times 3866}{924}x - \frac{247}{924}x^2$$

$$\text{or } y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$$

$$\text{Hence } y = -1000106.41 + 1034.29x - 0.267x^2.$$

PROBLEMS 24.2.

1. By the method of least squares, find the straight line that best fits the following data :

x :	1	2	3	4	5
y :	14	27	40	55	68

(U.P.T.U., 2008)

2. Fit a straight line to the following data :

Year x :	1961	1971	1981	1991	2001
Production y :	8	10	12	10	16

(in thousand tons)

and find the expected production in 2006.

3. A simply supported beam carries a concentrated load P (lb) at its mid-point. Corresponding to various values of P , the maximum deflection Y (in) is measured. The data are given below :

$P :$	100	120	140	160	180	200
$Y :$	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The results of measurement of electric resistance R of a copper bar at various temperatures $t^{\circ}\text{C}$ are listed below :

$t :$	19	25	30	36	40	45	50
$R :$	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined by you.

5. Find the best possible curve of the form $y = a + bx$, using method of least squares for the data :

$x :$	1	3	4	6	8	9	11	14
$y :$	1	2	4	4	5	7	8	9

(V.T.U., 2011)

6. Fit a straight line to the following data

(a) $x :$	1	2	3	4	5	6	7	8	9
$y :$	9	8	10	12	11	13	14	16	5
(b) $x :$	6	7	7	8	8	8	9	9	10
$y :$	5	5	4	5	4	3	4	3	3

(Bhopal, 2008) (J.N.T.U., 2008)

7. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations :

$x :$	-3	-2	-1	0	1	2	3
$y :$	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(V.T.U., 2006; J.N.T.U., 2000 S)

8. Fit a parabola $y = a + bx + cx^2$ to the following data :

$x :$	2	4	6	8	10
$y :$	3.07	12.85	31.47	57.38	91.29

(V.T.U., 2003 S)

9. Fit a second degree parabola to the following data :

$x :$	1	2	3	4	5	6	7	8	9	10
$y :$	124	129	140	159	228	289	315	302	263	210

(U.P.T.U., 2009)

10. The following table gives the results of the measurements of train resistances ; V is the velocity in miles per hour. R is the resistance in pounds per ton :

$V :$	20	40	60	80	100	120
$R :$	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a , b , and c . (U.P.T.U., 2002)

11. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$. Find the best values of a , b and c for the following table :

$T :$	1	2	3	4	5	6	7
$V :$	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(U.P.T.U., MCA, 2010)

24.6 FITTING OF OTHER CURVES

$$(1) y = ax^b$$

Taking logarithms, $\log_{10} y = \log_{10} a + b \log_{10} x$

$$\text{i.e., } Y = A + bX \quad \text{where } X = \log_{10} x, Y = \log_{10} y \text{ and } A = \log_{10} a. \quad (i)$$

∴ The normal equations for (i) are : $\Sigma Y = nA + b\Sigma X$, $\Sigma XY = A\Sigma X + b\Sigma X^2$

from which A and b can be determined. Then a can be calculated from $A = \log_{10} a$.

$$(2) y = ae^{bx}$$

(Exponential curve)

Taking logarithms, $\log_{10} y = \log_{10} a + bx \log_{10} e$

$$\text{i.e., } Y = A + BX \text{ where } Y = \log_{10} y, A = \log_{10} a \text{ and } B = b \log_{10} e$$

Here the normal equations are : $\Sigma Y = nA + B\Sigma x$, $\Sigma xY = A\Sigma x + B\Sigma x^2$

from which A , B can be found and consequently a , b can be calculated.

$$(3) xy^n = b \text{ (or } p v^y = k)$$

(Gas equation)

$$\text{Taking logarithms, } \log_{10} x + a \log_{10} y = \log_{10} b \quad \text{or} \quad \log_{10} y = \frac{1}{a} \log_{10} b - \frac{1}{a} \log_{10} x.$$

This is of the form $Y = A + BX$

where $X = \log_{10} x$, $Y = \log_{10} y$, $A = \frac{1}{a} \log_{10} b$, $B = -\frac{1}{a}$.

Here also the problem reduces to finding a straight line of best fit through the given data.

Example 24.8. Find the least squares fit of the form $y = a_0 + a_1 x^2$ to the following data :

$x :$	-1	0	1	2
$y :$	2	5	3	0

(U.P.T.U., 2008)

Solution. Putting $x^2 = X$, we have $y = a_0 + a_1 X$

\therefore the normal equations are : $\Sigma y = 4a_0 + a_1 \Sigma X$; $\Sigma Y = a_0 \Sigma X + a_1 \Sigma X^2$.

The values of ΣX , ΣX^2 etc. are calculated below :

x	y	X	X^2	XY
-1	2	1	1	2
0	5	0	0	0
1	3	1	1	3
2	0	4	16	0
	$\Sigma y = 10$	$\Sigma X = 10$	$\Sigma X^2 = 18$	$\Sigma XY = 5$

\therefore the normal equations become $10 = 400 + 6a_1$; $5 = 600 + 18a_1$

Solving these equations we get, $a_0 = 4.167$, $a_1 = -1.111$.

Hence the curve of best fit is

$$y = 4.167 - 1.111X \quad i.e., \quad y = 4.167 - 1.111x^2.$$

Example 24.9. An experiment gave the following values :

v (ft/min) :	350	400	500	600
t (min) :	61	26	7	26

It is known that v and t are connected by the relation $v = at^b$. Find the best possible values of a and b .

Solution. We have $\log_{10} v = \log_b a + b \log_{10} t$

or $y = A + bX$, where $X = \log_{10} t$, $y = \log_{10} v$, $A = \log_{10} a$

\therefore the normal equations are

$$\Sigma Y = 4A + b \Sigma X \quad \dots(i)$$

$$\Sigma XY = A \Sigma X + b \Sigma X^2 \quad \dots(ii)$$

Now ΣX etc. are calculated as in the following table :

v	t	$X = \log_{10} t$	$y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	26	0.4150	2.7782	1.153	0.172
Total		4.4604	10.6234	11.658	6.075

\therefore Equations (i) and (ii) become

$$4A + 4.46b = 10.623; 4.46A + 6.075b = 11.658$$

Solving these, $A = 2.845$, $b = -0.1697$

$\therefore a = \text{antilog } A = \text{antilog } 2.845 = 699.8$.

Example 24.10. Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data :

Altitude (x) :	50	450	780	1200	4400	4800	5300
Dose of radiation (y) :	28	30	32	36	51	58	69

(S.V.T.U., 2007; J.N.T.U., 2003)

Solution. Let $y = ab^x$ be the exponential curve.

Then $\log_{10} y = \log_{10} a + x \log_{10} b$

or $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$, $B = \log_{10} b$

\therefore the normal equations are

$$\Sigma Y = 7A + B \Sigma x \quad \dots(i)$$

$$\Sigma x Y = A \Sigma x + B \Sigma x^2 \quad \dots(ii)$$

Now Σx etc. are calculated as follows :

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
$\Sigma = 16980$		11.295579	29502.305	72743400

\therefore equations (i) and (ii) become

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Solving these equations, we get $A = 1.4521015$, $B = 0.0000666289$

$$\log_{10} y = Y = 1.4521015 + 0.0000666289x$$

Hence y (at $x = 3000$) = 44.874 i.e. 44.9 approx.

Example 24.11. The pressure and volume of a gas are related by the equation $pv^\gamma = k$, γ and k being constants. Fit this equation to the following set of observations :

p (kg/cm^2)	: 0.5	1.0	1.5	2.0	2.5	3.0	
v (litres)	: 1.62	1.00	0.75	0.62	0.52	0.46	(V.T.U., 2011)

Solution. We have $\log_{10} p + \gamma \log_{10} v = \log_{10} k$

$$\text{or } \log_{10} v = \frac{1}{\gamma} \log_{10} k - \frac{1}{\gamma} \log_{10} p \quad \text{or } Y = A + BX$$

$$\text{where } X = \log_{10} p, Y = \log_{10} v, A = \frac{1}{\gamma} \log_{10} k, B = -\frac{1}{\gamma}.$$

\therefore the normal equations are

$$\Sigma Y = 6A + B\Sigma X \quad \dots(i)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \dots(ii)$$

Now ΣX etc. are calculated as follows :

p	v	$X = \log_{10} p$	$Y = \log_{10} v$	XY	X^2
.5	1.62	-0.3010	0.2095	-0.0630	0.0906
1.0	1.00	0.0000	0.0000	-0.0000	0.0000
1.5	0.75	0.1761	-0.1249	-0.0220	0.0310
2.0	0.62	0.3010	-0.2076	-0.0625	0.0906
2.5	0.52	0.3979	-0.2840	-0.1130	0.1583
3.0	0.46	0.4771	-0.3372	-0.1609	0.2276
Total		1.0511	-0.7442	-0.4214	0.5981

\therefore equations (i) and (ii) become

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving these, we get $A = 0.0132$, $B = -0.7836$.

$\therefore \gamma = -1/B = 1.1276$ and $k = \text{antilog}(Ay) = \text{antilog}(0.0168) = 1.039$.

Hence the equation of best fit is $pv^{1.276} = 1.039$.

PROBLEMS 24.3

1. If V (km/hr) and R (kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table :

$V :$	10	20	30	40	50
$R :$	8	10	15	21	30

(Indore, 2008)

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations :

$x :$	1	2	3	4	5
$y :$	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data :

$x :$	1	2	3	4	5	6	7	8
$y :$	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(U.P.T.U., 2010)

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data :

$x :$	1	2	3	4
$y :$	3	1.5	6	7.5

(J.N.T.U., 2003)

5. Find the least square curve $y = ax + b/x$ for the following data :

$x :$	1	2	3	4
$y :$	-1.5	0.99	3.88	7.66

(Madras, 2003)

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data :

$x :$	1	2	3	4	5	6
$y :$	298	4.26	5.21	6.10	6.80	7.50

(J.N.T.U., 2003)

7. Fit the curve of the form $y = ae^{bx}$ to the following data :

$x :$	77	100	185	239	285
$y :$	2.4	3.4	7.0	11.1	19.6

(V.T.U., 2011 S ; J.N.T.U., 2006)

8. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the data :

$x :$	0.1	0.2	0.3	0.4
$f(t) :$	0.76	0.58	0.44	0.35

(U.P.T.U., 2008)

9. The voltage v across a capacitor at time t seconds is given by the following table :

$t :$	0	2	4	6	8
$v :$	150	63	28	12	5.6

Use the method of least squares to fit a curve of the form $v = ae^{kt}$ to this data.

10. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data :

$x :$	2	3	4	5	6
$y :$	144	172.8	207.4	248.8	298.5

(Tiruchirapalli, 2001)

24.7 METHOD OF GROUP AVERAGES

Let the straight line,
fit the set of n observations

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n) \text{ quite closely.} \quad (\text{Fig. 24.5})$$

...(1)

When $x = x_1$, the observed (or experimental) value of $y = y_1 = L_1 P_1$ and from (1),

$$y = a + bx_1 = L_1 M_1,$$

which is known as the expected (or calculated) value of y at L_1 .

Then e_1 = observed value at L_1 – expected value at L_1

$$= y_1 - (a + bx_1) = M_1 P_1,$$

which is called the error (or residual) at x_1 . Similarly the errors for the other observations are

$$e_2 = y_2 - (a + bx_2) = M_2 P_2$$

$$\dots$$

$$e_n = y_n - (a + bx_n) = M_n P_n$$

Some of these errors may be positive and others negative.

The method of group averages is based on the assumption that the sum of the residuals is zero. To find the constants a and b is (1), we require two equations. As such we divide the data into two groups : the first containing k observations

$$(x_1, y_1), (x_2, y_2) \dots (x_k, y_k);$$

and the second group having the remaining $n - k$ observations

$$(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \dots, (x_n, y_n).$$

Assuming that the sum of the errors in each group is zero, we get

$$(y_1 - (a + bx_1)) + (y_2 - (a + bx_2)) + \dots + (y_k - (a + bx_k)) = 0$$

$$(y_{k+1} - (a + bx_{k+1})) + (y_{k+2} - (a + bx_{k+2})) + \dots + (y_n - (a + bx_n)) = 0$$

On simplification, we obtain

$$\frac{y_1 + y_2 + \dots + y_k}{k} = a + b \frac{x_1 + x_2 + \dots + x_k}{k} \quad \dots(2)$$

$$\frac{y_{k+1} + y_{k+2} + \dots + y_n}{n-k} = a + b \frac{x_{k+1} + x_{k+2} + \dots + x_n}{n-k} \quad \dots(3)$$

In (2), $\frac{1}{k}(x_1 + x_2 + \dots + x_k)$ and $\frac{1}{k}(y_1 + y_2 + \dots + y_k)$ are simply the average values of x 's and y 's of the first group. Hence the equations (2) and (3) are obtained from (1) by replacing x and y by their respective averages of the two groups. Solving (2) and (3), we get a and b .

Obs. The main drawback of this method is that a different grouping of the observations will give different values of a and b . In practice, we divide the data in such a way that each group contains almost an equal number of observations.

Example 24.12. The latent heat of vaporisation of steam r , is given in the following table at different temperatures t :

t :	40	50	60	70	80	90	100	110
r :	1069.1	1063.6	1058.2	1052.7	1049.3	1041.8	1036.3	1030.8

For this range of temperature, a relation of the form $r = a + bt$ is known to fit the data. Find the values of a and b by the method of group averages. (Madras, 2003)

Solution. Let us divide the data into two groups each containing four readings. Then we have

t	r	t	r
40	1069.1	80	1049.3
50	1063.6	90	1041.8
60	1058.2	100	1036.3
70	1052.7	110	1030.8
$\Sigma t = 220$	$\Sigma r = 4243.6$	$\Sigma t = 380$	$\Sigma r = 4158.2$

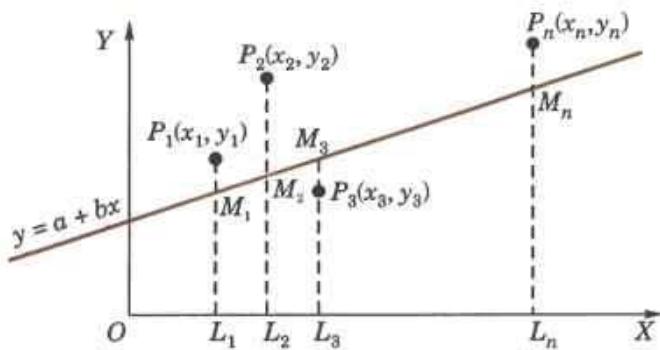


Fig. 24.5

Substituting the averages of t 's and r 's of the two groups in the given relation, we get

$$\frac{4243.6}{4} = a + b \frac{220}{4} \quad i.e., 1060.9 = a + 55b \quad \dots(i)$$

$$\frac{4158.2}{4} = a + b \frac{380}{4} \quad i.e., 1039.55 = a + 95b \quad \dots(ii)$$

Solving (i) and (ii), we obtain

$$a = 1090.26, b = -0.534.$$

24.8 FITTING A PARABOLA

We have applied the method of averages to *linear law* involving two constants only. To fit the parabola

$$y = a + bx + cx^2 \quad \dots(1)$$

which contains three constants, to a set of observations, we proceed as follows :

Let (x_1, y_1) be a point on (1) satisfying the given data so that

$$y_1 = a + bx_1 + cx_1^2$$

$$\text{Then } y - y_1 = b(x - x_1) + c(x^2 - x_1^2)$$

$$\text{or } \frac{y - y_1}{x - x_1} = b + c(x + x_1)$$

Putting $x + x_1 = X$ and $(y - y_1)/(x - x_1) = Y$, it takes the linear form

$$Y = b + cX.$$

Now b and c can be found as before.

Example 24.13. The corresponding values of x and y are given by the following table :

$x :$	87.5	84.0	77.8	63.7	46.7	36.9
$y :$	292	283	270	235	197	181

Solution. Taking $x = 84$, $y = 283$ as a particular point on $y = a + bx + cx^2$,

$$\text{we get } 283 = a + b(84) + c(84)^2 \quad \dots(i)$$

$$\therefore y - 283 = b(x - 84) + c[x^2 - (84)^2]$$

$$\text{or } \frac{y - 283}{x - 84} = b + c(x + 84)$$

$$\text{i.e., } Y = b + cX \quad \dots(ii)$$

where $X = x + 84$, $Y = (y - 283)/(x - 84)$.

Now we have the following table of values :

x	y	$X = x + 84$	$Y = (y - 283)/(x - 84)$
87.5	292	171.5	2.571
84.0	283	—	—
77.8	270	161.8	2.097
		<u>$\Sigma X = 333.3$</u>	<u>$\Sigma Y = 4.668$</u>
63.7	235	147.7	2.364
46.7	197	130.7	2.306
36.9	181	120.9	2.166
		<u>$\Sigma X = 399.3$</u>	<u>$\Sigma Y = 6.836$</u>

Substituting the averages of X and Y in (ii), we get

$$\frac{4.668}{2} = b + c \frac{333.3}{2} \quad i.e., 2.33 = b + 166.65 c \quad \dots(iii)$$

$$\frac{6.836}{3} = b + c \frac{399.3}{3} \quad i.e., 2.28 = b + 131.1 c \quad \dots(iv)$$

(iv)–(iii) gives $c = 0.0014$
 and (iii) gives $b = 2.0967$ i.e., 2.1 nearly
 From (i), we get $a = 96.9988$ i.e., 97 nearly.
 Hence the parabola of fit is

$$y = 97 + 2.1x + .0014x^2.$$

Example 24.14. The train resistance R (lbs/ton) is measured for the following values of its velocity V (km/hr) :

V :	20	40	60	80	100
R :	5	9	14	25	36

If R is related to V by the formula $R = a + bV^n$, find a , b , and n .

Solution. To find a , we take the following three values of v which are in G.P. :

$$\begin{aligned} v_1 &= 20, & v_2 &= 40, & v_3 &= 80 \\ \text{Then } R_1 &= 5, & R_2 &= 9, & R_3 &= 25 \\ \therefore (R_1 - a)(R_3 - a) &= (R_2 - a)^2 \end{aligned}$$

$$\text{whence } a = \frac{R_1 R_3 - R_2^2}{R_1 + R_3 - 2R_2} = 3.67$$

$$\begin{aligned} \text{Thus } R - 3.67 &= bV^n \quad \text{or} \quad \log_{10}(R - 3.67) = \log_{10}b + n \log_{10}V \\ \text{i.e., } Y &= k + nX \end{aligned} \quad \dots(i)$$

where $X = \log_{10}V$, $Y = \log_{10}(R - 3.67)$, $k = \log_{10}b$.

Now we have the following table of values :

V	R	$X = \log_{10}V$	$Y = \log_{10}(R - 3.67)$
20	5	1.3010	0.1238
40	9	1.6021	0.7267
60	14	1.7782	1.0141
		$\Sigma X = 4.6813$	$\Sigma Y = 1.8646$
80	25	1.9031	1.3290
100	36	2.0000	1.5096
		$\Sigma X = 3.9031$	$\Sigma Y = 2.8396$

Substituting the averages of X 's and Y 's in (i), we obtain

$$\frac{1.8646}{2} = k + n \frac{4.6813}{2} \quad \text{i.e., } 0.6215 = k + 1.5604 n \quad \dots(ii)$$

$$\frac{2.8396}{2} = k + n \frac{3.9031}{2} \quad \text{i.e., } 1.4193 = k + 1.9516 n \quad \dots(iii)$$

Solving (ii) and (iii), we get $n = 2.04$, $k = -2.56$ approx.

$$b = \text{antilog } k = \text{antilog } (-2.56) = 0.0028.$$

PROBLEMS 24.4

1. Fit a straight line of the form $y = a + bx$ to the following data by the method of group averages :

x :	0	5	10	15	20	25	
y :	12	15	17	22	24	30	(Tiruchirapalli, 2001)

2. The weights of a calf taken at weekly intervals are given below :

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Find a straight line of best fit.

3. Using the method of averages, fit a parabola $y = ax^2 + bx + c$ to the following data :

$x :$	20	40	60	80	100	120
$y :$	5.5	9.1	14.9	22.8	33.3	46.0

4. While testing a centrifugal pump, the following data is obtained. It is assumed to fit the equation $y = a + bx + cx^2$, where x is the discharge in litre/sec and y , head in metres of water. Find the values of the constants a, b, c by the method of group averages.

$x :$	2	2.5	3	3.5	4	4.5	5	5.5	6
$y :$	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

5. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data :

$x :$	5	15	20	30	35	40
$y :$	10	14	25	40	50	62

(Madras, 2002)

24.9 METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$ be the set of n observations such that

$$x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = h \text{ (say)}$$

We define the moments of the observed values of y as follows :

$$m_1, \text{ the 1st moment} = h \sum y$$

$$m_2, \text{ the 2nd moment} = h \sum xy$$

$$m_3, \text{ the 3rd moment} = h \sum x^2 y \text{ and so on.}$$

Let the curve fitting the given data be $y = f(x)$. Then the moments of the calculated values of y are

$$\mu_1, \text{ the 1st moment} = \int y dx$$

$$\mu_2, \text{ the 2nd moment} = \int xy dx$$

$$\mu_3, \text{ the 3rd moment} = \int x^2 y dx \text{ and so on.}$$

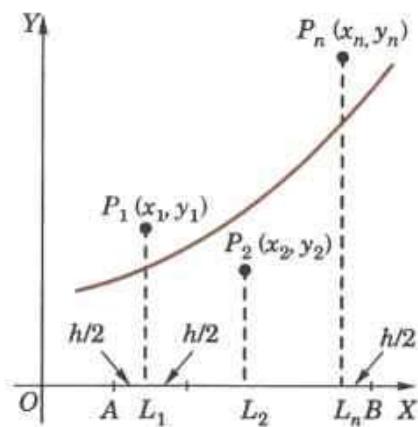


Fig. 24.6

This method is based on the assumption that the moment of the observed values of y are respectively equal to the moments of the calculated values of y i.e., $m_1 = \mu_1, m_2 = \mu_2, m_3 = \mu_3$ etc. These equations (known as observation equations) are used to determine the constants in $f(x)$.

m 's are calculated from the tabulated values of x and y while μ 's are computed as follows :

In Fig. 24.6, y_1 the ordinate of $P_1 (x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1 - h/2, x_1 + h/2)$. Similarly, y_n , the ordinate of $P_n (x = x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points such that

$$OA = x_1 - h/2 \text{ and } OB = x_n + h/2,$$

then

$$\mu_1 = \int y dx = \int_{x_1 - h/2}^{x_n + h/2} f(x) dx$$

$$\mu_2 = \int_{x_1 - h/2}^{x_n + h/2} xf(x) dx$$

and

$$\mu_3 = \int_{x_1 - h/2}^{x_n + h/2} x^2 f(x) dx.$$

Example 24.15. Fit a straight line $y = a + bx$ to the following data by the method of moments :

$x :$	1	2	3	4
$y :$	16	19	23	26

(Madras, 2001 S)

Solution. Since only two constants a and b are to be found, it is sufficient to calculate the first two moments in each case. Here $h = 1$.

$$m_1 = h \sum y = 1(16 + 19 + 23 + 26) = 84$$

$$m_2 = h \sum xy = 1(1 \times 16 + 2 \times 19 + 3 \times 23 + 4 \times 26) = 227$$

To compute the moments of calculated values of $y = a + bx$, the limits of integration will be $1 - h/2$ and $4 + h/2$ i.e., 0.5 to 4.5

$$\therefore \mu_1 = 2 \int_{0.5}^{4.5} (a + bx) dx = \left| ax + b \frac{x^2}{2} \right|_{0.5}^{4.5} = 4a + 10b$$

$$\mu_2 = \int_{0.5}^{4.5} x(a + bx) dx = 10a + \frac{91}{3}b.$$

Thus, the observation equations $m_r = \eta_r$ ($r = 1, 2$) are $4a + 10b = 84$; $10a + \frac{91}{3}b = 227$

Solving these, $a = 13.02$ and $b = 3.19$.

Hence the required equation is $y = 13.02 + 3.19x$.

Example 24.16. Given the following data :

$x :$	0	1	2	3	4
$y :$	1	5	10	22	38

find the parabola of best fit by the method of moments.

Solution. Let the parabola of best fit be $y = a + bx + cx^2$... (i)

Since three constants are to be found, we calculate the first three moments in each case. Here $h = 1$.

$$m_1 = h \Sigma y = 1(1 + 5 + 10 + 22 + 38) = 76$$

$$m_2 = h \Sigma xy = 1(0 + 5 + 20 + 66 + 152) = 243$$

$$m_3 = h \Sigma x^2 y = 1(0 + 5 + 40 + 198 + 608) = 851$$

For computing the moments of calculated values of (i), the limits of integration will be $0 - h/2$ and $4 + h/2$ i.e., -0.5 and 4.5.

$$\therefore \mu_1 = \int_{-0.5}^{4.5} (a + bx + cx^2) dx = 5a + 10b + 30.4c$$

$$\mu_2 = \int_{-0.5}^{4.5} x(a + bx + cx^2) dx = 10a + 30.4b + 102.5c$$

$$\mu_3 = \int_{-0.5}^{4.5} x^2(a + bx + cx^2) dx = 30.4a + 102.5b + 369.1c$$

Thus the observation equations $m_r = \mu_r$ ($r = 1, 2, 3$) are

$$5a + 10b + 30.4c = 76; 10a + 30.4b + 102.5c = 243; 30.4a + 102.5b + 369.1c = 851$$

Solving these equations, we get $a = 0.4$, $b = 3.15$, $c = 1.4$.

Hence the parabola of best fit is $y = 0.4 + 3.15x + 1.4x^2$.

PROBLEMS 24.5

1. Use the method of moments to fit the straight line $y = a + bx$ to the data :

$x :$	1	2	3	4
$y :$	0.17	0.18	0.23	0.32

2. Fit a straight line to the following data, using the method of moments :

$x :$	1	3	5	7	9
$y :$	1.5	2.8	4.0	4.7	6.0

(Madras, 2001)

3. Fit a parabola of the form $y = a + bx + cx^2$ to the data :

$x :$	1	2	3	4
$y :$	1.7	1.8	2.3	3.2

by the method of moments.

4. By using the method of moments, fit a parabola to the following data :

$x :$	1	2	3	4
$y :$	0.30	0.64	1.32	5.40

(Madras, 2000 S)

24.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 24.6

Fill up the blanks or choose the correct answer in the following problems :

- The law $y = ax^2 + bx$ converted to linear form is
- The gas equation $pv^n = k$ can be reduced to $y = a + bx$ where $a = \dots$ and $b = \dots$.
- The principle of 'least squares' states that
- $y = ax^b + c$ in linear form is
- To fit the straight line $y = mx + c$ to n observations, the normal equations are
 - $\Sigma y = n \Sigma x + \Sigma cm$, $\Sigma xy = c \Sigma x^2 + c \Sigma n$.
 - $\Sigma y = m \Sigma x + nc$, $\Sigma xy = m \Sigma x^2 + c \Sigma x$.
 - $\Sigma y = c \Sigma x + m \Sigma n$, $\Sigma xy = c \Sigma x^2 + m \Sigma x$.
- To fit $y = ab^x$ by least square method, normal equations are
- The observation equations for fitting a straight line by *method of moments* are
- The *method of group averages* is based on the assumption that the sum of the residuals is
- $y = ax^2 + b \log_{10} x$ reduced to linear law takes the form
- Given $\begin{bmatrix} x: & 0 & 1 & 2 \\ y: & 0 & 1.1 & 2.1 \end{bmatrix}$ then the straight line of best fit is
- The *method of moments* is based on the assumption that
- In $y = a + bx$, $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then $a = \dots$, $b = \dots$.
- $y = x/(ax + b)$ in linear form is
- If $y = a + bx + cx^2$ and

$x :$	0	1	2	3	4
$y :$	1	1.8	1.3	2.5	7.3

 then the first normal equation is :

(α) $15 = 5a + 10b + 29c$,	(β) $15 = 5a + 10b + 31c$
(γ) $12.9 = 5a + 10b + 30c$	(δ) $34 = 5a + 10b + 27c$.
- If $y = 2x + 5$ is the best fit for 8 pairs of values (x, y) by the method of least squares and $\Sigma y = 120$, then $\Sigma X =$

(a) 35	(b) 40	(c) 45	(d) 30.
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Statistical Methods

1. Introduction. 2. Collection and classification of data. 3. Graphical representation. 4. Comparison of frequency distributions. 5. Measures of central tendency. 6. Measures of dispersion. 7. Coefficient of variation; Relations between measures of dispersion. 8. Standard deviation of the combination of two groups. 9. Moments. 10. Skewness. 11. Kurtosis. 12. Correlation. 13. Coefficient of correlation. 14. Lines of regression. 15. Standard error of estimate. 16. Rank correlation. 17. Objective Type of Questions.

25.1 INTRODUCTION

Statistics deals with the methods for collection, classification and analysis of numerical data for drawing valid conclusions and making reasonable decisions. It has meaningful applications in production engineering, in the analysis of experimental data, etc. The importance of statistical methods in engineering is on the increase. As such we shall now introduce the student to this interesting field.

25.2 (1) COLLECTION OF DATA

The collection of data constitutes the starting point of any statistical investigation. Data may be collected for each and every unit of the whole lot (*population*), for it would ensure greater accuracy. But complete enumeration is prohibitively expensive and time consuming. As such out of a very large number of items, a few of them (*a sample*) are selected and conclusions drawn on the basis of this sample are taken to hold for the population.

(2) **Classification of data.** The data collected in the course of an inquiry is not in an easily assimilable form. As such, its proper classification is necessary for making intelligent inferences. The classification is done by dividing the raw data into a convenient number of groups according to the values of the variable and finding the frequency of the variable in each group.

Let us, for example, consider the raw data relating to marks obtained in Mechanics by a group of 64 students :

79	88	75	60	93	71	59	85
84	75	82	68	90	62	88	76
65	75	87	74	62	95	78	63
78	82	75	91	77	69	74	68
67	73	81	72	63	76	75	85
80	73	57	88	78	62	76	53
62	67	97	78	85	76	65	71
78	89	61	75	95	60	79	83

This data can conveniently be grouped and shown in a tabular form as follows :

Class	Frequency	Cumulative frequency
50—54	1	1
55—59	2	3
60—64	9	12
65—69	7	19
70—74	8	27
75—79	17	44
80—84	6	50
85—89	8	58
90—94	3	61
95—99	3	64
Total = 64		

It would be seen from the above table that there is one student getting marks between 50—54, two students getting marks between 55—59, nine students getting marks between 60—64 and so on. Thus the 64 figure have been put into only 10 groups, called the **classes**. The width of the class is called the **class interval** and the number in that interval is called the **frequency**. The mid-point or the mid-value of the class is called the **class mark**. The above table showing the classes and the corresponding frequencies is called a *frequency table*. Thus a set of raw data summarised by distributing it into a number of classes alongwith their frequencies is known as a **frequency distribution**.

While forming a frequency distribution, the number of classes should not ordinarily exceed 20, and should not, in general, be less than 10. As far as possible, the class intervals should be of equal width.

(3) **Cumulative frequency.** In some investigations, we require the number of items less than a certain value. We add up the frequencies of the classes upto that value and call this number as the *cumulative frequency*. In the above table, the third column shows the cumulative frequencies, i.e., the number of students, getting less than 54 marks, less than 59 marks and so on.

25.3 GRAPHICAL REPRESENTATION

A convenient way of representing a sample frequency distribution is by means of graphs. It gives to the eyes the general run of the observations and at the same time makes the raw data readily intelligible. We give below the important types of graphs in use :

(1) **Histogram.** A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies themselves (Fig. 25.1).

(2) **Frequency polygon.** A frequency polygon for an ungrouped data can be obtained by joining points plotted with the variable values as the abscissae and the frequencies as the ordinates. For a grouped distribution, the abscissae of the points will be the mid-values of the class intervals. In case the intervals are equal, the frequency polygon can be obtained by joining the middle points of the upper sides of the rectangles of the histogram by straight lines (shown by dotted lines in Fig. 25.1). If the class intervals become very very small, the frequency polygon takes the form of a smooth curve called the *frequency curve*.

(3) **Cumulative frequency curve-Ogive.** Very often, it is desired to show in a diagrammatic form, not the relative frequencies in the various intervals, but the cumulative frequencies above or below a given value. For example, we may wish to read off from a diagram the number or proportions of people whose income is not less than any given amount, or proportion of people whose height does not exceed any stated value. Diagrams of

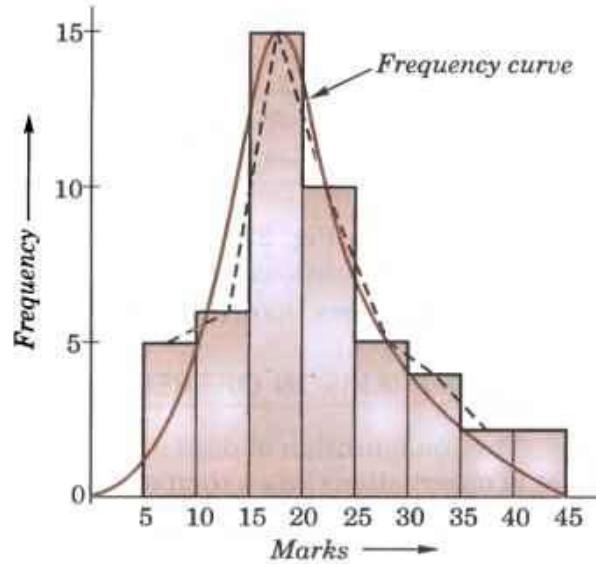


Fig. 25.1

this type are known as *cumulative frequency curves* or *ogives*. These are of two kinds 'more than' or 'less than' and typically they look somewhat like a long drawn S (Fig. 25.2).

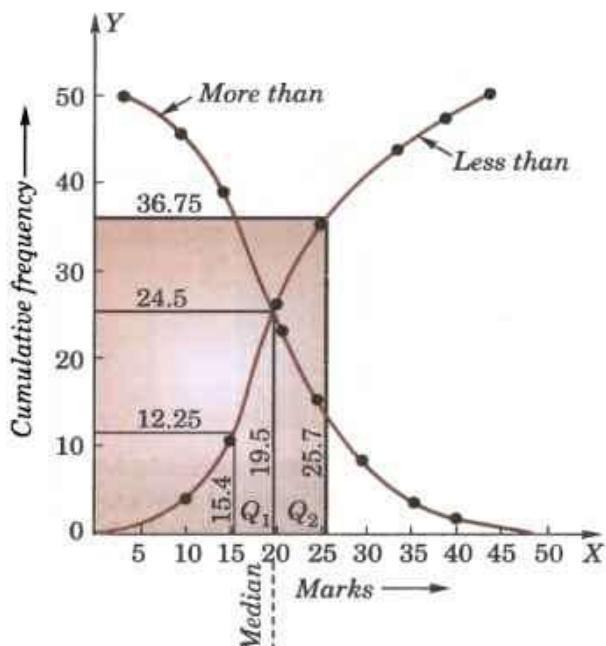


Fig. 25.2

Example 25.1. Draw the histogram, frequency polygon, frequency curve and the ogive 'less than' and 'more than' from the following distribution of marks obtained by 49 students :

Class (Marks group)	Frequency (No. of students)	Cumulative frequency	
		(Less than)	(More than)
5—10	5	5	49
10—15	6	11	44
15—20	15	26	38
20—25	10	36	23
25—30	5	41	13
30—35	4	45	8
35—40	2	47	4
40—45	2	49	2

Solution. In Fig. 25.1, the rectangles show the *histogram*; the dotted polygon represents the *frequency polygon* and the smooth curve is the *frequency curve*.

The *ogives 'less than'* and '*more than*' are shown in Fig. 25.2.

25.4 COMPARISON OF FREQUENCY DISTRIBUTIONS

The condensation of data in the form of a frequency distribution is very useful as far as it brings a long series of observations into a compact form. But in practice, we are generally interested in comparing two or more series. The inherent inability of the human mind to grasp in its entirety even the data in the form of a frequency distribution compels us to seek for certain constants which could concisely give an insight into the important characteristics of the series. The chief constants which summarise the fundamental characteristics of the frequency distributions are (i) *Measures of central tendency*, (ii) *Measures of dispersion* and (iii) *Measures of skewness*.

25.5 MEASURES OF CENTRAL TENDENCY

A frequency distribution in general, shows clustering of the data around some central value. Finding of this central value or the average is of importance, as it gives a most representative value of the whole group.

Different methods give different averages which are known as the *measures of central tendency*. The commonly used measures of central value are *Mean, Median, Mode, Geometric mean and Harmonic mean*.

(1) Mean. If $x_1, x_2, x_3, \dots, x_n$ are a set of n values of a variate, then the *arithmetic mean* (or simply *mean*) is given by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}, \text{ i.e. } \frac{\sum x_i}{n} \quad \dots(1)$$

In a *frequency distribution*, if x_1, x_2, \dots, x_n be the mid-values of the class-intervals having frequencies f_1, f_2, \dots, f_n respectively, we have

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \dots + f_n x_n}{f_1 + f_2 + \dots + f_n} = \frac{\sum f_i x_i}{\sum f_i} \quad \dots(2)$$

Calculation of mean. Direct method of computing especially when applied to grouped data involves heavy calculations and in order to avoid these, the following formulae are generally used :

$$\text{I. Short-cut method} \quad \bar{x} = A + \frac{\sum f_i d_i}{\sum f_i} \quad \dots(3)$$

$$\text{II. Step-deviation method} \quad \bar{x} = A + h \frac{\sum f_i u_i}{\sum f_i} \quad \dots(4)$$

where $d = x - A$ and $u = (x - A)/h$, A being an arbitrary origin and h the equal class interval.

Proof. If x_1, x_2, \dots, x_n are the mid-values of the classes with frequencies f_1, f_2, \dots, f_n , we have

$$\sum f_i x_i = \sum f_i (A + d_i) = A \sum f_i + \sum f_i d_i$$

$$\therefore \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = A + \frac{\sum f_i d_i}{\sum f_i}$$

Further $u_i = d_i/h$ or $d_i = hu_i$. Substituting this value in (3), we get (4).

Obs. The algebraic sum of the deviations of all the variables from their mean is zero, for

$$\sum f_i (x_i - \bar{x}) = \sum f_i x_i - \bar{x} \sum f_i = \sum f_i x_i - \frac{\sum f_i x_i}{\sum f_i} \cdot \sum f_i = 0.$$

Cor. If \bar{x}_1, \bar{x}_2 be the means of two samples of size n_1 and n_2 , then the mean \bar{x} of the combined sample of size $n_1 + n_2$ is given by

$$\bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

For $n_1 \bar{x}_1$ = sum of all observations of the first sample,

and $n_2 \bar{x}_2$ = sum of all observations of the second sample.

\therefore sum of the observations of the combined sample = $n_1 \bar{x}_1 + n_2 \bar{x}_2$.

Also number of the observations in the combined sample = $n_1 + n_2$.

$$\therefore \text{mean of the combined sample} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}.$$

Example 25.2. The following is the frequency distribution of a random sample of weekly earnings of 509 employees :

Weekly earnings : 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40

No. of employees : 3 6 10 15 24 42 75 90 79 55 36 26 19 13 9 7

Calculate the average weekly earnings.

Solution. The calculations are arranged in the following table. The arbitrary origin is generally taken as the value corresponding to the maximum frequency.

By direct method, we have

$$\text{Mean } \bar{x} = \frac{\sum f x}{\sum f} = \frac{13,315}{509} = 26.16$$

By step-deviation method, we have

$$\begin{aligned} \bar{x} &= A + h \frac{\sum f u}{\sum f} = 25 + 2 \times \frac{295}{509} \\ &= 25 + 1.16 = 26.16, \text{ which is same as found above.} \end{aligned}$$

Weekly earnings	Mid value <i>x</i>	No. of employees <i>f</i>	Step deviations		
			<i>f</i> × <i>x</i>	<i>u</i> = (<i>x</i> - 25)/2	<i>f</i> × <i>u</i>
10—12	11	3	33	-7	-21
12—14	13	6	78	-6	-36
14—16	15	10	150	-5	-50
16—18	17	15	255	-4	-60
18—20	19	24	456	-3	-72
20—22	21	42	882	-2	-84
22—24	23	75	1725	-1	-75
24—26	25	90	2250	0	-398
26—28	27	79	2133	1	79
28—30	29	55	1595	2	110
30—32	31	36	1116	3	108
32—34	33	26	858	4	104
34—36	35	19	665	5	95
36—38	37	13	481	6	78
38—40	39	9	351	7	63
40—42	41	7	287	8	56
				+ 693	
			$\Sigma f = 509$	$\Sigma fx = 13,815$	$\Sigma fu = 295$

(2) Median. If the values of a variable are arranged in the ascending order of magnitude, the median is the middle item if the number is odd and is the mean of the two middle items if the number is even. Thus the median is equal to the mid-value, i.e., the value which divides the total frequency into two equal parts.

For the grouped data,

$$\text{Median} = L + \frac{\left(\frac{1}{2}N - C\right)}{f} \times h$$

where L = lower limit of the median class, N = total frequency,

f = frequency of the median class, h = width of the median class,

and C = cumulative frequency upto the class preceding the median class.

Quartiles. Quartiles are those values which divide the frequency into four equal parts, when the values are arranged in the ascending order of magnitude. The **lower quartile** (Q_1) is mid-way between the lower extreme and the median. The **upper quartile** (Q_3) is midway between the median and the upper extreme.

For the grouped data, these are calculated by the formulae :

$$Q_1 = L + \frac{\left(\frac{1}{4}N - C\right)}{f} \times h$$

$$Q_3 = L + \frac{\left(\frac{3}{4}N - C\right)}{f} \times h$$

where L = lower limit of the class in which Q_1 or Q_3 lies, f = frequency of this class, h = width of the class

and C = cumulative frequency upto the class preceding the class in which Q_1 or Q_3 lies.

The difference between the upper and lower quartiles, i.e., $Q_3 - Q_1$ is called the **inter-quartile range**.

Obs. The ogives give a ready method of marking on the curve the values of the median and the quartiles. The two ogives 'less than' and 'more than' cut each other at the median (Fig. 25.2).

(3) Mode. The mode is defined as that value of the variable which occurs most frequently, i.e., the value of the maximum frequency.

For a grouped distribution, it is given by the formula

$$\text{Mode} = L + \frac{\Delta_1}{\Delta_1 + \Delta_2} h$$

where L = lower limit of the class containing the mode,

Δ_1 = excess of modal frequency over frequency of preceding class,

Δ_2 = excess of modal frequency over following class,

and h = size of modal class.

For a frequency curve (Fig. 25.1), the abscissa of the highest ordinate determines the value of the mode. There may be one or more modes in a frequency curve. Curves having a single mode are termed as *unimodal*, those having two modes as *bi-modal* and those having more than two modes as *multi-modal*.

Obs. In a symmetrical distribution, the mean, median and mode coincide. For other distributions, however, they are different and are known to be connected by the empirical relationship :

$$\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median}).$$

Example 25.3. Calculate median and the lower and upper quartiles from the distribution of marks obtained by 49 students of example 25.1. Find also the semi-interquartile range and the mode.

Solution. Median (or 49/2) falls in the class (15—20) and is given by

$$15 + \frac{(49/2) - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5 \text{ marks.}$$

Lower quartile Q_1 (or 49/4) = 12.25 also falls in the class 15—20.

$$\therefore Q_1 = 15 + \frac{(49/4) - 11}{15} \times 5 = 15 + \frac{12.5}{3} = 15.4 \text{ marks}$$

Upper quartile (or $\frac{3}{4} \times 49 = 36.75$) falls in the class 25—30.

$$\therefore Q_3 = 25 + \frac{36.75 - 36}{5} \times 5 = 25.75 \text{ marks.}$$

$$\text{Semi-interquartile range} = \frac{1}{2}(Q_3 - Q_1) = \frac{25.75 - 15.4}{2} = \frac{10.35}{2} = 5.175.$$

Mode. It is seen that the mode value falls in the class 15—20. Employing the formula for the grouped distribution, we have

$$\text{Mode} = 15 + \frac{15 - 6}{(15 - 6) + (15 - 10)} \times 5 = 18.2 \text{ marks.}$$

Obs. In Fig. 25.2, the ogives meet at a point whose abscissa is 19.5 which is the *median* of the distribution. The values for the lower and upper quartiles are similarly seen to be 15.4 (for frequency 12.25) and 25.7 (for frequency 36.75).

Example 25.4. Given below are the marks obtained by a batch of 20 students in a certain class test in Physics and Chemistry.

Roll No. of students	Marks in Physics	Marks in Chemistry	Roll No. of students	Marks in Physics	Marks in Chemistry
1	53	58	11	25	10
2	54	55	12	42	42
3	52	25	13	33	15
4	32	32	14	48	46
5	30	26	15	72	50
6	60	85	16	51	64
7	47	44	17	45	39
8	46	80	18	33	38
9	35	33	19	65	30
10	28	72	20	29	36

In which subject is the level of knowledge of the students higher?

Solution. The subject for which the value of the median is higher will be the subject in which the level of knowledge of the students is higher. To find the median in each case, we arrange the marks in ascending order of magnitude :

Sr. No.	Marks in Physics	Marks in Chemistry	Sr. No.	Marks in Physics	Marks in Chemistry
1	25	10	11	46	42
2	28	15	12	47	44
3	29	25	13	48	46
4	30	26	14	51	50
5	32	30	15	52	55
6	33	32	16	53	58
7	33	33	17	54	64
8	35	36	18	60	72
9	42	38	19	65	80
10	45	39	20	72	85

Median marks in Physics = A.M. of marks of 10th and 11th terms

$$= \frac{45 + 46}{2} = 45.5$$

Median marks in Chemistry = A.M. of marks of 10th and 11th items.

$$= \frac{39 + 42}{2} = 40.5$$

Since the median marks in Physics is greater than the median marks in Chemistry; the level of knowledge in Physics is higher.

Example 25.5. An incomplete frequency distribution is given as below :

Variable : 10–20 20–30 30–40 40–50 50–60 60–70 70–80

Frequency : 12 30 ? 65 ? 25 18

Given that the total frequency is 229 and median is 46, find the missing frequencies.

Solution. Let f_1, f_2 be the missing frequencies of the classes 30–40 and 50–60 respectively.

Since the median lies in the class 40–50,

$$\therefore 46 = 40 + \frac{229/2 - (12 + 30 + f_1)}{65} \times 10$$

which gives $f_1 = 33.5$ which can be taken as 34.

$$\therefore f_2 = 229 - (12 + 30 + 34 + 65 + 25 + 18) = 45.$$

(4) Geometric mean. If x_1, x_2, \dots, x_n are a set of n observations, then the geometric mean is given by

$$G.M. = (x_1 x_2 \dots x_n)^{1/n}$$

or $\log G.M. = \frac{1}{n} (\log x_1 + \log x_2 + \dots + \log x_n)$... (1)

In a frequency distribution, let x_1, x_2, \dots, x_n be the central values with corresponding frequencies f_1, f_2, \dots, f_n , we have

$$G.M. = [(x_1)^{f_1} \cdot (x_2)^{f_2} \cdots (x_n)^{f_n}]^{1/n} \quad \text{where } n = \sum f_i.$$

or $\log G.M. = \frac{1}{n} [f_1 \log x_1 + f_2 \log x_2 + \dots + f_n \log x_n]$... (2)

Hence (1) and (2) show that logarithm of G.M. = A.M. of logarithms of the values.

(5) Harmonic mean. If x_1, x_2, \dots, x_n be a set of n observations, then the harmonic mean is defined as the reciprocal of the (arithmetic) mean of the reciprocals of the quantities. Thus

$$H.M. = \frac{1}{\frac{1}{n} \left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)}$$

In a frequency distribution, $H.M. = \frac{1}{\frac{1}{n} \left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \dots + \frac{f_n}{x_n} \right)}$ where $n = \sum f_i$.

Example 25.6. Three cities A, B, C are equidistant from each other. A motorist travels from A to B at 30 km/hr, from B to C at 40 km/hr, from C to A at 50 km/hr. Determine the average speed.

Solution. Let $AB = BC = CA = s$ km

Time taken to travel from A to B = $s/30$

Time taken to travel from B to C = $s/40$

Time taken to travel from C to A = $s/50$

$$\therefore \text{average time taken} = \frac{1}{3} \left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50} \right)$$

$$\text{Thus the average speed} = \frac{s}{\frac{1}{3} \left(\frac{s}{30} + \frac{s}{40} + \frac{s}{50} \right)}$$

In other words, the average speed is the harmonic mean of 30, 40, 50 km/hr.

$$\text{Hence the average speed} = \frac{1}{\frac{1}{3} \left(\frac{1}{30} + \frac{1}{40} + \frac{1}{50} \right)} = 38.3 \text{ km/hr.}$$

Obs. Of the various measures of central tendency, the mean is the most important for it can be computed easily. The median, though more easily calculable, cannot be applied with ease to theoretical analysis. Median is of advantage when there are exceptionally large and small values at the ends of the distribution.

The mode, though most easily calculated, has the least significance. It is particularly misleading in distributions which are small in numbers or highly unsymmetrical.

The geometrical mean though difficult to compute, finds application in cases like populations where we are concerned with a quantity whose changes tend to be directly proportional to the quantity itself.

The harmonic mean is useful in limited situations where time and rate or prices are involved.

PROBLEMS 25.1

1. Draw the histogram and frequency polygon for the following distribution. Also calculate the arithmetic mean :

Class interval	0—99	100—199	200—299	300—399	400—499	500—599	600—699	700—799
Frequency	10	54	184	264	246	40	1	1

2. The following marks were given to a batch of candidates :

66	62	45	79	32	51	56	60	51	49
25	42	54	54	58	70	43	58	50	52
38	67	50	59	48	65	71	30	46	55
82	51	63	45	53	40	35	56	70	52
67	55	57	30	63	42	74	58	44	55

Draw a cumulative frequency curve.

Hence find the proportion of candidates securing more than 50 marks. Also mark off the median, the first and third quartiles.

3. Find the mean, median and mode for the following :

Mid Value	15	20	25	30	35	40	45	50	55
Frequency	2	22	19	14	3	4	6	1	1

(Kerala, 1990)

4. Calculate mean, median and mode of the following data relating to weight of 120 articles :

Weight (in gm)	0—10	10—20	20—30	30—40	40—50	50—60
No. of articles	14	17	22	26	23	18

5. The population of a country was 300 million in 1971. It became 520 million in 1989. Calculate the percentage compound rate of growth per annum.

[Hint. Use $P_n = P_0(1+r)^n$, r being the growth rate.]

6. The number of divorces per 1000 marriages in the United States increased from 84 in 1970 to 108 in 1990. Find the annual increase of the divorce rate for the period 1970 to 1990.

7. An aeroplane flies along the four sides of a square at speeds of 100, 200, 300 and 400 km/hr, respectively. What is the average speed of the plane in its flight around the square.

8. A man having to drive 90 km, wishes to achieve an average speed of 30 km/hr. For the first half of the journey, he averages only 20 km/hr. What must be his average speed for the second half of the journey if his overall average is to be 30 km/hr.

9. Following table gives the cumulative frequency of the age of a group of 199 teachers. Find the mean and median age of the group.
 Age in years : 20–25 25–30 30–35 35–40 40–45 45–50 50–55 55–60 60–65 65–70
 Cum. frequ. : 21 40 90 130 146 166 176 186 195 199
10. Recast the following cumulative table in the form of an ordinary frequency distribution and determine the median and the mode :

No. of days absent	No. of students	No. of days absent	No. of students
Less than 5	29	Less than 30	644
Less than 10	224	Less than 35	650
Less than 15	465	Less than 40	653
Less than 20	582	Less than 45	655
Less than 25	634		

25.6 MEASURES OF DISPERSION

Although measures of central tendency do exhibit one of the important characteristics of a distribution, yet they fail to give any idea as to how the individual values differ from the central value, i.e., whether they are closely packed around the central value or widely scattered away from it. Two distributions may have the same mean and the same total frequency, yet they may differ in the extent to which the individual values may be spread about the average (See Fig. 25.3). The magnitude of such a variation is called *dispersion*. The important measures of dispersion are given below :

(1) **Range.** This is the simplest measure of dispersion and is given by the difference between the greatest and the least values in the distribution. If the weekly wages of a group of labourers are

₹ 21 23 28 25 35 42 39 48

then range = Max. value – Min. value = 48 – 21 = ₹ 27.

(2) **Quartile deviation or semi-interquartile range.** One half of the interquartile range is called *quartile deviation*, or *semi-interquartile range*. If Q_1 and Q_3 are the first and third quartiles, the semi-interquartile range

$$Q = \frac{1}{2}(Q_3 - Q_1).$$

(3) **Mean deviation.** The mean deviation is the mean of the absolute differences of the values from the mean, median or mode. Thus *mean deviation (M.D.)*

$$= \frac{1}{n} \sum f_i |x_i - A|$$

where A is either the mean or the median or the mode. As the positive and negative differences have equal effects, only the absolute value of differences is taken into account.

(4) **Standard deviation.** The most important and the most powerful measure of dispersion is the *standard deviation (S.D.)* : generally denoted by σ . It is computed as the square root of the mean of the squares of the differences of the variate values from their mean.

Thus *standard deviation (S.D.)*

$$\sigma = \sqrt{\left[\frac{\sum f_i (x_i - \bar{x})^2}{N} \right]} \quad \dots(1)$$

where N is the total frequency $\sum f_i$.

If however, the deviations are measured from any other value, say A , instead of \bar{x} , it is called the *root-mean-square deviation*.

The square of the standard deviation is known as the *variance*.

Calculation of S.D. The change of origin and the change of scale considerably reduces the labour in the calculation of standard deviation. The formulae for the computation of σ are as follows :

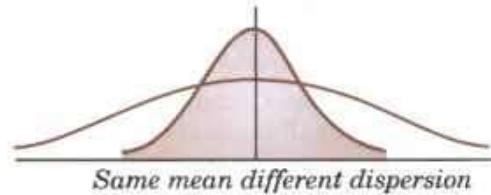


Fig. 25.3

I. Short-cut method

$$\sigma = \sqrt{\left[\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i} \right)^2 \right]} \quad \dots(2)$$

II. Step-deviation method

$$\sigma = h \sqrt{\left[\frac{\sum f_i d'_i{}^2}{\sum f_i} - \left(\frac{\sum f_i d'_i}{\sum f_i} \right)^2 \right]} \quad \dots(3)$$

where $d_i = x_i - A$ and $d'_i = (x_i - A)/h$, being the assumed mean and h the equal class interval.

Proof. We know that $x_i - \bar{x} = (x_i - A) - (\bar{x} - A)$

$$\begin{aligned} \therefore \sum f_i (x_i - \bar{x})^2 &= \sum f_i [d_i - (\bar{x} - A)]^2 = \sum f_i d_i^2 + (\bar{x} - A)^2 \sum f_i - 2(\bar{x} - A) \sum f_i d_i \\ &= \sum f_i d_i^2 - \frac{(\sum f_i d_i)^2}{\sum f_i} \end{aligned} \quad \left[\because \bar{x} = A + \frac{\sum f_i d_i}{\sum f_i} \right]$$

Hence

$$\sigma^2 = \frac{\sum f_i (x - \bar{x})^2}{\sum f_i} = \frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i} \right)^2$$

Further $d'_i = (x_i - A)/h = d_i/h$ or $d_i = h d'_i$, then substituting this value in (2), we get (3).

Obs. The root mean square deviation is least when measured from the mean.

The root mean square deviation is given by

$$s^2 = \frac{\sum f_i d_i^2}{\sum f_i} \quad \text{and} \quad \frac{\sum f_i d_i}{\sum f_i} = \left[A + \frac{\sum f_i d_i}{\sum f_i} \right] - A = \bar{x} - A$$

∴ from (2), we have $s^2 = \sigma^2 + (\bar{x} - A)^2$...(4)

This shows that s^2 is always $> \sigma^2$ and the least value of $s^2 = \sigma^2$. This occurs when $A = \bar{x}$.

25.7 (1) COEFFICIENT OF VARIATION

The ratio of the standard deviation to the mean, is known as the *coefficient of variation*. As this is a ratio having no dimension, it is used for comparing the variations between the two groups with different means. It is often expressed as a percentage.

$$\therefore \text{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100$$

(2) Relations between measures of dispersion

(i) Quartile deviation = $2/3$ (standard deviation)

(ii) Mean deviation = $4/5$ (standard deviation)

25.8 STANDARD DEVIATION OF THE COMBINATION OF TWO GROUPS

If m_1, σ_1 be the mean and S.D. of a sample of size n_1 and m_2, σ_2 be those for a sample of size n_2 , then the S.D. σ of the combined sample of size $n_1 + n_2$ is given by

$$(n_1 + n_2) \sigma^2 = n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

where $D_i = m_i - m$, m being the mean of combined sample.

From (4), we have $ns^2 = n\sigma^2 + n(\bar{x} - A)^2$ where n is the size of the sample.

i.e. sum of the squares of the deviations from $A = n\sigma^2 + n(\bar{x} - A)^2$.

Now let us apply this result to the first given sample taking A at m . Then, sum of the squares of the deviations of n_1 items from $m = n_1 \sigma_1^2 + n_1(m_1 - m)^2$...(5)

Similarly for the second given sample taking A at m , sum of the squares of the deviations of n_2 items from $m = n_2 \sigma_2^2 + n_2(m_2 - m)^2$...(6)

Adding (5) and (6), sum of the squares of the deviations of $n_1 + n_2$ items from m

$$= n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_1(m_1 - m)^2 + n_2(m_2 - m)^2$$

$$\therefore (n_1 + n_2) \sigma^2 = n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

This result can be extended to the combination of any number of samples, giving a result of the form

$$(\sum n_i) \sigma^2 = \sum (n_i \sigma_i^2) + \sum (n_i D_i^2)$$

Example. 25.7. Calculate the mean and standard deviation for the following :

Size of item : 6	7	8	9	10	11	12
Frequency : 3	6	9	13	8	5	4

(V.T.U., 2001)

Solution. The calculations are arranged as follows :

Size of item x	Frequency f	Deviation $d = x - 9$	$f \times d$	$f \times d^2$
6	3	-3	-9	27
7	6	-2	-12	24
8	9	-1	-9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	36
$\Sigma f = 48$			$\Sigma fd = 0$	$\Sigma fd^2 = 124$

$$\therefore \text{mean} = 9 + \frac{\Sigma fd}{\Sigma f} = 9$$

$$\text{Standard deviation} = \sqrt{\left[\frac{\Sigma f(x - \bar{x})^2}{\Sigma f} \right]} = \sqrt{\left(\frac{\Sigma fd^2}{\Sigma f} \right)} = \sqrt{\left(\frac{124}{48} \right)} = 1.607.$$

Example 24.8. Calculate the mean and standard deviation of the following frequency distribution :

Weekly wages in ₹	No. of men
4.5—12.5	4
12.5—20.5	24
20.5—28.5	21
28.5—36.5	18
36.5—44.5	5
44.5—52.5	3
52.5—60.5	5
60.5—68.5	8
68.5—76.5	2

Solution. The calculations are arranged in the table below :

Wages class ₹	Mid value x	No. of men f	Step deviation		
			$d' = \frac{x - 32.5}{8}$	fd'	fd'^2
4.5—12.5	8.5	4	-3	-12	36
12.5—20.5	16.5	24	-2	-48	96
20.5—28.5	24.5	21	-1	-21	21
28.5—36.5	32.5	18	0	0	0
36.5—44.5	40.5	5	1	5	5
44.5—52.5	48.5	3	2	6	12
52.5—60.5	56.5	5	3	15	45
60.5—68.5	64.5	8	4	32	128
68.5—76.5	72.5	2	5	10	50
$\Sigma f = 90$			$\Sigma fd' = -13$		$\Sigma fd'^2 = 393$

$$\therefore \text{mean wage} = 32.5 + 8 \times \frac{\Sigma fd'}{\Sigma f} = 32.5 + 8 \left(\frac{-13}{90} \right) = ₹ 31.35$$

$$\text{Standard deviation} = 8 \sqrt{\frac{\Sigma fd'^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f} \right)^2} = 8 \sqrt{\frac{393}{90} - \left(\frac{-13}{90} \right)^2} = ₹ 16.64.$$

Example 25.9. The following are scores of two batsmen A and B in a series of innings :

A :	12	115	6	73	7	19	119	36	84	29
B :	47	12	16	42	4	51	37	48	13	0

Who is the better score getter and who is more consistent ?

(V.T.U., 2004)

Solution. Let x denote score of A and y that of B.

Taking 51 as the origin, we prepare the following table :

x	$d (=x - 51)$	d^2	y	$\delta (=y - 51)$	δ^2
12	-39	1521	47	-4	16
115	64	4096	12	-39	1521
6	-45	2025	16	-35	1225
73	22	484	42	-9	81
7	-44	1936	4	-47	2209
19	-32	1024	51	0	0
119	68	4624	37	-14	196
36	-15	225	48	-3	9
84	33	1089	13	-38	1444
29	-22	484	0	-51	2601
Total	-10	17508		-240	9302

$$\text{For } A, \quad \text{A.M. } \bar{x} = 51 + \frac{\sum d}{n} = 51 - \frac{10}{10} = 50$$

$$\text{S.D. } \sigma_1 = \sqrt{\left\{ \frac{\sum d^2}{n} - \left(\frac{\sum d}{n} \right)^2 \right\}} = \sqrt{[1750.8 - (-1)^2]} = 41.8$$

$$\therefore \text{coefficient of variation} = \frac{\sigma_1}{\bar{x}} \times 100 = \frac{41.8}{50} \times 100 = 83.6\%$$

$$\text{For } B, \quad \text{A.M. } \bar{y} = 51 + \frac{\sum \delta}{n} = 51 - \frac{240}{10} = 27$$

$$\text{S.D. } \sigma_2 = \sqrt{\left\{ \frac{\sum \delta^2}{n} - \left(\frac{\sum \delta}{n} \right)^2 \right\}} = \sqrt{[930.2 - (-24)^2]} = 18.8$$

$$\therefore \text{coefficient of variation} = \frac{\sigma_2}{\bar{y}} \times 100 = \frac{18.8}{27} \times 100 = 69.6\%$$

Since the A.M. of A > A.M. of B, it follows that A is a better score getter (i.e., more efficient) than B.

Since the coefficient of variation of B < the coefficient of variation of A, it means that B is more consistent than A. Thus even though A is a better player, he is less consistent.

Example 25.10. The numbers examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.

Med. Exam.	No. Examined	Mean Wt. (lbs.)	S.D. (lbs.)
A	50	113	6
B	60	120	7
C	90	115	8

Solution. We have $n_1 = 50, \bar{x}_1 = 113, \sigma_1 = 6$

$$n_2 = 60, \bar{x}_2 = 120, \sigma_2 = 7$$

$$n_3 = 90, \bar{x}_3 = 115, \sigma_3 = 8.$$

If \bar{x} is the mean of the entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90} = \frac{23200}{200} = 116 \text{ lb.}$$

If σ is the S.D. of the entire data,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1D_1^2 + n_2D_2^2 + n_3D_3^2$$

where $N = n_1 + n_2 + n_3 = 200$, $D_1 = \bar{x}_1 - \bar{x} = -3$, $D_2 = \bar{x}_2 - \bar{x} = 4$ and $D_3 = \bar{x}_3 - \bar{x} = -1$.

$$\begin{aligned}\therefore 200\sigma^2 &= 50 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1 \\ &= 1800 + 2940 + 5760 + 450 + 960 + 90\end{aligned}$$

$$\sigma^2 = \frac{12000}{200} = 60. \text{ Hence } \sigma = \sqrt{60} = 7.746 \text{ lb.}$$

PROBLEMS 25.2

1. The crushing strength of 8 cement concrete experimental blocks, in metric tonnes per sq. cm., was 4.8, 4.2, 5.1, 3.8, 4.4, 4.7, 4.1 and 4.5. Find the mean crushing strength and the standard deviation.

2. Show that the variance of the first n positive integers is $\frac{1}{12}(n^2 - 1)$. (V.T.U., 2003)

3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. (V.T.U., 2002)

4. For the distribution

x :	5	6	7	8	9	10	11	12	13	14	15
f :	18	15	34	47	68	90	80	62	35	27	11

find the mean, median and lower and upper quartiles, variance and the standard deviation.

5. The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation :

Marks obtained :	1—10	11—20	21—30	31—40	41—50	51—60
No. of candidates :	3	16	26	31	16	8

(Osmania, 2003 S ; V.T.U., 2003 S)

6. Compute the quartile deviation and standard deviation for the following :

x :	100—109	110—119	120—129	130—139	140—149	150—159	160—169	170—179
f :	15	44	133	150	125	82	35	16

7. Calculate (i) mean deviation about the mean, (ii) mean deviation about the median for the following distribution :

Class :	3—4.9	5—6.9	7—8.9	9—10.9	11—12.9	13—14.9	15—16.9
f :	5	8	30	82	45	24	6

(Madras, 2002)

8. Two observers bring the following two sets of data which represent measurements of the same quantity :

I.	105.1	103.4	104.2	104.7	104.8	105.0	104.9
II.	105.3	105.1	104.8	105.2	106.7	102.9	103.1

Calculate the standard deviation in each case. Which set of data is more reliable ? Can the same conclusion be reached by calculating the mean deviation ?

Obs. The smaller the coefficient of variation, the greater is the reliability or consistency in the data.

9. The heights and weights of the 10 armymen are given below. In which characteristics are they more variable ?

Height in cm.	170	172	168	177	179	171	173	178	173	179
Weight in kg.	75	74	75	76	77	73	76	75	74	75

10. The index number of prices of two articles A and B for six consecutive weeks are given below :

A :	314	326	336	368	404	412
B :	330	331	320	318	321	330

Find which has a more variable price ?

11. The scores of two golfers A and B in 12 rounds are given below. Who is the better player and who is the more consistent player ?

A :	74	75	78	72	78	77	79	81	79	76	72	71
B :	87	84	80	88	89	85	86	82	82	79	86	80

12. The scores obtained by two batsmen A and B in 10 matches are given below :

A :	30	44	66	62	60	34	80	46	20	38
B :	34	46	70	38	55	48	60	34	45	30

Calculating mean, S.D. and coefficient of variation for each batsman, determine who is more efficient and who is more consistent.

13. Find the mean and standard deviation of the following two samples put together :

Sample No.	Size	Mean	S.D.
1	50	158	5.1
2	60	164	4.6

14. A distribution consists of three components with frequencies 200, 250 and 300 having means 25, 10 and 15 and S.D.s. 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its S.D. is 7.2 approximately.

25.9 (1) MOMENTS

The r th moment about the mean \bar{x} of a distribution is denoted by μ_r and is given by

$$\mu_r = \frac{1}{N} \sum f_i (x_i - \bar{x})^r \quad \dots(1)$$

The corresponding moment about any point a is defined as

$$\mu'_r = \frac{1}{N} \sum f_i (x_i - a)^r \quad \dots(2)$$

In particular, we have $\mu_0 = \mu'_0 = 1$...(3)

$$\mu_1 = \frac{1}{N} \sum f_i (x_i - \bar{x}) = 0; \mu'_1 = \frac{1}{N} \sum f_i (x_i - a) = \bar{x} - a = d, \text{ say} \quad \dots(4)$$

$$\mu_2 = \frac{1}{N} \sum f_i (x_i - \bar{x})^2 = \sigma^2. \quad \dots(5)$$

(2) Moments about the mean in terms of moments about any point.

We have

$$\begin{aligned} \mu_r &= \frac{1}{N} \sum f_i (x_i - \bar{x})^r = \frac{1}{N} \sum f_i [(x_i - a) - (\bar{x} - a)]^r \\ &= \frac{1}{N} \sum f_i (X_i - d)^r \quad \text{where } X_i = x_i - a, d = \bar{x} - a. \\ &= \frac{1}{N} [\sum f_i X_i^r - {}^r C_1 d \sum f_i X_i^{r-1} + {}^r C_2 d^2 \sum f_i X_i^{r-2} - \dots] \\ &= \mu'_r - {}^r C_1 d \mu'_{r-1} + {}^r C_2 d^2 \mu'_{r-2} - \dots \end{aligned} \quad \dots(6)$$

In particular,

$$\mu_2 = \mu'_2 - \mu'^2_1 \quad \dots(7)$$

$$\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu'^3_1 \quad \dots(8)$$

$$\mu_4 = \mu'_4 - 4\mu'_3 \mu'_1 + 6\mu'_2 \mu'^2_1 - 3\mu'^4_1 \quad \dots(9)$$

These three results should be committed to memory. It should be noted that in each of these relations, the sum of the coefficients of the various terms on the right side is zero. Also each term on the right side is of the same dimension as the term on the left.

25.10 SKEWNESS

Skewness measures the degree of asymmetry or the departure from symmetry. If the frequency curve has a longer 'tail' to the right, i.e., the mean is to the right of the mode [as in Fig. 25.4 (a)], then the distribution is said to have *positive skewness*. If the curve is more elongated to the left, then it is said to have *negative skewness* [Fig. 25.4 (b)].

The following three measures of skewness deserve mention :

$$(i) \text{ Pearson's* coefficient of skewness} = \frac{\text{mean} - \text{mode}}{\sigma}$$

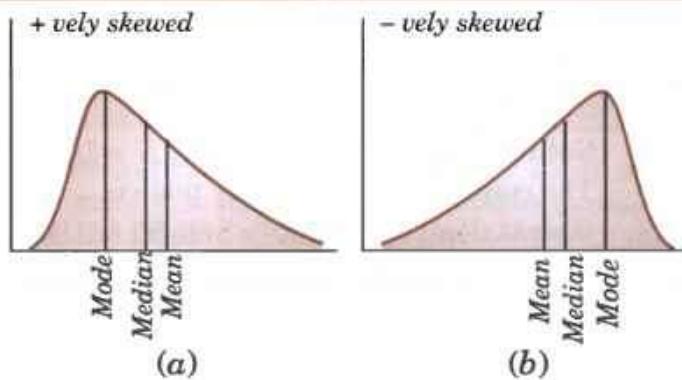


Fig. 25.4

* After the English statistician and biologist Karl Pearson (1857–1936) who did pioneering work and found the English school of statistics.

$$(ii) \text{Quartile coefficient of skewness} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$$

Its value always lies between -1 and +1.

$$(iii) \text{Coefficient of skewness based on third moment } \gamma_1 = \sqrt{\beta_1}.$$

where $\beta_1 = \mu_3^2/\mu_2^3$

Thus $\gamma_1 = \sqrt{\beta_1}$ gives the simplest measure of skewness.

25.11 KURTOSIS

Kurtosis measures the degree of peakedness of a distribution and is given by $\beta_2 = \mu_4/\mu_2^2$.

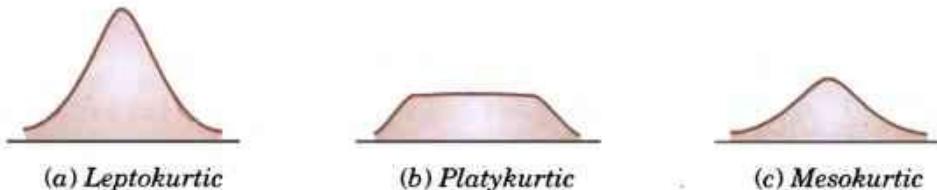


Fig. 25.5

$\gamma_2 = \beta_2 - 3$ gives the excess of Kurtosis. The curves with $\beta_2 > 3$ are called *Leptokurtic* and those with $\beta_2 < 3$ as *Platykurtic*. The normal curve for which $\beta_2 = 3$, is called *Mesokurtic* [Fig. 25.5].

Example 25.11. The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1 , β_2 and comment upon the skewness and kurtosis of the distribution. (V.T.U., 2005 S)

Solution. The first four moments about the arbitrary origin 28.5 are $\mu'_1 = 0.294$, $\mu'_2 = 7.144$, $\mu'_3 = 42.409$, $\mu'_4 = 454.98$.

$$\therefore \mu'_1 = \frac{1}{N} \sum f_i(x_i - 28.5) = \frac{1}{N} \sum f_i x_i - 28.5 = \bar{x} - 28.5 = 0.294 \text{ or } \bar{x} = 28.794$$

$$\mu_2 = \mu'_2 - \mu'_1^2 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'_1^3 = 42.409 - 3(7.144)(0.294) + 2(0.294)^3 = 36.151.$$

$$\begin{aligned} \mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'_1^2 - 3\mu'_1^4 \\ &= 454.98 - 4(42.409) \times (0.294) + 6(7.144)(0.294)^2 - 3(0.294)^4 = 408.738 \end{aligned}$$

Now $\beta_1 = \mu_3^2/\mu_2^3 = (36.151)^2/(7.058)^3 = 3.717$

$$\beta_2 = \mu_4/\mu_2^2 = 408.738/(7.058)^2 = 8.205.$$

$\therefore \gamma_1 = \sqrt{\beta_1} = 1.928$, which indicates considerable skewness of the distribution.

$$\gamma_2 = \beta_2 - 3 = 5.205 \text{ which shows that the distribution is leptokurtic.}$$

Example 25.12. Calculate the median, quartiles and the quartile coefficient of skewness from the following data :

Weight (lbs)	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
No. of persons	12	18	35	42	50	45	20	8

Solution. Here total frequency $N = \sum f_i = 230$.

The cumulative frequency table is

Weight (lbs) :	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
f :	12	18	35	42	50	45	20	8
cum. f. :	12	30	65	107	157	202	222	230

Now $N/2 = 230/2 = 115$ th item which lies in 110–120 group.

$$\therefore \text{median or } Q_2 = L + \frac{N/2 - C}{f} \times h = 110 + \frac{115 - 107}{50} \times 10 = 111.6$$

Also $N/4 = 230/4 = 57.5$ i.e. Q_1 is 57.5th or 58th item which lies in 90–100 group.

$$\therefore Q_1 = L + \frac{N/4 - C}{f} \times h = 90 + \frac{57.5 - 30}{35} \times 10 = 97.85$$

Similarly, $3N/4 = 172.5$ i.e. Q_3 is 173rd item which lies in 120–130 group.

$$\therefore Q_3 = L + \frac{3N/4 - C}{f} \times h = 120 + \frac{172.5 - 157}{45} \times 10 = 123.44$$

$$\text{Hence quartile coefficient of skewness} = \frac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$$

$$= \frac{97.85 + 123.44 - 2 \times 111.6}{123.44 - 97.85} = -0.07 \text{ (approx.)}$$

PROBLEMS 25.3

1. Calculate the first four moments of the following distribution about the mean :

x :	0	1	2	3	4	5	6	7	8
f :	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2 .

(V.T.U., 2004 ; Madras, 2003)

2. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficients of variation and skewness.

(V.T.U., 2001)

Monthly wages (in ₹)	No. of workers	Monthly wages (in ₹)	No. of workers
12.5–17.5	2	37.5–42.5	4
17.5–22.5	22	42.5–47.5	6
22.5–27.5	19	47.5–52.5	1
27.5–32.5	14	52.5–57.5	1
32.5–37.5	3		

3. Find Pearson's coefficient of skewness for the following data :

Class	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency	5	9	14	20	25	15	8	4

(V.T.U., 2000 S)

4. Compute the quartile coefficient of skewness for the following distribution :

x :	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f :	2	108	580	175	80	32	18	5

(Madras, 2002 ; V.T.U., 2000)

Also compute the measure of skewness based on the third moment.

5. The first three moments of a distribution about the value 2 of the variable are 1, 16 and –40. Show that the mean = 3, the variance = 15 and $\mu_3 = -86$.
(V.T.U., 2003 S)
6. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45.
(Coimbatore, 1999)
7. In a certain distribution, the first four moments about a point are –1.5, 17, –30 and 108. Calculate the moments about the mean, β_1 and β_2 ; and state whether the distribution is leptokurtic or platykurtic ?

25.12 CORRELATION

So far we have confined our attention to the analysis of observations on a single variable. There are, however, many phenomenae where the changes in one variable are related to the changes in the other variable. For instance, the yield of a crop varies with the amount of rainfall, the price of a commodity increases with the reduction in its supply and so on. Such a simultaneous variation, i.e. when the changes in one variable are associated or followed by changes in the other, is called *correlation*. Such a data connecting two variables is called *bivariate population*.

If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is said to be *positive*. If the increase (or decrease) in one corresponds to the decrease (or increase) in the other, the correlation is said to be *negative*. If there is no relationship indicated between the variables, they are said to be *independent or uncorrelated*.

To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph, taking one of the variables along the x -axis and the other along the y -axis. (Fig. 25.6).

Let the origin be shifted to (\bar{x}, \bar{y}) , where \bar{x}, \bar{y} are the means of x 's and y 's that the new co-ordinates are given by

$$X = x - \bar{x}, \quad Y = y - \bar{y}.$$

Now the points (X, Y) are so distributed over the four quadrants of XY -plane that the product XY is positive in the first and third quadrants but negative in the second and fourth quadrants. The algebraic sum of the products can be taken as describing the trend of the dots in all the quadrants.

\therefore (i) If ΣXY is positive, the trend of the dots is through the first and third quadrants,

(ii) if ΣXY is negative the trend of the dots is in the second and fourth quadrants, and

(iii) if ΣXY is zero, the points indicate no trend i.e. the points are evenly distributed over the four quadrants.

The ΣXY or better still $\frac{1}{n} \Sigma XY$, i.e., the average of n products may be taken as a measure of correlation. If we put X and Y in their units, i.e., taking σ_x as the unit for x and σ_y for y , then

$$\frac{1}{n} \sum \frac{X}{\sigma_x} \cdot \frac{Y}{\sigma_y}, \text{ i.e., } \frac{\Sigma XY}{n \sigma_x \sigma_y}$$

is the *measure of correlation*.

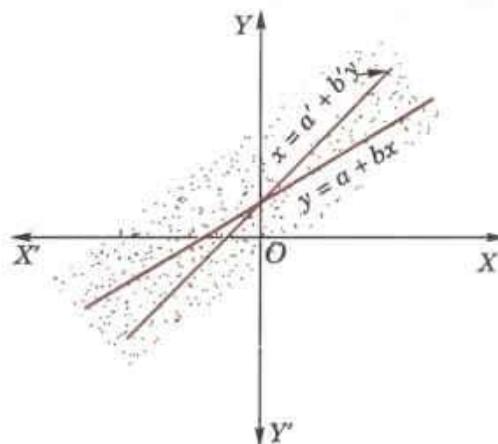


Fig. 25.6

25.13 COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the *coefficient of correlation* and is defined by the relation

$$r = \frac{\Sigma XY}{n \sigma_x \sigma_y}$$

where X = deviation from the mean $\bar{x} = x - \bar{x}$, Y = deviation from the mean $\bar{y} = y - \bar{y}$,

σ_x = S.D. of x -series, σ_y = S.D. of y -series and n = number of values of the two variables.

Methods of calculation :

(a) *Direct method*. Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2)(\Sigma Y^2)}} \quad \dots(1)$$

Another form of the formula (1) which is quite handy for calculation is

$$r = \frac{n \Sigma xy - \Sigma x \Sigma y}{\sqrt{[(n \Sigma x^2 - (\Sigma x)^2) \times (n \Sigma y^2 - (\Sigma y)^2)]}} \quad \dots(2)$$

(b) *Step-deviation method*. The direct method becomes very lengthy and tedious if the means of the two series are not integers. In such cases, use is made of assumed means. If d_x and d_y are step-deviations from the assumed means, then

$$r = \frac{n \Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[(n \Sigma d_x^2 - (\Sigma d_x)^2) \times (n \Sigma d_y^2 - (\Sigma d_y)^2)]}} \quad \dots(3)$$

where $d_x = (x - a)/h$ and $d_y = (y - b)/k$.

Obs. The change of origin and units do not alter the value of the correlation coefficient since r is a pure number.

(c) *Co-efficient of correlation for grouped data*. When x and y series are both given as frequency distributions, these can be represented by a two-way table known as the *correlation-table*. It is double-entry table with one series along the horizontal and the other along the vertical as shown on page 848. The co-efficient of correlation for such a *bivariate frequency distribution* is calculated by the formula.

$$r = \frac{n(\Sigma fd_x d_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[(n\Sigma fd_x)^2 - (\Sigma fd_x)^2] \times [n\Sigma fd_y^2 - (\Sigma fd_y)^2]}} \quad \dots(4)$$

where d_x = deviation of the central values from the assumed mean of x -series,
 d_y = deviation of the central values from the assumed mean of y -series,
 f is the frequency corresponding to the pair (x, y)
and $n (= \Sigma f)$ is the total number of frequencies.

Example 25.13. Psychological tests of intelligence and of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.). Calculate the co-efficient of correlation.

Student	A	B	C	D	E	F	G	H	I	J
I.R.	105	104	102	101	100	99	98	96	93	92
E.R.	101	103	100	98	95	96	104	92	97	94

(Andhra, 2000)

Solution. We construct the following table :

Student	Intelligence ratio		Engineering ratio		X^2	Y^2	XY
	x	$x - \bar{x} = X$	y	$y - \bar{y} = Y$			
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	-3	1	9	-3
F	99	0	96	-2	0	4	0
G	98	-1	104	6	1	36	-6
H	96	-3	92	-6	9	36	18
I	93	-6	97	-1	36	1	6
J	92	-7	94	-4	49	16	28
Total	990	0	980	0	170	140	92

From this table, mean of x , i.e., $\bar{x} = 990/10 = 99$ and mean of y , i.e. $\bar{y} = 980/10 = 98$.

$$\Sigma X^2 = 170, \Sigma Y^2 = 140 \text{ and } \Sigma XY = 92.$$

Substituting these values in the formula (1) p. 744, we have

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2)(\Sigma Y^2)}} = \frac{92}{\sqrt{(170)(140)}} = 92/154.3 = 0.59.$$

Example 25.14. The correlation table given below shows that the ages of husband and wife of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of the wife. (J.N.T.U., 2003)

Age of husband	Age of wife						Total
	15-25	25-35	35-45	45-55	55-65	65-75	
15-25	1	1	-	-	-	-	2
25-35	2	12	1	-	-	-	15
35-45	-	4	10	1	-	-	15
45-55	-	-	3	6	1	-	10
55-65	-	-	-	2	4	2	8
65-75	-	-	-	-	1	2	3
Total	3	17	14	9	6	4	53

Solution.

Age of husband			Age of wife x-series							Suppose $d_x = \frac{x - 40}{10}$ $d_y = \frac{y - 40}{10}$		
			15–25	25–35	35–45	45–55	55–65	65–75	Total f			
Years		Mid pt. x	20	30	40	50	60	70	fd _y	fd _y ²	fd _x d _y	
Age group	Mid pt. y		-20	-10	0	10	20	30				
			d_x	d_y	-2	-1	0	1	2	3		
15–25	20	-20	-2	4	2				2	-4	8	6
25–35	30	-10	-1	4	12	0			15	-15	15	16
35–45	40	0	0		0	0	0		15	0	0	0
45–55	50	10	1			0	6	2	10	10	10	8
55–65	60	20	2				4	16	12			32
65–75	70	30	3				2	4	2	8	16	32
Total f			3	17	14	9	6	4	53 = n	16	92	86
$\sum fd_x$			-6	-17	0	9	12	12	10	Thick figures in small sqs. stand for $fd_x d_y$		
$\sum fd_x^2$			12	17	0	9	24	36	98			
$\sum fd_x d_y$			8	14	0	10	24	30	86			

With the help of the above correlation table, we have

$$\begin{aligned}
 r &= \frac{n(\sum fd_x d_y) - (\sum fd_x)(\sum fd_y)}{\sqrt{[(n \sum fd_x^2) - (\sum fd_x)^2] \times [(n \sum fd_y^2) - (\sum fd_y)^2]}} \\
 &= \frac{53 \times 86 - 10 \times 16}{\sqrt{[(53 \times 98) - 100] \times [(53 \times 92) - 256]}} = \frac{4398}{\sqrt{5094 \times 4620}} = \frac{4398}{4850} = 0.91 \text{ (approx.)}.
 \end{aligned}$$

25.14 LINES OF REGRESSION

It frequently happens that the dots of the scatter diagram generally, tend to cluster along a well defined direction which suggests a linear relationship between the variables x and y . Such a line of best-fit for the given distribution of dots is called the *line of regression* (Fig. 25.6). In fact there are two such lines, one giving the best possible mean values of y for each specified value of x and the other giving the best possible mean values of x for given values of y . The former is known as the *line of regression of y on x* and the latter as the *line of regression of x on y* .

Consider first the line of regression of y on x . Let the straight line satisfying the general trend of n dots in a scatter diagram be

$$y = a + bx \quad \dots(1)$$

We have to determine the constants a and b so that (1) gives for each value of x , the best estimate for the average value of y in accordance with the *principle of least squares* (page 816), therefore, the normal equations for a and b are

$$\Sigma y = na + b\Sigma x \quad \dots(2)$$

and

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \dots(3)$$

$$(2) \text{ gives } \frac{1}{n} \Sigma y = a + b \cdot \frac{1}{n} \Sigma x \text{ i.e., } \bar{y} = a + b\bar{x}.$$

This shows that (\bar{x}, \bar{y}) , i.e., the means of x and y , lie on (1).

Shifting the origin to (\bar{x}, \bar{y}) , (3) takes the form

$$\Sigma(x - \bar{x})(y - \bar{y}) = a\Sigma(x - \bar{x}) + b\Sigma(x - \bar{x})^2, \text{ but } a\Sigma(x - \bar{x}) = 0,$$

$$\therefore b = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} = \frac{\Sigma XY}{n\sigma_x^2} = r \frac{\sigma_y}{\sigma_x} \quad \left[\because r = \frac{\Sigma XY}{n\sigma_x \sigma_y} \right]$$

$$\text{Thus the line of best fit becomes } y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \quad \dots(4)$$

which is the equation of the line of regression of y on x . Its slope is called the regression coefficient of y on x .

Interchanging x and y , we find that the line of regression of x on y is

$$x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \quad \dots(5)$$

$$\text{Thus the regression coefficient of } y \text{ on } x = r\sigma_y/\sigma_x \quad \dots(6)$$

and

$$\text{the regression coefficient of } x \text{ on } y = r\sigma_x/\sigma_y \quad \dots(7)$$

Cor. The correlation coefficient r is the geometric mean between the two regression coefficients.

$$\text{For } r \frac{\sigma_y}{\sigma_x} \times r \frac{\sigma_x}{\sigma_y} = r^2.$$

Example 25.15. The two regression equations of the variables x and y are $x = 19.13 - 0.87y$ and $y = 11.64 - 0.50x$. Find (i) mean of x 's, (ii) mean of y 's and (iii) the correlation coefficient between x and y .

(V.T.U., 2004; Anna, 2003; Burdwan, 2003)

Solution. Since the mean of x 's and the mean of y 's lie on the two regression lines, we have

$$\bar{x} = 19.13 - 0.87 \bar{y} \quad \dots(i)$$

$$\bar{y} = 11.64 - 0.50 \bar{x} \quad \dots(ii)$$

Multiplying (ii) by 0.87 and subtracting from (i), we have

$$[1 - (0.87)(0.50)] \bar{x} = 19.13 - (11.64)(0.87) \text{ or } 0.57 \bar{x} = 9.00 \text{ or } \bar{x} = 15.79$$

$$\therefore \bar{y} = 11.64 - (0.50)(15.79) = 3.74$$

∴ regression coefficient of y on x is -0.50 and that of x on y is -0.87 .

Now since the coefficient of correlation is the geometric mean between the two regression coefficients.

$$\therefore r = \sqrt{(-0.50)(-0.87)} = \sqrt{0.43} = -0.66.$$

[−ve sign is taken since both the regression coefficients are −ve]

Example 25.16. In the following table are recorded data showing the test scores made by salesmen on an intelligence test and their weekly sales :

Salesmen	1	2	3	4	5	6	7	8	9	10
Test scores	40	70	50	60	80	50	90	40	60	60
Sales (000)	2.5	6.0	4.5	5.0	4.5	2.0	5.5	3.0	4.5	3.0

Calculate the regression line of sales on test scores and estimate the most probable weekly sales volume if a salesman makes a score of 70.

Solution. With the help of the table below, we have

$$\bar{x} = \text{mean of } x \text{ (test scores)} = 60 + 0/10 = 60$$

$$\bar{y} = \text{mean of } y \text{ (sales)} = 4.5 + (-4.5)/10 = 4.05.$$

Regression line of sales (y) on scores (x) is given by

$$y - \bar{y} = r(\sigma_y / \sigma_x)(x - \bar{x})$$

where

$$\begin{aligned} r \frac{\sigma_y}{\sigma_x} &= \frac{\Sigma XY}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{(\sigma_x)^2} = \left[\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n} \right] / \left[\Sigma d_x^2 - (\Sigma d_x)^2 / n \right] \\ &= \frac{140 - \frac{0 \times (-4.5)}{10}}{2400 - 0^2 / 10} = \frac{140}{2400} = 0.06 \end{aligned}$$

∴ the required regression line is

$$y - 4.05 = 0.06(x - 60) \quad \text{or} \quad y = 0.06x + 0.45.$$

For $x = 70, y = 0.06 \times 70 + 0.45 = 4.65$.

Thus the most probable weekly sales volume for a score of 70 is 4.65.

Test scores	Sales	Deviation of x from assumed mean ($= 60$)	Deviation of y from assumed average ($= 4.5$)	$d_x \times d_y$	d_x^2	d_y^2
x	y	d_x	d_y			
40	2.5	-20	-2	40	400	4
70	6.0	10	1.5	15	100	2.25
50	4.5	-10	0	0	100	0
60	5.0	0	0.5	0	0	0.25
80	4.5	20	0	0	400	0
50	2.0	-10	-2.5	25	100	6.25
90	5.5	30	1	30	900	1.00
40	3.0	-20	-1.5	30	400	2.25
60	4.5	0	0	0	0	0
60	3.0	0	-1.5	0	0	2.25
		$\Sigma d_x = 0$	$\Sigma d_y = -4.5$	$\Sigma d_x d_y$ $= 140$	Σd_x^2 $= 2400$	Σd_y^2 $= 18.25$

Example 25.17. If θ is the angle between the two regression lines, show that

$$\tan \theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Explain the significance when $r = 0$ and $r = \pm 1$.

(U.P.T.U., 2007 ; V.T.U., 2007)

Solution. The equations to the line of regression of y on x and x on y are

$$y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \text{ and } x - \bar{x} = r \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

∴ their slopes are $m_1 = r \sigma_y / \sigma_x$ and $m_2 = \sigma_x / r \sigma_y$

$$\text{Thus } \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_x / r \sigma_y - r \sigma_y / \sigma_x}{1 + \sigma_x^2 / \sigma_y^2} = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\tan \theta \rightarrow \infty$ or $\theta = \pi/2$ i.e. when the variables are independent, the two lines of regression are perpendicular to each other.

When $r = \pm 1$, $\tan \theta = 0$ i.e., $\theta = 0$ or π . Thus the lines of regression coincide i.e., there is perfect correlation between the two variables.

Example 25.18. In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate \bar{x} , \bar{y} and the coefficient of correlation between x and y .
 (S.V.T.U., 2009; U.P.T.U., 2009; V.T.U., 2005)

Solution. Since the regression lines pass through (\bar{x}, \bar{y}) , therefore,

$$4\bar{x} - 5\bar{y} + 33 = 0, \quad 20\bar{x} - 9\bar{y} = 107.$$

Solving these equations, we get $\bar{x} = 13$, $\bar{y} = 17$.

Rewriting the line of regression of y on x as $y = \frac{4}{5}x + \frac{33}{5}$, we get

$$b_{yx} = r \frac{\sigma_y}{\sigma_x} = \frac{4}{5} \quad \dots(i)$$

Rewriting the line of regression of x on y as $x = \frac{9}{20}y + \frac{107}{9}$, we get

$$b_{xy} = r \frac{\sigma_x}{\sigma_y} = \frac{9}{20} \quad \dots(ii)$$

Multiplying (i) and (ii), we get

$$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36 \quad \therefore \quad r = 0.6$$

Hence $r = 0.6$, the positive sign being taken as b_{yx} and b_{xy} both are positive.

Example 25.19. Establish the formula $r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

Hence calculate r from the following data :

x :	21	23	30	54	57	58	72	78	87	90
y :	60	71	72	83	110	84	100	92	113	135

(U.P.T.U., 2002)

Solution. (a) Let $z = x - y$ so that $\bar{z} = \bar{x} - \bar{y}$.

$$\therefore z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

$$\text{or } (z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for n terms, we have

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

$$\text{or } \frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - 2 \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$$

i.e.,

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \quad \left[\because r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y} \right]$$

which is the required result.

(b) To find r , we have to calculate σ_x , σ_y and σ_{x-y} . We make the following table :

x	$X = x - 54$	X^2	y	$Y = y - 100$	Y^2	$y - x$	$(x - y)^2$
21	-33	1089	60	-40	1600	39	1521
23	-31	961	71	-29	841	48	2304
30	-24	576	72	-28	784	42	1764
54	0	0	83	-17	289	29	841
57	3	9	110	10	100	53	2809
58	4	16	84	-16	256	26	676
72	18	324	100	0	0	28	784
78	24	576	92	-8	64	14	196
87	33	1089	113	13	169	26	676
90	36	1296	135	35	1225	45	2025
Total	30	5936		-80	5328	350	13596

$$\therefore \sigma_x^2 = \frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N} \right)^2 = \frac{5636}{10} - \left(\frac{30}{10} \right)^2 = 593.6 - 9 = 584.6$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N} \right)^2 = \frac{5328}{10} - \left(\frac{-80}{10} \right)^2 = 532.8 - 64 = 468.8$$

$$\sigma_{x-y}^2 = \frac{\Sigma(x-y)^2}{N} - \left\{ \frac{\Sigma(x-y)}{N} \right\}^2 = 1359.6 - 1225 = 134.6$$

From the above formula,

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{584.6 + 468.8 - 134.6}{2 \times 24.18 \times 23.85} = 0.876.$$

Example 25.20. While calculating correlation coefficient between two variables x and y from 25 pairs of observations, the following results were obtained : $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.

Later it was discovered at the time of checking that the pairs of values $\begin{array}{|c|c|} \hline x & y \\ \hline 8 & 12 \\ \hline 6 & 8 \\ \hline \end{array}$ were copied down as $\begin{array}{|c|c|} \hline x & y \\ \hline 6 & 14 \\ \hline 8 & 6 \\ \hline \end{array}$.

Obtain the correct value of correlation coefficient.

(V.T.U., 2011 S ; S.V.T.U., 2009)

Solution. To get the correct results, we subtract the incorrect values and add the corresponding correct values.

\therefore The correct results would be

$$\Sigma n = 25, \Sigma x = 125 - 6 - 8 + 6 = 125, \Sigma x^2 = 650 - 6^2 - 8^2 + 6^2 = 650$$

$$\Sigma y = 100 - 14 - 6 + 12 + 8 = 100, \Sigma y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436$$

$$\Sigma xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[n\Sigma x^2 - (\Sigma x)^2][n\Sigma y^2 - (\Sigma y)^2]}} = \frac{25 \times 520 - 125 \times 100}{\sqrt{[25 \times 650 - (125)^2][25 \times 436 - (100)^2]}}$$

$$= \frac{20}{\sqrt{(25 \times 36)}} = \frac{2}{3}.$$

25.15 STANDARD ERROR OF ESTIMATE

The sum of the squares of the deviations of the points from the line of regression of y on x is

$$\Sigma(y - a - bx)^2 = \Sigma(Y - bX)^2, \text{ where } X = x - \bar{x}, Y = y - \bar{y}$$

$$\begin{aligned} &= \sum \left(Y - r \frac{\sigma_y}{\sigma_x} X \right)^2 = \Sigma Y^2 - 2r(\sigma_y/\sigma_x) \Sigma XY + r^2(\sigma_y^2/\sigma_x^2) \Sigma X^2 \\ &= n\sigma_y^2 - 2r(\sigma_y/\sigma_x) r \cdot n\sigma_x\sigma_y + r^2(\sigma_y^2/\sigma_x^2) \cdot n\sigma_x^2 = n\sigma_y^2(1 - r^2). \end{aligned}$$

Denoting this sum of squares by nS_y^2 , we have $S_y = \sigma_y \sqrt{1 - r^2}$... (1)

Since S_y is the root mean square deviation of the points from the regression line of y on x , it is called the standard error of estimate of y . Similarly the standard error of estimate of x is given by

$$S_x = \sigma_x \sqrt{1 - r^2} \quad \dots (2)$$

Since the sum of the squares of deviations cannot be negative, it follows that

$$r^2 \leq 1 \quad \text{or} \quad -1 \leq r \leq 1.$$

i.e., correlation coefficient lies between -1 and 1 .

(J.N.T.U., 2006)

If $r = 1$ or -1 , the sum of the squares of deviations from either line of regression is zero. Consequently each deviation is zero and all the points lie on both the lines of regression. These two lines coincide and we say that the correlation between the variables is perfect. The nearer r^2 is to unity the closer are the points to the lines of

regression. Thus the departure of r^2 from unity is a measure of departure from linearity of the relationship between the variables.

25.16 RANK CORRELATION

A group of n individuals may be arranged in order to merit with respect to some characteristic. The same group would give different orders for different characteristics. Considering the orders corresponding to two characteristics A and B , the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

Let x_i, y_i be the ranks of the i th individuals in A and B respectively. Assuming that no two individuals are bracketed equal in either case, each of the variables taking the values $1, 2, 3, \dots, n$, we have

$$\bar{x} = \bar{y} = \frac{1+2+3+\dots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

If X, Y be the deviations of x, y from their means, then

$$\begin{aligned}\Sigma X_i^2 &= \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 + n(\bar{x})^2 - 2\bar{x}\Sigma x_i = \Sigma n^2 + \frac{n(n+1)^2}{4} - 2\frac{n+1}{2} \cdot \Sigma n \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{n(n+1)^2}{2} = \frac{1}{12}(n^3 - n)\end{aligned}$$

$$\text{Similarly } \Sigma Y_i^2 = \frac{1}{12}(n^3 - n)$$

$$\begin{aligned}\text{Now let } d_i &= x_i - y_i \text{ so that } d_i = (x_i - \bar{x}) - (y_i - \bar{y}) = X_i - Y_i \\ \therefore \Sigma d_i^2 &= \Sigma X_i^2 + \Sigma Y_i^2 - 2\Sigma X_i Y_i\end{aligned}$$

$$\text{or } \Sigma X_i Y_i = \frac{1}{2}(\Sigma X_i^2 + \Sigma Y_i^2 - \Sigma d_i^2) = \frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2.$$

Hence the correlation coefficient between these variables is

$$r = \frac{\Sigma X_i Y_i}{\sqrt{(\Sigma X_i^2 \Sigma Y_i^2)}} = \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\Sigma d_i^2}{n^3 - n}$$

This is called the *rank correlation coefficient* and is denoted by ρ .

Example 25.21. Ten participants in a contest are ranked by two judges as follows :

$$\begin{array}{cccccccccc}x : & 1 & 6 & 5 & 10 & 3 & 2 & 4 & 9 & 7 & 8 \\y : & 6 & 4 & 9 & 8 & 1 & 2 & 3 & 10 & 5 & 7\end{array}$$

Calculate the rank correlation coefficient ρ . (V.T.U., 2002)

Solution. If $d_i = x_i - y_i$, then $d_i = -5, 2, -4, 2, 2, 0, 1, -1, 2, 1$
 $\therefore \Sigma d_i^2 = 25 + 4 + 16 + 4 + 4 + 0 + 1 + 1 + 4 + 1 = 60$

$$\text{Hence } \rho = 1 - \frac{6\Sigma d_i^2}{n^3 - n} = 1 - \frac{6 \times 60}{990} = 0.6 \text{ nearly.}$$

Example 25.22. Three judges, A, B, C, give the following ranks. Find which pair of judges has common approach

$$\begin{array}{cccccccccc}A : & 1 & 6 & 5 & 10 & 3 & 2 & 4 & 9 & 7 & 8 \\B : & 3 & 5 & 8 & 4 & 7 & 10 & 2 & 1 & 6 & 9 \\C : & 6 & 4 & 9 & 8 & 1 & 2 & 3 & 10 & 5 & 7\end{array}$$

(J.N.T.U., 2003)

Solution. Here $n = 10$.

<i>A</i> (= x)	<i>B</i> (= y)	<i>C</i> (= z)	d_1 $x - y$	d_2 $y - z$	d_3 $z - x$	d_1^2	d_2^2	d_3^2
1	3	6	-2	-3	5	4	9	25
6	5	4	1	1	-2	1	1	4
5	8	9	-3	-1	4	9	1	16
10	4	8	6	-4	-2	36	16	4
3	7	1	-4	6	-2	16	36	4
2	10	2	-8	8	0	64	64	0
4	2	3	2	-1	-1	4	1	1
9	1	10	8	-9	1	64	81	1
7	6	5	1	1	-2	1	1	4
8	9	7	-1	2	-1	1	4	1
Total			0	0	0	200	214	60

$$\therefore \rho(x, y) = 1 - \frac{6 \sum d_1^2}{n(n^2 - 1)} = 1 - \frac{6 \times 200}{10 \times 99} = -0.2$$

$$\rho(y, z) = 1 - \frac{6 \sum d_2^2}{n(n^2 - 1)} = 1 - \frac{6 \times 214}{10 \times 99} = -0.3$$

$$\rho(z, x) = 1 - \frac{6 \sum d_3^2}{n(n - 1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.6$$

Since $\rho(z, x)$ is maximum, the pair of judges A and C have the nearest common approach.

PROBLEMS 25.4

1. Find the correlation co-efficient and the regression lines of y and x and x on y for the following data :

x :	1	2	3	4	5					(V.T.U., 2010)
y :	2	5	3	8	7					

2. Find the correlation coefficient between x and y from the given data :

x :	78	89	97	69	59	79	68	57		
y :	125	137	156	112	107	138	123	108		(J.N.T.U., 2005)

3. Find the co-efficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons) :	55	56	58	59	60	60	62			
Exports (in crore tons) :	35	38	38	39	44	43	45			(Madras, 2000)

4. Ten people of various heights as under, were requested to read the letters on a car at 25 yards distance. The number of letters correctly read is given below :

Height (in feet) :	5.1	5.3	5.6	5.7	5.8	5.9	5.10	5.11	6.0	6.1
No. of letters :	11	17	19	14	8	15	20	6	8	12

Is there any correlation between heights and visual power ?

5. Using the formula $r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$, find r from the following data :

x :	92	89	87	86	83	77	71	63	53	50
y :	86	88	91	77	68	85	52	82	37	57

6. Find the correlation between x (marks in Mathematics) and y (marks in Engineering Drawing) given in the following data :

<i>x</i>	10—40	40—70	70—100	Total
<i>y</i>				
0—30	5	20	—	25
30—60	—	28	2	30
60—90	—	32	13	45
Total	5	80	15	100

7. Find two lines of regression and coefficient of correlation for the data given below :

$$n = 18, \Sigma x = 12, \Sigma y = 18, \Sigma x^2 = 60, \Sigma y^2 = 96, \Sigma xy = 48. \quad (\text{U.P.T.U., MCA, 2009})$$

8. If the coefficient of correlation between two variables *x* and *y* is 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2} \sigma_y$. (V.T.U., 2004)

9. For two random variables *x* and *y* with the same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-\alpha}{1-\alpha}$. Find also the common mean. (U.P.T.U., 2010)

10. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between *x* and *y*. (Madras, 2002)

11. The regression equations of two variables *x* and *y* are $x = 0.7y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them. (Osmania, 2002)

12. In a partially destroyed laboratory data, only the equations giving the two lines of regression of *y* on *x* and *x* on *y* are available and are respectively, $7x - 16y + 9 = 0$, $5y - 4x - 3 = 0$. Calculate the co-efficient of correlation, \bar{x} and \bar{y} .

13. The following results were obtained from records of age (*x*) and blood pressure (*y*) of a group of 10 men :

$$\left. \begin{array}{cc} x & y \\ \text{Mean} & 53 & 142 \\ \text{Variance} & 130 & 165 \end{array} \right\} \text{ and } \Sigma(x - \bar{x})(y - \bar{y}) = 1220.$$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.

14. Compute the standard error of estimate S_x for the respective heights of the following 12 couples :

Height *x* of husband (inches) : 68 66 68 65 69 66 68 65 71 67 68 70

Height *y* of wife (inches) : 65 63 67 64 68 62 70 66 68 67 69 71

15. Calculate the rank correlation coefficient from the following data showing ranks of 10 students in two subjects :

Maths : 3 8 9 2 7 10 4 6 1 5

Physics : 5 9 10 1 8 7 3 4 2 6

16. Find the rank correlation for the following data :

x : 56 42 72 36 63 47 55 49 38 42 68 60

y : 147 125 160 118 149 128 150 145 115 140 152 155

(S.V.T.U., 2009 ; J.N.T.U., 2003)

25.17 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 25.5

Select the correct answer or fill up the blanks in each of the following questions :

- The median of the numbers 11, 10, 12, 13, 9 is
 (a) 12.5 (b) 12 (c) 10.5 (d) 11.
- The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is
 (a) 11 (b) 12 (c) 7 (d) 7 and 11.

3. S.D. is defined as

(a) $\sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}}$

(b) $\frac{\sum f(x - \bar{x})}{\sum f}$

(c) $\frac{\sum f(x - \bar{x})^2}{\sum f}$.

4. Coefficient of variation is

(a) $\frac{\sigma}{\bar{x}} \times 100$

(b) $\frac{\sigma}{x}$

(c) $\sqrt{\frac{\sigma^2}{x}} \times 100$.

5. Average scores of three batsman A, B, C are respectively 40, 45 and 55 and their S.D.s are respectively 9, 11, 16. Which batsman is more consistent?

(a) A

(b) B

(c) C.

6. The equations of regression lines are $y = 0.5x + a$ and $x = 0.4y + b$. The correlation coefficient is

(a) $\sqrt{0.2}$

(b) 0.45

(c) $-\sqrt{0.2}$.

7. If the correlation coefficient is 0, the two regression lines are

(a) parallel

(b) perpendicular

(c) coincident

(d) inclined at 45° to each other.

8. If r_1 and r_2 are two regression coefficients, then signs of r_1 and r_2 depend on

9. Regression coefficient of y on x is 0.7 and that of x on y is 3.2. Is the correlation coefficient r consistent?

10. The standard deviation of the numbers 24, 48, 64, 36, 53 is

11. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\bar{x} =$, $\bar{y} =$ and $r =$

12. If the two regression lines are perpendicular to each other, then their coefficient of correlation is

13. Quartile deviation is defined as

14. The minimum value of correlation coefficient is

15. Prediction error of Y is defined as

16. If X and Y are independent, then the correlation coefficient between X and Y is

17. The point of intersection of the two regression lines is

18. The smaller the coefficient of variation, the greater is the in the data.

19. The moment coefficient of skewness is given by

20. Kurtosis measures the of a distribution.

21. The equation of the line of regression of y on x is

22. Coefficient of variation =

23. The angle between two regression lines is given by

24. A frequency curve is said to be Mesokurtic when β_2 is

25. Correlation coefficient is the geometrical mean between

26. When the variables are independent, the two lines of regression are

27. Arithmetic mean of the coefficients of regression is than the coefficient of correlation.

28. If two regression lines coincide then the coefficient of correlation is

29. The rank coefficient is given by

30. The ratio of the standard deviation to the mean is known as

31. The value of $\sum f(x - \bar{x}) =$

32. The value of coefficient of correlation lies between and

33. If the two regression coefficients are -0.4 and -0.9 , then the correlation coefficient is

34. A distribution with the following constants is positively skew : $Q_1 = 25.8$, median = 49.0, $Q_3 = 64.2$.

(True or False)

35. Quartile coefficient of skewness is $\frac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$.

(True or False)

36. Skewness indicates peakedness of the frequency distribution.

(True or False)

Probability and Distributions

1. Introduction, Principle of counting, Permutations and Combinations. 2. Basic terminology, Definition of probability. 3. Probability and Set notations. 4. Addition law of probability. 5. Independent events — Multiplication law of probability. 6. Baye's theorem. 7. Random variable. 8. Discrete probability distribution. 9. Continuous probability distribution. 10. Expectation, Variance, Moments. 11. Moment generating function. 12. Probability generating function. 13. Repeated trials. 14. Binomial distribution. 15. Poisson distribution. 16. Normal distribution. 17. Probable error. 18. Normal approximation to Binomial distribution. 19. Some other distributions. 20. Objective Type of Questions.

26.1 (1) INTRODUCTION

We often hear such statements : 'It is likely to rain today', 'I have a fair chance of getting admission', and 'There is an even chance that in tossing a coin the head may come up'. In each case, we are not certain of the outcome, but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions and is essential in every decision making process. Before defining probability, let us explain a few terms :

(2) Principle of counting. If an event can happen in n_1 ways and thereafter for each of these events a second event can happen in n_2 ways, and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \dots n_m$.

(3) Permutations. A permutation of a number of objects is their arrangement in some definite order. Given three letters a, b, c , we can permute them two at a time as "bc, cb ; ca, ac; ab, ba" yielding 6 permutations. The combinations or groupings are only 3, i.e., bc, ca, ab. Here the order is immaterial.

The number of permutations of n different things taken r at a time is

$$n(n-1)(n-2)\dots(n-r+1), \text{ which is denoted by } {}^nP_r$$

Thus
$${}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with repetitions. The number of permutations of n objects of which n_1 are alike, n_2 are alike and n_3 are alike is $\frac{n!}{n_1! n_2! n_3!}$.

(4) Combinations. The number of combinations of n different objects taken r at a time is denoted by nC_r . If we take any one of the combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the combinations is ${}^nP_r = {}^nC_r \cdot r!$.

Thus
$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Also
$${}^nC_{n-r} = {}^nC_r$$

e.g.,
$${}^{25}P_4 = 25 \times 24 \times 23 \times 22; {}^{25}C_{21} = {}^{25}C_4 = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}.$$

Example 26.1. In how many ways can one make a first, second, third and fourth choice among 12 firms leasing construction equipment. (J.N.T.U., 2003)

Solution. First choice can be made from any of the 12 firms. Thereafter the second choice can be made from among the remaining 11 firms. Then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the principle of counting, the number of ways in which first, second, third and fourth choice can be affected = $12 \times 11 \times 10 \times 9 = 11880$.

Example 26.2. Find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.

$$\text{Solution. (i)} \quad n = 9, n_1(m, m) = 2, n_2(t, t) = 2, n_3(e, e) = 2$$

$$\therefore \text{no. of permutations} = \frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{9!}{2! \cdot 2! \cdot 2!} = 45360.$$

$$\text{(ii)} \quad n = 11, n_1(e's) = 3, n_2(g, g) = 2, n_3(i, i) = 2, n_4(n's) = 3$$

$$\therefore \text{no. of permutations} = \frac{11!}{3! 2! 2! 3!} = 277200.$$

Example 26.3. From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction. (ii) two particular engineers must be included. (iii) one particular architect must be excluded.

$$\text{Solution. (i) Number of committees } {}^6C_3 \times {}^5C_2 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \times \frac{5 \cdot 4}{2 \cdot 1} = 200.$$

(ii) Here we have to choose one engineer from the remaining four engineers.

$$\therefore \text{no. of committees} = {}^4C_1 \times {}^5C_2 = 4 \times \frac{5 \cdot 4}{2 \cdot 1} = 40$$

(iii) Here we have to choose two architects from the remaining four architects.

$$\therefore \text{no. of committees} = {}^6C_3 \times {}^4C_2 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \times \frac{4 \cdot 3}{2 \cdot 1} = 120.$$

PROBLEMS 26.1

- If a test consists of 12 true-false questions, in how many different ways can a student make the test paper with one answer to each question. (J.N.T.U., 2003)
- How many 4-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9, without repetition? How many of these are less than 500?
- A student has to answer 9 out of 12 questions. How many choices has he (i) if he must answer first two questions (ii) if he must answer at least four of the first five questions.
- How many car number plates can be made if each plate contains two different letters followed by three different digits? Solve the problem (a) with repetitions and (b) without repetitions.

26.2 (I) BASIC TERMINOLOGY

(i) **Exhaustive events.** A set of events is said to be *exhaustive*, if it includes all the possible events. For example, in tossing a coin there are two exhaustive cases either head or tail and there is no third possibility.

(ii) **Mutually exclusive events.** If the occurrence of one of the events precludes the occurrence of all other, then such a set of events is said to be *mutually exclusive*. Just as tossing a coin, either head comes up or the tail and both can't happen at the same time, i.e., these are two mutually exclusive cases.

(iii) **Equally likely events.** If one of the events cannot be expected to happen in preference to another then such events are said to be *equally likely*. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Thus when a die* is thrown, the turning up of the six different faces of the die are exhaustive, mutually exclusive and equally likely.

(iv) **Odds in favour of an event.** If the number of ways favourable to an event A is m and the number of ways not favourable to A is n then $\text{odds in favour of } A = m/n$ and $\text{odds against } A = n/m$.

(2) **Definition of probability.** If there are n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event A , then probability (p) of the happening of A is

$$P(A) = \frac{m}{n}.$$

As there are $n - m$ cases in which A will not happen (denoted by A'), the chance of A not happening is q or $P(A')$ so that

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

i.e., $P(A') = 1 - P(A)$ so that $P(A) + P(A') = 1$,

i.e., if an event is certain to happen then its probability is unity, while if it is certain not to happen, its probability is zero.

Obs. This definition of probability fails when

(i) number of outcomes is infinite (not exhaustive) and (ii) outcomes are not equally likely.

(3) **Statistical (or Empirical) definition of probability.** If in n trials, an event A happens m times, then the probability (p) of happening of A is given by

$$p = P(A) = \lim_{n \rightarrow \infty} \frac{m}{n}$$

Example 26.4. Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.

Solution. (a) There are six possible ways in which the die can fall and of these there is only one way of throwing 4. Thus the required chance = $\frac{1}{6}$.

(b) There are six possible ways in which the die can fall. Of these there are only 3 ways of getting 2, 4 or 6. Thus the required chance = $3/6 = \frac{1}{2}$.

Example 26.5. What is the chance that a leap year selected at random will contain 53 Sundays?

(Madras, 2003)

Solution. A leap year consists of 366 days, so that there are 52 full weeks (and hence 52 Sundays) and two extra days. These two days can be (i) Monday, Tuesday (ii) Tuesday, Wednesday, (iii) Wednesday, Thursday (iv) Thursday, Friday (v) Friday, Saturday (vi) Saturday, Sunday (vii) Sunday, Monday.

Of these 7 cases, the last two are favourable and hence the required probability = $\frac{2}{7}$.

Example 26.6. A five figure number is formed by the digits 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.

Solution. The five digits can be arranged in $5!$ ways, out of which $4!$ will begin with zero.

∴ total number of 5-figure numbers formed = $5! - 4! = 96$.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, i.e., numbers ending in 04, 12, 20, 24, 32, 40.

Now numbers ending in 04 = $3! = 6$, numbers ending in 12 = $3! - 2! = 4$,

numbers ending in 20 = $3! = 6$, numbers ending in 24 = $3! - 2! = 4$,

numbers ending in 32 = $3! - 2! = 4$, and numbers ending in 40 = $3! = 6$.

[The numbers having 12, 24, 32 in the extreme right are $(3! - 2!)$ since the numbers having zero on the extreme left are to be excluded.]

* Die is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. The outcome of throwing a die is the number of dots on its upper face.

∴ total number of favourable ways = $6 + 4 + 6 + 4 + 4 + 6 = 30$.

$$\text{Hence the required probability} = \frac{30}{96} = \frac{5}{16}.$$

Example 26.7. A bag contains 40 tickets numbered 1, 2, 3, ... 40, of which four are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Find the probability of t_3 being 25?

Solution. Here exhaustive number of cases = ${}^{40}C_4$

If $t_3 = 25$, then the tickets t_1 and t_2 must come out of 24 tickets numbered 1 to 24. This can be done in ${}^{24}C_2$ ways.

Then t_4 must come out of the 15 tickets (numbering 25 to 40) which can be done in ${}^{15}C_1$ ways.

$$\therefore \text{favourable number of cases} = {}^{24}C_2 \times {}^{15}C_1$$

$$\text{Hence the probability of } t_3 \text{ being } 25 = \frac{{}^{24}C_2 \times {}^{15}C_1}{{}^{40}C_4} = \frac{414}{9139}.$$

Example 26.8. An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls?

Solution. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.

The number of ways of drawing 2 red balls is 5C_2 and corresponding to each of these 5C_2 ways of drawing a red ball, there are ${}^{10}C_6$ ways of drawing 6 black balls.

∴ the total number of ways in which 2 red and 6 black balls can be drawn is ${}^5C_2 \times {}^{10}C_6$.

$$\therefore \text{the required probability} = \frac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8} = \frac{140}{429}.$$

Example 26.9. A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, (iii) the three belong to the same class? (V.T.U., 2002 S)

Solution. (i) The total number of ways of choosing 3 students out of 9 is 9C_3 , i.e., 84.

A student can be removed from 1st year students in 2 ways, from 2nd year in 3 ways and from 3rd year in 4 ways, so that the total number of ways of removing three students, one from each group is $2 \times 3 \times 4$.

$$\text{Hence the required chance} = \frac{2 \times 3 \times 4}{{}^9C_3} = \frac{24}{84} = \frac{2}{7}.$$

(ii) The number of ways of removing two from 1st year students and one from others
 $= {}^2C_2 \times {}^7C_1$.

The number of ways of removing two from 2nd year students and one from others
 $= {}^3C_2 \times {}^6C_1$.

The number of ways of removing 2 from 3rd year students and one from others
 $= {}^4C_2 \times {}^5C_1$.

∴ the total number of ways in which two students of the same class and third from the others may be removed

$$= {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1 = 7 + 18 + 30 = 55.$$

$$\text{Hence, the required chance} = \frac{55}{84}.$$

(iii) Three students can be removed from 2nd year group in 3C_3 , i.e. 1 way and from 3rd year group in 4C_3 , i.e., 4 ways.

∴ the total number of ways in which three students belong to the same class = $1 + 4 = 5$.

$$\text{Hence the required chance} = \frac{5}{84}.$$

Example 26.10. A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks ; compare the probability of A's success to that of B's success.

Solution. A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which A can get a prize is clearly 1.

$$\therefore \text{the probability of } A\text{'s success} = \frac{1}{3}.$$

Again B can draw a ticket in ${}^9C_3 = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

$$\text{The number of ways in which } B \text{ gets all blanks} = {}^6C_3 = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

$$\therefore \text{the number of ways of getting a prize} = 84 - 20 = 64.$$

$$\text{Thus the probability of } B\text{'s success} = 64/84 = 16/21.$$

$$\text{Hence } A\text{'s probability of success : } B\text{'s probability of success} = \frac{1}{3} : \frac{16}{21} = 7 : 16.$$

26.3 PROBABILITY AND SET NOTATIONS

(1) Random experiment. Experiments which are performed essentially under the same conditions and whose results cannot be predicted are known as *random experiments*. e.g., Tossing a coin or rolling a die are random experiments.

(2) Sample space. The set of all possible outcomes of a random experiment is called *sample space* for that experiment and is denoted by S .

The elements of the sample space S are called the *sample points*.

e.g., On tossing a coin, the possible outcomes are the head (H) and the tail (T). Thus $S = \{H, T\}$.

(3) Event. The outcome of a random experiment is called an *event*. Thus every subset of a sample space S is an *event*.

The null set ϕ is also an event and is called an *impossible event*. Probability of an impossible event is zero i.e., $P(\phi) = 0$.

(4) Axioms

(i) The numerical value of probability lies between 0 and 1.

i.e., for any event A of S , $0 \leq P(A) \leq 1$.

(ii) The sum of probabilities of all sample events is unity i.e., $P(S) = 1$.

(iii) Probability of an event made of two or more sample events is the sum of their probabilities.

(5) Notations

(i) Probability of happening of events **A or B** is written as $P(A + B)$ or $P(A \cup B)$.

(ii) Probability of happening of **both the events A and B** is written as $P(AB)$ or $P(A \cap B)$.

(iii) 'Event A implies (\Rightarrow) event B' is expressed as $A \subset B$.

(iv) 'Events A and B are mutually exclusive' is expressed as $A \cap B = \phi$.

(6) For any two events A and B,

$$P(A \cap B') = P(A) - P(A \cap B)$$

Proof. From Fig. 26.1,

$$(A \cap B') \cup (A \cap B) = A$$

$$\therefore P[(A \cap B') \cup (A \cap B)] = P(A)$$

or

$$P(A \cap B') + P(A \cap B) = P(A)$$

or

$$P(A \cap B') = P(A) - P(A \cap B)$$

Similarly, $P(A' \cap B) = P(B) - P(A \cap B)$

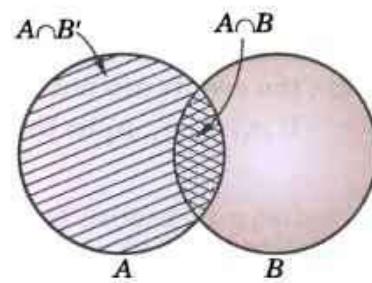


Fig. 26.1

26.4 ADDITION LAW OF PROBABILITY or THEOREM OF TOTAL PROBABILITY

(1) If the probability of an event A happening as a result of a trial is $P(A)$ and the probability of a **mutually exclusive** event B happening is $P(B)$, then the probability of either of the events happening as a result of the trial is $P(A + B)$ or $P(A \cup B) = P(A) + P(B)$.

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B . Then the number of cases favourable to A or B is $m_1 + m_2$. Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

(2) If A, B , are any two events (**not mutually exclusive**), then

$$P(A + B) = P(A) + P(B) - P(AB)$$

or

If the events A and B are any two events then, there are some outcomes which favour both A and B . If m_3 be their number, then these are included in both m_1 and m_2 . Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3.$$

Thus the probability of occurrence of A or B or both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

Hence

$$P(A + B) = P(A) + P(B) - P(AB)$$

or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Obs. When A and B are **mutually exclusive** $P(AB)$ or $P(A \cap B) = 0$ and we get

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B).$$

In general, for a number of **mutually exclusive** events A_1, A_2, \dots, A_n , we have

$$P(A_1 + A_2 + \dots + A_n) \text{ or } P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

(3) If A, B, C are any three events, then

$$P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).$$

or

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Proof. Using the above result for any two events, we have

$$\begin{aligned} P(A \cup B \cup C) &= P[(A \cup B) \cup C] \\ &= P(A \cup B) + P(C) - P[(A \cup B) \cap C] \\ &= [P(A) + P(B) - P(A \cap B)] + P(C) - P[(A \cap C) \cup (B \cap C)] && \text{(Distributive Law)} \\ &= P(A) + P(B) + P(C) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)] \\ &\quad [\because (A \cap C) \cap (B \cap C) = A \cap B \cap C] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) [\because A \cap C = C \cap A.] \end{aligned}$$

Example 26.11. In a race, the odds in favour of the four horses H_1, H_2, H_3, H_4 are $1 : 4, 1 : 5, 1 : 6, 1 : 7$ respectively. Assuming that a dead heat is not possible, find the chance that one of them wins the race.

Solution. Since it is not possible for all the horses to cover the same distance in the same time (a dead heat), the events are **mutually exclusive**.

If p_1, p_2, p_3, p_4 be the probabilities of winning of the horses H_1, H_2, H_3, H_4 respectively, then

$$p_1 = \frac{1}{1+4} = \frac{1}{5} \quad [\because \text{Odds in favour of } H_1 \text{ are } 1 : 4]$$

and

$$p_2 = \frac{1}{6}, p_3 = \frac{1}{7}, p_4 = \frac{1}{8}.$$

Hence the chance that one of them wins = $p_1 + p_2 + p_3 + p_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}.$$

Example 26.12. A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.

Solution. Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in 8C_2 ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^8C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly 2 red balls out of 6 can be drawn in 6C_2 ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^6C_2}{{}^{14}C_2} = \frac{15}{91}.$$

Hence the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 26.13. Find the probability of drawing an ace or a spade or both from a deck of cards*?

Solution. The probability of drawing an ace from a deck of 52 cards = $4/52$.

Similarly the probability of drawing a card of spades = $13/52$, and the probability of drawing an ace of spades = $1/52$.

Since the two events (*i.e.*, a card being an ace and a card being of spades) are not mutually exclusive, therefore, the probability of drawing an ace or a spade

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}.$$

26.5 (1) INDEPENDENT EVENTS

Two events are said to be *independent*, if happening or failure of one does not affect the happening or failure of the other. Otherwise the events are said to be *dependent*.

For two dependent events *A* and *B*, the symbol $P(B/A)$ denotes the probability of occurrence of *B*, when *A* has already occurred. It is known as the **conditional probability** and is read as a 'probability of *B* given *A*'.

(2) **Multiplication law of probability or Theorem of compound probability.** If the probability of an event *A* happening as a result of trial is $P(A)$ and after *A* has happened the probability of an event *B* happening as a result of another trial (*i.e.*, **conditional probability of *B* given *A***) is $P(B/A)$, then the probability of **both** the events *A* and *B* happening as a result of two trials is $P(AB)$ or $P(A \cap B) = P(A) \cdot P(B/A)$.

Proof. Let *n* be the total number of outcomes in the first trial and *m* be favourable to the event *A* so that $P(A) = m/n$.

Let n_1 be the total number of outcomes in the second trial of which m_1 are favourable to the event *B* so that $P(B/A) = m_1/n_1$.

Now each of the *n* outcomes can be associated with each of the n_1 outcomes. So the total number of outcomes in the combined trial is nn_1 . Of these mm_1 are favourable to both the events *A* and *B*. Hence

$$P(AB) \text{ or } P(A \cap B) = \frac{mm_1}{nn_1} = P(A) \cdot P(B/A).$$

Similarly, the **conditional probability of *A* given *B*** is $P(A/B)$.

$$\therefore P(AB) \text{ or } P(A \cap B) = P(B) \cdot P(A/B)$$

$$\text{Thus } P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B).$$

(3) If the events *A* and *B* are **independent**, *i.e.*, if the happening of *B* does not depend on whether *A* has happened or not, then $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$$\therefore P(AB) \text{ or } P(A \cap B) = P(A) \cdot P(B).$$

$$\text{In general, } P(A_1 A_2 \dots A_n) \text{ or } P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \dots \cdot P(A_n).$$

* Cards : A pack of cards consists of four suits *i.e.*, Hearts, Diamonds, Spades and Clubs. Each suit has 13 cards : an Ace, a King, a Queen, a Jack and nine cards numbered 2, 3, 4, ..., 10. Hearts and Diamonds are *red* while Spades and Clubs are *black*.

Cor. If p_1, p_2 be the probabilities of happening of two independent events, then

- the probability that the first event happens and the second fails is $p_1(1 - p_2)$.
- the probability that both events fail to happen is $(1 - p_1)(1 - p_2)$.
- the probability that at least one of the events happens is

$1 - (1 - p_1)(1 - p_2)$. This is commonly known as their **cumulative probability**.

In general, if $p_1, p_2, p_3, \dots, p_n$ be the chances of happening of n independent events, then their cumulative probability (i.e., the chance that at least one of the events will happen) is

$$1 - (1 - p_1)(1 - p_2)(1 - p_3) \dots (1 - p_n).$$

Example 26.14. Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) replaced, (ii) not replaced.

Solution. (i) The probability of drawing a king = $\frac{4}{52} = \frac{1}{13}$.

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen is $1/13$.

The two events being independent, the probability of drawing both cards in succession = $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

(ii) The probability of drawing a king = $\frac{1}{13}$.

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is $4/51$.

Hence the probability of drawing both cards = $\frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$.

Example 26.15. A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once, (b) at least once (c) twice.
(Kurukshestra, 2009 S.; V.T.U., 2004)

Solution. In a single toss of two dice, the sum 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) i.e., in 6 ways, so that the probability of getting 7 = $6/36 = 1/6$.

Also the probability of not getting 7 = $1 - 1/6 = 5/6$.

(a) The probability of getting 7 in the first toss and not getting 7 in the second toss = $1/6 \times 5/6 = 5/36$.

Similarly, the probability of not getting 7 in the first toss and getting 7 in the second toss = $5/6 \times 1/6 = 5/36$.

Since these are mutually exclusive events, addition law of probability applies.

$$\therefore \text{ required probability} = \frac{5}{36} + \frac{5}{36} = \frac{5}{18}.$$

$$(b) \text{ The probability of not getting 7 in either toss} = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

$$\therefore \text{ the probability of getting 7 at least once} = 1 - \frac{25}{36} = \frac{11}{36}.$$

$$(c) \text{ The probability of getting 7 twice} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

Example 26.16. There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately

Solution. Prob. of turning up 3 or 5 = $\frac{2}{6} = \frac{1}{3}$.

Prob. of selecting an engg. subject from first group = $\frac{3}{8}$

\therefore Prob of selecting an engg. subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8} \quad \dots(i)$$

Now prob. of not turning 3 or 5 = $1 - \frac{1}{3} = \frac{2}{3}$.

Prob. of selecting an engg. subject from second group = $\frac{5}{8}$

\therefore prob. of selecting an engg. subject from second group on turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \quad \dots(ii)$$

Thus the prob. of selecting an engg. subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}. \quad \text{[From (i) and (ii)]}$$

Example 26.17. A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white. (V.T.U., 2004)

Solution. The probability of drawing a white ball from box B will depend on whether the transferred ball is black or white.

If a black ball is transferred, its probability is $4/6$. There are now 5 white and 8 black balls in the box B. Then the probability of drawing white ball from box B is $\frac{5}{13}$.

Thus the probability of drawing a white ball from urn B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly the probability of drawing a white ball from urn B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

$$\text{Hence required probability} = \frac{10}{39} + \frac{2}{13} = \frac{16}{39}.$$

Example 26.18. (a) A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd. (Mumbai, 2006)

(b) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. (Madras, 2000 S)

Solution. (a) Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

$$\begin{aligned} &= \text{Probability of getting head in the 1st toss} \\ &\quad + \text{Probability of getting head in the 3rd toss} \\ &\quad + \text{Probability of getting head in the 5th toss} + \dots \infty \\ &= p + qp + qqqp + \dots \infty \\ &= p(1 + q^2 + q^4 + \dots) = p \cdot \frac{1}{1 - q^2} \quad (q < 1) \\ &= p \cdot \frac{1}{(1 - q)(1 + q)} = p \cdot \frac{1}{p(1 + q)} = \frac{1}{1 + q}. \end{aligned}$$

(b) Probability of getting a head = $1/2$. Then A can win in 1st, 3rd, 5th, ... throws.

$$\begin{aligned} \therefore \text{the chances of A's winning} &= \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \left(\frac{1}{2}\right)^6 \frac{1}{2} + \dots \\ &= \frac{1/2}{1 - (1/2)^2} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}. \end{aligned}$$

Hence the chance of B's winning = $1 - 2/3 = 1/3$.

Example 26.19. Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if

- the two cards are drawn together.
- the two cards are drawn one after the other without replacement.
- the two cards are drawn one after the other with replacement.

(J.N.T.U., 2003)

Solution. (i) Two cards out of 10 can be selected in ${}^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being 5 odd numbers (1, 3, 5, 7, 9) and 5 even numbers (2, 4, 6, 8, 10), an odd and an even number is chosen in $5 \times 5 = 25$ ways.

Thus
$$p = \frac{25}{45} = \frac{5}{9}.$$

(ii) Two cards out of 10 can be selected one after the other *without replacement* in $10 \times 9 = 90$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways

Thus
$$p = \frac{25 + 25}{90} = \frac{5}{9}.$$

(iii) Two cards can be selected one after the other *with replacement* in $10 \times 10 = 100$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways.

Thus
$$p = \frac{25 + 25}{100} = \frac{1}{2}.$$

Example 26.20. Given $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, evaluate $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$.

Solution. (i) Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B) \text{ or } P(A \cap B) = \frac{1}{12}$$

Thus
$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}.$$

(ii)
$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}.$$

(iii)
$$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

(iv)
$$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}.$$

Example 26.21. The odds that a book will be reviewed favourably by three independent critics are 5 to

Finally, prob. that all the three are favourable = $\frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$

Since they are mutually exclusive events, the required prob.

$$= \frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}.$$

Example 26.22. I can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) atleast two shots hit?

(A.M.I.E.T.E., 2003 ; Madras, 2000 S)

Solution. Prob. of A hitting the target = 3/5, prob. of B hitting the target = 2/5

Prob. of C hitting the target = 3/4.

(i) In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45.$$

(ii) In order that at least two shots may hit the target, we must also consider the case of all A, B, C hitting the target [in addition to the three cases of (i)] for which

$$p_4 = \text{chance that } A, B, C \text{ all hit} = \frac{3}{5} \times \frac{2}{3} \times \frac{3}{4} = \frac{18}{100}$$

Since all these are mutually exclusive events, the probability of atleast two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.63.$$

Example 26.23. A problem in mechanics is given to three students A, B, and C whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved. (V.T.U., 2004)

Solution. The probability that A can solve the problem is 1/2.

The probability that A cannot solve the problem is $1 - \frac{1}{2}$.

Similarly the probabilities that B and C cannot solve the problem are $1 - \frac{1}{3}$ and $1 - \frac{1}{4}$.

∴ the probability that A, B and C cannot solve the problem is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$.

Hence the probability that the problem will be solved, i.e., at least one student will solve it

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

Example 26.24. The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if

- (i) the class consists of 4 boys and 3 girls.
- (ii) the class consists of 3 boys and 3 girls.

(J.N.T.U., 2003)

Solution. (i) As there are 7 students in the class, the first examined must be a boy.

\therefore prob. that first is a boy = $\frac{4}{7}$

Then the prob. that the second is a girl = $\frac{3}{6}$.

\therefore prob. of the next boy = $\frac{3}{5}$

Similarly the prob. that the fourth is a girl = $\frac{2}{4}$,

the prob. that the fifth is a boy = $\frac{2}{3}$,

the prob. that the sixth is a girl = $\frac{1}{2}$

and the last is a boy = $\frac{1}{1}$.

Thus

$$p = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{35}.$$

(ii) The first student is a boy and the first student is a girl are two mutually exclusive cases. If the first student is a boy, then the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{20}.$$

If the first student is a girl, then the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{20}.$$

Thus the required prob. $p = p_1 + p_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$.

Example 26.25. (Huyghen's problem) A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.

(Madras, 2006; J.N.T.U., 2003)

Solution. The sum 6 can be obtained as follows : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), i.e., in 5 ways.

The probability of A's throwing 6 with 2 dice is $\frac{5}{36}$.

\therefore the probability of A's not throwing 6 is $31/36$.

Similarly the probability of B's throwing 7 is $6/36$, i.e., $\frac{1}{6}$.

\therefore the probability of B's not throwing 7 is $5/6$.

Now A can win if he throws 6 in the first, third, fifth, seventh etc. throws.

\therefore the chance of A's winning

$$\begin{aligned} &= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \dots \\ &= \frac{5}{36} \left[1 + \left(\frac{31}{36} \times \frac{5}{6} \right) + \left(\frac{31}{36} \times \frac{5}{6} \right)^2 + \left(\frac{31}{36} \times \frac{5}{6} \right)^3 + \dots \right] \\ &= \frac{5}{36} \cdot \frac{1}{1 - (31/36) \times (5/6)} = \frac{5}{36} \times \frac{36 \times 6}{61} = \frac{30}{61}. \end{aligned}$$

PROBLEMS 26.2

1. (i) Given $P(A) = 1/2$, $P(B) = 1/3$ and $P(AB) = 1/4$, find the value $P(A+B)$.

(Burdwan, 2003)

- (ii) Let A and B be two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A|B)$, $P(A \cup B)$, $P(A'|B')$.

(Kurukshetra, 2009; V.T.U., 2003 S)

2. In a single throw with two dice, what is the chance of throwing
(a) two aces ? (b) 7 ? Is this probability the same as that for getting 7 in two throws of a single die ?
3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.
4. Find the probability that a non-leap year should have 53 Saturdays? (Madras, 2003)
5. When a coin is tossed four times, find the probability of getting (i) exactly one head, (ii) at most three heads and (iii) at least two heads ? (V.T.U., 2000 S)
6. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads. (P.T.U., 2003)
7. If all the letters of word 'ENGINEER' be written at random, what is the probability that all the letters E are found together.
8. A ten digit number is formed using the digits from zero to nine, every digit being used only once. Find the probability that the number is divisible by 4.
9. Four cards are drawn from a pack of 52 cards. What is the chance that
(i) no two cards are of equal value ? (ii) each belongs to a different suit ?
10. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red what is the probability that all of them are hearts? (Mumbai, 2005)
11. Out of 50 rare books, 3 of which are especially valuable, 5 are stolen at random by a thief. What is the probability that
(a) none of the 3 is included ? (b) 2 of the 3 are included ?
12. Five men in a company of twenty are graduates. If 3 men are picked out of 20 at random, what is the probability that
(a) they are all graduates ? (b) at least one is graduate ?
13. From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probability that it is marked with a multiple of 3 or 5.
14. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn ?
15. The probability of n independent events are $p_1, p_2, p_3, \dots, p_n$. Find the probability that at least one of the events will happen. Use this result to find the chance of getting at least one six in a throw of 4 dice.
16. Find the probability of drawing 4 white balls and 2 black balls without replacement from a bag containing 1 red, 4 black and 6 white balls.
17. A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that one of them is black and the other white ?
18. A purse contains 2 silver and 4 copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin ? (Osmania, 2002)
19. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random and then a ball is drawn from it : (i) what is the probability that the ball drawn will be white ? (ii) Given that the ball drawn is white, what is the probability that it came from box I. (Mumbai, 2006)
20. A party of n persons take their seats at random at a round table ; find the probability that two specified persons do not sit together.
21. A speaks the truth in 75% cases, and B in 80% of the cases. In what percentage of cases, are they likely to contradict each other in stating the same fact ? (V.T.U., 2002 S)
22. The probability that Sushil will solve a problem is $1/4$ and the probability that Ram will solve it is $2/3$. If Sushil and Ram work independently, what is the probability that the problem will be solved by (a) both of them, (b) at least one of them ?
23. A student takes his examination in four subjects, P, Q, R, S. He estimates his chances of passing in P as $4/5$, in Q as $3/4$, in R as $5/6$ and in S as $2/3$. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies ? (Madras, 2000 S)
24. The probability that a 50 year old man will be alive at 60 is 0.83 and the probability that a 45 year old women will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence ?
25. If on an average one birth in 80 is a case of twins, what is the probability that there will be at least one case of twins in a maternity hospital on a day when 20 births occur ?
26. Two persons A and B fire at a target independently and have a probability 0.6 and 0.7 respectively of hitting the target. Find the probability that the target is destroyed.
27. A and B throw alternately with a pair of dice. The one who throws 9 first wins. Show that the chances of their winning are 9 : 8.

26.6 BAYE'S THEOREM

An event A corresponds to a number of exhaustive events B_1, B_2, \dots, B_n . If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i) P(A/B_i)}{\sum P(B_i) P(A/B_i)}.$$

Proof. By the multiplication law of probability,

$$P(AB_i) = P(A) P(B_i/A) = P(B_i) P(A/B_i) \quad \dots(1)$$

$$\therefore P(B_i/A) = \frac{P(B_i) P(A/B_i)}{P(A)} \quad \dots(2)$$

Since the event A corresponds to B_1, B_2, \dots, B_n , we have by the addition law of probability,

$$P(A) = P(AB_1) + P(AB_2) + \dots + P(AB_n) = \sum P(AB_i) = \sum P(B_i) P(A/B_i) \quad [\text{By (1)}]$$

$$\text{Hence from (2), we have } P(B_i/A) = \frac{P(B_i) P(A/B_i)}{\sum P(B_i) P(A/B_i)}$$

which is known as the *theorem of inverse probability*.

Obs. The probabilities $P(B_i)$, $i = 1, 2, \dots, n$ are called *apriori probabilities* because these exist before we get any information from the experiment.

The probabilities $P(A/B_i)$, $i = 1, 2, \dots, n$ are called *posteriori probabilities*, because these are found after the experiment results are known.

Example 26.26. Three machines M_1, M_2 and M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output?

Solution. Let the event of drawing a faulty item from any of the machines be A , and the event that an item drawn at random was produced by M_i be B_i . We have to find $P(B_i/A)$ for which we proceed as follows :

	M_1	M_2	M_3	Remarks
$P(B_i)$	0.25	0.30	0.45	∴ sum = 1
$P(A/B_i)$	0.05	0.04	0.03	
$P(B_i) P(A/B_i)$	0.0125	0.012	0.0135	sum = 0.38
$P(B_i/A)$	0.0125	0.012	0.0135	by Baye's theorem
	0.038	0.038	0.038	

The highest output being from M_3 , the required probability = $0.0135/0.038 = 0.355$.

Example 26.27. There are three bags : first containing 1 white, 2 red, 3 green balls ; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.

(J.N.T.U., 2003)

Solution. Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two balls are white and red.

$$\text{Now } P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$\begin{aligned} P(A/B_1) &= P \quad (\text{a white and a red ball are drawn from first bag}) \\ &= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15} \end{aligned}$$

$$\text{Similarly } P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \frac{2}{5}, P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \frac{1}{5}$$

$$\begin{aligned} \text{By Baye's theorem, } P(B_2/A) &= \frac{P(B_2) P(A/B_2)}{P(B_1) P(A/B_1) + P(B_2) P(A/B_2) + P(B_3) P(A/B_3)} \\ &= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}. \end{aligned}$$

PROBLEMS 26.3

- In a certain college, 4% of the boys and 1% of girls are taller than 1.8 m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m., what is the probability that the student is a girl?
- In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B or C? (V.T.U., 2006; Rohtak, 2005; Madras, 2000 S)
- In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D? (Hissar, 2007; J.N.T.U., 2003)
- The contents of three urns are: 1 white, 2 red, 3 green balls; 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. (Kurukshetra, 2007)

26.7 RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a *random variable* or a *stochastic variable* or simply a *variate*. For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the value 2, 3, 4, ..., 12 depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \leq c$ is written as $P(X \leq c)$.

If a random variable takes a finite set of values, it is called a *discrete variate*. On the other hand, if it assumes an infinite number of uncountable values, it is called a *continuous variate*.

26.8 (1) DISCRETE PROBABILITY DISTRIBUTION

Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i , is p_i , then

$$P(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2, \dots$$

where (i) $p(x_i) \geq 0$ for all values of i , (ii) $\sum p(x_i) = 1$

The set of values x_i with their probabilities p_i constitute a **discrete probability distribution** of the discrete variate X .

For example, the discrete probability distribution for X , the sum of the numbers which turn on tossing a pair of dice is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

[\because There are $6 \times 6 = 36$ equally likely outcomes and therefore, each has the probability $1/36$. We have $X = 2$ for one outcome, i.e. (1, 1); $X = 3$ for two outcomes (1, 2) and (2, 1); $X = 4$ for three outcomes (1, 3), (2, 2) and (3, 1) and so on.]

(2) Distribution function. The distribution function $F(x)$ of the discrete variate X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^x p(x_i) \text{ where } x \text{ is any integer. The graph of } F(x) \text{ will be}$$

stair step form (Fig. 26.2). The distribution function is also sometimes called *cumulative distribution function*.

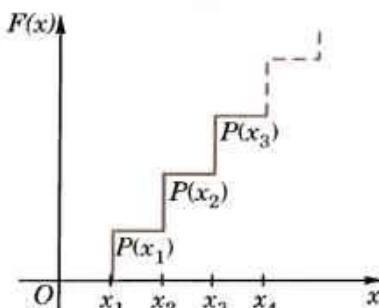


Fig. 26.2

Example 26.28. A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and variance of the number of successes. (V.T.U., 2011 S ; Rohtak, 2004)

Solution. Probability of a success = $\frac{2}{6} = \frac{1}{3}$, Probability of failure = $1 - \frac{1}{3} = \frac{2}{3}$.

$$\therefore \text{prob. of no success} = \text{Prob. of all 3 failures} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$\text{Probability of one successes and 2 failures} = 3c_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

$$\text{Probability of two successes and one failure} = 3c_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

$$\text{Probability of three successes} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$\text{Now } \begin{array}{cccc} x_i & = 0 & 1 & 2 & 3 \\ p_i & = 8/27 & 4/9 & 2/9 & 1/27 \end{array}$$

$$\therefore \text{mean } \mu = \sum p_i x_i = 0 + \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1.$$

$$\text{Also } \sum p_i x_i^2 = 0 + \frac{4}{9} + \frac{8}{9} + \frac{9}{27} = \frac{5}{3}$$

$$\therefore \text{variance } \sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}.$$

Example 26.29. The probability density function of a variate X is

$$\begin{array}{cccccccc} X & : & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ p(X) & : & k & 3k & 5k & 7k & 9k & 11k & 13k \end{array}$$

$$(i) \text{Find } P(X < 4), P(X \geq 5), P(3 < X \leq 6).$$

(V.T.U., 2010)

$$(ii) \text{What will be the minimum value of } k \text{ so that } P(X \leq 2) > 3.$$

Solution. (i) If X is a random variable, then

$$\sum_{i=0}^6 p(x_i) = 1 \text{ i.e., } k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \text{ or } k = 1/49.$$

$$\therefore P(X < 4) = k + 3k + 5k + 7k = 16k = 16/49.$$

$$P(X \geq 5) = 11k + 13k = 24k = 24/49.$$

$$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = 33/49.$$

$$(ii) P(X \leq 2) = k + 3k + 5k = 9k > 0.3 \text{ or } k > 1/30$$

Thus minimum value of $k = 1/30$.

Example 26.30. A random variable X has the following probability function :

$$\begin{array}{cccccccc} x & : & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ p(x) & : & 0 & k & 2k & 2k & 3k & k^2 & 2k^2 & 7k^2 + k \end{array}$$

$$(i) \text{Find the value of the } k$$

$$(ii) \text{Evaluate } P(X < 6), P(X \geq 6)$$

$$(iii) P(0 < X < 5).$$

(W.B.T.U., 2005 ; J.N.T.U., 2003)

Solution. (i) If X is a random variable, then

$$\sum_{i=0}^7 p(x_i) = 1, \text{ i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\text{i.e., } 7k^2 + 9k - 1 = 0 \text{ i.e. } (10 - k)(k + 1) = 0 \text{ i.e., } k = \frac{1}{10}$$

$$(ii) P(X < 6) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 0 + k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = P(X = 6) + P(X = 7) = 2k^2 + 7k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

$$(ii) P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ = k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5}.$$

26.9 (1) CONTINUOUS PROBABILITY DISTRIBUTION

When a variate X takes every value in an interval, it gives rise to *continuous distribution* of X . The distributions defined by the variates like heights or weights are continuous distributions.

A major conceptual difference, however, exists between discrete and continuous probabilities. When thinking in discrete terms, the probability associated with an event is meaningful. With continuous events, however, where the number of events is infinitely large, the probability that a specific event will occur is practically zero. For this reason, continuous probability statements must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval.

Thus the probability distribution of a continuous variate x is defined by a function $f(x)$ such that the probability of the variate x falling in the small interval $x - \frac{1}{2} dx$ to $x + \frac{1}{2} dx$ is $f(x) dx$. Symbolically it can be expressed as $P\left(x - \frac{1}{2} dx \leq x \leq x + \frac{1}{2} dx\right) = f(x) dx$. Then $f(x)$ is called the *probability density function* and the continuous curve $y = f(x)$ is called the *probability curve*.

The range of the variable may be finite or infinite. But even when the range is finite, it is convenient to consider it as infinite by supposing the density function to be zero outside the given range. Thus if $f(x) = \phi(x)$ be the density function denoted for the variate x in the interval (a, b) , then it can be written as

$$\begin{aligned} f(x) &= 0, & x < a \\ &= \phi(x), & a \leq x \leq b \\ &= 0, & x > b. \end{aligned}$$

The density function $f(x)$ is always positive and $\int_{-\infty}^{\infty} f(x) dx = 1$ (i.e., the total area under the probability curve and the x -axis is unity which corresponds to the requirements that the total probability of happening of an event is unity).

(2) Distribution function

$$\text{If } F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx,$$

then $F(x)$ is defined as the **cumulative distribution function** or simply the **distribution function** of the continuous variate X . It is the probability that the value of the variate X will be $\leq x$. The graph of $F(x)$ in this case is as shown in Fig. 26.3(b).

The distribution function $F(x)$ has the following properties :

(i) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

(ii) $F(-\infty) = 0$; (iii) $F(\infty) = 1$

$$(iv) P(a \leq x \leq b) = \int_a^b f(x) dx = \int_{-\infty}^b f(x) dx - \int_{-\infty}^a f(x) dx = F(b) - F(a).$$

Example 26.31. (i) Is the function defined as follows a density function?

$$\begin{aligned} f(x) &= e^{-x}, & x \geq 0 \\ &= 0, & x < 0, \end{aligned}$$

(ii) If so, determine the probability that the variate having this density will fall in the interval $(1, 2)$?

(iii) Also find the cumulative probability function $F(2)$?

Solution. (i) $f(x)$ is clearly ≥ 0 for every x in $(1, 2)$ and

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 \cdot dx + \int_0^{\infty} e^{-x} dx = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

$$(ii) \text{ Required probability} = P(1 \leq x \leq 2) = \int_1^2 e^{-x} dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233.$$

This probability is equal to the shaded area in Fig. 26.3 (a).

(iii) Cumulative probability function $F(2)$

$$\int_{-\infty}^2 f(x) dx = \int_{-\infty}^0 0 \cdot dx + \int_0^2 e^{-x} dx = 1 - e^{-2} = 1 - 0.135 = 0.865$$

which is shown in Fig. 26.3 (b).

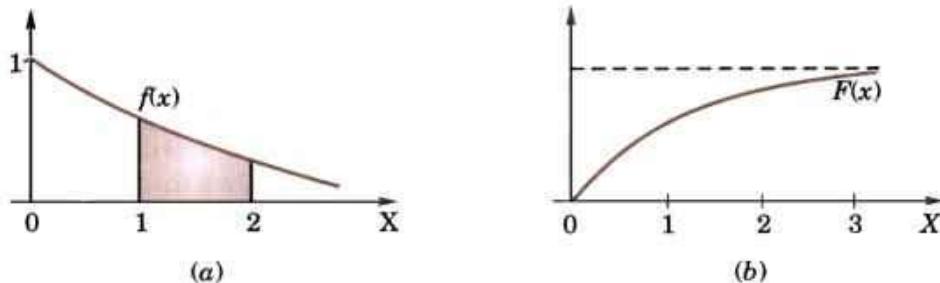


Fig. 26.3

26.10 (1) EXPECTATION

The mean value (μ) of the probability distribution of a variate X is commonly known as its **expectation** and is denoted by $E(X)$. If $f(x)$ is the probability density function of the variate X , then

$$\sum_i x_i f(x_i) \quad (\text{discrete distribution})$$

or
$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad (\text{continuous distribution})$$

In general, expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \sum_i \phi(x_i) f(x_i) \quad (\text{discrete distribution})$$

or
$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx \quad (\text{continuous distribution})$$

(2) **Variance of a distribution** is given by

$$\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i) \quad (\text{discrete distribution})$$

or
$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \quad (\text{continuous distribution})$$

where σ is the **standard deviation** of the distribution.

(3) **The rth moment about the mean** (denoted by μ_r) is defined by

$$\mu_r = \sum_i (x_i - \mu)^r f(x_i) \quad (\text{discrete distribution})$$

or
$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x) dx \quad (\text{continuous distribution})$$

(4) **Mean deviation from the mean** is given by

$$\sum |x_i - \mu| f(x_i) \quad (\text{discrete distribution})$$

or by
$$\int_{-\infty}^{\infty} |x - \mu| f(x) dx \quad (\text{continuous distribution})$$

Example 26.32. In a lottery, m tickets are drawn at a time out of n tickets numbered from 1 to n . Find the expected value of the sum of the numbers on the tickets drawn.

Solution. Let x_1, x_2, \dots, x_n be the variables representing the numbers on the first, second, ..., n th ticket. The probability of drawing a ticket out of n tickets being in each case $1/n$, we have

$$E(x_i) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \dots + n \cdot \frac{1}{n} = \frac{1}{2} (n+1)$$

\therefore expected value of the sum of the numbers on the tickets drawn

$$\begin{aligned} &= E(x_1 + x_2 + \dots + x_m) = E(x_1) + E(x_2) + \dots + E(x_m) \\ &= mE(x_i) = \frac{1}{2} m (n+1). \end{aligned}$$

Example 26.33. X is a continuous random variable with probability density function given by

$$\begin{aligned} f(x) &= kx \quad (0 \leq x < 2) \\ &= 2k \quad (2 \leq x < 4) \\ &= -kx + 6k \quad (4 \leq x < 6) \end{aligned}$$

Find k and mean value of X .

(J.N.T.U., 2003)

Solution. Since the total probability is unity

$$\therefore \int_0^6 f(x) dx = 1$$

$$\text{i.e., } \int_0^2 kx dx + \int_2^4 2kdx + \int_4^6 (-kx + 6k) dx = 1$$

$$\text{or } k \left| \frac{x^2}{2} \right|_0^2 + 2k \left| x \right|_2^4 + \left(-kx^2/2 + 6kx \right)_4^6 = 1$$

$$\text{or } 2k + 4k + (-10k + 12k) = 1 \text{ i.e., } k = 1/8.$$

$$\begin{aligned} \text{Mean of } X &= \int_0^6 x f(x) dx \\ &= \int_0^2 kx^2 dx + \int_2^4 2kx dx + \int_4^6 x (-kx + 6k) dx \\ &= k \left| \frac{x^3}{3} \right|_0^2 + 2k \left| \frac{x^2}{2} \right|_2^4 + \left(-k \left| \frac{x^3}{3} \right|_4^6 + 6k \left| \frac{x^2}{2} \right|_4^6 \right) \\ &= k(8/3) + k(12) - k(152/3) + 3k(20) = \frac{1}{8}(24) = 3. \end{aligned}$$

Example 26.34. A variate X has the probability distribution

x	:	-3	6	9
$P(X=x)$:	1/6	1/2	1/3

Find $E(X)$ and $E(X^2)$. Hence evaluate $E(2X+1)^2$.

Solution.

$$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = 11/2.$$

$$E(X^2) = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = 93/2$$

$$\begin{aligned} \therefore E(2X+1)^2 &= E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1 \\ &= 4(93/2) + 4(11/2) + 1 = 209. \end{aligned}$$

Example 26.35. The frequency distribution of a measurable characteristic varying between 0 and 2 is as under

$$\begin{aligned} f(x) &= x^3, \quad 0 \leq x \leq 1 \\ &= (2-x)^3, \quad 1 \leq x \leq 2, \end{aligned}$$

Calculate the standard deviation and also the mean deviation about the mean.

Solution. Total frequency $N = \int_0^1 x^3 dx + \int_1^2 (2-x)^3 dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$\therefore \mu'_1 \text{ (about the origin)} = \frac{1}{N} \left[\int_0^1 x \cdot x^3 dx + \int_1^2 x(2-x)^3 dx \right]$$

$$= 2 \left\{ \left| \frac{x^5}{5} \right|_0^1 + \left| -x \cdot \frac{(2-x)^4}{4} \right|_1^2 - \left| \frac{(2-x)^5}{20} \right|_1^2 \right\} = 2 \left(\frac{1}{5} + \frac{1}{4} + \frac{1}{20} \right) = 1$$

$$\mu'_2 \text{ (about the origin)} = \frac{1}{N} \left[\int_0^1 x^2 \cdot x^3 dx + \int_1^2 x^2 (2-x)^3 dx \right]$$

$$= 2 \left\{ \left| \frac{x^6}{6} \right|_0^1 + \left| -x^2 \cdot \frac{(2-x)^4}{4} \right|_1^2 + \frac{1}{2} \int_1^2 x(2-x)^4 dx \right\}$$

$$= 2 \left\{ \frac{1}{6} + \frac{1}{4} + \frac{1}{2} \left[\frac{1}{5} + \frac{1}{30} \right] \right\} = \frac{16}{15}$$

Hence

$$\sigma^2 = \mu_2' - (\mu_1')^2 = \frac{1}{15}$$

i.e., standard deviation $\sigma = \frac{1}{\sqrt{15}}$.

Mean deviation about the mean

$$\begin{aligned} &= \frac{1}{N} \left\{ \int_0^1 |x-1| x^3 dx + \int_1^2 |x-1| (2-x)^3 dx \right\} \\ &= 2 \left\{ \int_0^1 (1-x)x^3 dx + \int_1^2 (x-1)(2-x)^3 dx \right\} \\ &= 2 \left\{ \left(\frac{1}{4} - \frac{1}{5} \right) + \left(0 + \frac{1}{20} \right) \right\} = \frac{1}{5}. \end{aligned}$$

26.11 MOMENT GENERATING FUNCTION

(1) The moment generating function (m.g.f.) of the discrete probability distribution of the variate X about the value $x = a$ is defined as the expected value of $e^{t(x-a)}$ and is denoted by $M_a(t)$. Thus

$$M_a(t) = \sum p_i e^{t(x_i - a)} \quad \dots(1)$$

which is a function of the parameter t only.

Expanding the exponential in (1), we get

$$\begin{aligned} M_a(t) &= \sum p_i + t \sum p_i (x_i - a) + \frac{t^2}{2!} \sum p_i (x_i - a)^2 + \dots + \frac{t^r}{r!} \sum p_i (x_i - a)^r + \dots \\ &= 1 + t\mu'_1 + \frac{t^2}{2!} \mu'_2 + \dots + \frac{t^r}{r!} \mu'_r + \dots \quad \dots(2) \end{aligned}$$

where μ'_r is the moment of order r about a . Thus $M_a(t)$ generates moments and that is why it is called the moment generating function. From (2), we find

$$\mu'_r = \text{coefficient of } t^r/r! \text{ in the expansion of } M_a(t).$$

Otherwise differentiating (2) r times with respect to t and then putting $t = 0$, we get

$$\mu'_r = \left[\frac{d^r}{dt^r} M_a(t) \right]_{t=0} \quad \dots(3)$$

Thus the moment about any point $x = a$ can be found from (2) or more conveniently from the formula (3).

Rewriting (1) as

$$M_a(t) = e^{-at} \sum p_i e^{tx_i} \quad \text{or} \quad M_a(t) = e^{-at} M_0(t) \quad \dots(4)$$

Thus the m.g.f. about the point $a = e^{-at}$ (m.g.f. about the origin).

Obs. The m.g.f. of the sum of two independent variables is the product of their m.g.f.s.

... (5)

(2) If $f(x)$ is the density function of a continuous variate X , then the moment generating function of this continuous probability distribution about $x = a$ is given by

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x) dx.$$

Example 26.36. Find the moment generating function of the exponential distribution

$$f(x) = \frac{1}{c} e^{-x/c}, \quad 0 \leq x \leq \infty, \quad c > 0. \quad \text{Hence find its mean and S.D.}$$

(Kurukshetra, 2009)

Solution. The moment generating function about the origin is

$$\begin{aligned}
 M_0(t) &= \int_0^\infty e^{tx} \cdot \frac{1}{c} e^{-x/c} dx = \frac{1}{c} \int_0^\infty e^{(t-1/c)x} dx \\
 &= \frac{1}{c} \frac{\left[e^{(t-1/c)x} \right]_0^\infty}{|t-1/c|} = (1-ct)^{-1} = 1 + ct + c^2 t^2 + c^3 t^3 + \dots \\
 \mu'_1 &= \left[\frac{d}{dt} M_0(t) \right]_{t=0} = (c + 2c^2 t + 3c^3 t^2 + \dots)_{t=0} = c \\
 \mu'_2 &= \left[\frac{d^2}{dt^2} M_0(t) \right]_{t=0} = 2c^2, \text{ and } \mu_2 = \mu'_2 - (\mu'_1)^2 = 2c^2 - c^2 = c^2.
 \end{aligned}$$

Hence the mean is c and S.D. is also c .

26.12 PROBABILITY GENERATING FUNCTION

The probability generating function (p.g.f.) $P_x(t)$ for a random variable x which takes integral values 0, 1, 2, 3, ... only, is defined by

$$P_x(t) = p_0 + p_1 t + p_2 t^2 + \dots = \sum_{n=0}^{\infty} p_n t^n = E(t^x)$$

The coefficient of t^n in the expansion of $P(t)$ in powers of t gives $P(t)_{x=n}$

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} np_n t^{n-1} \quad \text{or} \quad \left(\frac{\partial P}{\partial t} \right)_{t=1} = \sum n p_n = \mu_1'$$

$$\frac{\partial^2 P}{\partial t^2} = \sum_{n=0}^{\infty} n(n-1) p_n t^{n-2} \quad \text{or} \quad \left(\frac{\partial^2 P}{\partial t^2} \right)_{t=1} = \sum n(n-1) p_n = \mu_2' - \mu_1'^2$$

$$= \mu_2 + \mu_1'^2 - \mu_1' \text{ and so on}$$

$$\text{Also } \left(\frac{\partial^k P}{\partial t^n} \right)_{t=0} = n! p_n, k = 1, 2, \dots, n.$$

For integral valued variates, we have

$$P_x(e^t) = E(e^{tx}) = m.g.f. \text{ for } x.$$

Obs. The p.g.f. of the sum of two independent random variables is the product of their p.g.f's.

Example 26.37. If x be a random variable with probability generating function $P_x(t)$, find the probability generating function of

- $$(i) x + 2 \quad (ii) 2x.$$

Solution. We have $P_x(t) = \sum_{k=0}^{\infty} p_k t^k$

(i) Probability generating function of $x + 2 = \sum_{k=0}^{\infty} p_k t^{k+2} = t^2 \sum_{k=0}^{\infty} p_k t^k = t^2 P_x(t)$.

(ii) Probability generating function of $2x = \sum_{k=0}^{\infty} p_k t^{2k} = \sum_{k=0}^{\infty} p_k (t^2)^k = P(t^2)$.

PROBLEMS 26.4

1. A random variable x has the following probability function :

Values of x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k and calculate mean and variance. (S.V.T.U., 2007; V.T.U., 2004; Madras, 2003)

2. Find the standard deviation for the following discrete distribution :

x :	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12

3. Obtain the distribution function of the total number of heads occurring in three tosses of an unbiased coin.

4. Show that for any discrete distribution $\beta_2 \geq 1$.

5. From an urn containing 3 red and 2 white balls, a man is to draw 2 balls at random without replacement, being promised Rs. 20 for each red ball he draws and Rs. 10 for each white one. Find his expectation.

6. Four coins are tossed. What is the expectation of the number of heads ?

7. The diameter of an electric cable is assumed to be a continuous variate with p.d.f. $f(x) = 6x(1-x)$, $0 \leq x \leq 1$. Verify that the above is a p.d.f. Also find the mean and variance.

8. A random variable gives measurements X between 0 and 1 with a probability function

$$\begin{aligned} f(x) &= 12x^3 - 21x^2 + 10x, \quad 0 \leq x \leq 1 \\ &= 0 \end{aligned}$$

(i) Find $P\left(X \leq \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$

(ii) Find a number k such that $P(X \leq k) = \frac{1}{2}$ (J.N.T.U., 2003)

9. The power reflected by an aircraft that is received by a radar can be described by an exponential random variable X .

The probability density of X is given by $f(x) = \begin{cases} \frac{1}{x_0} e^{-x/x_0}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

where x_0 is the average power received by the radar.

- (i) What is the probability that the radar will receive power larger than the power received on the average ? (ii) What is the probability that the radar will receive power less than the power received on the average ?

(Mumbai, 2006)

10. A function is defined as follows :

$$\begin{aligned} f(x) &= 0, \quad x < 2 \\ &= \frac{1}{18} (2x+3), \quad 2 \leq x \leq 4 \\ &= 0, \quad x > 4. \end{aligned}$$

Show that it is a density function. Find the probability that a variate having this density will fall in the interval $2 \leq x \leq 3$?

11. A continuous distribution of a variable x in the range $(-3, 3)$ is defined as

$$\begin{aligned} f(x) &= \frac{1}{16} (3+x)^2, \quad -3 \leq x < -1 \\ &= \frac{1}{16} (2-6x^2), \quad -1 \leq x < 1 \\ &= \frac{1}{16} (3-x)^2, \quad 1 \leq x \leq 3. \end{aligned}$$

Verify that the area under the curve is unity. Show that the mean is zero.

(Kurukshetra, 2005)

12. The frequency function of a continuous random variable is given by

$$f(x) = y_0 x (2 - x), \quad 0 \leq x \leq 2.$$

Find the value of y_0 , mean and variance of x .

(Kerala, 2005; J.N.T.U., 2003)

13. The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = y_0 e^{-|x|}, \quad -\infty < x < \infty.$$

Prove that $y_0 = 1/2$. Find the mean and variance of the distribution.

(S.V.T.U., 2008; Kurukshetra, 2007; V.T.U., 2004)

14. If $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

represents the density of a random variable X , find $E(X)$ and $\text{Var}(X)$.

15. A function is defined as under :

$$\begin{aligned} f(x) &= 1/k, \quad x_1 \leq x \leq x_2 \\ &= 0, \quad \text{elsewhere.} \end{aligned}$$

Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.

26.13 REPEATED TRIALS

We know that the probability of getting a head or a tail on tossing a coin is $\frac{1}{2}$. If the coin is tossed thrice, the probability of getting one head and two tails can be combined as $H-T-T, T-H-T, T-T-H$. The probability of each one of these being $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, i.e., $\left(\frac{1}{2}\right)^3$, their total probability shall be $3(1/2)^3$.

Similarly if a trial is repeated n times and if p is the probability of a success and q that of a failure, then the probability of r successes and $n - r$ failures is given by $p^r q^{n-r}$.

But these r successes and $n - r$ failures can occur in any of the ${}^n C_r$ ways in each of which the probability is same.

Thus the probability of r successes is ${}^n C_r p^r q^{n-r}$.

Cor. The probabilities of at least r successes in n trials

$$\begin{aligned} &= \text{the sum of the probabilities of } r, r+1, \dots, n \text{ successes} \\ &= {}^n C_r p^r q^{n-r} + {}^n C_{r+1} p^{r+1} q^{n-r-1} + \dots + {}^n C_n p^n. \end{aligned}$$

26.14 (1) BINOMIAL DISTRIBUTION*

It is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial p is the probability of a success and q that of a failure, then the probability of r successes in a series of n trials is given by ${}^n C_r p^r q^{n-r}$, where r takes any integral value from 0 to n . The probabilities of 0, 1, 2, ..., r , ..., n successes are, therefore, given by

$$q^n, {}^n C_1 p q^{n-1}, {}^n C_2 p^2 q^{n-2}, \dots, {}^n C_r p^r q^{n-r}, \dots, p^n.$$

The probability of the number of successes so obtained is called the **binomial distribution** for the simple reason that the probabilities are the successive terms in the expansion of the binomial $(q+p)^n$.

\therefore the sum of the probabilities

$$= q^n + {}^n C_1 p q^{n-1} + {}^n C_2 p^2 q^{n-2} + \dots + p^n = (q+p)^n = 1.$$

(2) Constants of the binomial distribution. The moment generating function about the origin is

$$\begin{aligned} M_0(t) &= E(e^{tx}) = \sum {}^n C_x p^x q^{n-x} e^{tx} \\ &= \sum {}^n C_x (pe^t)^x q^{n-x} = (q + pe^t)^n \end{aligned}$$

[By (1) § 26.11]

* It was discovered by a Swiss mathematician Jacob Bernoulli and was published posthumously in 1713.

Differentiating with respect to t and putting $t = 0$ and using (3) § 26.11, we get the mean

$$\mu'_1 = np.$$

Since $M_a(t) = e^{-at} M_0(t)$, the m.g.f. of the binomial distribution about its mean (m) = np , is given by

$$M_m(t) = e^{-npt} (q + pe^t)^n = (qe^{-pt} + pe^{qt})^n$$

$$= \left(1 + pq \frac{t^2}{2!} + pq(q^2 - p^2) \frac{t^3}{3!} + pq(q^3 + p^3) \frac{t^4}{4!} + \dots \right)^n$$

or $1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \mu_4 \frac{t^4}{4!} + \dots$

$$= 1 + npq \frac{t^2}{2!} + npq(q-p) \frac{t^3}{3!} + npq [1 + 3(n-2)pq] \frac{t^4}{4!} + \dots$$

Equating the coefficients of like powers of t on either side, we have

$$\mu_2 = npq, \mu_3 = npq(q-p), \mu_4 = npq [1 + 3(n-2)pq].$$

Also $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$

Thus mean = np , standard deviation = $\sqrt{(npq)}$.

skewness = $(1-2p)/\sqrt{(npq)}$, kurtosis = β_2 .

Obs. The skewness is positive for $p < \frac{1}{2}$ and negative for $p > \frac{1}{2}$. When $p = \frac{1}{2}$, the skewness is zero, i.e., the probability curve of the binomial distribution will be symmetrical (bell-shaped).

As n the number of trials increase indefinitely, $\beta_1 \rightarrow 0$, and $\beta_2 \rightarrow 3$.

(3) Binomial frequency distribution. If n independent trials constitute one experiment and this experiment be repeated N times, then the frequency of r successes is $N^n C_r p^r q^{n-r}$. The possible number of successes together with these expected frequencies constitute the *binomial frequency distribution*.

(4) Applications of Binomial distribution. This distribution is applied to problems concerning :

(i) Number of defectives in a sample from production line,

(ii) Estimation of reliability of systems,

(iii) Number of rounds fired from a gun hitting a target,

(iv) Radar detection.

Example 26.38. The probability that a pen manufactured by a company will be defective is $1/10$. If 12 such pens are manufactured, find the probability that

(a) exactly two will be defective. (b) at least two will be defective.

(c) none will be defective.

(V.T.U., 2004 ; Burdwan, 2003)

Solution. The probability of a defective pen is $1/10 = 0.1$

∴ The probability of a non-defective pen is $1 - 0.1 = 0.9$

(a) The probability that exactly two will be defective

$$= {}^{12}C_2 (0.1)^2 (0.9)^{10} = 0.2301$$

(b) The probability that at least two will be defective

$$= 1 - (\text{prob. that either none or one is non-defective})$$

$$= 1 - [{}^{12}C_0 (0.9)^{12} + {}^{12}C_1 (0.1) (0.9)^{11}] = 0.3412$$

(c) The probability that none will be defective

$$= {}^{12}C_{12} (0.9)^{12} = 0.2833.$$

Example 26.39. In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails.

(J.N.T.U., 2003)

Solution. $P(\text{head}) = \frac{1}{2}$ and $P(\text{tail}) = \frac{1}{2}$

By binomial distribution, probability of 8 heads and 4 tails in 12 trials is

$$P(X = 8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = \frac{12!}{8! 4!} \cdot \frac{1}{2^{12}} = \frac{495}{4096}$$

∴ the expected number of such cases in 256 sets

$$= 256 \times P(X = 8) = 256 \cdot \frac{495}{4096} = 30.9 = 31 \text{ (say).}$$

Example 26.40. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts. (V.T.U., 2004)

Solution. Mean number of defectives = $2 = np = 20p$.

∴ The probability of a defective part is $p = 2/20 = 0.1$.

and the probability of a non-defective part = 0.9

∴ The probability of at least three defectives in a sample of 20.

$$\begin{aligned} &= 1 - (\text{prob. that either none, or one, or two are non-defective parts}) \\ &= 1 - [{}^{20}C_0(0.9)^{20} + {}^{20}C_1(0.1)(0.9)^{19} + {}^{20}C_2(0.1)^2(0.9)^{18}] \\ &= 1 - (0.9)^{18} \times 4.51 = 0.323. \end{aligned}$$

Thus the number of samples having at least three defective parts out of 1000 samples

$$= 1000 \times 0.323 = 323.$$

Example 26.41. The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to these data :

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f:$	6	20	28	12	8	6	0	0	0	0	0

Solution. Here $n = 10$ and $N = \sum f_i = 80$

$$\therefore \text{mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{20 + 56 + 36 + 32 + 30}{80} = \frac{174}{80} = 2.175$$

Now the mean of a binomial distribution = np

$$\text{i.e., } np = 10p = 2.175 \quad \therefore p = 0.2175, q = 1 - p = 0.7825$$

Hence the binomial distribution to be fitted is

$$\begin{aligned} N(q+p)^n &= 80(0.7825 + 0.2175)^{10} \\ &= 80 \cdot {}^{10}C_0(0.7825)^{10} + 80 \cdot {}^{10}C_1(0.7825)^9(0.2175)^1 + {}^{10}C_2(0.7825)^8(0.2175)^2 + \\ &\quad \dots + {}^{10}C_9(0.7825)^1(0.2175)^9 + {}^{10}C_{10}(0.2175)^{10} \\ &= 6.885 + 19.13 + 23.94 + \dots + 0.0007 + 0.00002 \end{aligned}$$

∴ the successive terms in the expansion give the expected or theoretical frequencies which are

$x:$	0	1	2	3	4	5	6	7	8	9	10
$f:$	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

PROBLEMS 26.5

- Determine the binomial distribution for which mean = 2 (variance) and mean + variance = 3. Also find $P(X \leq 3)$. (Kerala, 2005)
- An ordinary six-faced die is thrown four times. What are the probabilities of obtaining 4, 3, 2, 1 and 0 faces?
- If the chance that one of the ten telephone lines is busy at an instant is 0.2.
 - What is the chance that 5 of the lines are busy?
 - What is the most probable number of busy lines and what is the probability of this number?
 - What is the probability that all the lines are busy? (V.T.U., 2002 S)
- If the probability that a new-born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys. (Kurukshetra, 2005)

5. If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely. (P.T.U., 2005)
6. The probability that a bomb dropped from a plane will strike the target is $1/5$. If six bombs are dropped, find the probability that (i) exactly two will strike the target, (ii) at least two will strike the target.
7. A sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that (i) one plane does not return (ii) at the most 5 planes do not return, and (iii) what is the most probable number of returns? (Hissar, 2007)
8. The probability that an entering student will graduate is 0.4. Determine the probability that out of 5 students (a) none (b) one and (c) at least one will graduate.
9. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys, (b) 5 girls, (c) either 2 or 3 boys? Assume equal probabilities for boys and girls. (V.T.U., 2004)
10. If 10 per cent of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (i) none will be defective, (ii) one will be defective, and (iii) at least two will be defective.
11. In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target. (V.T.U., 2003 S)
12. A product is 0.5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives?
13. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.
14. 500 articles were selected at random out of a batch containing 10,000 articles, and 30 were found to be defective. How many defectives articles would you reasonably expect to have in the whole batch? (J.N.T.U., 2003)
15. Fit a binomial distribution for the following data and compare the theoretical frequencies with the actual ones :

$x :$	0	1	2	3	4	5	
$f :$	2	14	20	34	22	8	

 (Bhopal, 2006)
16. Fit a binomial distribution to the following frequency distribution :

$x :$	0	1	2	3	4	5	6	
$f :$	13	25	52	58	32	16	4	

 (Kurukshetra, 2009 ; S.V.T.U., 2007)

26.15 (1) POISSON DISTRIBUTION*

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by making n very large and p very small, keeping np fixed (= m , say).

The probability of r successes in a binomial-distribution is

$$\begin{aligned} P(r) &= {}^nC_r p^r q^{n-r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} p^r q^{n-r} \\ &= \frac{np(np-p)(np-2p)\cdots(np-r-1p)}{r!} (1-p)^{n-r} \end{aligned}$$

As $n \rightarrow \infty$, $p \rightarrow 0$ ($np = m$), we have

$$P(r) = \frac{m^r}{r!} \underset{n \rightarrow \infty}{\text{Lt}} \frac{(1-m/n)^n}{(1-m/n)^r} = \frac{m^r}{r!} e^{-m}$$

so that the probabilities of 0, 1, 2..., r ... successes in a Poisson distribution are given by

$$e^{-m}, me^{-m}, \frac{m^2 e^{-m}}{2!}, \dots, \frac{m^r e^{-m}}{r!}, \dots$$

The sum of these probabilities is unity as it should be.

* It was discovered by a French mathematician S.D. Poisson in 1837.

(2) Constants of the Poisson distribution. These constants can easily be derived from the corresponding constants of the binomial distribution simply by making $n \rightarrow \infty$, $p \rightarrow 0$, ($q \rightarrow 1$) and noting that $np = m$

$$\text{Mean} = \text{Lt}(np) = m$$

$$\mu_2 = \text{Lt}(npq) = m \text{ Lt}(q) = m$$

$$\therefore \text{Standard deviation} = \sqrt{m}$$

$$\text{Also } \mu_3 = m, \mu_4 = m + 3m^2$$

$$\therefore \text{Skewness} (= \sqrt{\beta_1}) = 1/m, \text{Kurtosis} (= \beta_2) = 3 + 1/m.$$

Since μ_3 is positive, Poisson distribution is positively skewed and since $\beta_2 > 3$, it is *Leptokurtic*.

(3) Applications of Poisson distribution. This distribution is applied to problems concerning :

(i) Arrival pattern of 'defective vehicles in a workshop', 'patients in a hospital' or 'telephone calls'.

(ii) Demand pattern for certain spare parts.

(iii) Number of fragments from a shell hitting a target.

(iv) Spatial distribution of bomb hits.

Example 26.42. If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than two will get a bad reaction. (V.T.U., 2008 ; Kottayam, 2005)

Solution. It follows a Poisson distribution as the probability of occurrence is very small.

$$\text{Mean } m = np = 2000(0.001) = 2$$

Probability that more than 2 will get a bad reaction

$$= 1 - [\text{prob. that no one gets a bad reaction} + \text{prob. that one gets a bad reaction} + \text{prob. that two get bad reaction}]$$

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} \right] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2} \right] \quad [\because m = 2]$$

$$= 1 - \frac{5}{e^2} = 0.32. \quad [\because e = 2.718]$$

Example 26.43. In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10, use Poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets. (Kurukshetra, 2009 S; Madras, 2006 ; V.T.U., 2004)

Solution. We know that $m = np = 10 \times 0.002 = 0.02$

$$e^{-0.02} = 1 - 0.02 + \frac{(0.02)^2}{2!} - \dots = 0.9802 \text{ approximately}$$

$$\text{Probability of no defective blade is } e^{-m} = e^{-0.02} = 0.9802$$

\therefore no. of packets containing no defective blade is

$$10,000 \times 0.9802 = 9802$$

$$\text{Similarly the number of packets containing one defective blade} = 10,000 \times me^{-m}$$

$$= 10,000 \times (0.02) \times 0.9802 = 196$$

$$\text{Finally the number of packets containing two defective blades}$$

$$= 10,000 \times \frac{m^2 e^{-m}}{2!} = 10,000 \times \frac{(0.02)^2}{2!} \times 0.9802 = 2 \text{ approximately.}$$

Example 26.44. Fit a Poisson distribution to the set of observations :

$$x: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$f: \quad 122 \quad 60 \quad 15 \quad 2 \quad 1$$

(Bhopal, 2007 S ; V.T.U., 2004 ; U.P.T.U., 2003)

Solution. Mean = $\frac{\sum f_i x_i}{\sum f_i} = \frac{60 + 36 + 6 + 4}{200} = 0.5$.

\therefore mean of Poisson distribution i.e., $m = 0.5$.

Hence the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(0.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

\therefore the theoretical frequencies are

$x :$	0	1	2	3	4	
$f :$	121	61	15	2	0	$(\because e^{-0.5} = 0.61)$

PROBLEMS 26.6

- If a random variable has a Poisson distribution such that $P(1) = P(2)$, find
(i) mean of the distribution. (ii) $P(4)$. (V.T.U., 2003)
- X is a Poisson variable and it is found that the probability that $X = 2$ is two-thirds of the probability that $X = 1$. Find the probability that $X = 0$ and the probability that $X = 3$. What is the probability that X exceeds 3?
- For Poisson distribution, prove that $m \mu_2 \gamma_1 \gamma_2 = 1$, where symbols have their usual meanings. (S.V.T.U., 2008)
- A certain screw making machine produces on average of 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws. (Kurukshetra, 2006)
- A manufacturer knows that the condensers he makes contain on the average 1% defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers?
- A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand, (ii) on which demand is refused. ($e^{-1.5} = 0.2231$). (Bhopal, 2008 S; J.N.T.U., 2003)
- The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, five or more will suffer from it?
- The frequency of accidents per shift in a factory is as shown in the following table:
Accidents per shift : 0 1 2 3 4
Frequency : 180 92 24 3 1
Calculate the mean number of accidents per shift and the corresponding Poisson distribution and compare with actual observations.
- A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimetre equal to 3. Ten 1 c.c., test-tubes are filled with the liquid. Assuming that Poisson distribution is applicable, calculate the probability that all the test-tubes will show growth i.e., contain atleast 1 bacterium each.
- Find the expectation of the function $\phi(x) = xe^{-x}$ in a Poisson distribution. (V.T.U., 2003)

[Hint : If m be the mean of the Poisson distribution, then expectation of

$$\phi(x) = \sum_{x=0}^{\infty} \frac{\phi(x) \cdot m^x e^{-m}}{x!} = m \exp. m (e^{-1} - m - 1)$$

- Fit a Poisson distribution to the following:

$x :$	0	1	2	3	4	
$f :$	46	38	22	9	1	(Kurukshetra, 2009; Bhopal, 2008; V.T.U., 2003 S)

- Fit a Poisson distribution to the following data given the number of yeast cells per square for 400 squares:

No. of cells per sq. :	0	1	2	3	4	5	6	7	8	9	10
No. of squares :	103	143	98	42	8	4	2	0	0	0	0

(S.V.T.U., 2007)

26.16 (1) NORMAL DISTRIBUTION*

Now we consider a continuous distribution of fundamental importance, namely the normal distribution. Any quantity whose variation depends on random causes is distributed according to the normal law. Its importance lies in the fact that a large number of distributions approximate to the normal distribution.

* In 1924, Karl Pearson found this distribution which Abraham De Moivre had discovered as early as 1733. See footnote p. 843 and 647.

Let us define a variate $z = \frac{x - np}{\sqrt{(npq)}}$... (1)

where x is a binomial variate with mean np and S.D. $\sqrt{(npq)}$ so that z is a variate with mean zero and variance unity. In the limit as n tends to infinity, the distribution of z becomes a continuous distribution extending from $-\infty$ to ∞ .

It can be shown that the limiting form of the binomial distribution (1) for large values of n when neither p nor q is very small, is the normal distribution. The normal curve is of the form

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \dots (2)$$

where μ and σ are the mean and standard deviation respectively.

(2) Properties of the normal distribution

I. The normal curve (2) is bell-shaped and is symmetrical about its mean. It is unimodal with ordinates decreasing rapidly on both sides of the mean (Fig. 26.3). The maximum ordinate is $1/\sigma\sqrt{2\pi}$, found by putting $x = \mu$ in (2).

As it is symmetrical, its mean, median and mode are the same. Its points of inflexion (found by putting $d^2y/dx^2 = 0$ and verifying that at these points $d^3y/dx^3 \neq 0$) are given by $x = \mu \pm \sigma$, i.e., these points are equidistant from the mean on either side.

II. Mean deviation from the mean μ

$$\begin{aligned} &= \int_{-\infty}^{\infty} |x - \mu| \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \quad [\text{Put } z = (x - \mu)/\sigma] \\ &= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz \\ &= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{-\infty}^0 -ze^{-z^2/2} dz + \int_0^{\infty} ze^{-z^2/2} dz \right] = \frac{2\sigma}{\sqrt{2\pi}} \int_0^{\infty} ze^{-z^2/2} dz \\ &= \frac{2\sigma}{\sqrt{2\pi}} \left| -e^{-z^2/2} \right|_0^{\infty} = -\sqrt{\left(\frac{2}{\pi}\right)} \sigma(0 - 1) = 0.7979 \sigma = (4/5) \sigma \end{aligned}$$

III. Moments about the mean

$$\begin{aligned} \mu_{2n+1} &= \int_{-\infty}^{\infty} (x - \mu)^{2n+1} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-z^2/2} dz \text{ where } z = (x - \mu)/\sigma \\ &= 0, \text{ since the integral is an odd function.} \end{aligned}$$

Thus all odd order moments about the mean vanish.

$$\begin{aligned} \mu_{2n} &= \int_{-\infty}^{\infty} (x - \mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{-z^2/2} \cdot zdz \quad [\text{Integrate by parts}] \\ &= \frac{\sigma^{2n}}{\sqrt{2\pi}} \left[\left| -z^{2n-1} e^{-z^2/2} \right|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (2n-1)z^{2n-2} e^{-z^2/2} dz \right] \\ &= \frac{\sigma^{2n}}{\sqrt{2\pi}} (0 - 0) + (2n-1) \sigma^2 \mu_{2n-2} \end{aligned}$$

Repeated application of this reduction formula, gives

$$\mu_{2n} = (2n-1)(2n-3)\dots 3 \cdot 1 \sigma^{2n}$$

In particular, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$.

$$\text{Hence } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = 3$$

i.e., the coefficient of skewness is zero (i.e. the curve is symmetrical) and the Kurtosis is 3. This is the basis for the choice of the value 3 in the definitions of platykurtic and leptokurtic (page 844).

IV. The probability of x lying between x_1 and x_2 is given by the area under the normal curve from x_1 to x_2 , i.e., $P(x_1 \leq x \leq x_2)$

$$\begin{aligned} &= \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2} dz \text{ where } z = (x - \mu)/\sigma, dz = dx/\sigma \text{ and } z_1 = (x_1 - \mu)/\sigma, z_2 = (x_2 - \mu)/\sigma. \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_0^{z_2} e^{-z^2/2} dz - \int_0^{z_1} e^{-z^2/2} dz \right\} = P_2(z) - P_1(z) \end{aligned}$$

The values of each of the above integrals can be found from the table III–Appendix 2, which gives the values of

$$P(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$$

for various values of z . This integral is called the *probability integral* or the *error function* due to its use in the theory of sampling and the theory of errors.

Using this table, we see that the area under the normal curve from $z = 0$ to $z = 1$, i.e. from $x = \mu$ to $\mu + \sigma$ is 0.3413.

∴ (i) The area under the normal curve between the ordinates $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.6826, ~ 68% nearly. Thus approximately 2/3 of the values lie within these limits.

(ii) The area under the normal curve between $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$ is 0.9544 ~ 95.5%, which implies that about $4\frac{1}{2}\%$ of the values lie outside these limits.

(iii) 99.73% of the values lie between $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$ i.e., only a quarter % of the whole lies outside these limits.

(iv) 95% of the values lie between $x = \mu - 1.96\sigma$ and $x = \mu + 1.96\sigma$ i.e., only 5% of the values lie outside these limits.

(v) 99% of the values lie between $x = \mu - 2.58\sigma$ and $x = \mu + 2.58\sigma$ i.e., only 1% of the values lie outside these limits.

(vi) 99.9% of the values lie between $x = \mu - 3.29\sigma$ and $x = \mu + 3.29\sigma$.

In other words, a value that deviates more than σ from μ occurs about once in 3 trials. A value that deviates more than 2σ or 3σ from μ occurs about once in 20 or 400 trials. Almost all values lie within 3σ of the mean.

The shape of the standardised normal curve is

$$y = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ where } z = (x - \mu)/\sigma \quad \dots(3)$$

and the respective areas are shown in Fig. 26.4. 'z' is called a *normal variate*.

(3) Normal frequency distribution. We can fit a normal curve to any distribution. If N be the total frequency, μ the mean and σ the standard deviation of the given distribution then the curve

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \dots(4)$$

will fit the given distribution as best as the data will permit. The frequency of the variate between x_1 and x_2 as given by the fitted curve, will be the area under (1) from x_1 to x_2 .

(4) Applications of normal distribution. This distribution is applied to problems concerning :

(i) Calculation of errors made by chance in experimental measurements.

(ii) Computation of hit probability of a shot.

(iii) Statistical inference in almost every branch of science.

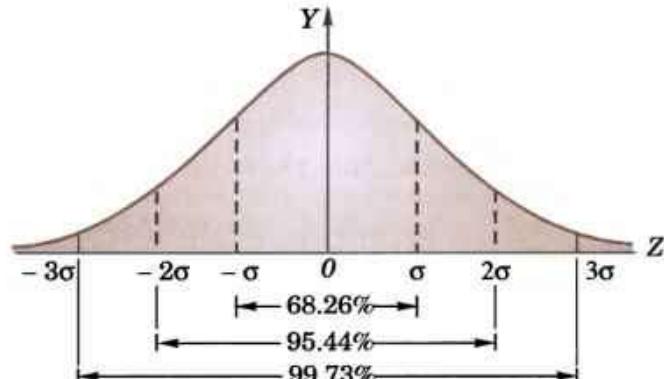


Fig. 26.4

26.17 PROBABLE ERROR

Any lot of articles manufactured to certain specifications is subject to small errors. In fact, measurement of any physical quantity shows slight error. In general, these errors of manufacture or experiment are of random nature and therefore, follow a normal distribution. While quoting a specification of an experimental result, we usually mention the *probable error* (λ). It is such that the probability of an error falling within the limits $\mu - \lambda$ and $\mu + \lambda$ is exactly equal to the chance of an error falling outside these limits, i.e. the chance of an error lying

within $\mu - \lambda$ and $\mu + \lambda$ is $\frac{1}{2}$.

$$\therefore \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\lambda}^{\mu+\lambda} e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{2}$$

$$\text{or } \frac{1}{\sqrt{2\pi}} \int_0^{\lambda/\sigma} e^{-z^2/2} dz = \frac{1}{4} \quad \left[z = \frac{x-\mu}{\sigma} \right]$$

The table V, (Appendix 2) gives $\lambda/\sigma = 0.6745$

Hence the probable error $\lambda = 0.6745\sigma \sim \frac{2}{3}\sigma$.

$$\text{Obs. Quartile deviation} = \frac{1}{2} (Q_3 - Q_1) - \frac{2}{3}\sigma; \text{ Mean deviation} = \frac{4}{5}\sigma$$

$$\therefore Q.D. : M.D. : S.D. = 10 : 12 : 15.$$

[p. 839]

(Madras, 2003)

Example 26.45. X is a normal variate with mean 30 and S.D. 5, find the probabilities that (i) $26 \leq X \leq 40$, (ii) $X \geq 45$ and (iii) $|X - 30| > 5$. (J.N.T.U., 2005)

Solution. We have $\mu = 30$ and $\sigma = 5$

$$\therefore z = \frac{X - \mu}{\sigma} = \frac{X - 30}{5}$$

$$(i) \text{ When } X = 26, z = -0.8; \text{ when } X = 40, z = 2$$

$$\begin{aligned} \therefore P(26 \leq X \leq 40) &= P(-0.8 \leq z \leq 2) \\ &= P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2) \\ &= P(0 \leq z \leq 0.8) + 0.4772 \\ &= 0.2881 + 0.4772 = 0.7653 \end{aligned}$$

[Using Table III]

$$(ii) \text{ When } X = 45, z = 3$$

$$\begin{aligned} \therefore P(X \geq 45) &= P(z \geq 3) = 0.5 - P(0 \leq z \leq 3) \\ &= 0.5 - 0.4986 = 0.0014 \end{aligned}$$

$$\begin{aligned} (iii) \quad P(|X - 30| \leq 5) &= P[25 \leq X \leq 35] \\ &= P(-1 \leq z \leq 1) = 2P(0 \leq z \leq 1) \\ &= 2 \times 0.3413 = 0.6826 \end{aligned}$$

$$\therefore P(|X - 30| > 5) = 1 - P(|X - 30| \leq 5) \\ = 1 - 0.6826 = 0.3174.$$

Example 26.46. A certain number of articles manufactured in one batch were classified into three categories according to a particular characteristic, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation for this batch if 60%, 35% and 5% were found in these categories.

Solution. Let μ be the mean (at $z = 0$) and σ the standard deviation of the normal curve (Fig. 26.5).

Now 60% of the articles have the characteristic below 50, 35% between 50 and 60 and only 5% greater than 60.

Let the area to the left of the ordinate PQ be 60% and that between the ordinates PQ and ST be 35% so that the areas to the left of PQ ($z = z_1$) and ST ($z = z_2$) are 0.6 and 0.95 respectively, i.e., the area $OPQR = 0.6 - 0.5 = 0.1$ and the area $OSTR = 0.45$.

\therefore area corresponding to $z_1 \left(= \frac{50 - \mu}{\sigma} \right) = 0.1$

and that corresponding to $z_2 \left(= \frac{60 - \mu}{\sigma} \right) = 0.45$

From the table III, we have

$$(50 - \mu)/\sigma = 0.2533 \quad \text{and} \quad (60 - \mu)/\sigma = 1.645$$

whence $\sigma = 7.543$ and $\mu = 48.092$.

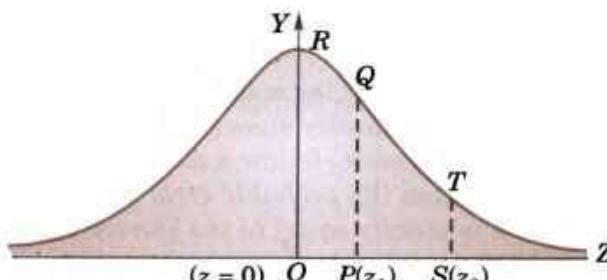


Fig. 26.5

Example 26.47. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and S.D. of the distribution. (V.T.U., 2009; S.V.T.U., 2008; Kurukshetra, 2007 S)

Solution. Let \bar{x} be the mean and σ the S.D. 31% of the items are under 45 means area to the left of the ordinate $x = 45$. (Fig. 26.6)

$$\text{When } x = 45, \text{ let } z = z_1 \text{ so that } z_1 = \frac{45 - \bar{x}}{\sigma} \quad \dots(i)$$

$$\therefore \int_{-\infty}^{z_1} \phi(z) dz = 0.31 \quad \text{or} \quad \int_{-\infty}^0 \phi(z) dz - \int_{z_1}^0 \phi(z) dz = 0.31$$

$$\text{Hence } \int_{z_1}^0 \phi(z) dz = \int_{-\infty}^0 \phi(z) dz - 0.31 = 0.5 - 0.31 = 0.19$$

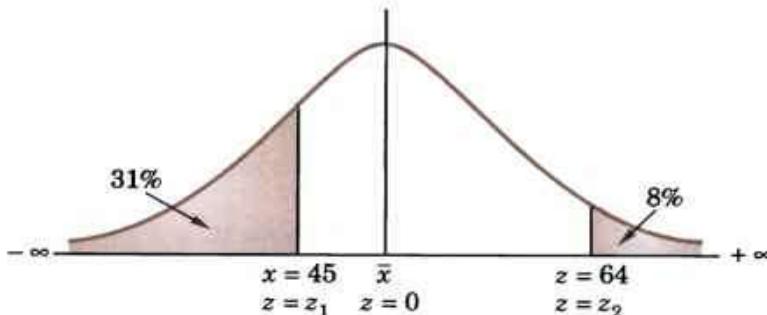


Fig. 26.6

From table III, $z_1 = -0.5$...(ii)

When $x = 64$, let $z = z_2$ so that $z_2 = (64 - \bar{x})/\sigma$...(iii)

$$\therefore \int_{z_2}^{\infty} \phi(z) dz = 0.08 \quad \text{or} \quad \int_0^{\infty} \phi(z) dz - \int_0^{z_2} \phi(z) dz = 0.08$$

$$\text{Hence } \int_0^{z_2} \phi(z) dz = \int_0^{\infty} \phi(z) dz - 0.08 = 0.5 - 0.08 = 0.42$$

From table III, $z_2 = 1.4$...(iv)

From (i) and (ii), $45 - \bar{x} = -0.5\sigma$

From (iii) and (iv), $64 - \bar{x} = 1.4\sigma$

Solving these equations, we get $\bar{x} = 50$ and $\sigma = 10$.

Example 26.48. In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for

(a) more than 2150 hours, (b) less than 1950 hours and

(c) more than 1920 hours and but less than 2160 hours.

(Bhopal, 2008 S; U.P.T.U., 2008)

Solution. Here $\mu = 2040$ hours and $\sigma = 60$ hours.

(a) For $x = 2150$, $z = \frac{x - \mu}{\sigma} = 1.833$.

\therefore area against $z = 1.83$ in the table III = 0.4664.

We, however, require the area to the right of the ordinate at $z = 1.83$. This area = $0.5 - 0.4664 = 0.0336$.
Thus the number of bulbs expected to burn for more than 2150 hours
 $= 0.0336 \times 2000 = 67$ approximately.

$$(b) \text{ For } x = 1950, z = \frac{x - \mu}{\sigma} = -1.5$$

The area required in this case is to the left of $z = -1.33$

$$\begin{aligned} \text{i.e.,} \\ &= 0.5 - 0.4082 \text{ (table value for } z = 1.33) \\ &= 0.0918. \end{aligned}$$

\therefore the number of bulbs expected to burn for less than 1950 hours
 $= 0.0918 \times 2000 = 184$ approximately.

$$(c) \text{ When } x = 1920, z = \frac{1920 - 2040}{60} = -2$$

$$\text{When } x = 2160, z = \frac{2160 - 2040}{60} = 2.$$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area from the table for $z = 2$, i.e., $= 2 \times 0.4772 = 0.9544$.

Thus the required number of bulbs $= 0.9544 \times 2000 = 1909$ nearly.

Example 26.49. If the probability of committing an error of magnitude x is given by

$$y = \frac{h}{\sqrt{\pi}} e^{-hx^2};$$

compute the probable error from the following data :

$$\begin{aligned} m_1 &= 1.305; & m_2 &= 1.301; & m_3 &= 1.295; & m_4 &= 1.286; \\ m_5 &= 1.318; & m_6 &= 1.321; & m_7 &= 1.283; & m_8 &= 1.289; \\ m_9 &= 1.300; & m_{10} &= 1.286. \end{aligned}$$

(Kurukshetra, 2005)

Solution. From the given data which is normally distributed, we have

$$\text{mean} = \frac{1}{10} \sum m_i = \frac{12.984}{10} = 1.2984$$

and

$$\begin{aligned} \sigma^2 &= \frac{1}{10} \sum (m_i - \text{mean})^2 \\ &= \frac{1}{10} [(0.007)^2 + (0.003)^2 + (0.003)^2 + (0.012)^2 + (0.02)^2 + (0.023)^2 \\ &\quad + (0.015)^2 + (0.009)^2 + (0.002)^2 + (0.012)^2] \\ &= 0.0001594 \text{ whence } \sigma = 0.0126. \end{aligned}$$

$$\therefore \text{probable error} = \frac{2}{3} \sigma = 0.0084 \text{ approx.}$$

Example 26.50. Fit a normal curve to the following distribution.

$x:$	2	4	6	8	10
$f:$	1	4	6	4	1

(V.T.U., 2001)

Solution.

$$\text{Mean} = \frac{\Sigma f x}{\Sigma f} = \frac{2 + 16 + 36 + 32 + 10}{16} = 6$$

$$\text{S.D.} = \sqrt{\left[\frac{\Sigma f x^2}{\Sigma f} - \left(\frac{\Sigma f x}{\Sigma f} \right)^2 \right]} = \sqrt{(40 - 36)} = 2$$

Taking $\mu = 6$, $\sigma = 2$ and $N = 16$, the equation of the normal curve is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \text{ or } y = \frac{1}{2\sqrt{2\pi}} e^{-(x-6)^2/8} \quad \dots(i)$$

Area under (i) in (x_1, x_2) or (z_1, z_2)

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-z^2/2} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz \quad \text{where } z = \frac{x-6}{2}$$

To evaluate these integrals, we refer to table III.

Calculations :

Mid x	(x_p, x_q)	(z_p, z_q)	Area under (i) in (z_p, z_q)	Expected frequency
2	(1, 3)	(-2.5, -1.5)	0.4938 - 0.4332	$16 \times 0.606 = 0.97$
4	(3, 5)	(-1.5, -0.5)	0.4332 - 0.1915	$16 \times 0.2417 = 3.9$
6	(5, 7)	(-0.5, 0.5)	0.1915 + 0.1915	$16 \times 0.383 = 6.1$
8	(7, 9)	(0.5, 1.5)	0.4332 - 0.1915	$16 \times 0.2417 = 3.9$
10	(9, 11)	(1.5, 2.5)	0.4938 - 0.4332	$16 \times 0.606 = 0.97$

Hence the expected (theoretical) frequencies corrected to nearest integer are 1, 4, 6, 4, 1 which agree with the observed frequencies. This shows that the normal curve (i) is a proper fit to the given distribution.

PROBLEMS 26.7

- Show that the standard deviation for a normal distribution is approximately 25% more than the mean deviation.
- For a normally distributed variate with mean 1 and S.D. 3, find the probabilities that
(i) $3.43 \leq x \leq 6.19$ (ii) $-1.43 \leq x \leq 6.19$.
- If z is normally distributed with mean 0 and variance 1, find
(i) $P_z \{z \leq -1.64\}$; (ii) z_1 if $P_z \{z \geq z_1\} = 0.84$.
- In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to be normal). (Kottayam, 2005)
- A manufacturer of air-mail envelopes knows from experience that the weight of the envelopes is normally distributed with mean 1.95 gm and standard deviation 0.05 gm. About how many envelopes weighing (i) 2 gm or more; (ii) 2.05 gm or more can be expected in a given packet of 100 envelopes.
- The mean height of 500 students is 151 cm. and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students' heights lie between 120 and 155 cm. (Burdwan, 2003)
- The mean and standard deviation of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.
- In an examination taken by 500 candidates, the average and the standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately
 - how many will pass, if 50% is fixed as a minimum?
 - what should be the minimum if 350 candidates are to pass?
 - how many have scored marks above 60%?

- The mean inside diameter of a sample of 200 washers produced by a machine is 5.02 mm and the standard deviation is 0.05 mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.

[Hint. 4.96 in standard units = $(4.96 - 5.02)/0.05 = -1.2$

5.08 in standard units = $(5.08 - 5.02)/0.05 = 1.2$

Proportion of non-defective washers = 2 (area between $z = 0$ and $z = 1.2$)

= 0.7698 or 77% nearly.

∴ percentage of defective washers = $100 - 77 = 23\%$.]

- Assuming that the diameters of 1000 brass plugs taken consecutively from a machine, form a normal distribution with mean 0.7515 cm. and standard deviation 0.0020 cm., how many of the plugs are likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.? (Bhopal, 2002)

11. It is given that the age of thermostats of a particular make follow the normal law with mean 5 years and S.D. 2 years. 1000 units are sold out every month. How many of them will have to be replaced at the end of the second year.
12. The income of a group of 10,000 persons was found to be normally distributed with mean Rs. 750 p.m., and standard deviation of Rs. 50. Show that, of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100. (U.P.T.U., 2004 S)
13. Find the equation of the best fitting normal curve to the following distribution :

$x :$	0	1	2	3	4	5
$y :$	13	23	34	15	11	4

14. Obtain the equation of the normal probability curve that may be fitted to the following data :

Variable :	4	6	8	10	12	14	16	18	20	22	24
Frequency:	1	7	15	22	35	43	38	20	13	5	1

15. A factory turns out an article by mass production and it is found that 10% of the product is rejected. Find the S.D. of the number of rejects and the equation to the normal curve to represent the number of rejects.

[Hint. $p = 0.1, q = 0.9, n = 100$.

\therefore binomial distribution of rejects gives mean $= np = 10$, S.D. $= \sqrt{(npq)} = 3$

If this binomial distribution is approximated by a normal distribution, then the equation to the normal curve is

$$y = \frac{100}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where } \mu = 10, \sigma = 3.$$

16. Given that the probability of committing an error of magnitude x is

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \text{ show that the probable error is } 0.4769/h.$$

26.18 NORMAL APPROXIMATION TO BINOMIAL DISTRIBUTION

If the number of successes in a Binomial distribution range from x_1 to x_2 , then the probability of getting these successes

$$= \sum_{r=x_1}^{x_2} {}^n C_r p^r q^{n-r}$$

As the number of trials increases, the Binomial distribution becomes approximated to the Normal distribution. The mean np and the variance npq of the binomial distribution will be quite close to the mean and standard deviation of the approximated normal distribution. Thus for n sufficiently large (≥ 30), the binomial distribution with probability of success p , is approximated by the normal distribution with $\mu = np$, $\sigma = \sqrt{npq}$.

We must however, be careful to get the correct values of z . For any success x , real class interval is $(x - 1/2, x + 1/2)$. Hence

$$z_1 = \frac{x_1 - \frac{1}{2} - \mu}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}; z_2 = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$

so that $P(x_1 < x < x_2) = P(z_1 < z < z_2) = \int_{z_1}^{z_2} \phi(z) dz$ which can be calculated by using table III–Appendix 2.

Example 26.51. In a referendum 60% of voters voted in favour. A random sample of 200 voters was selected. What is the probability that in the sample

- (a) more than 130 voted in favour ?
- (b) between 105 and 130 inclusive voted in favour ?
- (c) 120 voted in favour ?

Solution. Here $n = 200$, $p = 0.6$, $q = 0.4$

$$\therefore \mu = np = 200 \times 0.6 = 120; \sigma = \sqrt{npq} = \sqrt{48} = 6.928$$

$$(a) P(x > 130) = P(x > 130.5) = P\left(x > \frac{130.5 - 120}{\sqrt{48}}\right) = P(z > 1.516) = 0.0648$$

$$(b) P(105 < x < 130) = P(105.5 < x < 129.5)$$

$$= P\left(\frac{105.5 - 120}{\sqrt{48}} < z < \frac{129.5 - 120}{\sqrt{48}}\right) = P(-2.09 < z < 1.37) = 0.8964$$

$$(c) P(x = 120) = P(119.5 < x < 120.5) \\ = P(-0.072 < z < 0.072) = 0.0575.$$

PROBLEMS 26.8

1. A pair of unbiased dice are rolled 180 times and their score recorded. Find
(a) $P(x \leq 20)$, (b) $P(20 \leq x \leq 40)$, (c) $P(20 < x \leq 30)$.
2. A marksman has a probability of 0.9 of hitting a target on a single shot. If the marksman has 40 shots, what is the probability that he hits the target (a) at least 35 times; (b) between 34 and 36 times; (c) 37 times.
3. A certain drug is effective in 72% of cases. Given 2000 people are treated with the drug, what is the probability that it will be effective for (a) at least 1400 patients, (b) less than 1390 patients, (c) 1420 patients.

26.19 SOME OTHER DISTRIBUTIONS

Discrete distributions

(1) Geometric distribution. If p be the probability of success and k be the numbers of failures preceding the first success then this distribution is

$$P(k) = q^k p, \quad k = 0, 1, 2, \dots, q = 1 - p.$$

$$\text{Obviously } \sum_{k=0}^{\infty} P(k) = p \sum_{k=0}^{\infty} q^k = p \cdot \frac{1}{1-q} = 1.$$

It can easily be shown that mean = q/p , and variance = q/p^2 .

(2) Negative binomial distribution. This distribution gives the probability that the event occurs for the k th time on the r th trial ($r \geq k$). If p be the probability of occurrence of an event then

$$P(k, r) = {}^{r-1}C_{k-1} p^k q^{r-k}.$$

It contains two parameters p and k . If $k = 1$, the Negative binomial distribution reduces to the geometric distribution.

(3) Hypergeometric distribution. Suppose a bag contains m white and n black balls. If r balls are drawn one at a time (*with replacement*), then the probability that k of them will be white is

$$P(k) = {}^m C_k {}^n C_{r-k} / {}^{m+n} C_r, \quad k = 0, 1, \dots, r, r \leq m, r \leq n.$$

This distribution is known as *Hypergeometric distribution*.

$$\text{For } \sum_{k=0}^r P(k) = 1, \text{ since } \sum_{k=0}^r {}^m C_k {}^n C_{r-k} = {}^{m+n} C_r$$

This can be proved by equating the coefficient of t^r in

$$(1+t)^m (t+1)^n = (1+t)^{m+n}$$

Continuous distributions

(4) Uniform (or Rectangular) distribution. A random variable X is said to be uniformly distributed over the interval $-\infty < a < b < \infty$, if its density is given by

$$f(x) = \frac{1}{b-a}, \quad a < x < b \quad \dots(i)$$

The distribution given by (i) is called a *uniform distribution*. In this distribution, X takes the values with the same probability.

$$\text{Its mean } \mu = \int_a^b x \cdot f(x) dx = \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b = \frac{a+b}{2}.$$

$$\text{and variance } \sigma^2 = \mu'_2 - (\mu)^2 = \int_a^b x^2 \cdot \frac{1}{b-a} dx - \left(\frac{a+b}{2} \right)^2 = \frac{1}{12} (b-a)^2.$$

(5) Gamma distribution. This continuous distribution is given by $f(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}$ for all $x \geq 0$, where r and λ (both > 0) are called the parameters of the *gamma distribution*. Its mean = r/λ and variance = r/λ^2 .

Gamma distribution tends to normal distribution as the parameter r tends to infinity.

(6) Exponential distribution. This distribution is a special case of gamma distribution when $r = 1$ so that $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is a parameter.

It can be seen that mean = $1/\lambda$, standard deviation = $1/\lambda$.

This distribution plays an important role in the reliability and queuing theory.

(7) Weibull distribution*. This distribution is given by

$$f(x) = \frac{\alpha}{c} x^{\alpha-1} e^{-x^{\alpha}/c}, x > 0, c > 0$$

where c is a scale parameter and α a shape parameter.

Initially this distribution was used to describe experimentally observed variation in the fatigue resistance of steel and its elastic limits. But it has also been employed to study the variation of length of service of radio service equipment.

Example 26.52. A die is cast until 6 appears. What is the probability that it must be cast more than 5 times?

Solution. Here probability of getting 6 is $p = \frac{1}{6}$. Then $q = \frac{5}{6}$.

If X is the number of tosses required for the first success, then

$$P(X = x) = q^{x-1} p \text{ for } x = 1, 2, 3, \dots$$

$$\therefore \text{required probability} = P(X > 5) = 1 - P(X \leq 5)$$

$$= 1 - \sum_{x=1}^5 \left(\frac{5}{6}\right)^{x-1} \cdot \left(\frac{1}{6}\right) = 1 - \frac{1}{6} \left\{ 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^4 \right\} = \left(\frac{5}{6}\right)^5.$$

Example 26.53. A random variable X has a uniform distribution over $(-3, 3)$, find k for which

$$P(X > k) = \frac{1}{3}.$$

Also evaluate $P(X < 2)$ and $P(|X - 2| < 2)$.

Solution. (i) Density of $X = f(x) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$

$$\begin{aligned} \therefore P(X > k) &= 1 - P(X \leq k) = 1 - \int_{-3}^k f(x) dx \\ &= 1 - \frac{1}{6} \int_{-3}^k dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3} \end{aligned} \quad (\text{given})$$

This gives $k = 1$.

$$(ii) \quad P(X < 2) = \int_{-3}^2 f(x) dx = \frac{1}{6} \int_{-3}^2 dx = \frac{5}{6}.$$

$$(iii) \quad P(|X - 2| < 2) = P[2 - 2 < X < 2 + 2] = P[0 < x < 4] = \int_0^3 f(x) dx = \frac{1}{6} \int_0^3 dx = \frac{1}{2}.$$

PROBLEMS 26.9

- Show that the mode of the geometric distribution $P(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \dots$, is unity.
- Show that for the rectangular distribution $f(x) = 1$, $0 \leq x \leq 1$,

$$\text{mean} = \frac{1}{2}, \text{variance} = \frac{1}{12} \text{ and mean deviation} = \frac{1}{4}.$$

* It was first used by Swedish scientist Weibull in 1951.

3. Find the mean and variance of the *uniform distribution* given by $f(x) = 1/n$, $x = 1, 2, \dots, n$.
4. Show that for the *exponential distribution*
 $dP = y_0 e^{-x/\sigma}$, $0 \leq x \leq \infty$,
the mean and S.D. are both equal to σ .
5. Find the mean and variance of the *exponential distribution* $f(x) = \frac{1}{b} e^{-(x-a)/b}$, $x > a$. (Mumbai, 2005)
6. Find the moment generating function for the *triangular distribution* given by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x \leq 2. \end{cases}$$
7. Show that for the *Gamma distribution* $f(x) = \frac{e^{-x} x^{l-1}}{\Gamma(l)}$, $0 < x < \infty$, the mean and variance are both equal to l .
8. Find the moment generating function of the *Gamma distribution* $f(x) = \frac{1}{\Gamma(\frac{1}{4})} e^{-x} x^{-\frac{3}{4}}$, $x \geq 0$, at the origin. (J.N.T.U., 2006 ; Madras, 2000 S)

Chebyshev's inequality*. If x is a continuous random variable with mean μ and variance σ^2 , then for any positive real parameter t ,

$$P(|x - \mu| \geq t) \leq \sigma^2/t^2 \text{ or } P(|x - \mu| \leq t) \geq 1 - \sigma^2/t^2.$$

This result is known as *Chebyshev's inequality*. It gives limits to the probability that the value of the variate chosen at random will differ from mean by more than t .

9. For the points on a symmetrical die, prove that *Chebyshev's inequality* gives

$$P(|x - \bar{x}| > 2.5) < 0.478,$$

while the actual probability is zero.

10. For the *Geometrical distribution* $P(x) = 2^{-x}$, $x = 1, 2, 3, \dots$, prove that *Chebyshev's inequality* gives

$$P(|x - 2| < 2) > \frac{1}{2},$$

while the actual probability is $15/16$.

26.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 26.10

Select the correct answer or fill up the blanks in each of the following problems:

- The probability that A happens is $1/3$. The odds against happening of A are
(a) $2 : 1$ (b) $2 : 3$ (c) $3 : 2$ (d) $5 : 2$.
- The odds in favour of an event A are 5 to 4. The probability of success of A is
(a) $4/5$ (b) $5/9$ (c) $4/9$.
- The probability that A passes a test is $2/3$ and the probability that B passes the same test is $3/5$. The probability that only one of them passes is
(a) $2/5$ (b) $4/15$ (c) $2/15$ (d) $7/15$.
- A buys a lottery ticket in which the chance of winning is $1/10$; B has a ticket in which his chance of winning is $1/20$. The chance that atleast one of them wins is
(a) $1/200$ (b) $29/200$ (c) $30/200$ (d) $170/200$.
- The probability that a non-leap year should have 53 Tuesdays is ...
- The probability of getting 2 or 3 or 4 from a throw of single dice is ...
- The mean of the Binomial distribution with n observations and probability of success p , is
(a) pq (b) np (c) \sqrt{np} (d) \sqrt{pq} .
- If the mean of a Poisson distribution is m , then S.D. of this distribution is
(a) m^2 (b) \sqrt{m} (c) m (d) none of these.

* See footnote on page 571.

9. The S.D. of the Binomial distribution is
 (a) \sqrt{npq} (b) \sqrt{np} (c) npq (d) pq .
10. In a Poisson distribution if $2P(x=1) = P(x=2)$, then the variance is
 (a) 0 (b) 1 (c) 4 (d) 2.
11. If the probability of hitting a target by one shot be $p = 0.8$, then the probability that out of ten shots, seven will hit the target is ...
12. For a Poisson variate $x : P(x=1) = P(x=2)$, then the mean of x is ...
13. If $P(A) = 0.35$, $P(B) = 0.73$ and $P(A \cap B) = 0.14$, then $P(A \cap B') = \dots$
14. If A and B are independent, $P(B) = 0.14$ and $P(A/B) = 0.24$, then $P(A) = \dots$
15. The probability distribution of the number of heads, when two coins are tossed, is ...
16. The multiplication law of probability states that ...
17. The area under the standard normal curve which lies between $z = 0.90$ and $z = -1.85$ is ...
 [Given $P(0 < z < 1.85) = 0.4678$, $P(0 < z < 0.9) = 0.3159$]
18. The mean, median and mode of a normal distribution are ...
19. The mean and variance of a Poisson distribution are ...
20. If A and B are two mutually exclusive events, then $P(A \cup B) = \dots$
21. For a normal distribution $\beta_1 = \dots$ and $\beta_2 = \dots$
22. The number of ways in which five people can be lined up to get on a bus are ...
23. A shipment of 10 television sets contains 3 defective sets. The number of ways in which one can purchase 4 of these sets and receive 2 defective sets are ...
24. The probability of getting a total of 5 when a pair of dice is tossed is ...
25. If $P(B) = 0.81$ and $P(A \cap B) = 0.18$, then $P(A/B) = \dots$
26. If two unbiased dice are thrown simultaneously, the probability that the sum of the numbers on them is at least 10, is
27. If X is a Poisson variate such that $P(X=2) = P(X=3)$, then $P(X=0) = \dots$
28. An unbiased die is tossed twice, then the probability of obtaining the sum 6, is ...
29. The variance of Poisson distribution with parameter $\lambda = 2$ is ...
30. The distribution in which mean, median, mode are equal is ...
31. For the Poisson variate, probability of getting at least one success is ...
32. Total number of events in rolling of an ideal die is ...
33. If X be normal with mean 10 and variance 4, then $P(X < 11) = \dots$
34. If X is a binomial variate with parameters n and p , then its m.g.f. about the origin is ...
35. In a normal distribution, mean deviation : standard deviation = ...
36. If A and B are independent and $P(A) = 1/2$, $P(B) = 1/3$ then $P(A \cap B) = \dots$
37. If X is the random variable representing the outcome of the roll of an ideal die, then $E(X) = \dots$
38. If X is a binomial variate with $p = 1/5$ for the experiment of 50 trials, then the standard deviation is ...
39. The area under the whole normal curve is ...
40. Given $X = B(n, p)$, then the conditions under which X tends to a Poisson distribution, are ...
41. If A and B are mutually exclusive events then $P(A \cup B) = \dots$
42. The probability of selecting x white balls from a bag containing y white and z red balls is ...
43. The mean of the binomial distribution is ...
44. If A and B are mutually exclusive events, $P(A) = 0.29$, $P(B) = 0.43$, then $P(A \cup B) = \dots$ and $P(A \cap B') = \dots$
45. If the mean and variance of a binomial variate are 12 and 4, then the distribution is ...
46. If x is a Poisson variable such that $P(x=2) = 9 P(x=4) + 90 P(x=6)$, then the mean = ...
47. μ_r' the r th moment about the origin in terms of the m.g.f. is ...
48. The chance of throwing 7 in a single throw with two dice is ...
49. If A and B are any two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$, then $P(A/B) = \dots$
50. In the roll of an ideal die, the probability of getting a prime number is ...
51. If A and B are mutually exclusive events, $P(A \cup B) = 0.6$, $P(B) = 0.4$, then $P(A) = \dots$
52. The probability that a leap year should have 53 Sundays is ...
53. The probability density function of a binomial distribution is ...
54. The probable error is ... times S.D. approximately.
55. To fit a normal distribution, the parameters required are ...

Sampling & Inference

1. Introduction.
2. Sampling distribution ; Standard error.
3. Testing of Hypothesis ; Errors.
4. Level of significance ; Tests of significance.
5. Confidence limits.
6. Simple sampling of attributes.
7. Test of significance for large samples.
8. Comparison of large samples.
9. Sampling of variables.
10. Central limit theorem.
11. Confidence limits for unknown means.
12. Test of significance for means of two large samples.
13. Sampling of variables—small samples.
14. Student's t -distribution.
15. Significance test of a sample mean.
16. Significance test of difference between sample means.
17. Chi-square test.
18. Goodness of fit.
19. F-distribution.
20. Fisher's z-distribution.
21. Objective Type of Questions.

27.1 (1) INTRODUCTION

We know that a small section selected from the population is called a *sample* and the process of drawing a sample is called *sampling*. It is essential that a sample must be a *random* selection so that each member of the population has the same chance of being included in the sample. Thus the fundamental assumption underlying theory of sampling is *Random sampling*.

A special case of random sampling in which each event has the same probability p of success and the chance of success of different events are independent whether previous trials have been made or not, is known as *simple sampling*.

The statistical constants of the population such as mean (μ), standard deviation (σ) etc. are called the *parameters*. Similarly, constants for the *sample* drawn from the given population *i.e.*, mean (\bar{x}), standard deviation (S) etc. are called the *statistic*. The population parameters are in general, not known and their estimates given by the corresponding sample statistic are used. We use the Greek letters to denote the population parameters and Roman letters for sample statistic.

(2) Objectives of sampling. Sampling aims at gathering the maximum information about the population with the minimum effort, cost and time. The object of sampling studies is to obtain the best possible values of the parameters under specific conditions. Sampling determines the reliability of these estimates. The logic of the sampling theory is the logic of induction in which we pass from a particular (sample) to general (population). Such a generalisation from sample to population is called **Statistical Inference**.

27.2 SAMPLING DISTRIBUTION

Consider all possible samples of size n which can be drawn from a given population at random. For each sample, we can compute the mean. The means of the samples will not be identical. If we group these different means according to their frequencies, the frequency distribution so formed is known as *sampling distribution of the mean*. Similarly we can have *sampling distribution of the standard deviation* etc.

While drawing each sample, we put back the previous sample so that the parent population remains the same. This is called *sampling with replacement* and all the subsequent formulae will pertain to sampling with replacement.

(2) Standard error. The standard deviation of the sampling distribution is called the *standard error* (*S.E.*). Thus the standard error of the sampling distribution of means is called standard error of means. The standard error is used to assess the difference between the expected and observed values. The reciprocal of the standard error is called *precision*.

If $n \geq 30$, a sample is called *large* otherwise *small*. The sampling distribution of large samples is assumed to be normal.

27.3 (1) TESTING A HYPOTHESIS*

To reach decisions about populations on the basis of sample information, we make certain assumptions about the populations involved. Such assumptions, which may or may not be true, are called *statistical hypothesis*. By testing a hypothesis is meant a process for deciding whether to accept or reject the hypothesis. The method consists in assuming the hypothesis as correct and then computing the probability of getting the observed sample. If this probability is less than a certain preassigned value the hypothesis is rejected.

(2) Errors. If a hypothesis is rejected while it should have been accepted, we say that a *Type I error* has been committed. On the other hand, if a hypothesis is accepted while it should have been rejected, we say that a *Type II error* has been made. The statistical testing of hypothesis aims at limiting the Type I error to a preassigned value (say : 5% or 1%) and to minimize the Type II error. The only way to reduce both types of errors is to increase the sample size, if possible.

(3) Null hypothesis. The hypothesis formulated for the sake of rejecting it, under the assumption that it is true, is called the *null hypothesis* and is denoted by H_0 . To test whether one procedure is better than another, we assume that there is no difference between the procedures. Similarly to test whether there is a relationship between two variates, we take H_0 that there is no relationship. By accepting a null hypothesis, we mean that on the basis of the statistic calculated from the sample, we do not reject the hypothesis. It however, does not imply that the hypothesis is proved to be true. Nor its rejection implies that it is disproved.

27.4 (1) LEVEL OF SIGNIFICANCE

The probability level below which we reject the hypothesis is known as the *level of significance*. The region in which a sample value falling is rejected, is known as the *critical region*. We generally take two critical regions which cover 5% and 1% areas of the normal curve. The shaded portion in the figure corresponds to 5% level of significance. Thus the *probability of the value of the variate falling in the critical region is the level of significance*.

Depending on the nature of the problem, we use a *single-tail test* or *double-tail test* to estimate the significance of a result. In a double-tail test, the areas of both the tails of the curve representing the sampling distribution are taken into account whereas in the single tail test, only the area on the right of an ordinate is taken into consideration. For instance, to test whether a coin is biased or not, double-tail test should be used, since a biased coin gives either more number of heads than tails (which corresponds to right tail), or more number of tails than heads (which corresponds to left tail only).

(2) Tests of significance. The procedure which enables us to decide whether to accept or reject the hypothesis is called the *test of significance*. Here we test whether the differences between the sample values and the population values (or the values given by two samples) are so large that they signify evidence against the hypothesis or these differences are so small as to account for fluctuations of sampling.

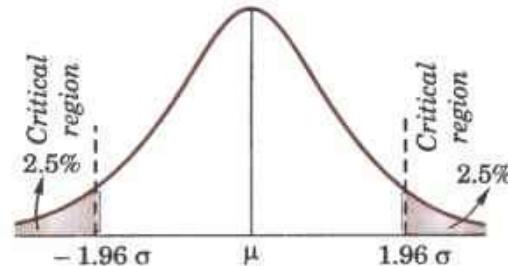


Fig. 27.1

27.5 CONFIDENCE LIMITS**

Suppose that the sampling distribution of a statistic S is normal with mean μ and standard deviation σ . As in the Fig. 27.1 the sample statistic S can be expected to lie in the interval $(\mu - 1.96\sigma, \mu + 1.96\sigma)$ for 95% times i.e., we can be confident of finding μ in the interval $(S - 1.96\sigma, S + 1.96\sigma)$ in 95% cases. Because of this, we call

*The American statistician J. Neyman (1894–1981) and the English statistician E.S. Pearson (1895–1980)—son of Karl Pearson (See footnote p. 843), developed a systematic theory of tests around 1930.

**J. Neyman developed the modern theory and terminology of confidence limits.

$(S - 1.96\sigma, S + 1.96\sigma)$ the 95% confidence interval for estimation of μ . The ends of this interval (i.e. $S \pm 1.96\sigma$) are called 95% confidence limits (or fiducial limits) for S . Similarly $S \pm 2.58\sigma$ are 99% confidence limits. The numbers 1.96, 2.58 etc. are called confidence coefficients. The values of confidence coefficients corresponding to various levels of significance can be found from the normal curve area table VI – Appendix 2.

27.6 SIMPLE SAMPLING OF ATTRIBUTES

The sampling of attributes may be regarded as the selection of samples from a population whose members possess the attribute K or not K . The presence of K may be called a success and its absence a failure.

Suppose we draw a simple sample of n items. Clearly it is same as a series of n independent trials with the same probability p of success. The probabilities of 0, 1, 2, ..., n successes are the terms in the binomial expansion of $(q + p)^n$ where $q = 1 - p$.

We know that the mean of this distribution is np and standard deviation is $\sqrt{(npq)}$ i.e., the expected value of success in a sample of size n is np and the standard error is $\sqrt{(npq)}$.

If we consider the proportion of successes, then

(i) mean proportion of successes = $np/n = p$.

(ii) standard error of the proportion of successes

$$= \sqrt{n \cdot \frac{p}{n} \cdot \frac{q}{n}} = \sqrt{\left(\frac{pq}{n}\right)}$$

and (iii) precision of the proportion of successes = $\sqrt{(n/pq)}$, which varies as \sqrt{n} , since p and q are constants.

27.7 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

We know that the binomial distribution tends to normal for large n . Suppose we wish to test the hypothesis that the probability of success in such trial is p . Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of number of successes are np and $\sqrt{(npq)}$ respectively.

For a normal distribution, only 5% of the members lie outside $\mu \pm 1.96\sigma$ while only 1% of the members lie outside $\mu \pm 2.58\sigma$.

If x be the observed number of successes in the sample and z is the standard normal variate then $z = (x - \mu)/\sigma$.

Thus we have the following test of significance :

(i) If $|z| < 1.96$, difference between the observed and expected number of successes is not significant.

(ii) If $|z| > 1.96$, difference is significant at 5% level of significance.

(iii) If $|z| > 2.58$, difference is significant at 1% level of significance.

Example 27.1. A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased at 5% level of significance. (V.T.U., 2007)

Solution. Suppose the coin is unbiased.

Then the probability of getting the head in a toss = $\frac{1}{2}$

\therefore expected number of successes = $\frac{1}{2} \times 400 = 200$

and the observed value of successes = 216

Thus the excess of observed value over expected value = $216 - 200 = 16$

Also S.D. of simple sampling = $\sqrt{npq} = \sqrt{\left(400 \times \frac{1}{2} \times \frac{1}{2}\right)} = 10$

Hence $z = \frac{x - np}{\sqrt{npq}} = \frac{16}{10} = 1.6$

As $z < 1.96$, the hypothesis is accepted at 5% level of significance i.e., we conclude that the coin is unbiased at 5% level of significance.

Example 27.2. A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate an unbiased die? (V.T.U., 2010)

Solution. Suppose the die is unbiased.

Then the probability of throwing 5 or 6 with one die = $\frac{1}{3}$

The expected number of successes = $\frac{1}{3} \times 9000 = 3000$

and the observed value of successes = 3240

Thus the excess of observed value over expected value $3240 - 3000 = 240$

Also S.D. of simple sampling = $\sqrt{npq} = \sqrt{\left(9000 \times \frac{1}{3} \times \frac{2}{3}\right)} = 44.72$

Hence $z = \frac{x - np}{\sqrt{(npq)}} = \frac{240}{44.72} = 5.4$ nearly.

As $z > 2.58$, the hypothesis has to be rejected at 1% level of significance and we conclude that the die is biased.

Example 27.3. In a locality containing 18000 families, a sample of 840 families was selected at random. Of these 840 families, 206 families were found to have a monthly income of ₹ 250 or less. It is desired to estimate how many out of 18,000 families have a monthly income of ₹ 250 or less. Within what limits would you place your estimate?

Solution. Here $p = \frac{206}{840} = \frac{103}{420}$ and $q = \frac{317}{420}$

∴ standard error of the population of families having a monthly income of ₹ 250 or less

$$= \sqrt{\left(\frac{pq}{n}\right)} = \sqrt{\left(\frac{103}{420} \times \frac{317}{420} \times \frac{1}{840}\right)} = .015 = 1.5\%$$

Hence taking $\frac{103}{420}$ (or 24.5%) to be the estimate of families having a monthly income of ₹ 250 or less in the locality, the limits are $(24.5 \pm 3 \times 1.5)\%$ i.e., 20% and 29% approximately.

27.8 COMPARISON OF LARGE SAMPLES

Two large samples of sizes n_1, n_2 are taken from two populations giving proportions of attributes A's as p_1, p_2 respectively.

(a) On the hypothesis that the populations are similar as regards the attribute A, we combine the two samples to find an estimate of the common value of proportion of A's in the populations which is given by

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

If e_1, e_2 be the standard errors in the two samples then

$$e_1^2 = \frac{pq}{n_1} \text{ and } e_2^2 = \frac{pq}{n_2}$$

If e be the standard error of the difference between p_1 and p_2 , then

$$e^2 = e_1^2 + e_2^2 = pq \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \quad \therefore z = \frac{p_1 - p_2}{e}$$

If $z > 3$, the difference between p_1 and p_2 is real one.

If $z < 2$, the difference may be due to fluctuations of simple sampling.

But if z lies between 2 and 3, then the difference is significant at 5% level of significance.

(b) If the proportions of A's are not the same in the two populations from which the samples are drawn, but p_1 and p_2 are the true values of proportions then S.E. e of the difference $p_1 - p_2$ is given by

$$e^2 = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

If $z = \frac{p_1 - p_2}{e} < 3$, the difference could have arisen due to fluctuations of simple sampling.

Example 27.4. In a city A 20% of a random sample of 900 school boys had a certain slight physical defect. In another city B, 18.5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant? (V.T.U., 2003 S)

Solution. We have $n_1 = 900, n_2 = 1600$

and $p_1 = \frac{20}{100} = \frac{1}{5}, p_2 = \frac{18.5}{100}$

$$\therefore p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{180 + 296}{900 + 1600} = 0.19$$

and $q = 1 - 0.19 = 0.81$

Thus $e^2 = pq \left(\frac{1}{n_1} + \frac{1}{n_2} \right) = 0.19 \times 0.81 \left(\frac{1}{900} + \frac{1}{1600} \right) = 0.0017$

giving $e = 0.04$ nearly.

Also $p_1 - p_2 = \frac{1.5}{100} = 0.015 \quad \therefore z = \frac{p_1 - p_2}{e} = \frac{.015}{.04} = 0.37$

As $z < 1$, the difference between the proportions is not significant.

Example 27.5. In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations?

(Coimbatore, 2001)

Solution. Here $p_1 = 0.3, p_2 = 0.25$ so that $p_1 - p_2 = 0.05$.

$$\therefore e^2 = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2} = \frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}$$

so that $e = 0.0195$

$$\therefore z = \frac{p_1 - p_2}{e} = \frac{0.05}{0.0195} = 2.5 \text{ nearly}$$

Hence it is unlikely that the real difference will be hidden.

PROBLEMS 27.1

- A die is tossed 960 times and it falls with 5 upwards 184 times. Is the die biased? (V.T.U., 2006)
- 12 dice are thrown 3086 times and a throw of 2, 3, 4 is reckoned as a success. Suppose that 19142 throws of 2, 3, 4 have been made out. Do you think that this observed value deviates from the expected value? If so, can the deviation from the expected value be due to fluctuations of simple sampling?
- Balls are drawn from a bag containing equal number of black and white balls, each ball being replaced before drawing another. In 2250 drawings 1018 black and 1232 white balls have been drawn. Do you suspect some bias on the part of the drawer?
- A sample of 1000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district?
- In a group of 50 first cousins there were found to be 27 males and 23 females. Ascertain if the observed proportions are inconsistent with the hypothesis that the sexes should be in equal proportion.
- A random sample of 500 apples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of the bad apples in the consignment as well as the standard error of the estimate. Deduce that the percentage of bad apples in the consignment almost certainly lies between 8.5 and 17.5.
- 400 children are chosen in an industrial town and 150 are found to be under weight. Assuming the conditions of simple sampling, estimate the percentage of children who are under weight in the industrial town and assign limits within which the percentage probably lies?

8. A machine produces 16 imperfect articles in a sample of 500. After machine is overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved? (Rohtak, 2005; Madras, 2003)
9. One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of a total of 200 flights. Is there a significant difference in the two types of aircrafts so far as engine defects are concerned?
10. In a sample of 600 men from a certain city, 450 are found smokers. In another sample of 900 men from another city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the habit of smoking among men? (J.N.T.U., 2003)
11. In a sample of 500 people from a state 280 take tea and rest take coffee. Can we assume that tea and coffee are equally popular in the state at 5% level of significance?

27.9 (1) SAMPLING OF VARIABLES

We now consider sampling of a variable such as weight, height, etc. Each member of the population gives a value of the variable and the population is a frequency distribution of the variable. Thus a random sample of size n from the population is same as selecting n values of the variables from those of the distribution.

(2) Sampling distribution of the mean. If a population is distributed normally with mean μ and standard deviation σ , then the means of all positive random samples of size n , are also distributed normally with mean μ and standard error σ/\sqrt{n} . This result shows how the precision of a sample mean increases as the sample size increases.

27.10 CENTRAL LIMIT THEOREM

This is a very important theorem regarding the distribution of the mean of a sample if the parent population is non-normal and the sample size is large.

If the variable X has a non-normal distribution with mean μ and standard deviation σ , then the limiting distribution of

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \text{ as } n \rightarrow \infty, \text{ is the standard normal distribution (i.e., with mean 0 and unit S.D.)}$$

There is no restriction upon the distribution of X except that it has a finite mean and variance. This theorem holds well for a sample of 25 or more which is regarded as large.

Thus if the population is normal, the sampling distribution of the mean is also normal with mean μ and S.E. σ/\sqrt{n} , while for large samples the same result holds even if the distribution of the population is non-normal. This property is of universal use and throws light on the importance of normal distribution in statistical theory.

27.11 CONFIDENCE LIMITS FOR UNKNOWN MEAN

Let the population from which a random sample of size n is drawn, have mean μ and S.D. σ . If μ is not known, there will be a range of values of μ for which observed mean \bar{x} of the sample is not significant at any assigned level of probability. The relative deviation of \bar{x} from μ is $(\bar{x} - \mu)/\sqrt{\sigma}$.

If \bar{x} is not significant at 5% level of probability, then

$$|(\bar{x} - \mu)/\sqrt{\sigma}| < 1.96 \quad \text{i.e. } \bar{x} - 1.96\sigma/\sqrt{n} < \mu < \bar{x} + 1.96\sigma/\sqrt{n}$$

Thus 95% confidence or fiducial limits for the mean of the population corresponding to given sample are $\bar{x} \pm 1.96\sigma/\sqrt{n}$.

Similarly 99% confidence limits for μ are $\bar{x} \pm 2.58\sigma/\sqrt{n}$.

Example 27.6. A sample of 900 members is found to have a mean of 3.4 cm. Can it be reasonably regarded as a truly random sample from a large population with mean 3.25 cm and S.D. 1.61 cm.

Solution. Here $\bar{x} = 3.4$ cm, $n = 900$, $\mu = 3.25$ and $\sigma = 1.61$ cm

$$\therefore z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.4 - 3.25}{1.61/30} = 2.8$$

As $z > 1.96$, the deviation of the sample mean from the mean of the population is significant at 5% level of significance. Hence it cannot be regarded as a random sample.

Example 27.7. The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative?

Solution. If μ be the mean and σ the S.D. of the distribution, then

$$\mu = \text{S.E. of the sample means} = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$$

$$\text{Also for a sample of size 25, we have } z = \frac{\bar{x} - \mu}{\sigma/\sqrt{25}} = \frac{\bar{x} - \sigma/10}{6/5} = \frac{5\bar{x} - \sigma}{\sigma} - \frac{1}{2}$$

Since \bar{x} is negative, $z < -\frac{1}{2}$.

∴ the probability that a normal variate $z < -\frac{1}{2}$

$$\begin{aligned} &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{1}{2}z^2} dz \\ &= 0.5 - 0.915 = 0.3085, \text{ from the tables.} \end{aligned}$$

Example 27.8. An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0.49 and 0.51. Find the smallest value of n that will ensure this result with 90% confidence.

Solution. S.E. of the proportion of heads = $\sqrt{\frac{pq}{n}} = \sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{n}} = \frac{1}{2\sqrt{n}}$

90% of confidence = 45% or .45 of the total area under the normal curve on each side of the mean.

∴ the corresponding value of $z = 1.645$, from the tables.

Thus $p \mp 1.645\sigma = 0.49 \text{ or } 0.51$.

i.e., $0.5 - 1.645 \cdot \frac{1}{2\sqrt{n}} = 0.49 \text{ and } 0.5 + 1.645 \cdot \frac{1}{2\sqrt{n}} = 0.51$

whence $\frac{1.645}{2\sqrt{n}} = 0.01 \text{ or } \sqrt{n} = \frac{329}{4} \text{ or } n = 6765 \text{ approximately.}$

27.12 TEST OF SIGNIFICANCE FOR MEANS OF TWO LARGE SAMPLES

(a) Suppose two random samples of sizes n_1 and n_2 have been drawn from the same population with S.D. σ . We wish to test whether the difference between the sample means \bar{x}_1 and \bar{x}_2 is significant or is merely due to fluctuations of sampling.

If the samples are independent, then the standard error e of the difference of their means is given by

$$e^2 = e_1^2 + e_2^2$$

where $e_1 = \sigma/\sqrt{n_1}$, $e_2 = \sigma/\sqrt{n_2}$ are the S.E.s of the means of the two samples.

$$\therefore e = \sigma \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2} \right)}. \quad \text{Hence } z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma \sqrt{(1/n_1 + 1/n_2)}}$$

is normally distributed with mean zero and S.D. 1.

Test of significance (n_1, n_2 being large):

If $z > 1.96$, then the difference is significant at 5% level of significance.

If $z > 3$, it is highly probable that either the samples have not been drawn from the same population or the sampling is not simple.

(b) If the samples are known to be drawn from different populations with means μ_1 , μ_2 and standard deviations σ_1 and σ_2 . Then the standard error e of their means is given by

$$e = \sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

Assuming that the two populations have the same mean (i.e., $\mu_1 = \mu_2$), the difference of the means of the samples will be normally distributed with mean zero and S.D. e . Now the same procedure of test of significance is applied.

Example 27.9. The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm. (Madras, 2002)

Solution. We have $\bar{x}_1 = 67.5$, $\bar{x}_2 = 68.0$
 $n_1 = 1000$, $n_2 = 2000$.

On the hypothesis, that the samples are drawn from the same population of S.D. $\sigma = 2.5$, we get

$$\begin{aligned} z &= \frac{\bar{x}_1 - \bar{x}_2}{\sigma \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{67.5 - 68.0}{2.5 \sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}} \\ &= \frac{0.5}{2.5 \times 0.0387} = \frac{0.5}{0.09675} = 5.1 \end{aligned}$$

Hence the difference between the sample means i.e., 5.1 is very much greater than 1.96 and is therefore significant. Thus, the samples cannot be regarded as drawn from the same population.

Example 27.10. A sample of height of 6400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches while a simple sample of heights of 1600 sailors has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers?

Solution. Here $\bar{x}_1 = 67.85$, $\sigma_1 = 2.56$, $n_1 = 6400$
 $\bar{x}_2 = 68.55$, $\sigma_2 = 2.52$, $n_2 = 1600$.

∴ S.E. of the difference of the mean heights is

$$\begin{aligned} e &= \sqrt{\left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right]} = \sqrt{\left[\frac{(2.56)^2}{6400} + \frac{(2.52)^2}{1600}\right]} \\ &= \sqrt{[.001024 + .003969]} = 0.005 \text{ nearly.} \end{aligned}$$

Also difference between the means = $\bar{x}_2 - \bar{x}_1 = 0.7$, which $> 10e$. This is highly significant. Hence the data indicates that the sailors are on the average taller than the soldiers.

PROBLEMS 27.2

- A sample of 400 items is taken from a normal population whose mean is 4 and variance 4. If the sample mean is 4.45, can the samples be regarded as a simple sample?
- To know the mean weights of all 10-year old boys in Delhi, a sample of 225 is taken. The mean weight of the sample is found to be 67 pounds with a S.D. of 12 pounds. Can you draw any inference from it about the mean weight of the population?
- A normal population has a mean 0.1 and a S.D. of 2.1. Find the probability that the mean of simple sample of 900 members will be negative.
- If the mean breaking strength of copper wire is 575 lbs. with a standard deviation of 8.3 lbs., how large a sample must be used in order that there be one chance in 100 that the mean breaking strength of the sample is less than 572 lbs.?

[Hint. $|z| = \left| \frac{\bar{x} - \mu}{\sigma} \sqrt{n} \right| = \frac{3}{8.3} \sqrt{n}$

Also from table IV, $z = 2.33$. Hence $n = 42$ nearly.]

5. A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that sample mean will not differ from the true mean by more than 25% of the S.D. How large a sample should be taken?
6. The density function of a random variable x is $f(x) = ke^{-2x^2 + 10x}$. Find the upper 5% point of the distribution of the means of the random sample of size 25 from the above population.
7. The means of two large samples of 1000 and 2000 members are 168.75 cms. and 170 cms. respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms.
8. If 60 new entrants in a given university are found to have a mean height of 68.60 inches and 50 seniors a mean height of 69.51 inches; is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants? Assume the standard deviation of height to be 2.48 inches.
9. A sample of 100 electric bulbs produced by manufacturer A showed a mean life time of 1190 hours and a standard deviation of 90 hours. A sample of 75 bulbs produced by manufacturer B showed a mean life time of 1230 hours, with a standard deviation of 120 hours. Is there a difference between the mean life time of two brands at a significance level of (i) 0.05 (ii) 0.01.
10. A random sample of 1000 men from North India shows that their mean wage is ₹ 5 per day with a S.D. of ₹ 1.50. A sample of 1500 men from South India gives a mean wage of ₹ 4.50 per day with a standard deviation of ₹ 2. Does the mean rate of wages varies as between the two regions?

27.13 SAMPLING OF VARIABLES—SMALL SAMPLES

In case of large samples, sampling distribution approaches a normal distribution and values of sample statistic are considered best estimates of the parameters in a population. It will no longer be possible to assume that statistics computed from small samples are normally distributed. As such, a new technique has been devised for small samples which involves the concept of 'degrees of freedom' which we explain below.

Number of degrees of freedom is the number of values in a set which may be assigned arbitrarily. For instance, if $x_1 + x_2 + x_3 = 15$ and we assign any values of two of the variables (say : x_1, x_2), then the values of x_3 will be known. The two variables are therefore, free and independent choices for finding the third. Hence these are the degrees of freedom. If there are n observations, the degrees of freedom (d.f.) are $(n - 1)$. In other words, while finding the mean of a small sample, one degree of freedom is used up and $(n - 1)$ d.f. are left to estimate the population variance.

27.14 (1) STUDENT'S t-DISTRIBUTION

Consider a small sample of size n , drawn from a normal population with mean μ and s.d. σ . If \bar{x} and σ_s be the sample mean and s.d., then the statistic, 't' is defined as

$$t = \frac{\bar{x} - \mu}{\sigma} \sqrt{n} \quad \text{or} \quad t = \frac{\bar{x} - \mu}{\sigma_s} \sqrt{(n-1)},$$

where $v = n - 1$ denotes the df. of t . If we calculate t for each sample, we obtain the sampling distribution for t . This distribution known as *Student's t-distribution**, is given by

$$y = \frac{y_0}{(1 + t^2/v)^{(v+1)/2}} \quad \dots(1)$$

where y_0 is constant such that the area under the curve is unity.

(2) Properties of t-distribution.

I. This curve is symmetrical about the line $t = 0$, like the normal curve, since only even powers of t appear in (1). But it is more peaked than the normal curve with the same S.D. The t -curve approaches the horizontal axis less rapidly than the normal curve. Also t -curve attains its maximum value at $t = 0$ so that its mode coincides with the mean. (Fig. 27.2)

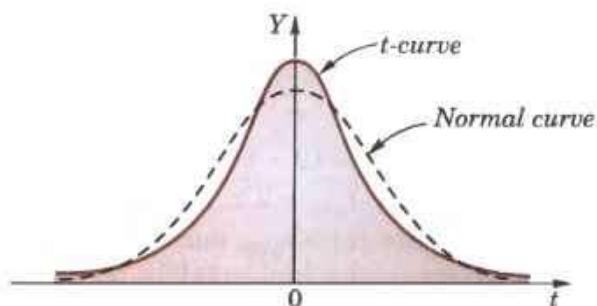


Fig. 27.2

*This distribution was first found by the English statistician W.S. Gosset in 1908 who wrote under the pen-name of 'Student'. R.A. Fisher defined t correctly and found its distribution in 1926.

II. The limiting form of t -distribution when $v \rightarrow \infty$ is given by $y = y_0 e^{-\frac{1}{2}t^2}$ which is a normal curve. This shows that t is normally distributed for large samples.

III. The probability P that the value of t will exceed t_0 is given by

$$P = \int_{t_0}^{\infty} y dx$$

The values of t_0 have been tabulated for various values of P for various values of v from 1 to 30 (Table IV – Appendix 2).

IV. Moments about the mean

All the moments of odd order about the mean are zero, due to its symmetry about the line $t = 0$.

Even order moments about the mean are

$$\mu_2 = \frac{v}{v-2}, \quad \mu_4 = \frac{3v^2}{(v-2)(v-4)}, \dots$$

The t -distribution is often used in tests of hypothesis about the mean when the population standard deviation σ is unknown.

27.15 SIGNIFICANCE TEST OF A SAMPLE MEAN

Given a random small sample $x_1, x_2, x_3, \dots, x_n$ from a normal population, we have to test the hypothesis that mean of the population is μ . For this, we first calculate $t = (\bar{x} - \mu) \sqrt{n}/\sigma_s$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \quad \sigma_s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Then find the value of P for the given df from the table.

If the calculated value of $t > t_{0.05}$, the difference between \bar{x} and μ is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis that μ is the mean of the population.

Example 27.11. A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure : 5, 2, 8, -1, 3, 0, -2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure. (V.T.U., 2007)

Solution. Let us assume that the stimulus administered to all the 12 patients will increase the B.P. Taking the population to be normal with mean $\mu = 0$ and S.D. σ ,

$$\bar{d} = \frac{5 + 2 + 8 - 1 + 3 + 0 - 2 + 1 + 5 + 0 + 4 + 6}{12} = 2.583$$

$$\begin{aligned} \sigma^2 &= \frac{\Sigma d^2}{n} - \bar{d}^2 = \frac{1}{12} [5^2 + 2^2 + 8^2 + (-1)^2 + 3^2 + 0^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2 + 6^2] - (2.583)^2 \\ &= 8.744. \quad \therefore \quad \sigma = 2.9571 \end{aligned}$$

$$\text{Now } t = \frac{\bar{d} - \mu}{\sigma_s} \sqrt{(n-1)} = \frac{2.583 - 0}{2.9571} \sqrt{(12-1)} = 2.897$$

Here $d.f. \gamma = 12 - 1 = 11$.

For $\gamma = 11$, $t_{0.05} = 2.2$, from table IV.

Since the $|t| > t_{0.05}$, our assumption is rejected i.e., the stimulus does not increase the B.P.

Example 27.12. The nine items of a sample have the following values : 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5 ? (V.T.U., 2010)

Solution. We find the mean and standard deviation of the sample as follows :

x	$d = x - 48$	d^2
45	-3	9
47	-1	1
50	2	4
52	2	4
48	0	0
47	-1	1
49	1	1
53	5	25
51	3	9
Total	10	66

$$\therefore \bar{x} = \text{mean} = 48 + \frac{\sum d}{9} = 48 + \frac{10}{9} = 49.1$$

$$\sigma_s^2 = \frac{\sum d^2}{9} - \left(\frac{\sum d}{9} \right)^2 = \frac{66}{9} - \frac{100}{81} = \frac{494}{81}$$

$$\therefore \sigma_s = 2.47$$

$$\text{Hence } t = \frac{\bar{x} - \mu}{\sigma_s} \sqrt{(n-1)} = \frac{49.1 - 47.5}{2.47} \sqrt{8} = 1.83$$

$$\text{Here } d.f. v = 9 - 1 = 8$$

$$\text{For } v = 8, \text{ we get from table IV, } t_{0.05} = 2.31.$$

As calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance which implies that there is no significant difference between \bar{x} and μ . Thus the test provides no evidence against the population mean being 47.5.

Example 27.13. A machinist is making engine parts with axle diameter of 0.7 inch. A random sample of 10 parts shows mean diameter 0.742 inch with a standard deviation of 0.04 inch. On the basis of this sample, would you say that the work is inferior ? (V.T.U., 2009)

Solution. Here we have $\mu = 0.700$, $\bar{x} = 0.742$, $\sigma_x = 0.040$, $n = 10$.

Taking the hypothesis that the product is not inferior i.e., there is no significant difference between \bar{x} and μ .

$$\therefore t = \frac{\bar{x} - \mu}{\sigma_x} \sqrt{(n-1)} = \frac{0.742 - 0.700}{0.040} \sqrt{(10-1)} = \frac{0.126}{0.040} = 3.16$$

Degrees of freedom $v = 10 - 1 = 9$.

For $v = 9$, we get from table IV, $t_{0.05} = 2.262$.

As the calculated value of $t > t_{0.05}$, the value of t is significant at 5% level of significance. This implies that \bar{x} differs significantly from μ and the hypothesis is rejected. Hence the work is inferior. In fact, the work is inferior even at 2% level of significance.

Example 27.14. Show that 95% confidence limits for the mean μ of the population are $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$.

Deduce that for a random sample of 16 values with mean 41.5 inches and the sum of the squares of the deviations from the mean 135 inches² and drawn from a normal population, 95% confidence limits for the mean of the population are 39.9 and 43.1 inches.

Solution. (a) The probability P that $t \leq t_{0.05}$ is 0.95. Hence the 95% confidence limits for μ are given by

$$\left| \frac{\bar{x} - \mu}{\sigma_s} \sqrt{n} \right| \leq t_{0.05}$$

$$\text{or } |\bar{x} - \mu| \leq \frac{\sigma_s}{\sqrt{n}} t_{0.05} \quad \text{or} \quad \bar{x} - \frac{\sigma_s}{\sqrt{n}} t_{0.05} \leq \mu \leq \bar{x} + \frac{\sigma_s}{\sqrt{n}} t_{0.05}$$

We can, therefore, say with a confidence coefficient 0.95 that the confidence interval $\bar{x} \pm \frac{\sigma_s}{\sqrt{n}} t_{0.05}$ contains the population mean μ .

$$(b) \text{ Here, } n = 16, v = n - 1 = 15, \sigma_s = \sqrt{\frac{135}{15}} = 3.$$

Also from table IV, $t_{0.05}$ (for $v = 15$) = 2.13

$$\therefore \frac{\sigma_s}{\sqrt{n}} t_{0.05} = \frac{3}{4} \times 2.13 = 1.6 \text{ approx.}$$

Hence the required confidence limits are 41.5 ± 1.6 i.e., 39.9 and 43.1 inches.

27.16 SIGNIFICANCE TEST OF DIFFERENCE BETWEEN SAMPLE MEANS

Given two independent samples $x_1, x_2, x_3, \dots, x_{n_1}$ and y_1, y_2, \dots, y_{n_2} with means \bar{x} and \bar{y} and standard deviations σ_x and σ_y from a normal population with the same variance, we have to test the hypothesis that the population means μ_1 and μ_2 are the same.

For this, we calculate $t = \frac{\bar{x} - \bar{y}}{\sigma \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$... (1)

where $\bar{x} = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i, \bar{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i$

and $\sigma_s^2 = \frac{1}{n_1 + n_2 - 2} \left[(n_1 - 1)\sigma_x^2 + (n_2 - 1)\sigma_y^2 \right] = \frac{1}{n_1 + n_2 - 2} \left\{ \sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2 \right\}$

It can be shown that the variate t defined by (1) follows the t -distribution with $n_1 + n_2 - 2$ degrees of freedom.

If the calculated value of $t > t_{0.05}$, the difference between the sample means is said to be significant at 5% level of significance.

If $t > t_{0.01}$, the difference is said to be significant at 1% level of significance.

If $t < t_{0.05}$, the data is said to be consistent with the hypothesis, that $\mu_1 = \mu_2$.

Cor. If the two samples are of the same size and the data are paired, then t is defined by

$$t = \frac{\bar{d} - 0}{(\sigma/\sqrt{n})} \quad \text{where} \quad \sigma^2 = \frac{1}{n-1} \sum_{i=1}^n (d_i - \bar{d})^2$$

d_i = difference of the i th members of the samples ;

\bar{d} = mean of the differences = $\sum d_i / n$; and the number of d.f. = $n - 1$.

Example 27.15. Eleven students were given a test in statistics. They were given a month's further tuition and a second test of equal difficulty was held at the end of it. Do the marks give evidence that the students have benefitted by extra coaching ?

Boys	:	1	2	3	4	5	6	7	8	9	10	11
Marks I test	:	23	20	19	21	18	20	18	17	23	16	19
Marks II test	:	24	19	22	18	20	22	20	20	23	20	17

(V.T.U., 2011 S)

Solution. We compute the mean and the S.D. of the difference between the marks of the two tests as under :

$$\bar{d} = \text{mean of } d\text{'s} = \frac{11}{11} = 1; \sigma_s^{-2} = \frac{\sum (d - \bar{d})^2}{n-1} = \frac{50}{10} = 5 \quad \text{i.e., } \sigma_s = 2.24$$

Assuming that the students have not been benefited by extra coaching, it implies that the mean of the difference between the marks of the two tests is zero i.e., $\mu = 0$.

Then $t = \frac{\bar{d} - \mu}{\sigma_s} \sqrt{n} = \frac{1 - 0}{2.24} \sqrt{11} = 1.48$ nearly and $df v = 11 - 1 = 10$.

Students	x_1	x_2	$d = x_2 - x_1$	$d - \bar{d}$	$(d - \bar{d})^2$
1	23	24	1	0	0
2	20	19	-1	-2	4
3	19	22	3	2	4
4	21	18	-3	-4	16
5	18	20	2	1	1
6	20	22	2	1	1
7	18	20	2	1	1
8	17	20	3	2	4
9	23	23	—	-1	1
10	16	20	4	3	9
11	19	17	-2	-3	9
			$\sum d = 11$		$\sum (d - \bar{d})^2 = 50$

From table IV, we find that $t_{0.05}$ (for $v = 10$) = 2.228. As the calculated value of $t < t_{0.05}$, the value of t is not significant at 5% level of significance i.e., the test provides no evidence that the students have benefited by extra coaching.

Example 27.16. From a random sample of 10 pigs fed on diet A, the increases in weight in a certain period were 10, 6, 16, 17, 13, 12, 8, 14, 15, 9 lbs. For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs. Test whether diets A and B differ significantly as regards their effect on increases in weight?

Solution. We calculate the means and standard deviations of the samples as follows :

	Diet A		Diet B		
x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	y_i	$y_i - \bar{y}$	$(y_i - \bar{y})^2$
10	-2	4	7	-8	64
6	-6	36	13	-2	4
16	4	16	22	7	49
17	5	25	15	0	0
13	1	1	12	-3	9
12	0	0	14	-1	1
8	-4	16	18	3	9
14	2	4	8	-7	49
15	3	9	21	6	36
9	-3	9	23	8	64
			10	-5	25
			17	2	4
120	0	120	180	0	314

$$\bar{x} = \frac{120}{10} = 12 \text{ lbs.}, \bar{y} = \frac{180}{12} = 15 \text{ lbs.}$$

$$\begin{aligned}\sigma_s^2 &= [\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2]/(n_1 + n_2 - 2) \\ &= (120 + 314)/(10 + 12 - 2) = (434/20) = 21.1\end{aligned}$$

$$\therefore \sigma_s = 4.65$$

Assuming that the samples do not differ in weight so far as the two diets are concerned i.e., $\mu_1 - \mu_2 = 0$.

Hence $t = \frac{(\bar{y} - \bar{x}) - 0}{\sigma_s \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{15 - 12}{4.65 \sqrt{\left(\frac{1}{10} + \frac{1}{12}\right)}} = \frac{3}{4.65} \sqrt{\frac{120}{22}} = 1.6$ nearly

Here $d.f. v = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$.

For $v = 20$, we find $t_{0.05} = 2.09$

[From table IV]

\therefore the calculated value of $t < t_{0.05}$.

Hence the difference between the sample means is not significant i.e., the two diets do not differ significantly as regards their effect on increase in weight.

PROBLEMS 27.3

1. Find the student's t for the following variable values in a sample of eight : -4, -2, -2, 0, 2, 2, 3, 3 ; taking the mean of the universe to be zero.

2. A random sample of 10 boys had the following I.Q. :

70, 120, 110, 101, 88, 83, 95, 98, 107, 100.

Do these data support the assumption of a population mean I.Q. of 100 (at 5% level of significance) ?

(V.T.U., 2006 ; Coimbatore, 2001)

3. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation 0.15 cm. Find 95% confidence limits for the actual diameter.

4. A random sample of size 25 from a normal population has the mean $\bar{x} = 47.5$ and s.d. $S = 8.4$. Does this information refute the claim that the mean of the population is $\mu = 42.1$. (J.N.T.U., 2003)

5. A process for making certain bearings is under control if the diameter of the bearings have the mean 0.5 cm. What can we say about this process if a sample of 10 of these bearings has a mean diameter of 0.506 cm. and s.d. of 0.004 cm ?

6. A machine is supposed to produce washers of mean thickness 0.12 cm. A sample of 10 washers was found to have a mean thickness of 0.128 cm and standard deviation 0.008. Test whether the machine is working in proper order at 5% level of significance.

7. Find out the reliability of the sample mean of the following data : *Breaking strength of 10 specimens of 1.04 cms diameter hard-drawn copper wire* :

Specimen	:	1	2	3	4	5	6	7	8	9	10
Breaking Strength (kgs)	:	578	572	570	568	572	570	570	572	526	584

8. Test runs with 6 models of an experiment. 1 engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of fuel. If the probability of a Type I error is at the most 0.01, is this evidence against a hypothesis that on the average this kind of engine will operate for atleast 29 minutes per gallon of the same fuel. Assume normality.

(J.N.T.U., 2003)

9. Two horses *A* and *B* were tested according to the time (in seconds) to run a particular race with the following results :

Horse A :	28	30	32	33	33	29	and 34
Horse B :	29	30	30	24	27	and 29	

Test whether you can discriminate between two horses ?

(Rohtak, 2005 ; Coimbatore, 2001)

10. A group of 10 rats fed on a diet *A* and another group of 8 rats fed on a different diet *B*, recorded the following increase in weights :

Diet A :	5	6	8	1	12	4	3	9	6	10	gm
Diet B :	2	3	6	8	10	1	2	8	8	gm.	

Does it show that superiority of diet *A* over that of *B* ?

(Madras, 2003)

11. A group of boys and girls were given an intelligence test. The mean score, S.D.s and numbers in each group are as follows :

	Boys	Girls
Mean	124	121
S.D.	12	10
<i>n</i>	18	14

Is the mean score of boys significantly different from that of girls ?

12. The means of two random samples of sizes 9 and 7 are 196.42 and 198.82 respectively. The sum of the squares of the deviations from the mean are 26.94 and 18.73 respectively. Can the sample be considered to have been drawn from the same normal population ?

(Mumbai, 2004)

27.17 (1) CHI-SQUARE (χ^2) TEST

When a fair coin is tossed 80 times, we expect from theoretical considerations that heads will appear 40 times and tail 40 times. But this never happens in practice i.e., the results obtained in an experiment do not agree exactly with the theoretical results. The magnitude of discrepancy between observation and theory is given by the quantity χ^2 (pronounced as chi-square). If $\chi^2 = 0$, the observed and theoretical frequencies completely agree. As the value of χ^2 increases, the discrepancy between the observed and theoretical frequencies increases.

(1) Definition. If O_1, O_2, \dots, O_n be a set of observed (experimental) frequencies and E_1, E_2, \dots, E_n be the corresponding set of expected (theoretical) frequencies, then χ^2 is defined by the relation

$$\begin{aligned}\chi^2 &= \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \dots + \frac{(O_n - E_n)^2}{E_n} \\ &= \sum \frac{(O_i - E_i)^2}{E_i}\end{aligned}\quad \dots(1)$$

with $n - 1$ degrees of freedom.

$[\Sigma O_i = \Sigma E_i = n \text{ the total frequency}]$

(2) Chi-square distribution*

If x_1, x_2, \dots, x_n be n independent normal variates with mean zero and s.d. unity, then it can be shown that $x_1^2 + x_2^2 + \dots + x_n^2$, is a random variate having χ^2 -distribution with ndf .

The equation of the χ^2 -curve is

$$y = y_0 e^{-\chi^2/2} (\chi^2)^{(v-1)/2} \quad \dots(2)$$

where $v = n - 1$ (Fig. 27.3).

(3) Properties of χ^2 -distribution

I. If $v = 1$, the χ^2 -curve (2) reduces to $y = y_0 e^{-\chi^2/2}$, which is the exponential distribution.

II. If $v > 1$, this curve is tangential to x -axis at the origin and is positively skewed as the mean is at v and mode at $v - 2$.

III. The probability P that the value of χ^2 from a random sample will exceed χ_0^2 is given by

$$P = \int_{\chi_0^2}^{\infty} y dx.$$

The values of χ_0^2 have been tabulated for various values of P and for values of v from 1 to 30. (Table-V-Appendix 2)

For $v > 30$, the χ^2 -curve approximates to the normal curve and we should refer to normal distribution tables for significant values of χ^2 .

IV. Since the equation of χ^2 -curve does not involve any parameters of the population, this distribution does not depend on the form of the population and is therefore, very useful in a large number of problems.

V. Mean = γ and variance = 2γ .

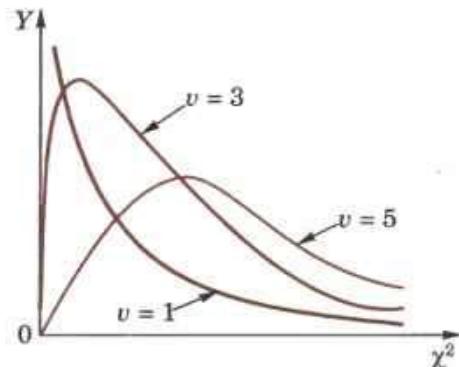


Fig. 27.3

27.18 GOODNESS OF FIT

The value of χ^2 is used to test whether the deviations of the observed frequencies from the expected frequencies are significant or not. It is also used to test how well a set of observations fit a given distribution, χ^2 therefore, provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed frequency distribution. As a test of goodness of fit, it can be used to study the correspondence between theory and fact.

This is a non-parametric distribution-free test since in this we make no assumption about the distribution of the parent population.

*Hamlet discovered this distribution in 1875. Karl Pearson rediscovered it independently in 1900 and applied it to test 'goodness of fit'.

Procedure to test significance and goodness of fit.

(i) Set up a 'null hypothesis' and calculate

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

(ii) Find the df and read the corresponding values of χ^2 at a prescribed significance level from Table V.

(iii) From χ^2 -table, we can also find the probability P corresponding to the calculated values of χ^2 for the given $d.f.$

(iv) If $P < 0.05$, the observed value of χ^2 is significant at 5% level of significance.

If $P < 0.01$, the value is significant at 1% level.

If $P > 0.05$, it is a good fit and the value is not significant.

Example 27.17. In experiments on pea breeding, the following frequencies of seeds were obtained :

Round and yellow	Wrinkled and yellow	Round and green	Wrinkled and green	Total
315	101	108	32	556

Theory predicts that the frequencies should be in proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and experiment.

Solution. The corresponding frequencies are

$$\frac{9}{16} \times 556, \frac{3}{16} \times 556, \frac{3}{16} \times 556, \frac{1}{16} \times 556 \text{ i.e., } 313, 104, 104, 35.$$

$$\begin{aligned} \text{Hence } \chi^2 &= \frac{(315 - 313)^2}{313} + \frac{(101 - 104)^2}{104} + \frac{(108 - 104)^2}{104} + \frac{(32 - 35)^2}{35} \\ &= \frac{4}{313} + \frac{9}{104} + \frac{16}{104} + \frac{9}{35} = 0.51 \quad \text{and } df v = 4 - 1 = 3. \end{aligned}$$

For $v = 3$, we have $\chi^2_{0.05} = 7.815$

[From Table V]

Since the calculated value of χ^2 is much less than $\chi^2_{0.05}$, there is a very high degree of agreement between theory and experiment.

Example 27.18. A set of five similar coins is tossed 320 times and the result is

No. of heads :	0	1	2	3	4	5
Frequency :	6	27	72	112	71	32

Test the hypothesis that the data follow a binomial distribution.

(Kottayam, 2005 ; P.T.U., 2005 ; V.T.U., 2004)

Solution. For $v = 5$, we have $\chi^2_{0.05} = 11.07$.

p , probability of getting a head = $\frac{1}{2}$; q , probability of getting a tail = $\frac{1}{2}$.

Hence the theoretical frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion $320(p+q)^5$

$$\begin{aligned} &= 320 [p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5] \\ &= 320 \left[\frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32} \right] = 10 + 50 + 100 + 100 + 50 + 10 \end{aligned}$$

Thus the theoretical frequencies are 10, 50, 100, 100, 50, 10.

Hence

$$\begin{aligned} \chi^2 &= \frac{(6 - 10)^2}{10} + \frac{(27 - 50)^2}{50} + \frac{(72 - 100)^2}{100} + \frac{(112 - 100)^2}{100} + \frac{(71 - 50)^2}{50} + \frac{(32 - 10)^2}{10} \\ &= \frac{1}{100} (160 + 1058 + 784 + 144 + 882 + 4840) = \frac{7868}{100} = 78.68 \end{aligned}$$

and

$$df v = 6 - 1 = 5.$$

Since the calculated value of χ^2 is much greater than $\chi^2_{0.05}$, the hypothesis that the data follow the binomial law is rejected.

Example 27.19. Fit a Poisson distribution to the following data and test for its goodness of fit at level of significance 0.05.

$x:$	0	1	2	3	4
$f:$	419	352	154	56	19

$$\begin{aligned}\chi^2 &= \frac{(5 - 4.13)^2}{4.13} + \frac{(18 - 20.68)^2}{20.68} + \frac{(42 - 38.92)^2}{38.92} + \frac{(27 - 27.71)^2}{27.71} + \frac{(8 - 7.43)^2}{7.43} \\ &= 0.1833 + 0.3473 + 0.2437 + 0.0182 + 0.0437 = 0.8362\end{aligned}$$

As regards the number of degrees of freedom (γ), there are three constraints (i) discrepancy between total observed and total estimated frequencies (ii) and (iii) mean (m) and standard deviation (σ) have been estimated from the sample data. $\therefore r = 5 - 3 = 2$.

For $\gamma = 2$, $\chi^2_{0.05} = 0.103$ from table V.

Since $\chi^2 = 0.8362 > 0.103$. Hence the fit is not good.

PROBLEMS 27.4

1. Five dice were thrown 96 times and the number of times 4, 5 or 6 were thrown were :

No. of dice showing 4, 5 or 6 :	5	4	3	2	1	0
Frequency	8	18	35	24	10	1

Find the probability of getting this result by chance ?

2. Genetic theory states that children having one parent of blood type M and other blood type N will always be one of the three types M, MN, N and that the proportions of these types will on average be $1 : 2 : 1$. A report states that out of 300 children having one M parent and one N parent, 30% were found to be of type M , 45% of type MN and remainder of type N . Test the hypothesis by χ^2 test.

3. A die was thrown 60 times and the following frequency distribution was observed :

Faces :	1	2	3	4	5	6
f_0 :	15	6	4	7	11	17

Test whether the die is unbiased ?

4. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week ?

Days	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents	14	16	8	12	11	9	14	84 (Hissar, 2005)

5. Fit a binomial distribution to the data :

x :	0	1	2	3	4	5
f :	38	144	342	287	164	25

and test for goodness of fit, at the level of significance 0.05.

6. In 1000 extensive sets of trials for an event of small probability, the frequencies f_0 of the number x of successes proved to be :

x :	0	1	2	3	4	5	6	7
f_0 :	305	366	210	80	28	9	2	1

Fit a Poisson distribution to the data and test the goodness of fit.

7. The frequencies of localities according to the number of deaths due to cholera during eight years in 1000 localities is as follows :

No. of deaths	0	1	2	3	4	5	6	7
No. of localities	314	355	204	86	29	9	3	0

Fit a suitable distribution to the data and test the goodness of fit.

8. Obtain the equation of the normal curve that may be fitted to the data and test the goodness of fit.

x :	4	6	8	10	12	14	16	18	20	22	24	Total
$f(x)$:	1	7	15	22	35	43	38	20	13	5	1	200

27.19 (1) F-DISTRIBUTION*

Let x_1, x_2, \dots, x_{n_1} and y_1, y_2, \dots, y_{n_2} be the values of two independent random samples drawn from the normal populations σ^2 having equal variances.

* This distribution was introduced by the English statistician Prof. R.A. Fisher (1890–1962) who had greatly influenced the development of modern statistics.

Let \bar{x}_1 and \bar{x}_2 be the sample means and $s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2$, $s_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (y_i - \bar{y})^2$ be the sample variances.

Then we define F by the relation

$$F = \frac{s_1^2}{s_2^2} \quad (s_1^2 > s_2^2)$$

This gives F -distribution (also known as variance ratio distribution) with $\gamma_1 = n_1 - 1$ and $\gamma_2 = n_2 - 1$ degrees of freedom. *The larger of the variances is placed in the numerator.*

(2) Properties. I. The F -distribution curve lies entirely in the first quadrant and is *unimodal*.

II. The F -distribution is independent of the population variance σ^2 and depends on γ_1 and γ_2 only.

III. $F_\alpha(\gamma_1, \gamma_2)$ is the value of F for γ_1 and γ_2 of such that the area to the right of F_α is α .

IV. It can be shown that the mode of F -distribution is less than unity.

(3) Significance test. Snedecor's F -tables give 5% and 1% points of significance for F . (Table VI – Appendix 2). 5% points of F mean that area under the F -curve to the right of the ordinate at a value of F , is 0.05. Clearly value of F at 5% significance is lower than that at 1%. F -distribution is very useful for testing the equality of population means by comparing sample variances. As such it forms the basis of *analysis of variance*.

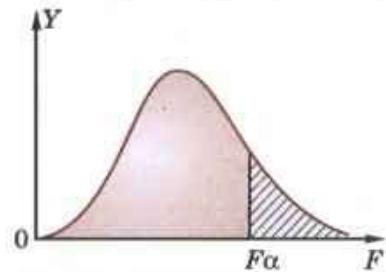


Fig. 27.4

Example 27.21. Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches² and 91 inches² respectively. Can these be regarded as drawn from the same normal population? (V.T.U., 2002)

Solution. We have $\Sigma(x - \bar{x})^2 = 160$ and $\Sigma(y - \bar{y})^2 = 91$

$$\therefore s_1^2 = \frac{160}{8} = 20$$

$$\text{and} \quad s_2^2 = \frac{91}{7} = 13.$$

$$\text{Hence} \quad F = \frac{s_1^2}{s_2^2} = \frac{20}{13} = 1.54 \text{ nearly.}$$

For $\gamma_1 = 8$, $\gamma_2 = 7$, we have $F_{0.05} = 3.73$.

[From Table VI]

Since the calculated value of $F < F_{0.05}$, the population variances are not significantly different. Thus the two samples can be regarded as drawn from two normal populations with the same variance. If the two populations are to be same, their means should also be the same which can be verified by applying t -test provided the sample means are known.

Example 27.22. Measurements on the length of a copper wire were taken in 2 experiments A and B as under :

A's measurements (mm) : 12.29, 12.25, 11.86, 12.13, 12.44, 12.78, 12.77, 11.90, 12.47.

B's measurements (mm) : 12.39, 12.46, 12.34, 12.22, 11.98, 12.46, 12.23, 12.06.

Test whether B's measurements are more accurate than A's. (The readings taken in both cases being unbiased)

Solution. Readings in both cases being unbiased, B's measurements will be taken more accurate if its population variance is less than that of A's measurements.

Under the hypothesis that the two populations have the same variance (i.e. $\sigma_1^2 = \sigma_2^2$), we have

$$F = \frac{s_1^2}{s_2^2}$$

with $\gamma_1 = n_1 - 1 = 8$ and $\gamma_2 = n_2 - 1 = 7$.

We calculate the s.d.'s of the two series as follows :

A's measurements			B's measurements		
x	$u = 100(x - 12)$	u^2	y	$v = 100(y - 12)$	v^2
12.29	29	841	12.39	39	1521
12.25	25	625	12.46	46	2116
11.86	-14	196	12.34	34	1156
12.13	13	169	12.22	22	484
12.44	44	1936	11.98	-2	4
12.78	78	6084	12.46	46	2116
12.77	77	5929	12.23	23	529
11.90	-10	100	12.06	6	36
12.47	47	2209			
	289	18089		214	7962

$$\therefore s_1^2 = \frac{1}{n_1 - 1} \left[18089 - \frac{(289)^2}{n_1} \right] = \frac{1}{8} (18089 - 9280) = 1101.1$$

$$s_2^2 = \frac{1}{n_2 - 1} \left[7962 - \frac{(214)^2}{n_2} \right] = \frac{1}{7} (7962 - 5724) = 319.7$$

$$\therefore F = \frac{s_1^2}{s_2^2} = \frac{1101.1}{319.7} = 3.44$$

For $\gamma_1 = 8$ and $\gamma_2 = 7$, from table VI, $F_{0.05} = 3.73$ and $F_{0.01} = 6.84$.

Since the calculated value of F is less than both $F_{0.05}$ and $F_{0.01}$, the result is insignificant at both 5% and 1% level.

Hence there is no reason to say that B's measurements are more accurate than those of A's.

27.20 (1) FISHER'S z-DISTRIBUTION

Changing the variable F to z by the substitution $z = \frac{1}{2} \log_e F$ or $F = e^{2z}$ in the F -distribution, we get the Fisher's z -distribution.

It is more nearly symmetrical than F -distribution. Table showing the values of z that will be exceeded in simple sampling with probabilities 0.05 and 0.01 have been prepared for various values of v_1 and v_2 .

(2) Significance test. As z -table give only critical values corresponding to right hand tail areas, therefore 5% (or 1%) points of z imply that the area to the right of the ordinate at z is 0.05 (or 0.01). In other words, 5% and 1% points of z correspond to 10% and 2% levels of significance respectively.

Example 27.23. Two gauge operations are tested for precision in making measurements. One operator completes a set of 26 readings with a standard deviations of 1.34 and the other does 34 readings with a standard deviations of 0.98. What is the level of significance of this difference.

(Given that for $v_1 = 25$ and $v_2 = 33$, $z_{0.05} = 0.305$, $z_{0.01} = 0.432$)

Solution. We have $n_1 = 26$, $\sigma_x = 1.34$; $n_2 = 34$, $\sigma_y = 0.98$

$$\therefore s_1^2 = \frac{n_1}{n_1 - 1} \cdot \sigma_x^2 = \frac{26}{25} (1.34)^2 = (1.34)^2 \quad \text{and} \quad s_2^2 = \frac{n_2}{n_2 - 1} \cdot \sigma_y^2 = \frac{34}{33} (0.98)^2 = (0.98)^2$$

$$\text{Hence } F = \left(\frac{1.34}{0.98} \right)^2 = 1.8696 \quad \text{and} \quad z = \frac{1}{2} \log_e F = 1.1513 \log_{10} 1.8696 = 0.3129$$

Since the calculated value of z is just greater than $z_{0.05}$ and less than $z_{0.01}$, the difference between the standard deviation is just significant at 5% level and insignificant at 1% level.

PROBLEMS 27.5

- Two samples of 9 and 7 individuals have variances 4.8 and 9.6 respectively. Is the variance 9.6 significantly greater than the variance 4.6?
- Test for breaking strength were carried out on two lots of 5 and 9 steel wires. The variance of one lot was 230 and that of other was 492. Is there a significant difference in their variability?
- Show how you would use Fisher's t -test to decide whether the two sets of observations 17, 27, 18, 25, 27, 29, 27, 23, 17 and 16, 16, 20, 16, 20, 17, 15, 21, indicate samples from the same universe.
- In two groups of ten children each, the increase in weight due to different diets during the same period, were in pounds
 3, 7, 5, 6, 5, 4, 4, 5, 3, 6
 8, 5, 7, 8, 3, 2, 7, 6, 5, 7.

Is there a significant difference in their variability?

- The mean diameter of rivets produced by two firms A and B are practically the same but their standard deviations are different. For 16 rivets manufactured by firm A, the S.D. is 3.8 mm while for 22 rivets manufactured by firm B is 2.9 mm. Do you think products of firm B are of better quality than those of firm A?
- The I.Q.'s of 25 students from one college showed a variance of 16 and those of an equal number from the other college had a variance of 8. Discuss whether there is any significant difference in variability of intelligence.
- Two random samples from two normal populations are given below :

Sample I :	16	26	27	23	24	22
Sample II :	33	42	35	32	28	31

Do the estimates of population variances differ significantly?

Degrees of freedom :	(5, 5)	(5, 6)	(6, 5)
5% value of F :	5.05	4.39	4.95

- Two independent samples of sizes 7 and 6 have the following values :

Sample A :	28	30	32	33	33	29	34
Sample B :	29	30	30	24	27	29	.

Examine whether the samples have been drawn from normal populations having the same variance?

[Given that the values of F at 5% level for (6, 5) d.f. is 4.95 and for (5, 6) d.f. is 4.39].

27.21 OBJECTIVE TYPE OF QUESTIONS**PROBLEMS 27.6**

Select the correct answer or fill up the blanks in each of the following questions :

- The 'null hypothesis' implies that
- Type I and type II errors are such that
- Control limit theorem states that
- A hypothesis is true, but is rejected. Then this is an error of type
- If the standard deviation of a χ^2 distribution is 10, then its degree of freedom is
- Range of F -distribution is
- A hypothesis is false but accepted, then there is an error of type
- The mean and variance of a χ^2 distribution with 8 degrees of freedom are and respectively.
- In a t -distribution of sample size n , the degrees of freedom are
- The test statistic $F = \frac{s_1^2}{s_2^2}$ is used when
 (i) $s_2^2 > s_1^2$ (ii) $s_2^2 < s_1^2$ (iii) $s_1^2 = s_2^2$ (iv) none of these.
- The t -test is applicable to samples for which n is
- The two main uses of χ^2 -test are
- Range of t -distribution is
- If two samples are taken from two populations of unequal variances, we can apply t -test to test the difference of means.
 (True or False)
- The Chi-square distribution is continuous.
 (True or False)

Numerical Solution of Equations

1. Introduction.
2. Solution of algebraic and transcendental equations—Bisection method, Method of false position, Newton's method.
3. Useful deductions from the Newton-Raphson Formula.
4. Approximate solution of equations—Horner's method.
5. Solution of linear simultaneous equations.
6. Direct methods of solution—Gauss elimination method, Gauss-Jordan method, Factorization method.
7. Iterative methods of solution—Jacobi's method, Gauss-Seidal method, Relaxation method.
8. Solution of non-linear simultaneous equations—Newton-Raphson method.
9. Determination of eigen values by iteration.
10. Objective Type of Questions.

28.1 INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. As seen in § 1.8, the graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensable tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We shall close the chapter by describing an iterative method for the solution of eigen-value problem. For a detailed study of these topics, the reader should refer to author's book '*Numerical Methods in Engineering & Science*'.

28.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

To find the roots of an equation $f(x) = 0$, we start with a known approximate solution and apply any of the following methods :

(1) **Bisection method.** This method consists in locating the root of the equation $f(x) = 0$ between a and b . If $f(x)$ is continuous between a and b , and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b . For definiteness, let $f(a)$ be negative and $f(b)$

be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

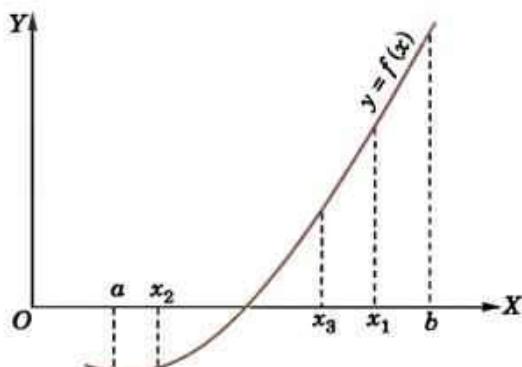


Fig. 28.1

In the Fig. 28.1, $f(x_1)$ is +ve, so that the root lies between a and x_1 . Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is - ve, the root lies between x_1 and x_2 . Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Example 28.1. (a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places. (Mumbai, 2003)

(b) Using bisection method, find the negative root of the equation $x^3 - 4x + 9 = 0$. (J.N.T.U., 2009)

Solution. (a) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is - ve and $f(3)$ is + ve, a root lies between 2 and 3

\therefore first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ i.e., - ve

\therefore the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ i.e., + ve

\therefore the root lies between x_1 and x_2 . Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ i.e., - ve

\therefore the root lies between x_2 and x_3 . Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$$x_5 = 2.71875,$$

$$x_6 = 2.70313,$$

$$x_7 = 2.71094$$

$$x_8 = 2.70703,$$

$$x_9 = 2.70508,$$

$$x_{10} = 2.70605$$

$$x_{11} = 2.70654,$$

$$x_{12} = 2.70642$$

Hence the root is 2.7064

(b) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$
 \therefore the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is - 2.7064.

Example 28.2. By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.

(V.T.U., 2003 S.)

Solution. Let $f(x) = x \sin x - 1$. We know that $1^\circ = 57.3^\circ$.

Since $f(1) = 1 \times \sin(1) - 1 = \sin(57.3^\circ) - 1 = -0.15849$

and $f(1.5) = 1.5 \times \sin(1.5) - 1 = 1.5 \times \sin(85.95^\circ) - 1 = 0.49625$;

a root lies between 1 and 1.5.

\therefore first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25) \sin(1.25) - 1 = 1.25 \sin(71.625^\circ) - 1 = 0.18627$ and $f(1) < 0$.

\therefore a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46^\circ) - 1 = 0.01509$ and $f(1) < 0$.

\therefore a root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$
and $f(x_2) > 0$, i.e. now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

∴ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,
i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

∴ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

∴ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

∴ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

(2) Method of false position or Regula-falsi

method. This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e., the graph of $y = f(x)$ crosses the x -axis between these points (Fig. 28.2). This indicates that a root lies between x_0 and x_1 consequently $f(x_0)f(x_1) < 0$.

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \quad \dots(1)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x -axis as an approximation to the root. So the abscissa of the point where the chord cuts the x -axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \dots(2)$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 . So replacing x_1 by x_2 in (2), we obtain the next approximation x_3 . (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (1) is known as the *method of false position*.

Example 28.3. Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places. (Manipal, 2005)

Solution. Let

$$f(x) = x^3 - 2x - 5$$

so that

$$f(2) = -1 \text{ and } f(3) = 16 \text{ i.e., A root lies between 2 and 3.}$$

∴ taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \quad \dots(i)$$

Now

$$f(x_2) = f(2.0588) = -0.3908 \text{ i.e., the root lies between 2.0588 and 3.}$$

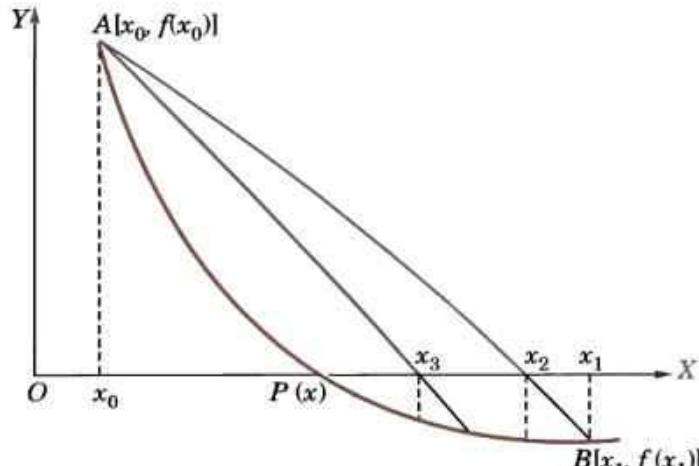


Fig. 28.2

∴ taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3908$, $f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908} (-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 28.4. Find the root of the equation $\cos x = xe^x$ using the regula-falsi method correct to four decimal places. (Bhopal, 2009)

Solution. Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1, f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

∴ taking $x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \quad \dots(i)$$

Now $f(0.31467) = 0.51987$

i.e., the root lies between 0.31467 and 1.

∴ taking $x_0 = 0.31467, x_1 = 1, f(x_0) = 0.51987, f(x_1) = -2.17798$ in (i), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$

i.e., the root lies between 0.44673 and 1.

∴ taking $x_0 = 0.44673, x_1 = 1, f(x_0) = 0.20356, f(x_1) = -2.17798$ in (i), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995, \quad x_6 = 0.51520, \quad x_7 = 0.51692$$

$$x_8 = 0.51748, \quad x_9 = 0.51767, \quad x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 28.5. Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places. (V.T.U., 2010; J.N.T.U., 2008; Kottayam, 2005)

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -\text{ve}, f(2) = -\text{ve}$ and $f(3) = +\text{ve}$.

∴ a root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3, f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \quad \dots(i)$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e., the root lies between 2.72102 and 3.

∴ taking $x_0 = 2.72102, x_1 = 3, f(x_0) = -0.01709$

and $f(x_1) = 0.23136$ in (i), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024, x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 28.6. Use the method of false position, to find the fourth root of 32 correct to three decimal places.

Solution. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, i.e., a root lies between 2 and 3.

∴ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \quad \dots(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ i.e. the root lies between 2.2462 and 3.

∴ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (i), we get $x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438} (-6.5438) = 2.335$

Now $f(x_3) = f(2.335) = -2.2732$ i.e. the root lies between 2.335 and 3.

∴ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (i), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732} (-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

(3) Newton-Raphson method*. Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

∴ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0$$

Since h is small, neglecting h^2 and higher powers of h , we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \quad \dots(1)$$

∴ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1 , a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$...(2)

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

Obs. 1. *Newton's method is useful in cases of large values of $f'(x)$ i.e. when the graph of $f(x)$ while crossing the x -axis is nearly vertical.*

Obs. 2. *Newton's method has a second order of quadratic convergence.* Suppose x_n differs from the root α by a small quantity ϵ_n so that $x_0 = \alpha + \epsilon_n$ and $x_{n+1} = \alpha + \epsilon_{n+1}$.

Then (2) becomes $\alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$

i.e., $\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)} = \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{1}{2!} \epsilon_n^2 f''(\alpha) + \dots}{f'(\alpha) + \epsilon_n f''(\alpha) + \dots}$

[By Taylor's expansion.]

$$= \epsilon_n - \frac{\epsilon_n f'(\alpha) + \frac{1}{2} \epsilon_n^2 f''(\alpha) + \dots}{f'(\alpha) + \epsilon_n f''(\alpha) + \dots} \quad [\because f(\alpha) = 0]$$

$$= \frac{\epsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \epsilon_n f''(\alpha)]} \approx \frac{\epsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}. \quad \left[\begin{array}{l} \text{neglecting third and} \\ \text{higher powers of } \epsilon_n \end{array} \right]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic.

(P.T.U., 2005)

Obs. 3. Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 28.3). Then the equation of the tangent at $A_0(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

*See footnote p. 466. Named after the English mathematician Joseph Raphson (1648–1715) who suggested a method similar to Newton's method.

It cuts the x -axis at $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ which is a first approximation

to the root α . If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x -axis at x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence the method consists in replacing the part of the curve between the point A_0 and the x -axis by means of the tangent to the curve at A_0 .]

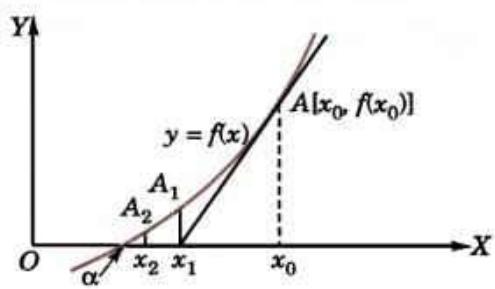


Fig. 28.3

Example 28.7. Find the positive root of $x^4 - x - 10 = 0$ correct to three decimal places, using Newton-Raphson method. (J.N.T.U., 2008; Madras, 2006)

Solution. Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 = -\text{ve}, f(2) = 16 - 2 - 10 = 4 = +\text{ve}$

\therefore a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \dots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)} \\ &= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856 \end{aligned}$$

Putting $n = 2$ in (i), the third approximation is

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1} \\ &= 1.856 - \frac{0.010}{24.574} = 1.856 \end{aligned}$$

Here $x_2 = x_3$. Hence the desired is 1.856 correct to three decimal places.

Example 28.8. Find the Newton's method, the real root of the equation $3x = \cos x + 1$.

(V.T.U., 2009; S.V.T.U., 2007)

Solution. Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}$.

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

\therefore Newton's iteration formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\ &= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad \dots(i) \end{aligned}$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6) \sin (0.6) + \cos (0.6) + 1}{3 \sin (0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (i), the second approximation is

$$\begin{aligned}x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} \\&= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \quad \text{Clearly, } x_1 = x_2.\end{aligned}$$

Hence the desired root is 0.6071 correct to four decimal places.

Example 28.9. Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places. (V.T.U., 2005; Mumbai, 2004; Burdwan, 2003)

Solution. Let $f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2 = \text{ve}, f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = \text{ve}$$

$$\text{and } f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = \text{+ ve}$$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

$$\text{Also } f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$$

∴ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \quad \dots(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (i), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) Iterative formula to find $1/N$ is $x_{n+1} = x_n (2 - Nx_n)$

(2) Iterative formula to find \sqrt{N} is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

(3) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)$

(4) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1}]$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n) \quad (\text{Madras, 2006})$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. Evaluate the following (correct to four decimal places) by Newton's iteration method :

(i) $\sqrt[3]{31}$ (ii) $\sqrt{5}$ (Anna, 2007)

(iii) $1/\sqrt{14}$ (iv) $\sqrt[3]{24}$ (Madras, 2003) (v) $(30)^{-1/5}$

Solution. (i) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

$$\text{Then } x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321$$

$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$$

$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226$$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(ii) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

$$\text{Then } x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$$

$$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$$

$$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(iii) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$

$$\text{Then } x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$$

$$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$$

$$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(iv) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

$$\text{Then } x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(v) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

$$\text{Then } x_1 = \frac{x_0}{5}(6 - 30x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 28.1

- Find a root of the following equations, using the bisection method correct to three decimal places :
 - $x^3 - 2x - 5 = 0$ (P.T.U., 2005)
 - $x^3 - x^2 - 1 = 0$ (J.N.T.U., 2009)
 - $x^3 - x - 11 = 0$ which lies between 2 and 3
 - $2x^3 + x^2 - 20x + 12 = 0$.
- Using the bisection method, find a real root of the following equations correct to three decimal places :
 - $\cos x = xe^x$ (Mumbai, 2004)
 - $x \log_{10} x = 1.2$ lying between 2 and 3
 - $e^x - x = 2$ lying between 1 and 1.4
 - $e^x = 4 \sin x$.
- Find a real root of the following equations correct to three decimal places by the method of false position :
 - $x^3 + x - 1 = 0$
 - $x^3 - 4x - 9 = 0$ (V.T.U., 2007)
 - $x^3 + x - 1 = 0$ near $x = 1$
 - $x^6 - x^4 - x^3 - 1 = 0$. (Nagarjuna, 2001)
- Using regula-falsi method, compute the real root of the following equations correct to three decimal places :
 - $xe^x = 2$ (S.V.T.U., 2007)
 - $\cos x = 3x - 1$
 - $x \tan x - 1 = 0$
 - $2x - \log x = 7$ (J.N.T.U., 2006)
 - $xe^x = \sin x$. (P.T.U., 2005)
- Find the fourth root of 12 correct to three decimal places using the method of false position.
- Find by Newton's method, a root of the following equations correct to 3 decimal places :
 - $x^3 - 3x + 1 = 0$ (Bhopal, 2009)
 - $x^3 - 2x - 5 = 0$ (P.T.U., 2005)
 - $x^3 - 5x + 3 = 0$ (Mumbai, 2004)
 - $3x^3 - 9x^2 + 8 = 0$ lying between 1 and 2. (Madras, 2003)
- Find a root of the following equations correct to three significant figures using Newton's iterative method :
 - $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3. (Madras, 2003)
 - $x^5 - 5x^2 + 3 = 0$.
- Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.
- Using Newton-Raphson method, find a root of the following equations correct to the three decimal places :
 - $xe^x - 2 = 0$ (V.T.U., 2005)
 - $x^2 + 4 \sin x = 0$ (Hazaribagh, 2009)
 - $x \tan x + 1 = 0$ which is near $x = \pi$ (J.N.T.U., 2006; V.T.U., 2006)
 - $e^x = x^2 + \cos 25x$ which is near $x = 4.5$. (V.T.U., 2007)
- Find by Newton's method, the root of the equations :
 - $\cos x = xe^x$ (J.N.T.U., 2009; V.T.U., 2003)
 - $x \log_{10} x = 12.34$ (Anna, 2004)
 - $10^x + x - 4 = 0$
 - $x + \log_{10} x = 3.375$ (Rohtak, 2003)
- Develop a recurrence formula for finding \sqrt{N} , using Newton-Raphson method and hence compute to three decimal places
 - $\sqrt{13}$ (U.P.T.U., 2008)
 - $\sqrt{10}$ (J.N.T.U., 2008)

12. Find the cube root of 41, using Newton-Raphson method. (Madras, 2003)
13. Develop an algorithm using N-R method, to find the fourth root of a positive number N and hence find $(32)^{1/4}$. (W.B.T.U., 2005)
14. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method :
 (i) $1/18$ J.N.T.U., 2004 (ii) $1/\sqrt{15}$ (iii) $(28)^{-1/4}$.

28.4 APPROXIMATE SOLUTION OF EQUATIONS—HORNER'S METHOD

This is the best method of finding approximate values of both rational and irrational roots of a numerical equation. Horner's method consists in diminution of the root of an equation by successive digits occurring in the roots.

If the root of an equation lies between a and $a + 1$, then the value of this root will be $a . bcd \dots$, where $b, c, d \dots$ are digits in its decimal part. To obtain these, we proceed as follows :

- (i) Diminish the roots of the given equation by a so that the root of the new equation is $0 . bcd \dots$
- (ii) Then multiply the roots of the transformed equation by 10 so that the root of the new equation is $b . cd \dots$
- (iii) Now diminish the root by b and multiply the roots of the resulting equation by 10 so that the root is $c . d \dots$
- (iv) Next diminish the root by c and so on. By continuing this process, the root may be evaluated to any desired degree of accuracy digit by digit. The method will be clear from the following example.

Example 28.11. Find by Horner's method, the positive root of the equation $x^3 + x^2 + x - 100 = 0$ correct to three decimal places.

Solution. Step I. Let $f(x) = x^3 + x^2 + x - 100$

By Descartes' rule of signs, there is only one positive root. Also $f(4) = -\text{ve}$ and $f(5) = +\text{ve}$, therefore, the root lies between 4 and 5.

Step II. Diminish the roots of given equation by 4 so that the transformed equation is

$$x^3 + 13x^2 + 57x - 16 = 0 \quad \dots(ii)$$

Its root lies between 0 and 1. (We draw a zig-zag line above the set of figures 13, 57, -16 which are the coefficients of the terms in (i) as shown below. Now multiply the roots of (i) by 10 for which multiply the second term by 10, the third term by 100 and the fourth term by 1000 (i.e. attach one zero to the second term, two zeros to the third term and three zeros to the fourth term). Then we get the equation

$$f_1(x) = x^3 + 130x^2 + 5700x - 16000 = 0 \quad \dots(ii)$$

1	1	1	- 100 (4.264)
4	20	84	
5	21	- 16000	
4	36	11928	
9	5700	- 4072000	
4	264	3788376	
130	5964	- 283624000	
2	268		
132	623200		
2	8196		
134	631396		
2	8232		
1360	63962800		
6			
1366			
6			
1372			
6			
13780			

Its root lies between 0 and 10.

Clearly $f_1(2) = -\text{ve}, f_1(3) = +\text{ve}$

\therefore the root of (ii) lies between 2 and 3 i.e., first figure after decimal is 2.

Step III. Diminish the roots of $f_1(x) = 0$ by 2 so that the next transformed equation is

$$x^3 + 136x^2 + 6232x - 4072 = 0 \quad \dots(iii)$$

Its root lies between 0 and 1. (We draw the second zig-zag line above the set of figures 136, 6232, -4072).

Multiply the roots of (iii), by 10, i.e. attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 1360x^2 + 623200x - 4072000 = 0$$

Its root lies between 0 and 10, which is nearly $= \frac{4072000}{623200} = 6$

Hence second figure after decimal place is 6.

Step IV. Diminish the roots of $f_2(x) = 0$ by 6, so that the transformed equation is

$$x^3 + 1378x^2 + 639628x - 283624 = 0.$$

Its root lies between 0 and 1. (We draw the third zig-zag line above the set of figures 1378, 639628, -283624.) As before multiply its roots by 10, i.e. attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then the equation becomes

$$f_3(x) = x^3 + 13780x^2 + 63962800x - 283624000 = 0$$

Its root lies between 0 and 10, which is nearly $= \frac{283624000}{63962800} = 4$. Thus the roots of $f_3(x) = 0$ are to be diminished by 4 i.e. the third figure after decimal place is 4. But there is no need to proceed further as the root is required correct to three decimal places only. Hence the root is 4.264.

Obs. 1. After two steps of diminishing, we apply the principle of trial divisor in which we divide the last coefficient by last but one coefficient to get the next integer by which the roots are to be diminished. These last two coefficients should have opposite signs.

Obs. 2. At any stage if the trial divisor suggests the next integer to be zero, then we should again multiply the roots by 10 and write zero in decimal place of the root.

Example 28.12. Find the cube root of 30 correct to 3 decimal places, using Horner's method.

Solution. **Step I.** Let $x = \sqrt[3]{30}$ i.e. $f(x) = x^3 - 30 = 0$

Now $f(3) = -3$ (-ve), $f(4) = 34$ (+ve)

\therefore the root lies between 3 and 4.

Step II. Diminish the roots of the given equation by 3 so that the transformed equation is

$$x^3 + 9x^2 + 27x - 3 = 0 \quad \dots(i)$$

Its roots lie between 0 and 1. (We draw a zig-zag line above the set of numbers 9, 27, -3 which are the coefficients of the terms in (i)). Now multiply the roots of (i) by 10 for which attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then we get the equation

$$f_1(x) = x^3 + 90x^2 + 2700x - 3000 = 0 \quad \dots(ii)$$

Its roots lie between 0 and 10.

Clearly $f_1(1) = -\text{ve}, f_1(2) = +\text{ve}$

\therefore the root of (ii) lies between 1 and 2 i.e., first figure after decimal place is 1.

Step III. Diminish the roots of $f_1(x) = 0$ by 1, so that the next transformed equation is

$$x^3 + 93x^2 + 2883x - 209 = 0 \quad \dots(iii)$$

Its root lies between 0 and 1. (We draw a second zig-zag line above the set of figures 93, 2883, -209). Multiply the roots of (iii) by 10 i.e., attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 930x^2 + 288300x - 209000 = 0$$

Its root lies between 0 and 10, which is nearly

$$= 209000/288300 = 0.724 > 0 \text{ and } < 1.$$

Hence second figure after decimal place is 0.

1	0	0	-30	(3.107)
3	9	27		
3	9	-30000		
3	18	2791		
6	2700	-209000000		
3	91			
90	2791			
1	92			
91	28830000			
1				
92				
1				
9300				

Step IV. Diminish the root of $f_2(x) = 0$ by 0 and then multiply its roots by 10 so that

$$f_3(x) = x^3 + 9300x^2 + 28830000x - 209000000 = 0.$$

Its root lies between 0 and 10, which is nearly $= 209000000/28830000 = 7.2 > 7$ and < 8 . Thus the roots of $f_3(x) = 0$ are to be diminished by 7 i.e., the third figure after decimal is 7. Hence the required root is 3.107.

PROBLEMS 28.2

- Find by Horner's method, the root (correct to three decimal places) of the equation
 - $x^3 - 3x + 1 = 0$ which lies between 1 and 2
 - $x^3 + x - 1 = 0$ (Coimbatore, 1997)
 - $x^3 - 6x - 13 = 0$ (Madras, 2000 S)
 - $x^3 - 3x^2 + 2.5 = 0$ which lies between 1 and 2.
- Using Horner's method, find the largest real root of $x^3 - 4x + 2 = 0$ correct to three decimal places.
- Show that the root of the equation $x^4 + x^3 - 4x^2 - 16 = 0$ lies between 2 and 3. Find its value correct to two decimal places by Horner's method.
- Find the negative root of the equation $x^3 - 9x^2 + 18 = 0$ correct to two decimal places by Horner's method.
- Find the cube root of 25 by Horner's method correct to 3 decimal places.

28.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method (§ 2.10). But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. We now explain some direct and iterative methods of solution.

28.6 DIRECT METHODS OF SOLUTION

(1) Gauss elimination method*. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\} \quad \dots(1)$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

*See footnote p. 37.

$$\left. \begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ b'_2y + c'_2z = d'_2 \\ b'_3y + c'_3z = d'_3 \end{array} \right\} \quad \dots(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b'_2 \neq 0$, we eliminate y from the third equation of (2), by subtracting (b'_3/b'_2) times the second equation from the third equation. We thus, get the new system

$$\left. \begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ b'_2y + c'_2z = d'_2 \\ c''_3z = d''_3 \end{array} \right\} \quad \dots(3)$$

Here the second equation is the *pivotal equation* and b'_2 is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Obs. 1. On writing the given equations as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ i.e., $AX = D$, this method consists in *transforming the coefficient matrix A to upper triangular matrix* by elementary row transformations only.

Obs. 2. Clearly the method will fail if any one of the pivots a_1, b'_2 or c''_3 becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Obs. 3. *Partial and complete pivoting.* In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x . In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the *second pivot* by interchanging the second equation with the equation having the largest coefficient of y' . This process is continued till we arrive at the equation with the single variable. This modified procedure is called *partial pivoting*.

If we are not taken about the elimination of x, y, z in a specified order, then we choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting*. It is more complicated and does not appreciably improve the accuracy.

Example 28.13. Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.
(Mumbai, 2009)

	Check sum	
Solution. We have	$x + 4y - z = -5$	-1
	$x + y - 6z = -12$	-16
	$3x - y - z = 4$	5

...(i)
...(ii)
...(iii)

Step I. Operate (ii) - (i) and (iii) - 3(i) to eliminate x :

	Check sum	
	$-3y - 5z = -7$	-15
	$-13y + 2z = 19$	8

...(iv)
...(v)

Step II. Operate (v) - $\frac{13}{3}$ (iv) to eliminate y :

	Check sum	
	$\frac{71}{3}z = \frac{148}{3}$	73

...(vi)

Step III. By back-substitution, we get

$$\text{From (vi): } z = \frac{148}{71} = 2.0845$$

$$\text{From (iv): } y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$$

From (i) : $x = -5 - 4 \left(-\frac{81}{71} \right) + \frac{148}{71} = \frac{117}{71} = 1.6479$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise : We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$

and

$$x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$.

Example 28.14. Solve $10x - 7y + 3z + 5u = 6, -6x + 8y - z - 4u = 5, 3x + y + 4z + 11u = 2, 5x - 9y - 2z + 4u = 7$ by Gauss elimination method. (S.V.T.U., 2007)

Check sum		
Solution. We have	$10x - 7y + 3z + 5u = 6$	17
	$-6x + 8y - z - 4u = 5$	2
	$3x + y + 4z + 11u = 2$	21
	$5x - 9y - 2z + 4u = 7$	5

Step I. To eliminate x , operate $\left[(ii) - \left(\frac{-6}{10} \right) (i) \right], \left[(iii) - \left(\frac{3}{10} \right) (i) \right], \left[(iv) - \left(\frac{5}{10} \right) (i) \right]$:

Check sum

$$3.8y + 0.8z - u = 8.6 \quad 12.2 \quad ... (v)$$

$$3.1y + 3.1z + 9.5u = 0.2 \quad 15.9 \quad ... (vi)$$

$$-5.5y - 3.5z + 1.5u = 4 \quad -3.5 \quad ... (vii)$$

Step II. To eliminate y , operate $\left[(vi) - \left(\frac{3.1}{3.8} \right) (v) \right], \left[(vii) - \left(\frac{-5.5}{3.8} \right) (v) \right] :$

$$2.4473684z + 10.315789u = -6.8157895 \quad ... (viii)$$

$$-2.3421053z + 0.0526315u = 16.447368 \quad ... (ix)$$

Step III. To eliminate z , operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684} \right) (viii) \right] :$

$$9.9249319u = 9.9245977$$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

(2) Gauss-Jordan method*. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately reducing the system to a diagonal matrix form i.e., each equation involving only one unknown. From these equations the unknowns x, y, z can be obtained readily.

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

*See footnote p. 37.

Example 28.15. Apply Gauss-Jordan method to solve the equations

$$x + y + z = 9; 2x - 3y + 4z = 13; 3x + 4y + 5z = 40.$$

(V.T.U., 2009; P.T.U., 2005)

Solution. We have

$$x + y + z = 9 \quad \dots(i)$$

$$2x - 3y + 4z = 13 \quad \dots(ii)$$

$$3x + 4y + 5z = 40 \quad \dots(iii)$$

Step I. Operate (ii) - 2(i) and (iii) - 3(i) to eliminate x from (ii) and (iii).

$$x + y + z = 9 \quad \dots(iv)$$

$$-5y + 2z = -5 \quad \dots(v)$$

$$y + 2z = 13 \quad \dots(vi)$$

Step II. Operate (iv) + $\frac{1}{5}$ (v) and (vi) + $\frac{1}{5}$ (v) to eliminate y from (iv) and (vi) :

$$x + \frac{7}{5}z = 8 \quad \dots(vii)$$

$$-5y + 2z = -5 \quad \dots(viii)$$

$$\frac{12}{5}z = 12 \quad \dots(ix)$$

Step III. Operate (vii) - $\frac{7}{12}$ (ix) and (viii) - $\frac{5}{6}$ (ix) to eliminate z from (vii) and (viii) :

$$x = 1$$

$$-5y = -15$$

$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise : Rewriting the equations as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

Operate $R_2 - 2R_1, R_3 - 3R_1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

Operate $R_3 + \frac{1}{5}R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$$

Operate $-R_2, 5R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$$

Operate $R_2 + \frac{1}{6}R_3, \frac{1}{12}R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$$

Operate $\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$$

Operate $R_1 - R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Hence, $x = 1, y = 3, z = 5$.

Obs. Here the process of elimination of variables amounts to reducing the given coefficient metric to a diagonal matrix by elementary row transformations only.

Example 28.16. Solve the equations of example 28.14, by Gauss-Jordan method.

Solution. We have

$10x - 7y + 3z + 5u = 6$...(i)
$-6x + 8y - z - 4u = 5$...(ii)
$3x + y + 4z + 11u = 2$...(iii)
$5x - 9y - 2z + 4u = 7$...(iv)

Step I. To eliminate x , operate $\left[(ii) - \left(\frac{-6}{10} \right) (i) \right]$, $\left[(iii) - \left(\frac{3}{10} \right) (i) \right]$, $\left[(iv) - \left(\frac{5}{10} \right) (i) \right]$:

$10x - 7y + 3z + 5u = 6$...(v)
$3.8y + 0.8z - u = 8.6$...(vi)
$3.1y + 3.1z + 9.5u = 0.2$...(vii)
$-5.5y - 3.5z + 1.5u = 4$...(viii)

Step II. To eliminate y , operate $\left[(v) - \left(\frac{-7}{3.8} \right) (vi) \right]$, $\left[(vii) - \left(\frac{3.1}{3.8} \right) (vi) \right]$, $\left[(viii) - \left(\frac{-5.5}{3.8} \right) (vi) \right]$:

$10x + 4.4736842z + 3.1578947u = 21.842105$...(ix)
$3.8y + 0.8z - u = 8.6$...(x)
$2.4473684z + 10.315789u = -6.8157895$...(xi)
$-2.3421053x + 0.0526315u = 16.447368$...(xii)

Step III. To eliminate z , operate $\left[(ix) - \left(\frac{4.473684}{2.4473684} \right) (xi) \right]$,

$\left[(x) - \left(\frac{0.8}{2.4473684} \right) (xi) \right]$, $\left[(xii) - \left(\frac{-2.3421053}{2.4473684} \right) (xi) \right]$:
$10x - 15.698923u = 34.301075$
$3.8y - 4.3720429u = 10.827957$
$2.4473684z + 10.315789u = -6.8157895$
$9.9247309u = 9.9245975$

Step IV. From the last equation $u = 1$ nearly.

Substitution of $u = 1$ in the above three equations gives $x = 5$, $y = 4$, $z = -7$.

(3) Factorization method*. This method is based on the fact that every matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, i.e., if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

which can be written as

$$AX = B \quad \dots(1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let

$$A = LU, \quad \dots(2)$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

*Another name given to this decomposition is Doolittle's method.

Then (1) becomes $LUX = B$... (3)

Writing $UX = V$, ... (4)

(3) becomes $LV = B$ which is equivalent to the equations

$$v_1 = b_1; l_{21}v_1 + v_2 = b_2; l_{31}v_1 + l_{32}v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3 , we know V . Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1; u_{22}x_2 + u_{23}x_3 = v_2; u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by back-substitution.

To compute the matrices L and U , we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

$$(i) \quad u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$$

$$(ii) \quad l_{21}u_{11} = a_{21} \quad \text{or} \quad l_{21} = a_{21}/a_{11}$$

$$l_{31}u_{11} = a_{31} \quad \text{or} \quad l_{31} = a_{31}/a_{11}$$

$$(iii) \quad l_{21}u_{12} + u_{22} = a_{22} \quad \text{or} \quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$l_{21}u_{13} + u_{23} = a_{23} \quad \text{or} \quad u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

$$(iv) \quad l_{31}u_{12} + l_{32}u_{22} = a_{32} \quad \text{or} \quad l_{32} = \frac{1}{u_{22}} \left[a_{32} - \frac{a_{31}}{a_{11}} a_{12} \right]$$

$$(v) \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33} \quad \text{which gives } u_{33}.$$

Thus we compute the elements of L and U in the following set order :

- (i) First row of U ,
- (ii) First column of L ,
- (iii) Second row of U ,
- (iv) Second column of L ,
- (v) Third row of U .

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and for finding the inverse of a matrix. Among the direct methods, Factorization method is also preferred as the software for computers.

Example 28.17. Apply factorization method to solve the equations :

$$3x + 2y + 7z = 4; 2x + 3y + z = 5; 3x + 4y + z = 7.$$

(Madras, 2000 S)

Solution. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (i.e. A),

so that

- (i) $u_{11} = 3, u_{12} = 2, \quad u_{13} = 7.$
- (ii) $l_{21}u_{11} = 2, \quad \therefore l_{21} = 2/3$
- $l_{31}u_{11} = 3, \quad \therefore l_{31} = 1.$
- (iii) $l_{21}u_{12} + u_{22} = 3, \quad \therefore u_{22} = 5/3,$
 $l_{21}u_{13} + u_{23} = 1, \quad \therefore u_{23} = -11/3.$
- (iv) $l_{31}u_{12} + l_{32}u_{22} = 4, \quad \therefore l_{32} = 6/5.$
- (v) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$
 $\therefore u_{33} = -8/5$

$$\text{Thus } A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Writing $UX = V$, the given system becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Solving this system, we have $v_1 = 4$,

$$\begin{aligned} \frac{2}{3}v_1 + v_2 &= 5 & \text{or} & & v_2 &= \frac{7}{3} \\ v_1 + \frac{6}{5}v_2 + v_3 &= 7 & \text{or} & & v_3 &= \frac{1}{5} \end{aligned}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$$\text{i.e., } 3x + 2y + 7z = 4 ; \frac{5}{3}y - \frac{11}{3}z = \frac{7}{3} ; -\frac{8}{5}z = \frac{1}{5}$$

By back-substitution, we have $z = -1/8$, $y = 9/8$ and $x = 7/8$.

Example 28.18. Solve the equations of Example 28.14 by factorization method.

Solution. Let $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$ (i.e., A)

so that

- (i) R_1 of U : $u_{11} = 10$, $u_{12} = -7$, $u_{13} = 3$, $u_{14} = 5$
- (ii) C_1 of L : $l_{21} = -0.6$, $l_{31} = 0.3$, $l_{41} = 0.5$
- (iii) R_2 of U : $u_{22} = 3.8$, $u_{23} = 0.8$, $u_{24} = -1$
- (iv) C_2 of L : $l_{32} = 0.81579$, $l_{42} = -1.44737$
- (v) R_3 of U : $u_{33} = 2.44737$, $u_{34} = 10.31579$
- (vi) C_3 of L : $l_{43} = -0.95699$
- (vii) R_4 of U : $u_{44} = 9.92474$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6, v_2 = 8.6, v_3 = -6.81579, v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

$$\text{i.e., } 10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6, \\ 2.44737z + 10.31579u = -6.81579, \quad u = 1.$$

By back-substitution, we get $u = 1$, $z = -7$, $y = 4$, $x = 5$.

PROBLEMS 28.3

Solve the following equations by Gauss elimination method :

1. $2x + y + z = 10 ; 3x + 2y + 3z = 18 ; x + 4y + 9z = 16.$ (P.T.U., 2005)
2. $2x + 2y + z = 12 ; 3x + 2y + 2z = 8 ; 5x + 10y - 8z = 10.$ (W.B.T.U., 2004)
3. $2x - y + 3z = 9 ; x + y + z = 6 ; x - y + z = 2.$ (Bhopal, 2009)
4. $2x_1 + 4x_2 + x_3 = 3 ; 3x_1 + 2x_2 - 2x_3 = -2 ; x_1 - x_2 + x_3 = 6.$ (Marathwada, 2008)
5. $5x_1 + x_2 + x_3 + x_4 = 4 ; x_1 + 7x_2 + x_3 + x_4 = 12 ;$
 $x_1 + x_2 + 6x_3 + x_4 = -5 ; x_1 + x_2 + x_3 + 4x_4 = -6.$

Solve the following equations by Gauss-Jordan method :

6. $2x + 5y + 7z = 52 ; 2x + y - z = 0 ; x + y + z = 9.$ (V.T.U., 2010)
7. $2x - 3y + z = -1 ; x + 4y + 5z = 25 ; 3x - 4y + z = 2.$ (Kerala, 2003)
8. $x + 3y + 3z = 16 ; z + 4y + 3z = 18 ; x + 3y + 4z = 19.$ (Anna, 2005)
9. $2x + y + z = 10 ; 3x + 2y + 3z = 18 ; x + 4y + 9z = 16.$ (V.T.U., 2008)
10. $2x_1 + x_2 + 5x_3 + x_4 = 5 ; x_1 + x_2 - 3x_3 + 4x_4 = -1 ;$
 $3x_1 + 6x_2 - 2x_3 + x_4 = 8 ; 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2.$

Solve the following equations by factorization method :

11. $10x + y + z = 12 ; 2x + 10y + z = 13 ; 2x + 2y + 10z = 14.$ (Andhra, 2004 ; P.T.U., 2003)
12. $x + 2y + 3z = 14 ; 2x + 3y + 4z = 20 ; 3x + 4y + z = 14.$
13. $2x + 3y + z = 9 ; x + 2y + 3z = 6 ; 3x + y + 2z = 8.$
14. $2x_1 - x_2 + x_3 = -1 ; 2x_2 - x_3 + x_4 = 1 ; x_1 + 2x_3 - x_4 = -1 ; x_1 + x_2 + 2x_4 = 5.$

15. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ by Crout's method.

28.7 ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods :

(1) **Jacobi's iteration method***. Consider the equations

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\} \quad \dots(1)$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left. \begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned} \right\} \quad \dots(2)$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z . Substituting these on the right, we get the first approximations $x_1 = k_1, y_1 = k_2, z_1 = k_3$.

Substituting these on the right-hand sides of (2), the second approximations are given by

$$\begin{aligned} x_2 &= k_1 - l_1y_1 - m_1z_1 \\ y_2 &= k_2 - l_2x_1 - m_2z_1 \\ z_2 &= k_3 - l_3x_1 - m_3y_1 \end{aligned}$$

This process is repeated till the difference between two consecutive approximations is negligible.

*See footnote p. 215.

Example 28.19. Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places. (Anna, 2007)

Solution. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z), y = \frac{1}{10}(28.08 - x - z), z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration $x_1 = \frac{11.19}{10} = 1.119, y_1 = \frac{28.08}{10} = 2.808, z_1 = \frac{35.61}{10} = 3.561$

Second iteration $x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration $x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$$y_3 = \frac{1}{10}(28.03 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration $x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$$y_4 = \frac{1}{10}(28.03 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration $x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$.

Example 28.20. Solve, by Jacobi's iteration method, the equations

$$20x + y - 2z = 17; 3x + 20y - z = -18; 2x - 3y + 20z = 25.$$

(Bhopal, 2009)

Solution. We write the given equations in the form

$$\left. \begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned} \right\} \quad \dots(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (i), we get

$$x_1 = \frac{17}{20} = 0.85; y_1 = -\frac{18}{20} = -0.9; z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right of the equations (i), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20} (-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20} (25 - 2x_1 + 3y_1) = 1.1515$$

Substituting these values in the right sides of the equations (i), we have

$$x_3 = \frac{1}{20} (17 - y_2 + 2z_2) = 1.0134$$

$$y_3 = \frac{1}{20} (-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20} (25 - 2x_2 + 3y_2) = 1.0032$$

Substituting these values, we get

$$x_4 = \frac{1}{20} (17 - y_3 + 2z_3) = 1.0009$$

$$y_4 = \frac{1}{20} (-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20} (25 - 2x_3 + 3y_3) = 0.9993$$

Putting these values, we have

$$x_5 = \frac{1}{20} (17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20} (-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20} (25 - 2x_4 + 3y_4) = 0.9996$$

Again substituting these values, we get

$$x_6 = \frac{1}{20} (17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20} (-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20} (25 - 2x_5 + 3y_5) = 1.0000$$

The values in the 5th and 6th iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

(2) Gauss-Seidel iteration method*. This is a modification of the Jacobi's iteration method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0, z = z_0$ in the first of the equations (2) on page 837, we get

$$x_1 = k_1$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2) on page 837, we have

$$y_1 = k_2 - l_2 x_1 - m_2 z_0$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2) on page 837, we obtain

$$z_1 = k_3 - l_3 x_1 - m_3 y_1$$

and so on, i.e., as soon as new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Obs 1. Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Obs 2. Gauss-Seidel method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

*See footnote p. 37. After Philipp Ludwig Von Seidel (1821–1896) who also suggested a similar method.

Example 28.21. Apply Gauss-Seidel iteration method to solve the equations of Ex. 28.20.

(V.T.U., 2011 ; Rohtak, 2005 ; Madras, 2003)

Solution. We write the given equation in the form

$$x = \frac{1}{20} (17 - y + 2z); y = \frac{1}{20} (-18 - 3x + z); z = \frac{1}{20} (25 - 2x + 3y) \quad \dots(i)$$

We start from the approximation $x_0 = y_0 = z_0 = 0$. Substituting $y = y_0, z = z_0$ in the right side of the first of equations (i), we get

$$x_1 = \frac{1}{20} (17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1, z = z_0$ in the second of the equations (i), we have

$$y_1 = \frac{1}{20} (-18 - 3x_1 + z_0) = -1.0275$$

Putting $x = x_1, y = y_1$ in the last of the equations (i), we obtain

$$z_1 = \frac{1}{20} (25 - 2x_1 + 3y_1) = 1.0109$$

For the second iteration, we have

$$x_2 = \frac{1}{20} (17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20} (-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20} (25 - 2x_2 + 3y_2) = 0.9998$$

For the third iteration, we get

$$x_3 = \frac{1}{20} (17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20} (-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20} (25 - 2x_3 + 3y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

Example 28.22. Solve the equations :

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidal iteration method.

(Bhopal, 2009 ; J.N.T.U., 2004)

Solution. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \dots(i)$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \dots(ii)$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \dots(iii)$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad \dots(iv)$$

First iteration

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (i), we get $x_1 = 0.3$

Putting $x_1 = 0.3, x_3 = 0, x_4 = 0$ in (ii), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3, x_2 = 1.56, x_4 = 0$ in (iii), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3, x_2 = 1.56, x_3 = 2.886$ in (iv), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56, x_3 = 2.886, x_4 = -0.1368$ in (i), we obtain

$$x_1 = 0.8869$$

Putting $x_1 = 0.8869, x_3 = 2.886, x_4 = -0.1368$ in (ii), we obtain

$$x_2 = 1.9523$$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_4 = -0.1368$ in (iii), we have

$$x_3 = 2.9566$$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_3 = 2.9566$ in (iv), we get

$$x_4 = -0.0248.$$

Third iteration

Putting $x_2 = 1.9523, x_3 = 2.9566, x_4 = -0.0248$ in (i), we obtain

$$x_1 = 0.9836$$

Putting $x_1 = 0.9836, x_3 = 2.9566, x_4 = -0.0248$ in (ii), we obtain

$$x_2 = 1.9899$$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_4 = -0.0248$ in (iii), we get

$$x_3 = 2.9924$$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_3 = 2.9924$ in (iv), we get

$$x_4 = -0.0042.$$

Fourth iteration. Proceeding as above

$$x_1 = 0.9968, x_2 = 1.9982, x_3 = 2.9987, x_4 = -0.0008.$$

Fifth iteration is

$$x_1 = 0.9994, x_2 = 1.9997, x_3 = 2.9997, x_4 = -0.0001.$$

Sixth iteration is

$$x_1 = 0.9999, x_2 = 1.9999, x_3 = 2.9999, x_4 = -0.0001.$$

Hence the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0$.

(3) Relaxation method*. Consider the equations

$$a_1x + b_1y + c_1z = d_1; a_2x + b_2y + c_2z = d_2; a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z; R_y = d_2 - a_2x - b_2y - c_2z; R_z = d_3 - a_3x - b_3y - c_3z \quad \dots(1)$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step by giving increments to the variables. For this purpose, we construct the following *operation table*:

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; e.g., to reduce R_x by p , x should be increased by p/a_1 .

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Example 28.23. Solve, by Relaxation method, the equations :

$$9x - 2y + z = 50, x + 5y - 3z = 18, -2x + 2y + 7z = 19.$$

(Madras, 2000 S)

*This method was originally developed by R.V. Southwell in 1935, for application to structural engineering problems.

Solution. The residuals are given by

$$R_x = 50 - 9x + 2y - z; R_y = 18 - x - 5y + 3z; R_z = 19 + 2x - 2y - 7z$$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	3	-7

The relaxation table is

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	...(i)
$\delta x = 5$	5	13	29	...(ii)
$\delta z = 4$	1	25	1	...(iii)
$\delta y = 5$	11	0	-9	...(iv)
$\delta x = 1$	2	-1	-7	...(v)
$\delta z = -1$	3	-4	0	...(vi)
$\delta y = -0.8$	1.4	0	1.6	...(vii)
$\delta z = 0.23$	1.17	0.69	-0.09	...(viii)
$\delta x = 0.13$	0	0.56	0.17	...(ix)
$\delta y = 0.112$	0.224	0	-0.054	...(x)

$$\Sigma \delta x = 6.13, \Sigma \delta y = 4.31, \Sigma \delta z = 3.23$$

Thus $x = 6.13, y = 4.31, z = 3.23$.

[Explanation. In (i), the largest residual is 50. To reduce it, we give an increment $\delta x = 5$ and the resulting residuals are shown in (ii). Of these $R_x = 29$ is the largest and we give an increment $\delta z = 4$ to get the results in (iii). In (vi), $R_y = -4$ is the (numerically) largest and we give an increment $\delta y = -4/5 = -0.8$ to obtain the results in (vii). Similarly the other steps have been carried out.]

Example 28.24. Solve by Relaxation method, the equations :

$$10x - 2y - 3z = 205; -2x + 10y - 2z = 154; -2x - y + 10z = 120. (\text{V.T.U., 2011 S; Rohtak, 2005})$$

Solution. The residuals are given by

$$R_x = 205 - 10x + 2y + 3z; R_y = 154 + 2x - 10y + 2z; R_z = 120 + 2x + y - 10z.$$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-10	2	2
$\delta y = 1$	2	-10	-1
$\delta z = 1$	3	2	-10

The relaxation table is :

	R_x	R_y	R_z
$x = y = z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta z = 1$	-2	10	-1
$\delta y = 1$	0	0	0

$$\Sigma \delta x = 32, \Sigma y = 26, \Sigma z = 21.$$

Hence

$$x = 32, y = 26, z = 21.$$

PROBLEMS 28.4

1. Solve by Jacobi's method, the equations : $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$. Start with the solution $(2, 3, 0)$.
 2. Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.

by (a) Jacobi's method (b) Gauss-Seidel method.

(Anna, 2006)

Solve the following equations by Gauss-Seidel method :

3. $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$.
 4. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$.
 5. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 104$.
 6. $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$.
 7. $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$; $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$; $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$
 8. $1.2x + 2.1y + 4.2z = 9.9$; $5.3x + 6.1y + 4.7z = 21.6$; $9.2x + 8.3y + z = 15.2$.

$$9. \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

Solve by Relaxation method, the following sets of equations :

10. $3x + 9y - 2z = 11$; $4x + 2y + 13z = 24$; $4x - 4y + 3z = -8$.
 11. $10x - 2y - 2z = 6$; $-x + 10y - 2z = 7$; $-x - y + 10z = 8$.
 12. $-9x + 3y + 4z + 100 = 0$; $x - 7y + 3z + 80 = 0$; $2x + 3y - 5z + 60 = 0$.
 13. $54x + y + z = 110$; $2x + 15y + 6z = 72$; $-x + 6y + 27z = 85$.

(Bhopal, 2003)

28.8 SOLUTION OF NON-LINEAR SIMULTANEOUS EQUATIONS—NEWTON-RAPHSON METHOD

Consider the equations

$$f(x, y) = 0, g(x, y) = 0 \quad \dots(1)$$

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be obtained as follows :

Let $x_1 = x_0 + h$, $y_1 = y_0 + k$, so that $f(x_0 + h, y_0 + k) = 0$, $g(x_0 + h, y_0 + k) = 0$ $\dots(2)$

Expanding each of the functions in (2) by Taylor's series to first degree terms, we get approximately

$$\left. \begin{aligned} f_0 + h \frac{\partial f}{\partial x_0} + k \frac{\partial f}{\partial y_0} &= 0 \\ g_0 + h \frac{\partial g}{\partial x_0} + k \frac{\partial g}{\partial y_0} &= 0 \end{aligned} \right\} \quad \dots(3)$$

where $f_0 = f(x_0, y_0)$, $\frac{\partial f}{\partial x_0} = \left(\frac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.

Solving the equations (3) for h and k , we get a new approximation to the root as

$$x_1 = x_0 + h, y_1 = y_0 + k$$

This process is repeated till we get the values to the desired accuracy.

Example 28.25. Solve the system of non-linear equations :

$$x^2 + y = 11, y^2 + x = 7.$$

(Pune, 2000)

Solution. An initial approximation to the solution is obtained from a rough graph of the given equations, as $x_0 = 3.5$ and $y_0 = -1.8$.

We have $f = x^2 + y - 11$ and $g = y^2 + x - 7$ so that

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = 1, \frac{\partial g}{\partial y} = 2y.$$

Then Newton-Raphson's equations (3) above will be

$$7h + k = 0.55, h - 3.6k = 0.26$$

Solving these, we get $h = 0.0855$, $k = -0.0485$

∴ the better approximation to the root is

$$x_1 = x_0 + h = 3.5855, y_1 = y_0 + k = -1.8485$$

Repeating the above process, replacing (x_0, y_0) by (x_1, y_1) , we obtain $x_2 = 3.5844, y_2 = -1.8482$.

PROBLEMS 28.5

- Solve the equations $x^2 + y = 5 ; y^2 + x = 3$.
- Solve the non-linear equations $x = 2(y + 1), y^2 = 3xy - 7$ correct to three decimals.
- Use Newton-Raphson method to solve the equations $x = x^2 + y^2, y = x^2 - y^2$ correct to two decimals, starting with the approximation (0.8, 0.4).
- Solve the non-linear equations $x^2 - y^2 = 4, x^2 + y^2 = 16$ numerically with $x_0 = y_0 = 2.828$ using N.R. method. Carry out two iterations. (V.T.U., MCA, 2007)
- Solve the equations $2x^2 + 3xy + y^2 = 3 ; 4x^2 + 2xy + y^2 = 30$. Correct to three decimal places, using Newton-Raphson method, given that $x_0 = -3$, and $y_0 = 2$.

28.9 DETERMINATION OF EIGEN VALUES BY ITERATION

In § 2.14, we came across equations of the type

$$\left. \begin{array}{l} (a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 = 0 \end{array} \right\} \quad \dots(1)$$

which in matrix form, may be written as $[A - \lambda I]X = 0$ or $AX = \lambda X$... (2)

where $A = [a_{ij}]$ and X is the column matrix $[x_i]$.

Equation (1) will have a non-trivial solution if the coefficient matrix vanishes e.g.,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

This gives a cubic in λ whose roots are *eigen values* of (2) and corresponding to each *eigen value*, we have a non-zero solution $X = [x_1, x_2, x_3]$ which is called an *eigen vector*. Such an equation can ordinarily be solved easily.

In some applications, it is required to compute the numerically largest *eigen value* and the corresponding *eigen vector*. In such cases, the following iterative method is more convenient which is also well-suited for computing machines.

If X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$, then an arbitrary column vector can be written as $X = k_1X_1 + k_2X_2 + k_3X_3$

Then $AX = k_1AX_1 + k_2AX_2 + k_3AX_3 = k_1\lambda_1X_1 + k_2\lambda_2X_2 + k_3\lambda_3X_3$

Similarly $A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + k_3\lambda_3^2X_3$

and $A^rX = k_1\lambda_1^rX_1 + k_2\lambda_2^rX_2 + k_3\lambda_3^rX_3$

If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A , it becomes nearer to the eigen vector X_1 . Then we make the largest component of the resulting column vector unity to avoid the factor k_1 .

Thus we start with a column vector X which is as near the solution as possible and evaluate AX which is written as $\lambda^{(1)}X^{(1)}$ after normalisation. This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector. Similarly we evaluate $AX^{(1)} = \lambda^{(2)}X^{(2)}$ which gives the second approximation. We repeat this process till $|X^{(r)} - X^{(r-1)}|$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value of (1) and $X^{(r)}$, the corresponding eigen vector.

This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.*

*After the English mathematician and physicist John William Strut known as Lord Rayleigh (1842–1919) who made important contributions to the theory of waves, elasticity and hydrodynamics. He was professor at Cambridge and London.

Example 28.26. Determine the largest eigen value and the corresponding eigen vector of the matrices using the power method :

$$(i) A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(V.T.U., 2007)

Solution. (i) Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)} X^{(1)}$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8 \begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)} X^{(2)}$

Thus the second approximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen-vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

Now $AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966 \begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)} X^{(3)}$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994 \begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999 \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

(ii) Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$

So the first approximation to the eigen value is $\lambda^{(1)} = 2$ and the corresponding eigen vector $X^{(1)} = [1, -0.5, 0]'$.

Hence $AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$

Repeating the above process, we get

$$AX^{(2)} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)} ; AX^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)} ; AX^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)} ; AX^{(6)} = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

PROBLEMS 28.6

1. Find by power method, the larger eigen-value of the matrices :

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (Anna, 2005)

(b) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Obtain the largest eigen-value and the corresponding eigen-vector for the equations

$$(2 - \lambda)x_1 - x_2 = 0; -x_1 + (2 - \lambda)x_2 - x_3 = 0; -x_2 + (2 - \lambda)x_3 = 0$$

by Rayleigh Quotient method.

3. Find the dominant eigen value and the corresponding eigen vector of the following matrices using the power method :

(a) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ (V.T.U., 2011)

(b) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (V.T.U., 2011 S)

4. Find the largest eigen-value and the corresponding eigen-vector of the matrices :

(a) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ (Anna, 2005)

(b) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ (V.T.U., 2008)

(c) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $[1, 1, 0]^T$ (Madras, 2006)

28.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 28.7

Fill up the blanks or select the correct answer to each of the following problems :

- Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for
- If x_n is the n th iterate, then the Newton-Raphson formula is
- In the Regula-falsi method of finding the real root of an equation, the curve AB is replaced by
- Newton's iterative formula to find the value of \sqrt{N} is
- Newton-Raphson formula converges when
- In solving simultaneous equations by Gauss-Jordan method, the coefficient matrix is reduced to matrix.
- In the case of bisection method, the convergence is
 - (a) linear
 - (b) quadratic
 - (c) very slow.
- The order of convergence in Newton-Raphson method is
 - (a) 2
 - (b) 3
 - (c) 0
 - (d) none.
- The Newton-Raphson algorithm for finding the cube root of N is
- The bisection method for finding the root of an equation $f(x) = 0$ is
- In Regula-falsi method, the first approximation is given by
- The order of convergence in Newton-Raphson method is
 - (a) 2
 - (b) 3
 - (c) 0
 - (d) none.
- The iterative formula for finding the reciprocal of N is $x_{n+1} = \dots$
- As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
 - (a) Gauss-Jordan method
 - (b) Gauss-Seidal method
 - (c) Jacobi's method
 - (d) Relaxation method.
- Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for
- The difference between direct and iterative methods of solving simultaneous linear equations is
- To which form the coefficient matrix is transformed when $AX = B$ is solved by Gauss elimination method.
- Jacobi's iteration method can be used to solve a system of non-linear equations. (True or False)
- The convergence in the Gauss-Seidal method is thrice as fast as in Jacobi's method. (True or False)
- By Gauss elimination method, solve $x + y = 2$ and $2x + 3y = 5$. (Anna, 2007)

Finite Differences and Interpolation

1. Finite differences. 2. Differences of a polynomial. 3. Factorial notation. 4. Relations between the operators. 5. To find one or more missing terms. 6. Newton's interpolation formulae. 7. Central difference interpolation formulae—Gauss's interpolation formulae; Stirling's formula; Bessel's formula; Everett's formula. 8. Choice of an interpolation formula. 9. Interpolation with unequal intervals. 10. Lagrange's formula. 11. Divided differences. 12. Newton's divided difference formula. 13. Inverse interpolation. 14. Objective Type of Questions.

29.1 FINITE DIFFERENCES

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

$$\begin{array}{cccc} x : & x_0 & x_1 & x_2 \dots x_n \\ y : & y_0 & y_1 & y_2 \dots y_n \end{array}$$

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation* is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called *extrapolation*. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss. For a detailed study, the reader should refer to author's book '*Numerical Methods in Engineering and Science*'.

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$ giving $y = y_0, y_1, y_2, \dots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x , the following three types of differences are found useful:

(1) **Forward differences.** The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

$$\text{In general, } \Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

defines the *pth forward differences*.

These differences are systematically set out as follows in what is called a *Forward Difference Table*.

In a difference table, x is called the *argument* and y the *function* or the *entry*. y_0 , the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences*.

Obs. Any higher order forward difference can be expressed in terms of the entries.

$$\text{We have } \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficient, we have in general,

$$\Delta^n y_0 = y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2} - \dots + (-1)^n y_0$$

Forward Difference Table

Value of x	Value of y	1st. diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
$x_0 + h$	y_1	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	
$x_0 + 2h$	y_2	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$		
$x_0 + 3h$	y_3	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_1$	
$x_0 + 4h$	y_4	Δy_3	$\Delta^2 y_3$	$\Delta^3 y_3$		
$x_0 + 5h$	y_5	Δy_4				

(2) **Backward differences.** The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ respectively, are called the *first backward differences* where ∇ is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have

$$\begin{aligned}\nabla y_r &= y_r - y_{r-1}, \quad \nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, \\ \nabla^3 y_r &= \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}\end{aligned}$$

The differences are exhibited in the following :

Backward Difference Table

Value of x	Value of y	1st. diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
$x_0 + h$	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$	$\nabla^4 y_4$	
$x_0 + 2h$	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$		
$x_0 + 3h$	y_3	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_5$	$\nabla^4 y_5$	
$x_0 + 4h$	y_4	∇y_4	$\nabla^2 y_5$			
$x_0 + 5h$	y_5	∇y_5				

(3) **Central differences.** Sometimes it is convenient to employ another system of differences known as *central differences*. In this system, the *central difference operator* δ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \dots, \quad y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\begin{aligned}\delta y_{3/2} - \delta y_{1/2} &= \delta^2 y_1, \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, \dots, \\ \delta^2 y_2 - \delta^2 y_1 &= \delta^3 y_{3/2} \text{ and so on.}\end{aligned}$$

These differences are shown in the following :

Central Difference Table

Value of x	Value of y	1st. diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
x_0	y_0					
$x_0 + h$	y_1	$\delta y_{1/2}$	$\delta^2 y_1$	$\delta^3 y_{3/2}$	$\delta^4 y_2$	
$x_0 + 2h$	y_2	$\delta y_{3/2}$	$\delta^2 y_2$	$\delta^3 y_{5/2}$		
$x_0 + 3h$	y_3	$\delta y_{5/2}$	$\delta^2 y_3$	$\delta^3 y_{7/2}$	$\delta^4 y_3$	
$x_0 + 4h$	y_4	$\delta y_{7/2}$	$\delta^2 y_4$	$\delta^3 y_{9/2}$		
$x_0 + 5h$	y_5	$\delta y_{9/2}$				

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ . Thus

$$\mu \delta y_1 = \frac{1}{2} (\delta y_{1/2} + \delta y_{3/2}), \mu \delta^2 y_{3/2} = \frac{1}{2} (\delta^2 y_1 + \delta^2 y_2) \text{ etc.}$$

Obs. The reader should note that it is only the notation which changes and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

Of all the interpolation formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 29.1. Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$ (iii) $\Delta(x^2 / \cos 2x)$ (iv) $\Delta^2 \cos 2x$. (P.T.U., 2001)

Solution. (i) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1} \left\{ \frac{x+h-x}{1+(x+h)x} \right\} = \tan^{-1} \left\{ \frac{h}{1+hx+x^2} \right\}$$

$$\begin{aligned} \text{(ii)} \quad \Delta(e^x \log 2x) &= e^{x+h} \log 2(x+h) - e^x \log 2x \\ &= e^{x+h} \log 2(x+h) - e^{x+h} \log 2x + e^{x+h} \log 2x - e^x \log 2x \\ &= e^{x+h} \log \frac{x+h}{x} + (e^{x+h} - e^x) \log 2x \\ &= e^x \left[e^h \log \left(1 + \frac{h}{x} \right) + (e^h - 1) \log 2x \right] \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad \Delta \left(\frac{x^2}{\cos 2x} \right) &= \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} = \frac{(x+h)^2 \cos 2x - x^2 \cos 2(x+h)}{\cos 2(x+h) \cos 2x} \\ &= \frac{[(x+h)^2 - x^2] \cos 2x + x^2 [\cos 2x - \cos 2(x+h)]}{\cos 2(x+h) \cos 2x} \\ &= \frac{(2hx + h^2) \cos 2x + 2x^2 \sin(h) \sin(2x+h)}{\cos 2(x+h) \cos 2x} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad \Delta^2 \cos 2x &= \Delta[\cos 2(x+h) - \cos 2x] \\ &= \Delta \cos 2(x+h) - \Delta \cos 2x \\ &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x] \\ &= -2 \sin(2x+3h) \sin h + 2 \sin(2x+h) \sin h \\ &= -2 \sin h [\sin(2x+3h) - \sin(2x+h)] \\ &= -2 \sin h [2 \cos(2x+2h) \sin h] = -4 \sin^2 h \cos(2x+2h). \end{aligned}$$

Example 29.2. Evaluate (i) $\Delta^2 \left(\frac{5x+12}{x^2+5x+16} \right)$ (Mumbai, 2003) (ii) $\Delta^2(ab^x)$ (iii) $\Delta^n(e^x)$ interval of differencing being unity. (Rohtak, 2003)

$$\begin{aligned} \text{Solution. (i)} \quad \Delta^2 \left(\frac{5x+12}{x^2+5x+16} \right) &= \Delta^2 \left\{ \frac{5x+12}{(x+2)(x+3)} \right\} = \Delta^2 \left\{ \frac{2}{x+2} + \frac{3}{x+3} \right\} \\ &= \Delta \left\{ \Delta \left(\frac{2}{x+2} \right) + \Delta \left(\frac{3}{x+3} \right) \right\} = \Delta \left\{ 2 \left(\frac{1}{x+3} - \frac{1}{x+2} \right) + 3 \left(\frac{1}{x+4} - \frac{1}{x+3} \right) \right\} \\ &= -2 \Delta \left\{ \frac{1}{(x+2)(x+3)} \right\} - 3 \Delta \left\{ \frac{1}{(x+3)(x+4)} \right\} \\ &= -2 \left\{ \frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)} \right\} - 3 \left\{ \frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} = \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)} \\
 (ii) \quad \Delta(ab^x) &= a \Delta(b^x) = a(b^{x+1} - b^x) = ab^x(b-1) \\
 \Delta^2(ab^x) &= \Delta[\Delta ab^x] = a(b-1) \Delta(b^x) \\
 &= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 b^x. \\
 (iii) \quad \Delta e^x &= e^{x+1} - e^x = (e-1)e^x \\
 \Delta^2 e^x &= \Delta(\Delta e^x) = \Delta[(e-1)e^x] \\
 &= (e-1)\Delta e^x = (e-1)(e-1)e^x = (e-1)^2 e^x \\
 \text{Similarly } \Delta^3 e^x &= (e-1)^3 e^x, \Delta^4 e^x = (e-1)^4 e^x, \dots \text{ and } \Delta^n e^x = (e-1)^n e^x.
 \end{aligned}$$

29.2 DIFFERENCES OF A POLYNOMIAL

The n th differences of a polynomial of the n th degree are constant and all higher order differences are zero.

Let the polynomial of the n th degree in x , be

$$\begin{aligned}
 f(x) &= ax^n + bx^{n-1} + cx^{n-2} + \dots + k(x+h) + l \\
 \therefore \Delta f(x) &= f(x+h) - f(x) \\
 &= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \dots + kh \\
 &= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \dots + k'x + l' \quad \dots(1)
 \end{aligned}$$

where b' , c' , ..., l' are new constant coefficients.

Thus the first differences of a polynomial of the n th degree is a polynomial of degree $(n-1)$.

$$\begin{aligned}
 \text{Similarly } \Delta^2 f(x) &= \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x) \\
 &= anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \dots + k'h \\
 &= an(n-1)h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \dots + k'', \quad [\text{by (1)}]
 \end{aligned}$$

\therefore the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the n th differences we get a polynomial of degree zero i.e.

$$\Delta^n f(x) = an(n-1)(n-2)\dots 1 \cdot h^n = an! h^n \quad \dots(2)$$

which is a constant. Hence the $(n+1)$ th and higher differences of a polynomial of n th degree will be zero.

Obs. The converse of this theorem is also true i.e. if the n th differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n . This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of n th degree, if its n th order differences become nearly constant.

Example 29.3. Evaluate $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

$$\begin{aligned}
 \text{Solution. } \Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] &= \Delta^{10}[abcd x^{10} + (\)x^9 + (\)x^8 + \dots + 1] \\
 &= abcd \Delta^{10}(x^{10}) \quad [\because \Delta^{10}(x^n) = 0 \text{ for } n < 10] \\
 &= abcd (10!).
 \end{aligned}$$

29.3 (1) FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)\dots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial**.

In particular $[x] = x$, $[x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\dots(x-n+1)$

In case, the interval of differencing is h , then

$$[x]^n = x(x-h)(x-2h)\dots(x-(n-1)h)$$

which is called a *Factorial polynomial or function*.

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

The result of differencing $[x]^r$ is analogous to that of differentiating x^r .

(2) To express a polynomial in the factorial notation

- (i) arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros;
- (ii) using detached coefficients divide by $x, x - 1, x - 2, \text{etc.}$ successively.

Obs. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice versa.

Example 29.4. Express $y = 2x^3 - 3x^2 + 3x - 10$ in a factorial notation and hence show that $\Delta^3y = 12$.

(Bhopal, 2007; P.T.U., 2005)

Solution. First method : Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Then

	x^3	x^2	x	
1	2	-3	3	$-10 = D$
	—	2	-1	
2	2	-1		$2 = C$
	—	4		
3	2	3 = B		
	—			
			2 = A	

Hence

$$y = 2[x]^3 + 3[x]^2 + 2[x] - 10$$

∴

$$\Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$$

$$\Delta^2 y = 6 \times 2[x] + 6$$

$\Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

Obs. The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.

Second method (Direct method) :

Let

$$\begin{aligned} y &= 2x^3 - 3x^2 + 3x - 10 \\ &= 2x(x-1)(x-2) + Bx(x-1) + Cx + D \end{aligned}$$

Putting $x = 0, -10 = D$

Putting $x = 1, 2 - 3 + 3 - 10 = C + D$

∴ $C = -8 - D = -8 + 10 = 2$

Putting $x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

∴ $B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3$.

Hence $y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10 = 2[x]^3 + 3[x]^2 + 2[x] - 10$

∴ $\Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12$.

Example 29.5. Find the missing values in the following table :

$x :$	45	50	55	60	65
$y :$	3.0	—	2.0	—	-2.4

(Bhopal, 2007; V.T.U., 2001)

Solution. The difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
50	y_1	$y_1 - 3$	$5 - 2y_1$	$3y_1 + y_3 - 9$
55	$y_2 = 2$	$2 - y_1$	$y_1 + y_3 - 4$	$3.6 - y_1 - 3y_3$
60	y_3	$-2.4 - y_3$	$-0.4 - 2y_3$	
65	$y_4 = -2.4$			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$$\therefore \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

$$\text{i.e.,} \quad 3y_1 + y_3 = 9; \quad y_1 + 3y_3 = 3.6$$

Solving these, we get $y_1 = 2.925, y_3 = 0.225$.

Otherwise : As only three entries $y_0 = 3, y_2 = 2, y_4 = -2.4$ are given, the function y can be represented by a second degree polynomial.

$$\therefore \Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

$$\text{i.e.,} \quad (E - 1)^3 y_0 = 0 \quad \text{and} \quad (E - 1)^3 y_1 = 0$$

$$\text{i.e.,} \quad (E^3 - 3E^2 + 3E - 1)y_0 = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1)y_1 = 0$$

$$y_3 - 3y_2 + 3y_1 - y_0 = 0$$

$$y_4 - 3y_3 + 3y_2 - y_1 = 0$$

$$\text{i.e.,} \quad y_3 + 3y_1 = 9; \quad 3y_3 + y_1 = 3.6$$

Solving these, we get $y_1 = 2.925, y_3 = 0.225$.

Example 29.6. Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values :

$x :$	0	1	2	3	4	5	6	7
$y :$	1	-1	1	-1	1	—	—	—

Solution. We construct the following difference table from the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$	—	—	—	—
1	$y_1 = -1$	-2	4	—	—
2	$y_2 = 1$	2	-4	-8	16
3	$y_3 = -1$	-2	4	8	16
4	$y_4 = 1$	2	$\Delta^2 y_3$	$\Delta^3 y_2$	16
5	y_5	Δy_4	$\Delta^2 y_4$	$\Delta^2 y_3$	16
6	y_6	Δy_5	$\Delta^2 y_5$	$\Delta^3 y_4$	16
7	y_7	Δy_6	—	—	—

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

\therefore The other fourth order differences must also be 16. Thus

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

$$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$

$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$

$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

$$\text{and} \quad y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$, we get

$$\Delta^3 y_3 = 40, \Delta^2 y_4 = 68, \Delta y_5 = 98, y_6 = 129.$$

Starting with $\Delta^4 y_3 = 16$, we obtain

$$\Delta^3 y_4 = 56, \Delta^2 y_5 = 124, \Delta y_6 = 222, y_7 = 351.$$

PROBLEMS 29.1

1. Construct the table of differences for the data below :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

2. If $u_0 = 3, u_1 = 12, u_2 = 18, u_3 = 2000, u_4 = 100$, calculate Δu_0 .

3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.

4. Form the table of backward differences of the function

$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$

5. Form a table of differences for the function

$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$

Continue the table to obtain $f(6)$.

6. Extend the following table to two more terms on either side by constructing the difference table :

x :	-2	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

7. Show that

$$(i) \Delta \left[\frac{1}{f(x)} \right] = \frac{-\Delta f(x)}{f(x) f(x+1)} ; \quad (Raipur, 2005) \quad (ii) \Delta \log f(x) = \log \left\{ 1 + \frac{\Delta f(x)}{f(x)} \right\}.$$

8. Evaluate :

$$(i) \Delta (x + \cos x) \quad (ii) \Delta \tan^{-1} \left(\frac{n-1}{n} \right) \quad (iii) \Delta \left\{ \frac{1}{x(x+4)(x+6)} \right\} \quad (Madras, 2001)$$

$$(iv) \Delta^2 \left(\frac{1}{x^2 + 5x + 6} \right) \quad (P.T.U., 2001)$$

9. Evaluate :

$$(i) \Delta(e^{ax} \log 2x) \quad (ii) \Delta(2^x/x!) \quad (iii) \Delta^n(a^x) \quad (Burdwan, 2003) \quad (iv) \Delta^n \left(\frac{1}{x} \right).$$

10. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression.

(Mumbai, 2003)

11. Prove that

$$(i) y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0 \quad (ii) \nabla^2 y_8 = y_8 - 2y_7 + y_6 ; \quad \delta^2 y_5 = y_6 - 2y_5 + y_4.$$

12. Evaluate :

$$(i) \Delta^3 [(1-x)(1-2x)(1-3x)]$$

$$(ii) \Delta^{10} [(1-x)(1-2x^2)(1-3x^3)(1-4x^4)], \text{ if the interval of differencing is 2.}$$

13. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$

(P.T.U., 2001)

14. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.

15. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.

16. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.

17. Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.

18. If $u(x)$ and $v(x)$ be two functions of x , prove that

$$(i) \Delta[u(x)v(x)] = u(x)\Delta v(x) + v(x+1)\Delta u(x). \quad (ii) \Delta \left\{ \frac{u(x)}{v(x)} \right\} = \frac{v(x)\Delta u(x) - u(x)\Delta v(x)}{v(x)v(x+1)}.$$

29.4 (1) OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ , ∇ and δ . Besides these, there are the operators E and μ , which we define below :

- (i) **Shift operator E** is the operation of increasing the argument x by h so that

$$Ef(x) = f(x+h), E^2 f(x) = f(x+2h), E^3 f(x) = f(x+3h) \text{ etc.}$$

The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}, E^{-1}y_x = y_{x-h}, E^n y_x = y_{x+nh}$,

where n may be any real number.

(ii) **Averaging operator** μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2})$

Obs. In the difference calculus, Δ and E are regarded as the fundamental operators and ∇, δ, μ can be expressed in terms of these.

(2) Relations between the operators. We shall now establish the following identities :

$$(i) \Delta = E - 1$$

$$(ii) \nabla = 1 - E^{-1}$$

$$(iii) \delta = E^{1/2} - E^{-1/2}$$

$$(iv) \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

$$(v) \Delta = EV = \nabla E = \delta E^{1/2}$$

$$(vi) E = e^{hD}$$

Proofs. (i) $\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E - 1) y_x$.

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

$$(ii) \nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1}) y_x$$

$$\therefore \nabla = 1 - E^{-1} \quad \text{or} \quad E = (1 - \nabla)^{-1}$$

$$(iii) \delta y_x = y_{x+h/2} - y_{x-h/2} = E^{1/2} y_x - E^{-1/2} y_x = (E^{1/2} - E^{-1/2}) y_x$$

$$\therefore \delta = E^{1/2} - E^{-1/2}.$$

$$(iv) \mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2}) = \frac{1}{2}(E^{1/2} y_x + E^{-1/2} y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2}) y_x$$

$$\therefore \mu = \frac{1}{2}(E^{1/2} + E^{-1/2}).$$

$$(v) EV y_x = E(y_x - y_{x-h}) = E y_x - E y_{x-h} = y_{x+h} - y_x = \Delta y_x \quad \therefore EV = \Delta$$

$$\text{Also } \nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+h/2} = y_{x+h/2} - y_{x+h/2-h/2} = y_{x+h} - y_x = \Delta y_x$$

$$\therefore \delta E^{1/2} = \Delta$$

$$\text{Hence } \Delta = EV = \nabla E = \delta E^{1/2}.$$

$$(vi) Ef(x) = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots$$

[By Taylor's series]

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \dots = \left(1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \dots \right) f(x) = e^{hD} f(x)$$

$$\therefore E = e^{hD}$$

$$\text{Cor. 1. } E = 1 + \Delta = e^{hD}.$$

$$2. D = \frac{1}{h} \log(1 + \Delta) = \frac{1}{h} \left(\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \dots \right)$$

(Burdwan, 2003)

Note. A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E .

(3) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E		$\Delta + 1$	$(1 + \nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1 - \nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1 +)^{-1} - 1$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1 + \Delta)^{-1/2}$	$\nabla(1 - \nabla)^{-1/2}$	—	$2 \sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1 + \Delta/2)(1 + \Delta)^{-1/2}$	$(1 + \nabla/2)(1 + \nabla)^{-1/2}$	$\sqrt{(1 + \delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1 + \Delta)$	$\log(1 -)^{-1}$	$2 \sinh^{-1}(\delta/2)$	

Example 29.7. Prove that

$$e^x = \left(\frac{\Delta^2}{E} \right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}, \text{ the interval of differencing being } h. \quad (\text{Bhopal, 2009})$$

Solution. Since $\left(\frac{\Delta^2}{E} \right) e^x = \Delta^2 \cdot E^{-1} e^x = \Delta^2 e^{x-h} = \Delta^2 e^x \cdot e^{-h} = e^{-h} \Delta^2 e^x$

$$\therefore \text{R.H.S.} = e^{-h} \Delta^2 e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} \cdot e^{x+h} = e^x.$$

Example 29.8. Prove with the usual notations, that

$$(i) hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta) \quad (\text{Rohtak, 2005})$$

$$(ii) (E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = 2 + \Delta \quad (\text{Bhopal, 2009; U.P.T.U., 2009})$$

$$(iii) \Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$$

$$(iv) \Delta^3 y_2 = \nabla^3 y_5.$$

Solution. (i) We know that $e^{hD} = E = 1 + \Delta \quad \therefore hD = \log(1 + \Delta)$

$$\text{Also } hD = \log E = -\log(E^{-1}) = -\log(1 - \nabla) \quad [\because E^{-1} = 1 - \nabla]$$

$$\text{We have proved that } \mu = \frac{1}{2}(E^{1/2} + E^{-1/2}) \text{ and } \delta = E^{1/2} - E^{-1/2}$$

$$\therefore \mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

i.e.

$$hD = \sinh^{-1}(\mu\delta).$$

$$\text{Hence } hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$$

$$(ii) (E^{1/2} + E^{-1/2})(1 + \Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta.$$

$$(iii) \frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$$

$$\begin{aligned} &= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]} \\ &= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]} \\ &= \frac{1}{2}(E + E^{-1} - 2) + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2}) \\ &= \frac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \frac{1}{2}(2E - 2) = E - 1 = \Delta. \end{aligned}$$

$$(iv) \Delta^3 y_2 = (E - 1)^3 y_2 \quad [\because \Delta = E - 1] \quad (1)$$

$$= (E^3 - 3E^2 + 3E - 1)y_2 = y_5 - 3y_4 + 3y_3 - y_2$$

$$\begin{aligned} \nabla^3 y_5 &= (1 - E^{-1})^3 y_5 \quad [\because \Delta = 1 - E^{-1}] \\ &= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5 = y_5 - 3y_4 + 3y_3 - y_2 \end{aligned} \quad (2)$$

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

29.5 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method : We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b . These give the missing term/terms.

Second method : If n entries of y are given, $f(x)$ can be represented by a $(n-1)$ th degree polynomial i.e., $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E - 1)^n y = 0$. Now expanding $(E - 1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 29.9. Find the missing term in the table :

$x :$	2	3	4	5	6
$y :$	45.0	49.2	54.1	...	67.4

(U.P.T.U., 2008)

Solution. Let the missing term be a . Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	45.0 ($= y_0$)		4.2		
3	49.2 ($= y_1$)		49.9	0.7	$a - 59.7$
4	54.1 ($= y_2$)		$a - 54.1$	$a - 59.0$	$240.2 - 4a$
5	a ($= y_3$)			$121.5 - a$	
6	67.4 ($= y_4$)		$67.4 - a$		

We know that $\Delta^4 y = 0$ i.e., $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise: As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore \Delta^3 y = \text{constant}$ or $\Delta^4 y = 0$ i.e., $(E - 1)^4 = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1) = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \text{ or } -4a = -240.2$$

Hence $a = 60.05$.

Example 29.10. Find the missing values in the following data :

x :	45	50	55	60	65
y :	3.0	...	2.0	...	-2.4

(Bhopal, 2007)

Solution. Let the missing value be a, b . Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3($= y_0$)		$a - 3$	
50	a ($= y_1$)			$5 - 2a$
55	2($= y_2$)		$2 - a$	$3a + b - 9$
60	b ($= y_3$)		$b - 2$	$3.6 - a - 3b$
65	-2.4 ($= y_4$)		$-2.4 - b$	

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $3a + b = 9, a + 3b = 3.6$

Solving these, we get $a = 2.925, b = 0.0225$.

Otherwise. As only three entries $y_0 = 3, y_2 = 2, y_4 = -2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore \Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

$(E - 1)^3 y_0 = 0$ and $(E - 1)^3 y_1 = 0$

i.e., $(E^3 - 3E^2 + 3E - 1) y_0 = 0 ; (E^3 - 3E^2 + 3E - 1) . y_1 = 0$

or $y_3 - 3y_2 + 3y_1 - y_0 = 0 ; y_4 - 3y_3 + 3y_2 - y_1 = 0$

or $y_3 + 3y_1 = 9 ; 3y_3 + y_1 = 3.6$

Solving three, we get $y_1 = 2.925, y_2 = 0.0225$.

Example 29.11. If $y_{10} = 3, y_{11} = 6, y_{12} = 11, y_{13} = 18, y_{14} = 27$, find y_4

(Mumbai, 2005)

Solution. Taking y_{14} as u_0 , we are required to find y_4 i.e., u_{-10} . Then the difference table is

x	u	Δu	$\Delta^2 u$
x_{-4}	$y_{10} = u_{-4} = 3$	3	
x_{-3}	$y_{11} = u_{-3} = 6$	5	2
x_{-2}	$y_{12} = u_{-2} = 11$	7	2
x_{-1}	$y_{13} = u_{-1} = 18$	9	0
x_0	$y_{14} = u_0 = 27$		

Then

$$\begin{aligned} y_4 &= u_{-10} = (E^{-1})^{10} u_0 = (1 - \nabla)^{10} u_0 \\ &= \left(1 - 10\nabla + \frac{10 \cdot 9}{2} \nabla^2 - \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \nabla^3 + \dots \right) u_0 \\ &= u_0 - 10\nabla u_0 + 45\nabla^2 u_0 - 120\nabla^3 u_0 \\ &= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27. \end{aligned}$$

Example 29.12. If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845, y_2 + y_6 = 686, y_3 + y_5 = 1088$, find y_4

(Mumbai, 2004)

Solution. Starting with y_1 instead of y_0 , we note that $\Delta^6 y_1 = 0$

[$\because \Delta^5 y_1$ is constant.]

i.e.,

$$(E - 1)^6 y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_1 = 0$$

$$\therefore y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$$

or

$$(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$$

i.e.

$$\begin{aligned} y_4 &= \frac{1}{20} [(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)] \\ &= \frac{1}{20} [-784 - 6(686) + 15(1088)] = 571. \end{aligned}$$

Example 29.13. Prove the following identities :

$$(i) u_1 x + u_2 x^2 + u_3 x^3 + \dots = \frac{x}{1-x} u_1 + \left(\frac{x}{1-x} \right)^2 \Delta u_1 + \left(\frac{x}{1-x} \right)^3 \Delta^2 u_1 + \dots$$

$$(ii) u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \dots = e^x \left(u_0 + x \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \frac{x^3}{3!} \Delta^3 u_0 + \dots \right).$$

$$\text{Solution. (i) L.H.S.} = xu_1 + x^2 Eu_1 + x^3 E^2 u_1 + \dots = x(1 + xE + x^2 E^2 + \dots) u_1$$

[$\because u_{x+h} = E^h u_x$]

$$= x \cdot \frac{1}{1-xE} u_1, \text{ taking sum of infinite G.P.}$$

$$= x \left[\frac{1}{1-x(1+\Delta)} \right] u_1 \quad [\because E = 1 + \Delta]$$

$$= x \left(\frac{1}{1-x-x\Delta} \right) u_1 = \frac{x}{1-x} \left(1 - \frac{x\Delta}{1-x} \right)^{-1} u_1 = \frac{x}{1-x} \left(1 + \frac{x\Delta}{1-x} + \frac{x^2 \Delta^2}{(1-x)^2} + \dots \right) u_1$$

$$= \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^2}{(1-x)^3} \Delta^2 u_1 + \dots = \text{R.H.S.}$$

$$\begin{aligned}
 (ii) \quad \text{L.H.S.} &= u_0 + \frac{x}{1!} Eu_0 + \frac{x^2}{2!} E^2 u_0 + \frac{x^3}{3!} E^3 u_0 + \dots \\
 &= \left(1 + \frac{xE}{1!} + \frac{x^2 E^2}{2!} + \frac{x^3 E^3}{3!} + \dots \right) u_0 = e^{xE} u_0 = e^{x(1+\Delta)} u_0 \\
 &= e^x \cdot e^{x\Delta} u_0 = e^x \left(1 + \frac{x\Delta}{1!} + \frac{x^2 \Delta^2}{2!} + \frac{x^3 \Delta^3}{3!} + \dots \right) u_0 \\
 &= e^x \left(u_0 + \frac{x}{1!} \Delta u_0 + \frac{x^2}{2!} \Delta^2 u_0 + \frac{x^3}{3!} \Delta^3 u_0 + \dots \right) = \text{R.H.S.}
 \end{aligned}$$

PROBLEMS 29.2

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{Eu_x}$. (Madras, 2003)
2. Evaluate taking h as the interval of differencing :
- (i) $\left(\frac{\Delta^2}{E}\right) \sin x$ (ii) $\left(\frac{\Delta^2}{E}\right) x^4$, ($h = 1$) (W.B.T.U., 2005)
 - (iii) $\left(\frac{\Delta^2}{E}\right) \sin(x+h) + \frac{\Delta^2 \sin(x+h)}{E \sin(x+h)}$ (iv) $(\Delta + \nabla)^2 (x^2 + x)$, ($h = 1$). (Mumbai, 2005)
3. With the usual notations, show that
- (i) $\nabla = 1 - e^{-hD}$ (ii) $D = \frac{2}{h} \sinh^{-1}\left(\frac{\delta}{2}\right)$ (U.P.T.U., 2009)
 - (iii) $(1 + \Delta)(1 - \nabla) = 1$. (iv) $\Delta - \nabla = \nabla \Delta = \delta^2$. (Mumbai, 2005)
4. Prove that
- (i) $\delta = \Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2}$ (ii) $\mu^2 = 1 + \frac{\delta^2}{4}$ (U.P.T.U., 2009)
 - (iii) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$ (iv) $\nabla = \Delta E^{-1} = E^{-1} \Delta = 1 - E^{-1}$ (U.P.T.U., MCA, 2008)
5. Show that (i) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (ii) $1 + \delta^2/2 = \sqrt{(1 + \delta^2/\mu^2)}$ (U.P.T.U., MCA, 2008)
- (iii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (iv) $\nabla^2 = h^2 D^2 - h^3 D^3 + \frac{7}{12} h^4 D^4 - \dots$ (U.P.T.U., 2009)
6. Prove that
- (i) $\nabla^r f_k = \Delta^r f_{k-r}$ (ii) $\Delta f_k^2 = (f_k + f_{k+1}) \Delta f_k$ (J.N.T.U., MCA, 2006)
 - (iii) $\Delta = \frac{1}{2} \delta^2 + \delta \sqrt{1 + \frac{\delta^2}{4}}$ (iv) $E^{1/2} = (1 + \delta^2/4)^{1/2} + \delta/2$. (S.V.T.U., 2007)
7. Prove that $\nabla y_{n+1} = h \left(1 + \frac{1}{2} \nabla + \frac{5}{12} \nabla^2 + \dots \right) y'_n$.
8. The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.
- | | | | | | | | |
|-------|---|---|----|-----|------|------|------|
| $x :$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $y :$ | 1 | 2 | 33 | 254 | 1025 | 3126 | 7777 |
- (Mumbai, 2004)
9. Estimate the missing term in the following table :
- | | | | | | |
|----------|---|---|---|---|----|
| $x :$ | 0 | 1 | 2 | 3 | 4 |
| $f(x) :$ | 1 | 3 | 9 | — | 81 |
- (S.V.T.U., 2007)
10. Find the missing terms of the following data :
- | | | | | | | | |
|----------|---|-----|----|-----|---|-----|---|
| $x :$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $f(x) :$ | 6 | ? | 10 | 20 | ? | 1.5 | 5 |
- (U.P.T.U., 2010)

11. Find the missing values in the following table :

$x :$	0	1	2	3	4	5	6
$y :$	5	11	22	40	...	140	...

(V.T.U., 2006)

12. If $u_{13} = 1, u_{14} = -3, u_{15} = -1, u_{16} = 13$ find u_8 .

(Mumbai, 2004)

13. Evaluate y_4 from the following data (stating the assumptions you make) :

$$y_0 + y_8 = 1.9243, y_1 + y_7 = 1.9590, y_2 + y_6 = 1.9823, y_3 + y_5 = 1.9956.$$

(Mumbai, 2003)

14. Using the method of separation of symbols, prove that

$$(i) u_0 + u_1 + u_2 + \dots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta^2 u_0 + \dots + {}^{n+1}C_{n+1} \Delta^n u_0.$$

$$(ii) y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \dots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}.$$

15. Using the method of finite differences, sum the following series :

$$(i) 2.5 + 5.8 + 8.11 + 11.14 + \dots \text{ to } n \text{ terms.}$$

$$(ii) 1.2.3 + 2.3.4 + 3.4.5 + \dots \text{ to } n \text{ terms.}$$

16. Prove that $u_0 + u_1 x + u_2 x^2 + \dots = \frac{u_0}{1-x} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \dots \infty$

Hence sum the series $1.2 + 2.3x + 3.4x^2 + \dots \infty$.

29.6 NEWTON'S INTERPOLATION FORMULAE*

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

(1) **Newton's forward interpolation formula.** Let the function $y = f(x)$ take the values y_0, y_1, y_2, \dots corresponding to the values $x_0, x_0 + h, x_0 + 2h, \dots$ of x . Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p , we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$\therefore y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0 \quad [\because E = 1 + \Delta]$$

$$= \left\{ 1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right\} y_0 \quad [\text{Using Binomial theorem}]$$

$$\text{i.e., } y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \quad \dots(1)$$

It is called **Newton's forward interpolation formula** as (1) contains y_0 and the forward differences of y_0 .

Obs. This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0 .

(2) **Newton's backward interpolation formula.** Let the function $y = f(x)$ take the values y_0, y_1, y_2, \dots corresponding to the values $x_0, x_0 + h, x_0 + 2h, \dots$ of x . Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then we have

$$\begin{aligned} y_p &= f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \quad [\because E^{-1} = 1 - \nabla] \\ &= \left[1 + p\nabla + \frac{p(p+1)}{2!} \nabla^2 + \frac{p(p+1)(p+2)}{3!} \nabla^3 + \dots \right] y_n \quad [\text{Using Binomial theorem}] \end{aligned}$$

$$\text{i.e., } y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots \quad \dots(2)$$

It is called **Newton's backward interpolation formula** as (2) contains y_n and backward differences of y_n .

Obs. This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of y_n .

Example 29.14. The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :

$x = \text{height} :$	100	150	200	250	300	350	400
$y = \text{distance} :$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) $x = 218$ ft (Madras, 2003 S) (ii) 410 ft.

(V.T.U., 2002)

*See foot note p.466.

Solution. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63	2.40			
150	13.03	2.01	-0.39	0.15	
200	15.04	1.77	-0.24	0.08	-0.07
250	16.81	1.61	-0.16	0.03	-0.05
300	18.42	1.48	-0.13	0.02	-0.01
350	19.90	1.37	-0.11		
400	21.27				

(i) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

$$\text{Since } x = 218 \text{ and } h = 50, \therefore p = \frac{x - x_0}{h} = \frac{18}{50} = 0.36$$

∴ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3} \Delta^3 y_0 + \dots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2} (-0.16) + \frac{0.36(-0.64)(-1.64)}{6} (0.03) + \dots \\ = 15.04 + 0.637 + 0.018 + 0.001 + \dots = 15.696 \text{ i.e., 15.7 nautical miles}$$

(ii) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$$\therefore \text{taking } x_n = 400, p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$$

Using the line of backward differences

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

∴ newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2} \nabla^2 y_{400} + \frac{p(p+1)(p+2)}{1.2.3} \nabla^3 y_{400} + \dots \\ = 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2} (-0.11) + \dots = 21.53 \text{ nautical miles.}$$

Example 29.15. From the following table, estimate the number of students who obtained marks between 40 and 45 :

Marks	: 30—40	40—50	50—60	60—70	70—80
No. of Students :	31	42	51	35	31

(V.T.U., 2011 S ; S.V.T.U., 2007 ; Madras, 2006)

Solution. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of Students (y_x) :	31	73	124	159	190

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31	42			
50	73	51	9	-25	
60	124	35	-16	12	37
70	159	31	-4		
80	190				

We shall find y_{45} i.e. number of students with marks less than 45.

Taking $x_0 = 40$, $x = 45$, we have $p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$ [∴ $h = 10$]

∴ using Newton's forward interpolation formula, we get

$$\begin{aligned}y_{45} &= y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\Delta^3 y_{40} + \dots \\&= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37 \\&= 47.87, \text{ on simplification.}\end{aligned}$$

∴ the number of students with marks less than 45 is 47.87 i.e., 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 - 31 = 17.

Example 29.16. Find the cubic polynomial which takes the following values :

x	: 0	1	2	3
$f(x)$: 1	2	1	10

Hence or otherwise evaluate $f(4)$.

(Bhopal, 2009 ; Rohtak, 2005 ; W.B.T.U., 2005)

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1		1	
1	2	-1		-2
2	1	9		12
3	10			

We take $x_0 = 0$ and $p = \frac{x - 0}{h} = x$

[∴ $h = 1$]

∴ using Newton's forward interpolation formula, we get

$$\begin{aligned}f(x) &= f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1 \cdot 2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3}\Delta^3 f(0) \\&= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12) \\&= 2x^3 - 7x^2 + 6x + 1, \text{ which is the required polynomial.}\end{aligned}$$

To compute $f(4)$, we take $x_n = 3$, $x = 4$ so that $p = \frac{x - x_n}{h} = 1$

[∴ $h = 1$]

Using Newton's backward interpolation formula, we get

$$\begin{aligned}f(4) &= f(3) + p\nabla f(3) + \frac{p(p+1)}{1 \cdot 2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1 \cdot 2 \cdot 3}\nabla^3 f(3) \\&= 10 + 9 + 10 + 12 + 41.\end{aligned}$$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.

Example 29.17. In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, 2007)

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
4	8.4	3.6			
5	14.5	6.1	2.5	0.5	
6	23.6	9.1	3.0	0.5	0
7	36.2	12.6	3.5	0.5	0
8	52.8	16.6	4.0	0.5	0
9	73.9	21.1	4.5		

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1, h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9, x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

PROBLEMS 29.3

1. Using Newton's forward formula, find the value of $f(1.6)$, if

$x :$	1	1.4	1.8	2.2		
$f(x) :$	3.49	4.82	5.96	6.5		(J.N.T.U., 2006)

2. State Newton's interpolation formula and use it to calculate the value of $\exp(1.85)$, given the following table :

$x :$	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x) :$	5.474	6.050	6.686	7.389	8.166	9.025	9.974

3. If $f(1.15) = 1.0723, f(1.20) = 1.0954, f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.

4. Given $\sin 45^\circ = 0.7071, \sin 50^\circ = 0.7660, \sin 55^\circ = 0.8192, \sin 60^\circ = 0.8660$, find $\sin 52^\circ$, using Newton's forward formula.

5. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age	:	45	50	55	60	65	
Premium (in rupees)	:	114.84	96.16	83.32	74.48	68.48	(U.P.T.U., 2010)

6. The area A of a circle of diameter d is given for the following values :

$d :$	80	85	90	95	100	
$A :$	5026	5674	6362	7088	7854	(V.T.U., 2010)

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(22)$ and $f(42)$ from the following available data :

$x :$	20	25	30	35	40	45	
$f(x) :$	354	332	291	260	231	204	(J.N.T.U., 2007)

8. From the following table :

$x^\circ :$	10	20	30	40	50	60	70	80
$\cos x :$	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate $\cos 25^\circ$ and $\cos 73^\circ$ using Gregory Newton formulae.

(U.P.T.U., 2006)

9. Find the number of men getting wages below Rs. 15 from the following data :

Wages in Rs. : 0—10	10—20	20—30	30—40
Frequency : 9	30	35	42

(Nagarjuna, 2001)

10. Find the polynomial interpolating the data :

x : 0	1	2
f(x) : 0	5	2

(U.P.T.U., 2008)

11. Construct Newton's forward interpolation polynomial for the following data :

x : 4	6	8	10
y : 1	3	8	16

(Madras, 2006)

Hence evaluate y for x = 5.

12. Construct the difference table for the following data :

x : 0.1	0.3	0.5	0.7	0.9	1.1	1.3
f(x) : 0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate f(0.6)

(J.N.T.U., 2007)

13. Estimate from following table f(3.8) to three significant figures using Gregory Newton backward interpolation formula:

x : 0	1	2	3	4
f(x) : 1	1.5	2.2	3.1	4.6

(U.P.T.U., 2009)

14. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year : 1941	1951	1961	1971	1981	1991
Population (in thousands) : 12	15	20	27	39	52

(U.P.T.U., 2009)

15. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x : 3	4	5	6	7	8	9
y : 2.7	6.4	12.5	21.6	34.3	51.2	72.9

(P.T.U., 2001)

16. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4 .

(Nagarjuna, 2003 S)

17. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3 .

18. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

29.7 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding section, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0 , $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2} , y_{-1} , y_0 , y_1 , y_2 , then we can write the difference table in the two notations as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
$x_0 - 2h$	y_{-2}		$\Delta y_{-2} (= \delta y_{-3/2})$		
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$	$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta y_{-1} (= \delta y_{-1/2})$	$\Delta^2 y_{-1} (\delta^2 y_0)$	$\Delta^4 y_{-2} (= \delta^4 y_0)$
$x_0 + h$	y_1	$\Delta y_0 (= \delta y_{1/2})$	$\Delta^2 y_0 (= \delta^2 y_1)$	$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + 2h$	y_2	$\Delta y_1 (= \delta y_{3/2})$			

- (1) **Gauss's forward interpolation formula.** The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1 \cdot 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \Delta^3 y_0 + \dots \quad \dots(1)$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$... (2)
i.e., $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$... (2)

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$... (3)

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc. ... (4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$
i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$
 Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ... (5)

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$... from (2), (3), (4) ... in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1 \cdot 2} (\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} (\Delta^3 y_{-1} + \Delta^4 y_{-1}) \\ + \frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4} (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots$$

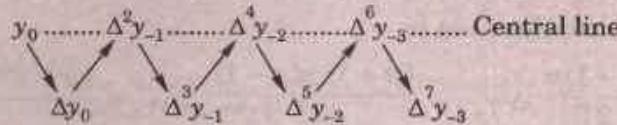
Hence $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$ [Using (5)]

which is called *Gauss's forward interpolation formula*.

Cor. In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!} \delta^2 y_0 + \frac{(p+1)p(p-1)}{3!} \delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!} \delta^4 y_0 + \dots$$

Obs. 1. It employs odd differences just below the central line and even difference on the central line as shown below:



Obs. 2. This formula is used to interpolate the values of y for p ($0 < p < 1$) measured forwardly from the origin.

(2) Gauss's backward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1 \cdot 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \Delta^3 y_0 + \dots \quad \dots (1)$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$... (2)
i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$... (2)

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$... (3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. ... (4)

Also $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$
i.e., $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$... (5)

Similarly $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ... (6)

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$ from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1 \cdot 2} (\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} (\Delta^3 y_{-1} + \Delta^4 y_{-1}) \\ + \frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4} (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots \\ = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1 \cdot 2} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1 \cdot 2 \cdot 3} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 y_{-1} \\ + \frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^5 y_{-1} + \dots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1 \cdot 2} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1 \cdot 2 \cdot 3} (\Delta^3 y_{-2} + \Delta^4 y_{-2}) \\ + \frac{(p+1)p(p-1)(p-2)}{1 \cdot 2 \cdot 3 \cdot 4} (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots \quad [\text{Using (5) and (6)}]$$

Hence $y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} + \dots$

which is called *Gauss's backward interpolation formula*.

Cor. In the central differences notation, this formula will be

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!} \delta^2 y_0 + \frac{(p+1)p(p-1)}{3!} \delta^3 y_{-1/2} + \frac{(p+2)(p+1)p(p-1)}{4!} \delta^4 y_0 + \dots$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below :



Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

(3) Stirling's formula.* Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(3)$$

which is called *Stirling's formula*.

Cor. In the central differences notation, (3) takes the form

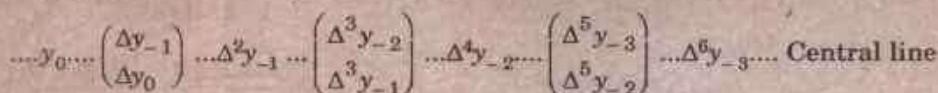
$$y_p = y_0 + p \mu \delta y_0 + \frac{p^2}{2!} \delta^2 y_0 + \frac{p(p^2+1^2)}{3!} \mu \delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!} \delta^4 y_0 + \dots \quad \dots(4)$$

for

$$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu \delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu \delta^3 y_0 \text{ etc.}$$

Obs. This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below :



(4) Bessel's formula.** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} \dots \quad \dots(1)$$

*Named after the Scottish mathematicians James Stirling (1692-1770).

**See footnote p. 550.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

$$\text{i.e., } \Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \quad \dots(2)$$

Similarly $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$ etc. $\dots(3)$

Now (1) can be written as

$$\begin{aligned} y_p &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \left(\frac{1}{2} \Delta^2 y_{-1} + \frac{1}{2} \Delta^2 y_{-1} \right) + \frac{p(p^2-1)}{3!} \Delta^3 y_{-1} \\ &\quad + \frac{p(p^2-1)(p-2)}{4!} \left(\frac{1}{2} \Delta^4 y_{-2} + \frac{1}{2} \Delta^4 y_{-2} \right) + \dots \\ &= y_0 + p\Delta y_0 + \frac{1}{2} \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{1}{2} \frac{p(p-1)}{2!} (\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!} \Delta^3 y_{-1} \\ &\quad + \frac{1}{2} \frac{p(p^2-1)(p-2)}{4!} \Delta^4 y_{-2} + \frac{1}{2} \frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \dots \quad [\text{Using (2), (3) etc.}] \\ &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!} \times \left(\frac{p+1}{3} - \frac{1}{2} \right) \Delta^3 y_{-1} \\ &\quad + \frac{p(p^2-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \end{aligned}$$

$$\text{Hence } y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)p(p-1)}{3!} \Delta^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \quad \dots(4)$$

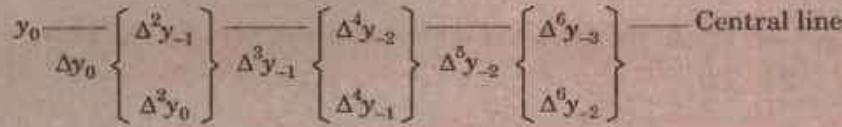
which is known as the *Bessel's formula*.

Cor. In the central differences notation, (4) becomes

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!} \mu \delta^2 y_{1/2} + \frac{(p-1/2)p(p-1)}{3!} \delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!} \mu \delta^4 y_{1/2} + \dots \quad \dots(5)$$

$$\text{for } \frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = y \delta^2 y_{1/2}, \frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu \delta^4 y_{1/2} \text{ etc.}$$

Obs. This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below this line as shown below :



(5) Everett's formula. Gauss's forward interpolation formula is

$$\begin{aligned} y_p &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} \\ &\quad + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \Delta^5 y_{-2} + \dots \quad \dots(1) \end{aligned}$$

We eliminate the odd difference in (1) by using the relations

$$\Delta y_0 = y_1 - y_0, \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}, \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (1) becomes

$$\begin{aligned} y_p &= y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} (\Delta^2 y_0 - \Delta^2 y_{-1}) \\ &\quad + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \dots \\ &= (1-p)y_0 + py_1 - \frac{p(p-1)(p-2)}{3!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^2 y_0 \end{aligned}$$

$$-\frac{(p+1)p(p-1)(p-2)(p-3)}{5!} \Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \Delta^4 y_{-1} - \dots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!} \Delta^4 y_{-2} + \dots + py_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_0 \\ + \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!} \Delta^4 y_{-1} + \dots$$

This is known as *Everett's formula*.

Obs. This formula is extensively used and involves only even differences on and below the central line as shown below :

$$\begin{array}{ccccccc} y_0 & \Delta^2 y_{-1} & \Delta^4 y_{-2} & \Delta^6 y_{-3} & \text{Central line} \\ \hline y_1 & \Delta^2 y_0 & \Delta^4 y_{-1} & \Delta^6 y_{-2} & \end{array}$$

29.8 CHOICE OF AN INTERPOLATION FORMULA

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As much, whenever possible, *central difference formulae should be used in preference to Newton's formulae*.

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. To find a tabulated value near the beginning of the table, use Newton's forward formula.
2. To find a value near the end of the table, use Newton's backward formula.
3. To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.

If interpolation is required for p lying between $-1/4$ and $1/4$, prefer Stirling's formula.

If interpolation is desired for p lying between $1/4$ and $3/4$, use Bessel's or Everett's formula.

Example 29.18. Find $f(22)$ from the Gauss forward formula :

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

Solution. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

$$\text{i.e., for } p = \frac{x - x_0}{h} = \frac{22 - 25}{5} = -0.6$$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	-1	354 ($= y_{-1}$)					
			-22				
25	0	332 ($= y_0$)		-19			
			-41		29		
30	1	291 ($= y_1$)		10		-37	
			-31		-8	45	
35	2	260 ($= y_2$)		2		8	
			-29		0		
40	3	231 ($= y_3$)		2			
			-27				
45	4	204 ($= y_4$)					

Gauss forward formula is

$$\begin{aligned}
 y^p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} \\
 + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + (p+1)(p-1)(p-2)(p+2) \Delta^5 y_{-2} \\
 \therefore f(22) = 332 + (0.6)(-41) + \frac{(-0.6)(-0.6-1)}{2!} (-19) + \frac{(-0.6+1)(-0.6)(-0.6-1)}{3!} (-8) \\
 + \frac{(-0.6-1)(-0.6)(-0.6-1)(-0.6-2)}{4!} (-37) \\
 + \frac{(-0.6+1)(-0.6)(-0.6-1)(-0.6-2)(-0.6+2)}{5!} (45) \\
 = 332 + 24.6 - 9.12 + 1.5392 - 0.5241
 \end{aligned}$$

Hence $f(22) = 347.983$.

Example 29.19. Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :

Year	:	1939	1949	1959	1969	1979	1989
Population (in thousands)	:	12	15	20	27	39	52

(Kottayam, 2005; Madras, 2003)

Solution. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for $p = \frac{1974 - 1969}{10} = 0.5$.

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	-3	12		3			
1949	-2	15		5	2		
1959	-1	20		7	2	0	
1969	0	27		12	5	3	-10
1979	1	39			1	-7	
1989	2	52		13			

Gauss's backward formula is

$$\begin{aligned}
 y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+1)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} \\
 + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!} \Delta^5 y_{-3} + \dots
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 y_{.5} &= 27 + (0.5)(7) + \frac{(1.5)(.5)}{2}(5) + \frac{(1.5)(.5)(-.5)}{6}(3) + \frac{(2.5)(1.5)(-.5)}{24}(-7) \\
 &\quad + \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{120}(-10) \\
 &= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.345 \text{ thousands approx.}
 \end{aligned}$$

Example 29.20. Given

0°	:	0	5	10	15	20	25	30
$\tan \theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of $\tan 16^\circ$.

(Anna, 2005)

Solution. Taking the origin at $\theta = 15^\circ$, $h = 5^\circ$ and $p = \frac{\theta - 15}{5}$, we have the following central difference table :

P							
-3	0.0000		0.08575				
-2	0.0875		0.0888	0.0013			
-1	0.1763		0.0916	0.0028			
0	0.2679		0.0961	0.0045	0.0015	0.0002	-0.0002
1	0.3640		0.1023	0.0062	0.0017	0.0000	0.0009
2	0.4663		0.1111	0.0088	0.0026	0.0009	
3	0.5774						

$$\text{At } \theta = 16^\circ, \quad p = \frac{16 - 15}{5} = 0.2$$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2 - 1)}{4!} \Delta^4 y_{-2} + \dots$$

$$\therefore y_{0.2} = 0.2679 + (0.2) \left(\frac{0.0916 + 0.0961}{2} \right) + \frac{(0.2)^2}{2!} (0.0045) + \dots$$

$$= 0.2679 + 0.01877 + 0.00009 + \dots = 0.28676$$

Hence $\tan 16^\circ = 0.28676$.

Example 29.21. Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$) :

x°	:	10	11	12	13	14	
$10^5 y_x$:	23,967	28,060	31,788	35,209	38,368	(V.T.U., 2004)

Solution. Taking the origin at $x_0 = 12^\circ$, $h = 1$ and $p = x - 12$, we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
-2	0.23967		0.04093		
-1	0.28060		0.03728	-0.00365	0.00058
0	0.31788		0.034121	-0.06307	-0.00045
1	0.35209		0.03159	-0.00062	-0.00013
2	0.38368				

At $x = 12.2$, $p = 0.2$. (As p lies between $-1/4$ and $1/4$, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2 - 1)}{4!} \Delta^4 y_{-2} + \dots$$

When $p = 0.2$, we have

$$\begin{aligned} \therefore y_{0.2} &= 0.31788 + 0.2 \left(\frac{0.03728 + 0.03421}{2} \right) + \frac{(0.2)^2}{2} (-0.00307) \\ &\quad + \frac{(0.2)[(0.2)^2 - 1]}{6} \left(\frac{0.00058 - 0.00045}{2} \right) + \frac{(0.2)^2[(0.2)^2 - 1]}{24} (-0.00013) \\ &= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497. \end{aligned}$$

Example 29.22. Apply Bessel's formula to obtain y_{25} , given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.
(S.V.T.U., 2007; V.T.U., 2000 S)

Solution. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2854			
0	3162	308		
1	3544	382	74	
2	3992	448	66	-8

At $x = 25$, $p = (25 - 24)/4 = 1/4$. (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)p(p-1)}{3!} \Delta^3 y_{-1} + \dots \quad \dots(1)$$

When $p = 0.25$, we have

$$\begin{aligned} y_p &= 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2} \left(\frac{74 + 66}{2} \right) + \frac{(-0.25)0.25(-0.75)}{6} (-8) \\ &= 3162 + 95.5 - 6 - 5625 - 0.0625 = 3250.875 \text{ approx.} \end{aligned}$$

Example 29.23. Apply Bessel's formula to find the value of $f(27.5)$ from the table :

x :	25	26	27	28	29	30	
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333	(U.P.T.U., 2009)

Solution. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	-2	4.000		-0.154		
26	-1	3.846	-0.142	0.012	-0.003	
27	0	3.704	-0.133	0.009	-0.001	0.004
28	1	3.571	-0.123	0.010	-0.002	-0.001
29	2	3.448	-0.115	0.008		
30	3	3.333				

At $x = 27.5, p = 0.5$ (As p lies between $1/4$ and $3/4$, the use of Bessel's formula will yield accurate result)
Bessel's formula is

$$\begin{aligned} y_p &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \right) + \frac{\left(p - \frac{1}{2} \right) p(p-1)}{3!} \Delta^3 y_{-1} \\ &\quad + \frac{(p+1)p(p-1)(p-2)}{4!} \left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} \right) + \dots \end{aligned}$$

When $p = 0.5$, we have

$$\begin{aligned} y_p &= 3.704 - \frac{(0.5)(0.5-1)}{2} \left(\frac{0.009 + 0.010}{2} \right) + 0 \\ &\quad + \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24} \left(\frac{-0.001 - 0.004}{2} \right) \\ &= 3.704 - 0.11875 - 0.00006 = 3.585 \end{aligned}$$

Hence $f(27.5) = 3.585$.

Example 29.24. Given the table

x	310	320	330	340	350	360
$\log x$	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of $\log 337.5$ by Everett's formula.

Solution. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \frac{x - 330}{10}$

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2.49136	0.01379				
-1	2.50515	0.01336	-0.00043	0.00004		
0	2.51851	0.01297	-0.00039	0.00001	-0.00003	0.00004
1	2.53148	0.01259	-0.00038	0.00002	0.00001	
2	2.54407	0.01223	-0.00036			
3	2.55630					

To evaluate $\log 337.5$ i.e. for $x = 337.5, p = \frac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)

Everett's formula is

$$\begin{aligned} y_p &= qy_0 + \frac{q(q^2 - 1^2)}{3!} \Delta^2 y_{-1} + \frac{q(q^2 - 1^2)(q^2 - 2^2)}{5!} \Delta^4 y_{-2} + \dots + py_1 + \frac{p(p^2 - 1^2)}{3!} \Delta^2 y_0 \\ &\quad + \frac{p(p^2 - 1^2)(p^2 - 2^2)}{5!} \Delta^4 y_{-1} + \dots \\ &= 0.25 \times 2.51851 + \frac{0.25(0.0625 - 1)}{6} \times (-0.00039) + \frac{0.25(0.0625 - 1)(0.0625 - 4)}{120} \\ &\quad \times (-0.00003) + 0.75 \times 2.53148 + \frac{0.75(0.5625 - 1)}{6} \times (-0.00038) \\ &\quad + \frac{0.75(0.5625 - 1)(0.5625 - 4)}{120} \times (0.00001) \\ &= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001 = 2.52828 \text{ nearly.} \end{aligned}$$

PROBLEMS 29.4

1. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

$x :$	2.5	3.0	3.5	4.0	4.5	5.0	
$y :$	24.145	22.043	20.225	18.644	17.262	16.047	(Bhopal, 2002; Madras, 2000)

2. Using Gauss's backward difference formula, find $y(8)$ from the following table :

$x :$	0	5	10	15	20	25	
$y :$	7	11	14	18	24	32	(J.N.T.U., 2007)

3. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (₹) :	Below 40	40–60	60–80	80–100	100–120	
No. of persons : (in thousands)	250	120	100	70	50	(Madras, 2000)

4. From the following table :

$x :$	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$e^x :$	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

Find $e^{1.17}$, using Gauss forward formula.

5. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $v = 25$,

$v :$	10	20	30	40
$p :$	1.1	2	4.4	7.9

6. Using Stirling's formula find y_{35} , given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. (Nagarjuna, 2003 S)

7. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

$x :$	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$F :$	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here? Which is more appropriate? Give reasons.

8. Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the following table :

$x :$	0	1	2	3	
$f(x) :$	3	6	12	15	(U.P.T.U., 2008)

9. Apply Everett's formula to obtain u_{25} , given $u_{20} = 854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$. (S.V.T.U., 2007)

10. Using Everett's formula, evaluate $f(30)$, if $f(20) = 2854$, $f(28) = 3162$, $f(36) = 7088$, $f(44) = 7984$. (U.P.T.U., 2006)

11. Given the table :

$x :$	310	320	330	340	350	360
$\log x :$	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Find the value of $\log 337.5$ by Gauss's, Stirling's and Bessel's formulae.

29.9 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantages of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x . Now we shall study two such formulae :

(i) Lagrange's interpolation formula

(ii) Newton's general interpolation formula with divided differences.

29.10 LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value y_0, y_1, \dots, y_n corresponding to $x = x_0, x_1, \dots, x_n$, then

$$f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} y_1 + \dots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} y_n \quad \dots (1)$$

This is known as Lagrange's interpolation formula for unequal intervals.

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y , we can represent $f(x)$ by a polynomial in x of degree n . Let this polynomial be of the form

$$\begin{aligned} y = f(x) &= a_0(x - x_1)(x - x_2) \dots (x - x_n) + a_1(x - x_0)(x - x_2) \dots (x - x_n) \\ &\quad + a_2(x - x_0)(x - x_1)(x - x_3) \dots (x - x_n) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \end{aligned} \quad \dots(2)$$

Putting $x = x_0, y = y_0$, in (2), we get

$$\begin{aligned} y_0 &= a_0(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n) \\ a_0 &= y_0 / [(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)] \end{aligned}$$

Similarly putting $x = x_1, y = y_1$ in (2), we have $a_1 = y_1 / [(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)]$

Proceeding the same way, we find a_2, a_3, \dots, a_n

Substituting the values of a_0, a_1, \dots, a_n in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \dots (x - x_n)$, we get

$$\begin{aligned} \frac{f(x)}{(x - x_0)(x - x_1) \dots (x - x_n)} &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} \cdot \frac{1}{x - x_0} \\ &\quad + \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} \cdot \frac{1}{x - x_1} + \dots + \frac{y_n}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} \cdot \frac{1}{x - x_n}. \end{aligned}$$

Example 29.25. Given the values

$x :$	5	7	11	13	17
$f(x) :$	150	392	1492	2366	5202

evaluate $f(9)$, using (i) Lagrange's formula.

(Anna, 2006)

Solution. (i) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$.

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$\begin{aligned} f(9) &= \frac{(9 - 7)(9 - 11)(9 - 13)(9 - 17)}{(5 - 7)(5 - 11)(5 - 13)(5 - 17)} \times 150 + \frac{(9 - 5)(9 - 11)(9 - 13)(9 - 17)}{(7 - 5)(7 - 11)(7 - 13)(7 - 17)} \times 392 \\ &\quad + \frac{(9 - 5)(9 - 7)(9 - 13)(9 - 17)}{(11 - 5)(11 - 7)(11 - 13)(11 - 17)} \times 1452 + \frac{(9 - 5)(9 - 7)(9 - 11)(9 - 17)}{(13 - 5)(13 - 7)(13 - 11)(13 - 17)} \times 2366 \\ &\quad + \frac{(9 - 5)(9 - 7)(9 - 11)(9 - 13)}{(17 - 5)(17 - 7)(17 - 11)(17 - 13)} \times 5202 = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810. \end{aligned}$$

Example 29.26. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

$x :$	0	1	2	5
$f(x) :$	2	3	12	147

(Anna, 2005)

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$

Lagrange's formula is

$$\begin{aligned} y &= \frac{(x - x_1)(x - x_2) \dots (x - x_3)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2) \dots (x - x_3)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_3)} y_1 \\ &\quad + \frac{(x - x_0)(x - x_1) \dots (x - x_3)}{(x_2 - x_0)(x_2 - x_1) \dots (x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_2) \dots (x - x_5)}{(x_3 - x_0)(x_3 - x_2) \dots (x_3 - x_5)} y_3 \\ &= \frac{(x - 1)(x - 2)(x - 5)}{(0 - 1)(0 - 2)(0 - 5)} (2) + \frac{(x - 0)(x - 2)(x - 5)}{(1 - 0)(1 - 2)(1 - 5)} (3) \\ &\quad + \frac{(x - 0)(x - 1)(x - 5)}{(2 - 0)(2 - 1)(2 - 5)} (12) + \frac{(x - 0)(x - 1)(x - 2)}{(5 - 0)(5 - 1)(5 - 2)} (147) \end{aligned}$$

Hence $f(x) = x^3 + x^2 - x + 2$
 $\therefore f(3) = 27 + 9 - 3 + 2 = 35.$

Example 29.27. A curve passes through the point $(0, 18), (1, 10), (3, -18)$ and $(6, 90)$. Find the slope of the curve at $x = 2$. (J.N.T.U., 2009)

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$\begin{aligned} y &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 \\ &\quad + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3 \\ &= \frac{(x - 1)(x - 3)(x - 6)}{(0 - 1)(0 - 2)(0 - 6)} (18) + \frac{(x - 0)(x - 3)(x - 6)}{(1 - 0)(1 - 3)(1 - 6)} (10) \\ &\quad + \frac{(x - 0)(x - 1)(x - 6)}{(3 - 0)(3 - 1)(3 - 6)} (-18) + \frac{(x - 0)(x - 1)(x - 3)}{(6 - 0)(6 - 1)(6 - 3)} (90) \\ &= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x) \end{aligned}$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $(x = 2) = \left(\frac{dy}{dx}\right)_{x=2}$
 $= (6x^2 - 20x)_{x=2} = -16.$

Example 29.28. Using Lagrange's formula, express the function $\frac{3x^2 + x + 1}{(x - 1)(x - 2)(x - 3)}$ as a sum of partial fractions.

Solution. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2$ and $x = 3$

These values are

$x :$	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
$y :$	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$\begin{aligned} y &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2 \\ &= \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} (5) + \frac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} (15) + \frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)} (31) \end{aligned}$$

Substituting the above values, we get

$$\begin{aligned} &= \frac{(x - 2)(x - 3)}{(1 - 2)(1 - 3)} (5) + \frac{(x - 1)(x - 3)}{(2 - 1)(2 - 3)} (15) + \frac{(x - 1)(x - 2)}{(3 - 1)(3 - 2)} (31) \\ &= 2.5(x - 2)(x - 3) - 15(x - 1)(x - 3) + 15.5(x - 1)(x - 2) \end{aligned}$$

Thus $\frac{3x^2 + x + 1}{(x - 1)(x - 2)(x - 3)} = \frac{2.5(x - 2)(x - 3) - 15(x - 1)(x - 3) + 15.5(x - 1)(x - 2)}{(x - 1)(x - 2)(x - 3)}$
 $= \frac{2.5}{x - 1} - \frac{15}{x - 2} + \frac{15.5}{x - 3}.$

Example 29.29. Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time versus velocity data is as follows :

$t :$	0	1	3	4
$v :$	21	15	12	10

Solution. Since the values of t are not equispaced, we use Lagrange's formula :

$$\begin{aligned}
 v &= \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} v_1 \\
 &\quad + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} v_2 + \frac{(t-t_0)(t-t_2)(t-t_5)}{(t_3-t_0)(t_3-t_2)(t_3-t_2)} v_3 \\
 v &= \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)} (21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)} (15) + \frac{t(t-1)(t-4)}{(3)(2)(-1)} (12) + \frac{t(t-1)(t-3)}{(4)(3)(1)} (10)
 \end{aligned}$$

i.e., $v = \frac{1}{12} (-5t^3 + 38t^2 - 105t + 252)$

$$\begin{aligned}
 \therefore \text{Distance moved } s &= \int_0^4 v dt = \frac{1}{12} \int_0^4 (-5t^3 + 38t^2 - 105t + 252) dt \quad \left[\because v = \frac{ds}{dt} \right] \\
 &= \frac{1}{12} \left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t \right)_0^4 \\
 &= \frac{1}{12} \left(-320 + \frac{2432}{3} - 840 + 1008 \right) = 54.9
 \end{aligned}$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2} (-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12} (-15(16) + 76(4) - 105) = -3.4$.

PROBLEMS 29.5

- Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given:
 $x : 5 \quad 6 \quad 9 \quad 11$
 $y : 12 \quad 13 \quad 14 \quad 16$ (U.P.T.U., 2009 ; J.N.T.U., 2008)
- Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. (Hazaribagh, 2009)
- The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.
 $T : 1.2 \quad 2.0 \quad 2.5 \quad 3.0$
 $I : 1.36 \quad 0.58 \quad 0.34 \quad 0.20$
Using Lagrange's formula, find I at $T = 16$. (J.N.T.U., 2009)
- Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year	: 1997	1999	2001	2002
Profit in Lakhs of ₹	: 43	65	159	248

(Anna, 2004)
- Use Lagrange's formula to find the form of $f(x)$, given
 $x : 0 \quad 2 \quad 3 \quad 6$
 $f(x) : 648 \quad 704 \quad 729 \quad 792$ (Madras, 2003 S)
- If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. (V.T.U., 2006)
- Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$. (Nagarjuna, 2003)
- Find the missing term in the following table using interpolation
 $x : 1 \quad 2 \quad 4 \quad 5 \quad 6$
 $y : 14 \quad 15 \quad 5 \quad \dots \quad 9$
- Using Lagrange's formula, express the function $\frac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as sum of partial fractions.

29.11 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. The labour of recomputing the interpolation

coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots$ be given points, then the *first divided difference* for the arguments, x_0, x_1 is defined by the relation $[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \frac{[x_1 - x_2] - [x_0, x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$ and so on.

Obs. 1. The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.

$$\begin{aligned} \text{For it is easy to write } [x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} = [x_1, x_0] [x_0, x_1, x_2] \\ &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \cdot \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \\ &= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1] \text{ and so on.} \end{aligned}$$

Obs. 2. The n th divided differences of a polynomial of the n th degree are constant.

Let the arguments be equally spaced so that, $x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = h$. Then

$$\begin{aligned} [x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h} \\ [x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h} \left[\frac{\Delta y_1 - \Delta y_0}{h} \right] \\ &= \frac{1}{2!h^2} \Delta^2 y_0 \text{ and in general, } [x_0, x_1, x_2, \dots, x_n] = \frac{1}{n!h^n} \Delta^n y_0. \end{aligned}$$

If the tabulated function is a n th degree polynomial, then $\Delta^n y_0$ will be constant. Hence the n th divided differences will also be constant.

29.12 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let y_0, y_1, \dots, y_n be the values of $y = f(x)$ corresponding to the arguments x_0, x_1, \dots, x_n . Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that

$$y = y_0 + (x - x_0) [x, x_0] \quad \dots(1)$$

Again

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives

$$[x, x_0] = [x_0, x_1] + (x - x_1) + (x - x_1) [x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0) [x_0, x_1] + (x - x_0)(x - x_1) [x, x_0, x_1] \quad \dots(2)$$

Also

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2) [x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0) [x_0, x_1] + (x - x_0)(x - x_1) [x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) [x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$\begin{aligned}y = f(x) &= y_0 + (x - x_0) [x_0, x_1] + (x - x_0)(x - x_1) [x_0, x_1, x_2] \\&\quad + (x - x_0)(x - x_1)(x - x_2) [x_0, x_1, x_2, x_3] + \dots \\&\quad + (x - x_0)(x - x_1) \dots (x - x_n) [x, x_0, x_1, \dots, x_n]\end{aligned}\quad \dots(3)$$

which is called *Newton's general interpolation formula with divided differences*.

Example 29.30. Given the values

$x :$	5	7	11	13	17
$f(x) :$	150	392	1452	2366	5202,

evaluate $f(9)$, using Newton's divided difference formula.

(V.T.U., 2010 ; P.T.U., 2005)

Solution. The divided difference table is

x	y	1st divided differences	2nd divided differences	3rd divided differences
5	150			
7	392	$\frac{392 - 150}{7 - 5} = 121$	$\frac{265 - 121}{11 - 5} = 24$	
11	1452	$\frac{1452 - 392}{11 - 7} = 265$	$\frac{457 - 265}{13 - 7} = 32$	$\frac{32 - 24}{13 - 5} = 1$
13	2366	$\frac{2366 - 1452}{13 - 11} = 457$	$\frac{709 - 457}{17 - 11} = 42$	$\frac{42 - 32}{17 - 7} = 1$
17	5202	$\frac{5202 - 2366}{17 - 13} = 709$		

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$\begin{aligned}f(9) &= 150 + (9 - 5) \times 121 + (9 - 5)(9 - 7) \times 24 + (9 - 5)(9 - 7)(9 - 11) \times 1 \\&= 150 + 484 + 192 - 16 = 810.\end{aligned}$$

Example 29.31. Determine $f(x)$ as a polynomial in x for the following data :

$x :$	-4	-1	0	2	5
$f(x) :$	1245	33	5	9	1335

Solution. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
-4	1245				
-1	33	-404			
0	5	-28	94		-14
2	9	2	10	13	3
5	1335	442	88		

Applying Newton's divided difference formula

$$\begin{aligned}
 f(x) &= f(x_0) + (x - x_0) [x_0, x_1] + (x - x_0)(x - x_1) [x_0, x_1, x_2] + \dots \\
 &= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) \\
 &\quad + (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2) (3) \\
 &= 3x^4 - 5x^3 + 6x^2 - 14x + 5.
 \end{aligned}$$

PROBLEMS 29.6

- Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. (U.P.T.U., 2005)
- Use Newton's divided difference method to compute $f(5.5)$ from the following data :

x :	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(U.P.T.U., 2010)
- Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(U.P.T.U., MCA, 2009, V.T.U., 2008)
- Obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(U.P.T.U., 2007)
- Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	-1	0	1	3
$f(x)$:	2	1	0	-1

(Anna, 2007)
- For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x :	5	6	9	11
$f(x)$:	12	13	14	16
- Using the following table, find $f(x)$ as a polynomial in

x :	-1	0	3	6	7
$f(x)$:	3	-6	39	822	1611

(U.P.T.U., 2009)
- Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

29.13 INVERSE INTERPOLATION

So far, given a set of values of x and y , we have been finding the values of y corresponding to a certain value of x . On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the *inverse interpolation*.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$\begin{aligned}
 x &= \frac{(y - y_1)(y - y_2)\dots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\dots(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)\dots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\dots(y_1 - y_n)} x_1 \\
 &\quad + \dots + \frac{(y - y_0)(y - y_1)\dots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\dots(y_n - y_{n-1})} x_n \quad \dots(1)
 \end{aligned}$$

which is used for inverse interpolation.

Example 29.32. The following table gives the values of x and y :

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to $y = 12$, using Lagrange's technique.

(V.T.U., 2009)

Solution. Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$
and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking $y = 12$, the above formula (1) gives

$$\begin{aligned}
 x &= \frac{(12 - 6.8)(12 - 9.8)(12 - 13.4)(12 - 15.5)(12 - 19.6)}{(4.2 - 6.8)(4.2 - 9.8)(4.2 - 13.4)(4.2 - 15.5)(4.2 - 19.6)} \times 1.2 \\
 &\quad + \frac{(12 - 4.2)(12 - 9.8)(12 - 13.4)(12 - 15.5)(12 - 19.6)}{(6.8 - 4.2)(6.8 - 9.8)(6.8 - 13.4)(6.8 - 15.5)(6.8 - 19.6)} \times 2.1 \\
 &\quad + \frac{(12 - 4.2)(12 - 6.8)(12 - 13.4)(12 - 15.5)(12 - 19.6)}{(9.8 - 4.2)(9.8 - 6.8)(9.8 - 13.4)(9.8 - 15.5)(9.8 - 19.6)} \times 2.8 \\
 &\quad + \frac{(12 - 4.2)(12 - 6.8)(12 - 9.8)(12 - 15.5)(12 - 19.6)}{(13.4 - 4.2)(13.4 - 6.8)(13.4 - 9.8)(13.4 - 15.5)(13.4 - 19.6)} \times 4.1 \\
 &\quad + \frac{(12 - 4.2)(12 - 6.8)(12 - 9.8)(12 - 13.4)(12 - 19.6)}{(15.5 - 4.2)(15.5 - 6.8)(15.5 - 9.8)(15.5 - 13.4)(15.5 - 19.6)} \times 4.9 \\
 &\quad + \frac{(12 - 4.2)(12 - 6.8)(12 - 9.8)(12 - 13.4)(12 - 15.5)}{(19.6 - 4.2)(19.6 - 6.8)(19.6 - 9.8)(19.6 - 13.4)(19.6 - 15.5)} \times 6.2 \\
 &= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.
 \end{aligned}$$

Example 29.33. Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f(42) = 18$. (V.T.U., 2009 S)

Solution. Here $x_0 = 30$, $x_1 = 34$, $x_2 = 38$, $x_3 = 42$
and $y_0 = -30$, $y_1 = -13$, $y_2 = 3$, $y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$\begin{aligned}
 x &= \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)} x_0 + \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y - y_2)(y_1 - y_3)} x_1 \\
 &\quad + \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)} x_2 + \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)} x_3 \\
 &= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34 + \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42 \\
 &= -0.782 + 6.532 + 33.682 - 2.202 = 37.23
 \end{aligned}$$

Hence the desired root of $f(x) = 0$ is 37.23.

PROBLEMS 29.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(Madras, 2000)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

29.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 29.8

Select the correct answer or fill up the blanks in the following problems :

- Newton's backward interpolation formula is
- Bessel's formula is most appropriate when p lies between
 (a) -0.25 and 0.25 (b) 0.25 and 0.75 (c) 0.75 and 1.00.

3. From the divided difference table for the following data :

$x :$	5	15	22
$y :$	7	36	160

4. Interpolation is the technique of estimating the value of a function for any

5. Bessel's formula for interpolation is

6. The 4th divided differences for $x_0, x_1, x_2, x_3, x_4 = \dots$

7. Stirling's formula is best suited for p lying between

8. Newton's divided differences formula is

9. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, Lagrange's interpolation formula is

10. If $f(0) = 1, f(2) = 5, f(3) = 10$ and $f(x) = 14$, then $x = 0$

11. Gauss forward interpolation formula involves

- (a) even differences above the central line and odd differences on the central line
- (b) even differences below the central line and odd differences on the central line
- (c) odd differences below the central line and even differences on the central line
- (d) odd differences above the central line and even differences on the central line.

12. If $y(1) = 4, y(3) = 12, y(4) = 19$ and $y(x) = 7$ find x using Lagrange's formula.

13. Extrapolation is defined as

14. The second divided difference of $f(x) = 1/x$, with arguments, a, b, c , is.....

(Anna, 2007)

15. Gauss-forward interpolation formula is used to interpolate values of y for

- | | |
|-----------------------|-------------------------|
| (a) $0 < p < 1$ | (b) $-1 < p < 0$ |
| (c) $0 < p < -\alpha$ | (d) $-\alpha < p < 0$. |

16. Given

$x :$	0	1	3	4
$y :$	-12	0	6	12

Using Lagrange's formula, a polynomial that can be fitted to the data is

17. The n th divided difference of a polynomial of degree n is

- | | |
|----------------|--------------------|
| (a) zero | (b) a constant |
| (c) a variable | (d) none of these. |

18. If h is the interval of differencing, $\Delta^2 x^3 = \dots$

Numerical Differentiation & Integration

1. Numerical differentiation. 2. Formulae for derivatives. 3. Maxima and minima of a tabulated function. 4. Numerical integration. 5. Newton-Cotes quadrature formula. 6. Trapezoidal rule. 7. Simpson's 1/3rd rule. 8. Simpson's 3/8th rule. 9. Boole's rule. 10. Weddle's rule. 11. Objective Type of Questions.

30.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i) . To compute dy/dx , we first replace the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx , is desired.

If the values of x are equi-spaced and dy/dx , is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx , is calculated by means of Stirling's or Bessel's formula. If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

30.2 FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values x_i ($= x_0 + ih$), $i = 0, 1, 2, \dots, n$.

(1) Derivatives using forward difference formula. Newton's forward interpolation formula (p. 958) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Differentiating both sides w.r.t. p , we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 + \dots$$

Since $p = \frac{(x - x_0)}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 + \dots + \frac{4p^3 - 18p^2 + 22p - 6}{4!} \Delta^4 y_0 + \dots \right] \quad \dots(1)$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \frac{1}{5} \Delta^5 y_0 - \frac{1}{6} \Delta^6 y_0 + \dots \right] \quad \dots(2)$$

Again differentiating (1) w.r.t. x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dp} \left(\frac{dy}{dp} \right) \frac{dp}{dx} \\ &= \frac{1}{h} \left[\frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \frac{12p^2-36p+22}{4!} \Delta^4 y_0 + \dots \right] \frac{1}{h}\end{aligned}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \frac{137}{180} \Delta^6 y_0 + \dots \right] \quad \dots(3)$$

$$\text{Similarly } \left(\frac{d^3y}{dx^3} \right)_{x_0} = \frac{1}{h^3} \left[\Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \dots \right] \quad \dots(4)$$

Otherwise : We know that $1 + \Delta = E = e^{hD}$

$$\therefore hD = \log(1 + \Delta) = \Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \frac{1}{4} \Delta^4 + \dots$$

$$\text{or } D = \frac{1}{h} \left[\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \frac{1}{4} \Delta^4 + \dots \right]$$

$$\text{and } D^2 = \frac{1}{h^2} \left[\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \frac{1}{4} \Delta^4 + \dots \right]^2 = \frac{1}{h^2} \left[\Delta^2 - \Delta^3 + \frac{11}{12} \Delta^4 + \dots \right]$$

$$\text{and } D^3 = \frac{1}{h^3} \left[\Delta^3 - \frac{3}{2} \Delta^4 + \dots \right]$$

Now applying the above identities to y_0 , we get

$$Dy_0 \text{ i.e., } \left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \frac{1}{5} \Delta^5 y_0 - \frac{1}{6} \Delta^6 y_0 + \dots \right]$$

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 + \frac{137}{180} \Delta^6 y_0 - \dots \right]$$

$$\text{and } \left(\frac{d^3y}{dx^3} \right)_{x_0} = \frac{1}{h^3} \left[\Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \dots \right]$$

which are the same as (2), (3) and (4) respectively.

(2) Derivatives using backward difference formula. Newton's backward interpolation formula (p. 958) is

$$y = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots$$

Differentiating both sides w.r.t. p , we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!} \nabla^2 y_n + \frac{3p^2+6p+2}{3!} \nabla^3 y_n + \dots$$

Since $p = \frac{x-x_n}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \left[\nabla y_n \frac{2p+1}{2!} \nabla^2 y_n + \frac{3p^2+6p+2}{3!} \nabla^3 y_n + \dots \right] \quad \dots(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx} \right)_{x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \frac{1}{5} \nabla^5 y_n + \frac{1}{6} \nabla^6 y_n + \dots \right] \quad \dots(6)$$

Again differentiating (5) w.r.t. x , we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp} \left(\frac{dy}{dx} \right) \frac{dp}{dx} = \frac{1}{h^2} \left[\nabla^2 y_n + \frac{6p+6}{3!} \nabla^3 y_n + \frac{6p^2+18p+11}{12} \nabla^4 y_n + \dots \right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \frac{137}{180} \nabla^6 y_n + \dots \right] \quad \dots(7)$$

Similarly, $\left(\frac{d^3y}{dx^3} \right)_{x_0} = \frac{1}{h^3} \left[\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right] \quad \dots(8)$

Otherwise : We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore -hD = \log(1 - \nabla) = -\left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{1}{4} \nabla^4 + \dots \right]$$

or $D = \frac{1}{h} \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{1}{4} \nabla^4 + \dots \right]$

$$\therefore D^2 = \frac{1}{h^2} \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \dots \right]^2 = \frac{1}{h^2} \left[\nabla^2 + \nabla^3 + \frac{11}{12} \nabla^4 + \dots \right]$$

Similarly, $D^3 = \frac{1}{h^3} \left[\nabla^3 + \frac{3}{2} \nabla^4 + \dots \right]$

Applying these identities to y_n , we get

Dy_n , i.e., $\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \frac{1}{5} \nabla^5 y_n + \frac{1}{6} \nabla^6 y_n + \dots \right]$

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \frac{137}{180} \nabla^6 y_n + \dots \right)$$

and $\left(\frac{d^3y}{dx^3} \right)_{x_0} = \frac{1}{h^3} \left[\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right]$

which are the same as (6), (7) and (8).

(3) Derivatives using central difference formulae. Stirling's formula (p. 964) is

$$y_p = y_0 + \frac{p}{1!} \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2 - 1^2)}{4!} \Delta^4 y_{-2} + \dots$$

Differentiating both sides w.r.t. p , we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{2p}{2!} \Delta^2 y_{-1} + \frac{3p^2 - 1}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{4p^3 - 2p}{4!} \Delta^4 y_{-2} + \dots$$

Since $p = \frac{x - x_0}{h}$, $\therefore \frac{dp}{dx} = \frac{1}{h}$.

Now $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + p \Delta^2 y_{-1} + \frac{3p^2 - 1}{6} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{2p^3 - p}{12} \Delta^4 y_{-2} + \dots \right]$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6} \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30} \frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \dots \right] \quad \dots(9)$$

Similarly $\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \frac{1}{90} \Delta^6 y_{-3} - \dots \right] \quad \dots(10)$

Obs. We can similarly use any other interpolation formula for computing the derivatives.

Example 30.1. Given that

$x :$	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$y :$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) $x = 1.1$

(b) $x = 1.6$.

(V.T.V., 2006; Madras, 2003 S)

(Rohtak, 2006; J.N.T.U., 2004 S)

Solution. (a) The difference table is :

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
1.1	8.403	0.414					
1.2	8.781	0.378	-0.036				
1.3	9.129	0.348	-0.030	0.006			
1.4	9.451	0.322	-0.026	0.004	-0.002	0.001	
1.5	9.750	0.299	-0.023	0.003	-0.001	0.003	0.002
1.6	10.031	0.281	-0.018	0.005			

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \frac{1}{5} \Delta^5 y_0 - \frac{1}{6} \Delta^6 y_0 + \dots \right] \quad \dots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \frac{5}{6} \Delta^5 y_0 - \dots \right] \quad \dots(ii)$$

Here $h = 0.1$, $x_0 = 1.1$, $\Delta y_0 = 0.378$, $\Delta^2 y_0 = -0.03$ etc.

Substituting these values in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1} \left[0.378 - \frac{1}{2} (-0.03) + \frac{1}{3} (0.004) - \frac{1}{4} (-0.001) + \frac{1}{5} (0.003) \right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2} \left[-0.03 - (0.004) + \frac{11}{12} (-0.001) - \frac{5}{6} (0.003) \right] = -3.74$$

(b) We use the above difference table and the backward difference operator ∇ instead of Δ .

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \frac{1}{5} \nabla^5 y_n + \frac{1}{6} \nabla^6 y_n + \dots \right] \quad \dots(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \frac{137}{180} \nabla^6 y_n + \dots \right] \quad \dots(ii)$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_n = 0.281$, $\nabla^2 y_n = -0.018$ etc.

Putting these values in (i) and (ii), we get

$$\left[\frac{dy}{dx}\right]_{1.6} = \frac{1}{0.1} \left[0.281 + \frac{1}{2} (-0.018) + \frac{1}{3} (0.005) + \frac{1}{4} (0.002) + \frac{1}{5} (0.003) + \frac{1}{6} (0.002) \right] = 2.75$$

$$\begin{aligned} \left(\frac{d^2y}{dx^2}\right)_{1.6} &= \frac{1}{(0.1)^2} \left[-0.018 + 0.005 + \frac{11}{12} (0.002) + \frac{5}{6} (0.003) + \frac{137}{180} (0.002) \right] \\ &= -0.715. \end{aligned}$$

Example 30.2. The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data :

Time t (sec)	:	0	5	10	15	20
Velocity v (m/sec)	:	0	3	14	69	228

(Anna, 2004)

Solution. The difference table is :

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
5	3	3			
10	14	11	8	36	
15	69	55	44	60	24
20	228	159	104		

An initial acceleration (i.e. $\frac{dv}{dt}$) at $t = 0$ is required, we use Newton's forward formula :

$$\left(\frac{dv}{dt} \right)_{t=0} = \frac{1}{h} \left(\Delta v_0 - \frac{1}{2} \Delta^2 v_0 + \frac{1}{3} \Delta^3 v_0 - \frac{1}{4} \Delta^4 v_0 + \dots \right)$$

$$\therefore \left(\frac{dv}{dt} \right)_{t=0} = \frac{1}{5} \left[3 - \frac{1}{2} (8) + \frac{1}{3} (36) - \frac{1}{4} (24) \right] = \frac{1}{5} (3 - 4 + 12 - 6) = 1$$

Hence the initial acceleration is 1 m/sec².

Example 30.3. A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when $t = 0.3$ seconds.

$t =$	0	0.1	0.2	0.3	0.4	0.5	0.6	
$x =$	30.13	31.62	32.87	33.64	33.95	33.81	33.24	(V.T.U, 2009)

Solution. The difference table is :

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
0.1	31.62	1.49					
0.2	32.87	1.25	-0.24				
0.3	33.64	0.77	-0.48	0.02			
0.4	33.95	0.31	-0.46	0.01	-0.01		
0.5	33.81	-0.14	-0.45	0.02	0.01		
0.6	33.24	-0.57	-0.43				

As the derivatives are required near the middle of the table, we use Stirling's formulae :

$$\left(\frac{dx}{dt} \right)_{t_0} = \frac{1}{h} \left(\frac{\Delta x_0 + \Delta x_{-1}}{2} \right) - \frac{1}{6} \left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2} \right) + \frac{1}{30} \left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2} \right) + \dots \quad \dots(i)$$

$$\left(\frac{d^2x}{dt^2} \right)_{t_0} = \frac{1}{h^2} \left[\Delta^2 x_{-1} - \frac{1}{12} \Delta^4 x_{-2} + \frac{1}{90} \Delta^6 x_{-3} \dots \right] \quad \dots(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (i) and (ii), we get

$$\left(\frac{dx}{dt} \right)_{0.3} = \frac{1}{0.1} \left[\frac{0.31 + 0.77}{2} - \frac{1}{6} \left(\frac{0.01 + 0.02}{2} \right) + \frac{1}{30} \left(\frac{0.02 - 0.27}{2} \right) - \dots \right] = 5.33$$

$$\left(\frac{d^2x}{dt^2} \right)_{0.3} = \frac{1}{(0.1)^2} \left[-0.46 - \frac{1}{12} (-0.01) + \frac{1}{90} (0.29) - \dots \right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is -45.6 cm/sec².

Example 30.4. Using Bessel's formula, find $f'(7.5)$ from the following table :

x :	7.47	7.48	7.49	7.50	7.51	7.52	7.53	
$f(x)$:	0.193	0.195	0.198	0.201	0.203	0.206	0.208	(J.N.T.U., 2006)

Solution. Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \frac{x - x_0}{h} = \frac{x - 7.50}{0.01}$

The difference table is :

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	-3	0.193		0.002				
7.48	-2	0.195	0.003	0.001				
7.49	-1	0.198	0.003	0.000	-0.001		0.000	
7.50	0	0.201	0.003	-0.001	0.003		0.003	
7.51	1	0.203	0.002	0.001	0.002	-0.004	-0.007	-0.01
7.52	2	0.206	0.003	-0.001	-0.002			
7.53	3	0.208	0.002					

Bessel's formula (p. 550) is

$$\begin{aligned}
 y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2} \right) p(p-1)}{3!} \cdot \Delta^3 y_{-1} \\
 + \frac{(p+1)p(p-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{\left(p - \frac{1}{2} \right) (p+1)p(p-1)(p-2)}{5!} \cdot \Delta^5 y_{-2} \\
 + \frac{(p+2)p(p+1)p(p-1)(p-2)(p-3)}{6!} \cdot \frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \dots \quad \dots(i)
 \end{aligned}$$

Since $p = \frac{x - x_0}{h}$, $\therefore \frac{dp}{dx} = \frac{1}{h}$ and $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \frac{dy}{dp}$

Differentiating (i) w.r.t. p and putting $p = 0$, we get

$$\begin{aligned}
 \left(\frac{dy}{dx} \right)_{7.5} = \frac{1}{h} \left(\frac{dy}{dp} \right)_{p=0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{h} (\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12} \Delta^3 y_{-1} + \frac{1}{24} (\Delta^4 y_{-2} + \Delta^4 y_{-1}) \right. \\
 \left. - \frac{1}{120} \Delta^5 y_{-2} - \frac{1}{240} (\Delta^6 y_{-3} + \Delta^6 y_{-2}) \right]
 \end{aligned}$$

$$\begin{aligned} \left(\frac{dy}{dx} \right)_{7.5} &= \frac{1}{0.01} \left[0.002 - \frac{1}{4} (-0.001 + 0.001) + \frac{1}{12} (0.002)^1 \right. \\ &\quad \left. + \frac{1}{24} (-0.004 + 0.003) - \frac{1}{120} (-0.007) \frac{-1}{240} (0.010 + 0) \right] \\ &= 0.2 + 0 + 0.01666 - 0.00583 + 0.00416 = 0.223. \end{aligned}$$

Example 30.5. Find $f'(0)$ from the following data :

x :	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606

Solution. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	1st div. diff.	2nd div. diff.	3rd div. diff.	4th div. diff.
3	-13				
5	23	18			
11	899	146	16	0.998	
27	17315	1025	39.96	1.003	0.0002
34	35606	2613	69.04		

Fifth difference being zero, Newton's divided difference formula is

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f(x_0 - x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1) \\ &\quad \times (x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

Differentiating it w.r.t. x , we get

$$\begin{aligned} f'(x) &= f(x_0, x_1) + (2x - x_0 - x_1) f(x_0, x_1, x_2) \\ &\quad + [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0 x_1 + x_1 x_2 + x_2 x_0)] \times f(x_0, x_1, x_2, x_3) \\ &\quad + [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 + x_1 x_3 + x_0 x_2) \\ &\quad - x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_0 x_1 x_3] f(x_0, x_1, x_2, x_3, x_4) \end{aligned}$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(x) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

30.3 MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward interpolation formula is

$$y = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Differentiating it w.r.t. p , we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 + \dots \quad \dots(1)$$

For maxima or minima, $dy/dp = 0$. Hence equating the right hand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 = 0$$

$$\left(\frac{1}{2} \Delta^3 y_0 \right) p^2 + (\Delta^2 y_0 - \Delta^3 y_0) p + \left(\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 \right) = 0$$

i.e.,

Substituting the values of Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for p . Then the corresponding values of $x = x_0 + ph$ at which y is maximum or minimum.

Example 30.6. Find the maximum and minimum value of y from the following data :

x :	-2	-1	0	1	2	3	4	
y :	2	-0.25	0	-0.25	2	15.75	56	(Anna, 2004)

Solution. The difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2		-2.25			
-1	-0.25	0.25	2.5	-3		
0	0	-0.25	-0.5	3	6	0
1	-0.25	0.25	2.5	9	6	0
2	2	13.75	11.5	15	6	
3	15.75		26.5			
4	56	40.25				

Taking $x_0 = 0$, we have $y_0 = 0$, $\Delta y_0 = -0.25$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 9$, $\Delta^4 y_0 = 6$.

Newton's forward difference formula for the first derivative gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{h} \left[\Delta y_0 - \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 - \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 - \dots \right] \\ &= \frac{1}{1} \left[-0.25 + \frac{2x-1}{2} (2.5) + \frac{1}{6} (3x^2 - 6x + 2)(9) + \frac{1}{24} (4x^2 - 18x^2 + 22x - 6)(6) \right] \\ &= \frac{1}{1} [-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5] = x^3 - x \end{aligned}$$

For y to be maximum or minimum, $\frac{dy}{dx} = 0$ i.e., $x^3 - x = 0$

i.e.,

$$x = 0, 1, -1$$

Now $\frac{d^2y}{dx^2} = 3x^2 - 1 = -ve$ for $x = 0$
 $= +ve$ for $x = 1$
 $= +ve$ for $x = -1$

Since $y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!} \Delta^2 y_0 + \dots, y(0) = 0$

Thus y is maximum for $x = 0$, and maximum value = $y(0) = 0$.

Also y is minimum for $x = 1$ and minimum value = $y(0) = -0.25$.

PROBLEMS 30.1

1. Find $y'(0)$ and $y''(0)$ from the following table :

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

2. Find the first and second derivatives of $f(x)$ at $x = 1.5$ if

x :	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

(S.V.T.U., 2007)

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$:

$x :$	1.0	1.2	1.4	1.6	1.8	2.0
$y :$	0	0.128	0.544	1.296	2.432	4.000

(U.P.T.U., 2010; Bhopal, 2009)

4. Given the following table of values of x and y

$x :$	1.00	1.05	1.10	1.15	1.20	1.25	1.30
$y :$	1.000	1.025	1.049	1.072	1.095	1.118	1.140

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) $x = 1.05$ (b) $x = 1.25$ (c) $x = 1.15$.

(V.T.U., 2008)

5. For the following values of x and y , find the first derivative at $x = 4$.

$x :$	1	2	4	8	10
$y :$	0	1	5	21	27

(J.N.T.U., 2009)

6. From the following table, find the values of dy/dx and d^2y/dx^2 at $x = 2.03$.

$x :$	1.96	1.98	2.00	2.02	2.04
$y :$	0.7825	0.7739	0.7651	0.7563	0.7473

(Anna, 2005)

7. Find the value of $\cos 1.74$ from the following table:

$x :$	1.7	1.74	1.78	1.82	1.86
$\sin x :$	0.9916	0.9857	0.9781	0.9691	0.9584

(J.N.T.U., 2009)

8. The distance covered by an athlete for the 50 metre is given in the following table:

<i>Time (sec)</i> :	0	1	2	3	4	5	6
<i>Distance (metre)</i> :	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of the athlete at $t = 5$ sec. correct to two decimals.

(U.P.T.U., 2009)

9. The following data gives corresponding values of pressure and specific volume of a superheated stream.

$v :$	2	4	6	8	10
$p :$	105	42.7	25.3	16.7	13

Find the rate of change of

- (i) pressure with respect to volume when $v = 2$,
(ii) volume with respect to pressure when $p = 105$.

10. The table below reveals the velocity v of a body during the specific time t , find its acceleration at $t = 1.1$?

$t :$	1.0	1.1	1.2	1.3	1.4
$v :$	43.1	47.7	52.1	56.4	60.8

(J.N.T.U., 2009)

11. The elevation above a datum line of 7 points of a road is given below:

$x :$	0	300	600	900	1200	1500	1800
$y :$	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

12. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

$t :$	0	0.2	0.4	0.6	0.8	1.0	1.2
$\theta :$	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second.

(V.T.U., 2004)

13. Find the value of $f'(x)$ at $x = 0.4$ from the following table using Bessel's formula

$x :$	0.01	0.02	0.03	0.04	0.05	0.06
$f(x) :$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

14. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 be neglected.

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{3}{4h} \left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3}) \right] \quad (\text{U.P.T.U., 2006})$$

[Hint: Differentiate Stirling's formula w.r.t. x , and put $x = 0$]

15. Find the value of $f'(8)$ from the table given below:

$x :$	6	7	9	12
$f(x) :$	1.556	1.690	1.908	2.158

(Anna, 2007)

16. Find the $f'(6)$ from the following data:

$x :$	0	2	3	4	7	8
$f(x) :$	4	26	58	112	466	922

(J.N.T.U., 2009; U.P.T.U., 2008)

17. Find the maximum and minimum values of y from the following table :

x :	0	1	2	3	4	5
$f(x)$:	0	0.25	0	2.25	16	56.25

18. Find the value of x for which $f(x)$ is minimum, using the table

x :	9	10	11	12	13	14
$f(x)$:	1330	1340	1320	1250	1120	980

Also find the maximum value of $f(x)$?

30.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values only.

30.5 NEWTON-COTES QUADRATURE FORMULA

Let $I = \int_a^b f(x) dx$

where $f(x)$ takes the values $y_0, y_1, y_2, \dots, y_n$ for $x = x_0, x_1, x_2, \dots, x_n$. (Fig. 30.1)

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0 + nh} f(x) dx = h \int_0^n f(x_0 + rh) dr,$$

putting $x = x_0 + rh, dx = hdr$

$$\begin{aligned} &= h \int_0^n \left[y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 \right. \\ &\quad + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!} \Delta^5 y_0 \\ &\quad \left. + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!} \Delta^6 y_0 + \dots \right] dr \end{aligned}$$

[By Newton's forward interpolation formula]

Integrating term by term, we obtain

$$\begin{aligned} \int_{x_0}^{x_0 + nh} f(x) dx &= nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 \right] \\ &\quad + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right) \frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n \right) \frac{\Delta^5 y_0}{5!} \\ &\quad + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n \right) \frac{\Delta^6 y_0}{6!} + \dots \end{aligned} \quad \dots(A)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3, \dots$

30.6 TRAPEZOIDAL RULE

Putting $n = 1$ in (A) § 30.5 and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e. a polynomial of first order so that differences of order higher than first become zero, we get

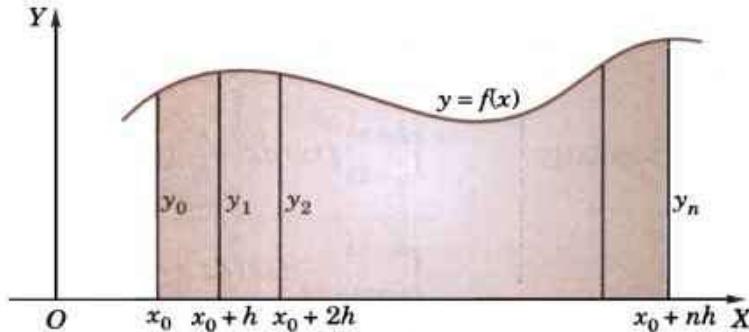


Fig. 30.1

$$\int_{x_0}^{x_0+h} f(x) dx = h \left(y_0 + \frac{1}{2} \Delta y_0 \right) = \frac{h}{2} (y_0 + y_1)$$

Similarly $\int_{x_0}^{x_0+2h} f(x) dx = h \left(y_1 + \frac{1}{2} \Delta y_1 \right) = \frac{h}{2} (y_1 + y_2)$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x) dx = \frac{h}{2} (y_{(n-1)} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

This is known as the **trapezium rule**.

Obs. The area of each strip (trapezium) is found separately. Then the area under the curve and the *ordinates* at x_0 and $x_0 + nh$ is approximately equal to the areas of the trapezia.

30.7 SIMPSON'S ONE-THIRD RULE

Putting $n = 2$ in (A) above and taking the curve through $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) as a parabola i.e., a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x) dx = 2h (y_0 + \Delta y_0 + \frac{1}{6} \Delta^2 y_0) \frac{h}{3} (y_0 + 4y_1 + y_2)$$

Similarly, $\int_{x_0+2h}^{x_0+nh} f(x) dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$ when

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x) dx = \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n), n \text{ being even.}$$

Adding all these integrals, we have (when n is even)

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3) + \dots + y_{n-1}] + 2(y_2 + y_4 + \dots + y_{n-2})]$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

Obs. While applying *Simpson's 1/3rd rule*, the given interval must be divided into even number of equal subintervals, since we find the area of two strips at a time.

30.8 SIMPSON'S THREE-EIGHTH RULE

Putting $n = 3$ in (A) above and taking the curve through (x_i, y_i) : $i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\begin{aligned} \int_{x_0}^{x_0+3h} f(x) dx &= 3h \left(y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right) \\ &= \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + y_3) \end{aligned}$$

Similarly,

$$\int_{x_0+3h}^{x_0+nh} f(x) dx = \frac{3h}{8} (y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

which is known as *Simpson's three-eighth rule*.

Obs. While applying *Simpson's 3/8th rule*, the number of sub-intervals should be taken as multiple of 3.

30.9 BOOLE'S RULE

Putting $n = 4$ in (A) above and neglecting all differences above the fourth, we obtain

$$\begin{aligned}\int_{x_0}^{x_0+4h} f(x) dx &= 4h \left(y_0 + 2\Delta y_0 + \frac{5}{3} \Delta^2 y_0 + \frac{2}{3} \Delta^3 y_0 + \frac{7}{90} \Delta^4 y_0 \right) \\ &= \frac{2h}{45} (7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)\end{aligned}$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x) dx = \frac{2h}{45} (7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{2h}{45} (7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \dots)$$

This is known as *Boole's rule*.

Obs. While applying *Boole's rule*, the number of sub-intervals should be taken as a multiple of 4.

30.10 WEDDLE'S RULE

Putting $n = 6$ in (A) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x) dx = \left(y_0 + 3\Delta y_0 + \frac{9}{2} \Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60} \Delta^4 y_0 + \frac{11}{20} \Delta^5 y_0 + \frac{1}{6} \cdot \frac{41}{140} \Delta^6 y_0 \right)$$

If we replace $\frac{41}{140} \Delta^6 y_0$ by $\frac{3}{10} \Delta^6 y_0$, the error made will be negligible.

$$\therefore \int_{x_0}^{x_0+6h} f(x) dx = \frac{3h}{10} (y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x) dx = \frac{3h}{10} (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{3h}{10} (y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \dots)$$

This is known as *Weddle's rule*.

Obs. While applying *Weddle's rule* the number of sub-intervals should be taken as a multiple of 6. *Weddle's rule* is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

Example 30.7. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule,

(i) Simpson's 1/3 rule,

(Mumbai, 2005)

(ii) Simpson's 3/8 rule,

(J.N.T.U., 2008)

(iii) Weddle's rule and compare the results with its actual value.

(V.T.U., 2008)

Solution. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
$= y$	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal rule,

$$\begin{aligned}\int_0^6 \frac{1}{1+x^2} dx &= \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)] \\ &= \frac{1}{2} [(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.\end{aligned}$$

(ii) By Simpson's 1/3 rule,

$$\begin{aligned}\int_0^6 \frac{1}{1+x^2} dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{1}{3} [(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.\end{aligned}$$

(iii) By Simpson's 3/8 rule,

$$\begin{aligned}\int_0^6 \frac{1}{1+x^2} dx &= \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] \\ &= \frac{3}{8} [(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.3571.\end{aligned}$$

(iv) By Weddle's rule,

$$\begin{aligned}\int_0^6 \frac{1}{1+x^2} dx &= \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \\ &= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.3735.\end{aligned}$$

Also, $\int_0^6 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^6 = 1.4056$

Obs. This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3rd.

Example 30.8. Use the Trapezoidal rule to estimate the integral $\int_0^2 e^{x^2} dx$ taking 10 intervals.

(U.P.T.U., 2008)

Solution. Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows :

$x :$	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$y :$	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	

By Trapezoidal rule, we have

$$\begin{aligned}\int_0^1 e^{x^2} dx &= \frac{h}{2} [(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)] \\ &= \frac{0.2}{2} [(1 + 54.5981) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964 \\ &\quad + 2.178 + 4.2206 + 7.0993 + 12.9358 + 25.5337)]\end{aligned}$$

Hence $\int_0^2 e^{x^2} dx = 17.0621$.

Example 30.9. Use Simpson's 1/3rd rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking seven ordinates.

(V.T.U., 2011 ; Bhopal, 2009)

Solution. Divide the interval $(0, 0.6)$ into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3rd rule, we have

$$\begin{aligned} \int_0^{0.6} e^{-x^2} dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{0.1}{3} [(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)] \\ &= \frac{0.1}{3} [1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3} (16.0543) = 0.5351. \end{aligned}$$

Example 30.10. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x) dx$ using Simpson's $\frac{3}{8}$ th rule.

(Mumbai, 2005)

Solution. Let $y = \sin x - \log_e x + e^x$ and $h = 0.2$, $n = 6$.

The values of y are as given below :

$x :$	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$y :$	3.0295	2.7975	2.8976	3.1660	3.5597	4.0698	4.4042

By Simpson's $\frac{3}{8}$ th rule, we have

$$\begin{aligned} \int_{0.2}^{1.4} y dx &= \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)] \\ &= \frac{3}{8} (0.2) [7.7336 + 2(3.1660) + 3(13.3247)] = 4.053 \end{aligned}$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x) dx = 4.053$.

Obs. Applications of Simpson's rule. If the various ordinates in §30.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similar if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

Example 30.11. The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows :

$t :$	2	4	6	8	10	12	14	16	18	20
$y :$	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution. If s km be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore s \Big|_{t=0}^{20} = \int_0^{20} v dt = \frac{h}{3} [X + 4.O + 2E], \text{ by Simpson's rule}$$

Hence $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$

$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

$$\text{Hence the required distance} = \left| s \right|_{t=0}^{20} = \frac{2}{3} (0 + 4 \times 80 + 2 \times 72) \\ = 309.33 \text{ km.}$$

Example 30.12. The velocity v of a particle at distance s from a point on its linear path is given by the following table :

$s \text{ (m)} :$	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
$v \text{ (m/sec)} :$	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres, using Boole's rule.

(U.P.T.U. 2007)

Solution. If t sec be the time taken to traverse a distance s (m) then $\frac{ds}{dt} = v$

$$\text{or } \frac{dt}{ds} = \frac{1}{v} = y \text{ (say),}$$

$$\therefore \text{then } \left| t \right|_{s=0}^{s=20} = \int_0^{20} y ds$$

Here $h = 2.5$ and $n = 8$

$$\text{Also } y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}, y_7 = \frac{1}{11}, y_8 = \frac{1}{9}$$

\therefore by Boole's Rules, we have

$$\begin{aligned} \left| t \right|_{s=0}^{s=20} &= \int_0^{20} y ds = \frac{2h}{45} [7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8] \\ &= \frac{2(2.5)}{45} \left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{4}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right) \right. \\ &\quad \left. + 12\left(\frac{1}{13}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right) \right] \\ &= \frac{1}{9} (12.11776) = 1.35 \end{aligned}$$

Hence the required time = 1.35 sec.

Example 30.13. A solid of revolution is formed by rotating about the x -axis, the area between the x -axis the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates

$$x : \quad 0.00 \quad 0.25 \quad 0.50 \quad 0.75 \quad 1.00$$

$$y : \quad 1.0000 \quad 0.9896 \quad 0.9589 \quad 0.9089 \quad 0.8415$$

Estimate the volume of the solid formed using Simpson's rule.

(Raipur, 2000)

Solution. Here $h = 0.25$, $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$, etc.

\therefore Required volume of the solid generated

$$\begin{aligned} &= \int_0^1 \pi y^2 dx = \pi \cdot \frac{h}{3} [(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2] \\ &= 0.25 \frac{\pi}{3} [(1 + (0.8415)^2) + 4[(0.9896)^2 + (0.9089)^2] + 2(0.0589)^2] \\ &= \frac{0.25 \times 3.1416}{3} [1.7081 + 7.2216 + 1.839] = 0.2618 (10.7687) = 2.8192. \end{aligned}$$

PROBLEMS 30.2

1. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying
 (i) Trapezoidal rule (J.N.T.U., 2009) (ii) Simpson's 1/3rd rule
 (iii) Simpson's 3/8th rule. (Mumbai, 2004)
2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal rule taking $h = 1/4$
 (ii) Simpson's 1/3rd rule taking $h = 1/4$. (J.N.T.U., 2008)
 (iii) Simpson's 3/8th rule taking $h = 1/6$. (U.P.T.U., 2010; V.T.U., 2007)
 (iv) Weddle's rule taking $h = 1/6$. (Bhopal, 2009)
 Hence compute an approximate value of π in each case.
3. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (Anna, 2005)
4. Evaluate $\int_0^6 x \sec x dx$ using eight intervals by Trapezoidal rule. (U.P.T.U., 2009)
5. Evaluate using Simpson's $\frac{1}{3}$ rd rule (i) $\int_0^6 \frac{e^x}{1+x} dx$ (U.P.T.U., 2006)
 (ii) $\int_0^2 e^{-x^2} dx$ (Take $h = 0.25$). (J.N.T.U., 2007)
6. Evaluate using Simpson's 1/3rd rule $\int_0^1 \frac{dx}{x^3 + x + 1}$, choose step length 0.25. (U.P.T.U., 2009)
7. Evaluate using Simpson's 1/3rd rule, (i) $\int_0^\pi \sin x dx$ using 11 ordinates.
 (ii) $\int_0^{\pi/2} \sqrt{\cos \theta} d\theta$ taking 9 ordinates. (V.T.U., 2009)
8. Evaluate correct to 4 decimal places, by Simpson's $\frac{3}{8}$ th rule
 (i) $\int_0^9 \frac{dx}{1+x^3}$ (U.P.T.U., M. Tech., 2010) (ii) $\int_0^{\pi/2} e^{\sin x} dx$ (U.P.T.U., 2007)
9. Given that
 $x :$ 4.0 4.2 4.4 4.6 4.8 5.0 5.2
 $\log x :$ 1.3863 1.4351 1.4816 1.5261 1.5686 1.6094 1.6487
 evaluate $\int_4^{5.2} \log x dx$
 (a) Trapezoidal rule (b) Simpson's 1/3rd rule, (Kerala, 2003)
 (c) Simpson's 3/8 rule (d) Weddle's rule (V.T.U., 2008)
10. Use Boole's rule to compute $\int_0^{\pi/2} \sqrt{\sin x} dx$. (U.P.T.U., 2008)
11. The displacement $s(t)$ as a function of time t is given below:
 t 5 6 7
 $s(t)$ 78 70 60
 Using Simpson's rule estimate $\int_5^7 s(t) dt$. (J.N.T.U., 2007)
12. A curve is defined by the following table:
 $x :$ 3.5 4
 $y :$ 2.6 2.1
 Estimate the area bounded by the curve, the x-axis and the vertical lines $x = 3$ and $x = 4$. (Bhopal, 2007)

13. A river is 80 ft wide. The depth d in feet at a distance x ft. from one bank is given by the following table :

$x :$	0	10	20	30	40	50	60	70	80
$y :$	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section.

(Rohtak, 2005)

14. A curve is drawn to pass through the points given by following table :

$x :$	1	1.5	2	2.5	3	3.5	4
$y :$	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x -axis and the lines $x = 1, x = 4$. (V.T.U., 2011 S)

15. A body is in the form of a solid of revolution. The diameter D in cms of its sections at distances x cm. from the one end are given below. Estimate the volume of the solid.

$x :$	0	2.5	5.0	7.5	10.0	12.5	15.0
$D :$	5	5.5	6.0	6.75	6.25	5.5	4.0

16. The velocity v of a particle at distances s from a point on its path is given by the table :

s ft. :	0	10	20	30	40	50	60
v ft/sec. :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft. by using Simpson's 1/3 rule.

(U.P.T.U., 2007)

Compare the result with Simpson's 3/8 rule.

(Madras, 2003)

17. The following table gives the velocity v of a particle at time t :

t (second) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ sec.

(S.V.T.U., 2007)

18. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$ rd rule, find the velocity of the rocket at $t = 80$ seconds.

t sec :	0	10	20	30	40	50	60	70	80
f (cm/sec 2) :	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

(Mumbai, 2004)

19. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below :

h (ft.) :	10	11	12	13	14
A (sq.ft.) :	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

30.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 30.3

Select the correct answer or fill up the blanks in the following questions.

- The value of $\int_0^1 \frac{dx}{1+x}$ by Simpson's rule is
 (a) 0.96315 (b) 0.63915 (c) 0.69315 (d) 0.69..
- Using forward differences, the formula for $f'(\alpha) = \dots$
- In application of Simpson's 1/3 rule, the interval h for closer approximation should be ...
- $f(x)$ is given by

$$\begin{array}{lll} x & : & 0 & 0.5 & 1 \\ f(x) & : & 1 & 0.8 & 0.5, \end{array}$$
 then using Trapezoidal rule, the value of $\int_0^1 f(x) dx$ is ...

5. If x : 0 0.5 1 1.5 2
 $f(x)$: 0 0.25 1 2.25 4

then the value of $\int_0^2 f(x) dx$ by Simpson's 1/3rd rule is ...

6. Simpson's 3/8th rule states that ...

7. For the data :

t :	3	6	9	12
$y(t)$:	-1	1	2	3,

the value of $\int_3^{12} y(t) dt$ when computed by Simpson's $\frac{1}{3}$ rd rule is

- (a) 15 (b) 10 (c) 0 (d) 5.

8. While evaluating a definite integral by Trapezoidal rule, the accuracy can be increased by taking ...

9. The value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's 1/3rd rule (taking $n = 1/4$) is ...

10. For the data:

x :	2	4	6	8
$f(x)$:	3	5	6	7,

$\int_2^8 f(x) dx$ when found by the Trapezoidal rule is

- (a) 18 (b) 25 (c) 16 (d) 32.

11. The expression for $\left(\frac{dy}{dx}\right)_{x=x_0}$ using backward differences is ...

12. The number of strips required in Weddle's rule is ...

13. The number of strips required in Simpson's 3/8th rule is a multiple of

- (a) 1 (b) 2 (c) 3 (d) 6.

14. If $y_0 = 1, y_1 = \frac{16}{17}, y_2 = \frac{4}{5}, y_3 = \frac{16}{25}, y_4 = \frac{1}{2}$ and $h = \frac{1}{4}$, then using Trapezoidal rule, $\int_0^4 y dx = \dots$

15. Using Simpson's $\frac{1}{3}$ rd rule, $\int_0^1 \frac{dx}{x} = \dots$ (taking $n = 4$).

16. If $y_0 = 1, y_1 = 0.5, y_2 = 0.2, y_3 = 0.1, y_4 = 0.06, y_5 = 0.04$ and $y_6 = 0.03$, then $\int_0^4 y dx$ by Simpson's $\frac{3}{8}$ th rule is = ...

17. If $f(0) = 1, f(1) = 2.7, f(2) = 7.4, f(3) = 20.1, f(4) = 54.6$ and $h = 1$, then $\int_0^4 f(x) dx$ by Simpson's $\frac{1}{3}$ rd rule = ...

18. Simpson's 1/3rd rule and direct integration give the same result if ...

19. To evaluate $\int_{x_0}^{x_n} y dx$ by Simpson's 1/3rd rule as well as Simpson's 3/8th rule, the number of intervals should be and respectively.

20. Whenever Trapezoidal rule is applicable, Simpson's 1/3rd rule can also be applied.

(True or False)

Difference Equations

1. Introduction. 2. Definition. 3. Formation of difference equations. 4. Linear difference equations. 5. Rules for finding complementary function. 6. Rules for finding particular integral. 7. Simultaneous difference equations with constant coefficients. 8. Application to deflection of a loaded string. 9. Objective Type of Questions.

31.1 INTRODUCTION

Difference calculus also forms the basis of Difference equations. These equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such differences equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of Differential equations grew out Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

31.2 DEFINITION

(1) A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$... (1) and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$... (2)
are difference equations.

An alternative way of writing a difference equation is as under :

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$, therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \quad \dots (3)$$

Also since, $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$, therefore (2) takes the form :

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \quad \dots (4)$$

Quite often, difference equations are met under the name of *recurrence relations*.

(2) Order of a difference equation is the difference between the largest and the smallest arguments occurring in the difference equation divided by the unit of increment.

Thus (3) above is the second order, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2) - n}{1} = 2,$$

and (4) is of the third order, for $\frac{(n+2) - (n-1)}{1} = 3$.

Obs. While finding the order of a difference equation, it must always be expressed in a form free of Δs , for the highest power of Δ does not give order of the difference equation.

(3) Solution of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.

The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

A particular solution or **particular integral** is that solution which is obtained from the general solution by giving particular values to the constants.

31.3 FORMATION OF DIFFERENCE EQUATIONS

The following examples illustrate the way in which difference equations arise and are formed.

Example 31.1. Form the difference equation corresponding to the family of curves

$$y = ax + bx^2 \quad \dots(i)$$

Solution. We have $\Delta y = a\Delta(x) + b\Delta(x^2) = a(x+1-x) + b[(x+1)^2 - x^2]$
 $= a + b(2x+1) \quad \dots(ii)$

and $\Delta^2 y = 2b[(x+1)-x] = 2b \quad \dots(iii)$

To eliminate a and b , we have from (iii), $b = \frac{1}{2} \Delta^2 y$

and from (ii), $a = \Delta y - b(2x+1) = \Delta y - \frac{1}{2} \Delta^2 y (2x+1)$

Substituting these values of a and b in (i), we get

$$y = \left[\Delta y - \frac{1}{2} \Delta^2 y (2x+1) \right] x + \frac{1}{2} \Delta^2 y \cdot x^2$$

or $(x^2 + x) \Delta^2 y - 2x \Delta y + 2y = 0$

This is the desired difference equation which may equally well be written in terms of E as

or $(x^2 + x) y_{x+2} - (2x^2 + 4x) y_{x+1} + (x^2 + 3x + 2) y_x = 0.$

Example 31.2. From $y_n = A2^n + B(-3)^n$, derive a difference equation by eliminating the constants.

Solution. We have $y_n = A.2^n + B(-3)^n$, $y_{n+1} = 2A.2^n - 3B(-3)^n$

and $y_{n+2} = 4A.2^n + 9B(-3)^n.$

Eliminating A and B , we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0 \quad \text{or} \quad y_{n+2} + y_{n+1} - 6y_n = 0$$

which is the desired difference equation.

PROBLEMS 31.1

- Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.
- Assuming $\frac{\log(1-z)}{1+z} = y_0 + y_1 z + y_2 z^2 + \dots + y_n z^n, \dots$, find the difference equations satisfied by y_n .
- Form a difference equation by eliminating arbitrary constant from $u_n = a2^{n+1}$. (Anna, 2008)
- Find the difference equation satisfied by
 - $y = a/x + b$ (Tiruchirapalli, 2001)
 - $y = ax^2 - bx$.
- Derive the difference equations in each of the following cases :
 - $y_n = A.3^n + B.5^n$
 - $y_n = (A + Bx) 2^x$. (Madras, 2001)
- Form the difference equations generated by
 - $y_n = ax + b2^n$
 - $y_n = a2^n + b(-2)^n$
 - $y_x = a2^x + b3^x + c$.

31.4 LINEAR DIFFERENCE EQUATIONS

(1) **Def.** A **linear difference equation** is that in which y_{n+1} , y_{n+2} , etc. occur to the first degree only and are not multiplied together.

A **linear difference equation with constant coefficient** is of the form

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \quad \dots(1)$$

where a_1, a_2, \dots, a_r are constants.

Now we shall deal with linear difference equations with constant coefficients only. Their properties are analogous to those of linear differential equations with constant co-efficients.

(2) **Elementary properties.** If $u_1(n), u_2(n), \dots, u_r(n)$ be r independent solution of the equation

$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \quad \dots(2)$$

then its complete solution is $U_n = c_1 u_1(n) + \dots + c_r u_r(n)$

where c_1, c_2, \dots, c_r are arbitrary constants.

If V_n is a particular solution of (1), then the complete solution of (1) is $y_n = U_n + V_n$. The part U_n is called the **complementary function (C.F.)** and the part V_n is called the **particular integral (P.I.)** of (1).

Thus the **complete solution (C.S.)** of (1) is $y_n = C.F. + P.I.$

31.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

(i.e., rules to solve a linear difference equation with constant coefficients having right hand side zero).

(1) To begin with, consider the first order linear equation $y_{n+1} - \lambda y_n = 0$, where λ is a constant.

Rewriting it as $\frac{y_{n+1}}{\lambda^{n+1}} - \frac{y_n}{\lambda^n} = 0$, we have $\Delta \left(\frac{y_n}{\lambda^n} \right) = 0$, which gives $y_n / \lambda^n = c$, a constant.

Thus the solution of $(E - \lambda) y_n = 0$ is $y_n = c \lambda^n$.

(2) Now consider the second order linear equation $y_{n+2} + a y_{n+1} + b y_n = 0$ which in symbolic form is

$$(E^2 + aE + b)y_n = 0 \quad \dots(1)$$

Its symbolic co-efficient equated to zero i.e., $E^2 + aE + b = 0$

is called the **auxiliary equation**. Let its roots be λ_1, λ_2 .

Case I. If these roots are real and distinct, then (1) is equivalent to

$$(E - \lambda_1)(E - \lambda_2)y_n = 0 \quad \dots(2)$$

$$(E - \lambda_2)(E - \lambda_1)y_n = 0 \quad \dots(3)$$

If y_n satisfies the subsidiary equation $(E - \lambda_1)y_n = 0$, then it will also satisfy (3).

Similarly, if y_n satisfies the subsidiary equation $(E - \lambda_2)y_n = 0$, then it will also satisfy (2).

∴ it follows that we can derive two independent solutions of (1), by solving the two subsidiary equations

$$(E - \lambda_1)y_n = 0 \quad \text{and} \quad (E - \lambda_2)y_n = 0$$

Their solutions are respectively, $y_n = c_1(\lambda_1)^n$ and $y_n = c_2(\lambda_2)^n$

where c_1 and c_2 are arbitrary constants.

Thus the general solution of (1) is $y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

Case II. If the roots are real and equal (i.e., $\lambda_1 = \lambda_2$), then (2) becomes

$$(E - \lambda_1)^2 y_n = 0 \quad \dots(4)$$

Let

$$y_n = (\lambda_1)^n z_n$$

where z_n is a new dependent variable. Then (4) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2 \cdot (\lambda_1)^n z_n = 0$$

or $z_{n+2} - 2z_{n+1} + z_n = 0 \quad \text{i.e.,} \quad \Delta^2 z_n = 0$

∴ $z_n = c_1 + c_2 n$, where c_1, c_2 are arbitrary constants.

Thus the solution of (1) becomes $y_n = (c_1 + c_2 n)(\lambda_1)^n$.

Case III. If the roots are imaginary, (i.e. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$) then the solution of (1) is

$$\begin{aligned} y_n &= c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n & [\text{Put } \alpha = r \cos \theta \text{ and } \beta = r \sin \theta] \\ &= r^n [c_1(\cos n\theta + i \sin n\theta) + c_2(\cos n\theta - i \sin n\theta)] \end{aligned}$$

$$= r^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

where A_1, A_2 are arbitrary constants are $r = \sqrt{(\alpha^2 + \beta^2)}$, $\theta = \tan^{-1}(\beta/\alpha)$.

(3) In general, to solve the equation $y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0$ where a 's are constants :

(i) Write the equation in the symbolic form $(E^r + a_1 E^{r-1} + \dots + a_r) y_n = 0$.

(ii) Write down the auxiliary equation i.e., $E^r + a_1 E^{r-1} + \dots + a_r = 0$ and solve it for E .

(iii) Write the solution as follows :

Roots of A.E.	Solution, i.e. C.F.
1. $\lambda_1, \lambda_2, \lambda_3, \dots$ (real and distinct roots)	$c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + \dots$
2. $\lambda_1, \lambda_1, \lambda_3, \dots$ (2 real and equal roots)	$(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + \dots$
3. $\lambda_1, \lambda_1, \lambda_1, \dots$ (3 real and equal roots)	$(c_1 + c_2 n + c_3 n^2) (\lambda_1)^n + \dots$
4. $\alpha + i\beta, \alpha - i\beta, \dots$ (a pair of imaginary roots)	$r^n (c_1 \cos \theta + c_2 \sin n\theta)$ where $r = \sqrt{(\alpha^2 + \beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$

Example 31.3. Solve the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$.

Solution. Given equation in symbolic form is $(E^3 - 2E^2 - 5E + 6)u_n = 0$

\therefore its auxiliary equation is $E^3 - 2E^2 - 5E + 6 = 0$

or

$$(E - 1)(E + 2)(E - 3) = 0.$$

$$\therefore E = 1, -2, 3$$

Thus the complete solution is $u_n = c_1(1)^n + c_2(-2)^n + c_3(3)^n$.

Example 31.4. Solve $u_{n+2} - 2u_{n+1} + u_n = 0$.

Solution. Given difference equation in symbolic form is $(E^2 - 2E + 1)u_n = 0$.

\therefore its auxiliary equation is $E^2 - 2E + 1 = 0$

or

$$(E - 1)^2 = 0.$$

$$\therefore E = 1, 1$$

Thus the required solution is $u_n = (c_1 + c_2 n)(1)^n$, i.e., $u_n = c_1 + c_2 n$.

Example 31.5. Solve $y_{n+1} - 2y_n \cos \alpha + y_{n-1} = 0$.

Solution. This is a second order difference equation in y_{n-1} ; which in symbolic form is

$$(E^2 - 2E \cos \alpha + 1) y_n = 0$$

The auxiliary equation is $E^2 - 2E \cos \alpha + 1 = 0$

$$E = \frac{2 \cos \alpha \pm \sqrt{(4 \cos^2 \alpha - 4)}}{4} = \cos \alpha \pm i \sin \alpha$$

Thus the solution is $y_{n-1} = (1)^{n-1} [c_1 \cos(n-1)\alpha + c_2 \sin(n-1)\alpha]$

or

$$y_n = c_1 \cos n\alpha + c_2 \sin n\alpha.$$

Example 31.6. The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.

Solution. In this sequence, each number beyond the second, is the sum of its two previous number. If y_n be the n th number then $y_n = y_{n-1} + y_{n-2}$ for $n > 2$.

or $y_{n+2} - y_{n+1} - y_n = 0$ (for $n > 0$)

or $(E^2 - E - 1)y_n = 0$ is the difference equation.

Its A.E. is $E^2 - E - 1 = 0$ which gives $E = \frac{1}{2}(1 \pm \sqrt{5})$.

Thus the solution is $y_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$, for $n > 0$

When $n = 1, y = 0$

$$\therefore c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) = 0 \quad \dots(i)$$

When $n = 2, y_2 = 0$

$$\therefore c_1 \left(\frac{1+\sqrt{5}}{2} \right)^2 + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^2 = 0 \quad \dots(ii)$$

Solving (i) and (ii), we get

$$c_1 = \frac{5-\sqrt{5}}{10} \text{ and } c_2 = \frac{5+\sqrt{5}}{10}$$

Hence the complete solution is

$$y_n = \frac{5-\sqrt{5}}{10} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{5+\sqrt{5}}{10} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

PROBLEMS 31.2

Solve the following difference equations :

1. $u_{x+2} - 6u_{x+1} + 9u_x = 0.$
2. $y_{n+2} + y_{n+1} + 2y_n = 0.$
3. $\Delta^2 u_n + 2\Delta u_n + u_n = 0.$
4. $(\Delta^2 - 3\Delta + 2)y_n = 0.$
5. $4y_n - y_{n+2} = 0$ given that $y_0 = 0, y_1 = 2.$
6. $u_{k+3} - 3u_{k+2} + 4u_k = 0.$
7. $f(x+3) - 3f(x+1) - 2f(x) = 0.$
8. $u_{n+3} - 3u_{n+1} + 2u_n = 0,$ given $u_1 = 0, u_2 = 8$ and $u_3 = -2.$
9. $(E^3 - 5E^2 + 8E - 4)y_n = 0,$ given that $y_0 = 3, y_1 = 2, y_3 = 22.$
10. $u_{n+1} - 2u_n + 2u_{n-1} = 0.$
11. $y_{m+3} + 16y_{m-1} = 0.$

[Hint. $E^4 = -16 = 16 [\cos(2n+1)\pi + i \sin(2n+1)\pi]$; use De Moivre's theorem.]

12. Show that the difference equation $I_{m+1} - (2 + r_o/r) I_m + I_{m-1} = 0$ has the solution.

$$I_m = I_0 \sinh(n-m)\alpha / \sinh(n-1)\alpha, \text{ if } I = I_0 \text{ and } I_n = 0, \alpha \text{ being } = 2 \sinh^{-1} \frac{1}{2} (r_o/r)^{1/2}.$$

13. A series of values of y_n satisfy the relation, $y_{n+2} + ay_{n+1} + by_n.$

Given that $y_0 = 0, y_1 = 1, y_2 = y_3 = 2.$ Show that $y_n = 2^{n/2} \sin n\pi/4.$

14. A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two years old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking y_n to be the number of seeds produced at the end of the n th year, show that $y_{n+1} = 8y_n + 18(y_1 + y_2 + \dots + y_{n-1}).$
Hence show that $y_{n+2} - 9y_{n+1} - 10y_n = 0$ and find $y_n.$

31.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = f(n)$

which in symbolic form is $\phi(E)y_n = f(n) \quad \dots(1)$

where $\phi(E) = E^r + a_1 E^{r-1} + \dots + a_r$

Then the particular integral is given by P.I. = $\frac{1}{\phi(E)} f(n).$

Case I. When $f(n) = a^n$

$$\begin{aligned} \text{P.I.} &= \frac{1}{\phi(E)} a^n, \text{ put } E = a \\ &= \frac{1}{\phi(a)} a^n, \text{ provided } \phi(a) \neq 0 \end{aligned}$$

If $\phi(a) = 0,$ then for the equation

$$(i) (E - a)y_n = a^n, \quad \text{P.I.} = \frac{1}{E - a} a^n = na^{n-1}$$

$$(ii) (E - a)^2 y_n = a^n, \quad P.I. = \frac{1}{(E - a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$$

$$(iii) (E - a)^3 y_n = a^n, \quad P.I. = \frac{1}{(E - a)^3} a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$$

and so on.

Example 31.7. Solve $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.

Solution. Given equation in symbolic form is $(E^2 - 4E + 3)y_n = 5^n$

∴ The auxiliary equation is $E^2 - 4E + 3 = 0$

$$\text{or } (E - 1)(E - 3) = 0. \quad \therefore E = 1, 3$$

$$\therefore C.F. = c_1(1)^n + c_2(3)^n = c_1 + c_2 \cdot 3^n$$

and

$$\begin{aligned} P.I. &= \frac{1}{E^2 - 4E + 3} 5^n && [\text{Put } E = 5] \\ &= \frac{1}{25 - 4 \cdot 5 + 3} 5^n = \frac{1}{8} \cdot 5^n \end{aligned}$$

Thus the complete solution is $y_n = c_1 + c_2 \cdot 3^n + 5^n/8$.

Example 31.8. Solve $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.

Solution. Given equation in symbolic form is $(E^2 - 4E + 4)u_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 4 = 0. \quad \therefore E = 2, 2$.

$$C.F. = (c_1 + c_2 n) 2^n$$

$$P.I. = \frac{1}{(E - 2)^2} \cdot 2^n = \frac{n(n-1)}{2!} \cdot 2^{n-2} = n(n-1) 2^{n-3}$$

Hence the complete solution is $u_n = (c_1 + c_2 n) 2^n + n(n-1) 2^{n-3}$.

Case II. When $f(n) = \sin kn$.

$$P.I. = \frac{1}{\phi(E)} \sin kn = \frac{1}{\phi(E)} \left(\frac{e^{ikn} - e^{-ikn}}{2i} \right) = \frac{1}{2i} \left[\frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n \right]$$

where $a = e^{ik}$ and $b = e^{-ik}$.

Now proceed as in case I.

$$\begin{aligned} (2) \text{ When } f(n) = \cos kn \quad P.I. &= \frac{1}{\phi(E)} \cos kn = \frac{1}{\phi(E)} \left(\frac{e^{ikn} + e^{-ikn}}{2} \right) \\ &= \frac{1}{2} \left[\frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n \right] \text{ as before} \end{aligned}$$

Now proceed as in case I.

Example 31.9. Solve $y_{n+2} - 2 \cos \alpha \cdot y_{n+1} + y_n = \cos \alpha n$.

(Nagpur, 2008)

Solution. Given equation in symbolic form is $(E^2 - 2 \cos \alpha \cdot E + 1)y_n = \cos \alpha n$.

The auxiliary equation is $E^2 - 2 \cos \alpha \cdot E + 1 = 0$.

$$\therefore E = \frac{2 \cos \alpha \pm \sqrt{(4 \cos^2 \alpha - 4)}}{2} = \cos \alpha \pm i \sin \alpha$$

$$\therefore C.F. = (1)^n [c_1 \cos \alpha n + c_2 \sin \alpha n] \text{ i.e., } c_1 \cos \alpha n + c_2 \sin \alpha n$$

$$P.I. = \frac{1}{E^2 - 2E \cos \alpha + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left(\frac{e^{ian} + e^{-ian}}{2} \right)$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{ian} + \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{-ian} \right] \\
 &\quad [\text{Put } E = e^{i\alpha}] \qquad \qquad \qquad [\text{Put } E = e^{-i\alpha}] \\
 &= \frac{1}{2} \left[\frac{1}{(E - e^{i\alpha})} \cdot \frac{1}{e^{ia} - e^{-ia}} e^{ian} + \frac{1}{E - e^{-i\alpha}} \cdot \frac{1}{e^{-ia} - e^{ia}} e^{-ian} \right] \\
 &= \frac{1}{4i \sin \alpha} \left[\frac{1}{E - e^{i\alpha}} e^{ian} - \frac{1}{E - e^{-i\alpha}} e^{-ian} \right] = \frac{1}{4i \sin \alpha} [n \cdot e^{i\alpha(n-1)} - n \cdot e^{-i\alpha(n-1)}] \\
 &= \frac{n}{2 \sin \alpha} \left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i} \right] = \frac{n \sin(n-1)\alpha}{2 \sin \alpha}
 \end{aligned}$$

Hence the complete solution is

$$y_n = c_1 \cos \alpha n + c_2 \sin \alpha n + \frac{n \sin(n-1)\alpha}{2 \sin \alpha}.$$

Case III. When $f(n) = n^p$. P.I. = $\frac{1}{\phi(E)} n^p = \frac{1}{\phi(1+\Delta)} n^p$

- (1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by the Binomial theorem as far as the term in Δ^p .
- (2) Expand n^p in the factorial form (p. 950) and operate on it with each term of the expansion.

Example 31.10. Solve $y_{n+2} - 4y_n = n^2 + n - 1$.

(Madras, 1999)

Solution. Given equation is $(E^2 - 4)y_n = n^2 + n - 1$.

The auxiliary equation is $E^2 - 4 = 0$, $\therefore E = \pm 2$.

$$\therefore \text{C.F.} = c_1 (2)^n + c_2 (-2)^n.$$

$$\begin{aligned}
 \therefore \text{P.I.} &= \frac{1}{E^2 - 4} (n^2 + n - 1) = \frac{1}{(1 + \Delta)^2 - 4} [n(n-1) + 2n - 1] \\
 &= \frac{1}{\Delta^2 + 2\Delta - 3} ([n]^2 + 2[n] - 1) = -\frac{1}{3} \left[1 - \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3} \right) \right]^{-1} ([n]^2 + 2[n] - 1) \\
 &= -\frac{1}{3} \left[1 + \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3} \right) + \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3} \right) + \dots \right] ([n]^2 + 2[n] - 1) \\
 &= -\frac{1}{3} \left\{ 1 + \frac{2}{3}\Delta + \frac{7}{9}\Delta^2 + \dots \right\} ([n]^2 + 2[n] - 1) = -\frac{1}{3} \left\{ [n]^2 + 2[n] - 1 + \frac{2}{3}(2[n] + 2) + \frac{7}{9} \times 2 \right\} \\
 &= -\frac{1}{3} \left\{ [n]^2 + \frac{10}{3}[n] + \frac{17}{9} \right\} = -\frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}.
 \end{aligned}$$

Hence the complete solution is $y_n = c_1 2^n + c_2 (-2)^n - \frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}$.

Case IV. When $f(n) = a^n F(n)$, $F(n)$, being a polynomial of finite degree in n .

$$\text{P.I.} = \frac{1}{\phi(E)} a^n F(n) = a^n \frac{1}{\phi(aE)} F(a)$$

Now $F(n)$ being a polynomial in n , proceed as in case III.

Example 31.11. Solve $y_{n+2} - 2y_{n+1} + y_n = n^2 \cdot 2^n$.

(Nagpur, 2008)

Solution. Given equation is $(E^2 - 2E + 1)y_n = n^2 \cdot 2^n$.

$$\text{Its C.F.} = c_1 + c_2 n$$

and

$$\text{P.I.} = \frac{1}{(E-1)^2} 2^n \cdot n^2 = 2^n \frac{1}{(2E-1)^2} n^2 = 2^n \frac{1}{(1+2\Delta)^2} n^2$$

$$\begin{aligned}
 &= 2^n (1 + 2\Delta)^{-2} n(n-1) + n = 2^n (1 - 4\Delta + 12\Delta^2 - \dots) ([n]^2 + [n]) \\
 &= 2^n ([n]^2 + [n] - 4(2[n] + 1) + 12 \times 2) \\
 &= 2^n (([n]^2 - 7[n] + 20) = 2^n (n^2 - 8n + 20)
 \end{aligned}$$

Hence the complete solution is $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$.

PROBLEMS 31.3

Solve the following difference equations:

1. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n$, $y_0 = 0$, $y_1 = 1$.

(Madras, 2003)

2. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0 = y_1 = 0$.

(V.T.U., 2009)

3. $y_{p+3} - 3y_{p+2} - 3y_{p+1} - y_p = 1$.

(Kottayam, 2005)

4. $y_{n+2} - 2y_{n+1} + 4y_n = 6$, given that $y_0 = 0$ and $y_1 = 2$.

5. $(E^2 - 4E + 3)y = 3^x$.

6. $y_{x+2} - 4y_{x+1} + 4y_x = 3 \cdot 2^x + 5 \cdot 4^x$.

7. $u_{n+2} - u_n = \cos n/2$. (Madras, 2001 S)

8. $y_{p+2} - \left(2 \cos \frac{1}{2}\right) y_{p+1} + y_p = \sin p/2$.

9. $(E^2 - 4)y_x = x^2 - 1$.

10. $y_{n+3} + y_n = n^2 + 1$, $y_0 = y_1 = y_2 = 0$. (Tiruchirapalli, 2001)

11. $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n$.

(Nagpur, 2009)

12. $(4E^2 - 4E + 1)y = 2^n + 2^{-n}$. (Madras, 2001)

13. $y_{n+2} + 5y_{n+1} + 6y_n = n + 2^n$. (Nagpur, 2006)

14. $u_{x+2} + 6u_{x+1} + 9u_x = x2^x + 3^x + 7$.

15. $y_{n+3} + 8y_n = (2n+3)2^n$. (Nagpur, 2005)

16. $u_{n+2} - 4u_{n+1} - 4u_n = n^2 2^n$.

17. $(E^2 - 5E + 6)y_k = 4^k (k^2 - k + 5)$.

18. $(E^2 - 2E + 4)y_n = -2^n \left\{ 6 \cos \frac{n\pi}{3} + 2\sqrt{3} \sin \frac{n\pi}{3} \right\}$.

19. A beam of length l , supported at n points carries a uniform load w per unit length. The bending moments M_1, M_2, \dots, M_n at the supports satisfy the Clapeyron's equation :

$$M_{r+2} + 4M_{r+1} + M_r = -\frac{1}{2} wl^2$$

If a beam weighing 30 kg is supported at its ends and at two other supports dividing the beam into three equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

31.7 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

The method used for solving simultaneous differential equations with constant coefficients also applies to simultaneous difference equations with constants coefficients. The following example illustrates the technique.

Example 31.12. Solve the simultaneous difference equations

$$u_{x+1} + v_x - 3u_x = x, \quad 3u_x + v_{x+1} - 5v_x = 4^x$$

subject to the conditions $u_1 = 2$, $v_1 = 0$.

Solution. Given equation in symbolic form, are

$$(E - 3)u_x + v_x = x \quad \dots(i)$$

$$3u_x + (E - 5)v_x = 4^x \quad \dots(ii)$$

Operating the first equation with $E - 5$ and subtracting the second from it, we get

$$[(E - 5)(E - 3) - 3]u_x = (E - 5)x - 4^x$$

or

$$(E^2 - 8E + 12)u_x = 1 - 4x - 4^x$$

Its solution is $u_x = c_1 2^x + c_2 6^x - \frac{4}{5}x - \frac{19}{25} + \frac{4^x}{4}$ $\dots(iii)$

Substituting the value of u_x from (iii) in (i), we get

$$v_x = c_1 2^x - 3c_2 6^x - \frac{3x}{5} - \frac{34}{25} - \frac{4^x}{4} \quad \dots(iv)$$

Taking $u_1 = 2$, $v_1 = 0$, in (iii) and (iv), we obtain

$$2c_1 + 6c_2 = \frac{64}{25}, \quad 2c_1 - 18c_2 = \frac{74}{25}$$

when

$$c_1 = 1.33, \quad c_2 = -0.0167$$

Hence

$$u_x = 1.33 \cdot 2^x - 0.0167 \cdot 6^x - 0.8x - 0.76 + 4^{x-1}$$

$$u_x = 1.33 \cdot 2^x - 0.05 \cdot 6^x - 0.6x - 1.36 - 4^{x-1}.$$

PROBLEMS 31.4

Solve the following simultaneous difference equations :

1. $y_{x+1} - z_x = 2(x+1)$, $z_{x+1} - y_x = -2(x+1)$.
2. $y_{n+1} - y_n + 2z_{n+1} = 0$, $z_{n+1} - z_n - 2y_n = 2^n$.
3. $u_{n+1} + n = 3u_n + 2v_n$, $v_{n+1} - n = u_n + 2v_n$, given $u_0 = 0$, $v_0 = 3$.
4. $u_{x+1} + v_x + w_x = 1$, $u_x + v_{x+1} + w_x = x$, $u_x + v_x + w_{x+1} = 2x$.

31.8 APPLICATION TO DEFLECTION OF A LOADED STRING

Consider a light string of length l stretched tightly between A and B . Let the forces P_i be acting at its equispaced points x_i ($i = 1, 2, \dots, n-1$) and perpendicular to AB resulting in small transverse displacements y_i at these points (Fig. 31.1). Assuming the angle θ_i made by the portion between x_i and x_{i+1} with the horizontal, to be small, we have

$$\sin \theta_i = \tan \theta_i = \theta_i \text{ and } \cos \theta_i = 1$$

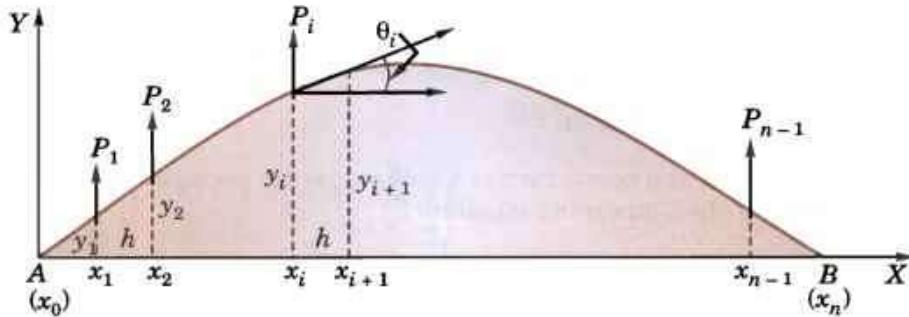


Fig. 31.1

If T be the tension of the string at x_i , then $T \cos \theta_i = T$
i.e., the tension may be taken as uniform.

Taking $x_{i+1} - x_i = h$, we have

$$y_{i+1} - y_i = h \tan \theta_i = h\theta_i \quad \dots(1)$$

$$y_i - y_{i-1} = h \tan \theta_{i-1} = h\theta_{i-1} \quad \dots(2)$$

Also resolving the forces in equilibrium at (x_i, y_i) \perp to AB , we get

$$T \sin \theta_i - T \sin \theta_{i-1} + P_i = 0 \text{ i.e. } T(\theta_i - \theta_{i-1}) + P_i = 0 \quad \dots(3)$$

Eliminating θ_i and θ_{i-1} from (1), (2) and (3), we obtain

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{hP_i}{T} \quad \dots(4)$$

which is a difference equation and its solution gives the displacements y_i . To obtain the arbitrary constants in the solution, we take $y_0 = y_n = 0$ as the boundary conditions, since the ends A and B of the string are fixed.

Example 31.13. A light string stretched between two fixed nails 120 cm apart, carries 11 loads of weight 5 gm each at equal intervals and the resulting tension is 500 gm wt. Show that the sag at the mid-point is 1.8 cm.

Solution. Taking $h = 10$ cm, $P_i = 5$ gm and $T = 500$ gm wt.,
the above equation (4) becomes $y_{i+1} - 2y_i + y_{i-1} = -1/10$

i.e.,

$$y_{i+2} - 2y_{i+1} + y_i = -\frac{1}{10}$$

Its A.E. is $(E - 1)^2 = 0$ i.e. $E = 1, 1$. \therefore C.F. = $c_1 + c_2 i$

and

$$\text{P.I.} = \frac{1}{(E-1)^2} \left(-\frac{1}{10} \right) = -\frac{1}{10} \frac{1}{(E-1)^2} (1)^i = -\frac{1}{10} \frac{i(i-1)}{2} = \frac{1}{20} (i - i^2)$$

Thus the C.S. is $y_i = c_1 + c_2 i + \frac{1}{20} (i - i^2)$

Since $y_0 = 0, \therefore c_1 = 0$

and $y_{12} = 0, \therefore c_2 = \frac{11}{20}$.

Hence $y_i = \frac{11}{20} i + \frac{1}{20} (i - i^2)$

At the mid-point $i = 6$, we get $y_6 = 1.8$ cm.

PROBLEMS 31.5

1. A light string of length $(n+1)l$ is stretched between two fixed points with a force P . It is loaded with n equal masses m at distance l . If the system starts rotating with angular velocity ω , find the displacement y_i of the i th mass.

31.9 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 31.6

Select the correct answer or fill up the blanks in the following questions :

1. $y_n = A 2^n + B 3^n$, is the solution of the difference equation
2. The solution of $(E-1)^3 u_n = 0$ is
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is
4. The solution of $y_{n+1} - y_n = 2^n$ is
5. The difference equation $y_{n+1} - 2y_n = n$ has $y_n = \dots$ as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is
7. The particular integral of the equation $(E-2) y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is
9. The equation $\Delta^2 u_{n+1} + \frac{1}{2} \Delta^2 u_n = 0$ is of order
10. The difference equation satisfied by $y = a + b/x$ is
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is
14. The difference equation generated by $u_n = (a + bn) 3^n$ is
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$.

(True or False)

Numerical Solution of Ordinary Differential Equations

1. Introduction.
2. Picard's method.
3. Taylor's series method.
4. Euler's method.
5. Modified Euler's method.
6. Runge's method.
7. Runge-Kutta method.
8. Predictor-corrector methods.
9. Milne's method.
10. Adams-Bashforth method.
11. Simultaneous first order differential equations.
12. Second order differential equations.
13. Boundary value problems.
14. Finite-difference method.
15. Objective Type of Questions.

32.1 INTRODUCTION

The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce numerical work considerably.

A number of numerical methods are available for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \quad \dots(1)$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y . The methods of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as *step-by-step methods*.

Euler and Runge-Kutta methods are used for computing y over a limited range of x -values whereas Milne and Adams-Bashforth methods may be applied for finding y over a wider range of x -values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (1) is specified at the point x_0 . Such problems in which all the initial conditions are given at the initial point only are called **initial value problems**. But there are problems involving second and higher order differential equations in which the conditions may be given at two or more points. These are known as **boundary value problems**. In this chapter, we shall first explain methods for solving initial value problems and then give a method of solving boundary value problems.

32.2 PICARD'S METHOD*

Consider the first order equation $dy/dx = f(x, y)$

...(1)

* Called after the French mathematician Emile Picard (1856—1941) who was professor in Paris since 1881 and is famous for his researches in the theory of functions.

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^y dy = \int_{x_0}^x f(x, y) dx \quad \text{or} \quad y = y_0 + \int_{x_0}^x f(x, y) dx \quad \dots(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

For a second approximation y_2 , we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx.$$

Similarly, a third approximation is $y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx$.

Continuing this process, a sequence of functions of x , i.e., $y_1, y_2, y_3 \dots$ is obtained each giving a better approximation of the desired solution than the preceding one.

Obs. Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 32.11 and 32.12).

Example 32.1. Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.

Solution. (a) We have $y = 1 + \int_0^x (y + x) dx$.

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x) dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2) dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} \right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \quad \dots(i)$$

(b) Given equation :

$$\frac{dy}{dx} - y = x \text{ is a Leibnitz's linear in } x.$$

Its I.F. being e^{-x} , the solution is

$$ye^{-x} = \int xe^{-x} dx + c = -xe^{-x} - \int (-e^{-x}) dx + c = -xe^{-x} - e^{-x} + c \quad [\text{Integrate by parts}]$$

$$\therefore y = ce^x - x - 1.$$

Since $y = 1$, when $x = 0$, $\therefore c = 2$.

Thus the desired particular solution is $y = 2e^x - x - 1$

... (ii)

Or using the series : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \infty,$

we get $y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \dots \infty$... (iii)

Comparing (i) and (iii), it is clear that (i) approximates to the exact particular solution (ii) upto the term in $x^5.$

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 32.2. Find the value of y for $x = 0.1$ by Picard's method, given that

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1. \quad (\text{P.T.U., 2002})$$

Solution. We have $y = 1 + \int_0^x \frac{y-x}{y+x} dx$

First approximation. Put $y = 1$ in the integrand, giving

$$\begin{aligned} y_1 &= 1 + \int_0^x \frac{1-x}{1+x} dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x} \right) dx \\ &= 1 + [-x + 2 \log(1+x)]_0^x = 1 - x + 2 \log(1+x) \end{aligned} \quad \dots(i)$$

Second approximation. Put $y = 1 - x + 2 \log(1+x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1-x+2 \log(1+x)-x}{1-x+2 \log(1+x)+x} dx = 1 + \int_0^x \left[1 - \frac{2x}{1+2 \log(1+x)} \right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain

$$y(0.1) = 1 - (0.1) + 2 \log 1.1 = 0.9828.$$

32.3 TAYLOR'S SERIES METHOD*

Consider the first order equation $dy/dx = f(x, y)$... (1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \quad \text{i.e.,} \quad y'' = f_x + f_y f' \quad \dots(2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0, (y'')_0, (y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots \quad \dots(3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example 32.3. Find by Taylor's series method the value of y at $x = 0.1$ and $x = .$ to five places of decimals from $dy/dx = x^2y - 1, y(0) = 1.$ (V.T.U., 2009, htak, 2005)

Solution. Here $(y)_0 = 1, y' = x^2y - 1, (y')_0 = -1$

\therefore Differentiating successively and substituting, we get

$$\begin{aligned} y'' &= 2xy + x^2y', & (y'')_0 &= 0 \\ y''' &= 2y + 4xy' + x^2y'', & (y''')_0 &= 2 \\ y^{iv} &= 6y' + 6xy'' + x^2y''', & (y^{iv})_0 &= -6 \text{ etc.} \end{aligned}$$

*See footnote p. 145.

Putting these values in the Taylor's series,

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \dots,$$

we have $y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 32.4. Employ Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

(V.T.U., 2009; P.T.U., 2003)

Solution. (a) We have $y' = 2y + 3e^x$ $y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0$, $y = 0$, we get

$$y'' = 2y' + 3e^x, \quad y''(0) = 2y'(0) + 3 = 9$$

$$y''' = 2y'' + 3e^x, \quad y'''(0) = 2y''(0) + 3 = 21$$

$$y^{iv} = 2y''' + 3e^x, \quad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$\begin{aligned} y(x) &= y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \dots \\ &= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \dots = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \dots \end{aligned}$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \dots = 0.8116$... (i)

(b) Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x .

Its I.F. being e^{-2x} , the solution is

$$ye^{-2x} = \int 3e^x \cdot e^{-2x} dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\therefore c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

$$\text{When } x = 0.2, y = 3(e^{0.4} - e^{0.2}) = 0.8112$$

... (ii)

Comparing (i) and (ii), it is clear that (i) approximates to the exact value upto 3 decimal places.

Example 32.5. Solve by Taylor's series method the equation $\frac{dy}{dx} = \log(xy)$ for $y(1.1)$ and $y(1.2)$, given $y(1) = 2$.

(Hazaribagh, 2009)

Solution. We have $y' = \log x + \log y$; $y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1$, $y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y'; \quad y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y} + y'' + y'\left(-\frac{1}{y^2}\right); \quad y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2!}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + \dots$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1 + \frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2) \log 2 + \frac{(0.2)^2}{2} \left(1 + \frac{1}{2} \log 2 \right) + \frac{(0.2)^3}{6} \left[-\frac{1}{2} + \frac{1}{4} \log 2 - \frac{1}{4} (\log 2)^2 \right] = 2.081.$$

PROBLEMS 32.1

32.4 EULER'S METHOD*

Consider the equation $\frac{dy}{dx} = f(x, y)$

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 32.1. Now we have to find the ordinate of any other point Q on this curve.

Let us divide LM into n sub-intervals each of width h at L_1, L_2, \dots so that h is quite small. In the interval LL_1 , we approximate the curve by the tangent at P . If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$\begin{aligned} y_1 &= L_1 P_1 = LP + R_1 P_1 \\ &= y_0 + PR_1 \tan \theta = y_0 + h \left(\frac{dy}{dx} \right)_P \\ &= y_0 + h f(x_0, y_0) \end{aligned}$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + h f(x_0 + h, y_1) \quad \dots(2)$$

Repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_n = y_{n-1} + h f(x_0 + \overline{n-1} h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e. by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section.

Example 32.6. Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$. (Mumbai. 2005 ; Rohtak. 2003)

*See footnote p. 302.

Solution. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	$Old\ y + 0.1(dy/dx) = new\ y$
0.0	1.00	1.00	$1.00 + 0.1(1.00) = 1.10$
0.1	1.10	1.20	$1.10 + 0.1(1.20) = 1.22$
0.2	1.22	1.42	$1.22 + 0.1(1.42) = 1.36$
0.3	1.36	1.66	$1.36 + 0.1(1.66) = 1.53$
0.4	1.53	1.93	$1.53 + 0.1(1.93) = 1.72$
0.5	1.72	2.22	$1.72 + 0.1(2.22) = 1.94$
0.6	1.94	2.54	$1.94 + 0.1(2.54) = 2.19$
0.7	2.19	2.89	$2.19 + 0.1(2.89) = 2.48$
0.8	2.48	3.89	$2.48 + 0.1(3.89) = 2.81$
0.9	2.81	3.71	$2.81 + 0.1(3.71) = 3.18$
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 32.1, the true value of y from its exact solution at $x = 1$ is 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 32.7. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

(P.T.U., 2001)

Solution. We divide the interval $(0, 0.1)$ into five steps i.e. we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	$(y-x)/(y+x) = dy/dx$	$Old\ y + 0.02(dy/dx) = new\ y$
0.00	1.0000	1.0000	$1.0000 + 0.02(1.0000) = 1.0200$
0.02	1.0200	0.9615	$1.0200 + 0.02(.9615) = 1.0392$
0.04	1.0392	0.926	$1.0392 + 0.02(.926) = 1.0577$
0.06	1.0577	0.893	$1.0577 + 0.02(.893) = 1.0756$
0.08	1.0756	0.862	$1.0756 + 0.02(.862) = 1.0928$
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

32.5 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 32.1) such that at P_1 , we have

$$y_1 = y_0 + h f(x_0, y_0) \quad \dots(1)$$

Then the slope of the curve of solution through P_1 [i.e. $(dy/dx)_{P_1} = f(x_0 + h, y_1)$] is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1 , i.e.

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0 + h, y_1)] \quad \dots(2)$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$. The equation (1) is therefore, called the *predictor* while (2) serves as the *corrector of y_1* .

Again the corrector is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval $L_1 L_2$.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from the predictor

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from the corrector

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which is a predictor-corrector method.

Example 32.8. Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.
(Rohtak, 2005; Bhopal, 2002 S; Delhi, 2002)

Solution. Taking $h = 0.1$, the various calculations are arranged as follows :

x	$x + y = y'$	Mean slope	$Old\ y + 0.1\ (mean\ slope) = new\ y$
0.0	0 + 1	—	$1.00 + 0.1(1.00) = 1.10$
0.1	.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	$1.00 + 0.1(1.1) = 1.11$
0.1	.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	$1.00 + 0.1(1.105) = 1.1105$
0.1	.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	$1.00 + 0.1(1.1052) = 1.1105$
0.1	1.2105	—	$1.1105 + 0.1(1.2105) = 1.2316$
0.2	.2 + 1.2316	$\frac{1}{2}(1.2105 + 1.4316)$	$1.1105 + 0.1(1.3211) = 1.2426$
0.2	.2 + 1.2426	$\frac{1}{2}(1.2105 + 1.4426)$	$1.1105 + 0.1(1.3266) = 1.2432$
0.2	.2 + 1.2432	$\frac{1}{2}(1.2105 + 1.4432)$	$1.1105 + 0.1(1.3268) = 1.2432$
0.2	1.4432	—	$1.2432 + 0.1(1.4432) = 1.3875$
0.3	.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	$1.2432 + 0.1(1.5654) = 1.3997$
0.3	.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	$1.2432 + 0.1(1.5715) = 1.4003$
0.3	.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	$1.2432 + 0.1(1.5718) = 1.4004$
0.3	.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	$1.2432 + 0.1(1.5718) = 1.4004$

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 32.6, the approximate value of y for $x = 0.3$ would be 1.53 whereas by modified Euler's method the corresponding value is 1.4004 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 32.9. Using modified Euler's method, find $y(0.2)$ and $y(0.4)$ given

$$y' = y + e^x, y(0) = 0.$$

(J.N.T.U., 2009)

Solution. We have $y' = y + e^x = f(x, y)$; $x = 0, y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$:

x	$y + e^x = y'$	Mean slope	$Old\ y + h\ (mean\ slope) = new\ y$
0.0	1	—	$0 + 0.2 (1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2 (1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2 (1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2 (1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2 (1.2341) = 0.2468$

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$:

x	$y + e^x = y'$	Mean slope	$Old\ y + h\ (Mean\ slope) = new\ y$
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2 (1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2 (1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + 0.2 (1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + 0.2 (1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + 0.2 (1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + 0.2 (1.7815) = 0.6031$

Since the last two value of y are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Example 32.10. Solve the following by Euler's modified method :

$$\frac{dy}{dx} = \log(x + y), y(0) = 2.$$

at $x = 1.2$ and 1.4 with $h = 0.2$.

(Bhopal, 2009 ; U.P.T.U., 2007)

Solution. The various calculations are arranged as follows :

x	$\log(x + y) = y'$	Mean slope	$Old\ y + 0.2\ (mean\ slope) = new\ y$
0.0	$\log(0 + 2)$	—	$2 + 0.2 (0.301) = 2.0602$
0.2	$\log(0.2 + 2.0602)$	$\frac{1}{2}(0.301 + 0.3541)$	$2 + 0.2 (0.3276) = 2.0655$
0.2	$\log(0.2 + 2.0655)$	$\frac{1}{2}(0.301 + 0.3552)$	$2 + 0.2 (0.3281) = 2.0656$
0.2	0.3552	—	$2.0656 + 0.2 (0.3552) = 2.1366$
0.4	$\log(0.4 + 2.1366)$	$\frac{1}{2}(0.3552 + 0.4042)$	$2.0656 + 0.2 (0.3797) = 2.1415$
0.4	$\log(0.4 + 2.1415)$	$\frac{1}{2}(0.3552 + 0.4051)$	$2.0656 + 0.2 (0.3801) = 2.1416$

x	$\log(x+y) = y'$	Mean slope	$Old\ y + 0.2\ (mean\ slope) = new\ y$
0.4	0.4051	—	$2.1416 + 0.2(0.4051) = 2.2226$
0.6	$\log(0.6 + 2.2226)$	$\frac{1}{2}(0.4051 + 0.4506)$	$2.1416 + 0.2(0.4279) = 2.2272$
0.6	$\log(0.6 + 2.2272)$	$\frac{1}{2}(0.4051 + 0.4514)$	$2.1416 + 0.2(0.4282) = 2.2272$
0.6	0.4514	—	$2.2272 + 0.2(0.4514) = 2.3175$
0.8	$\log(0.8 + 2.3175)$	$\frac{1}{2}(0.4514 + 0.4938)$	$2.2272 + 0.2(0.4726) = 2.3217$
0.8	$\log(0.8 + 2.3217)$	$\frac{1}{2}(0.4514 + 0.4943)$	$2.2272 + 0.2(0.4727) = 2.3217$
0.8	0.4943	—	$2.3217 + 0.2(0.4943) = 2.4206$
1.0	$\log(1 + 2.4206)$	$\frac{1}{2}(0.4943 + 0.5341)$	$2.3217 + 0.2(0.5142) = 2.4245$
1.0	$\log(1 + 2.4245)$	$\frac{1}{2}(0.4943 + 0.5346)$	$2.3217 + 0.2(0.5144) = 2.4245$
1.0	0.5346	—	$2.4245 + 0.2(0.5346) = 2.5314$
1.2	$\log(1.2 + 2.5314)$	$\frac{1}{2}(0.5346 + 0.5719)$	$2.4245 + 0.2(0.5532) = 2.5351$
1.2	$\log(1.2 + 2.5351)$	$\frac{1}{2}(0.5346 + 0.5723)$	$2.4245 + 0.2(0.5534) = 2.5351$
1.2	0.5723	—	$2.5351 + 0.2(0.5723) = 2.6496$
1.4	$\log(1.4 + 2.6496)$	$\frac{1}{2}(0.5723 + 0.6074)$	$2.5351 + 0.2(0.5898) = 2.6531$
1.4	$\log(1.4 + 2.6531)$	$\frac{1}{2}(0.5723 + 0.6078)$	$2.5351 + 0.2(0.5900) = 2.6531$

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 32.11. Using Euler's modified method, obtain a solution of the equation $dy/dx = x + |\sqrt{y}|$, with initial conditions $y = 1$ at $x = 0$, for the range $0 \leq x \leq 0.6$ in steps of 0.2. (V.T.U., 2007)

Solution. The various calculations are arranged as follows :

x	$x + \sqrt{y} = y'$	Mean slope	$Old\ y + .2\ (mean\ slope) = new\ y$
0.0	$0 + 1 = 1$	—	$1 + 0.2(1) = 1.2$
0.2	$0.2 + \sqrt{1.2} = 1.2954$	$\frac{1}{2}(1 + 1.2954) = 1.1477$	$1 + 0.2(1.1477) = 1.2295$
0.2	$0.2 + \sqrt{1.2295} = 1.3088$	$\frac{1}{2}(1 + 1.3088) = 1.1544$	$1 + 0.2(1.1544) = 1.2309$
0.2	$0.2 + \sqrt{1.2309} = 1.3094$	$\frac{1}{2}(1 + 1.3094) = 1.1547$	$1 + 0.2(1.1547) = 1.2309$
0.2	1.3094	—	$1.2309 + 0.2(1.3094) = 1.4927$
0.4	$0.4 + \sqrt{1.4927} = 1.6218$	$\frac{1}{2}(1.3094 + 1.6218) = 1.4654$	$1.2309 + 0.2(1.4654) = 1.5240$
0.4	$0.2 + \sqrt{1.524} = 1.6345$	$\frac{1}{2}(1.3094 + 1.6345) = 1.4718$	$1.2309 + 0.2(1.4718) = 1.5253$
0.4	$0.4 + \sqrt{1.5253} = 1.6350$	$\frac{1}{2}(1.3094 + 1.6350) = 1.4721$	$1.2309 + 0.2(1.4721) = 1.5253$

x	$x + \sqrt{y} = y'$	Mean slope	$Old\ y + .2\ (mean\ slope) = new\ y$
0.4	1.6350	—	$1.5253 + 0.2(1.635) = 1.8523$
0.6	$0.6 + \sqrt{(1.8523)} = 1.9610$	$\frac{1}{2}(1.635 + 1.961) = 1.798$	$1.5253 + 0.2(1.798) = 1.8849$
0.6	$0.6 + \sqrt{(1.8849)} = 1.9729$	$\frac{1}{2}(1.635 + 1.9729) = 1.8040$	$1.5253 + 0.2(1.804) = 1.8861$
0.6	$0.6 + \sqrt{(1.8861)} = 1.9734$	$\frac{1}{2}(1.635 + 1.9734) = 1.8042$	$1.5253 + 0.2(1.8042) = 1.8861$

Hence $y(0.6) = 1.8861$ approximately.

PROBLEMS 32.2

- Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). *(Kottayam, 2005)*
- Using simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
- Using Euler's method, find the approximate value of y when $dy/dx = x^2 + y^2$ and $y(0) = 1$ in five steps (i.e. $h = 0.2$). *(Mumbai, 2006)*
- Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$. *(Anna, 2005)*
- Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. *(J.N.T.U., 2007)*
- Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$ taking $h = 0.05$ by modified Euler's method. *(V.T.U., 2010)*
- Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (5 steps). *(V.T.U., 2007)*
- Given that $dy/dx = 2 + \sqrt{xy}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method. *(Anna, 2004)*

32.6 RUNGE'S METHOD*

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \quad \dots(1)$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 32.2).

Integrate both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0+k} dy = \int_{x_0}^{x_0+h} f(x, y) dx \quad \dots(2)$$

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (i.e. dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1 . The value of y_s is given by the point S_1 .

$$\therefore y_s = NS = LP + HS_1 = y_0 + PH \tan \theta$$

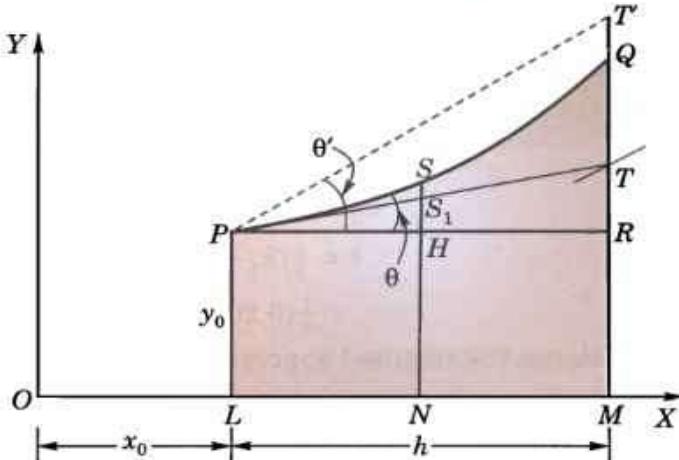


Fig. 32.2

* Called after the German mathematician Carl Runge (1856–1927) who was professor at Gottingen.

$$= y_0 + \frac{h}{2} (dy/dx)_P = y_0 + \frac{h}{2} f(x_0, y_0) \quad \dots(3)$$

Also $y_T = MT = LP + RT = y_0 + PR \tan \theta = y_0 + hf(x_0, y_0)$.

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P drawn with slope at $T(x_0 + h, y_T)$ meets MQ .

\therefore Slope at $T = \tan \theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore y_Q = MR + RT' = y_0 + PT \tan \theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \quad \dots(4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,

the value of $f(x, y)$ at $S = f(x_0 + h/2, y_S)$

and the value of $f(x, y)$ at $Q = f(x_0 + h, y_Q)$

where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$\begin{aligned} k &= \int_{x_0}^{x_0+h} f(x, y) dx = \frac{h}{6} [f_P + 4f_S + f_Q] && [\text{By Simpsons' rule (p. 1106)}] \\ &= \frac{h}{6} [f(x_0, y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \end{aligned} \quad \dots(5)$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working rule to solve (1) by Runge's method :

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

$$k_3 = hf(x_0 + h, y_0 + k')$$

and

$$\text{Finally compute, } k = \frac{1}{6}(k_1 + 4k_2 + k_3).$$

(Note that k is the weighted mean of k_1 , k_2 and k_3)

Example 32.12. Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution. Here we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$$\therefore k_1 = hf(x_0, y_0) = 0.2(1) = 0.200$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2f(0.1, 1.1) = 0.240$$

$$k' = hf(x_0 + h, y_0 + k_1) = 0.2f(0.2, 1.2) = 0.280$$

and

$$k_3 = hf(x_0 + h, y_0 + k') = 0.2f(0.1, 1.28) = 0.296$$

$$\begin{aligned} \therefore k &= \frac{1}{6}(k_1 + 4k_2 + k_3) \\ &= \frac{1}{6}(0.200 + 0.960 + 0.296) = 0.2426 \end{aligned}$$

Hence the required approximate value of y is 1.2426.

32.7 RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in h^r , where r differs from method to method and is called the *order of that method*. *Euler's method*, *Modified Euler's method* and *Runge's method* are the Runge-Kutta methods of the first, second and third order respectively.

* See footnote p. 1017. Named after *Wilhelm Kutta* (1867—1944).

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \text{ is as follows :}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

and

$$\text{Finally compute } k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4)

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 32.13. Apply Runge-Kutta fourth order method, to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$. (V.T.U., 2009; P.T.U., 2007; S.V.T.U., 2007)

Solution. Here

$$x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$$

$$\therefore k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888) = \frac{1}{6} \times (1.4568) = 0.2468.$$

Hence the required approximate value of y is 1.2428.

Example 32.14. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$, 0.4. (U.P.T.U., 2010; J.N.T.U., 2009; V.T.U., 2008)

Solution. We have $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$:

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2 f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence

$$y(0.2) = y_0 + k = 1.196.$$

To find $y(0.4)$:

Here

$$\begin{aligned}
 x_1 &= 0.2, y_1 = 1.196, h = 0.2 & \\
 k_1 &= hf(x_1, y_1) & = 0.1891 \\
 k_2 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) & = 0.1795 \\
 k_3 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) & = 0.1793 \\
 k_4 &= hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) & = 0.1688 \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
 &= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] & = 0.1792
 \end{aligned}$$

Hence

$$y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$$

Example 32.15. Apply Runge-Kutta method to find an approximate value of y for $x = 0.2$ in steps of 0.1, if $dy/dx = x + y^2$, given that $y = 1$, where $x = 0$. (V.T.U., 2009; Osmania, 2007; Madras, 2000)

Solution. Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$$\begin{aligned}
 \therefore k_1 &= hf(x_0, y_0) = 0.1f(0, 1) & = 0.1000 \\
 k_2 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.1f(0.05, 1.1) & = 0.1152 \\
 k_3 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.1f(0.05, 1.1152) & = 0.1168 \\
 k_4 &= hf(x_0 + h, y_0 + k_3) = 0.1f(0.1, 1.1168) & = 0.1347 \\
 \therefore k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
 &= \frac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) & = 0.1165
 \end{aligned}$$

giving

$$y(0.1) = y_0 + k = 1.1165.$$

Step II. $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$$\begin{aligned}
 \therefore k_1 &= hf(x_1, y_1) = 0.1f(0.1, 1.1165) & = 0.1347 \\
 k_2 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.1f(0.15, 1.1838) & = 0.1551 \\
 k_3 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.1f(0.15, 1.194) & = 0.1576 \\
 k_4 &= hf(x_1 + h, y_1 + k_3) = 0.1f(0.2, 1.1576) & = 0.1823 \\
 \therefore k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) & = 0.1571
 \end{aligned}$$

Hence

$$y(0.2) = y_1 + k = 1.2736.$$

Example 32.16. Using Runge-Kutta method of fourth order, solve for y at $x = 1.2, 1.4$ from $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1, y_0 = 0$. (Mumbai, 2008)

Solution. We have $f(x, y) = \frac{2xy + e^x}{x^2 + xe^x}$

To find $y(1.2)$:

Here

$$x_0 = 1, y_0 = 0, h = 0.2$$

$$\therefore k_1 = hf(x_0, y_0) = 0.2 \frac{0+e}{1+e} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \left\{ \frac{2(1+0.1)(0+0.073) + e^{1+0.1}}{(1+0.1)^2 + (1+0.1)e^{1+0.1}} \right\} = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \left\{ \frac{2(1+0.1)(0+0.07) + e^{1.1}}{(1+0.1)^2 + (1+0.1)e^{1.1}} \right\} = 0.1399$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \left\{ \frac{2(1.2)(0.1399) + e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}} \right\} = 0.1348$$

and

$$\begin{aligned} k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348] \\ &= 0.1402. \end{aligned}$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402.$

To find $y(1.4)$:

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2 f(1.2, 0) = 0.1348$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2 f(1.3, 0.2076) = 0.1303$$

$$k_3 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2 f(1.3, 0.2053) = 0.1301$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 f(1.3, 0.2703) = 0.1260$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260] = 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

PROBLEMS 32.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y, y(0) = 1$, taking $h = 0.1$.

(V.T.U., 2004)

3. Using Runge-Kutta method of order 4, compute $y(.2)$ and $(.4)$ from $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$, taking $h = 0.1$.

(Rohtak, 2003 ; Bhopal, 2002)

4. Use Runge-Kutta method to find y when $x = 1.2$ in steps of 0.1, given that :

$dy/dx = x^2 + y^2$ and $y(1) = 1.5$. (Mumbai, 2007)

5. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. (J.N.T.U., 2006)

6. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3, dy/dx = (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2 . (Anna, 2007)

7. Use fourth order Runge-Kutta method to find y at $x = 0.1$, given that $\frac{dy}{dx} = 3e^x + 2y, y(0) = 0$ and $h = 0.1$.

(V.T.U., 2006)

8. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{x+y}$. (S.V.T.U., 2007 S)

9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$. Take $h = 0.2$.

(V.T.U., 2011 S)

10. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5 .

(Delhi, 2002)

32.8 PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 32.4), we have

$$y_i = y_{i-1} + hf(x_0 + i-1 h, y_{i-1}) ; i = 1, 2, 3, \dots \quad \dots(1)$$

The modified Euler's method (§ 32.5), gives

$$y_i = y_{i-1} + \frac{h}{2} [f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \quad \dots(2)$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i . This value of y_i is again substituted in (2) to find a still better approximation of y_i . This step is repeated till two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as predictor-corrector method.* The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i .

In the methods so far explained, to solve a differential equation over an interval (x_i, x_{i+1}) only the value of y at the beginning of the interval was required. In the *predictor-corrector* methods, four prior values are required for finding the value of y at x_{i+1} . A predictor formula is used to predict the value of y at x_{i+1} and then a corrector formula is applied to improve this value.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

32.9 MILNE'S METHOD

Given $dy/dx = f(x, y)$ and $y = y_0, x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows :

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h), y_2 = y(x_0 + 2h), y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1), f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \frac{n(n-1)(n-2)}{6} \Delta^3 f_0 + \dots$$

in the relation $y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y) dx$

$$\begin{aligned} \therefore y_4 &= y_0 + \int_{x_0}^{x_0+4h} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \dots \right) dx && [\text{Put } x = x_0 + nh, dx = hdn] \\ &= y_0 + h \int_0^4 \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2} \Delta^2 f_0 + \dots \right) dn \\ &= y_0 + h \left(4f_0 + 8\Delta f_0 + \frac{20}{3} \Delta^2 f_0 + \frac{8}{3} \Delta^3 f_0 + \dots \right) \end{aligned}$$

Neglecting fourth and higher order differences and expressing Δf_0 , $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3) \text{ which is called a predictor.}$$

Having found y_4 , we obtain a first approximation to $f_4 = f(x_0 + 4h, y_4)$.

Then a better value of y_4 is found by Simpson's rule (p. 1106) as

$$y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4) \text{ which is called a corrector.}$$

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4 . We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the predictor as

$$y_5^{(p)} = y_1 + \frac{4h}{3} (2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the corrector as

$$y_5^{(c)} = y_3 + \frac{h}{3} (f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is Milne's predictor-corrector method. To ensure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

Example 32.17. Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \leq x \leq 1$ for the boundary conditions $y = 0$ at $x = 0$. (V.T.U., 2009, Anna, 2005, Rohtak, 2005)

Solution. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y) dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $y_1 = 0 + \int_0^x x dx = \frac{x^2}{2}$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $y_2 = \int_0^x \left(x - \frac{x^4}{4} \right) dx = \frac{x^2}{2} - \frac{x^5}{20}$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20} \right)^2 \right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \quad \dots(i)$$

Now let us determine the starting values of the Milne's method from (i), by choosing $h = 0.2$.

$$\begin{aligned} \therefore x_0 &= 0.0, & y_0 &= 0.0000, & f_0 &= 0.0000 \\ x_1 &= 0.2, & y_1 &= 0.020, & f_1 &= 0.1996 \\ x_2 &= 0.4, & y_2 &= 0.0795, & f_2 &= 0.3937 \\ x_3 &= 0.6, & y_3 &= 0.1762, & f_3 &= 0.5689 \end{aligned}$$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3)$

$$x = 0.8, \quad y_4^{(p)} = 0.3049, \quad f_4 = 0.7070$$

and the corrector, $y_4^{(c)} = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4)$, yields

$$y_4^{(c)} = 0.3046, \quad f_4 = 0.7072 \quad \dots(ii)$$

Again using the corrector, $y_4^{(c)} = 0.3046$, which is same as in (ii)

Now using the predictor, $y_5^{(p)} = y_1 + \frac{4h}{3} (2f_2 - f_3 + 2f_4)$,

$$x = 1.0, \quad y_5^{(p)} = 0.4554, \quad f_5 = 0.7926$$

and the corrector, $y_5^{(c)} = y_3 + \frac{h}{3} (f_3 + 4f_4 + f_5)$, gives

$$y_5^{(c)} = 0.4555, \quad f_5 = 0.7925$$

Again using the corrector,

$$y_5^{(c)} = 0.4555, \text{ a value which is the same as before.}$$

Hence, $y(1) = 0.4555$.

Example 32.18. Given $y' = x(x^2 + y^2) e^{-x}$, $y(0) = 1$, find y at $x = 0.1, 0.2$ and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method. (Anna, 2007)

Solution. Given

We have

$$y(0) = 1 \quad \text{and} \quad h = 0.1$$

$$y'(x) = x(x^2 + y^2)e^{-x};$$

$$y'(0) = 0$$

$$y''(x) = [(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)y']e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'];$$

$$y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy^2 - 2xyy']$$

$$y'''(0) = -2$$

Substitute these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \dots$$

$$\begin{aligned} y(0.1) &= 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \dots \\ &= 1 + 0.005 - 0.0003 = 1.0047 \quad i.e., \quad 1.005 \end{aligned}$$

Now taking

$$x = 0.1, y(0.1) = 1.005, h = 0.1$$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$\begin{aligned} y(0.2) &= y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \dots \\ &= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{3}(-1.247) + \dots \\ &= 1.018 \end{aligned}$$

Now taking

$$x = 0.2, y(0.2) = 1.018, h = 0.1$$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$\begin{aligned} y(0.3) &= y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \dots \\ &= 1.018 + 0.0176 + 0.0039 + 0.0001 = 1.04 \end{aligned}$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$$\begin{array}{lll} x_0 = 0.0 & y_0 = 1 & f_0 = y'_0 = 0 \\ x_1 = 0.1 & y_1 = 1.005 & f_1 = 0.092 \\ x_2 = 0.2 & y_2 = 1.018 & f_2 = 0.176 \\ x_3 = 0.3 & y_3 = 1.04 & f_3 = 0.26 \end{array}$$

$$\begin{aligned} \text{Using the predictor, } y_4^{(p)} &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \\ &= 1 + \frac{4(0.1)}{3}[2(0.092) - (0.176) + 2(0.26)] = 1.09 \end{aligned}$$

$$\therefore x = 0.4 \quad y_4^{(p)} = 1.09 \quad f_4 = y'(0.4) = 0.362$$

$$\text{Using the corrector, } y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$\therefore y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

$$\text{Hence } y(0.4) = 1.071.$$

Example 32.19. Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method.

(V.T.U., 2008; S.V.T.U., 2007; Madras, 2006)

Solution. We have $f(x, y) = xy + y^2$.

To find $y(0.1)$:

Here $x_0 = 0, y_0 = 1, h = 0.1$.

$$\begin{aligned}
 k_1 &= h f(x_0, y_0) = (0.1) f(0.1) &= 0.1000 \\
 k_2 &= h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 1.05) &= 0.1155 \\
 k_3 &= h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 1.0577) &= 0.1172 \\
 k_4 &= h f(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 1.1172) &= 0.13598 \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
 &= \frac{1}{6}(0.1 + 0.231 + 0.2348 + 0.13598) &= 0.11687
 \end{aligned}$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$.

To find $y(0.2)$:

Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$.

$$\begin{aligned}
 k_1 &= h f(x_1, y_1) = (0.1) f(0.1, 1.1169) &= 0.1359 \\
 k_2 &= h f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 1.1848) &= 0.1581 \\
 k_3 &= h f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 1.1959) &= 0.1609 \\
 k_4 &= h f(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 1.2778) &= 0.1888 \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) &= 0.1605
 \end{aligned}$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

To find $y(0.3)$:

Here $x_2 = 0.2, y_2 = 1.2773, h = 0.1$.

$$\begin{aligned}
 k_1 &= h f(x_2, y_2) = (0.1) f(0.2, 1.2773) &= 0.1887 \\
 k_2 &= h f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 1.3716) &= 0.2224 \\
 k_3 &= h f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 1.3885) &= 0.2275 \\
 k_4 &= h f(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 1.5048) &= 0.2716 \\
 k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) &= 0.2267
 \end{aligned}$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$.

Now the starting values of the Milne's method are :

$$\begin{array}{lll}
 x_0 = 0.0 & y_0 = 1.0000 & f_0 = 1.0000 \\
 x_1 = 0.1 & y_1 = 1.1169 & f_1 = 1.3591 \\
 x_2 = 0.2 & y_2 = 1.2773 & f_2 = 1.8869 \\
 x_3 = 0.3 & y_3 = 1.5049 & f_3 = 2.7132
 \end{array}$$

Using the predictor,

$$\begin{aligned}
 y_4^{(p)} &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \\
 x_4 &= 0.4 & y_4^{(p)} &= 1.8344 & f_4 &= 4.0988
 \end{aligned}$$

and the corrector,

$$\begin{aligned}
 y_4^{(c)} &= y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields} \\
 y_4^{(c)} &= 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098] \\
 &= 1.8386 & f_4 &= 4.1159
 \end{aligned}$$

Again using the corrector,

$$\begin{aligned} y_4^{(c)} &= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7132) + 4.1159] \\ &= 1.8391 \quad f_4 = 4.1182 \end{aligned} \quad \dots(i)$$

Again using the corrector

$$\begin{aligned} y_4^{(c)} &= 1.2773 + \frac{0.1}{3} [1.8869 + 4(2.7132) + 4.1182] \\ &= 1.8392 \text{ which is same as (i).} \end{aligned}$$

Hence $y(0.4) = 1.8392$.

PROBLEMS 32.4

1. Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. The value of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. Method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. (Anna, 2004)

2. Given $2\frac{dy}{dx} = (1+x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor-corrector method. (V.T.U., 2011 S ; Madras, 2003)

3. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula :

$\frac{dy}{dx} = x^2 + \frac{y}{2}$				
$x :$	1	1.1	1.2	1.3
$y :$	2	2.2156	2.4549	2.7514

(V.T.U., 2007)

4. Using Milne's method, find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$. (Anna, 2007)

5. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$ and $y(0.3) = 2.09$; find $y(0.4)$ using Milne's predictor-corrector method. (V.T.U., 2010)

6. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

32.10 ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series of Euler's method or Runge-Kutta method.

Next we calculate $f_{-1} = f(x_0 - h, y_{-1})$, $f_{-2} = f(x_0 - 2h, y_{-2})$, $f_{-3} = f(x_0 - 3h, y_{-3})$.

Then to find y_1 , we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n \nabla f_0 + \frac{n(n+1)}{2} \nabla^2 f_0 + \frac{n(n+1)(n+2)}{6} \nabla^3 f_0 + \dots$$

in

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y) dx \quad \dots(1)$$

$$\begin{aligned} \therefore y_1 &= y_0 + \int_{x_0}^{x_1} \left(f_0 + n \nabla f_0 + \frac{n(n+1)}{2} \nabla^2 f_0 + \dots \right) dx \quad [\text{Put } x = x_0 + nh, dx = hd] \\ &= y_0 + h \int_0^1 \left(f_0 + n \nabla f_0 + \frac{n(n+1)}{2} \nabla^2 f_0 + \dots \right) dn \\ &= y_0 + h \left(f_0 + \frac{1}{2} \nabla f_0 + \frac{5}{12} \nabla^2 f_0 + \frac{3}{8} \nabla^3 f_0 + \dots \right) \end{aligned}$$

Neglecting fourth and higher order differences and expressing ∇f_0 , $\nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \quad \dots(2)$$

This is called *Adams-Basforth predictor formula*.

Having found y_1 , we find $f_1 = f(x_0 + h, y_1)$.

Then to find a better value of y_1 , we derive a *corrector formula* by substituting Newton's backward formula at f_1 i.e.,

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2} \nabla^2 f_1 + \frac{n(n+1)(n+2)}{6} \nabla^3 f_1 + \dots \text{ in (1).}$$

$$\begin{aligned} \therefore y_1 &= y_0 + \int_{x_0}^{x_1} \left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2} \nabla^2 f_1 + \dots \right) dx \quad [\text{Put } x = x_1 + nh, dx = hdn] \\ &= y_0 + \int_1^0 \left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2} \nabla^2 f_1 + \dots \right) dn \\ &= y_0 + h \left(f_1 - \frac{1}{2} \nabla f_1 - \frac{1}{12} \nabla^2 f_1 - \frac{1}{24} \nabla^3 f_1 - \dots \right) \end{aligned}$$

Neglecting fourth and higher order differences and expressing ∇f_1 , $\nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24} (9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \quad \dots(3)$$

which is called a *Adams-Moulton corrector formula*.

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value of y_1 . This step is repeated till y_1 remains unchanged and then proceed to calculate y_2 as above.

Obs. To apply both Milne and Adams-Basforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

Example 32.20. Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams-Basforth method. (V.T.U., 2010 ; J.N.T.U., 2009 ; Anna, 2004)

Solution. Here $f(x, y) = x^2(1+y)$.

Starting values of the Adams-Basforth method with $h = 0.1$, are

$$\begin{aligned} x &= 1.0, y_{-3} = 1.000, f_{-3} = (1.0)^2(1+1.000) = 2.000 \\ &\cdots = 1.1, y_{-2} = 1.233, f_{-2} = 2.702 \\ &\cdots = 1.2, y_{-1} = 1.548, f_{-1} = 3.669 \\ &\cdots = 1.3, y_0 = 1.979, f_0 = 5.035 \end{aligned}$$

Using th

$$\frac{h}{24} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$73, f_1 = 7.004$$

U:

$$5f_{-1} + f_{-2})$$

$$19 \times 5.035 - 5 \times 3.669 + 2.702 = 2.575$$

Hence,

5.

Example 32.21. If $\frac{dy}{dx} = 2e^x y$, $y(0) = 0$, find $y(4)$ using Adams predictor-corrector formula by calculating $y(1)$, $y(2)$ and $y(3)$ using Euler's modified formula. (J.N.T.U., 2006)

Solution. We have $f(x, y) = 2e^x y$.

To find 0.1 :

x	$2e^x y = y'$	Mean slope	$Old\ y + h\ (Mean\ slope) = new\ y$
0.0	4	—	$2 + 0.1(4) = 2.4$
0.1	$2e^{0.1}(2.4) = 5.305$	$\frac{1}{2}(4 + 5.305) = 4.6524$	$2 + 0.1(4.6524) = 2.465$
0.1	$2e^{0.1}(2.465) = 5.449$	$\frac{1}{2}(4 + 5.449) = 4.7244$	$2 + 0.1(4.7244) = 2.472$
0.1	$2e^{0.1}(2.4724) = 5.465$	$\frac{1}{2}(4 + 5.465) = 4.7324$	$2 + 0.1(4.7324) = 2.473$
0.1	$2e^{0.1}(2.473) = 5.467$	$\frac{1}{2}(4 + 5.467) = 4.7333$	$2 + 0.1(4.7333) = 2.473$
0.1	5.467	—	$2 + 0.1(5.467) = 3.0199$
0.2	$2e^{0.2}(3.0199) = 7.377$	$\frac{1}{2}(5.467 + 7.377) = 6.422$	$2.473 + 0.1(6.422) = 3.1155$
0.2	7.611	$\frac{1}{2}(5.467 + 7.611) = 6.539$	$2.473 + 0.1(6.539) = 3.127$
0.2	7.639	$\frac{1}{2}(5.467 + 7.639) = 6.553$	$2.473 + 0.1(6.553) = 3.129$
0.2	7.643	$\frac{1}{2}(5.467 + 7.643) = 6.555$	$2.473 + 0.1(6.455) = 3.129$
0.2	7.643	—	$3.129 + 0.1(7.643) = 3.893$
0.3	$2e^{0.3}(3.893) = 10.51$	$\frac{1}{2}(7.643 + 10.51) = 9.076$	$3.129 + 0.1(9.076) = 4.036$
0.3	10.897	$\frac{1}{2}(7.643 + 10.897) = 9.266$	$3.129 + 0.1(9.2696) = 4.056$
0.3	10.949	$\frac{1}{2}(7.643 + 10.949) = 9.296$	$3.129 + 0.1(9.296) = 4.058$
0.3	10.956	$\frac{1}{2}(7.643 + 10.956) = 9.299$	$3.129 + 0.1(9.299) = 4.0586$

To find $y(0.4)$ by Adam's method, the starting values with $h = 0.1$ are

$$\begin{array}{lll} x = 0.0 & y_{-3} = 2.4 & f_{-3} = 4 \\ x = 0.1 & y_{-2} = 2.473 & f_{-2} = 5.467 \\ x = 0.2 & y_{-1} = 3.129 & f_{-1} = 7.643 \\ x = 0.3 & y_0 = 4.059 & f_0 = 10.956 \end{array}$$

Using the predictor formula

$$\begin{aligned} y_1^{(p)} &= y_0 + \frac{h}{24} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \\ &= 4.059 + \frac{0.1}{24} (55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4) \\ &= 5.383 \end{aligned}$$

$$\text{Now } x = 0.4 \quad y_1 = 5.383 \quad f_1 = 2e^{0.4} (5.383) = 16.061$$

Using the corrector formula,

$$\begin{aligned}y_1^{(c)} &= y_0 + \frac{h}{24} (9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \\&= 4.0586 + \frac{0.1}{24} (9 \times 6.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392\end{aligned}$$

Hence $y(0.4) = 5.392$.

Example 32.22. Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$. (P.T.U., 2003)

Solution. We have $f(x, y) = x - y^2$.

To find $y(0.1)$:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$\begin{aligned}\therefore k_1 &= hf(x_0, y_0) = (0.1)f(0, 1) &= -0.1000 \\k_2 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)f(0.05, 0.95) &= -0.08525 \\k_3 &= hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)f(0.05, 0.9574) &= -0.0867 \\k_4 &= hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 0.9137) &= -0.07341 \\k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) &= -0.0883\end{aligned}$$

$$\text{Thus } y(0.1) = y_1 = y_0 + k = 1 - 0.0883 = 0.9117$$

To find $y(0.2)$:

Here $x_1 = 0.1$, $y_1 = 0.9117$, $h = 0.1$.

$$\begin{aligned}\therefore k_1 &= hf(x_1, y_1) = (0.1)f(0.1, 0.9117) &= -0.0731 \\k_2 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)f(0.15, 0.8751) &= -0.0616 \\k_3 &= hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)f(0.15, 0.8809) &= -0.0626 \\k_4 &= hf(x_1 + h, y_1 + k_3) = (0.1)f(0.2, 0.8491) &= -0.0521 \\k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) &= -0.0623\end{aligned}$$

$$\text{Thus } y(0.2) = y_2 = y_1 + k = 0.8494.$$

To find $y(0.3)$:

Here $x_2 = 0.2$, $y_2 = 0.8494$, $y = 0.1$

$$\begin{aligned}k_1 &= hf(x_2, y_2) = (0.1)f(0.2, 0.8494) &= -0.0521 \\k_2 &= hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)f(0.25, 0.8233) &= -0.0428 \\k_3 &= hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)f(0.25, 0.828) &= -0.0436 \\k_4 &= hf(x_2 + h, y_2 + k_3) = (0.1)f(0.3, 0.8058) &= -0.0349 \\k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) &= -0.0438\end{aligned}$$

$$\text{Thus } y(0.3) = y_3 = y_2 + k = 0.8061$$

Now the starting values of Adam's method with $h = 0.1$ are :

$$\begin{array}{llll}x = 0.0 & y_{-3} = 1.0000 & f_{-3} = 0.0 - (1.0)^2 & = -1.0000 \\x = 0.1 & y_{-2} = 0.9117 & f_{-2} = 0.1 - (0.9117)^2 & = -1.7312 \\x = 0.2 & y_{-1} = 0.8494 & f_{-1} = 0.2 - (0.8494)^2 & = -0.5215 \\x = 0.3 & y_0 = 0.8061 & f_0 = 0.3 - (0.8061)^2 & = -0.3498\end{array}$$

Using the predictor,

$$\begin{aligned} y_1^{(p)} &= y_0 + \frac{h}{24} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \\ x = 0.4 &\quad y_1^{(p)} = 0.8061 + \frac{0.1}{24} [55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)] \\ &= 0.7789 \qquad \qquad \qquad f_1 = -0.2067 \end{aligned}$$

Using the corrector,

$$\begin{aligned} y_1^{(c)} &= y_0 + \frac{h}{24} (9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \\ &= 0.8061 + \frac{0.1}{24} [9(-0.2067) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785 \end{aligned}$$

Hence $y(0.4) = 0.7785$.

PROBLEMS 32.5

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

$x :$	0	0.2	0.4	0.6
$y :$	0	0.0200	0.0795	0.1762

(Bhopal, 2002)

2. Using Adams-Bashforth formulae, determine $y(0.4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data

$x :$	0	0.1	0.2	0.3
$y :$	1	1.0025	1.0101	1.0228

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0.4)$ using Adams-Bashforth method. (S.V.T.U., 2007)

4. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4, 1) = 1.0049$, $y(4, 2) = 1.0097$ and $y(4, 3) = 1.0143$.

5. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

$x :$	1	1.1	1.2	1.3
$y :$	1	1.233	1.548488	1.978921

(Indore, 2003 S)

6. Using Adams-Bashforth method, evaluate $y(1.4)$, if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. (Madras, 2003)

32.11 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \quad \dots(1)$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \quad \dots(2)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially by Picard's or Runge-Kutta methods.

(i) *Picard's method* gives

$$\begin{aligned} y_1 &= y_0 + \int f(x, y_0, z_0) dx, z_1 = z_0 + \int \phi(x, y_0, z_0) dx \\ y_2 &= y_0 + \int f(x, y_1, z_1) dx, z_2 = z_0 + \int \phi(x, y_1, z_1) dx \\ y_3 &= y_0 + \int f(x, y_2, z_2) dx, z_3 = z_0 + \int \phi(x, y_2, z_2) dx \end{aligned}$$

and so on.

(ii) *Taylor's series* method is used as follows :

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad \dots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \dots \quad \dots(4)$$

Differentiating (1) and (2) successively, we get y'', z'' , etc. So the values $y_0', y_0'', y_0''' \dots$ and $z_0', z_0'', z_0''' \dots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots \quad \dots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \dots \quad \dots(6)$$

Since y_1 and z_1 are known, we can calculate y_1', y_1'', \dots and z_1', z_1'', \dots . Substituting these in (5) and (6), we get y_2 and z_2 .

Proceeding further, we can calculate the other values of y and z step by step.

(iii) *Runge-Kutta method* is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\text{Hence } y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{and} \quad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

To compute y_2 and z_2 , we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 32.23. Using Picard's method find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $dy/dx = x + z$, $dz/dx = x - y^2$.

Solution. Here $x_0 = 0, y_0 = 2, z_0 = 1$,

$$\frac{dy}{dx} = f(x, y, z) = x + z; \quad \text{and} \quad \frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore y = y_0 + \int_{x_0}^x f(x, y, z) dx \quad \text{and} \quad z = z_0 + \int_{x_0}^x \phi(x, y, z) dx.$$

$$\text{First approximations} \quad y_1 = y_0 + \int_{x_0}^x \phi(x, y_0, z_0) dx = 2 + \int_0^x (x+1) dx = 2 + x + \frac{1}{2}x^2$$

$$z_1 = z_0 + \int_{x_0}^x \phi(x, y_0, z_0) dx = 1 + \int_0^x (x-4) dx = 1 - 4x + \frac{1}{2}x^2$$

$$\begin{aligned} \text{Second approximations} \quad y_2 &= y_0 + \int_{x_0}^x f(x, y_1, z_1) dx = 2 + \int_0^x \left(x+1 - 4x + \frac{1}{2}x^2\right) dx \\ &= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6} \end{aligned}$$

$$\begin{aligned} z_2 &= z_0 + \int_{x_0}^x \phi(x, y_1, z_1) dx \\ &= 1 + \int_{x_0}^x \left[x - \left(2 + x - \frac{1}{2}x^2\right)^2\right] dx = 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}. \end{aligned}$$

$$\begin{aligned} \text{Third approximations } y_3 &= y_0 + \int_{x_0}^x f(x, y_2, z_2) dx \\ &= 2 + x - \frac{3}{2} x^2 - \frac{1}{2} x^3 - \frac{1}{4} x^4 - \frac{1}{20} x^5 - \frac{1}{120} x^6 \\ z_3 &= z_0 + \int_{x_0}^x \phi(x, y_2, z_2) dx \\ &= 1 - 4x - \frac{3}{2} x^2 + \frac{5}{3} x^3 + \frac{7}{12} x^4 - \frac{31}{60} x^5 + \frac{1}{12} x^6 - \frac{1}{252} x^7 \end{aligned}$$

and so on.

When

$$\begin{aligned} x &= 0.1, \\ y_1 &= 2.105, y_2 = 2.08517, y_3 = 2.08447 \\ z_1 &= 0.605, z_2 = 0.58397, z_3 = 0.58672. \end{aligned}$$

$$\text{Hence } y(0.1) = 2.0845, z(0.1) = 0.5867$$

correct to four decimal places.

Example 32.24. Solve the differential equations

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy \text{ for } x = 0.3,$$

using fourth order Runge-Kutta method. Initial values are $x = 0, y = 0, z = 1$.

Solution. Here $f(x, y, z) = 1 + xz, \phi(x, y, z) = -xy$

$$x_0 = 0, y_0 = 0, z_0 = 1. \text{ Let us take } h = 0.3.$$

$$\therefore k_1 = h f(x_0, y_0, z_0) = 0.3 f(0, 0, 1) = 0.3 (1 + 0) = 0.3$$

$$l_1 = h \phi(x_0, y_0, z_0) = 0.3 (-0 \times 0) = 0$$

$$\begin{aligned} k_2 &= h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\ &= (0.3) f(0.15, 0.15, 1) = 0.3 (1 + 0.15) = 0.345 \end{aligned}$$

$$\begin{aligned} l_2 &= h \phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \\ &= 0.3 [-(0.15)(0.15)] = -0.00675. \end{aligned}$$

$$\begin{aligned} k_3 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= (0.3) f(0.15, 0.1725, 0.996625) \\ &= 0.3 [1 + 0.996625 \times 0.15] = 0.34485 \end{aligned}$$

$$\begin{aligned} l_3 &= h \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.3 [-(0.15)(0.1725)] = -0.007762 \end{aligned}$$

$$\begin{aligned} k_4 &= h f(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= (0.3) f(0.3, 0.34485, 0.99224) = 0.3893 \end{aligned}$$

$$\begin{aligned} l_4 &= h \phi(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= 0.3 [-(0.3)(0.34485)] = -0.03104 \end{aligned}$$

$$\text{Hence } y(x_0 + h) = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$\text{i.e., } y(0.3) = 0 + \frac{1}{6} [0.3 + 2(0.345) + 2(0.34485) + 0.3893] = 0.34483$$

$$\text{and } z(x_0 + h) = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$\text{i.e., } z(0.3) = 1 + \frac{1}{6} [0 + 2 + (-0.00675) + 2(-0.0077625) + (-0.03104)] = 0.98999$$

32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 32.25. Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0, y = 1, y' = 0$. (Delhi, 2002)

Solution. Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using k_1, k_2, \dots for $f(x, y, z)$ and l_1, l_2, \dots for $\phi(x, y, z)$, Runge-Kutta formulae become

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) \\ &= 0.2(0) = 0 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1) \\ &= 0.2(-0.1) = -0.02 \end{aligned}$$

$$\begin{aligned} k_3 &= hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2) \\ &= 0.2(-0.0999) = -0.02 \end{aligned}$$

$$\begin{aligned} k_4 &= hf(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= 0.2(-0.1958) = -0.0392 \end{aligned}$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$

$$\begin{aligned} l_1 &= h\phi(x_0, y_0, z_0) \\ &= 0.2(-1) = -0.2 \end{aligned}$$

$$\begin{aligned} l_2 &= h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1) \\ &= 0.2(-0.999) = -0.1998 \end{aligned}$$

$$\begin{aligned} l_3 &= h\phi(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2) \\ &= 0.2(-0.9791) = -0.1958 \end{aligned}$$

$$\begin{aligned} l_4 &= h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) \\ &= 0.2(0.9527) = -0.1905 \end{aligned}$$

$$\begin{aligned} l &= \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\ &= -0.1970 \end{aligned}$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $y' = z = z_0 + l = 0 - 0.1970 = -0.1970$.

Example 32.26. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$, obtain y for $x = 0(0.1)0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$. (Anna, 2004; Madras, 2003 S)

Solution. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, y' = z \quad \dots(i)$$

We employ Taylor's series method to find y .

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

$$\text{At } x = 0, (y_{n+2})_0 = -(n+1)(y_n)_0$$

$$\therefore y(0) = 1, \text{ gives } y_2(0) = -1, y_4(0) = 32, y_6(0) = -5 \times 3, \dots$$

$$\text{and } y_1(0) = 0 \text{ yields } y_3(0) = y_5(0) = \dots = 0.$$

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots$$

$$\therefore y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!} x^4 - \frac{5 \times 3}{6!} x^6 + \dots \quad \dots(ii)$$

$$\text{and } z(x) = y'(x) = -x + \frac{1}{2} x^3 - \frac{1}{8} x^5 = \dots = -xy \quad \dots(iii)$$

From (ii), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8} (0.1)^4 - \dots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \dots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \dots = 0.956$$

From (iii), we have

$$z(0.1) = -0.0995, z(0.2) = -0.196, z(0.3) = -0.2863.$$

Also from (i), $z'(x) = -(xz + y)$ $\therefore z'(0.1) = 0.985, z'(0.2) = -0.941, z'(0.3) = -0.87$.

Applying Milne's predictor formula, first to z and then to y , we obtain

$$\begin{aligned} z(0.4) &= z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\} \\ &= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692 \end{aligned}$$

and

$$\begin{aligned} y(0.4) &= y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\} \\ &= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \quad [\because y' = z] \end{aligned}$$

Also $z'(0.4) = -[x(0.4)z(0.4) + y(0.4)] = \{0.4(-0.3692) + 0.9231\} = -0.7754$.

Now applying Milne's corrector formula, we get

$$\begin{aligned} z(0.4) &= z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\} \\ &= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692 \end{aligned}$$

and

$$\begin{aligned} y(0.4) &= y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\} \\ &= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232 \end{aligned}$$

Hence $y(0.4) = 0.9232$ and $z(0.4) = -0.3692$.

PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of y and z , given that $dy/dx = z, dz/dx = x^2(y + z)$, given $y = 1, z = \frac{1}{2}$ when $x = 0$.

2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and 0.2 , $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy ; y(0) = 0$ and $z(0) = 1$.

3. Find $y(0.1), z(0.1), y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z, z' = x - y^2$ given $y(0) = 0, z(0) = 1$ using Runge-Kutta of 4th order. (J.N.T.U., 2009)

4. Using Picard's method, obtain the second approximation to the solution of

$$\frac{d^2y}{dx^2} = x^3 \frac{dy}{dx} + x^3 y \text{ so that } y(0) = 1, y'(0) = \frac{1}{2}.$$

5. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + y = 0$ and $y = 0.5, \frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy', y(0) = 1, y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2x} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (Anna, 2003)

8. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec². If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R . Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(i) $\frac{d^2y}{dx^2} + \lambda(x) \frac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$, $y(x_n) = b$.

(ii) $\frac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \dots \quad \dots(1)$$

and $y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \dots \quad \dots(2)$

Equation (1) gives $y'(x) = \frac{1}{h} [y(x+h) - y(x)] - \frac{h}{2}y''(x) - \dots$

i.e., $y'(x) = \frac{1}{h} [y(x+h) - y(x)] + O(h)$

which is the *forward difference approximation of $y'(x)$* with an error of the order h .

Similarly (2) gives $y'(x) = \frac{1}{h} [y(x) - y(x-h)] + O(h)$

which is the *backward difference approximation of $y'(x)$* with an error of the order h .

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h} [y(x+h) - y(x-h)] + O(h^2)$$

which is the *central-difference approximation of $y'(x)$* with an error of the order h^2 . Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2} [y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the *central difference approximation of $y''(x)$* . Similarly we can derive central difference approximations to higher derivatives.

Hence the working expressions for the central difference approximations to the first four derivatives of y_i are as under :

$$y'_i = \frac{1}{2h} (y_{i+1} - y_{i-1}) \quad \dots(3)$$

$$y''_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1}) \quad \dots(4)$$

$$y'''_i = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad \dots(5)$$

$$y^{iv}_i = \frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) \quad \dots(6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h , the accuracy improves but the number of equations to be solved also increases.

Example 32.27. Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$. (Calicut, 1999)

Solution. We divide the interval $(0, 1)$ into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2} [y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or

$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i, \quad i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$\begin{aligned} 16y_2 - 33y_1 &= \frac{1}{4} \\ 16y_3 - 33y_2 + 16y_1 &= \frac{1}{2} \\ -33y_3 + 16y_2 &= \frac{3}{4} \end{aligned}$$

Their solution gives

$$y_1 = -0.03488, y_2 = -0.05632, y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \frac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

x	Computed value $y(x)$	Exact value $y(x)$	Error
0.25	-0.03488	-0.03505	0.00017
0.5	-0.05632	-0.05659	0.00027
0.75	-0.05003	-0.05028	0.00025

Example 32.28. Determine values of y at the pivotal points of the interval $(0, 1)$, if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y -values are $y_0 (= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or

$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2, \quad i = 1, 2$$

$$\text{At } i = 1, \quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$$

$$\text{At } i = 2, \quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$$

$$\text{Using } y_0 = y_3 = 0, \text{ we get } -4y_2 + 7y_1 - 4y_0 = 1/9 \quad \dots(i)$$

$$y_4 + 7y_1 - 4y_1 = 4/9 \quad \dots(ii)$$

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$,

$$y''_0 = 9(y_1 - 2y_0 + y_{-1})$$

or

$$[\because y_0 = y''_0 = 0] \quad \dots(iii)$$

At $i = 3$,

$$y''_3 = 9(y_4 - 2y_3 + y_2)$$

$$y_4 = -y_2$$

$$[\because y_3 = y''_3 = 0] \quad \dots(iv)$$

Using (iii), the equation (i) becomes

$$-4y_2 + 6y_1 = 1/9 \quad \dots(v)$$

Using (iv), the equation (ii) reduces to

$$6y_2 - 4y_1 = 4/9 \quad \dots(vi)$$

Solving (v) and (vi), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 32.29. The deflection of a beam is governed by the equation

$$\frac{d^4 y}{dx^4} + 81y = \phi(x)$$

where $\phi(x)$ is given by the table

:	1/3	2/3	1,
$\phi(x)$:	81	162	243,

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y -values are $y_0 (= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4} (y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

$$\text{At } i = 1, \quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \quad \dots(i)$$

$$\text{At } i = 2, \quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \quad \dots(ii)$$

$$\text{At } i = 3, \quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \quad \dots(iii)$$

$$\text{We have } y_0 = 0 \quad \dots(iv)$$

$$\text{Since } y'_i = \frac{1}{2h} (y_{i+1} - y_{i-1})$$

$$\therefore \text{for } i = 0, \quad 0 = y'_0 = \frac{1}{2h} (y_1 - y_{-1}) \text{ i.e. } y_{-1} = y_1 \quad \dots(v)$$

$$\text{Since } y''_i = \frac{1}{h^2} (y_{i+1} - 2y_i + y_{i-1})$$

$$\therefore \text{for } i = 3, \quad 0 = y''_3 = \frac{1}{h^2} (y_4 - 2y_3 + y_2), \text{ i.e. } y_4 = 2y_3 - y_2 \quad \dots(vi)$$

$$\text{Also } y'''_i = \frac{1}{2h^3} (y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})$$

$$\therefore \text{for } i = 3, \quad 0 = y'''_3 = \frac{1}{2h^3} (y_5 - 2y_4 + 2y_2 - y_1)$$

$$y_5 = 2y_4 - 2y_2 + y_1 \quad \dots(vii)$$

Using (iv) and (v), the equation (i) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \quad \dots(viii)$$

Using (iv) and (vi), the equation (ii) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \quad \dots(ix)$$

Using (vi) and (vii), the equation (iii) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \quad \dots(x)$$

Solving (viii), (ix) and (x), we get

$$y_1 = 8/13, y_2 = 22/13, y_3 = 37/13.$$

Hence $y(1/3) = 0.6154, y(2/3) = 1.6923, y(1) = 2.8462$.

PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$:

$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \quad (\text{Take } n = 4)$$

2. Find an approximate solution of the boundary value problem:

$$y'' + 8(\sin^2 \pi y)y = 0, 0 \leq x \leq 1, y(0) = y(1) = 1. \quad (\text{Take } n = 4)$$

3. Solve the boundary value problem

$$xy'' + y = 0, y(1) = 1, y(2) = 2. \quad (\text{Take } n = 4)$$

4. Solve the equation

$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0, y(1) = -2, \text{ taking } n = 4.$$

5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.

6. Solve the boundary value problem

$$y'' + xy' + y = 3x^2 + 2, y(0) = 0, y(1) = 1.$$

7. The boundary value problem governing the deflection of a beam of length 3 metres is given by

$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$

The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.

8. Solve the boundary value problem,

$$\frac{d^4y}{dx^4} + 81y = 729x^2, y(0) = y'(0) = y''(1) = y'''(1) = 0. \quad (\text{Use } n = 3)$$

9. Solve the equation $y''' - y'' + y = x^2$ subject to the boundary conditions

$$y(0) = y'(0) = 0 \text{ and } y(1) = 2, y'(1) = 0. \quad (\text{Take } n = 5)$$

32.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 32.8

Select the correct answer or fill up the blanks in the following questions:

- Which of the following is a step by step method:
 - Taylor's
 - Adams-Bashforth
 - Picard's
 - None.
- The finite difference scheme for the equation $2y'' + y = 5$ is
- If $y'' = x + y, y(0) = 1$ and $y'(1) = 1 + x + x^2/2$, then by Picard's method, the value of $y''(x)$ is
- The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is
- Taylor's series for solution of first order ordinary differential equations is
- Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ is
 - 0.813
 - 0.825
 - 0.0825
 - none.
- Given y_0, y_1, y_2, y_3 , Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is
- The second order Runge-Kutta formula is
- Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is
- The multi-step methods available for solving ordinary differential equations are
- To predict Adam's method atleast values of y , prior to the desired value, are required.
- Taylor's series solution of $y' = -xy, y(0) = 1$ upto x^4 is

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y$, $y(0) = 1$ is
 (a) 0.809 (b) 0.909 (c) 0.0809 (d) none.
14. Milne's Predictor formula is
15. Adam's corrector formula is
16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$; gives $y(0.1) = \dots$.
17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations and
18. The formula for the 4th order Runge-Kutta method is
19. Taylor's series method will be useful to give some of Milne's method.
20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because
22. If $y' = x$, $y(0) = 1$ then by Picard's method, the value of $y(1)$ is
 (a) 0.915 (b) 0.905 (c) 0.981 (d) none.
23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is
24. If $y' = -y$, $y(0) = 1$, the Euler's method, the value of $y(1)$ is
 (a) 0.99 (b) 0.999 (c) 0.981 (d) none.
25. In Euler's method if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)
26. Runge-Kutta method is a self-starting method. (True or False)
27. Predictor-corrector methods are self-starting methods. (True or False)

Numerical Solution of Partial Differential Equations

1. Introduction. 2. Classification of second order equations. 3. Finite difference approximation to derivatives. 4. Elliptic equations. 5. Solution of Laplace's equation. 6. Solution of Poisson's equations. 7. Parabolic equations. 8. Solution of heat equation. 9. Hyperbolic equations. 10. Solution of wave equation. 11. Objective Type of Questions.

33.1 INTRODUCTION

There are many boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations. Of the various numerical methods available for solving these equations, the method of finite differences is most commonly used.

In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite-difference approximations. Then the given equation is changed to a difference equation which is solved by iterative procedures. This process is slow but gives good results of boundary value problems. An added advantage of this method is that the computation can be done by electronic computer. Here we shall apply this method to the solution of important applied partial differential equations. For a detailed study, the reader should refer to author's book '*Numerical Methods in Engineering and Science*'.

33.2 CLASSIFICATION OF SECOND ORDER EQUATIONS

The general linear partial differential equation of the second order in two independent variables is of the form.

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Such a partial differential equation is said to be

- (i) **elliptic**, if $B^2 - 4AC < 0$,
- (ii) **parabolic**, if $B^2 - 4AC = 0$,
- and (iii) **hyperbolic**, if $B^2 - 4AC > 0$.

Example 33.1. Classify the following equations :

$$(i) \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0 \quad (ii) x^2 \frac{\partial^2 u}{\partial x^2} + (1 - y^2) \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1$$

(Madras, 2003)

$$(iii) (1+x^2) \frac{\partial^2 u}{\partial x^2} + (5+2x^2) \frac{\partial^2 u}{\partial x \partial t} + (4+x^2) \frac{\partial^2 u}{\partial t^2} = 0.$$

Solution. (i) Comparing this equation with (1) above, we find that

$$A = 1, B = 4, C = 4$$

$$\therefore B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$$

So the equation is parabolic.

(ii) Here $A = x^2, B = 0, C = 1 - y^2$

$$\therefore B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$$

For all x between $-\infty$ and ∞ , x^2 is positive

For all y between -1 and 1 , $y^2 < 1 \quad \therefore B^2 - 4AC < 0$

Hence the equation is elliptic.

(iii) Here $A = 1 + x^2, B = 5 + 2x^2, C = 4 + x^2$

$$\therefore B^2 - 4AC = (5 + 2x^2)^2 - 4(1 + x^2)(4 + x^2) = 9 \text{ i.e. } > 0$$

So the equation is hyperbolic.

PROBLEMS 33.1

- What is the classification of the equation $f_{xx} + 2f_{xy} + f_{yy} = 0$?
- Determine whether the following equation is elliptic or hyperbolic?

$$(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0.$$

- Classify the equations (i) $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$

(Madras, 2000 S)

$$(ii) x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y}$$

(P.T.U., 2009 S)

$$(iii) \frac{3\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 6 \frac{\partial^2 u}{\partial y^2} - 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0.$$

(Anna, 2008)

- In which parts of the (x, y) plane is the following equation elliptic?

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial x \partial y + (x^2 + 4y^2) \partial^2 u / \partial y^2 = 2 \sin(xy).$$

33.3 FINITE-DIFFERENCE APPROXIMATIONS TO DERIVATIVES

Consider a rectangular region R in the x - y plane. Divide this region into a rectangular network of sides $\Delta x = h$ and $\Delta y = k$ as shown in Fig. 33.1. The points of intersection of the dividing lines are called *mesh points*, *nodal points* or *grid points*.

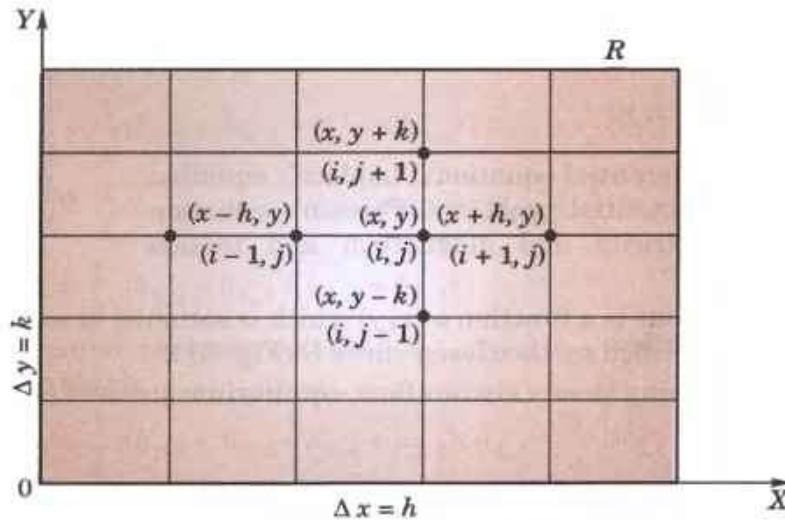


Fig. 33.1

Then we have the finite difference approximations for the partial derivatives in x -direction (§ 32.12) :

$$\frac{\partial u}{\partial x} = \frac{u(x+h, y) - u(x, y)}{h} + O(h)$$

$$= \frac{u(x, y) - u(x-h, y)}{h} + O(h) = \frac{u(x+h, y) - u(x-h, y)}{2h} + O(h^2)$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + O(h^2)$$

Writing $u(x, y) = u(ih, jk)$ as simply $u_{i,j}$, the above approximations become

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad \dots(1)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad \dots(2)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \quad \dots(3)$$

and

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \quad \dots(4)$$

Similarly we have the approximations for the derivatives w.r.t. y :

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad \dots(5)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad \dots(6)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad \dots(7)$$

and

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad \dots(8)$$

Replacing the derivatives in any partial differential equation by their corresponding difference approximations (1) to (8), we obtain the finite-difference analogues of the given equations.

33.4 ELLIPTIC EQUATIONS

The Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

are examples of elliptic partial differential equations. Laplace's equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

The solution of these equations is a function $u(x, y)$ which is satisfied at every point of a region R subject to certain boundary conditions specified on the closed curve C (Fig. 33.2).

In general, problems concerning steady viscous flow, equilibrium stresses in elastic structures etc., lead to elliptic type of equations.

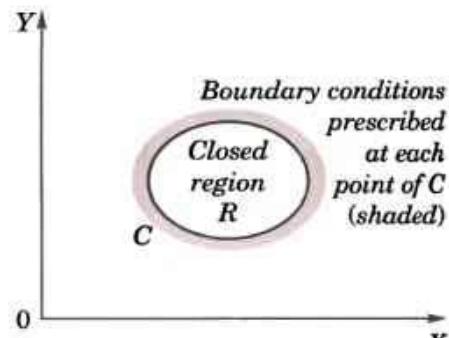


Fig. 33.2

33.5 SOLUTION OF LAPLACE'S EQUATION*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

* See p. 619

Consider a rectangular region R for which $u(x, y)$ is known at the boundary. Divide this region into a network of square mesh of side h , as shown in Fig. 33.3 (assuming that an exact sub-division of R is possible). Replacing the derivatives in (1) by their difference approximations, we have

$$\begin{aligned} \frac{1}{h^2} [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] \\ + \frac{1}{h^2} [u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] = 0 \\ \text{or } u_{i,j} = \frac{1}{4} [u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}] \quad \dots(2) \end{aligned}$$

This shows that the value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighbouring points to the left, right, above and below. (2) is called the **standard 5-point formula** which is exhibited in Fig. 33.4.

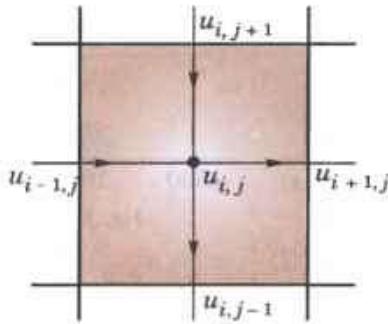


Fig. 33.4

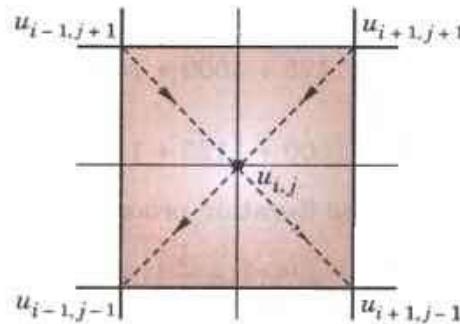


Fig. 33.5

Sometimes a formula similar to (2) is used which is given by

$$u_{i,j} = \frac{1}{4} (u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \quad \dots(3)$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neighbouring diagonal mesh points. (3) is called the **diagonal 5-point formula** which is represented in Fig. 33.5. Although (3) is less accurate than (2), yet it serves as a reasonably good approximation for obtaining the starting values at the mesh points.

Now to find the initial values of u at the interior mesh points, we first use diagonal five point formula (3) and compute $u_{3,3}, u_{2,4}, u_{4,4}, u_{4,2}$ and $u_{2,2}$, in this order. Thus we get,

$$\begin{aligned} u_{3,3} &= \frac{1}{4} (b_{1,5} + b_{5,1} + b_{5,5} + b_{1,1}); u_{2,4} = \frac{1}{4} (b_{1,5} + u_{3,3} + b_{3,5} + b_{1,3}) \\ u_{4,4} &= \frac{1}{4} (b_{3,5} + b_{5,3} + b_{5,5} + u_{3,3}); u_{4,2} = \frac{1}{4} (u_{3,3} + b_{5,1} + b_{3,1} + b_{5,3}) \\ u_{2,2} &= \frac{1}{4} (b_{1,3} + b_{3,1} + u_{3,3} + b_{1,1}) \end{aligned}$$

The values at the remaining interior points i.e. $u_{2,3}, u_{3,4}, u_{4,3}$ and $u_{3,2}$ are computed by the standard five-point formula (2). Thus, we obtain

$$\begin{aligned} u_{2,3} &= \frac{1}{4} (b_{1,3} + u_{3,3} + u_{2,4} + u_{2,2}), u_{3,4} = \frac{1}{4} (u_{2,4} + u_{4,4} + b_{3,5} + u_{3,3}) \\ u_{4,3} &= \frac{1}{4} (u_{3,3} + b_{5,3} + u_{4,4} + u_{4,2}), u_{3,2} = \frac{1}{4} (u_{2,2} + u_{4,2} + u_{3,3} + u_{3,1}) \end{aligned}$$

Having found all the nine values of $u_{i,j}$ once, their accuracy is improved by repeated application of (2) in the form

$$u^{(n+1)}_{i,j} = \frac{1}{4} [u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j+1} + u^{(n)}_{i,j-1}]$$

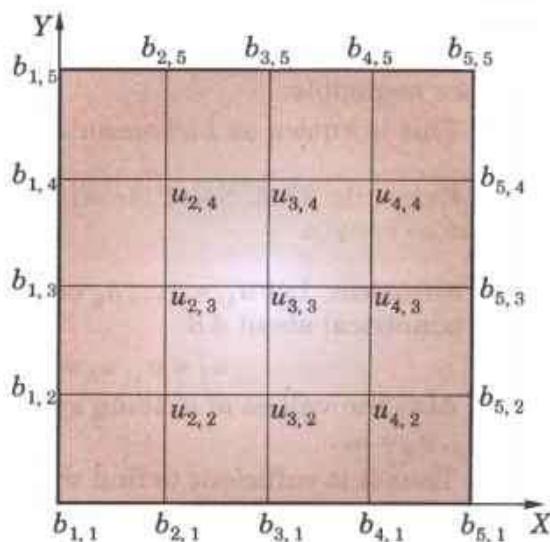


Fig. 33.3

This formula utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. This process is repeated till the difference of values in one round and the next becomes negligible.

This is known as *Liebmann's iteration process*.

Example 33.2. Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh of Fig. 33.6 with boundary values as shown.
(Rohtak, 2005; V.T.U., 2005)

Solution. Let u_1, u_2, \dots, u_9 be the values of u at the interior mesh-points. Since the boundary values of u are symmetrical about AB .

$$\therefore u_7 = u_1, u_8 = u_2, u_9 = u_3.$$

Also the values of u being symmetrical about CD , $u_3 = u_1$, $u_6 = u_4$, $u_9 = u_7$.

Thus it is sufficient to find the values u_1, u_2, u_4 and u_5 .

Now we find their initial value in the following order :

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad (\text{Std. formula})$$

$$u_1 = \frac{1}{4}(0 + 1500 + 1000 + 2000) = 1125 \quad (\text{Diag. formula})$$

$$u_2 = \frac{1}{4}(1125 + 1125 + 1000 + 1500) = 1188 \quad (\text{Std. formula})$$

$$u_4 = \frac{1}{4}(2000 + 1500 + 1125 + 1125) = 1438 \quad (\text{Std. formula})$$

We carry out the iteration process using the formulae :

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_1^{(n)} + 1000 + u_5^{(n)}]$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + u_5^{(n)} + u_1^{(n+1)} + u_1^{(n)}]$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_4^{(n)} + u_2^{(n+1)} + u_2^{(n)}]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4}(1000 + 1188 + 500 + 1438) = 1032$$

$$u_2^{(1)} = \frac{1}{4}(1032 + 1032 + 1000 + 1500) = 1141$$

$$u_4^{(1)} = \frac{1}{4}(2000 + 1500 + 1032 + 1032) = 1391$$

$$u_5^{(1)} = \frac{1}{4}(1391 + 1391 + 1141 + 1141) = 1266$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4}(1000 + 1141 + 500 + 1391) = 1008$$

$$u_2^{(2)} = \frac{1}{4}(1008 + 1008 + 1000 + 1266) = 1069$$

$$u_4^{(2)} = \frac{1}{4}(2000 + 1266 + 1008 + 1008) = 1321$$

$$u_5^{(2)} = \frac{1}{4}(1321 + 1321 + 1069 + 1069) = 1195$$

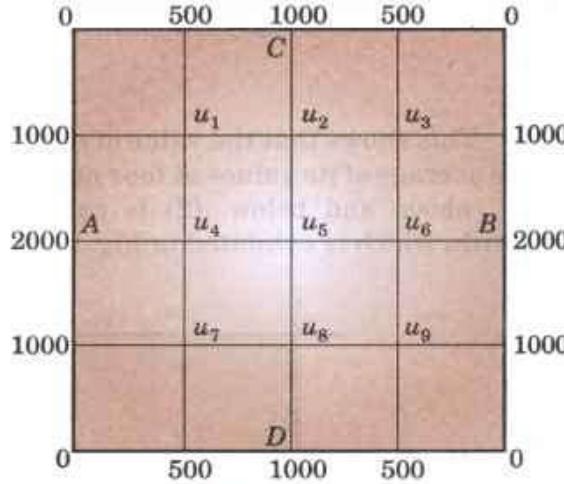


Fig. 33.6

Third iteration :

$$u_1^{(3)} = \frac{1}{4}(1000 + 1069 + 500 + 1321) = 973$$

$$u_2^{(3)} = \frac{1}{4}(973 + 973 + 1000 + 1195) = 1035$$

$$u_4^{(3)} = \frac{1}{4}(2000 + 1195 + 973 + 973) = 1288$$

$$u_5^{(3)} = \frac{1}{4}(1288 + 1288 + 1035 + 1035) = 1162$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4}(1000 + 1135 + 500 + 1288) = 956$$

$$u_2^{(4)} = \frac{1}{4}(956 + 956 + 1000 + 1162) = 1019$$

$$u_4^{(4)} = \frac{1}{4}(2000 + 1162 + 956 + 956) = 1269$$

$$u_5^{(4)} = \frac{1}{4}(1269 + 1269 + 1019 + 1019) = 1144$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4}(1000 + 1019 + 500 + 1269) = 947$$

$$u_2^{(5)} = \frac{1}{4}(947 + 947 + 1000 + 1144) = 1010$$

$$u_4^{(5)} = \frac{1}{4}(2000 + 1144 + 947 + 947) = 1260$$

$$u_5^{(5)} = \frac{1}{4}(1260 + 1260 + 1010 + 1010) = 1135$$

Similarly,

$$u_1^{(6)} = 942, u_2^{(6)} = 1005, u_4^{(6)} = 1255, u_5^{(6)} = 1130$$

$$u_1^{(7)} = 940, u_2^{(7)} = 1003, u_4^{(7)} = 1253, u_5^{(7)} = 1128$$

$$u_1^{(8)} = 939, u_2^{(8)} = 1002, u_4^{(8)} = 1252, u_5^{(8)} = 1127$$

$$u_1^{(9)} = 939, u_2^{(9)} = 1001, u_4^{(9)} = 1251, u_5^{(9)} = 1126$$

Thus there is negligible difference between the values obtained in the eighth and ninth iterations.

Hence $u_1 = 939, u_2 = 1001, u_4 = 1251$ and $u_5 = 1126$.

Example 33.3. Given the values of $u(x, y)$ on the boundary of the square in the Fig. 33.7, evaluate the function $u(x, y)$ satisfying the Laplace equation $\nabla^2 u = 0$ at the pivotal points of this figure.

(Bhopal, 2009 ; Madras, 2003)

Solution. To get the initial values of u_1, u_2, u_3, u_4 , we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \quad (\text{Diag. formula})$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \quad (\text{Std. formula})$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \quad (\text{Std. formula})$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \quad (\text{Std. formula})$$

We carry out the successive iterations, using the formulae

$$u_1^{(n+1)} = \frac{1}{4}[2000 + u_2^{(n)} + 1000 + u_3^{(n)}]$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}]$$

$$u_3^{(n+1)} = \frac{1}{4} [2000 + u_4^{(n)} + u_1^{(n+1)} + 500]$$

$$u_4^{(n+1)} = \frac{1}{4} [u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0]$$

First iteration : (put $n = 0$)

$$u_1^{(1)} = \frac{1}{4} (2000 + 625 + 1000 + 875) = 1125$$

$$u_2^{(1)} = \frac{1}{4} (1125 + 500 + 1000 + 375) = 750$$

$$u_3^{(1)} = \frac{1}{4} (2000 + 375 + 1125 + 500) = 1000$$

$$u_4^{(1)} = \frac{1}{4} (1000 + 0 + 750 + 0) = 438$$

Second iteration : (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4} (2000 + 750 + 1000 + 1000) = 1188$$

$$u_2^{(2)} = \frac{1}{4} (1188 + 500 + 1000 + 438) = 782$$

$$u_3^{(2)} = \frac{1}{4} (2000 + 438 + 1188 + 500) = 1032$$

$$u_4^{(2)} = \frac{1}{4} (1032 + 0 + 782 + 0) = 454$$

Third iteration : (put $n = 2$)

$$u_1^{(3)} = \frac{1}{4} (2000 + 782 + 1000 + 1032) = 1204$$

$$u_2^{(3)} = \frac{1}{4} (1204 + 500 + 1000 + 454) = 789$$

$$u_3^{(3)} = \frac{1}{4} (2000 + 454 + 1204 + 500) = 1040$$

$$u_4^{(3)} = \frac{1}{4} (1040 + 0 + 789 + 0) = 458$$

Similarly, $u_1^{(4)} = 1207, u_2^{(4)} = 791, u_3^{(4)} = 1041, u_4^{(4)} = 458$

and $u_1^{(5)} = 1208, u_2^{(5)} = 791.5, u_3^{(5)} = 1041.5, u_4^{(5)} = 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = 1208, u_2 = 792, u_3 = 1042$ and $u_4 = 458$.

Example 33.4. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ given that (Fig. 33.8).

Solution. We first find the initial values in the following order :

$$u_5 = \frac{1}{4} (0 + 17 + 21 + 12.1) = 12.5 \quad (\text{Std. formula})$$

$$u_1 = \frac{1}{4} (0 + 12.5 + 0 + 17) = 7.4 \quad (\text{Diag. formula})$$

$$u_3 = \frac{1}{4} (12.5 + 18.6 + 17 + 21) = 17.28 \quad (\text{Diag. formula})$$

$$u_7 = \frac{1}{4} (12.5 + 0 + 0 + 12.1) = 6.15 \quad (\text{Diag. formula})$$

$$u_9 = \frac{1}{4} (12.5 + 9 + 21 + 12.1) = 13.65 \quad (\text{Diag. formula})$$

$$u_2 = \frac{1}{4} (17 + 12.5 + 7.4 + 17.3) = 13.55 \quad (\text{Std. formula})$$

$$u_4 = \frac{1}{4} (7.4 + 6.2 + 0 + 12.5) = 6.52 \quad (\text{Std. formula})$$

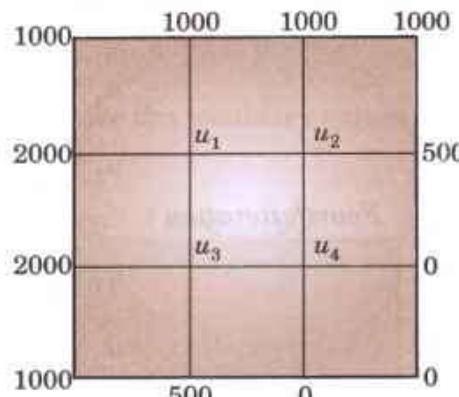


Fig. 33.7

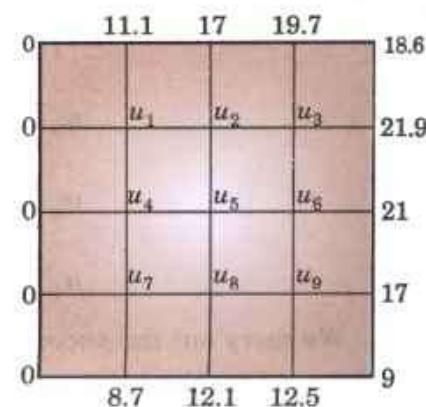


Fig. 33.8

$$u_6 = \frac{1}{4} (17.3 + 13.7 + 12.5 + 21) = 16.12 \quad (\text{Std. formula})$$

$$u_8 = \frac{1}{4} (12.5 + 12.1 + 6.2 + 13.7) = 11.12 \quad (\text{Std. formula})$$

Now we carry out the iteration process using the Standard formula :

$$u_1^{(n+1)} = \frac{1}{4} [(0 + 11.1 + u_4^{(n)} + u_2^{(n)})]$$

$$u_2^{(n+1)} = \frac{1}{4} [(u_1^{(n+1)} + 17 + u_5^{(n)} + u_3^{(n)})]$$

$$u_3^{(n+1)} = \frac{1}{4} [(u_2^{(n+1)} + 19.7 + u_6^{(n)} + 21.9)]$$

$$u_4^{(n+1)} = \frac{1}{4} [(u_1^{(n+1)} + 19.7 + u_7^{(n)} + u_5^{(n)})]$$

$$u_5^{(n+1)} = \frac{1}{4} [(u_4^{(n+1)} + u_2^{(n+1)} + u_8^{(n)} + u_6^{(n)})]$$

$$u_6^{(n+1)} = \frac{1}{4} [(u_5^{(n+1)} + u_3^{(n+1)} + u_9^{(n)} + 21)]$$

$$u_7^{(n+1)} = \frac{1}{4} [0 + (u_4^{(n+1)} + 8.7 + u_8^{(n)})]$$

$$u_8^{(n+1)} = \frac{1}{4} [(u_7^{(n+1)} + u_5^{(n+1)} + 12.1 + u_9^{(n)})]$$

$$u_9^{(n+1)} = \frac{1}{4} [(u_8^{(n+1)} + u_6^{(n)} + 12.8 + 17)]$$

First iteration (put $n = 0$, in the above results)

$$u_1^{(1)} = \frac{1}{4} (0 + 11.1 + u_4^{(0)} + u_2^{(0)}) = \frac{1}{4} (0 + 11.1 + 6.52 + 13.55) = 7.79$$

$$u_2^{(1)} = \frac{1}{4} (7.79 + 17 + 12.5 + 17.28) = 13.64$$

$$u_3^{(1)} = \frac{1}{4} (13.64 + 19.7 + 16.12 + 21.9) = 12.84$$

$$u_4^{(1)} = \frac{1}{4} (0 + 7.79 + 6.15 + 12.5) = 6.61$$

$$u_5^{(1)} = \frac{1}{4} (6.61 + 13.64 + 11.12 + 16.12) = 11.88$$

$$u_6^{(1)} = \frac{1}{4} (11.88 + 17.84 + 13.65 + 21) = 16.09$$

$$u_7^{(1)} = \frac{1}{4} (0 + 6.61 + 8.7 + 11.12) = 6.61$$

$$u_8^{(1)} = \frac{1}{4} (6.61 + 11.88 + 12.1 + 13.65) = 11.06$$

$$u_9^{(1)} = \frac{1}{4} (11.06 + 16.09 + 12.8 + 17) = 12.238$$

Second iteration (put $n = 1$)

$$u_1^{(2)} = \frac{1}{4} (0 + 11.1 + 6.61 + 13.64) = 7.84$$

$$u_2^{(2)} = \frac{1}{4} (7.84 + 17 + 11.88 + 17.84) = 16.64$$

$$u_3^{(2)} = \frac{1}{4} (13.64 + 19.7 + 16.09 + 21.9) = 17.83$$

$$u_4^{(2)} = \frac{1}{4} (0 + 7.84 + 6.61 + 11.88) = 6.58$$

$$u_5^{(2)} = \frac{1}{4} (6.58 + 13.64 + 11.06 + 16.09) = 11.84$$

$$u_6^{(2)} = \frac{1}{4} (11.84 + 17.83 + 14.24 + 21) = 16.23$$

$$u_7^{(2)} = \frac{1}{4} (0 + 6.58 + 8.7 + 11.06) = 6.58$$

$$u_8^{(2)} = \frac{1}{4} (6.58 + 11.84 + 12.1 + 14.24) = 11.19$$

$$u_9^{(2)} = \frac{1}{4} (11.19 + 16.23 + 12.8 + 17) = 14.30$$

Third iteration (put n = 2)

$$u_1^{(3)} = \frac{1}{4} (0 + 11.1 + 6.58 + 13.64) = 7.83$$

$$u_2^{(3)} = \frac{1}{4} (7.83 + 17 + 11.84 + 17.83) = 13.637$$

$$u_3^{(3)} = \frac{1}{4} (13.63 + 19.7 + 16.23 + 21.9) = 17.86$$

$$u_4^{(3)} = \frac{1}{4} (0 + 7.83 + 6.58 + 11.84) = 6.56$$

$$u_5^{(3)} = \frac{1}{4} (6.56 + 13.63 + 11.19 + 16.23) = 11.90$$

$$u_6^{(3)} = \frac{1}{4} (11.90 + 17.86 + 14.30 + 21) = 16.27$$

$$u_7^{(3)} = \frac{1}{4} (0 + 6.56 + 8.7 + 11.19) = 6.61$$

$$u_8^{(3)} = \frac{1}{4} (6.61 + 11.90 + 12.1 + 14.30) = 11.23$$

$$u_9^{(3)} = \frac{1}{4} (11.23 + 16.27 + 12.8 + 17) = 14.32$$

Similarly,

$$u_1^{(4)} = 7.82, u_2^{(4)} = 13.65, u_3^{(4)} = 17.88, u_4^{(4)} = 6.58, u_5^{(4)} = 11.94, u_6^{(4)} = 16.28,$$

$$u_7^{(4)} = 6.63, u_8^{(4)} = 11.25, u_9^{(4)} = 14.33$$

$$u_1^{(5)} = 7.83, u_2^{(5)} = 13.66, u_3^{(5)} = 17.89, u_4^{(5)} = 6.50, u_5^{(5)} = 11.95, u_6^{(5)} = 16.29,$$

$$u_7^{(5)} = 6.64, u_8^{(5)} = 11.25, u_9^{(5)} = 14.34$$

33.6 SOLUTION OF POISSON'S EQUATION*

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad \dots(1)$$

This is an *elliptic equation* which can be solved numerically at the interior mesh points of a square network when the boundary values are known. The standard 5-point formula for (1) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \quad \dots(2)$$

By applying (2) at each mesh-point, we arrive at linear equations in the pivotal values i, j . These equations can be solved by Gauss-Seidal iteration method (p. 938).

Example 33.5. Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$, $0 < x < 1$, $0 < y < 1$ given that $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 100$, $u(x, 1) = 100$ and $h = 1/3$. (Anna, 2005)

Solution. Here $h = 1/3$ $u_{i,j-1} u_{i,j}$ (Fig. 33.9)

* See p. 882.

The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) = h^2 [-81(ih, jh)] = h^4 (-81) ij = -ij \dots (i)$$

For u_1 ($i = 1, j = 2$), (i) gives $0 + u_2 + u_3 + 100 - 4u_1 = -2$

$$\text{i.e., } -4u_1 + u_2 + u_3 = -102 \dots (ii)$$

For u_2 ($i = 2, j = 2$), (i) gives $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

$$\text{i.e., } u_1 - 4u_2 + u_4 = -204 \dots (iii)$$

For u_3 ($i = 1, j = 1$), (i) gives $0 + u_4 + 0 + u_1 - 4u_3 = -1$

$$\text{i.e., } u_1 - 4u_3 + u_4 = -1 \dots (iv)$$

For u_4 ($i = 2, j = 1$), gives $u_3 + 100 + u_2 - 4u_4 = -2$

$$u_2 + u_3 - 4u_4 = -102 \dots (v)$$

Subtracting (v) from (ii), $-4u_1 + 4u_4 = 0$ i.e. $u_1 = u_4$

$$\text{Then (iii) becomes } 2u_1 - 4u_2 = -204 \dots (vi)$$

$$\text{and (iv) becomes } 2u_1 - 4u_3 = -1 \dots (vii)$$

$$\text{Now (4) } \times (ii) + (vi) \text{ gives } -14u_1 + 4u_3 = -612 \dots (viii)$$

$$(vii) + (viii) \text{ gives } -12u_1 = -613$$

$$\text{Thus } u_1 = 613/12 = 51.0833 = u_4.$$

$$\text{From (vi), } u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$$

$$\text{From (vii), } u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916.$$

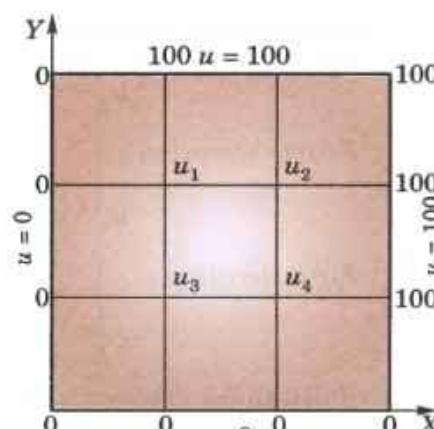


Fig. 33.9

Example 33.6. Solve the partial differential equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.

(Anna, 2007; P.T.U., 2007; Delhi, 2002)

Solution. Here $h = 1$ (Fig. 33.10).

∴ The standard 5-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(i^2 + j^2 + 10) \dots (i)$$

For u_1 ($i = 1, j = 2$), (i) gives

$$0 + u_2 + 0 + u_3 - 4u_1 = -10(1 + 4 + 10)$$

$$\text{i.e., } u_1 = \frac{1}{4}(u_2 + u_3 + 150) \dots (ii)$$

$$\text{For } u_2 \text{ } (i = 2, j = 2), \text{ (i) gives } u_2 = \frac{1}{4}(u_1 + u_4 + 180) \dots (iii)$$

$$\text{For } u_3 \text{ } (i = 1, j = 1), \text{ we have } u_3 = \frac{1}{4}(u_1 + u_4 + 120) \dots (iv)$$

$$\text{For } u_4 \text{ } (i = 2, j = 1), \text{ we have } u_4 = \frac{1}{4}(u_2 + u_3 + 150) \dots (v)$$

Equations (ii) and (v) show that $u_4 = u_1$. Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \dots (vi)$$

$$u_2 = \frac{1}{2}(u_1 + 90) \dots (vii)$$

$$u_3 = \frac{1}{2}(u_1 + 60) \dots (viii)$$

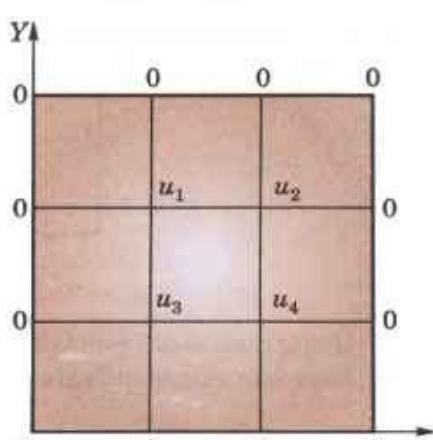


Fig. 33.10

Now let us solve these equations by Gauss-Seidal iteration method.

First iteration : Starting from the approximations $u_2 = 0$, $u_3 = 0$, we obtain

$$u_1^{(1)} = 37.5. \text{ Then } u_2^{(1)} = \frac{1}{2}(37.5 + 90) \approx 64; u_3^{(1)} = \frac{1}{2}(37.5 + 60) \approx 49$$

Second iteration :

$$u_1^{(2)} = \frac{1}{4} (64 + 49 + 150) = 66; u_2^{(2)} = \frac{1}{2} (66 + 90) = 78; u_3^{(2)} = \frac{1}{2} (66 + 60) = 63$$

Third iteration :

$$u_1^{(3)} = \frac{1}{4} (78 + 63 + 150) = 73; u_2^{(3)} = \frac{1}{2} (73 + 90) = 82; u_3^{(2)} = \frac{1}{2} (73 + 60) = 67$$

Fourth iteration :

$$u_1^{(4)} = \frac{1}{4} (82 + 67 + 150) = 75; u_2^{(4)} = \frac{1}{2} (75 + 90) = 82.5; u_3^{(4)} = \frac{1}{2} (75 + 60) = 67.5$$

Fifth iteration :

$$u_1^{(5)} = \frac{1}{4} (82.5 + 67.5 + 150) = 75; u_2^{(5)} = \frac{1}{2} (75 + 90) = 82.5; u_3^{(5)} = \frac{1}{2} (75 + 60) = 67.5$$

Since these values are the same as those of fourth iteration, we have

$$u_1 = 75, u_2 = 82.5, u_3 = 67.5 \text{ and } u_4 = 75.$$

PROBLEMS 33.2

- Solve the equation $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in Fig. 33.11. Iterate until the maximum difference between the successive values at any point is less than 0.001. (Delhi, 2002)
- Solve $\nabla^2 u = 0$ under the conditions ($h = 1, k = 1$), $u(0, y) = 0, u(4, y) = 12 + y$ for $0 \leq y \leq 4$; $u(x, 0) = 3x, u(x, 4) = x^2$ for $0 \leq x \leq 4$. (Cusat, 2008)
- Solve the elliptic equation $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.12. Iterate until the maximum difference between successive values at any point is less than 0.005.

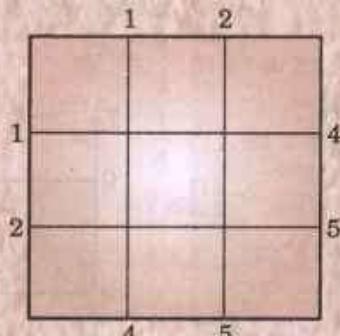


Fig. 33.11

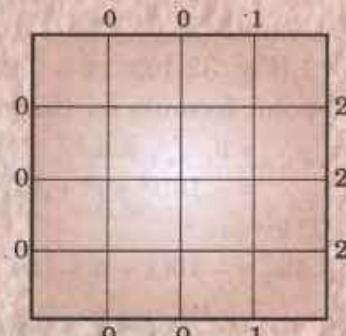


Fig. 33.12

- Using central-difference approximation solve $\nabla^2 u = 0$ at the nodal points of the square grid of Fig. 33.13 using the boundary values indicated. (V.T.U., 2000)

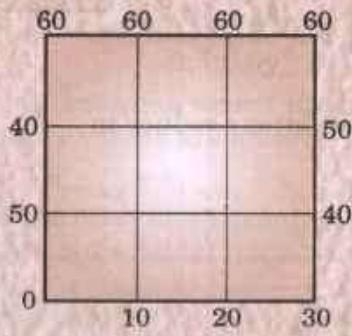


Fig. 33.13

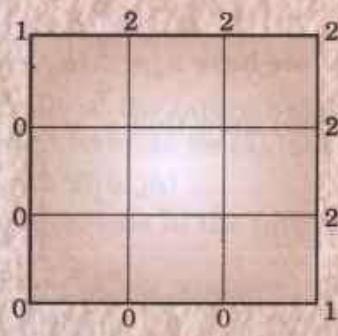


Fig. 33.14

- Solve $u_{xx} + u_{yy} = 0$ for the square mesh with boundary values as shown in Fig. 33.14. Iterate till the mesh values are correct to two decimal places.

6. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain of Fig. 33.15.

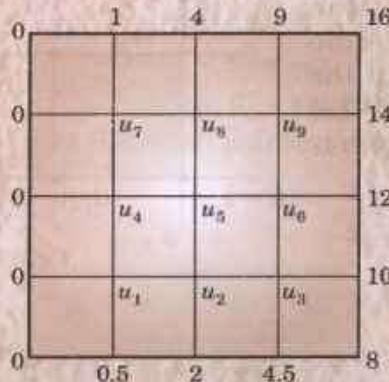


Fig. 33.15

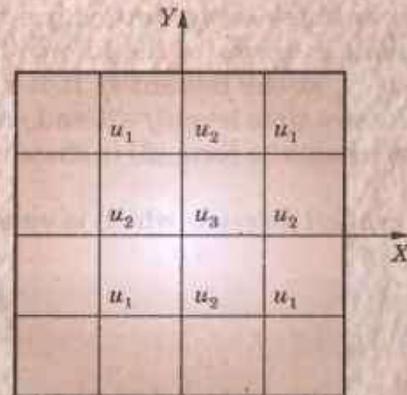


Fig. 33.16

7. Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mesh of Fig. 33.16 with $u(x, y) = 0$ on the boundary and mesh length = 1. (J.N.T.U., 2004 S)

Note. Solution of elliptic equations by *Relaxation method* is given in author's book "Numerical Methods in Engineering and Science".

33.7 PARABOLIC EQUATIONS

The one-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is a well-known example of parabolic partial differential equations. The solution of this equation is a temperature function $u(x, t)$ which is defined for values of x from 0 to l and for values of time t from 0 to ∞ . The solution is not defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions. (Fig. 33.17).

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of equations.

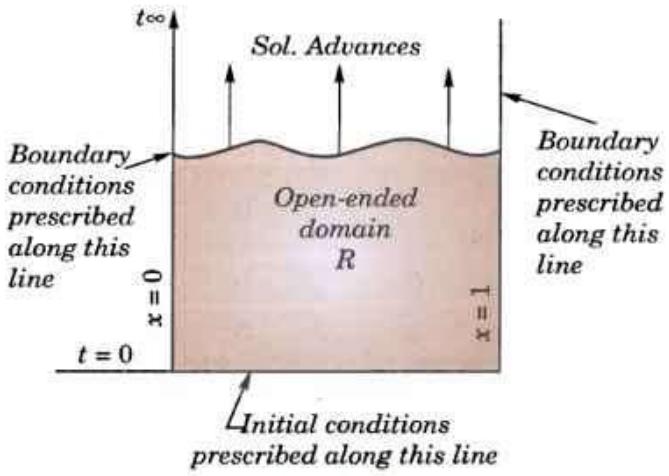


Fig. 33.17

33.8 SOLUTION OF HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

where $c^2 = k/\rho\sigma$ is the diffusivity of the substance ($\text{cm}^2/\text{sec.}$)

Consider a rectangular mesh in the $x-t$ plane with spacing h along x direction and k along time t direction. Denoting a mesh point $(x, t) = (ih, jk)$ as simply i, j , we have

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{k} \quad [\text{By (5) } \S \text{ 33.3}]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \quad [\text{By (4) } \S \text{ 33.3}]$$

and Substituting these in (1), we obtain

$$u_{i,j+1} - u_{i,j} = \frac{kc^2}{h^2} [u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

$$\text{or } u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j} \quad \dots(2)$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j+1)$ th mesh point in terms of the known function values at the points x_{i-1} , x_i and x_{i+1} at the instant t_j . It is a relation between the function values at the two time levels $j+1$ and j and is therefore, called a *2-level formula*. In schematic form (2) is shown in Fig. 33.18 which is

called the *Schmidt explicit formula* which is valid only for $0 < \alpha \leq \frac{1}{2}$.

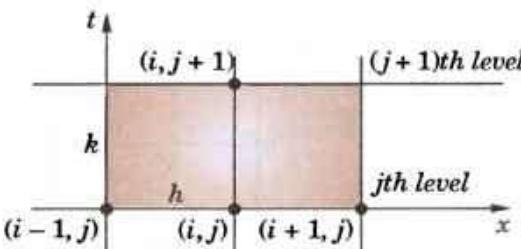


Fig. 33.18

Obs. In particular when $\alpha = \frac{1}{2}$, (2) reduces to

$$u_{i,j+1} = \frac{1}{2} (u_{i-1,j} + u_{i+1,j}) \quad \dots(3)$$

which shows that the value of u at x_i at time t_{j+1} is the mean of the u -values at x_{i-1} and x_{i+1} at time t_j . This relation, known as *Bendre-Schmidt recurrence relation*, gives the values of u at the internal mesh points with the help of boundary conditions.

Note. The other formulae (i.e. Crank-Nicolson formula and Du Fort-Frankel formula) are given in author's book 'Numerical Methods in Engineering and Science'.

Example 33.7. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0 < x < 5$, $t \geq 0$ given that $u(x, 0) = 20$, $u(0, t) = 0$, $u(5, t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson method. (Anna, 2006)

Solution. Here $c^2 = 1$ and $h = 1$.

Taking α (i.e., $c^2 k/h = 1$), we get $k = 1$

Also we have

	i	0	1	2	3	4	5
j	0	0	20	20	20	20	100
	1	0	u_1	u_2	u_3	u_4	100

Then Crank-Nicholson formula becomes

$$4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j} \quad \dots(1)$$

$$4u_1 = 0 + 20 + 0 + u_2 \quad \text{i.e.} \quad 4u_1 - u_2 = 20 \quad \dots(2)$$

$$4u_2 = 20 + 20 + u_1 + u_3 \quad \text{i.e.} \quad 4u_1 - 4u_2 + u_3 = -40 \quad \dots(3)$$

$$4u_3 = 20 + 20 + u_2 + u_4 \quad \text{i.e.} \quad u_2 - 4u_3 + u_4 = -40 \quad \dots(4)$$

$$4u_4 = 20 + 100 + u_3 + 100 \quad \text{i.e.} \quad 4u_3 - 4u_4 = -220 \quad \dots(5)$$

$$\text{Now (1) } - 4(2) \text{ gives } 15u_2 - 4u_3 = 180 \quad \dots(5)$$

$$4(3) + (4) \text{ gives } 4u_2 - 15u_3 = -380 \quad \dots(6)$$

$$\text{Then } 15(5) - 4(6) \text{ gives } 209u_2 = 4220 \quad \text{i.e.} \quad u_2 = 20.2$$

$$\text{From (5), we get } 4u_3 = 15 \times 20.2 - 180 \quad \text{i.e.} \quad u_3 = 30.75$$

$$\text{From (1),} \quad 4u_1 = 20 + 20.2 \quad \text{i.e.} \quad u_1 = 10.05$$

$$\text{From (4),} \quad 4u_4 = 220 + 30.75 \quad \text{i.e.} \quad u_4 = 62.69$$

Thus the required values are 10.05, 20.2, 30.75 and 62.69.

Example 33.8. Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin \pi x$, $0 \leq x \leq 1$ using Schmidt method (Take $h = 0.2$ and $\alpha = 1/2$). (Rohtak, 2003)

Solution. Since $h = 0.2$ and $\alpha = 1/2$

$$\therefore \alpha = \frac{k}{h^2} \text{ gives } k = 0.02$$

Since $\alpha = 1/2$, we use Bredre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \quad \dots(i)$$

We have

$$u(0, 0) = 0, u(0.2, 0) = \sin \pi/5 = 0.5875$$

$$u(0.4, 0) = \sin 2\pi/5 = 0.95111, u(0.6, 0) = \sin 3\pi/5 = 0.9511$$

$$u(0.8, 0) = \sin 4\pi/5 = 0.5875, u(1, 0) = \sin \pi = 0$$

The value of u at the mesh points can be obtained by using the recurrence relation (i) as shown in table below :

	$x \rightarrow 0$	0.2	0.4	0.6	0.8	1.0	
$t \downarrow$	$i \backslash j$	0	1	2	3	4	5
0	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Example 33.9. Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points $x = i : i = 0, 1, 2, \dots, 8$ and $t = \frac{1}{8}j : j = 0, 1, 2, \dots, 5$.

Solution. Here $c^2 = 4$, $h = 1$ and $k = 1/8$. Then $\alpha = c^2k/h^2 = 1/2$.

\therefore we have Bredre-Schmidt's recurrence relation

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \quad \dots(i)$$

Now since $u(0, t) = 0 = u(8, t)$

$\therefore u_{0,i} = 0$ and $u_{8,j} = 0$ for all values of j , i.e., the entries in the first and last column are zero.

Since $u(x, 0) = 4x - \frac{1}{2}x^2$

$$\therefore u_{i,0} = 4i - \frac{1}{2}i^2 = 0, 3.5, 6, 7.5, 8, 7.5, 6, 3.5$$

for $i = 0, 1, 2, 3, 4, 5, 6, 7$ at $t = 0$

These are the entries of the first row.

	$i \backslash j$	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0	
1	0	3	5.5	7	7.5	7	5.5	3	0	
2	0	2.75	5	6.5	7	6.5	5	2.75	0	
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0	
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0	
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0	

Putting $j = 0$ in (i), we have

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

Taking $i = 1, 2, \dots, 7$ successively, we get

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 8) = 7$$

$$u_{4,1} = 7.5, u_{5,1} = 7, u_{6,1} = 5.5, u_{7,1} = 3.$$

These are the entries in the second row.

Putting $j = 1$ in (i), the entries of the third row are given by

$$u_{i,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

Similarly putting $j = 2, 3, 4$ successively in (i), the entries of the fourth, fifth and sixth rows are obtained. Hence the values of $u_{i,j}$ are as given in the above table.

Example 33.10. Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(x, 0) = \sin \pi x, 0 \leq x \leq 1; u(0, t) = u(1, t) = 0$. Carry out computations for two levels, taking $h = 1/3, k = 1/36$. (V.T.U., 2005)

Solution. Here $c^2 = 1, h = 1/3, k = 1/36$ so that

$$\alpha = hc^2/h^2 = 1/4.$$

Also $u_{1,0} = \sin \pi/3 = \sqrt{3}/2, u_{2,0} = \sin 2\pi/3 = \sqrt{3}/2$

and all other boundary values are zero as shown in Fig. 33.19.

Schmidt's formula [(2) of § 33.8]

$$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha) u_{i,j} + \alpha u_{i+1,j}$$

becomes $u_{i,j+1} = \frac{1}{4}[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}]$

For $i = 1, 2; j = 0$:

$$u_{1,1} = \frac{1}{4}[u_{0,0} + 2u_{1,0} + u_{2,0}] = \frac{1}{4}(0 + 2 \times \sqrt{3}/2 + \sqrt{3}/2) = 0.65$$

$$u_{2,1} = \frac{1}{4}[u_{1,0} + 2u_{2,0} + u_{3,0}] = \frac{1}{4}(\sqrt{3}/2 + 2 \times \sqrt{3}/2 + 0) = 0.65$$

For $i = 1, 2, j = 1$:

$$u_{1,2} = \frac{1}{4}(u_{0,1} + 2u_{1,1} + u_{2,1}) = 0.49$$

$$u_{2,2} = \frac{1}{4}(u_{1,1} + 2u_{2,1} + u_{3,1}) = 0.49.$$

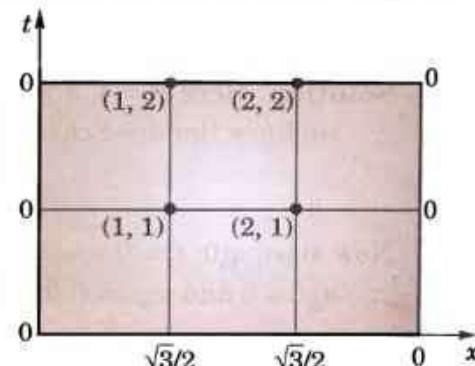


Fig. 33.19

PROBLEMS 33.3

- Find the solution of the parabolic equation $u_{xx} = 2u_t$ when $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4-x)$, taking $h = 1$. Find the values upto $t = 5$. (Madras, 2001)
- Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with the conditions $u(0, t) = 0, u(x, 0) = x(1-x)$ and $u(1, t) = 0$. Assume $h = 0.1$. Tabulate u for $t = k, 2k$ and $3k$ choosing an appropriate value of k . (Anna, 2004)

3. Given $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} = 0$; $f(0, t) = f(5, t) = 0$, $f(x, 0) = x^2(25 - x^2)$; find the values of f for $x = ih$ ($i = 0, 1, \dots, 5$) and $t = jk$ ($j = 0, 1, \dots, 6$) with $h = 1$ and $k = \frac{1}{2}$, using the explicit method.

Replacing the derivatives in (1) by their above approximations, we obtain

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2 k^2}{h^2} (u_{i-1,j} - 2u_{i,j} + u_{i,j+1})$$

or $u_{i,j+1} = 2(1 - \alpha^2 c^2) u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j} - u_{i,j-1})$... (4)

where $\alpha = k/h$.

Now replacing the derivative in (2) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x) \quad [\text{See (7) p. 1042}]$$

or $u_{i,j+1} = u_{i,j-1} + 2k g(x) \quad \text{at } t = 0 \quad \text{i.e. } u_{i,1} = u_{i,-1} + 2kg(x) \text{ for } j = 0$... (5)

Also initial condition $u = f(x)$ at $t = 0$ becomes $u_{i,0} = f(x)$... (6)

Combining (5) and (6), we have $u_{i,1} = f(x) + 2kg(x)$... (7)

Also (3) gives $u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$

Hence (4) gives the values of $u_{i,j+1}$ at the $(j+1)$ th level when the nodal values at $(j-1)$ th and j th levels are known from (6) and (7) as shown in Fig. 32.20. Thus (4) gives an **implicit scheme for the solution of the wave equation**.

A special case : The coefficient of $u_{i,j}$ in (4) will vanish if $\alpha = 1/c$ or $k = h/c$. Then (4) reduces to the *simple form*

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \dots (8)$$

This result provides an **explicit scheme for the solution of the wave equation**.

Obs. 1. For $\alpha = 1/c$, the solution of (4) is stable and coincides with the solution of (1).

For $\alpha < 1/c$, the solution is stable but inaccurate.

For $\alpha > 1/c$, the solution is unstable.

Obs. 2. The formula (4) converges for $\alpha \leq 1$ i.e. for $k \leq h$.

Example 33.11. Evaluate the pivotal values of the equation $u_{tt} = 16u_{xx}$, taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5-x)$. (Madras, 2006)

Solution. Here $c^2 = 16$.

∴ The difference equation for the given equation is

$$u_{i,j+1} = 2(1 - 16\alpha^2) u_{i,j} + 16\alpha^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{where } \alpha = k/h \quad \dots (i)$$

Taking $h = 1$ and choosing k so that the coefficient of $u_{i,j}$ vanishes, we have $16\alpha^2 = 1$, i.e., $k = h/4 = 1/4$.

∴ (i) reduces to $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$... (ii)

which gives a convergent solution (since $k/h < 1$). Its solution coincides with the solution of the given differential equation.

Now since $u(0, t) = u(5, t) = 0$.

∴ $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j

i.e. the entries in the first and last columns are zero.

Since $u_{(x,0)} = x^2(5-x)$

∴ $u_{i,0} = i^2(5-i) = 4, 12, 18, 16$ for $i = 1, 2, 3, 4$ at $t = 0$.

These are the entries of the *first row*.

Finally the initial condition $u_t(x, 0) = 0$, becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0, \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \quad \dots (iii)$$

Putting $j = 0$ in (ii), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$
 $= u_{i-1,0} + u_{i+1,0} - u_{i,1}$ using (iii)

or $u_{i,1} = \frac{1}{2} (u_{i-1,0} + u_{i+1,0}) \quad \dots (iv)$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,1} = \frac{1}{2} (u_{0,0} + u_{2,0}) = \frac{1}{2} (0 + 12) = 6$$

$$u_{2,1} = \frac{1}{2} (u_{1,0} + u_{3,0}) = \frac{1}{2} (4 + 18) = 11$$

$$u_{3,1} = \frac{1}{2} (u_{2,0} + u_{4,0}) = \frac{1}{2} (12 + 16) = 14$$

$$u_{4,1} = \frac{1}{2} (u_{3,0} + u_{5,0}) = \frac{1}{2} (18 + 0) = 9$$

These are the entries of the second row.

Putting $j = 1$ in (ii), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

Taking $i = 1, 2, 3, 4$ successively, we obtain

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the third row.

Similarly putting $j = 2, 3, 4$ successively in (ii), the entries of the fourth, fifth and sixth rows are obtained.

Hence the values of $u_{i,j}$ are as shown in the table below :

j	i	0	1	2	3	4	5
0	0	0	4	12	18	16	0
1	0	0	6	11	14	9	0
2	0	0	7	8	2	-2	0
3	0	0	2	-2	-8	-7	0
4	0	0	-9	-14	-11	-6	0
5	0	0	-16	-18	-12	-4	0

Example 33.12. The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2 u / \partial t^2 = 4 \partial^2 u / \partial x^2$, with boundary conditions $u = 0$ at $x = 0, t > 0$ and $u = 0$ at $x = 4, t > 0$ and initial conditions $u = x(4-x)$ and $\partial u / \partial t = 0$ at $t = 0, 0 \leq x \leq 4$. Solve this equation numerically for one half period of vibration, taking $h = 1$ and $k = 1/2$.

Solution. Here, $h/k = 2 = c$.

∴ the difference equation for the given equation is

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \dots(i)$$

which gives a convergent solution (since $k < h$).

Now since $u(0, t) = u(4, t) = 0$,

∴ $u_{0,j} = 0$ and $u_{4,j} = 0$ for all values of j .

i.e., the entries in the first and last columns are zero.

Since $u_{(x,0)} = x(4-x)$,

∴ $u_{i,0} = i(4-i) = 3, 4, 3$ for $i = 1, 2, 3$ at $t = 0$.

These are the entries of the first row.

Also $u_t(x, 0) = 0$ becomes

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \text{ when } j = 0, \text{ giving } u_{i,1} = u_{i,-1} \quad \dots(ii)$$

$$\begin{aligned} \text{Putting } j = 0 \text{ in (i), } u_{i,1} &= u_{i-1,0} + u_{i+1,0} - u_{i,-1} \\ &= u_{i-1,0} + u_{i+1,0} - u_{i,1}, \text{ using (ii)} \end{aligned}$$

or

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0}) \quad \dots(iii)$$

Taking $i = 1, 2, 3$ successively, we obtain

$$\begin{aligned} u_{1,1} &= \frac{1}{2}(u_{0,0} + u_{1,0}) = 2; u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = 3 \\ u_{3,1} &= \frac{1}{2}(u_{2,0} + u_{4,0}) = 2 \end{aligned}$$

These are the entries of the 2nd row.

Putting $j = 1$ in (i), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Taking $i = 1, 2, 3$, successively, we get

$$\begin{aligned} u_{1,2} &= u_{0,1} + u_{2,1} - u_{1,0} = 0 + 3 - 3 = 0 \\ u_{2,2} &= u_{1,1} + u_{3,1} - u_{2,0} = 2 + 2 - 4 = 0 \\ u_{3,2} &= u_{2,1} + u_{4,1} - u_{3,0} = 3 + 0 - 3 = 0 \end{aligned}$$

These are the entries of the 3rd row and so on.

Now the equation of the vibrating string of length l is $u_{tt} = c^2 u_{xx}$.

$$\therefore \text{Its period of vibration} = \frac{2l}{c} = \frac{2 \times 4}{2} = 4 \text{ sec.} \quad [\because l = 4 \text{ and } c = 2]$$

This shows that we have to compute $u_{(x,t)}$ upto $t = 2$.

i.e., similarly we obtain the values of $u_{i,2}$ (4th row) and $u_{i,3}$ (5th row).

Hence the values of $u_{i,j}$ are as shown in the table below :

$j \backslash i$	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	0	0	0	0
3	0	-2	-3	-2	0
4	0	-3	-4	-3	0

Example 33.13. Find the solution of the initial boundary value problem ; $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 \leq x \leq 1$; subject to

the initial conditions $u(x, 0) = \sin \pi x, 0 \leq x \leq 1$ and the boundary conditions $u(0, t) = 0, u(1, t) = 0, t > 0$; by using in the (a) the explicit scheme,

(b) the implicit scheme.

(Anna, 2007)

Solution. (a) *Explicit scheme*

$$\text{Take } h = 0.2, k = \frac{h}{c} = 0.2$$

$[\because c = 1]$

\therefore We use the formula, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$... (i)

Since $u(0, t) = 0, u(1, t) = 0, u_{0,j} = 0, u_{1,j} = 0$ for all values of j

i.e., the entries in the first and last columns are zero.

Since $u(x, 0) = \sin \pi x, u_{i,0} = \sin \pi x$

$\therefore u_{1,0} = 0, u_{2,0} = \sin(0.2\pi) = 0.5878, u_{3,0} = \sin(0.4\pi) = 0.9511, u_{4,0} = \sin(0.6\pi) = 0.5878$.

These are the entries of the first row.

Since $u_t(x, 0) = 0$ we have $\frac{1}{2}(u_{i,j+1} - u_{i,j-1}) = 0$, when $j = 0$

... (ii)

i.e.,

$$u_{i,1} = u_{i,-1}$$

Putting $j = 0$ in (i), $u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$

Using (ii) $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

Taking $i = 1, 2, 3, 4$ successively, we obtain the entries of the second row.

Putting $j = 1$ in (i), $u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$

Now taking $i = 1, 2, 3, 4$ successively, we get the entries of the third row.

Similarly taking $j = 2, j = 3, j = 4$ successively, we obtain the entries of the fourth, fifth and sixth rows respectively.

$j \backslash i$	0	1	2	3	4	5
0	0	0.5878	0.9511	0.9511	0.5878	0
1	0	0.4756	0.7695	0.9511	0.7695	0
2	0	0.1817	0.4756	0.5878	0.3633	0
3	0	0	0.0001	-0.1122	-0.1816	0
4	0	-0.1816	-0.5878	-0.7694	0.4755	0
5	0	-0.5878	-0.9511	-0.9511	-0.5878	0

(b) *Implicit scheme*

We have the formula :

$$u_{i,j+1} = 2(1 - \alpha^2 c^2) u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}, \text{ where } \alpha = k/h \quad \dots(i)$$

Here $c^2 = 1$. Take $h = 0.25$ and $k = 0.5$ so that $\alpha = k/h = 2$.

$\therefore (i)$ reduces to

$$u_{i,j+1} = -6u_{i,j} + 4(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \dots(ii)$$

Since

$$u_{i,0} = \sin \pi x$$

$$\therefore u_{(1,0)} = 0.7071, u_{(2,0)} = 0.5, u_{(3,0)} = 0.7071$$

There are the entries of the first row.

Since $u_t(x, 0) = 0$, we have $\frac{1}{2}(y_{i,i+1} - y_{i,i-1}) = 0$, where $j = 0$

$$\therefore y_{i,1} = y_{i,-1} \quad \dots(iii)$$

Put $j = 0$ and using (iii), (ii) reduces to

$$u_{i,1} = -3u_{i,0} + 2(u_{i-1,0} + 2u_{i+1,0})$$

Now taking

$$i = 1, u_{1,1} = -3u_{1,0} + 2(u_{0,0} + u_{2,0}) = 0.1213$$

$$i = 2, u_{2,1} = -3u_{2,0} + 2(u_{1,0} + u_{3,0}) = 0.1716$$

$$i = 3, u_{3,1} = -3u_{3,0} + 2(u_{2,0} + u_{4,0}) = 0.1213$$

These are the entries of the second row.

Putting $j = 1$, (ii) reduces to

$$u_{i,2} = -6u_{i,1} + 4(u_{i-1,2} + u_{i+1,2})$$

Now taking

$$i = 1, u_{1,2} = -6u_{1,1} + 4(u_{0,1} + u_{2,1}) = 0.414$$

$$i = 2, u_{2,2} = -6u_{2,1} + 4(u_{1,1} + u_{3,1}) = 0.0592$$

$$i = 3, u_{3,2} = -6u_{3,1} + 4(u_{2,1} + u_{4,1}) = 0.414$$

These are the entries of the third row.

Putting $j = 2$, (ii) reduces to

$$u_{i,3} = -6u_{i,2} + 4(u_{i-1,3} + u_{i+1,3}) - u_{i,1}$$

Now taking

$$i = 1, u_{1,3} = -6u_{1,2} + 4(u_{0,2} + u_{2,2}) - u_{1,1} = 0.1097$$

$$i = 2, u_{2,3} = -6u_{2,2} + 4(u_{1,2} + u_{3,2}) - u_{2,1} = 0.1476$$

$$i = 3, u_{3,3} = -6u_{3,2} + 4(u_{2,2} + u_{4,2}) - u_{4,1} = 0.1097$$

These are the entries of the third row.

Hence the value of $u_{i,j}$ are as tabulated below :

$j \backslash i$	0	1	2	3	4
0	0	0.7071	0.5	0.7071	0
1	0	-0.1213	-0.1716	-0.1213	0
2	0	0.0414	0.0592	0.0414	0

PROBLEMS 33.4

1. Solve the boundary value problem $u_{tt} = u_{xx}$ with the conditions $u(0, t) = u(1, t) = 0$, $u(x, 0) = \frac{1}{2}x(1-x)$ and $u_t(x, 0) = 0$, taking $h = k = 0.1$ for $0 \leq t \leq 0.4$. Compare your solution with the exact solution at $x = 0.5$ and $t = 0.3$. (V.T.U., 2000)
2. The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $\partial^2 u / \partial t^2 = 25 \partial^2 u / \partial x^2$, with the boundary conditions $u(0, t) = u(5, t) = 0$ and the initial conditions $u(x, 0) = \begin{cases} 20x & \text{for } 0 \leq x \leq 1 \\ 5(5-x) & \text{for } 1 \leq x \leq 5 \end{cases}$ and $u_t(x, 0) = 0$. Solve this equation numerically for one half period of vibration, taking $h = 1$, $k = 0.2$.
3. Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0, t) = 0$, $y(1, t) = 0$, $y_t(x, 0) = 0$, and $y(x, 0) = 10 + x(1-x)$. (Anna, 2004)
4. The function u satisfies the equation

$$\partial^2 u / \partial t^2 = \partial^2 u / \partial x^2$$

and the conditions : $u(x, 0) = \frac{1}{8} \sin \pi x$, $u_t(x, 0) = 0$ for $0 \leq x \leq 1$,

$$u(0, t) = u(1, t) = 0 \text{ for } t \geq 0.$$

Use the explicit scheme to calculate u for $x = 0(0.1) 1$ and $t = 0(0.1) 0.5$.

33.11 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 33.5

Fill up the blanks or select the correct answer from each of the following questions :

1. Which of the following equations is parabolic :
 (a) $f_{xy} - f_x = 0$ (b) $f_{xx} + 2f_{xy} + f_{yy} = 0$ (c) $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ (d) none
2. $u_y = \frac{1}{4} (u_{i+1,j} - u_{i-1,j} + u_{i,j+1} - u_{i,j-1})$ is Leibmann's five point formula. (True or False)
3. $u_{xx} + 3u_{xy} + u_{yy} = 0$ is classified as
 4. $\nabla^2 u = f(x, y)$ is known as
5. The simplest formula to solve $u_{tt} = \alpha^2 u_{xx}$ is
 6. The finite difference form of $\partial^2 u / \partial x^2$ is
7. Schmidt's finite difference scheme to solve $u_t = c^2 u_{xx}$ is
 8. The 5-point diagonal formula gives $u_{ij} = \dots$
 9. The partial differential equation $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$ is classified as
10. $u_{i,j+1} = \frac{1}{2} (u_{i+1,j} + u_{i-1,j})$ is called recurrence relation.
11. In terms of difference quotients $4u_{xx} = u_{tt}$ is
 12. Bredre-Schmidt recurrence relation for one dimensional heat equation is
13. The diagonal 5-point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$ is
14. In the parabolic equation $u_t = \alpha^2 u_{xx}$ if $\lambda = k\alpha^2/h^2$, where $k = \Delta t$, and $h = \Delta x$, then explicit method is stable if $\lambda = \dots$
15. $2 \frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial x^2} = 0$ is classified as (P.T.U., 2007)
16. The boundary conditions of one-dimensional wave equation are
17. The explicit formula for one-dimensional wave equation with $1 - \lambda^2 \alpha^2 = 0$ and $\lambda = k/h$ is
18. The general form of Poisson's equation in partial derivations is
19. If u satisfies Laplace equation and $u = 100$ on the boundary of a square, the value of u at an interior grid point is
20. The Laplace equation $u_{xx} + u_{yy} = 0$ in difference quotients is
21. The equation $yu_{xx} + u_{yy} = 0$ is hyperbolic in the region
22. To solve $\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$ by Bredre-Schmidt method with $h = 1$, the value of k is

Linear Programming

1. Introduction.
2. Formulation of the problem.
3. Graphical method.
4. Some exceptional cases.
5. General linear programming problem.
6. Canonical and standard forms of L.P.P.
7. Simplex method.
8. Working procedure of the simplex method.
9. Artificial variable techniques—M-method, Two phase method.
10. Exceptional cases—Degeneracy.
11. Duality concept.
12. Duality principle.
13. Dual simplex method.
14. Transportation problem.
15. Working procedure for transportation problems.
16. Degeneracy in transportation problems.
17. Assignment problem.
18. Working procedure to solve assignment problems.
19. Objective Type of Questions.

34.1 INTRODUCTION

Linear programming deals with the optimization (maximization or minimization) of linear functions subject to linear constraints. This technique has found its applications to important areas of product mix, blending problems and diet problems. Oil refineries, chemical industries, steel industries and food processing industry are also using linear programming with considerable success.

In this chapter, our purpose is to present the principles of linear programming and the techniques of its application in a manner that will suit the engineering students. Beginning with the graphical method which provides a great deal of insight into the basic concepts, the simplex method of solving linear programming problems is developed. Then the reader is introduced to the Duality concept. Finally a special class of linear programming problems namely : Transportation and Assignment problems, is taken up. For a detailed study, the student should refer to author's book '*Numerical Methods in Engineering and Science*'.

34.2 FORMULATION OF THE PROBLEM

To begin with, a problem is to be presented in a linear programming form which requires defining the variables involved, establishing relationships between them and formulating the objective function and the constraints. We illustrate this through a few examples.

Example 34.1. A manufacturer produces two types of models M_1 and M_2 . Each M_1 model requires 4 hours of grinding and 2 hours of polishing ; whereas each M_2 model requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is ₹ 3 and on an M_2 model is ₹ 4. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week.

Solution. Let x_1 be the number of M_1 models and x_2 , the number of M_2 models produced per week. Then the weekly profit (in ₹) is

$$Z = 3x_1 + 4x_2 \quad \dots(i)$$

To produce these number of models, the total number of grinding hours needed per week

$$= 4x_1 + 2x_2$$

and the total number of polishing hours required per week

$$= 2x_1 + 5x_2$$

Since the number of grinding hours available is not more than 80 and the number of polishing hours is not more than 180, therefore,

$$4x_1 + 2x_2 \leq 80 \quad \dots(ii)$$

$$2x_1 + 5x_2 \leq 180 \quad \dots(iii)$$

Also since the negative number of models are not produced, obviously we must have

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \quad \dots(iv)$$

Hence this allocation problem is, to find x_1, x_2 which

maximize $Z = 3x_1 + 4x_2$

subject to $4x_1 + 2x_2 \leq 80, 2x_1 + 5x_2 \leq 180, x_1, x_2 \geq 0.$

Obs. The variables that enter into the problem are called **decision variables**.

The expression (i) showing the relationship between the manufacturer's goal and the decision variables, is called the **objective function**.

The inequalities (ii), (iii) and (iv) are called the **constraints**.

The objective function and the constraints being all linear, it is a *linear programming problem (L.P.P.)*. This is an example of a real situation from industry.

Example 34.2. A firm making castings uses electric furnace to melt iron with the following specifications :

	Minimum	Maximum
Carbon	3.20%	3.40%
Silicon	2.25%	2.35%

Specifications and costs of various raw materials used for this purpose are given below :

Material	Carbon %	Silicon %	Cost (₹)
Steel scrap	0.4	0.15	850 /tonne
Cast iron scrap	3.80	2.40	900 /tonne
Remelt from foundry	3.50	2.30	500 /tonne

If the total charge of iron metal required is 4 tonnes, find the weight in kg of each raw material that must be used in the optimal mix at minimum cost. (J.N.T.U., 1999 S)

Solution. Let x_1, x_2, x_3 be the amounts (in kg) of these raw materials. The objective is to minimize the cost i.e.,

$$\text{minimize } Z = \frac{850}{1000} x_1 + \frac{900}{1000} x_2 + \frac{500}{1000} x_3 \quad \dots(i)$$

For iron melt to have a minimum of 3.2% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \geq 3.2 \times 4,000 \quad \dots(ii)$$

For iron melt to have a maximum of 3.4% carbon,

$$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 3.4 \times 4,000 \quad \dots(iii)$$

For iron melt to have a minimum of 2.25% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \geq 2.25 \times 4,000 \quad \dots(iv)$$

For iron melt to have a maximum of 2.35% silicon,

$$0.15x_1 + 2.41x_2 + 2.35x_3 \leq 2.35 \times 4,000 \quad \dots(v)$$

Also, since the materials added up must be equal to the full charge weight of 4 tonnes.

$$\therefore x_1 + x_2 + x_3 = 4,000 \quad \dots(vi)$$

Finally since the amounts of raw material cannot be negative

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \quad \dots(vii)$$

Thus the linear programming problem is to find x_1, x_2, x_3 which

minimize $Z = 0.85x_1 + 0.9x_2 + 0.5x_3$

subject to $0.4x_1 + 3.8x_2 + 3.5x_3 \geq 12,800$

$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 13,600$

$$\begin{aligned}0.15x_1 + 2.41x_2 + 2.35x_3 &\geq 9,000 \\0.15x_1 + 2.41x_2 + 2.35x_3 &\leq 9,400 \\x_1 + x_2 + x_3 &= 4,000 \\x_1, x_2, x_3 &\geq 0.\end{aligned}$$

PROBLEMS 34.1

1. A firm manufactures two items. It purchases castings which are then machined, bored and polished. Castings for items *A* and *B* cost ₹ 3 and ₹ 4 each and are sold at ₹ 6 and ₹ 7 each respectively. Running costs of these machines are ₹ 20, ₹ 14 and ₹ 17.50 per hour respectively. Formulate the problem so that the product mix maximizes the profit? Capacities of the machines are

	Part A	Part B
<i>Machining capacity</i>	25 per hr.	40 per hr.
<i>Boring capacity</i>	28 per hr.	35 per hr.
<i>Polishing capacity</i>	35 per hr.	25 per hr.

2. A firm manufactures 3 products *A*, *B* and *C*. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has two machines *M*₁ and *M*₂ and below is the required processing time in minutes for each machine on each product.

	Product			
	A	B	C	
<i>Machine</i>	$\left\{ \begin{array}{l} M_1 \\ M_2 \end{array} \right.$	4	3	5
		2	2	4

Machines *M*₁ and *M*₂ have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 *A*'s, 200 *B*'s and 50 *C*'s but not more than 150 *A*'s. Set up an L.P.P. to maximize profit. (Kurukshetra, 2009 S)

3. Three products are processed through three different operations. The time (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold for each product (in rupees) are as follows:

<i>Operation</i>	<i>Time per unit</i>			<i>Operation capacity</i>
	<i>Product I</i>	<i>Product II</i>	<i>Product III</i>	
1	3	4	3	42
2	5	0	3	45
3	3	6	2	41
<i>Profit (₹)</i>	3	2	1	

The zero time indicates that the product does not require the given operation. The problem is to determine the optimum daily production for three products that maximize the profit. Formulate this production planning problem as a linear programming problem assuming that all units produced are sold.

4. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. How many tickets of each class must be sold in order to maximize profit for the airline? Formulate the problem as an L.P. model.

(Rohtak, 2006)

5. A firm manufactures headache pills in two sizes *A* and *B*. Size *A* contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size *B* contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard L.P.P.

6. Consider the following problem faced by a production planner in a soft-drink plant. He has two bottling machines *A* and *B*. *A* is designed for 8-ounce bottles and *B* for 16-ounce bottles. However, each be can be used on both types with some loss of efficiency. The following data is available:

<i>Machine</i>	<i>8-ounce bottles</i>	<i>16-ounce bottles</i>
A	100/minute	40/minute
B	60/minute	75/minute

The machines can be run 8 hours per day, 5 days per week. Profit on a 8-ounce bottle is 15 paise and on a 16-ounce bottle is 25 paise. Weekly production of the drink cannot exceed 300,000 ounces and the market can absorb 25,000

8-ounce bottles and 7,000 16-ounce bottles per week. The planner wishes to maximise his profit subject, of course, to all the production and marketing restrictions. Formulate this as a L.P.P.

7. A dairy feed company may purchase and mix one or more of three types of grains containing different amounts of nutritional elements. The data is given in the table below. The production manager specifies that any feed mix for his live stock must meet at least minimum nutritional requirements and seeks the least costly among all three mixes.

Item	One unit weight of			Minimum requirement
	Grain 1	Grain 2	Grain 3	
<i>Nutritional ingredients</i>	A	2	3	7
	B	1	1	0
	C	5	3	0
	D	6	25	1
<i>Cost per weight of</i>	41	35	96	

Formulate the problem as a L.P. model.

8. A firm produces an alloy with the following specifications:

(i) specific gravity ≤ 0.97 ; (ii) chromium content $\geq 15\%$; (iii) melting temperature $\geq 494^{\circ}\text{C}$

The alloy requires three raw materials A, B and C whose properties are as follows:

Property	Properties of raw material		
	A	B	C
Sp. gravity	0.94	1.00	1.05
Chromium	10%	15%	17%
Melting pt.	470°C	500°C	520°C

Find the values of A, B, C to be used to make 1 tonne of alloy of desired properties, keeping the raw material costs at the minimum when they are ₹ 105/tonne for A, ₹ 245/tonne for B and ₹ 165/tonne for C. Formulate an L.P. model for the problem.

34.3 GRAPHICAL METHOD

Linear programming problems involving only two variables can be effectively solved by a graphical technique which provides a pictorial representation of the solution and one gets insight into the basic concepts used in solving large L.P.P.

Working procedure to solve a linear programming problem graphically:

Step 1. Formulate the given problem as a linear programming problem.

Step 2. Plot the given constraints as equalities on $x_1 - x_2$ coordinate plane and determine the convex region* formed by them.

Step 3. Determine the vertices of the convex region and find the value of the objective function at each vertex. The vertex which gives the optimal (maximum or minimum) value of the objective function gives the desired optimal solution to the problem.

Otherwise. Draw the dotted line through the origin representing the objective function with $Z = 0$. As Z is increased from zero, this line moves to the right remaining parallel to itself. We go on sliding this line (parallel to itself), till it is *farthest* away from the origin and passes through only one vertex of the convex region. This is the vertex where the maximum value of Z is attained.

* A region or a set of points is said to be **convex** if the line joining any two of its points lies completely in the region (or the set). Figs. 34.1 and 34.2 represent convex regions while Figs. 34.3 and 34.4 do not form convex sets.



Fig. 34.1

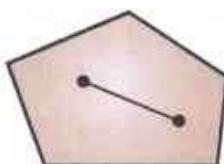


Fig. 34.2

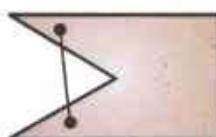


Fig. 34.3

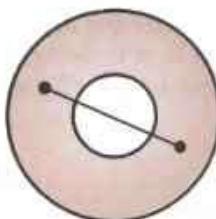


Fig. 34.4

When it is required to minimize Z , value of Z is increased till the dotted line passes through the nearest vertex of the convex region.

Example 34.3. Solve the L.P.P. of Ex. 34.1 graphically.

(V.T.U., 2003)

Solution. The problem is:

$$\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 4x_2 \\ \text{subject to} & 4x_1 + 2x_2 \leq 80 \\ & 2x_1 + 5x_2 \leq 180 \\ & x_1, x_2 \geq 0 \end{array} \quad \dots(i) \quad \dots(ii) \quad \dots(iii) \quad \dots(iv)$$

Consider $x_1 - x_2$ coordinate system as shown in Fig. 34.5. The non-negativity restrictions (iv) imply that the values of x_1, x_2 lie in the first quadrant only.

We plot the lines $4x_1 + 2x_2 = 80$ and $2x_1 + 5x_2 = 180$.

Then any point on or below $4x_1 + 2x_2 = 80$ satisfies (ii) and any point on or below $2x_1 + 5x_2 = 180$ satisfies (iii). This shows that the desired point (x_1, x_2) must be somewhere in the shaded convex region $OABC$. This region is called the *solution space* or *region of feasible solutions* for the given problem. Its vertices are $O(0, 0)$, $A(20, 0)$, $B(2.5, 35)$ and $C(0, 36)$.

The values of the objective function (i) at these points are

$$Z(O) = 0, Z(A) = 60, Z(B) = 147.5, Z(C) = 144.$$

Thus the maximum value of Z is 147.5 and it occurs at B . Hence the optimal solution to the problem is

$$x_1 = 2.5, x_2 = 35 \text{ and } Z_{\max} = 147.5.$$

Otherwise. Our aim is to find the point (or points) in the solution space which maximizes the profit function Z . To do this, we observe that on making $Z = 0$, (i) becomes $3x_1 + 4x_2 = 0$ which is represented by the dotted line LM through O . As the value of Z is increased, the line LM starts moving parallel to itself towards the right. Larger the value of Z , more will be the company's profit. In this way, we go on sliding LM till it is farthest away from the origin and passes through one of the corners of the convex region. This is the point where the maximum value of Z is attained. Just possible, such a line may be one of the edges of the solution space. In that case every point on that edge gives the same maximum value of Z .

Here Z_{\max} is attained at $B(2.5, 35)$. Hence the optimal solution is $x_1 = 2.5, x_2 = 35$ and $Z_{\max} = 147.5$.

Example 34.4. Find the maximum value of $Z = 2x + 3y$ subject to the constraints: $x + y \leq 30$, $y \geq 3$, $0 \leq y \leq 12$, $x - y \geq 0$, and $0 \leq x \leq 20$. (Rohtak, 2006)

Solution. Any point (x, y) satisfying the conditions $x \geq 0, y \geq 0$ lies in the first quadrant only. Also since $x + y \leq 30$, $y \geq 3$, $y \leq 12$, $x \geq y$ and $x \leq 20$, the desired point (x, y) lies within the convex region $ABCDE$ (shown shaded in Fig. 34.6). Its vertices are $A(3, 3)$, $B(20, 3)$, $C(20, 10)$, $D(18, 12)$, and $E(12, 12)$.

The values of Z at these five vertices are $Z(A) = 15$, $Z(B) = 49$, $Z(C) = 70$, $Z(D) = 72$, and $Z(E) = 60$.

Since the maximum value of Z is 72 which occurs at the vertex D , the solution to the L.P.P. is $x = 18, y = 12$ and maximum $Z = 72$.

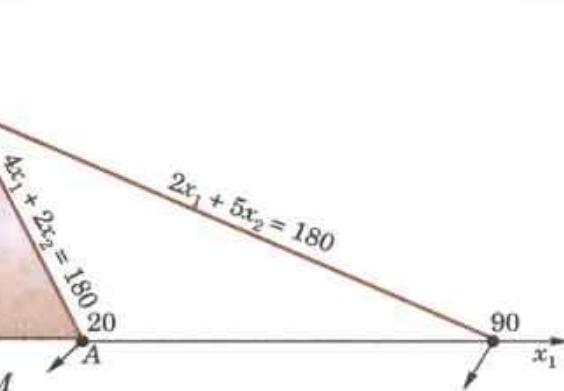


Fig. 34.5

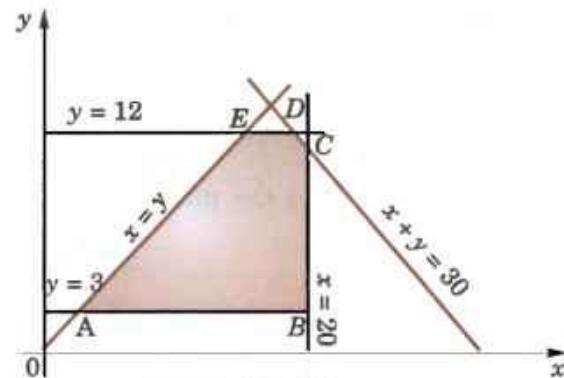


Fig. 34.6

Example 34.5. A company manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth requires 4 oz of red wool, 5 oz of green wool and 3 oz of yellow wool. One yard length of type B cloth requires 5 oz of red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacturer is 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool. The manufacturer can make a profit of ₹ 5 on one yard of type A cloth and ₹ 3 on one yard of type B cloth. Find the best combination of the quantities of type A and type B cloth which gives him maximum profit by solving the L.P.P. graphically.

Solution. Let the manufacturer decide to produce x_1 yards of type A cloth and x_2 yards of type B cloth. Then the total income in rupees, from these units of cloth is given by

$$Z = 5x_1 + 3x_2 \quad \dots(i)$$

To produce these units of two types of cloth, he requires

$$\text{red wool} = 4x_1 + 5x_2 \text{ oz},$$

$$\text{green wool} = 5x_1 + 2x_2 \text{ oz},$$

and

$$\text{yellow wool} = 3x_1 + 8x_2 \text{ oz}.$$

Since the manufacturer does not have more than 1000 oz of red wool, 1000 oz of green wool and 1200 oz of yellow wool, therefore

$$4x_1 + 5x_2 \leq 1000 \quad \dots(ii)$$

$$5x_1 + 2x_2 \leq 1000 \quad \dots(iii)$$

$$3x_1 + 8x_2 \leq 1200 \quad \dots(iv)$$

$$\text{Also } x_1 \geq 0, x_2 \geq 0. \quad \dots(v)$$

Thus the given problem is to maximize Z subject to the constraints (ii) to (v). (V.T.U., 2004)

Any point satisfying the condition (v) lies in the first quadrant only. Also the desired point satisfying the constraints (ii) to (iv) lies in the convex region $OABCD$ (Fig. 34.7). Its vertices are $O(0, 0)$, $A(200, 0)$, $B(3000/17, 1000/17)$, $C(2000/17, 1800/17)$ and $D(0, 150)$.

The values of Z at these vertices are given by $Z(O) = 0$, $Z(A) = 1000$, $Z(B) = 1057.6$, $Z(C) = 905.8$ and $Z(D) = 450$.

Since the maximum value of Z is 1058.8 which occurs at the vertex B , the solution to the given problem is $x_1 = 3000/17$, $x_2 = 1000/17$ and max. $Z = 1058.8$.

Hence the manufacturer should produce 176.5 yards of type A cloth, 58.8 yards of type B cloth, so as to get the maximum profit of ₹ 1058.8.

Example 34.6. A company making cold drinks has two bottling plants located at towns T_1 and T_2 . Each plant produces three drinks A, B and C and their production capacity per day is shown below:

Cold drinks	Plant at	
	T_1	T_2
A	6,000	2,000
B	1,000	2,500
C	3,000	3,000

The marketing department of the company forecasts a demand of 80,000 bottles of A, 22,000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 and T_2 are ₹ 6,000 and ₹ 4,000 respectively. Find (graphically) the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Solution. Let the plants at T_1 and T_2 be run for x_1 and x_2 days. Then the objective is to minimize the operation costs i.e.,

$$\min. Z = 6000x_1 + 4000x_2 \quad \dots(i)$$

Constraints on the demand for the three cold drinks are:

$$\text{for } A, 6,000x_1 + 2,000x_2 \geq 80,000 \text{ or } 3x_1 + x_2 \geq 40 \quad \dots(ii)$$

$$\text{for } B, 1,000x_1 + 2,500x_2 \geq 22,000 \text{ or } x_1 + 2.5x_2 \geq 22 \quad \dots(iii)$$

$$\text{for } C, 3,000x_1 + 3,000x_2 \geq 40,000 \text{ or } x_1 + x_2 \geq 40/3 \quad \dots(iv)$$

$$\text{Also } x_1, x_2 \geq 0 \quad \dots(v)$$

Thus the L.P.P. is to minimize (i) subject to constraints (ii) to (v). (V.T.U., 2000 S)

The solution space satisfying the constraints (ii) to (v) is shown shaded in Fig. 34.8. As seen from the direction of the arrows, the solution space is unbounded. The constraint (iv) is dominated by the constraints (ii) and (iii) and hence does not affect the solution space. Such a constraint as (iv) is called the *redundant constraint*.

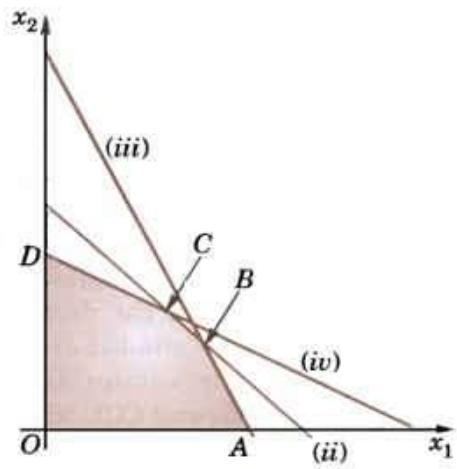


Fig. 34.7

The vertices of the convex region ABC are $A(22, 0)$, $B(12, 4)$ and $C(0, 40)$.

Values of the objective function (i) at these vertices are

$$Z(A) = 132,000, Z(B) = 88,000, Z(C) = 160,000.$$

Thus the minimum value of Z is ₹ 88,000 and it occurs at B . Hence the solution to the problem is $x_1 = 12$ days, $x_2 = 4$ days, $Z_{\min} = ₹ 88,000$.

Otherwise. Making $Z = 0$, (i) becomes $3x_1 + 2x_2 = 0$ which is represented by the dotted line LM through O . As Z is increased, the line LM moves parallel to itself, to the right. Since we are interested in finding the minimum value of Z , value of Z is increased till LM passes through the vertex nearest to the origin of the shaded region, i.e. $B(12, 4)$.

Thus the operating cost will be minimum for $x_1 = 12$ days, $x_2 = 4$ days and

$$Z_{\min} = 6000 \times 12 + 4000 \times 4 = ₹ 88,000.$$

Obs. The dotted line parallel to the line LM is called the *iso-cost line* since it represents all possible combinations of x_1, x_2 which produce the same total cost.

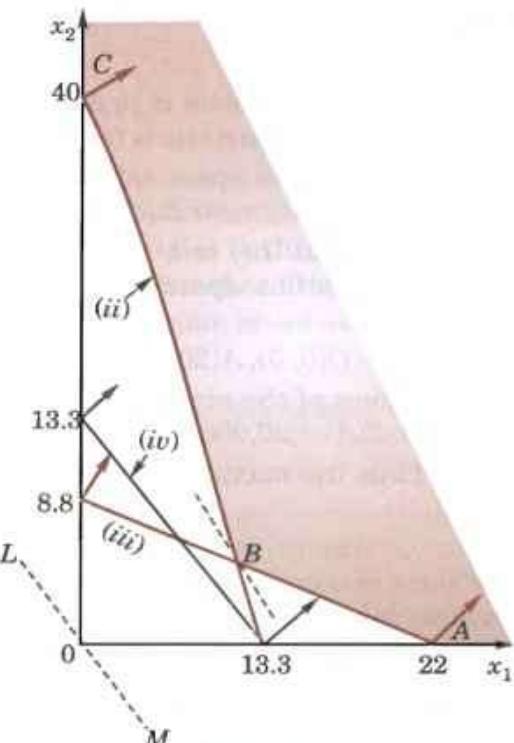


Fig. 34.8

34.4 SOME EXCEPTIONAL CASES

The constraints generally, give region of feasible solution which may be bounded or unbounded. In problems involving two variables and having a finite solution, we observed that the optimal solution existed at a vertex of the feasible region. In fact, this is true for all L.P. problems for which solutions exist. Thus it may be stated that *if there exists an optimal solution of an L.P.P., it will be at one of the vertices of the solution space.*

In each of the above examples, the optimal solution was unique. But it is not always so. In fact, L.P.P. may have

(i) a unique optimal solution, or (ii) an infinite number of optimal solutions, or (iii) an unbounded solution, or (iv) no solution.

We now give below a few examples to illustrate the exceptional cases (ii) to (iv).

Example 34.7. A firm uses milling machines, grinding machines and lathes to produce two motor parts. The machining times required for each part, the machining times available on different machines and the profit on each motor part are given below:

Type of machine	Machining time reqd. for the motor part (mts)		Max. time available per week (minutes)
	I	II	
Milling machines	10	4	2,000
Grinding machines	3	2	900
Lathes	6	12	3,000
Profit/unit (₹)	100	40	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

Solution. Let x_1, x_2 be the number of parts I and II manufactured per week. Then objective being to maximize the profit, we have maximize $Z = 100x_1 + 40x_2$... (i)

Constraints being on the time available on each machine, we obtain

$$\text{for milling machines, } 10x_1 + 4x_2 \leq 2,000 \quad \dots(ii)$$

$$\text{for grinding machines, } 3x_1 + 2x_2 \leq 900 \quad \dots(iii)$$

for lathes,

$$6x_1 + 12x_2 \leq 3,000 \quad \dots(iv)$$

Also

$$x_1, x_2 \geq 0 \quad \dots(v)$$

Thus the problem is to determine x_1, x_2 which maximize (i) subject to the constraints (ii) to (v).

The solution space satisfying (ii), (iii), (iv) and meeting the non-negativity restrictions (v) is shown shaded in Fig. 34.9.

Note that (iii) is a redundant constraint as it does not affect the solution space. The vertices of the convex region $OABC$ are

$$O(0, 0), A(200, 0), B(125, 187.5), C(0, 250).$$

Values of the objective function (i) at these vertices are $Z(O) = 0$, $Z(A) = 20,000$, $Z(B) = 20,000$ and $Z(C) = 10,000$.

Thus the maximum value of Z occurs at two vertices A and B .

\therefore Any point on the line joining A and B will also give the same maximum value of Z i.e., there are infinite number of feasible solutions which yield the same maximum value of Z .

Thus there is no unique optimal solution to the problem and any point on the line AB can be taken to give the profit of ₹ 20,000.

Obs. An L.P.P. having more than one optimal solution, is said to have alternative or multiple optimal solutions. It implies that the resources can be combined in more than one way to maximize the profit.

Example 34.8. Using graphical method, solve the following L.P.P.:

$$\text{Maximize} \quad Z = 2x_1 + 3x_2 \quad \dots(i)$$

$$\text{subject to} \quad x_1 - x_2 \leq 2 \quad \dots(ii)$$

$$x_1 + x_2 \geq 4 \quad \dots(iii)$$

$$x_1, x_2 \geq 0. \quad \dots(iv)$$

(Kurukshetra, 2005; V.T.U., 2003 S) ... (iv)

Solution. Consider $x_1 - x_2$ coordinate system. Any point (x_1, x_2) satisfying the restrictions (iv) lies in the first quadrant only. The solution space satisfying the constraints (ii) and (iii) is the convex region shown shaded in Fig. 34.10.

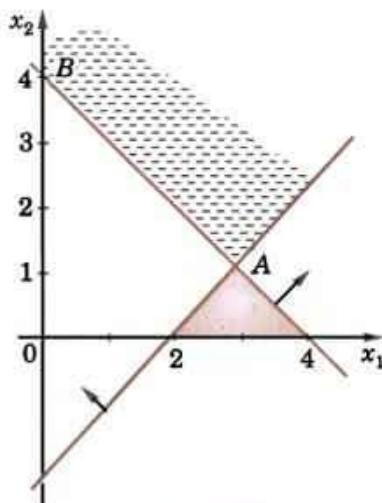


Fig. 34.10

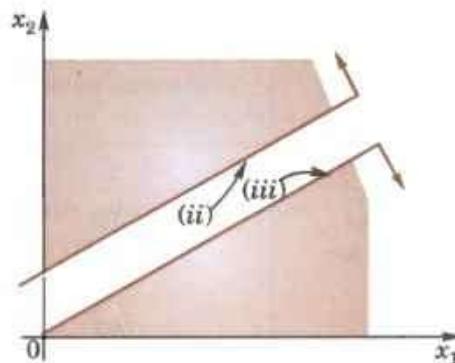


Fig. 34.11

Here the solution space is unbounded. The vertices of the feasible region (in the finite plane) are $A(3, 1)$ and $B(0, 4)$.

Values of the objective function (i) at these vertices are $Z(A) = 9$ and $Z(B) = 12$.

But there are points in this convex region for which Z will have much higher values. For instance, the point $(5, 5)$ lies in the shaded region and the value of Z thereat is 25. In fact, the maximum value of Z occurs at infinity. Thus the problem has an unbounded solution.

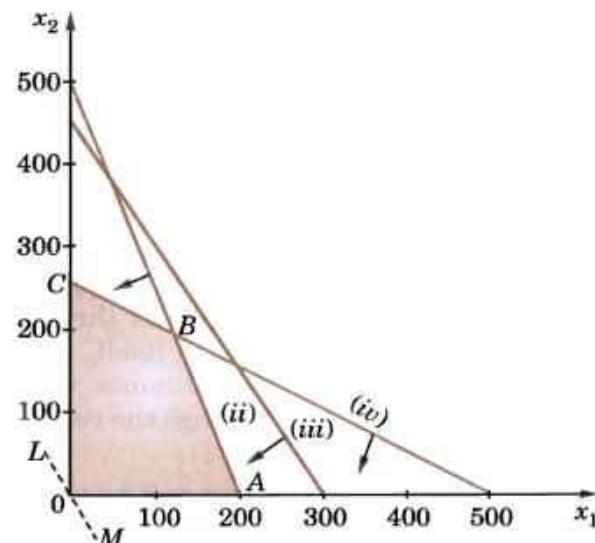


Fig. 34.9

Example 34.9. Solve graphically the following L.P.P.:

$$\begin{array}{ll} \text{Maximize} & Z = 4x_1 + 3x_2 \quad \dots(i) \\ \text{subject to} & x_1 - x_2 \leq -1, \quad \dots(ii) \\ & -x_1 + x_2 \leq 0 \quad \dots(iii) \\ \text{and} & x_1, x_2 \geq 0 \quad \dots(iv) \end{array}$$

Solution. Consider $x_1 - x_2$ coordinate system. Any point (x_1, x_2) satisfying (iv) lies in the first quadrant only. The two solution spaces, one satisfying (ii) and the other satisfying (iii) are shown shaded in Fig. 34.11.

There being no point (x_1, x_2) common to both the shaded regions, the problem cannot be solved. Hence the solution does not exist since the constraints are inconsistent.

Obs. The above problem had no solution because the constraints were incompatible. There may be cases in which the constraints are compatible but the problem may still have no feasible solution.

PROBLEMS 34.2

Using graphical method, solve the following L.P. problems:

1. Max. $Z = 3x_1 + 5x_2$
subject to $x_1 + 2x_2 \leq 200$, $x_1 + x_2 \leq 150$, $x_1 \leq 60$, $x_1, x_2 \geq 0$ (Rajasthan, 2003)

2. Max. $Z = 5x_1 + 7x_2$
subject to $x_1 + x_2 \leq 4$, $5x_1 + 8x_2 \leq 24$, $10x_1 + 7x_2 \leq 35$, and $x_1, x_2 \geq 0$.

3. Min. $Z = 20x_1 + 30x_2$
subject to $x_1 + 2x_2 \leq 40$, $3x_1 + x_2 \geq 30$, $4x_1 + 3x_2 \geq 60$, $x_1, x_2 \geq 0$.

(Kurukshetra, 2009 S; Mumbai, 2004; V.T.U., 2004)

4. Max. $Z = 3x + 5x_2$ subject to $x_1 + 2x_2 \leq 2000$, $x_1 + x_2 \leq 1500$, $x_2 \leq 600$ and $x_1 \geq 0$, $x_2 \geq 0$. (Rohtak, 2004)

5. A firm manufactures two products *A* and *B* on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M_1 and M_2 . Product *A* requires one minute of processing time on M_1 and 2 minutes on M_2 while *B* requires one minute on M_1 and one minute on M_2 . Machine M_1 is available for not more than 7 hours and 30 minutes while M_2 is available for 10 hours during any working day. Find the number of units of products *A* and *B* to be manufactured to get maximum profit.

6. Two spare parts *X* and *Y* are to be produced in a batch. Each one has to go through two processes *A* and *B*. The time required in hours per unit and total time available are given below:

	<i>X</i>	<i>Y</i>	Total hours available
Process <i>A</i>	3	4	24
Process <i>B</i>	9	4	36

Profits per unit of *X* and *Y* are ₹ 5 and ₹ 6 respectively. Find how many number of spare parts of *X* and *Y* are to be produced in this batch to maximize the profit. (Each batch is complete in all respects and one cannot produce fractional units and stop the batch).

7. A manufacturer has two products I and II both of which are made in steps by machines *A* and *B*. The process times per hundred for the two products on the two machines are:

Product	<i>M/c. A</i>	<i>M/c. B</i>
I	4 hrs.	5 hrs.
II	5 hrs.	2 hrs.

Set-up times are negligible. For the coming period machine *A* has 100 hrs, and *B* has 80 hrs. The contribution for product I is ₹ 10 per 100 units and for product II is ₹ 5 per 100 units. The manufacturer is in a market which can absorb both products as much as he can produce for the immediate period ahead. Determine graphically, how much of products I and II, he should produce to maximize his contribution.

8. A production manager wants to determine the quantity to be produced per month of products *A* and *B* manufactured by his firm. The data on resources required and availability of resources are given below:

Resources	Requirements		Available per month
	Product A	Product B	
Raw material (kg.)	60	120	12,000
Machine hrs/piece	8	5	600
Assembly man hrs.	3	4	500
Sale price/piece	₹ 30	₹ 40	

Formulate the problem as a standard L.P.P. Find product mix that would give maximum profit by graphical technique.

9. A pineapple firm produces two products: canned pineapple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

	Canned Juice	Pineapple	Available Resources
Labour (man hrs)	3	2.0	12.0
Equipment (m/c hrs)	1	2.3	6.9
Material (unit)	1	1.4	4.9

Assuming one unit each of canned juice and canned pineapple has profit margins of ₹ 2 and ₹ 1 respectively. Formulate this as L.P. problem and solve it graphically.

Solve the following L.P. problems graphically:

10. Maximize $Z = 6x + 4y$ subject to $2x + y \geq 1$, $3x + 4y \geq 1.5$ and $x, y \geq 0$. (Bombay, 2004)
11. Minimize $Z = 8x_1 + 12x_2$ subject to $60x_1 + 30x_2 \geq 240$, $30x_1 + 60x_2 \geq 300$, $30x_1 + 180x_2 \geq 540$, and $x_1, x_2 \geq 0$.
12. G.J. Breweries Ltd. have two bottling plants one located at 'G' and other at 'J'. Each plant produces three drinks: whiskey, beer and brandy. The number of bottles produced per day are as follows:

Drink	Plant at G	Plant at J
Whiskey	1500	1500
Beer	3000	1000
Brandy	2000	5000

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whiskey, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at G and J are ₹ 600 and ₹ 400. For how many days each plant be run in July so as to minimize the production cost, while still meeting the market demand. Solve graphically.

34.5 GENERAL LINEAR PROGRAMMING PROBLEM

Any L.P.P. problem involving more than two variables may be expressed as follows:

Find the values of the variables x_1, x_2, \dots, x_n which maximize (or minimize) the objective function

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \dots(i)$$

subject to the constraints

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \end{array} \right\} \quad \dots(ii)$$

and meet the non-negativity restrictions

$$x_1, x_2, \dots, x_n \geq 0. \quad \dots(iii)$$

Def. 1. A set of values x_1, x_2, \dots, x_n which satisfies the constraints of the L.P.P. is called its **solution**.

Def. 2. Any solution to a L.P.P. which satisfies the non-negativity restrictions of the problem is called its **feasible solution**.

Def. 3. Any feasible solution which maximizes (or minimizes) the objective function of the L.P.P. is called its **optimal solution**.

Some of the constraints in (ii) may be equalities, some others may be inequalities of (\leq) type and remaining ones inequalities of (\geq) type. The inequality constraints are changed to equalities by adding (or subtracting) non-negative variables to (from) the left hand side of such constraints.

Def. 4. If the constraints of a general L.P.P. be

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad (i = 1, 2, \dots, k) \text{ then the non-negative variables } s_i \text{ which satisfy}$$

$\sum_{j=1}^n a_{ij}x_j + s_i = b_i$ ($i = 1, 2, \dots, k$), are called **slack variables**.

Def. 5. If the constraints of a general L.P.P. be

$\sum_{j=1}^n a_{ij}x_j \geq b_i$, ($i = k, k+1, \dots$) then the non-negative variables s_i which satisfy

$\sum_{j=1}^n a_{ij}x_j - s_i = b_i$, ($i = k, k+1, \dots$), are called **surplus variables**.

34.6 CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of L.P.P., the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable form. As such, we explain its following two forms:

(1) Canonical form. The general L.P.P. can always be expressed in the following form:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i$; $i = 1, 2, \dots, m$

$$x_1, x_2, \dots, x_n \geq 0,$$

by making some elementary transformations. This form of the L.P.P. is called its **canonical form** and has the following characteristics:

(i) Objective function is of maximization type,

(ii) All constraints are of (\leq) type,

(iii) All variables x_i are non-negative.

The canonical form is a format for a L.P.P. which finds its use in the Duality theory.

(2) Standard form. The general L.P.P. can also be put in the following form:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to the constraints $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$; $i = 1, 2, \dots, m$

$$x_1, x_2, \dots, x_n \geq 0,$$

This form of the L.P.P. is called its **standard form** and has the following characteristics:

(i) Objective function is of maximization type;

(ii) All constraints are expressed as equations;

(iii) Right hand side of each constraint is non-negative;

(iv) All variables are non-negative.

Obs. Any L.P.P. can be expressed in the standard form.

$$\text{As minimize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

is equivalent to maximize $Z' (= -Z) = -c_1x_1 - c_2x_2 - \dots - c_nx_n$,

the objective function can always be expressed in the maximization form.

The inequality constraints can always be converted to equalities by adding (or subtracting) the slack (or surplus) variables to the left hand sides of such constraints.

So far, the decision variables x_1, x_2, \dots, x_n have been assumed to be all non-negative. In actual practice, these variables could also be zero or negative. If a variable is negative, it can always be expressed as the difference of two non-negative variables e.g. a variable x_i can be written as

$$x_i = x_i' - x_i'' \quad \text{where } x_i' \geq 0, x_i'' \geq 0.$$

Example 34.10. Convert the following L.P.P. to the standard form:

$$\text{Maximize } Z = 3x_1 + 5x_2 + 7x_3$$

$$\text{subject to } 6x_1 - 4x_2 \leq 5, 3x_1 + 2x_2 + 5x_3 \geq 11, 4x_1 + 3x_3 \leq 2, x_1, x_2 \geq 0.$$

Solution. As x_3 is unrestricted, let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$. Now the given constraints can be expressed as

$$6x_1 - 4x_2 \leq 5,$$

$$3x_1 + 2x_2 + 5x_3' - 5x_3'' \geq 11$$

$$\begin{aligned} 4x_1 + 3x_3' - 3x_3'' &\leq 2 \\ x_1, x_2, x_3', x_3'' &\geq 0. \end{aligned}$$

Introducing the slack/surplus variables, the problem in standard form becomes:

$$\begin{array}{ll} \text{Maximize} & Z = 3x_1 + 5x_2 + 7x_3' - 7x_3'' \\ \text{subject to} & 6x_1 - 4x_2 + s_1 = 5, \\ & 3x_1 + 2x_2 + 5x_3' - 5x_3'' - s_2 = 11, \\ & 4x_1 + 3x_3' - 3x_3'' + s_3 = 2, \\ & x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0. \end{array}$$

Example 34.11. Express the following problem in the standard form:

$$\begin{array}{ll} \text{Minimize} & Z = 3x_1 + 4x_2 \\ \text{subject to} & 2x_1 - x_2 - 3x_3 = -4, \quad 3x_1 + 5x_2 + x_4 = 10, \quad x_1 - 4x_2 = 12, \quad x_1, x_3, x_4 \geq 0. \end{array}$$

Solution. Here x_3, x_4 are the slack/surplus variables and x_1, x_2 are the decision variables. As x_2 is unrestricted, let $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$.

∴ The problem in standard form is

$$\begin{array}{ll} \text{Maximize} & Z' (= -Z) = -3x_1 - 4x_2' + 4x_2'' \\ \text{subject to} & -2x_1 + x_2' - x_2'' + 3x_3 = 4 \\ & 3x_1 + 5x_2' - 5x_2'' + x_4 = 10 \\ & x_1 - 4x_2' - 4x_2'' = 12 \\ & x_1, x_2', x_2'', x_3, x_4 \geq 0. \end{array}$$

34.7 SIMPLEX METHOD

(1) While solving an L.P.P. graphically, the region of feasible solutions was found to be convex, bounded by vertices and edges joining them. The optimal solution occurred at some vertex. If the optimal solution was not unique, the optimal points were on an edge. These observations also hold true for the general L.P.P. Essentially the problem is that of finding the particular vertex of the convex region which corresponds to the optimal solution. The most commonly used method for locating the optimal vertex is the **simplex method**. This method consists in moving step by step from one vertex to the adjacent one. Of all the adjacent vertices, the one giving better value of the objective function over that of the preceding vertex, is chosen. This method of jumping from one vertex to the other is then repeated. Since the number of vertices is finite, the simplex method leads to an optimal vertex in a finite number of steps.

(2) In simplex method, an infinite number of solutions is reduced to a finite number of promising solutions by using the following facts:

(i) When there are m constraints and $(m + n)$ variables (m being $\leq n$), the starting solution is found by setting n variables equal to zero and then solving the remaining m equations, provided the solution exists and is unique. The **n zero variables are known as non-basic variables** while the remaining m variables are called **basic variables** and they form a **basic solution**.

(ii) In an L.P.P., the variables must always be non-negative. Some of the basic solutions may contain negative variables. Such solutions are called **basic infeasible solutions** and should not be considered. To achieve this, we start with a basic solution which is non-negative. The next basic solution must always be non-negative. This is ensured by feasibility condition. Such a solution is known as **basic feasible solution**.

If all the variables in the basic feasible solution are positive, then it is called **non-degenerate solution** and if some of the variables are zero, it is called **degenerate solution**.

(iii) A new basic feasible solution may be obtained from the previous one by equating one of the basic variables to zero and replacing it by a new non-basic variable. The eliminated variable is called the **outgoing variable** while the new variable is known as the **incoming variable**.

The incoming variable must improve the value of the objective function which is ensured by the optimality condition. This process is repeated till no further improvement is possible. The resulting solution is called the **optimal basic feasible solution** or simply **optimal solution**.

(3) The simplex method is, therefore, based on the following two conditions:

I. Feasibility condition. It ensures that if the starting solution is basic feasible, the subsequent will also be basic feasible.

II. Optimality condition. It ensures that only improved solutions will be obtained.

(4) Now, we shall elaborate the above terms in relation to the general linear programming problem in standard form, i.e.,

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \dots(1)$$

$$\text{subject to} \quad \sum_{j=1}^n a_{ij}x_j + s_i = b_i, \quad i = 1, 2, \dots, m \quad \dots(2)$$

$$\text{and} \quad x_j \geq 0, \quad s_i \geq 0, \quad j = 1, 2, \dots, n \quad \dots(3)$$

(i) **Solution.** x_1, x_2, \dots, x_n is a solution of the general L.P.P. if it satisfies the constraints (2).

(ii) **Feasible solution,** x_1, x_2, \dots, x_n is a feasible solution of the general L.P.P. if it satisfies both the constraints (2) and the non-negativity restrictions (3). The set S of all feasible solutions is called the feasible region. A linear programme is said to be *infeasible* when the set S is empty.

(iii) **Basic solution** is the solution of the m basic variables when each of the n non-basic variables is equated to zero.

(iv) **Basic feasible solution** is that *basic solution* which also satisfies the non-negativity restrictions (3).

(v) **Optimal solution** is that basic feasible solution which also optimizes the objective function (1) while satisfying the conditions (2) and (3).

(vi) **Non-degenerate basic feasible solution** is that basic feasible solution which contains exactly m non-zero basic variables. If any of the basic variables becomes zero, it is called a *degenerate basic feasible solution*.

Example 34.12. Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables; $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$. (Mumbai, 2004 ; V.T.U., 2003 S)

Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic-feasible solution of the system.

Solution. Since there are $m + n = 3$ variables and there are $m = 2$ constraints in this problem, a basic solution can be obtained by setting any one variable equal to zero and then solving the resulting equations. Also the total number of basic solutions = ${}^{m+n}C_m = {}^3C_2 = 3$.

The characteristics of the various basic solutions are as given below:

No. of basic sol.	Basic variables	Nonbasic variables	Values of basic variables	Is the sol. feasible? (Are all $x_j > 0$?)	Is the sol. degenerate?
1.	x_1, x_2	x_3	$2x_1 + x_2 = 11$ $3x_1 + x_2 = 14$ $\therefore x_1 = 3, x_2 = 5$	Yes	No
2.	x_2, x_3	x_1	$x_2 + 4x_3 = 11$ $x_2 + 5x_3 = 14$ $\therefore x_2 = 3, x_3 = -1$	No	Yes
3.	x_1, x_3	x_2	$2x_1 + 4x_3 = 11$ $3x_1 + 5x_3 = 14$ $\therefore x_1 = 1/2, x_3 = 5/2$	Yes	No

The basic feasible solutions are:

(i) $x_1 = 3, x_2 = 5, x_3 = 0$; (ii) $x_1 = 1/2, x_2 = 0, x_3 = 5/2$

which are also non-degenerate basic solutions.

Example 34.13. Find an optimal solution to the following L.P.P. by computing all basic solutions and then finding one that maximizes the objective function:

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8, \quad x_1 - 2x_2 + 6x_3 - 7x_4 = -3, \quad x_1, x_2, x_3, x_4 \geq 0,$$

$$\text{Max. } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

Solution. Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions = ${}^4C_2 = 6$.

The characteristics of the various basic solutions are as given below:

No. of basic sol.	Basic variables	Non-basic variables	Values of basic variables	Is the sol. feasible? (Are all $x_j \geq 0$?)	Value of Z	Is the sol. optimal?
1.	x_1, x_2	$x_3, x_4 = 0$	$2x_1 + 3x_2 = 8$ $x_1 - 2x_2 = -3$ $\therefore x_1 = 1, x_2 = 2$	Yes	8	No
2.	x_1, x_3	$x_2, x_4 = 0$	$2x_1 - x_3 = 8$ $x_1 + 6x_3 = -3$ $\therefore x_1 = -14/13,$ $x_3 = -67/13$	No	—	—
3.	x_1, x_4	$x_2, x_3 = 0$	$2x_1 + 4x_4 = 8$ $x_1 - 7x_4 = -3$ $\therefore x_1 = 22/9,$ $x_4 = 7/9$	Yes	10.3	No
4.	x_2, x_3	$x_1, x_4 = 0$	$3x_2 - x_3 = 8$ $-2x_2 + 6x_3 = -3$ $\therefore x_2 = 45/16,$ $x_3 = 7/16$	Yes	10.2	No
5.	x_2, x_4	$x_1, x_3 = 0$	$3x_2 + 4x_4 = 8$ $-2x_2 - 7x_4 = -3$ $\therefore x_2 = 132/39,$ $x_4 = -7/13$	No	—	—
6.	x_3, x_4	$x_1, x_2 = 0$	$-x_3 + 4x_4 = 8$ $6x_3 - 7x_4 = -3$ $\therefore x_3 = 44/17,$ $x_4 = 45/17$	Yes	28.9	Yes

Hence the optimal basic feasible solution is

$$x_1 = 0, x_2 = 0, x_3 = 44/17, x_4 = 45/17 \text{ and the maximum value of } Z = 28.9.$$

PROBLEMS 34.3

1. Reduce the following problem to the standard form:

Determine $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ so as to

Maximize $Z = 3x_1 + 5x_2 + 8x_3$

subject to the constraints $2x_1 - 5x_2 \leq 6, 3x_1 + 2x_2 + x_3 \geq 5, 3x_1 + 4x_3 \leq 3$.

2. Express the following L.P.P. in the standard form

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to $-5x_1 + 2x_2 \leq 5, 2x_1 + 3x_2 + 4x_3 \geq 7, 2x_1 + 5x_3 \leq 3, x_1, x_2, x_3 \geq 0$.

(Kurukshetra, 2009)

3. Convert the following L.P.P. to standard form:

Maximize $Z = 3x_1 - 2x_2 + 4x_3$

subject to $x_1 + 2x_2 + x_3 \leq 8, 2x_1 - x_2 + x_3 \geq 2, 4x_1 - 2x_2 - 3x_3 = -6, x_1, x_2 \geq 0$.

(Kurukshetra, 2007 S)

4. Obtain all the basic solutions to the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4, 2x_1 + x_2 + 5x_3 = 5.$$

5. Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible:

$$2x_1 + x_2 - x_3 = 2, 3x_1 + 2x_2 + x_3 = 3.$$

(Kurukshetra, 2007 S)

6. Find all the basic solutions to the following problem:

$$\text{Maximize } Z = x_1 + 3x_2 + 3x_3,$$

$$\text{subject to } x_1 + 2x_2 + 3x_3 = 4, 2x_1 + 3x_2 + 5x_3 = 7 \text{ and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Which of the basic solutions are (a) non-degenerate basic feasible, (b) optimal basic feasible?

(Kurushetra, 2009 S; Mumbai, 2003)

34.8 WORKING PROCEDURE OF THE SIMPLEX METHOD

Assuming the existence of an initial basic feasible solution, an optimal solution to any L.P.P. by simplex method is found as follows:

Step 1. (i) Check whether the objective function is to be maximized or minimized.

$$\text{If } Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$$

is to be minimized, then convert it into a problem of maximization, by writing

$$\text{Minimize } Z = \text{Maximize } (-Z)$$

(ii) Check whether all b 's are positive.

If any of the b_i 's is negative, multiply both sides of that constraint by -1 so as to make its right hand side positive.

Step 2. Express the problem in the standard form.

Convert all inequalities of constraints into equations by introducing slack/surplus variables in the constraints giving equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + s_1 + 0s_2 + 0s_3 + \dots = b_1.$$

Step 3. Find an initial basic feasible solution.

If there are m equations involving n unknowns, then assign zero values to any $(n - m)$ of the variables for finding a solution. Starting with a basic solution for which $x_j : j = 1, 2, \dots, (n - m)$ are each zero, find all s_i . If all s_i are ≥ 0 , the basic solution is feasible and non-degenerate. If one or more of the s_i values are zero, then the solution is degenerate.

The above information is conveniently expressed in the following simplex table:

	c_j	c_1	c_2	$c_3 \dots 0$	0	0 ...
c_B	Basis	x_1	x_2	$x_3 \dots s_1$	s_2	$s_3 \dots b$
0	s_1	a_{11}	a_{12}	$a_{13} \dots 1$	0	$0 \dots b_1$
0	s_2	a_{21}	a_{22}	$a_{23} \dots 0$	1	$0 \dots b_2$
0	s_3	a_{31}	a_{32}	$a_{33} \dots 0$	0	$1 \dots b_3$
:	:	:	:	:	:	:
				Body matrix		Unit matrix

[The variables s_1, s_2, s_3 etc. are called *basic variables* and variables x_1, x_2, x_3 etc. are called *non-basic variables*. *Basis* refers to the basic variables $s_1, s_2, s_3 \dots c_j$ row denotes the coefficients of the variables in the objective function, while c_B -column denotes the coefficients of the basic variables only in the objective function. b -column denotes the values of the basic variables while remaining variables will always be zero. The coefficients of x 's (decision variables) in the constraint equations constitute the *body matrix* while coefficients of slack variables constitute the *unit matrix*.]

Step 4. Apply optimality test.

$$\text{Compute } C_j = c_j - Z_j; \text{ where } Z_j = \sum c_B a_{ij}$$

[C_j -row is called *net evaluation row* and indicates the per unit increase in the objective function if the variable heading the column is brought into the solution.]

If all C_j are negative, then the initial basic feasible solution is *optimal*.

If even one C_j is positive, then the current feasible solution is not optimal (*i.e.*, can be improved) and proceed to the next step.

Step 5. (i) Identify the incoming and outgoing variables.

If there are more than one positive C_j , then the *incoming variable* is the one that heads the column containing maximum C_j . The column containing it is known as the *key column* which is shown marked with an

arrow at the bottom. If more than one variable has the same maximum C_j , any of these variables may be selected arbitrarily as the incoming variable.

Now divide the elements under b -column by the corresponding elements of key column and choose the row containing the minimum positive ratio θ . Then replace the corresponding basic variable (by making its value zero). It is termed as the *outgoing variable*. The corresponding row is called the *key row* which is shown marked with an arrow on its right end. The element at the intersection of the key row and key column is called the *key element* which is shown bracketed. If all these ratios are ≤ 0 , the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an *unbounded solution* and no further iteration is required.

(ii) *Iterate towards an optimal solution.*

Drop the outgoing variable and introduce the incoming variable alongwith its associated value under c_B column. Convert the key element to unity by dividing the key row by the key element. Then make all other elements of the key column zero by subtracting proper multiples of key row from the other rows.

[This is nothing but the sweep-out process used to solve the linear equations. The operations performed are called *elementary row operations*.]

Step 6. Go to step 4 and repeat the computational procedure until either an optimal (or an unbounded) solution is obtained.

Example 34.14. Using simplex method

$$\text{Maximize } Z = 5x_1 + 3x_2$$

$$\text{subject to } x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10, 3x_1 + 8x_2 \leq 12, x_1, x_2 \geq 0.$$

(V.T.U., 2003 S)

Solution. Consists of the following steps :

Step 1. Check whether the objective function is to be maximized and all b 's are positive.

The problem being of maximization type and all b 's being ≥ 0 , this step is not necessary.

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3 , the problem in standard form becomes

$$\text{Max. } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2 \quad \dots(i)$$

$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10 \quad \dots(ii)$$

$$3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 = 12 \quad \dots(iii)$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

Step 3. Find an initial basic feasible solution.

There are three equations involving five unknowns and for obtaining a solution, we assign zero values to any two of the variables. We start with a basic solution for which we set $x_1 = 0$ and $x_2 = 0$. (This basic solution corresponds to the origin in the graphical method). Substituting $x_1 = x_2 = 0$ in (i), (ii) and (iii), we get the basic solution

$$s_1 = 2, s_2 = 10, s_3 = 12$$

Since all s_1, s_2, s_3 are positive, the basic solution is also feasible and non-degenerate.

\therefore The basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic) and } s_1 = 2, s_2 = 10, s_3 = 12 \text{ (basic)}$$

\therefore Initial basic feasible solution is given by the following table :

c_j	5	3	0	0	0			
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	(1)	1	1	0	0	2	2/1 ←
0	s_2	5	2	0	1	0	10	10/5
0	s_3	3	8	0	0	1	12	12/3
	$Z_j = \sum c_B a_{ij}$	0	0	0	0	0	0	
	$C_j = c_j - Z_j$	5	3	0	0	0		
		↑						

[For x_1 -column ($j = 1$), $Z_j = \sum c_B a_{i1} = 0(1) + 0(5) + 0(3) = 0$
 and for x_2 -column ($j = 2$), $Z_j = \sum c_B a_{i2} = 0(1) + 0(2) + 0(8) = 0$

Similarly $Z_j(b) = 0(2) + 0(10) + 0(12) = 0.$]

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal (i.e. can be improved) and we proceed to the next step.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_1 is the *incoming variable* as its incremental contribution $C_j (= 5)$ is maximum and the column in which it appears is the *key column* (shown marked by an arrow at the bottom).

Dividing the elements under b -column by the corresponding elements of key-column, we find minimum positive ratio θ is 2 in two rows. We, therefore, arbitrarily choose the row containing s_1 as the *key row* (shown marked by an arrow on its right end). The element at the intersection of key row and the key column i.e., (1), is the *key element*. s_1 is therefore, the *outgoing basic variable* which will now become non-basic.

Having decided that x_1 is to enter the solution, we have tried to find as to what maximum value x_1 could have without violating the constraints. So removing s_1 , the new basis will contain x_1 , s_2 and s_3 as the basic variables.

(ii) Iterate towards the optimal solution.

To transform the initial set of equations with a basic feasible solution into an equivalent set of equations with a different basic feasible solution, we make the key element unity. Here the key element being unity, we retain the key row as it is. Then to make all other elements in key column zero, we subtract proper multiples of key row from the other rows. Here we subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the third row. These become the second and the third rows of the next table. We also change the corresponding value under c_B column from 0 to 5, while replacing s_1 by x_1 under the basis. Thus the *second basic feasible solution* is given by the following table :

c_j	5	3	0	0	0	b	θ
c_B	Basis	x_1	x_2	s_1	s_2	s_3	
5	x_1	1	1	1	0	0	2
0	s_2	0	-3	-5	1	0	0
0	s_3	0	5	-3	0	1	6
$Z_j = \sum c_B a_{ij}$		5	5	5	0	0	10
$C_j = c_j - Z_j$		0	-2	-5	0	0	

As C_j is either zero or negative under all columns, the above table gives the optimal basic feasible solution. This optimal solution is $x_1 = 2$, $x_2 = 0$ and maximum $Z = 10$.

Example 34.15. A firm produces three products which are processed on three machines. The relevant data is given below :

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product A	Product B	Product C	
M_1	2	3	2	440
M_2	4	—	3	470
M_3	2	5	—	430

The profit per unit for products A, B, and C is ₹ 4, ₹ 3 and ₹ 6 respectively. Determine the daily number of units to be manufactured for each product. Assume that all the units produced are consumed in the market.

Solution. Let the firm decide to produce x_1 , x_2 , x_3 units of products A, B, C, respectively. Then the L.P. model for this problem is :

$$\text{Max. } Z = 4x_1 + 3x_2 + 6x_3$$

subject to $2x_1 + 3x_2 + 2x_3 \leq 440$, $4x_1 + 3x_3 \leq 470$, $2x_1 + 5x_2 \leq 430$, $x_1, x_2, x_3 \geq 0$. (V.T.U., 2004)

Step 1. Check whether the objective function is to be maximized and all b's are non-negative.

The problem being of maximization type and b's being ≥ 0 , this step is not necessary.

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3 , the problem in standard form becomes :

$$\begin{array}{ll} \text{Max.} & Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3 \\ \text{subject to} & 2x_1 + 3x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 440 \\ & 4x_1 + 0x_2 + 3x_3 + 0s_1 + s_2 + 0s_3 = 470 \\ & 2x_1 + 5x_2 + 0x_3 + 0s_1 + 0s_2 + s_3 = 430 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{array}$$

Step 3. Find an initial basic feasible solution.

The basic (non-degenerate) feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}$$

$$s_1 = 440, s_2 = 470, s_3 = 430 \text{ (basic)}$$

∴ Initial basic feasible solution is given by the following table :

c_B	c_j	4	3	6	0	0	0	b	0
	Basis	x_1	x_2	x_3	s_1	s_2	s_3		
0	s_1	2	3	2	1	0	0	440	440/2
0	s_2	4	0	(3)	0	1	0	470	470/3←
0	s_3	2	5	0	0	0	1	430	430/0
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0		
$C_j = c_j - Z_j$		4	3	6	0	0	0		

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal and we proceed to the next step.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_3 is the incoming variable while s_2 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

Drop s_2 and introduce x_3 with its associated value 6 under c_B column. Convert the key element to unity and make all other elements of key column zero. Then the second feasible solution is given by the table below :

c_B	c_j	4	3	6	0	0	0	b	0
	Basis	x_1	x_2	x_3	s_1	s_2	s_3		
0	s_1	-2/3	(3)	0	1	-2/3	0	380/3	380/9←
6	s_2	4/3	0	1	0	1/3	0	470/3	∞
0	s_3	2	5	0	0	0	1	430	86
Z_j		8	0	6	0	2	0	940	
C_j		-4	3	0	0	-2	0		
				↑					

Step 6. As C_j is positive under the second column, the solution is not optimal and we proceed further. Now x_2 is the incoming variable and s_1 is the outgoing variable and (3) is the key element for the next iteration.

Drop s_1 and introduce x_2 with its associated value 3 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the third basic feasible solution is given by the following table :

c_B	c_j	4	3	6	0	0	0	b	0
	Basis	x_1	x_2	x_3	s_1	s_2	s_3		
3	x_2	-2/9	1	0	1/3	-2/9	0	380/9	
6	x_3	4/3	0	1	0	1/3	0	470/3	
0	s_3	28/9	0	0	-5/3	10/9	0	1970/9	
Z_i		22/3	3	6	1	4/3	0	3200/3	
C_j		-10/3	0	0	-1	-4/3	0		

Now since each $C_j \leq 0$, therefore it gives the optimal solution

$$x_1 = 0, x_2 = 380/9, x_3 = 470/3 \text{ and } Z_{\max} = 3200/3 \text{ i.e., } 1066.67 \text{ rupees.}$$

Example 34.16. Solve the following L.P.P. by simplex method :

$$\text{Minimize } Z = x_1 - 3x_2 + 3x_3$$

$$\text{subject to } 3x_1 - x_2 + 2x_3 \leq 7, 2x_1 + 4x_2 \geq -12, -4x_1 + 3x_2 + 8x_3 \leq 10, x_1, x_2, x_3 \geq 0.$$

(Mumbai, 2004; V.T.U., 2003)

Solution. Consists of the following steps :

Step 1. Check whether objective function is to be maximized and all b's are non-negative.

As the problem is that of minimizing the objective function, converting it to the maximization type, we have Max. $Z' = -x_1 + 3x_2 - 3x_3$.

As the right hand side of the second constraint is negative, we write it as

$$-2x_1 - 4x_2 \leq 12$$

Step 2. Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3 , the problem in the standard form becomes

$$\text{Max. } Z' = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } 3x_1 - x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 7$$

$$-2x_1 - 4x_2 + 0s_3 + 0s_1 + s_2 + 0s_3 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + s_3 = 10$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Step 3. Find initial basic feasible solution.

The basic (non-degenerate) feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}; s_1 = 7, s_2 = 12, s_3 = 10 \text{ (basic)}$$

∴ Initial basic feasible solution is given by the table below :

c_B	c_j	-1	3	-3	0	0	0	b	θ
	Basis	x_1	x_2	x_3	s_1	s_2	s_3		
0	s_1	3	-1	2	1	0	0	7	$7/(-1)$
0	s_2	-2	-4	0	0	1	0	12	$12/(-4)$
0	s_3	-4	(3)	8	0	0	1	10	$10/3 \leftarrow$
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	0	
$C_j = c_j - Z_j$		-1	3	-3	0	0	0		
			↑						

Step 4. Apply optimality test.

As C_j is positive under second column, the initial basic feasible solution is not optimal and we proceed further.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_2 is the incoming variable, s_3 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

∴ Drop s_3 and introduce x_2 with its associated value 3 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second basic feasible solution is given by the following table :

c_B	c_j	-1	3	-3	0	0	0	b	θ
	Basis	x_1	x_2	x_3	s_1	s_2	s_3		
0	s_1	(5/3)	0	14/3	1	0	1/3	31/3	$31/5 \leftarrow$
0	s_2	-22/3	0	32/3	0	1	4/3	76/3	$-38/11$
3	x_2	-4/3	1	8.3	0	0	1/3	10/3	$-5/2$
	Z_j	-4	3	8	0	0	1	10	
	C_j	3	0	-11	0	0	-1		
		↑							

Step 6. As C_j is positive under first column, the solution is not optimal and we proceed further x_1 is the incoming variable, s_1 is the outgoing variable and (5/3) is the key element.

∴ Drop s_1 and introduce x_1 with its associated value - 1 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the *third basic feasible solution* is given by the table below :

c_B	c_j <i>Basis</i>	-1 x_1	3 x_2	-3 x_3	0 s_1	0 s_2	0 s_3	b
-1	x_1	1	0	14/5	3/5	0	1/5	31/5
0	s_2	0	0	156/5	22/5	1	14/5	354/5
3	x_2	0	1	32/5	4/5	0	3/5	58/5
	Z_j	-1	3	82/5	9/5	0	8/5	143/5
	C_j	0	0	-97/5	-9/5	0	-8/5	

Now since each $C_j \leq 0$, therefore it gives the optimal solution

$$x_1 = 31/5, x_2 = 58/5, x_3 = 0 \text{ (non-basic) and } Z'_{\max} = 143/5$$

$$\text{Hence } Z_{\min} = -143/5.$$

Example 34.17. Maximize $Z = 107x_1 + x_2 + 2x_3$,

subject to the constraints : $14x_1 + x_2 - 6x_3 + 3x_4 = 7$,

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5, 3x_1 - x_2 - x_3 \leq 0, x_1, x_2, x_3, x_4 \geq 0.$$

Solution. Consists of the following steps :

Step 1. Check whether objective function is to be maximized and all b's are non-negative.

This step is not necessary.

Step 2. Express the problem in the standard form.

Here x_4 is a slack variable. By introducing other slack variables s_1 and s_2 the problem in standard form becomes

$$\text{Max. } Z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$$

$$\text{subject to } \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 + 0s_1 + 0s_2 = \frac{7}{3}$$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 + 0x_4 + s_1 + 0s_2 = 5$$

$$3x_1 - x_2 - x_3 + 0x_4 + 0s_1 + s_2 = 0$$

$$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0.$$

Step 3. Find initial basic feasible solution.

The basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (non-basic)}; x_4 = 7/3, s_1 = 5, s_2 = 0 \text{ (basic)}$$

∴ Initial basic feasible solution is given in the table below :

c_B	c_j <i>Basis</i>	107 x_1	1 x_2	2 x_3	0 x_4	0 s_1	0 s_2	b	0
0	x_4	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{7}{3}$	$\frac{7}{3}/\frac{14}{3}$
0	s_1	16	$\frac{1}{2}$	-6	0	1	0	5	5/16
0	s_2	(3)	-1	-1	0	0	1	0	0/3 ←
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0		
$C_j = c_j - Z_j$		107	1	2	0	0	0		
		↑							

Step 4. Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal and we proceed further.

Step 5. (i) Identify the incoming and outgoing variables.

The above table shows that x_1 is the incoming variable, s_2 is the outgoing variable and (3) is the key element.

(ii) Iterate towards the optimal solution.

Drop s_2 and introduce x_1 with its associated value 107 under c_B column. Convert key element to unity and make all other elements of the key column zeros. Then the second basic feasible solution is given by the following table :

c_B	Basis	c_j	107	1	2	0	0	0	b	θ
0	x_4	x_1	0	x_2	x_3	x_4	s_1	s_2	7/3	-21/4
0	s_1	0	0	17/9	-4/9	1	0	14/9	5	-15/2
107	x_1	1	1	-1/3	-1/3	0	1	-16/3	0	0
	Z_j	107	-107/3	-107/3	0	0	0	107/3		
	C_j	0	110/3	113/3	0	0	0	-107/3		
				↑						

As C_j is positive under some columns, the solution is not optimal. Here 113/3 being the largest positive value of C_j , x_3 is the incoming variable. But all the values of θ being ≤ 0 , x_3 will not enter the basis. This indicates that the solution to the problem is unbounded.

[Remember that (i) the incoming variable is the non-basic variable corresponding to the largest positive value of C_j and

(ii) the outgoing variable is the basic-variable corresponding to the least positive ratio θ , obtained by dividing the b -column elements by the corresponding key-column elements.]

PROBLEMS 34.4

Using simplex method, solve the following L.P.P. (1-8) :

- Maximize $Z = x_1 + 3x_2$, subject to $x_1 + 2x_2 \leq 10$, $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 4$. (Kurushetra, 2009 ; V.T.U., 2003)
- Maximize $Z = 4x_1 + 10x_2$, subject to $2x_1 + x_2 \leq 50$, $2x_1 + 5x_2 \leq 100$, $2x_1 + 3x_2 \leq 90$, $x_1, x_2 \geq 0$. (Kurushetra, 2006)
- Maximize $Z = 4x_1 + 5x_2$, subject to $x_1 - 2x_2 \leq 2$, $2x_1 + x_2 \leq 6$, $x_1 + 2x_2 \leq 5$, $-x_1 + x_2 \leq 2$, $x_1, x_2 \geq 0$.
- Maximize $Z = 10x_1 + x_2 + 2x_3$, subject to $x_1 + x_2 - 2x_3 \leq 10$, $4x_1 + x_2 + x_3 \leq 20$, $x_1, x_2, x_3 \geq 0$.
- Maximize $Z = 3x_1 + 2x_2 + 5x_3$, subject to $x_1 + 2x_2 + x_3 \leq 430$, $3x_1 + 2x_3 \leq 460$, $x_1 + 4x_2 \leq 420$, $x_1, x_2, x_3 \geq 0$. (Mumbai, 2004)
- Minimize $Z = 3x_1 + 5x_2 + 4x_3$, subject to $2x_1 + 3x_2 \leq 8$, $2x_2 + 5x_3 \leq 10$, $3x_1 + 2x_2 + 4x_3 \leq 15$, $x_1, x_2, x_3 \geq 0$. (Mumbai, 2004 S)
- Minimize $Z = x_1 - 3x_2 + 2x_3$, subject to $3x_1 - x_2 + 2x_3 \leq 7$, $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$, $x_1, x_2, x_3 \geq 0$. (Madras, 2006)
- Maximize $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$, subject to $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 80$, $2x_1 + 2x_3 + x_4 \leq 60$, $3x_1 + 3x_2 + x_3 + x_4 \leq 80$, $x_1, x_2, x_3, x_4 \geq 0$.
- A firm produces products A and B and sells them at a profit of ₹ 2 and ₹ 3 each respectively. Each product is processed on machines G and H. Product A requires 1 minute on G and 2 minutes on H whereas product B requires 1 minute on each of the machines. Machine G is not available for more than 6 hrs. 40 min/day whereas the time constraint for machine H is 10 hrs. Solve this problem via simplex method for maximizing the profit.
- A company makes two types of products. Each product of the first type requires twice as much labour time as the second type. If all products are of second type only, the company can produce a total of 500 units a day. The market limits daily sales of the first and the second type to 150 and 250 units respectively. Assuming that the profits per

unit are ₹ 8 for type I and ₹ 5 for type II, determine the number of units of each type to be produced to maximize profit?

11. The owner of a dairy is trying to determine the correct blend of two types of feed. Both contain various percentages of four essential ingredients. With the following data determine the least cost blend?

Ingredient	% per kg of feed		Min requirement in kg.
	Feed 1	Feed 2	
1	40	20	4
2	10	30	2
3	20	40	3
4	30	10	6
Cost (₹/kg.)	5	3	

12. A manufacturing firm has discontinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering to devote their excess capacity to one or more of three products 1, 2, and 3. The available capacity on machines and the number of machine-hours required for each unit of the respective product, is given below :

Machine Type	Available Time (hrs/week)	Productivity (hrs/unit)		
		Product 1	Product 2	Product 3
Milling machine	250	8	2	3
Lathe	150	4	3	2
Grinder	50	2	—	1

The unit profit would be ₹ 20, ₹ 6 and ₹ 8 respectively for products 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit.

13. The following table gives the various vitamin contents of three types of food and daily requirements of vitamins alongwith cost per unit. Find the combination of food for minimum cost.

Vitamin (mg)	Food F	Food G	Food H	Minimum daily requirement (mg)
A	1	1	10	1
C	100	10	10	50
D	10	100	10	10
Cost/unit (₹)	10	15	5	

14. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs ₹ 100 for preparation, requires 7 man-days of work and yields a profit of ₹ 30. An acre of wheat costs ₹ 120 to prepare, requires 10 man-days of work and yields a profit of ₹ 40. An acre of soyabeans costs ₹ 70 to prepare, requires 8 man-days of work and yields a profit of ₹ 20. If the farmer has ₹ 100,000 for preparation and can count on 8,000 man-days of work, how many acres should be allocated to each crop to maximize profits?

34.9 ARTIFICIAL VARIABLE TECHNIQUES

So far we have seen that the introduction of slack/surplus variables provided the initial basic feasible solution. But there are many problems wherein at least one of the constraints is of (\geq) or (=) type and slack variables fail to give such a solution. There are two similar methods for solving such problems which we explain below :

(1) **M-method or Method of Penalties.** This method is due to A. Charnes and consists of the following steps :

Step 1. Express the problem in standard form.

Step 2. Add non-negative variables to the left hand side of all those constraints which are of (\geq) or (=) type. Such new variables are called *artificial variables* and the purpose of introducing these is just to obtain an initial basic feasible solution. But their addition causes violation of the corresponding constraints. As such, we would

like to get rid of these variables and would not allow them to appear in the final solution. For this purpose, we assign a very large penalty ($-M$) to these artificial variables in the objective function.

Step 3. Solve the modified L.P.P. by simplex method.

At any iteration of simplex method, one of the following three cases may arise :

(i) There remains no artificial variable in the basis and the optimality condition is satisfied. Then the solution is an optimal basic feasible solution to the problem.

(ii) There is at least one artificial variable in the basis at zero level (with zero value in b -column) and the optimality condition is satisfied. Then the solution is a degenerate optimal basic feasible solution.

(iii) There is at least one artificial variable in the basis at non-zero level (with positive value in b -column) and the optimality condition is satisfied. Then the problem has no feasible solution. The final solution is not optimal, since the objective function contains an unknown quantity M . Such a solution satisfies the constraints but does not optimize the objective function and is therefore, called *pseudo optimal solution*.

Step 4. Continue the simplex method until either an optimal basic feasible solution is obtained or an unbounded solution is indicated.

Obs. The artificial variables are only a computational device for getting a starting solution. Once an artificial variable leaves the basis, it has served its purpose and we forget about it i.e., the column for this variable is omitted from the next simplex table.

Example 34.18. Use Charnes' penalty method to

$$\text{Minimize } Z = 2x_1 + x_2$$

$$\text{subject to } 3x_1 + x_2 = 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 3, x_1, x_2 \geq 0. \quad (\text{Anna, M. Tech, 2006; V.T.U., 2000 S})$$

Solution. Consists of the following steps :

Step 1. Express the problem in standard form.

The second and third inequalities are converted into equations by introducing the surplus and slack variables s_1, s_2 respectively.

Also the first and second constraints being of (=) and (\geq) type, we introduce two artificial variables A_1, A_2 .

Converting the minimization problem to the maximization form, the L.P.P. can be rewritten as

$$\text{Max. } Z' = -2x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

$$\begin{aligned} \text{subject to} \quad & 3x_1 + x_2 + 0s_1 + 0s_2 + A_1 + 0A_2 = 3 \\ & 4x_1 + 3x_2 - s_1 + 0s_2 + 0A_1 + A_2 = 6 \\ & x_1 + 2x_2 + 0s_1 + s_2 + 0A_1 + 0A_2 = 3 \\ & x_1, x_2, s_1, s_2, A_1, A_2 \geq 0 \end{aligned}$$

Step 2. Obtain an initial basic feasible solution.

Surplus variable s_1 is not a basic variable since its value is -6 . As negative quantities are not feasible, s_1 must be prevented from appearing in the initial solution. This is done by taking $s_1 = 0$. By setting the other non-basic variables x_1, x_2 each = 0, we obtain the initial basic feasible solution as

$$x_1 = x_2 = 0, s_1 = 0; A_1 = 3, A_2 = 6, s_2 = 3$$

Thus the initial simplex table is

	c_j	-2	-1	0	0	$-M$	$-M$		
c_B	Basis	x_1	x_2	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	(3)	1	0	0	1	0	3	$3/3 \leftarrow$
$-M$	A_2	4	3	-1	0	0	1	6	$6/4$
0	s_2	1	2	0	1	0	0	3	$3/1$
$Z_j = \sum c_B a_{ij}$		$-7M$	$-4M$	M	0	$-M$	$-M$	$-9M$	
$C_j = c_j - Z_j$		$7M - 2$	$4M - 1$	$-M$	0	0	0		
↑									

Since C_j is positive under x_1 and x_2 columns, this is not an optimal solution.

Step 3. Iterate towards optimal solution.

Introduce x_1 , and drop A_1 from basis.

∴ The new simplex table is

	c_j	-2	-1	0	0	-M		
c_B	Basis	x_1	x_2	s_1	s_2	A_2	b	θ
-2	x_1	1	1/3	0	0	0	1	3
-M	A_2	0	(5/3)	-1	0	1	2	6/5 ←
0	s_2	0	5/3	0	1	0	2	6/5
	Z_j	-2	$-\frac{2}{3} - \frac{5M}{3}$	M	0	-M	$-2 - 2M$	
	C_j	0	$-\frac{1}{3} + \frac{5M}{3}$	-M	0	0		
			↑					

Since C_j is positive under x_2 column, this is not an optimal solution.

∴ Introduce x_2 and drop A_2 .

Then the revised simplex table is

	c_j	-2	-1	0	0		
c_B	Basis	x_1	x_2	s_1	s_2		
-2	x_1	1	0	1/5	0	3/5	
-1	x_2	0	1	-3/5	0	6/5	
0	s_2	0	0	1	1	0	
	Z_j	-2	-1	1/5	0	-12/5	
	C_j	0	0	-1/5	0		

Since none of C_j is positive, this is an optimal solution. Thus, an optimal basic feasible solution to the problem is

$$x_1 = 3/5, x_2 = 6/5, \text{Max. } Z' = -12/5.$$

Hence the optimal value of the objective function is

$$\text{Min. } Z = -\text{Max. } Z' = -(-12/5) = 12/5$$

Example 34.19. Maximize $Z = 3x_1 + 2x_2$

subject to the constraints : $2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12, x_1, x_2 \geq 0$.

Solution. Consists of the following steps :

Step 1. Express the problem in standard form.

The inequalities are converted into equations by introducing the slack and surplus variables s_1, s_2 respectively. Also the second constraint being of (\geq) type, we introduce the artificial variable A. Thus the L.P.P. can be rewritten as

$$\text{Max. } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA$$

$$\text{subject to } 2x_1 + x_2 + s_1 + 0s_2 + 0A = 2,$$

$$3x_1 + 4x_2 + 0s_1 - s_2 + A = 12,$$

$$x_1, x_2, s_1, A \geq 0$$

Step 2. Find an initial basic feasible solution.

Surplus variable s_2 is not a basic variable since its value is -12. Since a negative quantity is not feasible, s_2 must be prevented from appearing in the initial solution. This is done by letting $s_2 = 0$. By taking the other non-basic variables x_1 and x_2 each = 0, we obtain the initial basic feasible solution as

$$x_1 = x_2 = s_2 = 0, s_1 = 2, A = 12$$

∴ The initial simplex table is

	c_j	3	2	0	0	-M		
c_B	Basis	x_1	x_2	s_1	s_2	A	b	θ
0	s_1	2	(1)	1	0	0	2	2 ←
-M	A	3	4	0	-1	1	12	3
	$Z_j = \Sigma c_B a_{ij}$	-3M	-4M	0	M	-M	-12M	
	$C_j = c_j - Z_j$	3 + 3M	2 + 4M	0	-M	0		
			↑					

Since C_j is positive under some columns, this is not an optimal solution.

Step 3. Iterate towards optimal solution.

Introduce x_2 and drop s_1 .

∴ The new simplex table is

	c_j	3	2	0	0	$-M$	
c_B	Basis	x_1	x_2	s_1	s_2	A	b
2	x_2	2	1	1	0	0	2
$-M$	A	-5	0	-4	-1	1	4
Z_j		$4 + 5M$	2	$2 + 4M$	M	$-M$	$4 - 4M$
C_j		$-(1 + 5M)$	0	$-(2 + 4M)$	$-M$	0	

Here each C_j is negative and an artificial variable appears in the basis at non-zero level. Thus there exists a *pseudo optimal solution* to the problem.

(2) Two-phase method. This is another method to deal with the artificial variables wherein the L.P.P. is solved in two phases.

Phase I. Step 1. Express the given problem in the standard form by introducing slack, surplus and artificial variables.

Step 2. Formulate an artificial objective function

$$Z^* = -A_1 - A_2 \dots - A_m$$

by assigning (-1) cost to each of the artificial variables A_i and zero cost to all other variables.

Step 3. Maximize Z^* subject to the constraints of the original problem using the simplex method. Then three cases arise :

(a) Max. $Z^* < 0$ and at least one artificial variable appears in the optimal basis at a positive level

In this case, the original problem doesn't possess any feasible solution and the procedure comes to an end.

(b) Max. $Z^* = 0$ and no artificial variable appears in the optimal basis.

In this case, a basic feasible solution is obtained and we proceed to phase II for finding the optimal basic feasible solution to the original problem.

(c) Max. $Z^* = 0$ and at least one artificial variable appears in the optimal basis at zero level.

Here a feasible solution to the auxiliary L.P.P. is also a feasible solution to the original problem with all artificial variables set = 0.

To obtain a basic feasible solution, we prolong phase I for pushing all the artificial variables out of the basis (without proceeding on to phase II).

Phase II. The basic feasible solution found at the end of phase I is used as the starting solution for the original problem in this phase i.e., the final simplex table of phase I is taken as the initial simplex table of phase II and the artificial objective function is replaced by the original objective function. Then we find the optimal solution.

Example 34.20. Use two-phase method to

$$\text{Minimize } Z = 7.5x_1 - 3x_2$$

subject to the constraints $3x_1 - x_2 - x_3 \geq 3$, $x_1 - x_2 + x_3 \geq 2$,

$$x_1, x_2, x_3 \geq 0.$$

Phase I. Step 1. Express the problem in standard form.

Solution. Introducing surplus variables s_1 , s_2 and artificial variables A_1 , A_2 , the phase I problem in standard form becomes

$$\begin{array}{ll} \text{Max. } & Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2 \\ \text{subject to} & 3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3 \\ & x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2 \\ & x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0. \end{array}$$

Step 2. Find an initial basic feasible solution.

Setting $x_1 = x_2 = x_3 = s_1 = s_2 = 0$,
we have $A_1 = 3, A_2 = 2$ and $Z^* = -5$

∴ Initial simplex table is

c_B	$Basis$	c_j	0	0	0	0	0	-1	-1	b	0
-1	A_1	(3)	x_1	x_2	x_3	s_1	s_2	A_1	A_2	3	1←
-1	A_2	1	-1	1	0	-1	0	1	2	2	
$Z_j^* = \sum c_B a_{ij}$		-4	2	0	1	1	-1	-1	-5		
$C_j = c_j - Z_j^*$		4	-2	0	-1	-1	0	0			
		↑									

As C_j is positive under x_1 column, this solution is not optimal.

Step 3. Iterate towards an optimal solution.

Making key element (3) unity and replacing A_1 by x_1 , we have the new simplex table :

c_B	$Basis$	c_j	0	0	0	0	0	-1	-1	b	0
0	x_1	1		$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	1	-3
-1	A_2	0	$-\frac{2}{3}$	$\left(\frac{4}{3}\right)$	$\frac{1}{3}$	-1	$-\frac{1}{3}$	1	1	$\frac{3}{4} \leftarrow$	
Z_j^*		0	$\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	1	$\frac{1}{3}$	-1	-1		
C_j		0	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	-1	$-\frac{1}{3}$	0			
		↑									

Since C_j is positive under x_3 and s_1 columns, this solution is not optimal.

Making key element (4/3) unity and replacing A_2 by x_3 , we obtain the revised simplex table :

c_B	$Basis$	c_j	0	0	0	0	0	-1	-1	b
0	x_1	1		$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	$5/4$
0	x_2	0		$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	$3/4$
Z_j^*		0		0	0	0	0	0	0	0
C_j		0	0	0	0	0	0	-1	-1	
		↑								

Since all $C_j \leq 0$, this table gives the optimal solution. Also $Z_{\max}^* = 0$ and no artificial variable appears in the basis. Thus an optimal basic feasible solution to the auxiliary problem and therefore to the original problem, has been attained.

Phase II. Considering the actual costs associated with the original variables, the objective function is

$$\text{Max. } Z' = -15/2x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2 - 0A_1 - 0A_2$$

subject to

$$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3,$$

$$x_2 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2,$$

$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

The optimal initial feasible solution thus obtained, will be an optimal basic feasible solution to the original L.P.P.

Using final table of phase I, the initial simplex table of phase II is as follows :

c_B	$Basis$	c_j	-15/2	3	0	0	0	b
-15/2	x_1	1		$-1/2$	0	$-1/4$	$-1/4$	$5/4$
0	x_3	0		$-1/2$	1	$1/4$	$-3/4$	$3/4$
Z_j		-15/2		15/4	0	15/8	15/8	-75/8
C_j		0		$-3/4$	0	$-15/8$	$-15/8$	
		↑						

Since all $C_j \leq 0$, this solution is optimal.

Hence an optimal basic feasible solution to the given problem is

$$x_1 = 5/4, x_2 = 0, x_3 = 3/4 \text{ and } \min. Z = 75/8.$$

34.10 EXCEPTIONAL CASES

(1) Tie for the incoming variable. When more than one variable has the same largest positive value in C_j row (in maximization problem), a tie for the choice of incoming variable occurs. As there is no method to break this tie, we choose any one of the prospective incoming variables arbitrarily. Such an arbitrary choice does not in any way affect the optimal solution.

(2) Tie for the outgoing variable. When more than one variable has the same least positive ratio under then θ -column, a tie for the choice of outgoing variable occurs. If the equal values of said ratio are > 1 , choose any one of the prospective leaving variables arbitrarily. Such an arbitrary choice doesn't affect the optimal solution.

If the equal values of ratios are zero, the simplex method fails and we make use of the following degeneracy technique.

(3) Degeneracy. We know that a basic feasible solution is said to be degenerate if any of the basic variables vanishes. This phenomenon of getting a degenerate basic feasible solution is called *degeneracy* which may arise

- (i) at the initial state, when atleast one basic variable is zero in the initial basic feasible solution
 or (ii) at any subsequent stage, when the least positive ratios under θ -column are equal for two or more

In this case, an arbitrary choice of one of these basic variables may result in one or more basic variables becoming zero in the next iteration. At times, the same sequence of simplex iterations is repeated endlessly without improving the solution. These are termed as *cycling* type of problems. Cycling occurs very rarely. In fact, cycling has seldom occurred in practical problems.

To avoid cycling, we apply the following perturbation procedure :

- (i) Divide each element in the tied rows by the *positive coefficients* of the key column in that row.
 - (ii) Compare the resulting ratios (from left to right) first of unit matrix and then of the body matrix, column by column.
 - (iii) The outgoing variable lies in that row which first contains the smallest algebraic ratio.

Example 34.21. Maximize $Z = 5x_1 + 3x_2$

$$\text{subject to } x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10, 3x_1 + 8x_2 \leq 12; x_1, x_2 \geq 0$$

Solution. Consists of the following steps :

Step 1. Express the problem in the standard form.

Introducing the slack variables s_1, s_2, s_3 , the problem in the standard form is

$$\begin{aligned} \text{Max. } Z &= 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 \\ x_1 + x_2 + s_1 + 0s_2 + 0s_3 &= 2 \\ 5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 &= 10 \\ 3x_1 + 8x_2 + 0s_1 + 0s_2 + s_3 &= 12 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Step 2. Find the initial basic feasible solution

The initial basic feasible solution is

$$x_1 = x_2 = 0 \text{ (non-basic)}$$

∴ Initial simplex table is

	c_j	5	3	0	0	0	b	θ
c_B	Basis	x_1	x_2	s_1	s_2	s_3		
0	s_1	1	1	1	0	0	2	2/1
0	s_2	(5)	2	0	1	0	10	10/5←
0	s_3	3	8	0	0	1	12	12/3
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	
$C_j = c_j - Z_j$		5	3	0	0	0		
		↑						

As C_j is positive under x_2 columns, this solution is not optimal.

Step 3. Iterate towards optimal solution.

x_1 is the incoming variable. But the first two rows have the same ratio under θ -column. Therefore we apply *perturbation* method.

First column of the unit matrix has 1 and 0 in the tied rows. Dividing these by the corresponding elements of the key columns, we get 1/1 and 0/5, s_2 -row gives the smaller ratio and therefore s_2 is the first outgoing variable and (5) is the key element.

Thus the new simplex table is

	c_j	5	3	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	(3/5)	1	-1/5	0	0	0
5	x_1	1	2/5	0	1/5	0	2	5 ←
0	s_3	0	34/5	0	-3/5	1	6	15/17
Z_j		5	2	0	1	0	10	
C_j		0	1	0	-1	0		
			↑					

As C_j is positive under x_2 column, this solution is not optimal.

Making key element (3/5) unity and replacing s_1 by x_2 , we obtain the revised simplex table :

	c_j	5	3	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	
3	x_2	0	1	5/3	-1/3	0	0	
5	x_1	1	0	-2/3	1/3	0	2	
0	s_3	0	0	-34/3	5/3	1	6	
Z_j		5	3	5/3	2/3	0	10	
C_j		0	0	-5/3	-2/3	0		

As $C_j \leq 0$ under all columns, this table gives the optimal solution. Hence an optimal basic feasible solution is $x_1 = 2, x_2 = 0$ and $Z_{\max} = 10$.

PROBLEMS 34.5

Solve the following L.P. problems using M-method :

1. Maximize $Z = 3x_1 + 2x_2 + 3x_3$

subject to : $2x_1 + x_2 + x_3 \leq 2, 3x_1 + 4x_2 + 2x_3 \geq 8, x_1, x_2, x_3 \geq 0$.

2. Maximize $Z = 2x_1 + x_2 + 3x_3$

subject to : $x_1 + x_2 + 2x_3 \leq 5, 2x_1 + 3x_2 + 4x_3 = 12, x_1, x_2, x_3 \geq 0$.

3. Maximize $Z = 8x_2$,

subject to : $x_1 - x_2 \geq 0, 2x_1 + 3x_2 \leq -6, x_1, x_2$ unrestricted.

4. Maximize $Z = 5x_1 - 2x_2 + 3x_3$

subject to : $2x_1 + 2x_2 - x_3 \geq 2, 3x_1 - 4x_2 \leq 3, x_2 + 3x_3 \leq 5, x_1, x_2, x_3 \geq 0$. (Mumbai, 2004)

5. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to : $x_1 + 2x_2 + 3x_3 = 15, 2x_1 + x_2 + 5x_3 = 20,$

$x_1 + 2x_2 + x_3 + x_4 = 10, x_1, x_2, x_3, x_4 \geq 0$. (Madras, 2003)

Use two phase method to solve the following L.P. problems :

6. Minimize $Z = x_1 + x_2$

subject to : $2x_1 + x_2 \geq 4, x_1 + 7x_2 \geq 7,$

$x_1, x_2 \geq 0$. (Rajasthan, 2005)

7. Maximize $Z = 5x_1 + 3x_2$

subject to : $2x_1 + x_2 \leq 1, x_1 + 4x_2 \geq 6,$

$x_1, x_2 \geq 0$. (Kottayam, 2005)

8. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
 subject to : $2x_1 + 2x_2 - x_3 \geq 2$,
 $3x_1 - 4x_2 \leq 3$, $x_2 + x_3 \leq 5$,
 $x_1, x_2, x_3 \geq 0$. (Mumbai, 2009)

9. Maximize $Z = 5x_1 - 4x_2 + 3x_3$,
 subject to : $2x_1 + x_2 - 6x_3 = 20$,
 $6x_1 + 5x_2 + 10x_3 \leq 76$,
 $8x_1 - 3x_2 + 6x_3 \leq 50$,
 $x_1, x_2, x_3 \geq 0$.

Solve the following degenerate L.P. problems :

10. Maximize $Z = 9x_1 + 3x_2$
 subject to : $4x_1 + x_2 \leq 8$, $2x_1 + x_2 \leq 4$,
 $x_1, x_2 \geq 0$.

11. Maximize $Z = 2x_1 + 3x_2 + 10x_3$
 subject to : $x_1 + 2x_3 = 0$, $x_2 + x_3 = 1$,
 $x_1, x_2, x_3 \geq 0$.

34.11 (1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the *duality* theory. Every linear programming problem has associated with it, another linear programming problem involving the same data and closely related optimal solutions. Such two problems are said to be *duals* of each other. While one of these is called the *primal*, the other the *dual*.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it. Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions in the activities being programmed.

(2) Formulation of dual problem. Consider the following L.P.P. :

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n,$$

$$\begin{aligned} \text{subject to the constraints} \quad & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1, \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2, \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0. \end{aligned}$$

To construct the dual problem, we adopt the following guidelines :

- (i) The maximization problem in the primal becomes the minimization problem in the dual and *vice versa*.
- (ii) (\leq) type of constraints in the primal become (\geq) type of constraints in the dual and *vice versa*.
- (iii) The coefficients c_1, c_2, \dots, c_n in the objective function of the primal become b_1, b_2, \dots, b_m in the objective function of the dual.
- (iv) The constants b_1, b_2, \dots, b_m in the constraints of the primal become c_1, c_2, \dots, c_n in the constraints of the dual.
- (v) If the primal has n variables and m constraints, the dual will have m variables and n constraints i.e. the transpose of the body matrix of the primal problem gives the body matrix of the dual.
- (vi) The variables in both the primal and dual are non-negative.

Then the dual problem will be

$$\text{Minimize } W = b_1y_1 + b_2y_2 + \dots + b_my_m$$

$$\begin{aligned} \text{subject to the constraints} \quad & a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1, \\ & a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2, \\ & \dots \\ & a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n, \\ & y_1, y_2, \dots, y_m \geq 0. \end{aligned}$$

Example 34.22. Write the dual of the following L.P.P.:

$$\text{Minimize } Z = 3x_1 - 2x_2 + 4x_3$$

$$\begin{aligned} \text{subject to} \quad & 3x_1 + 5x_2 + 4x_3 \geq 7, \quad 6x_1 + x_2 + 3x_3 \geq 4, \quad 7x_1 - 2x_2 - x_3 \leq 10, \\ & x_1 - 2x_2 + 5x_3 \geq 3, \quad 4x_1 + 7x_2 - 2x_3 \geq 2, \quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution. Since the problem is of minimization, all constraints should be of \geq type. We multiply the third constraint throughout by -1 so that $-7x_1 + 2x_2 + x_3 \geq -10$.

Let y_1, y_2, y_3, y_4 and y_5 be the dual variables associated with the above five constraints. Then the dual problem is given by

$$\begin{array}{ll} \text{Maximize} & W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5 \\ \text{subject to} & 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3, 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2, \\ & 4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4, y_1, y_2, y_3, y_4, y_5 \geq 0. \end{array}$$

(3) Formulation of dual problem when the primal has equality constraints. Consider the problem

$$\begin{array}{ll} \text{Maximize} & Z = c_1x_1 + c_2x_2 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 = b_1, a_{21}x_1 + a_{22}x_2 \leq b_2, x_1, x_2 \geq 0. \end{array}$$

The equality constraint can be written as

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \geq b_1 \\ a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \leq -b_1, \end{array}$$

or
Then the above problem can be restated as

$$\begin{array}{ll} \text{Maximize} & Z = c_1x_1 + c_2x_2 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 \leq b_1, -a_{11}x_1 - a_{12}x_2 \leq -b_1, \\ & a_{21}x_1 + a_{22}x_2 \leq b_2, x_1, x_2 \geq 0. \end{array}$$

Now we form the dual using y_1', y_1'', y_2 as the dual variables. Then the dual problem is

$$\begin{array}{ll} \text{Minimize} & W = b_1(y_1' - y_1'') + b_2y_2, \\ \text{subject to} & a_{11}(y_1' - y_1'') + a_{21}y_2 \geq c_1, a_{12}(y_1' - y_1'') + a_{22}y_2 \geq c_2, y_1', y_1'', y_2 \geq 0. \end{array}$$

The term $(y_1' - y_1'')$ appears in both the objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. Then the new variable $y_1' - y_1'' (= y_1)$ becomes unrestricted in sign being the difference of two non-negative variables and the above dual problem takes the form.

$$\begin{array}{ll} \text{Minimize} & W = b_1y_1 + b_2y_2, \\ \text{subject to} & a_{11}y_1 + a_{21}y_2 \geq c_1, a_{12}y_1 + a_{22}y_2 \geq c_2, y_1 \text{ unrestricted in sign}, y_2 \geq 0. \end{array}$$

In general, if the primal problem is

$$\begin{array}{ll} \text{Maximize} & Z = c_1x_1 + c_2x_2 + \dots + c_nx_n, \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ & \dots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0, \end{array}$$

then the dual problem is

$$\begin{array}{ll} \text{Minimize} & W = b_1y_1 + b_2y_2 + \dots + b_my_m \\ \text{subject to} & a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \geq c_1, \\ & a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2, \\ & \dots \\ & a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n \\ & y_1, y_2, \dots, y_m \text{ all unrestricted in sign}. \end{array}$$

Thus the dual variables corresponding to equality constraints are unrestricted in sign. Conversely when the primal variables are unrestricted in sign, corresponding dual constraints are equalities.

Example 34.23. Construct the dual of the L.P.P.:

$$\begin{array}{ll} \text{Maximize} & Z = 4x_1 + 9x_2 + 2x_3, \\ \text{subject to} & 2x_1 + 3x_2 + 2x_3 \leq 7, 3x_1 - 2x_2 + 4x_3 = 5, x_1, x_2, x_3 \geq 0. \end{array}$$

Solution. Let y_1 and y_2 by the dual variables associated with the first and second constraints. Then the dual problem is

$$\begin{array}{ll} \text{Minimize} & W = 7y_1 + 5y_2, \\ \text{subject to} & 2y_1 + 3y_2 \leq 4, 3y_1 - 2y_2 \leq 9, 2y_1 + 4y_2 \leq 2, y_1 \geq 0, y_2 \text{ is unrestricted in sign}. \end{array}$$

PROBLEMS 34.6

Write the duals of the following problems (1 – 4) :

1. Maximize $Z = 10x_1 + 13x_2 + 19x_3$
subject to $6x_1 + 5x_2 + 3x_3 \leq 26, 4x_1 + 2x_2 + 5x_3 \leq 7, x_1, x_2, x_3 \geq 0.$
2. Minimize $Z = 2x_1 + 4x_2 + 3x_3$
subject to $3x_1 + 4x_2 + x_3 \geq 11, -2x_1 - 3x_2 + 2x_3 \leq -7, x_1 - 2x_2 - 3x_3 \leq -1$
 $3x_1 + 2x_2 + 2x_3 \geq 5, x_1, x_2, x_3 \geq 0.$
3. Maximize $Z = 3x_1 + 16x_2 + 7x_3$
subject to $x_1 - x_2 + x_3 \geq 3, -3x_1 + 2x_3 \leq 1, 2x_1 + x_2 - x_3 = 4, x_1, x_2, x_3 \geq 0.$
4. Minimize $Z = 3x_1 - 3x_2 + x_3$
subject to $2x_1 - 3x_2 + x_3 \leq 5, 4x_1 - 2x_2 \geq 9, -8x_1 + 4x_2 + 3x_3 = 8,$
 $x_1, x_2 \geq 0$ and x_3 is unrestricted.

(Mumbai, 2004)

5. Obtain the dual problem of the following L.P.P.

$$\begin{aligned} \text{Maximize } f(x) &= 2x_1 + 5x_2 + 6x_3 \\ \text{subject to} & \quad 5x_1 + 6x_2 - x_3 \leq 3, -2x_1 + x_2 + 4x_3 \leq 4, x_1 - 5x_2 + 3x_3 \leq 1, \\ & \quad -3x_1 - 3x_2 + 7x_3 \leq 6, x_1, x_2, x_3 \geq 0. \end{aligned}$$

Also verify that the dual of the dual problem is the primal problem.

34.12 (1) DUALITY PRINCIPLE

If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.,

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and vice-versa.

(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal) :

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

Rule I. If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the C_j row of the optimal dual simplex table and vice-versa.

Rule II. If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the C_j row of the optimal dual simplex table, after deleting the constant M and vice-versa.

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and vice-versa.

Now we shall workout two examples to demonstrate the primal dual relationships.

Example 34.24. Construct the dual of the following problem and solve both the primal and the dual :

$$\begin{aligned} \text{Maximize } Z &= 2x_1 + x_2 \\ \text{subject to} & \quad -x_1 + 2x_2 \leq 2, x_1 + x_2 \leq 4, x_1 \leq 3, x_1, x_2 \geq 0. \end{aligned}$$

(Rohtak, 2005)

Solution. Using the primal problem. Since only two variables are involved, it is convenient to solve the problem graphically.

In the x_1, x_2 -plane, the five constraints show that the point (x_1, x_2) lies within the shaded region $OABCD$ of Fig. 34.12. Values of the objective function $Z = 2x_1 + x_2$ at these corners are $Z(0) = 0$, $Z(A) = 6$, $Z(B) = 7$, $Z(C) = 6$ and $Z(D) = 1$. Hence the optimal solution is $x_1 = 3$, $x_2 = 1$ and max. (Z) = 7.

Solution. Using the dual problem. The dual problem of the given primal is :

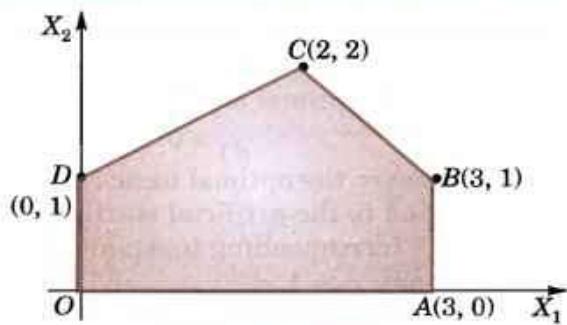


Fig. 34.12

Minimize $W = 2y_1 + 4y_2 + 3y_3$

subject to $-y_1 + y_2 + y_3 \geq 2, 2y_1 + y_2 \geq 1, y_1, y_2 \geq 0.$

Step 1. Express the problem in the standard form.

Introducing the slack and the artificial variables, the dual problem in the standard form is

$$\text{Max. } W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2$$

$$\text{subject to } -y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$$

$$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$$

Step 2. Find an initial basic feasible solution.

Setting the non-basic variables y_1, y_2, y_3, s_1, s_2 , each equal to zero, we get the initial basic feasible solution as

$$y_1 = y_2 = y_3 = s_1 = s_2 = 0 \text{ (non-basic); } A_1 = 2, A_2 = 1. \text{ (basic)}$$

∴ Initial simplex table is

c_B	c_j	-2	-4	-3	0	0	-M	-M		
	Basis	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	0
-M	A_1	-1	1	1	-1	0	1	0	2	2/1
-M	A_2	2	(1)	0	0	-1	0	1	1	1/1←
	Z_j	-M	-2M	-M	M	M	-M	-M	-3M	
	C_j	$M - 2$	$2M - 4$	$M - 3$	$-M$	$-M$	0	0		

↑

As C_j is positive under some columns, the initial solution is not optimal.

Step 3. Iterate towards an optimal solution.

(i) Introduce y_2 and drop A_2 . Then the new simplex table is

c_B	c_j	-2	-4	-3	0	0	-M	-M		
	Basis	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	0
-M	A_1	-3	0	(1)	-1	1	1	-1	1	1/1←
-4	y_2	2	1	0	0	-1	0	1	1	1/0
	Z_j	$3M - 8$	-4	-M	M	$4 - M$	-M	$M - 4$	$-M - 4$	
	C_j	$6 - 3M$	0	$M - 3$	$-M$	$M - 4$	0	$4 - 2M$		

↑

As C_j is positive under some columns, this solution is not optimal.

(ii) Now introduce y_3 and drop A_1 . Then the revised simplex table is

c_B	c_j	-2	-4	-3	0	0	-M	-M		
	Basis	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	
-3	y_3	-3	0	1	-1	1	1	-1	1	
-4	y_2	2	1	0	0	-1	0	1	1	
	Z_j	1	-4	-3	3	1	-3	-1	-7	
	C_j	-3	0	0	-3	-1	$3 - M$	$1 - M$		

As all $C_j \leq 0$, the optimal solution is attained.

Thus an optimal solution to the dual problem is

$$y_1 = 0, y_2 = 1, y_3 = 1, \text{ Min. } W = -\text{Max. } (W') = 7.$$

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables x_1, x_2 correspond to the artificial starting dual variables A_1, A_2 respectively. In the final simplex table of the dual problem, C_j corresponding to A_1 , and A_2 are 3 and 1 respectively after ignoring M . Thus by rule II, we get opt. $x_1 = 3$ and opt. $x_2 = 1$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1; \max. Z = 7.$$

Obs. The validity of the duality theorem is therefore checked since $\max. Z = \min. W = 7$ from both the methods.

Example 34.25. Using duality solve the following problem :

$$\text{Minimize } Z = 0.7x_1 + 0.5x_2$$

$$\text{subject to } x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18, x_1, x_2 \geq 0.$$

(V.T.U., 2004)

Solution. The dual of the given problem is $\max. W = 4y_1 + 6y_2 + 20y_3 + 18y_4$,

$$\text{subject to } y_1 + y_3 + 2y_4 \leq 0.7, y_2 + 2y_3 + y_4 \leq 0.5, y_1, y_2, y_3, y_4 \geq 0.$$

Step 1. Express the problem in the standard form.

Introducing slack variables, the dual problem in the standard form becomes

$$\max. W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2,$$

$$\text{subject to } y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7,$$

$$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5, y_1, y_2, y_3, y_4 \geq 0.$$

Step 2. Find an initial basic feasible solution.

Setting non-basic variables y_1, y_2, y_3, y_4 each equal to zero, the basic solution is

$$y_1 = y_2 = y_3 = y_4 = 0 \quad (\text{non-basic}) ; s_1 = 0.7, s_2 = 0.5 \quad (\text{basic})$$

Since the basic variables $s_1, s_2 > 0$, the initial basic solution is feasible and non-degenerate.

Initial simplex table is

	c_j	4	6	20	18	0	0		
c_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	0	1	2	1	0	0.7	0.7/1
0	s_2	0	1	(2)	1	0	1	0.5	0.5/2 ←
	Z_j	0	0	0	0	0	0	0	
	C_j	4	6	20	18	0	0		
				↑					

As C_j is positive in some columns, the initial basic solution is not optimal.

Step 3. Iterate towards an optimal solution.

(i) Introduce y_3 and drop s_2 . Then the new simplex table is

	c_j	4	6	20	18	0	0		
c_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	-1/2	0	(3/2)	1	-1/2	9/20	3/10 ←
20	y_3	0	1/2	1	1/2	0	1/2	1/4	1/2
	Z_j	0	10	20	10	0	10	5	
	C_j	4	-4	0	8	0	-10		
				↑					

As C_j is positive under some of the columns, this solution is not optimal.

(ii) Introduce y_4 and drop s_1 . Then the revised simplex table is

	c_j	4	6	20	18	0	0		
c_B	Basis	y_1	y_2	y_3	y_4	s_1	s_2	b	
18	y_4	2/3	-1/3	0	1	2/3	-1/3	3/10	
20	y_3	-1/3	2/3	1	0	-1/3	2/3	1/10	
	Z_j	16/3	22/3	20	18	16/3	22/3	74/10	
	C_j	-4/3	-4/3	0	0	-16/13	-22/3		

As all $C_j \leq 0$, this table gives the optimal solution.

Thus the optimal basic feasible solution is $y_1 = 0, y_2 = 0, y_3 = 20, y_4 = 18$, max. $W = 7.4$

Step 4. Derive optimal solution to the primal.

We note that the primal variables x_1, x_2 correspond to the slack starting dual variables s_1, s_2 respectively. In the final simplex table of the dual problem, C_j values corresponding to s_1 and s_2 are $-16/3$ and $-22/3$ respectively.

Thus, by rule I, we conclude that opt. $x_1 = 16/3$ and opt. $x_2 = 22/3$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16/3, x_2 = 22/3; \text{ min. } Z = 7.4.$$

Obs. To check the validity of the duality theorem, the student is advised to solve the given L.P.P. directly by simplex method and see that min. $Z = \text{max. } W = 7.4$.

PROBLEMS 34.7

Using duality solve the following problems (1 – 4) :

1. Minimize $Z = 2x_1 + 9x_2 + x_3$,
subject to $x_1 + 4x_2 + 2x_3 \geq 5, 3x_1 + x_2 + 2x_3 \geq 4$ and $x_1, x_2 \geq 0$. (J.N.T.U., 2001)
2. Maximize $Z = 2x_1 + x_2$,
subject to $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1, x_1, x_2 \geq 0$. (Andhra M. Tech., 2006)
3. Maximize $Z = 3x_1 + 2x_2$,
subject to $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \leq 10, x_2 \leq 3, x_1, x_2 \geq 0$.
4. Maximize $Z = 3x_1 + 2x_2 + 5x_3$,
subject to $x_1 + 2x_2 + x_3 \leq 430, 3x_1 + 2x_3 \leq 460, x_1 + 4x_2 \leq 420, x_1, x_2, x_3 \geq 0$.

34.13 (1) DUAL SIMPLEX METHOD

In § 34.9, we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using M -method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

(2) Working procedure for dual simplex method :

Step 1. (i) Convert the problem to maximization form, if it is not so.

(ii) Convert (\geq) type constraints, if any to (\leq) type by multiplying such constraints by -1 .

(iii) Express the problem in standard form by introducing slack variables.

Step 2. Find the initial basic solution and express this information in the form of dual simplex table.

Step 3. Test the nature of $C_j = c_j - Z_j$:

(a) If all $C_j \leq 0$ and all $b_i \geq 0$, then optimal basic feasible solution has been attained.

(b) If all $C_j \leq 0$ and at least one $b_i < 0$, then go to step 4.

(c) If any $C_j \geq 0$, the method fails.

Step 4. Mark the outgoing variable. Select the row that contains the most negative b_i . This will be the key row and the corresponding basic variable is the outgoing variable.

Step 5. Test the nature of key row elements :

(a) If all these elements are ≥ 0 , the problem does not have a feasible solution.

(b) If at least one element < 0 , find the ratios of the corresponding elements of C_j -row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the incoming variable.

Step 6. Iterate towards optimal feasible solution. Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

Example 34.26. Using dual simplex method :

$$\text{maximize} \quad -3x_1 - 2x_2$$

subject to $x_1 + x_2 \geq 1$, $x_1 + x_2 \leq 7$, $x_1 + 2x_2 \geq 10$, $x_2 \geq 3$, $x_1 \geq 0$, $x_2 \geq 0$.

(Mumbai, 2004)

Solution. Consists of the following steps :

Step 1. (i) Convert the first and third constraints into (\leq) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

(ii) Express the problem in standard form.

Introducing slack variables s_1, s_2, s_3, s_4 , the given problem takes the form

$$\text{Max. } Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$\text{subject to } -x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$$

$$x_0 + s_1 \equiv 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0,$$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 \equiv x_2 \equiv 0, s_1 \equiv -1, s_2 \equiv 7, s_3 \equiv -10, s_4 \equiv 3 \text{ and } Z \equiv 0.$$

\therefore Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
$Z_j = \sum c_B a_{ij}$		0	0	0	0	0	0	0
$C_j = c_j - Z_j$		-3	-2	0	0	0	0	
			↑					

Step 3. Test nature of C_r

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable:

Since b_3 is negative and numerically largest, the third row is the key row and s_2 is the outgoing variable.

Step 5. Calculate ratios of elements in C_1 -row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3$, $-2/-2 = 1$ (neglecting ratios corresponding to +ve or zero elements of key row).

Since the smaller ratio is 1, therefore, x_3 -column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value – 2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0	s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
-2	x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0	s_4	$(-\frac{1}{2})$	0	0	0	$\frac{1}{2}$	1	$-2 \leftarrow$
$Z_j = \Sigma c_B a_{ij}$		-1	-2	0	0	1	0	-10
$C_j = c_j - Z_j$		-2	0	0	0	-1	0	
		↑						

Since all C_j values are ≤ 0 and $b_4 = -2$, this solution is optimal but infeasible. We therefore proceed further.

(ii) *Mark the outgoing variable.*

Since b_4 is negative, the fourth row is the key row and s_4 is the outgoing variable.

(iii) *Calculate ratios of elements in C_j -row to the corresponding negative elements of the key row.*

This ratio is $-2/-\frac{1}{2} = 4$ (neglecting other ratios corresponding to +ve or 0 elements of key row).

$\therefore x_1$ -column is the key column and $\left(-\frac{1}{2}\right)$ is the key element.

(iv) *Drop s_4 and introduce x_1 with its associated value -3 under the c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :*

c_B	$Basis$	c_j	-3	-2	0	0	0	0	b
		x_1	x_2	s_1	s_2	s_3	s_4		
0	s_1	0	0	1	0	-1	-1	6	
0	s_2	0	0	0	1	1	1	0	
-2	x_2	0	1	0	0	0	1	3	
-3	x_1	1	0	0	0	-10	-2	4	
Z_j		-3	-2	0	0	3	4	-18	
C_j		0	0	0	0	-3	-4		

Since all C_j values are ≤ 0 and all b 's are ≥ 0 , therefore this solution is optimal and feasible. Thus the optimal solution is $x_1 = 4$, $x_2 = 3$ and $Z_{\max} = -18$.

Example 34.27. Using dual simplex method, solve the following problem :

$$\text{Minimize } Z = 2x_1 + 2x_2 + 4x_3$$

$$\text{subject to } 2x_1 + 3x_2 + 5x_3 \geq 2, 3x_1 + x_2 + 7x_3 \leq 3, x_1 + 4x_2 + 6x_3 \leq 5, x_1, x_2, x_3 \geq 0.$$

(Kurukshetra, 2009 ; Kerala, 2005)

Solution. Consists of the following steps :

Step 1. (i) Convert the given problem to maximization form by writing

$$\text{Maximize } Z' = -2x_1 - 2x_2 - 4x_3.$$

(ii) Convert the first constraint into (\leq) type. Thus it is equivalent to

$$-2x_1 - 3x_2 - 4x_3 \leq -2$$

(iii) Express the problem in standard form.

Introducing slack variables, s_1, s_2, s_3 , the given problem becomes

$$\text{Max. } Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } -2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$$

$$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$$

$$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Step 2. Find the initial basic solution.

Setting the decision variables x_1, x_2, x_3 each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0, s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

\therefore Initial solution is given by the table below :

c_B	$Basis$	c_j	-2	-2	-4	0	0	0	b
		x_1	x_2	x_3	s_1	s_2	s_3		
0	s_1	-2	(-3)	-5	1	0	0	-2 ←	
0	s_2	3	1	7	0	1	0	3	
0	s_3	1	4	6	0	0	1	5	
		Z_j	0	0	0	0	0	0	0
		C_j	-2	-2	-4	0	0	0	
					↑				

Step 3. Test nature of C_j

Since all C_j values are ≤ 0 and $b_1 = -2$, the initial solution is optimal but infeasible.

Step 4. Mark the outgoing variable.

Since $b_1 < 0$, the first row is the key row and s_1 is the outgoing variable.

Step 5. Calculate the ratio of elements of C_j -row to the corresponding negative elements of the key row.

These ratios are $-2/-2 = 1$, $-2/-3 = 0.67$, $-4/-5 = 0.8$.

Since 0.67 is the smallest ratio, x_2 -column is the key column and (-3) is the key element.

Step 6. Iterate towards optimal feasible solution.

Drop s_1 and introduce x_2 with its associated value -2 under c_B column. Then the revised dual simplex table is

	c_j	-2	-2	-4	0	0	0	
c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
-2	x_2	2/3	1	5/3	-1/3	0	0	2/3
0	s_2	7/3	0	16/3	1/3	1	0	7/3
0	s_3	-5/3	0	-2/3	4/3	0	1	7/3
	Z_j	-4/3	-2	-10/3	2/3	0	0	-4/3
	C_j	-2/3	0	-2/3	-2/3	0	0	

Since all $C_j \leq 0$ and all b_i 's are > 0 , this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 0 \text{ and } \max Z' = -4/3 \text{ i.e., min } Z = 4/3.$$

PROBLEMS 34.8

Using dual simplex method, solve the following problems :

- Maximize $Z = -3x_1 - x_2$

subject to $x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2; x_1, x_2 \geq 0$.

- Minimize $Z = 2x_1 + x_2$,

subject to $3x_1 + x_2 \geq 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 3, x_1, x_2 \geq 0$.

(Kurukshetra, 2007 S)

- Minimize $Z = x_1 + 2x_2 + 3x_3$,

subject to $2x_1 - x_2 + x_3 \geq 4, x_1 + x_2 + 2x_3 \leq 8, x_2 - x_3 \geq 2; x_1, x_2, x_3 \geq 0$.

- Minimize $Z = x_1 + 2x_2 + x_3 + 4x_4$

subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10, 3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$

$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15, x_1, x_2, x_3, x_4 \geq 0$.

34.14 (1) TRANSPORTATION PROBLEM

This is a special class of linear programming problems in which the objective is to transport a single commodity from various origins to different destinations at a minimum cost.

(2) Formulation of a transportation problem. There are m plant locations (origins) and n distribution centres (destinations). The production capacity of the i th plant is a_i and the number of units required at the j th destination is b_j . The transportation cost of one unit from the i th plant to the j th destination is c_{ij} . Our objective is to determine the number of units to be transported from the i th plant to j th destination so that the total transportation cost is minimum.

Let x_{ij} be the number of units shipped from i th plant to j th destination, then the general transportation problem is :

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to the constraints

$$x_{i1} + x_{i2} + \dots + x_{in} = a_i, \text{ for } i\text{th origin } (i = 1, 2, \dots, m)$$

$$x_{1j} + x_{2j} + \dots + x_{mj} = b_j, \text{ for } j\text{th destination } (j = 1, 2, \dots, n)$$

$$x_{ij} \geq 0.$$

Def. 1. The two sets of constraints will be consistent if

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

which is the condition for a transportation problem to have a *feasible solution*. Problems satisfying this condition are called *balanced transportation problems*.

2. A feasible solution to a transportation problem is said to be a *basic feasible solution* if it contains at the most $(m+n-1)$ strictly positive allocations, otherwise the solution will *degenerate*. If the total number of positive (non-zero) allocations is exactly $(m+n-1)$, then the basic feasible solution is said to be *non-degenerate*.

3. A feasible solution which minimizes the transportation cost is called an *optimal solution*. This problem is explicitly represented in the following *transportation table*:

		Distribution centres (Destinations)				Supply a_1
		1	2	j	n	
Plants (origins)	1					c_{1n}
	2	c_{11}	c_{12}	\dots	c_{1j}	\dots
	2					c_{2n}
	2	c_{21}	c_{22}	\dots	c_{2j}	\dots
	i					c_{in}
	i	c_{i1}	c_{i2}	\dots	c_{ij}	\dots
	m					c_{mn}
	m	c_{m1}	c_{m2}	\dots	c_{mj}	\dots
Demand		b_1	b_2	\dots	b_j	\dots
				\dots	b_n	$\Sigma a_i = \Sigma b_j$

The mn squares are called *cells*. The per unit cost c_{ij} of transporting from the i th origin to the j th destination is displayed in the *lower right side of the (i, j) th cell*. Any feasible solution is shown in the table by entering the value of x_{ij} in the *small square at the upper left side of the (i, j) th cell*. The various a 's and b 's are called *rim requirements*. The feasibility of a solution can be verified by summing the values of x_{ij} along the rows and down the columns.

Obs. 1. The special features of a transportation problem are that

- (i) the coefficients of all x_{ij} in the constraints are unity, and
- (ii) the total supply $\sum a_i = \text{total demand } \sum b_j$.

Obs. 2. The objective function and the constraints being all linear, the problem can be solved by simplex method. But the number of variables being large, there will be too many calculations. However, the coefficients of all x_{ij} in the constraints being unity, we can look for some technique which would be simpler than the simplex method.

34.15 WORKING PROCEDURE FOR TRANSPORTATION PROBLEMS

Step 1. Construct transportation table. Express the supply from the origins a_i , demand at destinations b_j and the unit shipping cost c_{ij} in the form of a matrix, known as transportation table. If the supply and demand are equal, the problem is *balanced*.

Step 2. Find the initial basic feasible solution. We find an initial allocation which satisfies the demand at each project site without violating the capacities of the plants (origins) and also meeting the non-negativity

restrictions. There are several methods for initial allocations e.g., North-West corner rule, Row minima method, Least cost method, Vogel's approximation method. *The Vogel's approximation method (VAM) takes into account not only the least cost c_{ij} but also the costs that just exceed the least cost c_{ij} and therefore yields a better initial solution than obtained from other methods.* As such we shall confine ourselves to VAM only which consists of the following steps :

- Display the difference between the least and the next to least costs in each row, by enclosing them in brackets to the right of the row. Similarly display the differences for each column within brackets below that column.
- Identify the row or column with the largest difference among all the rows and columns and allocate as much as possible under the rim requirements, to the lowest cost cell in that row or column. In case of a tie allocate to the cell associated with the lower cost.
If the greatest difference corresponds to i th row and c_{ij} is the lowest cost in the i th row, allocate as much as possible i.e., $\min(a_i, b_j)$ in the (i, j) th cell and cross off the i th row or the j th column.
- Recalculate the row and column differences for the reduced table and go to the previous step.
- Repeat the procedure till all the rim requirements are satisfied. Note the solution in the upper left corner small squares of the basic cells.

Step 3. Apply optimality check

In the above solution, the number of allocations must be ' $m + n - 1$ ', otherwise the basic solution degenerates.

Now to test for optimality, we apply the modified distribution (MODI) method and examine each unoccupied cell to determine whether making an allocation in it reduces the total transportation cost and then repeat this procedure until lowest possible transportation cost is obtained. This method consists of the following steps :

- Note the numbers u_i along the left and v_j along the top of the cost matrix such that their sums equal to the original costs of occupied cells i.e., solve the equations $[u_i + v_j = c_{ij}]$ starting initially with some $u_i = 0$.
- Compute the net evaluations $w_{ij} = u_i + v_j - c_{ij}$ for all the empty cells and enter them in upper right hand corners of the corresponding cells.
- Examine the sign of each w_{ij} . If all $w_{ij} \leq 0$, then the current basic feasible solution is optimal. If even one $w_{ij} > 0$, this solution is not optimal and we proceed further.

Step 4. Iterate towards optimal solution

- Choose the unoccupied cell with the largest w_{ij} and mark θ in it.
- Draw a closed path consisting of horizontal and vertical lines beginning and ending at θ -cell and having its other corners at the allocated cells.
- Add and subtract θ alternately to and from the transition cells of the loop subject to rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain non-negative. Now the basic cell whose allocation has been reduced to zero leaves the basis.

Step 5. Return to step 3 and repeat the process until an optimal basic feasible solution is obtained.

Example 34.28. Solve the following transportation problem :

		Destination				Availability
		A	B	C	D	
Source	I	21	16	25	13	11
	II	17	18	14	23	13
	III	33	27	18	41	19
Requirement		6	10	12	15	43

Solution. Consists of the following steps :

Step 1. Transportation table. Here the total availability and the total requirement being the same i.e. 43, the problem is balanced.

Step 2. Find the initial basic feasible solution. Following VAM, the differences between the smallest and next to the smallest costs in each row and each column are computed and displayed within brackets against the respective rows and columns (Table 1). The largest of these differences is (10) which is associated with the fourth column.

Table 1

21	16	25	11 13
17	18	14	23
32	27	18	41

6 10 12 15
(4) (2) (4) (10)

11(3)
13(3)
19(9)

Table 2

17	18	14	4 23	13(3)
32	27	18	41	19(9)
6 (15)	10 (9)	12 (4)	4 (18)	+

Since c_{14} (= 13) is the minimum cost, we allocate $x_{14} = \min(11, 15) = 11$. This exhausts the availability of first row and therefore we cross it.

Table 3

6	17	18	14	9(3)
32	27	18		19(9)
6 (15)	10 (9)	12 (4)		

Table 4

3	18	14	3(4)
	27	18	19(9)
	10 (9)	12 (4)	

Table 5

7	27	12 18	19
7	12	+	

The row and column differences are now computed for reduced table 2 and displayed within brackets. The largest of these is (18) which is against the fourth column. Since c_{14} (= 23) is the minimum cost, we allocate $x_{14} = \min(13, 4) = 4$.

This exhausts the availability of fourth column which we cross off. Proceeding in this way, the subsequent reduced transportation tables and differences for the remaining rows and columns are shown in Tables 3, 4 and 5.

Finally the initial basic feasible solution is as shown in Table 6.

Table 6

21	16	25	11 13
6	3		4 23
17	18	14	

32 27 12 41

Table 7

v_j	17	18	9	23
u_i	-10	(-)	(-)	11 13
-10	(-)	(-)	(-)	11 13
0	21	16	25	
9	6	3	(-) 4	23
	17	18	14	
	(-)	7	12	(-)
	32	27	18	41

Step 3. Apply optimality check

As the number of allocations = $m + n - 1$ (i.e., 6), we can apply MODI method.

(i) We have $u_2 + v_1 = 17$, $u_2 + v_2 = 18$, $u_3 + v_2 = 27$

$$u_3 + v_3 = 18, u_1 + v_4 = 13, u_2 + v_4 = 23$$

Let $u_2 = 0$, then $v_1 = 17$, $v_2 = 18$, $u_3 = 9$, $v_3 = 9$, $v_4 = 23$, $u_1 = -10$.

(ii) Net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ for all empty cells are

$$w_{11} = -14, w_{12} = -8, w_{13} = -26, w_{23} = -5, w_{31} = -6, w_{34} = -9.$$

(iii) Since all the net evaluations are negative, the current solution is optimal. Hence the optimal allocation is given by

$$x_{14} = 11, x_{21} = 6, x_{22} = 3, x_{24} = 4, x_{32} = 7 \text{ and } x_{33} = 12.$$

∴ The optimal (minimum) transportation cost

$$= 11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18 = ₹ 796.$$

Example 34.29. A company has three cement factories located in cities 1, 2, 3 which supply cement to four projects located in towns 1, 2, 3, 4. Each plant can supply 6, 1, 10 truck loads of cement daily respectively and the daily cement requirements of the projects are respectively 7, 5, 3, 2 truck loads. The transportation costs per truck load of cement (in hundreds of rupees) from each plant to each project site are as follows :

		Project sites			
		1	2	3	4
Factories	1	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

Determine the optimal distribution for the company so as to minimize the total transportation cost.

Solution. Consists of the following steps :

Step 1. Construct transportation table. Express the supply from the factories, demands at sites and the unit shipping cost in the form of the following transportation table (Table 1). Here the supply being equal to the demand, the problem is balanced.

Table 1

		Project sites				Supply
		1	2	3	4	
Factories	1	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10
Demand		7	5	3	2	17

Step 2. Find the initial basic feasible solution.

Using VAM, the initial basic feasible solution is as shown in Table 2. The transportation cost according to this route is given by

$$Z = ₹(1 \times 2 + 5 \times 3 + 1 \times 1 + 6 \times 5 + 3 \times 15 + 1 \times 9) \text{ times } 100 = ₹ 10,200.$$

Step 3. Apply optimality check.

As the numbers of allocations = $(m + n - 1)$ i.e., 6, we can apply MODI method.

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 3. Since the net evaluations in two cells are positive, a better solution can be found.

Table 2

1	5				6
2		3	11	7	
			1		1
1		0	6	1	
6		3	1		10
5		8	15	9	
7	5	3	2		

Table 3

$u_i \backslash v_j$	2	3	12	6
0	1	5	(+)	(-)
-5	2	3	11	7
1	(-)	(-)	(+)	1
3	1	0	6	1
6	6	(-)	3	1
5	8	15		9

Step 4. Iterate towards optimal solution.

First iteration :

(a) **Next basic feasible solution.**

(i) Choose the unoccupied cell with the maximum w_{ij} . In case of a tie, select the one with lower original cost. In Table 3, cells (1, 3) and (2, 3) each have $w_{ij} = 1$ and out of these all (2, 3) has lower original cost 6, therefore we take this as the next basic cell and note θ in it.

(ii) Draw a closed path beginning and ending at θ -cell. Add and subtract θ , alternately to and from the transition cells of the loop subject the rim requirements. Assign a maximum value to θ so that one basic variable becomes zero and the other basic variables remain ≥ 0 . Now the basic cell whose allocation has been reduced to zero leaves the basis. This gives the second basic feasible solution (Table 5).

Table 4

1	5			
2	3	11	7	
		θ	1	-θ
1	0	6	1	
6	3	-θ	1	+θ
5	8	15	9	

Table 5

1	5			
2	3	11	7	
		θ = 1	1 - 1	
1	0	6	1	
6	3	3 - 1	1 + 1	
5	8	15	9	

∴ Total transportation cost of this revised solution.

$$= ₹(1 \times 2 + 5 \times 3 + 1 \times 6 + 6 \times 5 + 2 \times 15 + 2 \times 9) \text{ times } 100 = ₹ 10,100.$$

(b) *Optimality check.* As the number of allocations in table 5 = $m + n - 1$ (i.e., 6), we can apply MODI method. We compute the net evaluations which are shown in Table 6. Since the cell (1, 3) has a positive value, the second basic feasible solution is not optimal.

Table 6

$u_i \backslash v_j$	2	3	12	6
1	2	5	(+)	(-)
0	3	11	7	
(-)	(-)	1	(-)	
-6	1	0	6	1
6	(-)	2	2	
3	5	8	15	9

Table 7

1 - 1	5	θ = 1		
2	3	11	7	
		1		
1	0	6	1	
6 + 1		2 - 1	2	
5	8	15	9	

Second iteration :

(a) *Next basic feasible solution.* In the second basic feasible solution introduce the cell (1, 3) taking $\theta = 1$ and drop the cell (1, 1) giving Table 7. Thus we obtain the third basic feasible solution (Table 8).

Table 8

	5	1		
2	3	11	7	
	1			
1	0	6	1	
7	1	2		
5	8	15	9	

Table 9

1	3	11	5
(-)	5	1	(-)
2	3	11	7
(-)	(-)	1	(-)
1	0	6	1
7	(-)	1	2
5	8	15	9

(b) *Optimality Check.* As the number of allocations in Table 8 = $m + n - 1$ (i.e., 6), we can apply MODI method.

We compute the net evaluations which are shown in Table 9. Since all the net evaluations are ≤ 0 , this basic feasible solution is optimal.

Thus the optimal transportation policy is as shown in Table 9 and the optimal transportation cost

$$= ₹[5 \times 3 + 1 \times 11 + 1 \times 6 + 7 \times 5 + 1 \times 15 + 2 \times 9] \text{ times } 100 = ₹ 10,000.$$

34.16 DEGENERACY IN TRANSPORTATION PROBLEMS

When the number of basic cells in a non-transportation table, is less than ' $m + n - 1$ ' the basic solution degenerates. To remove the degeneracy, we assign a small positive value ϵ to as many zero-valued variables as may be necessary to complete ' $m + n - 1$ ' basic variables. The cells containing ϵ are then treated like other basic cells and the problem is solved in the usual way. The ϵ 's are kept till the optimum solution is attained. Then we let each $\epsilon \rightarrow 0$.

Example 34.30. Solve the following transportation problem :

		To						
		9	12	9	6	9	10	5
From	7	3	7	7	5	5	6	
	6	5	9	11	3	11	2	
	6	8	11	2	2	10	9	
	4	4	6	2	4	2	22	
		4	4	6	2	4	2	22

Solution. Consists of the following steps :

Step 1. Transportation table. The total supply and total demand being equal, the transportation problem is balanced.

Step 2. Find the initial basic feasible solution.

Using VAM, the initial basic feasible solution is as shown in Table 1.

Step 3. Apply optimality check. Since the number of basic cells is 8 which is less than $m + n - 1 = 9$, the basic solution degenerates. In order to complete the basis and thereby remove degeneracy, we require only one more positive basic variable. We select the variable x_{23} and allocate a small positive quantity ε to the cell (2, 3).

Table 1

		5						+
9		12	9	6	9	10		5
	4		ε				2	
7		3	7	7	5	5		
1			1					2
6		5	9	11	3	11		
3				2	4			9
6		8	11	2	2	10		
4	4			2	4	2	2	+

We now compute the net evaluations $w_{ij} = (u_i + v_j) - c_{ij}$ which are exhibited in Table 2. Since all the net evaluations are ≤ 0 , the current solution is optimal. Hence the optimal allocation is

$$x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{33} = 1, x_{41} = 3, x_{44} = 2 \text{ and } x_{45} = 4.$$

\therefore The minimum (optimal) transportation cost

$$\begin{aligned} &= 5 \times 9 + 4 \times 3 + \varepsilon \times 7 + 2 \times 5 + 1 \times 6 + 1 \times 9 + 3 \times 6 + 2 \times 2 + 4 \times 2 \\ &= 112 + 7\varepsilon = ₹ 112 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Table 2

$u_i \backslash v_j$	4	3	7	0	0	5	
2	(-)	(-)	5	(-)	(-)	(-)	
9	12		9	6	9	10	
(-)	4	ε		(-)	(-)	2	
0	7	3	7	7	5	5	
1		(0)	1	(-)	(-)	(-)	
2	6	5	9	11	3	11	
3		(-)	(-)	2	4	(-)	
2	6	8	11	2	2	10	

PROBLEMS 34.9

1. Obtain an initial basic feasible solution to the following transportation problem :

	To					
	D	E	F	G		
From	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
	200	225	275	250		

2. Solve the following transportation problem :

Suppliers	A	B	C	Available
Consumers				
I	6	8	4	14
II	4	9	8	12
III	1	2	6	5
Required	6	10	15	31

3. Consider four bases of operations B_i and three targets T_j . The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table :

	T_j	1	2	3
B_i		8	6	5
1		6	6	6
2		10	8	4
3		8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all the three targets.

4. A company has factories F_1, F_2, F_3 which supply warehouses at W_1, W_2 and W_3 . Weekly factory capacities, weekly warehouse requirements and unit shipping costs (in rupees) are as follows :

Factories	Warehouses			Supply
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Determine the optimal distribution for this company to minimize shipping costs.

5. A company is spending ₹ 1,000 on transportation of its units from plants to four distribution centres. The supply and demand of units, with unit cost of transportation are given below :

Plants	Distribution centres				Availabilities
	D_1	D_2	D_3	D_4	
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirements	5	8	7	15	

What can be the maximum saving by optimal scheduling.

6. A departmental store wishes to stock the following quantities of a popular product in three types of containers :

Container type	1	2	3
Quantity	170	200	180
Dealer	1	2	3
Quantity	150	160	110

Tenders are submitted by four dealers who undertake to supply not more than the quantities shown below :

Dealer	1	2	3	4
Quantity	150	160	110	130

The store estimates that profit per unit will vary with the dealer as shown below :

Dealers → Container type ↓	1	2	3	4
1	8	9	6	3
2	6	11	5	10
3	3	8	7	9

Find the maximum profit of the store.

7. Obtain an optimum basic feasible solution to the following transportation problem :

From			To	Available		
			7	3	4	2
			2	1	3	3
			3	4	6	5
			4	1	5	10
			Demand			

8. A company has three plants at locations *A*, *B* and *C* which supply to warehouses located as *D*, *E*, *F*, *G* and *H*. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs in rupees are given below :

From		To				
		<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>
<i>A</i>	5	8	6	6	3	
<i>B</i>	4	7	7	6	6	
<i>C</i>	8	4	6	6	8	

Determine an optimum distribution for the company in order to minimize the total transportation cost.

34.17 (1) ASSIGNMENT PROBLEM

An assignment problem is a special type of transportation problem in which the objective is to assign a number of origins to an *equal* number of destinations at a minimum cost (or maximum profit).

(2) Formulation of an assignment problem. There are *n* new machines M_i ($i = 1, 2, \dots, n$) which are to be installed in a machine shop. There are *n* vacant spaces S_j ($j = 1, 2, \dots, n$) available. The cost of installing the machine M_i at space S_j is c_{ij} rupees. Let us formulate the problem of assigning machines to spaces so as to minimize the overall cost.

Let x_{ij} be the assignment of machine M_i to space S_j i.e., let x_{ij} be a variable such that

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th machine is installed at } j\text{th space} \\ 0, & \text{otherwise} \end{cases}$$

Since one machine can only be installed at each space, we have

$$x_{i1} + x_{i2} + \dots + x_{in} = 1, \text{ for machine } M_i \quad (i = 1, 2, \dots, n)$$

$$x_{1j} + x_{2j} + \dots + x_{nj} = 1, \text{ for space } S_j \quad (j = 1, 2, \dots, n)$$

Also the total installation cost is $\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$.

Thus the assignment problem can be stated as follows :

Determine $x_{ij} \geq 0$ ($j = 1, 2, \dots, n$) so as to

$$\text{minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

subject to the constraints $\sum_{i=1}^n x_{ij} = 1, j = 1, 2, \dots, n$ and $\sum_{j=1}^n x_{ij} = 1, i = 1, 2, \dots, n$.

This problem is explicitly represented by the following $n \times n$ cost matrix :

		Spaces				
		S_1	S_2	S_3	...	S_n
<i>Machines</i>	M_1	c_{11}	c_{12}	c_{13}	...	c_{1n}
	M_2	c_{21}	c_{22}	c_{23}	...	c_{2n}
	M_3	c_{31}	c_{32}	c_{33}	...	c_{3n}
	:	:	:	:	...	:
	M_n	c_{n1}	c_{n2}	c_{n3}	...	c_{nn}

Obs. This assignment problem constitutes $n!$ possible ways of installing n machines at n spaces. If we enumerate all these $n!$ alternatives and evaluate the cost of each one of them and select the one with the minimum cost, the problem would be solved. But this method would be very slow and time consuming, even for small value of n and hence it is not at all suitable. However, a much more efficient method of solving such problems is available. This is the **Hungarian method** for solution of assignment problems which we describe below.

34.18 WORKING PROCEDURE TO SOLVE AN ASSIGNMENT PROBLEM

Step 1. Reduce the matrix. Subtract the smallest element of each row (of the given cost matrix) from all elements of that row. See if each row contains at least one zero. If not, subtract the smallest element of each column (not containing zero) from all the elements of that column. This gives the *reduced matrix*.

Step 2. Assign the zeros

(a) Examine rows (of the reduced matrix) successively until a row with exactly one unmarked zero is found. Make an assignment to this single zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

(b) Now examine columns successively until a column with exactly one unmarked zero is found. Encircle this zero and make an assignment there. Then cross any other zero in its row. Continue in this way until all the columns have been examined.

In case, some rows or columns contain more than one unmarked zeros, encircle any unmarked zero arbitrarily and cross all other zeros in its row or column. Proceed in this way, till no zero is left unmarked.

Step 3. Apply optimality check.

Repeat step 2 (a) and (b) until one of the following occurs :

- (i) If no row or no column is without assignment (encircled zero), then the current assignment is optimal.
- (ii) If there is some row and/or column without an assignment, then the current assignment is not optimal and we go to next step.

Step 4. Find minimum number of lines crossing all zeros.

(a) Tick (\checkmark) the rows which do not have assignments.

(b) Tick (\checkmark) the columns (not already marked) which have zeros in the ticked row.

(c) Tick (\checkmark) the rows (not already marked) which have assignments in ticked columns.

Repeat (b) and (c) until no more marking is required.

(d) Draw lines through all unticked rows and ticked columns. If the number of these lines is equal to the order of the matrix then it is an optimal solution otherwise not.

Step 5. Iterate towards optimal solution.

Select the smallest element and subtract it from all uncovered elements. Add this smallest element to every element lying at the intersection of two lines. The resulting matrix is the second basic feasible solution.

Step 6. Go to step 2 and repeat the procedure until the optimal solution is attained.

Example 34.31. Four jobs are to be done on four different machines. The cost (in rupees) of producing i th job on the j th machine is given below :

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J_1	15	11	13	15
	J_2	17	12	12	13
	J_3	14	15	10	14
	J_4	16	13	11	17

Assign the jobs to different machines so as to minimize the total cost.

Solution. Consists of the following steps :

Step 1. Reduce the matrix. Subtract the smallest element 11 of row 1 from all its elements. Similarly subtract 12, 10 and 11 from rows 2, 3 and 4 respectively. The resulting matrix is as shown in Table 1. Columns 1 and 4 do not have any zero element. Subtract the smallest element 4 of Col. 1 from all its elements and element 1 from all elements of Col. 4. The reduced matrix is as given in Table 1.

Table 1

	M_1	M_2	M_3	M_4
J_1	4	0	2	4
J_2	5	0	0	1
J_3	4	5	0	4
J_4	5	2	0	6

Table 2

	M_1	M_2	M_3	M_4
J_1	0	0	2	3
J_2	1	X	X	0
J_3	0	5	X	3
J_4	1	2	0	5

Step 2. Assign the zeros. Row 4 has a single unmarked zero in Col. 3. Encircle it and cross all other zeros in Col. 3. Row 3 has a single unmarked zero in Col. 1. Encircle it and cross the other zero in col. 1. Row 1 has a single unmarked zero in Col. 2. Encircle it and cross the other zero in Col. 2. Finally row 2 has a single unmarked zero in Col. 4. Encircle it (Table 2).

Step 3. Apply optimality check. Since we have one encircled zero in each row and in each column, this gives the optimal solution.

∴ The optimal assignment policy is

Job 1 to machine 2, Job 2 to machine 4, Job 3 to machine 1, Job 4 to machine 3,
and the minimum assignment cost = ₹ (11 + 13 + 14 + 11) = ₹ 49.

Example 34.32. A marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows :

		Sales districts				
		A	B	C	D	E
Salesman	1	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find the assignment of salesmen to districts that will result in maximum sales.

(Madras, 2000)

Solution. Consists of the following steps :

Step 1. Reduce the matrix. Convert the given maximization problem into a minimization problem, by making all the profits negative, since $\max Z = \min (-Z)$. Then subtract the smallest element of each row from the elements of that row. Now subtract the smallest element of each col. (not containing zero) from the elements of that column. This gives the *reduced matrix* (Table 1).

Table 1

8	0	X	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	X	5
11	5	0	X	1

Table 2

12	0	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Step 2. Assign the zeros. Rows 2 and 3 have each a single unmarked zero in Col. 1. Encircle these. Columns 2 and 5 have each a single unmarked zero in row 1. Encircle these and cross the zero in row 1. Columns 3 and 4 have each unmarked zeros. Encircle the zeros in each of the rows 4 and 5 as shown in Table 1 and cross other zeros.

Step 3. Apply optimality check. As col. 4 is without assignment, this solution is not optimal. Therefore we go to next step.

Step 4. Find minimum number of lines crossing all zeros. Draw the least number of horizontal and vertical (dotted) lines which cover all the zeros. Since there are four dotted lines which are less than the order of the cost matrix (= 5), we got to step 5.

Step 5. Iterate towards optimal solution. Select the smallest element in the Table 1, not covered by the dotted lines. Such an element is 4 which lies at two different positions. Selecting the elements that lies at position (3, 5) arbitrarily, subtract it from all the uncovered elements of the cost matrix (Table 1) and add the same to the elements lying at the intersection of two dotted lines. Now draw more minimum number of dotted lines so as to cover the new zero. Here we draw such a line in Col. 5 (Table 2).

Table 3

	A	B	C	D	E
1		(0)	X		
2	(0)				X
3	X				(0)
4			(0)	X	
5			X	(0)	

Now, since the number of dotted lines is equal to the order for the cost matrix, the optimal solution is attained.

Finally, to determine this optimal assignment, we consider only the zero elements (Table 3) :

- (i) Examine successively the rows with exactly one zero. There is no such row.
- (ii) Examine successively the columns with exactly one zero. Col. 2 has one zero, encircle it and cross all zeros of row 1.

(iii) Encircle arbitrarily the zero in position (2, 1) and cross all zeros in row 2 and Col. 1. Then encircle the unmarked zero in row 3. Now encircle arbitrarily the zero in position (4, 3) and cross all zeros in row 4 and Col. 3. Finally encircle the remaining unmarked zero in row 5.

Now each row and each column has one encircled zero, therefore the optimal assignment policy is :

Salesman 1 to district B, 2 to A, 3 to E, 4 to C and 5 to D.

Hence the maximum sales = ₹ (38 + 40 + 37 + 41 + 35) × 100 = ₹ 19,100.

PROBLEMS 34.10

1. A firm plans to begin production of three new products on its three plants. The unit cost of producing i at plant j is as given below. Find the assignment that minimizes the total unit cost.

		Plant		
		1	2	3
Product	1	10	8	12
	2	18	6	14
	3	6	4	2

2. Solve the following assignment problem :

		1	2	3	4
A	10	12	19	11	
	5	10	7	8	
	12	14	13	11	
	8	15	11	9	

3. A machine tool company decides to make four sub-assemblies through four contractors. Each contractor is to receive only one sub-assembly. The cost of each sub-assembly is determined by the bids submitted by each contractor and is shown in table below (in hundreds of rupees). Assign different assemblies to contractors so as to minimize the total cost.

		Contractor			
		A	B	C	D
Sub-assembly	I	15	13	14	17
	II	11	12	15	13
	III	18	12	10	11
	IV	15	17	14	16

4. Four professors are each capable of teaching any one of the four different courses. Class preparations time in hours for different topics varies from professor to professor and is given in the table below. Each professor is assigned only one course. Find the assignment policy schedule so as to minimize the total course preparation time for all courses.

Prof.	L.P.	Quenig Theory	Dynamic Programming	Regression analysis
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	3	15	13	8

5. Consider the problem of assigning four working labour units to four jobs. The assignment costs in thousands of rupees are given by the following matrix.

Labour unit ↓	Job			
	I	II	III	IV
L_1	42	35	28	21
L_2	30	25	20	15
L_3	30	25	20	15
L_4	24	20	16	12

Find the optimal assignment.

6. A company has six jobs to be processed by six mechanics. The following table gives the return in rupees when the i th job is assigned to the j th mechanic. How should the jobs be assigned to the mechanics so as to maximize the over all return ?

Mechanic ↓	Job					
	I	II	III	IV	V	VI
1	9	22	58	11	19	27
2	43	78	72	50	63	48
3	41	28	91	37	45	33
4	74	42	27	49	39	32
5	36	11	57	22	25	18
6	13	56	53	31	17	28

34.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 34.11

Fill up the blanks in the following questions :

1. Infeasibility in a linear programming problem means
2. The significance of the $(Z_j - C_j)$ row in the simplex solution procedure is that
3. The duality principle states that
4. The difference between the transportation problem and the assignment problem is
5. The special features of a transportation problem are
6. The canonical form of an L.P.P. is such that
7. The dual problem of the L.P.P. :

Max. $Z = 4x_1 + 9x_2 + 2x_3$,
 subject to $2x_1 + 3x_2 + 2x_3 \leq 7$, $3x_1 - 2x_2 + 4x_3 = 5$, $x_1, x_2, x_3 \geq 0$, is

8. The optimality and feasibility conditions related with Dual simplex method are
9. Feasible and basic solutions related with a transportation problem are
10. A transportation problem is

Demand	Supply				
	2	3	11	4	15
	5	6	8	7	20
	10	5	12	8	

Its linear programming problem is

11. The basic feasible solutions of $2x_1 + x_2 + 4x_3 = 11$, $3x_1 + x_2 + 5x_3 = 14$ are
12. A slack variable is defined as
13. The advantage of dual simplex method is
14. If the total availability is equal to the total requirements, the transportation problem is called
15. An artificial variable is that
16. Two conditions on which the simplex method is based are
17. A feasible solution which minimizes the transportation cost is called an solution.
18. The dual problem of: Maximize $5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$, $-x_1 + 5x_2 \geq 3$, x_1 unrestricted and $x_2 \geq 0$, is
19. For a balanced transportation problem with 3 rows and 3 columns, the number of basic variables will be
20. Using graphical method, Max. $Z = 5x_1 + 3x_2$ subject to $5x_1 + 2x_2 \leq 10$, $3x_1 + 5x_2 \leq 15$, $x_1, x_2 \geq 0$, is
21. In a L.P. problem, unbounded solution is that
22. Degeneracy in a transportation problem is resolved by
23. A basic solution is said to be non-degenerate in L.P.P. when
24. The dual of the problem Max. $Z = 2x_1 + x_2$ subject to $-x_1 + 2x_2 \leq 2$, $x_1 + x_2 \leq 4$, $x_1 \leq 3$, $x_1, x_2 \geq 0$ is
25. The two methods used to find the initial solution of a transportation problem are
26. Constraints involving 'equal to sign' do not require use of or variables.

Calculus of Variations

1. Introduction.
2. Functionals.
3. Euler's equation.
4. Solutions of Euler's equation.
5. Geodesics.
6. Isoperimetric problems.
7. Several dependent variables.
8. Functionals involving higher order derivatives.
9. Approximate solution of boundary value problems—Rayleigh–Ritz method.
10. Weighted residual method—Galerkin's method.
11. Hamilton's principle.
12. Lagrange's equations.

35.1 INTRODUCTION

The calculus of variations is a powerful technique for the solution of problems in dynamics of rigid bodies, optimization of orbits and vibration problems. The subject primarily concerns with finding maximum or minimum value of a definite integral involving a certain function. It is something beyond finding stationary values of a given function. Only an elementary exposition of the subject is given here with the sole aim of introducing the student to a topic whose importance is fast growing in science and engineering.

Before proceeding further, the student should revise § 5.12 concerning maxima and minima of functions of several variables.

35.2 FUNCTIONALS

Consider the problem of finding a curve through two points (x_1, y_1) and (x_2, y_2) whose length is a minimum (Fig. 35.1). It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1$, $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$ is a minimum.

In general terms, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x, y, y')$,

$$\int_{x_1}^{x_2} f(x, y, y') dx \text{ is a stationery value or an extremum.} \quad \dots(1)$$

An integral such as (1), which assumes a definite value for functions of the type $y = y(x)$ is called a **functional**.

In differential calculus, we deal with the problems of maxima and minima of functions. The calculus of variations is however, concerned with maximizing or minimizing functionals.

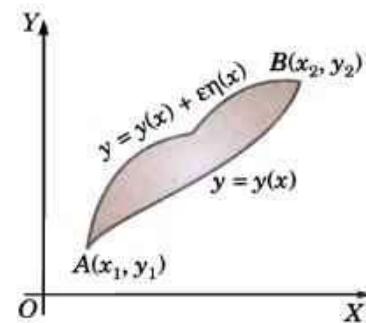


Fig. 35.1

35.3 EULER'S EQUATION

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y') dx \text{ to be an extremum is that}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}} - \frac{\mathbf{d}}{\mathbf{dx}} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}'} \right) = \mathbf{0}$$

This is called *Euler's equation*.

Proof. Let $y = y(x)$ be the curve joining points $A(x_1, y_1), B(x_2, y_2)$ which makes I an extremum. Let

$$y = y(x) + \varepsilon \eta(x) \quad \dots(1)$$

be a neighbouring curve joining these points so that at A , $\eta(x_1) = 0$ and at B , $\eta(x_2) = 0$. $\dots(2)$

The value of I along (1) is $I = \int_{x_1}^{x_2} f[x, y(x) + \varepsilon \eta(x), y'(x) + \varepsilon \eta'(x)] dx$

This being a function of ε , is a maximum or minimum for $\varepsilon = 0$, when

$$\frac{dI}{d\varepsilon} = 0 \text{ at } \varepsilon = 0 \quad \dots(3)$$

\therefore Differentiating I under the integral sign by Leibnitz's rule (p. 139), we have

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \varepsilon} \right) dx \quad \dots(4)$$

But ε being independent of x , $\frac{\partial x}{\partial \varepsilon} = 0$. Also from (1), $\frac{\partial y}{\partial \varepsilon} = \eta(x)$ and $\frac{\partial y'}{\partial \varepsilon} = \eta'(x)$.

Substituting these values in (4), we get $\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx$

Integrating the second term on the right by parts, we have

$$\begin{aligned} \frac{dI}{d\varepsilon} &= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta(x) dx + \left[\left| \frac{\partial f}{\partial y'} \eta(x) \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta(x) dx \right] \\ &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta(x) dx \end{aligned} \quad [\text{By (2)}]$$

Since this has to be zero by (3),

$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \dots(I)$$

which is the desired *Euler's equation*.

Obs. 1. Other forms of Euler's equation.

$$\begin{aligned} (a) \text{ Since } f \text{ is a function of } x, y, y', \text{ we have } \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \end{aligned} \quad \dots(5)$$

and

$$\frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) = y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{\partial f}{\partial y'} y'' \quad \dots(6)$$

$$\text{Subtracting (6) from (5), we get } \frac{df}{dx} - \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' - y' \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right)$$

or

$$\frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} = y' \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right\} = 0 \quad [\text{By (I)}]$$

$$\text{Hence } \frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} = 0 \quad \dots(II)$$

which is another form of (I).

$$(b) \text{ Again since } \frac{\partial f}{\partial y'} \text{ is also a function of } x, y, y', \text{ say : } \psi(x, y, y').$$

$$\therefore \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d\psi}{dx} = \frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} + \frac{\partial \psi}{\partial y'} \frac{dy'}{dx}$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y'} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y'} \right) y' + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y'} \right) y'' = \frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + y'' \frac{\partial^2 f}{\partial y'^2}$$

Substituting this in (I), we get $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y' \frac{\partial^2 f}{\partial y \partial y'} - y'' \frac{\partial^2 f}{\partial y'^2} = 0$ (V.T.U., 2001) ... (III)

which is an extended form of (I).

Obs. 2. The above problem can easily be extended to the integral

$$= \int_{x_1}^{x_2} f(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx$$

involving n functions y_1, y_2, \dots, y_n of x . Then the necessary condition for this integral to be stationary is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_i} \right) = 0, \quad i = 1, 2, \dots, n \quad \dots (IV)$$

These are Euler's equations for the n functions.

35.4 SOLUTIONS OF EULER'S EQUATION

Every solution of the Euler's equation which satisfies the boundary conditions, is called an *extremal* or a *stationary function* of the problem. The extremal can easily be obtained in the following cases :

(1) When f is independent of x

We have $\partial f / \partial x = 0$ and Euler's equation (II) above becomes $\frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) = 0$

Integrating, we get $f - y' \frac{\partial f}{\partial y'} = \text{constant}$. This directly gives a solution of Euler's equation.

(2) When f is independent of y

We have $\partial f / \partial y = 0$ and Euler's equation (I) reduces to $\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$

Integrating $\frac{\partial f}{\partial y'} = \text{constant}$ which gives a solution directly.

(3) When f is independent of y' .

We have $\partial f / \partial y' = 0$ and the equation (I) becomes $\frac{\partial f}{\partial y} = 0$ which gives the desired solution.

(4) When f is independent of x and y

We have $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial^2 f}{\partial x \partial y'} = 0, \frac{\partial^2 f}{\partial y \partial y'} = 0$.

Then the equation (III) above becomes $y'' \frac{\partial^2 f}{\partial y'^2} = 0$.

If $\frac{\partial^2 f}{\partial y'^2} \neq 0$, it reduces to $y'' = 0$ which gives a solution of the form $y = ax + b$.

Example 35.1. Find the extremals of the functional $\int_{x_0}^{x_1} (y'^2 / x^3) dx$. (V.T.U., 2003)

Solution. We have $f = y'^2 / x^3$ which is independent of y i.e., $\partial f / \partial y = 0$.

Also $\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d}{dx} \left(\frac{2y'}{x^3} \right) = 2 \frac{x^3 y'' - y' \cdot 3x^2}{x^6} = \frac{2}{x^4} (xy'' - 3y')$

\therefore Euler's equation reduces to $\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$

i.e.,

$$\frac{2}{x^4} (xy'' - 3y') = 0 \quad \text{or} \quad y''/y' = 3/x$$

Integrating both sides, $\int \frac{y''}{y} dy = 3 \int \frac{dx}{x} + \log c$
i.e., $\log y' = 3 \log x + \log c \quad \text{or} \quad y' = cx^3$
 Hence $y = cx^4/4 + c' \quad \text{or} \quad y = c_1 x^4 + c_2$
 This is the required extremal.

Example 35.2. Prove that the shortest distance between two points in a plane is a straight line.

(V.T.U., 2003 S; Bhopal, 2003)

Solution. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points and s the arc length of a curve connecting them (Fig. 35.2). Then

$$s = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(1+y'^2)} dx$$

Now s will be minimum if it satisfies Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

Here $f = \sqrt{(1+y'^2)}$ which is independent of y i.e., $\partial f / \partial y = 0$

$$\therefore \frac{d}{dx} \left\{ \frac{\partial}{\partial y'} \sqrt{(1+y'^2)} \right\} = 0 \quad \text{or} \quad \frac{d}{dx} \left\{ \frac{y'}{\sqrt{(1+y'^2)}} \right\} = 0.$$

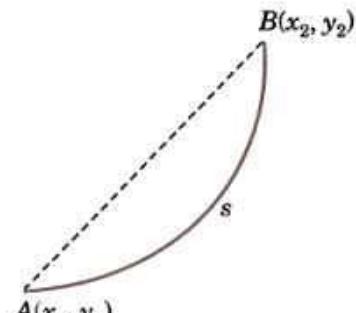


Fig. 35.2

∴ On integration, we have $y'/\sqrt{(1+y'^2)} = \text{constant}$ $y' = \text{constant}, m$ say.

Integrating, we get $y = mx + c$, which is a straight line, the constants m and c are determined from the fact that the straight line passes through A and B .

Example 35.3. Find the curve passing through the points (x_1, y_1) and (x_2, y_2) which when rotated about the x -axis gives a minimum surface area. (V.T.U., 2009)

Solution. In Fig. 35.3, the surface area = $\int_{x_1}^{x_2} 2\pi y \, ds$

$$= 2\pi \int_{x_1}^{x_2} y \sqrt{(1+y'^2)} dx. \text{ This has to be minimum.}$$

Since $f = y \sqrt{(1+y'^2)}$ is independent of x , therefore, Euler's equation reduces to

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}, c : \text{say} \quad [\text{By } \S 35.4 (1)]$$

$$\therefore y \sqrt{(1+y'^2)} - y' \frac{\partial}{\partial y'} \left\{ y \sqrt{(1+y'^2)} \right\} = c$$

$$\text{i.e.,} \quad y \sqrt{(1+y'^2)} - y' \left\{ \frac{y}{2} (1+y'^2)^{-1/2} \cdot 2y' \right\} = c$$

$$\text{or} \quad y/\sqrt{(1+y'^2)} = c \quad \text{or} \quad y' = \frac{dy}{dx} = \frac{\sqrt{(y^2 - c^2)}}{c}$$

Separating the variables and integrating, we have

$$\int \frac{dy}{\sqrt{(y^2 - c^2)}} = \int \frac{dx}{c} + c' \quad \text{or} \quad \cosh^{-1} \left(\frac{y}{c} \right) = \frac{x+a}{c}$$

$$\text{i.e.,} \quad y = c \cosh \left(\frac{x+a}{c} \right)$$

which is a *catenary*. The constants a and c are determined from the points (x_1, y_1) and (x_2, y_2) .

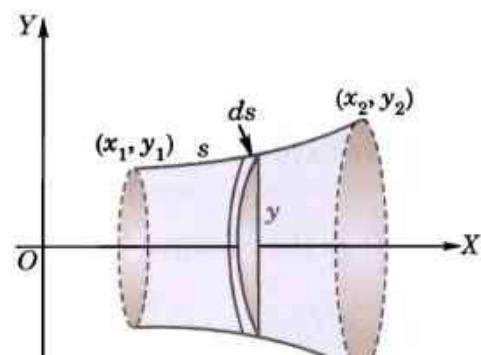


Fig. 35.3

Obs. This problem is also important in connection with *soap films* which are known to have shapes with minimum surface areas.

Example 35.4. Find the path on which a particle in the absence of friction, will slide from one point to another in the shortest time under the action of gravity. (V.T.U., 2004)

Solution. Let the particle start sliding on the curve OP_1 from O with zero velocity (Fig. 35.4). At time t , let the particle be $P(x, y)$ such that arc $OP = s$.

By the principle of work and energy, we have

K.E. at P – K.E. at O = Work done in moving the particle from O to P .

$$\frac{1}{2} m \left(\frac{ds}{dt} \right)^2 - 0 = mgy$$

or

$$ds/dt = \sqrt{(2gy)}$$

Fig. 35.4

...(i)

Thus the time taken by the particle to move from O to P_1 is

$$T = \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}} \int_0^{x_1} \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} dx$$

Here $f = \sqrt{(1+y'^2)/y}$ is independent of x .

\therefore Euler's equation reduces to $f - y' \frac{\partial f}{\partial y'} = \text{constant}, c$: say

$$\text{i.e., } \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \frac{\partial}{\partial y'} \left\{ \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} \right\} = c \quad \text{or} \quad \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \left\{ \frac{y'}{\sqrt{(1+y'^2)} \sqrt{y}} \right\} = c$$

or

$$\sqrt{[y(1+y'^2)]} = 1/c = \sqrt{a}, \text{ say.}$$

$$\text{Solving for } y', \text{ we have } y' = \frac{dy}{dx} = \sqrt{\left(\frac{a-y}{y} \right)}$$

Separating the variables and integrating, we get

$$\int_0^x dx = \int_0^y \sqrt{\left(\frac{y}{a-y} \right)} dy \quad [\text{Put } y = a \sin^2 \theta] \quad \dots(i)$$

or

$$\begin{aligned} x &= \int_0^\theta \sqrt{\left(\frac{a \sin^2 \theta}{a - a \sin^2 \theta} \right)} 2a \sin \theta \cos \theta d\theta \\ &= a \int_0^\theta 2 \sin^2 \theta d\theta = a \int_0^\theta (1 - \cos 2\theta) d\theta = \frac{a}{2} (2\theta - \sin 2\theta) \end{aligned} \quad \dots(ii)$$

Writing $a/2 = b$ and $2\theta = \phi$, equations (ii) and (i) become $x = b(\phi - \sin \phi)$, $y = b(1 - \cos \phi)$ which is a cycloid. The constant b is found from the fact that the curve goes through (x_1, y_1) .

Obs. This is the well-known **brachistochrone problem** which derives its name from the Greek words 'brachistos' meaning shortest and 'chronos' meaning time. It was proposed by John Bernoulli in 1696 and its solution formed the basis for the study of the 'Calculus of Variations'. (V.T.U., 2006)

Example 35.5. Find the curves on which the functional $\int_0^1 [(y')^2 + 12xy] dx$ with $y(0) = 0$ and $y(1) = 1$ can be extremised. (V.T.U., 2010)

Solution. We have

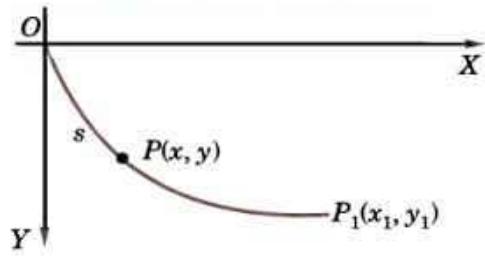
$$f = y'^2 + 12xy$$

$$\therefore \frac{\partial f}{\partial y} = 12x; \frac{\partial f}{\partial y'} = 2y'; \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 2y''$$

Hence the Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes

$$12x - 2y'' = 0 \quad \text{i.e.,} \quad y'' = 6x$$

...(i)



Integrating (i), $y' = 3x^2 + C$... (ii) and $y = x^3 + Cx + C'$... (iii)

Using the boundary conditions, when $x = 0, y = 0$ (iii) gives $C' = 0$.

When $x = 1, y = 1$, (iii) again gives $C = 0$.

Hence (iii) reduces to $y = x^3$ which is the only curve on which extremum can be attained.

Example 35.6. On which curve the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy) dy$ with $y(0) = 0$ and $y(\pi/2) = 0$, be extremized? (V.T.U., 2006)

Solution. Let $f = y'^2 - y^2 + 2xy$ so that $\frac{\partial f}{\partial y'} = 0 - 2y + 2x$

and

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{d}{dx} (2y') = 2y''$$

\therefore Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes
 $-2y + 2x - 2y'' = 0$ or $y'' + y = x$ or $(D^2 + 1)y = x$

Its A.E. $D^2 + 1 = 0$ gives $D = \pm i$.

$$\therefore C.F. = c_1 \cos x + c_2 \sin x$$

and

$$P.I. = \frac{1}{D^2 + 1} x = (1 + D^2)^{-1} x = (1 - D^2)x = x$$

Thus

$$y = c_1 \cos x + c_2 \sin x + x \quad \dots(i)$$

Using boundary conditions : when $x = 0, y = 0$, (i) gives $c_1 = 0$;

when $x = \pi/2, y = 0$, (i) gives $0 = c_2 + \pi/2$, i.e., $c_2 = -\pi/2$.

Hence (i) reduces to $y = x - \frac{\pi}{2} \sin x$, which is the only curve on which the given functional can be extremized.

Example 35.7. Solve the variational problem

$$\delta \int_1^2 [x^2(y')^2 + 2y(x+y)] dx = 0, \text{ given } y(1) = y(2) = 0. \quad (\text{V.T.U., 2006})$$

Solution. Let $f = x^2(y')^2 + 2xy + 2y^2$ so that $\frac{\partial f}{\partial y'} = 2x + 4y, \frac{\partial f}{\partial y} = 2x^2y'$

\therefore Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$ becomes

$$2x + 4y - \frac{d}{dx} (2x^2y') = 0 \quad \text{or} \quad 2x + 4y - (2x^2y'' + 4xy') = 0$$

or $x^2y'' + 2xy' - 2y = x$. This is Cauchy's homogeneous linear (§ 13.9)

Putting $x = e^t$, it reduces to $(D^2 + D - 2)y = e^t$.

Its solution is $y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} te^t$ or $y = c_1 x + \frac{c_2}{x^2} + \frac{1}{3} x \log x$... (i)

Since $y(1) = 0$, we have $c_1 + c_2 = 0$

and $y(2) = 0$ gives $0 = 2c_1 + \frac{1}{4}c_2 + \frac{2}{3} \log 2$

Solving these equations, we get $c_1 = -c_2 = \frac{-8}{21} \log 2$.

Putting the values of c_1 and c_2 in (i), we get

$$y = \frac{1}{21} \{8 \log 2 (x^{-2} - x) + 7x \log x\}$$

which is the required solution.

35.5 GEODESICS

A geodesic on a surface is a curve along which the distance between any two points of the surface is a minimum. To find the geodesics on a surface is a variational problem involving the conditional extremum. This problem was first studied by Jacob Bernoulli in 1698 and its general method of solution was given by Euler.

Example 35.8. Show that the geodesics on a plane are straight lines.

(V.T.U., 2009)

Solution. Let $y = y(x)$ be a curve joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the xy -plane. Then the length of a curve joining A and B is given by

$$s = \int_A^B \frac{ds}{dx} dx = \int_{x_1}^{x_2} \sqrt{[1 + (dy/dx)^2]} dx \quad \text{i.e., } s = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)} dx$$

The geodesic on the xy -plane is the curve $y = y(x)$ for which s is minimum.

We have $f(x, y, y') = \sqrt{(1 + y'^2)}$ which depends on y' only. Hence the Euler's equation.

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \text{ yields}$$

$$\frac{d}{dx} \left\{ \frac{2y'}{2\sqrt{(1 + y'^2)}} \right\} = 0 \quad \text{i.e., } y'' \sqrt{(1 + y'^2)} - \frac{y'_2 2y'y''}{2\sqrt{(1 + y'^2)}} = 0$$

$$\text{i.e. } y''(1 + y'^2) - y'^2 y'' = 0 \quad \text{i.e., } \frac{d^2 y}{dx^2} = 0$$

Integrating twice, we get $y = c_1 x + c_2$

which is a straight line.

Hence the geodesics on a plane are straight lines.

Example 35.9. Find the geodesics on a right circular cylinder of radius a .

Solution. In cylindrical coordinates (ρ, ϕ, z) we have

$$x = r \cos \phi, y = r \sin \phi, z = z. \quad (\text{p. 357})$$

∴ The element of arc on a right circular cylinder of radius a , is given by

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + (\rho d\phi)^2 + dz^2 = a^2 d\phi^2 + dz^2 \quad [\because \rho = a \text{ and } d\rho = 0]$$

or

$$ds = \sqrt{[a^2 + (dz/d\phi)^2]} \cdot d\phi \quad \text{or} \quad s = \int_{\phi_1}^{\phi_2} \sqrt{(a^2 + z'^2)} d\phi \quad \dots(i)$$

Now the geodesic for the given cylinder is the curve for which s is minimum. Here $f = \sqrt{(a^2 + z'^2)}$, which is a function of ϕ and z' while z is absent.

∴ Euler's equation for the functional (i) reduces to

$$\frac{\partial f}{\partial z'} = \text{constant} \quad \text{or} \quad \frac{z'}{\sqrt{(a^2 + z'^2)}} = c, \text{ say.}$$

This simplifies to $(z')^2 = \text{constant}$ or $dz/d\phi = c_1$, say.

Integrating, $z = c_1 \phi + c_2$

This is the desired geodesics on a circular cylinder which is a circular helix. (Example 8.3, p. 318)

Example 35.10. Show that the geodesics on a sphere of radius a are its great circles.

Solution. In spherical coordinates (r, θ, ϕ) , we have

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta. \quad (\text{p. 359}) \quad \dots(i)$$

∴ The arc element on a sphere of radius a , is given by

$$ds^2 = dr^2 + (r d\theta)^2 + (r \sin \theta d\phi)^2 = a^2 d\theta^2 + (a \sin \theta)^2 d\phi^2 \quad [\because r = a, dr = 0]$$

or

$$ds = a \sqrt{[1 + \sin^2 \theta (d\phi/d\theta)^2]} d\theta, \quad \text{or} \quad s = a \int_{\theta_1}^{\theta_2} \sqrt{[1 + \sin^2 \theta \cdot (\phi')^2]} d\theta$$

Now the geodesic on the sphere $r = a$ is the curve for which s is minimum. Here $f = a \sqrt{1 + \sin^2 \theta \cdot \phi'^2}$ which is a function of θ and ϕ' while ϕ is absent.

\therefore Euler's equation reduces to $\partial f / \partial \phi' = \text{constant}$.

$$\frac{\partial f}{\partial \phi'} = \frac{a \sin^2 \theta \cdot \phi'}{\sqrt{(1 + \sin^2 \theta \cdot \phi'^2)}} = \text{constant.}$$

or $\frac{\sin^2 \theta \cdot \phi'}{\sqrt{(1 + \sin^2 \theta \cdot \phi'^2)}} = c \text{ (say)} \quad \text{or} \quad \sin^2 \theta (\sin^2 \theta - c^2) \phi'^2 = c^2$

or $\frac{d\phi}{d\theta} = \frac{c}{\sin \theta \sqrt{(\sin^2 \theta - c^2)}} = \frac{c \operatorname{cosec}^2 \theta}{\sqrt{(1 - c^2) \operatorname{cosec}^2 \theta}}$

Integrating $\phi = \int \frac{c \operatorname{cosec}^2 \theta \cdot d\theta}{\sqrt{[(1 - c^2) - (c \cot \theta)^2]}} + c' = -\sin^{-1} \left\{ \frac{c \cot \theta}{\sqrt{(1 - c^2)}} \right\} + c'$

or $\cot \theta = A \cos \phi + B \sin \phi \quad \text{or} \quad a \cos \theta = Aa \sin \theta \cos \phi + Ba \sin \theta \sin \phi$

or $z = Ax + By \quad [\text{By (i) when } r = a]$

This is a plane through the centre $(0, 0, 0)$ of the sphere which cuts the sphere along a great circle. Hence the required geodesics are the arcs of the great circles.

PROBLEMS 35.1

1. Solve the Euler's equation for the following functionals :

(i) $\int_{x_0}^{x_1} (x + y') y' dx \quad$ (ii) $\int_{x_0}^{x_1} (1 + x^2 y') y' dx. \quad (\text{V.T.U., 2004})$

2. Show that the general solution of the Euler's equation for the integral

$$\int_a^b \frac{1}{y} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \text{ is } (x - h)^2 + y^2 = k^2.$$

Find the extremals of the following functionals :

3. $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x) dx. \quad (\text{V.T.U., 2004})$

4. $\int_0^{\pi/2} (y^2 + y'^2 - 2y \sin x) dx, y(0) = y(\pi/2) = 0. \quad (\text{V.T.U., 2008})$

5. $\int_0^\pi (y'^2 - y^2 + 4y \cos x) dx; y(0) = 0, y(\pi) = 0. \quad (\text{V.T.U., 2008 S})$

6. $\int_{x_0}^{x_1} \frac{1+y^2}{y'^3} dx. \quad$ 7. $\int_1^2 \frac{x^3}{y'^2} dx; y(1) = 0, y(2) = 3.$

8. $\int_1^2 \frac{\sqrt{1+y'^2}}{x} dx; y(1) = 0, y(2) = 1. \quad (\text{Madurai, M.E., 2000 S})$

9. Solve the variational problem $\delta \int_0^{\pi/2} [y^2 - (y')^2] dx$ under the conditions $y(0) = 0, y(\pi/2) = 2. \quad (\text{V.T.U., 2010})$

10. A heavy cable hangs freely under gravity between two fixed points. Show that the shape of the cable is a catenary.

11. A particle is moving with a force perpendicular to and proportional to its distance from the line of zero velocity. Show that the path of quickest descent (brachistochrone) is a circle.

12. Find the geodesics on a right circular cone of semi-vertical angle α .

35.6 ISOPERIMETRIC PROBLEMS

In certain problems, it is necessary to make a given integral.

$$I = \int_{x_1}^{x_2} f(x, y, y') dx \quad \dots(1)$$

maximum or minimum while keeping another integral

$$J = \int_{x_1}^{x_2} g(x, y, y') dx \quad \dots(2)$$

constant. Such problems involve one or more constraint conditions, just as $J = a$ constant. A typical example of this type is that of finding a closed curve of a given perimeter and maximum area. This being one of the earliest problems to engage attention, we often refer to problems of this type as *isoperimetric problems*.

Such problems are generally solved by the method of Lagrange multipliers. To extremize (1), we multiply (2) by λ and add to (1) where λ is the Lagrange multiplier. Then the necessary condition for the integral $\int_{x_1}^{x_2} H$

dx to be an extremum is $\frac{\partial H}{\partial y} - \frac{d}{dx} \left(\frac{\partial H}{\partial y'} \right) = 0$ where $H = f + \lambda g$. The values of the two constants of integration and the parameter λ are determined from the three conditions namely : the two boundary conditions and the integral J having given constant value.

Example 35.11. Find the plane curve of fixed perimeter and maximum area.

(V.T.U., 2000 S)

Solution. Let l be the fixed perimeter of a plane curve between the points with abscissae x_1 and x_2 (Fig. 35.5). Then

$$l = \int_{x_1}^{x_2} \sqrt{(1 + y'^2)} dx \quad \dots(i)$$

Also the area between the curve and the x -axis is

$$A = \int_{x_1}^{x_2} y dx \quad \dots(ii)$$

We have to maximize (ii) subject to constraint (i).

\therefore Taking $f = y$ and $g = \sqrt{(1 + y'^2)}$, we write $H = f + \lambda g = y + \lambda \sqrt{(1 + y'^2)}$

Now H must satisfy the Euler's equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx} \left(\frac{\partial H}{\partial y'} \right) = 0 \quad \therefore 1 - \frac{d}{dx} \left[\frac{\lambda y'}{\sqrt{(1 + y'^2)}} \right] = 0$$

Integrating w.r.t. x , we have $x - \lambda y' / \sqrt{(1 + y'^2)} = 0$

Solving for y' , we get $y' = \frac{x - a}{\sqrt{[\lambda^2 - (x - a)^2]}}$

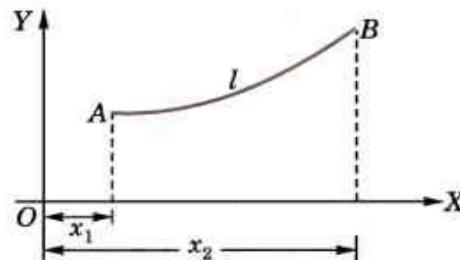


Fig. 35.5

Integrating again, $y = \sqrt{[\lambda^2 - (x - a)^2]} + b$ i.e. $(x - a)^2 + (y - b)^2 = \lambda^2$ which is a circle.

Example 35.12. Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume.

(V.T.U., 2006 ; Madras, 2000 S)

Solution. Consider the arc OPA of the curve which rotates about the x -axis as shown in Fig. 35.6.

$$\text{Then surface area } S = \int_{x=0}^a 2\pi y ds = \int_0^a 2\pi y \sqrt{(1 + y'^2)} dx$$

$$\text{and volume of the solid so formed } V = \int_0^a \pi y^2 dx.$$

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Here we have to maximize V subject to fixed S . Taking $f = \pi y^2$ and $g = 2\pi y \sqrt{1 + y'^2}$, we write $H = f + \lambda g = \pi y^2 + 2\pi \lambda y \sqrt{1 + y'^2}$.

Now H has to satisfy Euler's equation. But it does not contain x explicitly.

$$\therefore H - y' \frac{\partial H}{\partial y'} = \text{constant}, c : \text{say}$$

$$\text{i.e., } \pi y^2 + 2\pi \lambda y \sqrt{1 + y'^2} - y' \cdot 2\pi \lambda y \frac{y'}{\sqrt{1 + y'^2}} = c$$

$$\text{or } \pi y^2 + \frac{2\pi \lambda y}{\sqrt{1 + y'^2}} = c \quad \dots(i)$$

Since the curve passes through O and A for which $y = 0$, (i) gives $c = 0$.

$$\therefore y + 2\lambda \sqrt{1 + y'^2} = 0$$

$$\text{Solving for } y', \quad y' \left(= \frac{dy}{dx} \right) = \frac{\sqrt{(4\lambda^2 - y^2)}}{y}$$

Separating the variables and integrating, we get

$$\int dx = \int \frac{y dy}{\sqrt{(4\lambda^2 - y^2)}} + k \quad \text{or} \quad x = k - \sqrt{(4\lambda^2 - y^2)} \quad \dots(ii)$$

When $x = 0, y = 0 \quad \therefore k = 2\lambda$

\therefore (ii) becomes $(x - 2\lambda)^2 + y^2 = (2\lambda)^2$ which is a circle with centre $(2\lambda, 0)$ and radius 2λ .

Hence the figure formed by the revolution of given arc is a sphere.

PROBLEMS 35.2

- Find a function $y(x)$ for which $\int_0^1 (x^2 + y'^2) dx$ is stationary, given that $\int_0^1 y^2 dx = 2$; $y(0) = 0$, $y(1) = 0$.
(Madras, 2000 S)
- Find the extremals of the isoperimetric problem $v[y(x)] = \int_{x_0}^{x_1} y'^2 dx$ given that $\int_{x_0}^{x_1} y dx = c$, a constant.
- Show that the curve c of given length l which minimizes the curved surface area of the solid generated by the revolution of c about the x -axis is a catenary.
(V.T.U., 2000 S)
- Find the extremal of the functional $I = \int_0^\pi [(y')^2 - y^2] dx$ under the conditions $y(0) = 0$, $y(\pi) = 1$ and subject to the constraint $\int_0^\pi y dx = 1$.
- Prove that the extremal of the isoperimetric problem $v[y(x)] = \int_1^4 y'_2 dx$, $y(1) = 3$, $y(4) = 24$, subject to the condition $\int_1^4 y dx = 36$ is a parabola.
(Madras M.E., 2000)

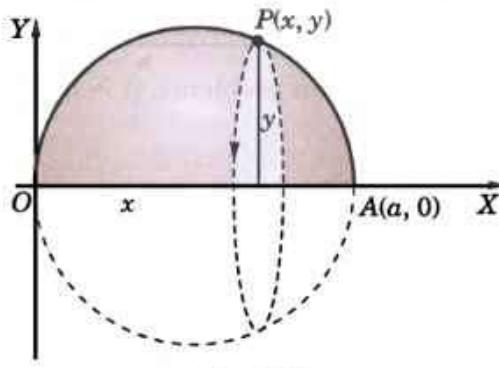


Fig. 35.6

35.7 SEVERAL DEPENDENT VARIABLES

We now extend the variational problem of § 35.3 to a problem with several variables as functions of a single independent variable i.e., A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx \quad \dots(1)$$

to be an extremum is that $\frac{\partial \mathbf{f}}{\partial \mathbf{y}_i} - \frac{\mathbf{d}}{\mathbf{dx}} \left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}'_i} \right) = \mathbf{0}, i = 1, 2, \dots, n$... (2)

Let y_1, y_2, \dots, y_n satisfy the boundary conditions $y_i(x_1) = y_{i1}, y_i(x_2) = y_{i2}$

Consider arbitrary functions $\eta_1(x), \eta_2(x), \dots, \eta_n(x)$ which are all zero on the boundary i.e.,

$$\eta_i(x_1) = 0 = \eta_i(x_2)$$

Replacing y_1, y_2, \dots by $y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots$ in (1), we get

$$I(\varepsilon) = \int_{x_1}^{x_2} f(x, y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots, y'_1 + \varepsilon_1 \eta'_1, y'_2 + \varepsilon_2 \eta'_2, \dots) dx \quad \dots (3)$$

This is a function of the parameters $\varepsilon_1, \varepsilon_2, \dots$ and reduces to (1) for $\varepsilon_1 = \varepsilon_2 = \dots = 0$.

To find the stationary value of (1), we find the stationary value of $I(\varepsilon)$ for $\varepsilon_1 = \varepsilon_2 = \dots = 0$, $I(\varepsilon)$ will have a stationary value when

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0, \frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0, \dots$$

Writing

$$f = f(x, y_1, y_2, \dots, y'_1, y'_2, \dots)$$

and

$$F = f(x, y_1 + \varepsilon_1 \eta_1, y_2 + \varepsilon_2 \eta_2, \dots, y'_1 + \varepsilon_1 \eta'_1, y'_2 + \varepsilon_2 \eta'_2, \dots).$$

(3) becomes $I(\varepsilon) = \int_{x_1}^{x_2} F dx$.

x_1, x_2 being constants independent of ε_1 , differentiating under the integral sign, we get

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon_1} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y_1} \eta_1 + \frac{\partial F}{\partial y'_1} \eta'_1 \right) dx$$

$$\therefore \frac{\partial I(\varepsilon)}{\partial \varepsilon_1} = 0, \text{ when } \varepsilon_1 = \varepsilon_2 = \dots = 0 \text{ gives}$$

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y_1} \eta_1 + \frac{\partial f}{\partial y'_1} \eta'_1 \right) dx = 0$$

Integrating by parts the second term, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_1} \eta_1 + \left| \frac{\partial f}{\partial y'_1} \eta_1 \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'_1} \right) \eta_1(x) dx = 0$$

$$\text{i.e., } \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_1} \right) \right\} \eta_1(x) dx = 0 \quad [\because \eta_1(x_1) = 0 = \eta_1(x_2)]$$

Since this equation must hold good for all values of $\eta_1(x)$, we get

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_1} \right) = 0$$

Similarly $\frac{\partial I(\varepsilon)}{\partial \varepsilon_2} = 0$ when $\varepsilon_1 = \varepsilon_2 = \dots = 0$, will give

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_2} \right) = 0 \text{ and so on.}$$

All these conditions give a system of Euler's equation (2). A solution of these equations leads to the desired curves.

Example 35.13. Show that the functional $\int_0^{\pi/2} \left\{ 2xy + \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} dt$ such that $x(0) = 0, x(\pi/2) = -1, y(0) = 0, y(\pi/2) = 1$ is stationary for $x = -\sin t, y = \sin t$.

Solution. Euler's equations are $\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial x'} \right) = 0$... (i)

$$\frac{\partial f}{\partial y} - \frac{d}{dt} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \dots (ii)$$

Here $f = 2xy + x'^2 + y'^2$. $\therefore \frac{\partial f}{\partial x} = 2y, \frac{\partial f}{\partial x'} = 2x', \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial y'} = 2y'$

$$(i) \text{ becomes } 2y - \frac{d}{dt}(2x') = 0 \quad i.e., \quad 2y - 2\frac{d^2x}{dt^2} = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} = y \quad \dots (iii)$$

$$(ii) \text{ becomes } 2x - \frac{d}{dt}(2y') = 0 \quad i.e., \quad 2x - 2\frac{d^2y}{dt^2} = 0 \quad \text{or} \quad \frac{d^2y}{dt^2} = x \quad \dots (iv)$$

Now to solve these simultaneous differential equations, we differentiate (iii) twice,

$$\frac{d^4x}{dt^4} = \frac{d^2y}{dt^2} = x \quad [\text{By (iv)}]$$

or $(D^4 - 1)x = 0$ which is a linear equation with constant coefficients.

$$\text{Its solution is } x = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \quad \dots (v)$$

$$\text{From (iii), } y = x'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x \quad \dots (vi)$$

$$\text{Since } x = 0 \text{ when } t = 0 \quad \therefore 0 = c_1 + c_2 + c_3 \quad \dots (vii)$$

$$y = 0 \text{ when } t = 0 \quad \therefore 0 = c_1 + c_2 - c_3 \quad \dots (viii)$$

$$\text{Also } x = -1 \text{ when } t = \pi/2 \quad \therefore -1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 \quad \dots (ix)$$

$$y = 1 \text{ when } t = \pi/2 \quad \therefore 1 = c_1 e^{\pi/2} + c_2 e^{-\pi/2} - c_4 \quad \dots (x)$$

$$\text{Adding (vii) and (viii), } c_1 + c_2 = 0$$

$$\text{Adding (ix) and (x), } c_1 e^{\pi/2} + c_2 e^{-\pi/2} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$.

From (viii), $c_3 = c_1 + c_2 = 0$. From (ix), $c_4 = -1$.

Hence from (v), $x = -\sin x$ and from (vi), $y = \sin x$.

35.8 FUNCTIONALS INVOLVING HIGHER ORDER DERIVATIVES

A necessary condition for

$$I = \int_{x_1}^{x_2} f(x, y, y', y'') dx \quad \dots (1)$$

to be extremum is $\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0$.

Let $y(x)$ be the function which makes (1) stationary and satisfies the boundary conditions

$$y(x_1) = y_1, y(x_2) = y_2, y'(x_1) = y'_1 \text{ and } y''(x_2) = y''_2.$$

Consider the differentiable function $\eta(x)$ such that

$$\eta(x_1) = 0 = \eta(x_2) \quad \text{and} \quad \eta'(x_1) = 0 = \eta'(x_2) \quad \dots (2)$$

Replacing y by $y + \epsilon\eta$ in (1), we get

$$I(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'') dx \quad \dots (3)$$

This is a function of the parameter ϵ and reduces to (1) for $\epsilon = 0$.

To find the stationary value of (1), we find the stationary value of $I(\epsilon)$ for $\epsilon = 0$. But $I(\epsilon)$ will have a stationary value when $dI(\epsilon)/d\epsilon = 0$.

Writing $f = f(x, y, y', y'')$ and $F = f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'')$.

$$(3) \text{ becomes } I(\epsilon) = \int_{x_1}^{x_2} F dx$$

x_1, x_2 being constants independent of ϵ , differentiating under the integral sign, we get

$$\begin{aligned}\frac{dI(\epsilon)}{d\epsilon} &= \int_{x_1}^{x_2} \frac{dF}{d\epsilon} dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' + \frac{\partial F}{\partial y''} \eta'' \right) dx \\ \therefore \frac{dI(\epsilon)}{d\epsilon} = 0 \text{ when } \epsilon = 0 \text{ gives } \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' + \frac{\partial f}{\partial y''} \eta'' \right) dx &= 0\end{aligned}$$

Integrating by parts once the second term and twice the third term, we get

$$\begin{aligned}\int_{x_1}^{x_2} \frac{\partial f}{\partial y} \eta dx + \left| \frac{\partial f}{\partial y'} \eta \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \cdot \eta dx + \left| \frac{\partial f}{\partial y''} \eta' - \frac{d}{dx} \left(\frac{\partial f}{\partial y''} \right) \cdot \eta \right|_{x_1}^{x_2} \\ + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \cdot \eta dx &= 0\end{aligned}$$

or $\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) \right] \eta(x) dx = 0$ [By (2)]

Since this equation must hold good for all values of $\eta(x)$, we get

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0$$

In general, a necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y', y'', \dots, y^{(n)}) dx$ to be stationary will be

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) - \dots + (-1)^n \frac{d^n}{dx^n} \left(\frac{\partial f}{\partial y^{(n)}} \right) = 0$$

which is called the *Euler-Poisson equation* and its solutions are called *extremals*.

Example 35.14. Show that the curve which extremizes the functional $I = \int_0^{\pi/4} (y''^2 - y^2 + x^2) dx$ under the conditions $y(0) = 0, y'(0) = 1, y(\pi/4) = y'(\pi/4) = 1/\sqrt{2}$ is $y = \sin x$. (Madras M.E., 2000 S)

Solution. The Euler-Poisson equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0 \quad \dots(i)$$

Here $f = (y'')^2 - y^2 + x^2, \frac{\partial f}{\partial y} = -2y, \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial y''} = 2y''$

$\therefore (i)$ becomes $-2y + \frac{d^2}{dx^2}(2y'') = 0 \quad \text{or} \quad y'' - y = 0 \quad \text{or} \quad (D^2 - 1)y = 0$

Its solution is $y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \quad \dots(ii)$

$\therefore y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x \quad \dots(iii)$

Applying the given boundary conditions to (ii) and (iii), we get

$$0 = y(0) = c_1 + c_2 + c_3, 1 = y'(0) = c_1 - c_2 + c_4$$

$$\frac{1}{\sqrt{2}} = y(\pi/4) = c_1 e^{\pi/4} + c_2 e^{-\pi/4} + \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4 \quad \dots(iv)$$

$$\frac{1}{\sqrt{2}} = y'(\pi/4) = c_1 e^{\pi/4} - c_2 e^{-\pi/4} - \frac{1}{\sqrt{2}} c_3 + \frac{1}{\sqrt{2}} c_4$$

Solving the equation (iv), we get $c_1 = c_2 = c_3 = 0, c_4 = 1$. Hence the required curve is $y = \sin x$.

PROBLEMS 35.3

1. Show that the functional $\int_0^1 \left\{ 2x + \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\} dt$, such that $x(0) = 1, y(0) = 1; x(1) = 1.5, y(1) = 1$ is stationary for $x = 1 + t^3/2, y = t$.
2. Find the extremals of the functional $v(y, z) = \int_{x_0}^{x_1} (2yz - 2y^2 + y'^2 - z'^2) dx$ (V.T.U., M.E., 2006)
3. Find a function $y(x)$ such that $\int_0^\pi y^2 dx = 1$ which makes $\int_0^\pi y'^2 dx$ a minimum if $y(0) = 0 = y(\pi), y''(0) = 0 = y''(\pi)$.
4. Find the extremals of the functional $\int_0^{\pi/2} (y'^2 - y^2 + x^2) dx$ that satisfies the conditions $y(0) = 1, y'(0) = 0, y(\pi/2) = 0, y'(\pi/2) = -1$. (Madras, M.E. 2000 S)
5. Find the extremals of the functional $\int_{-a}^a \left(\lambda y + \frac{1}{2} \mu y'^2 \right) dx$ which satisfies the boundary conditions $y(-a) = 0, y'(-a) = 0, y(a) = 0, y'(a) = 0$.
6. Find the extremals of the following functionals :
- (i) $v[y(x)] = \int_{x_0}^{x_1} (16y^2 - y'^2 + x^2) dx$ (Nagpur, 1997) (ii) $v[y(x)] = \int_{x_0}^{x_1} (2xy + y'^2) dx$.

35.9 APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS — Rayleigh-Ritz Method

In § 35.3, we have seen that the solution of Euler's differential equation alongwith boundary conditions amounts to extremising a certain definite integral. This fact provides a technique of solving a boundary value problem approximately by assuming a trial solution satisfying the given boundary conditions and then extremising the integral whose integrand is found from the given differential equation.

To solve a boundary value problem of Rayleigh-Ritz method, we try to write the given differential equation as the Euler's equation of some variational problem. Then we reduce this variational problem to a minimizing problem assuming an approximate solution in the form

$$\bar{y}(x) = y_0(x) + \sum c_i \phi_i(x) \quad \dots(1)$$

where the *trial functions* $\phi_i(x)$ satisfy the boundary conditions and $\phi_i(x) = 0$ on the boundary C of its region R .

$$\text{Let the integral to be extremised be } I = \int_a^b f(y, y', x) dx \quad \dots(2)$$

such that $y(a) = A$ and $y(b) = B$.

Substituting (1) in (2) by replacing y in \bar{y} in I , giving \bar{I} as a function of the unknowns c_i . Then c 's become parameters which are so determined as to extremise \bar{I} . This requires

$$\frac{\partial \bar{I}}{\partial c_i} = 0, \quad i = 1, 2, \dots$$

Solving these equations, we get the values of c_i , which when substituted in (1) give the desired solution.

Example 35.15. Solve the boundary value problem $y'' - y + x = 0$ ($0 \leq x \leq 1$), $y(0) = y(1) = 0$ by Rayleigh-Ritz method.

Solution. Given differential equation is $y'' - y + x = 0$...(i)

Its solution is equivalent to extremising the integral $I = \int_0^1 F(x, y, y') dx$

where

$$F(x, y, y') = 2xy - y^2 - y'^2, \quad \dots(ii)$$

since the Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$ gives (i).

Assume that the *trial function* is $\bar{y}(x) = c_0 + c_1x + c_2x^2$... (iii)

To satisfy $y(0) = 0, y(1) = 0$, we require $c_0 = 0, c_2 = -c_1$.

\therefore (iii) becomes $\bar{y}(x) = c_1x(1-x)$... (iv)

Substituting \bar{y} and \bar{y}' in I , we have

$$\begin{aligned}\bar{I} &= \int_0^1 [2x\bar{y} - \bar{y}^2 - (\bar{y}')^2] dx = \int_0^1 [2c_1(x^2 - x^3) - c_1^2(x - x^2)^2 - c_1^2(1 - x)^2] dx \\ &= \frac{1}{6}c_1 - \frac{11}{30}c_1^2\end{aligned}$$

Its stationary values are given by $dI/dc_1 = 0$. $\therefore \frac{1}{6} - \frac{11}{15}c_1 = 0$ i.e., $c_1 = \frac{5}{22}$.

Thus the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$... (v)

35.10 WEIGHTED RESIDUAL METHOD—Galerkin's Method

The starting point of this method is to guess a solution to the differential equation which satisfies the boundary conditions. This *trial solution* will contain certain parameters which can be adjusted to minimize the errors so that the trial solution is as close to the exact solution as possible.

Consider the boundary value problem

$$y'' = f(y, y', x) \text{ with } y(a) = A \text{ and } y(b) = B \quad \dots(1)$$

We write the differential equation as $R = \bar{y}'' - f(\bar{y}, \bar{y}', x)$... (2)

where R is the residual of the equation ($R = 0$ for the exact solution $y(x)$ only which will satisfy the boundary conditions).

Consider the trial solution as $\bar{y}(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots$

where $\bar{y}(a) = A$ and $\bar{y}(b) = B$. The trial solution is differentiated twice and is substituted in (2).

To find c_1, c_2, \dots , we weight the residual by trial functions $\phi_1(x), \phi_2(x), \dots$ and set the integrals to zero. Thus we have $\int_c R \phi_1(x) dx = 0, \int_c R \phi_2(x) dx = 0, \dots$

These lead to simultaneous equations in the unknowns.

Having found c_1, c_2, \dots , the approximate solution $\bar{y}(x)$ is obtained.

Example 35.16. Use Galerkin's method to solve the boundary value problem of Example 35.14.

Compare your approximate solution with the exact solution.

Solution. The residual is $R = \bar{y}'' - \bar{y} + x$... (i)

To find the trial solution which satisfies the boundary conditions, we derive a Lagrangian polynomial (§ 28.8) which passes through the points :

x	:	0	1/2	1
y	:	0	c	0

The resulting polynomial is $\bar{y}(x) = 4cx(1-x)$, so that $\phi(x) = x(1-x)$.

Substituting $\bar{y}(x), \bar{y}''(x)$ in (i), we get $R = 4cx^2 + (1-4c)x - 8c$

Thus $\int R \phi(x) dx = 0$ gives $\int_0^1 [4cx^2 + (1-4c)x - 8c] x(1-x) dx = 0$ whence $c = 5/88$.

Hence the approximate solution is $\bar{y}(x) = \frac{5}{22}x(1-x)$ which is same as found in Example 34.14.

Exact solution. Rewriting the given equation as $(D^2 - 1)y = -x$, we find its solution as $y = c_1e^x + c_2e^{-x} + x$

Since $y(0) = 0$ and $y(1) = 0$, therefore $c_2 = -c_1 = 1/(e - e^{-1})$.

Hence the exact solution is $y = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

The approximate and the exact solutions for some values of x are given below for comparison :

x	Approx. Sol.	Exact Sol.
0.25	0.043	0.035
0.50	0.057	0.057
0.75	0.043	0.05

Obs. To obtain a trial solution containing two unknown parameters, we derive a Lagrangian polynomial which passes through the points :

$$\begin{array}{l} x : 0 \quad 1/3 \quad 2/3 \quad 1 \\ y : 0 \quad c_1 \quad c_2 \quad 0 \end{array}$$

More the undetermined parameters, the more accurate is the solution, but it involves more labour to find their values.

Example 35.17. Find the approximate deflection of a simply supported beam under a uniformly distributed load w per unit length, using Galerkin's method.

Solution. The differential equation governing the deflection $y(x)$ of the beam is

$$EI \frac{d^4 y}{dx^4} - w = 0, \quad 0 < x < l \quad (\text{i}) \quad [\S 14.8]$$

The boundary conditions to be satisfied are

$$y(x=0) = y(x=l) = 0 \quad (\text{deflection is zero at ends}) \quad \dots(\text{ii})$$

$$\frac{d^2 y}{dx^2}(x=0) = \frac{d^2 y}{dx^2}(x=l) = 0 \quad (\text{bending moment zero at ends})$$

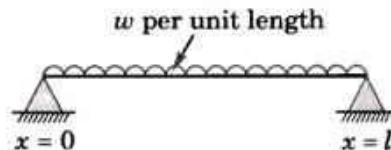


Fig. 35.7

... (iii)

We assume the trial solution $\bar{y}(x) = c_1 \sin(\pi x/l) + c_2 \sin(3\pi x/l)$, which satisfies the boundary conditions (ii) and (iii).

Substituting the trial solution in (i), we obtain the residual

$$R = EI c_1 (\pi/l)^4 \sin(\pi x/l) + EI c_2 (3\pi/l)^4 \sin(3\pi x/l) - w$$

$$\text{Thus } \int_0^l R \cdot \sin(\pi x/l) dx = 0 \quad \text{and} \quad \int_0^l R \cdot \sin(3\pi x/l) dx = 0$$

Computing these integrals, we get

$$EI c_1 (\pi/l)^4 l/2 - w \cdot 2l/\pi = 0, \quad EI c_2 (3\pi/l)^4 l/2 - w \cdot 2l/3\pi = 0$$

$$\text{Solving these, we obtain } c_1 = \frac{4wl^4}{\pi^5 EI} \text{ and } c_2 = \frac{4wl^4}{243\pi^5 EI}.$$

Hence the deflection of the beam is given by

$$\bar{y}(x) = \frac{4wl^4}{\pi^5 EI} \left\{ \sin\left(\frac{\pi x}{l}\right) + \frac{1}{243} \sin\left(\frac{3\pi x}{l}\right) \right\}.$$

PROBLEMS 35.4

1. Solve the boundary value problem :

$$y'' + y + x = 0 \quad (0 \leq x \leq 1), \quad y(0) = y(1) = 0 \text{ by}$$

(i) Rayleigh-Ritz method, (ii) Galerkin's method. Compare your solution with the exact solution.

2. Using Galerkin's method, solve the boundary value problem $y'' = 3x + 4y$; $y(0) = 0$, $y(1) = 1$.

3. Apply Galerkin's method to the boundary value problem $y'' + y + x = 0$ ($0 \leq x \leq 1$); $y(0) = y(1) = 0$, to find the coefficients of the approximate solution $\bar{y}(x) = c_1 x(1-x) + c_2 x^2(1-x)$.

[Hint. Substituting $\bar{y}(x)$, $\bar{y}'(x)$ in the given equation replacing y, y'' by \bar{y}, \bar{y}'' , we get the residual $R = -2c_1 + c_2(2 - 6x) + x(1-x)(c_1 + c_2x) + x$

$$\text{Thus } \int_0^1 R \cdot x(1-x) dx = 0 \text{ and } \int_0^1 R \cdot x^2(1-x) dx = 0.$$

Computing these integrals, we get

$$\frac{3}{10}c_1 + \frac{3}{20}c_2 = \frac{1}{12}, \quad \frac{3}{20}c_1 + \frac{13}{305}c_2 = \frac{1}{20}.$$

Solving these, we obtain $c_1 = 71/369$, $c_2 = 7/41$.]

4. Using Ritz method, find an approximate solution of the problem $y'' - y + 4xe^x = 0$, $y'(0) = y(0) = 1$, $y'(1) + y(1) = -e$.
5. Solve the boundary value problem : $y'' + (1+x^2)y + 1 = 0$, $y(-1) = y(1) = 0$, by taking the approximate solution $\bar{y}(x) = c_1(1-x^2) + c_2x^2(1-x^2)$ and using (i) Ritz method, (ii) Galerkin's method.
6. Given the boundary value problem : $y'' + \pi^2y = x$, $y(0) = 1$, $y(1) = -0.9$.

Use Galerkin's method to estimate $y(0.5)$, taking the trial solution :

$$y = 1 - 1.9x + c_1x(1-x) + c_2x^2(1-x).$$

7. Using Galerkin's method, obtain an approximate solution of the boundary value problem :

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) + y = x, \quad y(0) = 0, \quad y(1) = 1,$$

in the form $\bar{y}(x) = x + x(1-x)(c_1 + c_2x)$.

8. Of all the parabolas which pass through $(0, 0)$ and $(1, 1)$, determine the one, which when rotated about the x -axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$.

[Hint. Take the parabola as $y = x + cx(1-x)$.]

9. Using Rayleigh-Ritz method, find the potential at any point due to a charged sphere of radius a .

[Hint. Potential at a distance r from the centre of the sphere is $\phi = \phi_0(r/a)^k$, where ϕ_0 is the value of ϕ for $r = a$ and $k < 0$ so that $\phi \rightarrow 0$ as $r \rightarrow \infty$.

Electrostatic field due to charged sphere being conservative, electrostatic intensity $E = \nabla\phi$.

Also potential energy for unit volume = $\frac{1}{8\pi}E^2$

∴ Total potential energy of the field in the entire region R exterior to the given sphere is

$$\begin{aligned} V &= \frac{1}{8\pi} \iiint_R E^2 dx dy dz = \frac{1}{8\pi} \iiint_R \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right] dx dy dz \\ &= \frac{1}{8\pi} \int_a^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{\partial \phi}{\partial r} \right)^2 r^2 \sin \theta d\phi d\theta dr. \end{aligned}$$

Electrostatic field will be in stable equilibrium if V is minimum, i.e., $dV/dp = 0$ and $d^2V/dp^2 > 0$.

This gives $k = -1$. Hence $\phi = \phi_0 a/r$.]

35.11 HAMILTON'S PRINCIPLE*

An important concept of mathematical physics is *Hamilton's principle* which states that the time integral of the difference between the kinetic and potential energies of a dynamical system is stationary

Consider a particle of mass m moving from a fixed origin O under the effect of a force F (Fig. 35.8). At any time t , let its position vector be \mathbf{R} . Then by Newton's second law,

$$\frac{md^2\mathbf{R}}{dt^2} = \mathbf{F} \quad \dots(1)$$

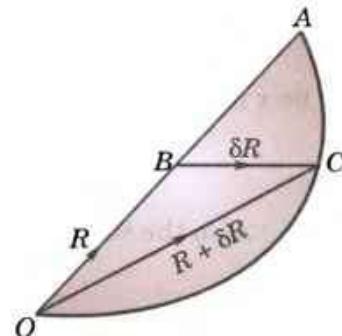


Fig. 35.8

* Named after the Irish mathematician William Rowan Hamilton (1805–1865) who is known for his contributions to dynamics.

Let the natural path OBA of the particle be changed to another path OCA , end points remaining the same. Let this variation in path, often called *virtual displacement*, be $\delta\mathbf{R}$. Then the work done during this displacement is

$$\delta W = \mathbf{F} \cdot \delta\mathbf{R} = \frac{md^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} \quad [\text{By (1)}]$$

Also the kinetic energy of the particle is

$$T = \frac{1}{2} m \left(\frac{d\mathbf{R}}{dt} \right)^2$$

$$\therefore \delta T = m \frac{d\mathbf{R}}{dt} \cdot \delta \left(\frac{d\mathbf{R}}{dt} \right) = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt} (\delta\mathbf{R})$$

$$\text{Thus } \delta T + \delta W = m \frac{d\mathbf{R}}{dt} \cdot \frac{d}{dt} (\delta\mathbf{R}) + m \frac{d^2\mathbf{R}}{dt^2} \cdot \delta\mathbf{R} = m \frac{d}{dt} \left(\frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right)$$

Integrating both sides w.r.t. t from t_0 to t_1 , we get

$$\int (\delta T + \delta W) dt = m \left| \frac{d\mathbf{R}}{dt} \cdot \delta\mathbf{R} \right|_{t_0}^{t_1} = 0 \quad \dots(2) \quad [\because \delta\mathbf{R} = 0 \text{ at } t_0 \text{ and } t_1]$$

If the force field is conservative, there exists a potential V such that $W = -V$. Then (2) takes the form

$$\int (\delta T - \delta V) dt = 0 \quad \text{or} \quad \delta \int (T - V) dt = 0 \quad i.e., \quad \int (T - V) dt \quad \dots(3)$$

is stationary. This proves the Hamilton's principle for a particle. Its derivation can be extended to a system of particles by summation and to a rigid body by integration. Hence the principle is true for any dynamical system.

Obs. It can be easily shown that the integral (3) is a minimum along the natural path as compared to that along any other path joining O to A .

Def. The energy difference $T - V = L$ is called the **kinetic potential** or the **Lagrangian function**.

35.12 LAGRANGE'S EQUATION

In a dynamical system, the position of a body can be specified by the quantities q_1, q_2, \dots, q_n which are called the *generalised coordinates*.

The potential energy V , being a function of position only depends on these generalised coordinates q_i . The kinetic energy T , however, depends upon q_i and the velocities dq_i/dt (*i.e.*, q_i) $i = 1, 2, \dots, n$. Therefore, Lagrangian function $L = T - V$ is also a function of q_i and q_i' , $i = 1, 2, \dots, n$.

Thus by Hamilton's principle, the system moves so that $\int_{t_0}^{t_1} L dt$ is stationary.

$$\therefore \text{Euler's equation must hold good, i.e., } \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i'} \right) = 0, i = 1, 2, \dots, n.$$

These are called *Lagrange's equations* which determine the motion of the system.

Example 35.18. A mass, suspended at the end of a light spring having spring constant k , is set into vertical motion. Use Lagrange's equation, to find the equation of motion of the mass.

Solution. At any time t , let the displacement of m from the equilibrium position O be x (Fig. 35.9). Then the kinetic energy of P is

$$T = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$$

Also the work done during its fall from O to P is

$$W = \int_0^x (mg - kx) dx = mgx - \frac{1}{2} kx^2.$$

If V is the potential energy of the mass at P , then

$$V = -W = \frac{1}{2} kx^2 - mgx$$

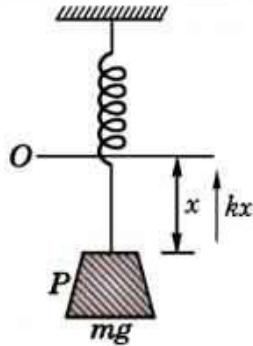


Fig. 35.9

$$\therefore \text{Lagrangian} \quad L = T - V = \frac{1}{2}m\dot{x}^2 + mgx - \frac{1}{2}kx^2.$$

Thus the Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

becomes

$$(mg - kx) - \frac{d}{dt}(m\dot{x}) = 0 \quad \text{or} \quad m\frac{d^2x}{dt^2} = mg - kx$$

which is the required equation of motion.

Example 35.19. Apply Lagrange's equations, to show that the equations of motion of the double pendulum of Fig. 35.10 are given by

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0$$

and

$$l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0$$

for small angles θ_1, θ_2

(Punjab, M.E., 1997)

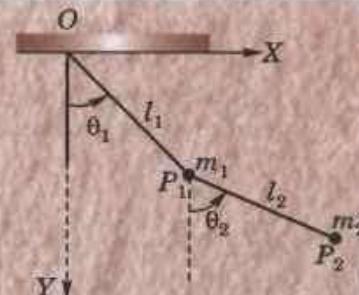


Fig. 35.10

Solution. At any time t , let the masses m_1, m_2 be at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 35.10) so that

$$\left. \begin{aligned} x_1 &= l_1 \sin \theta_1, y_1 = l_1 \cos \theta_1 \\ x_2 &= l_1 \sin \theta_1 + l_2 \sin \theta_2, y_2 = l_1 \cos \theta_1 + l_2 \cos \theta_2 \end{aligned} \right\} \quad \dots(i)$$

Then total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \end{aligned} \quad [\text{Using (i)}]$$

Also total potential energy is

$$V = m_1gl_1(1 - \cos \theta_1) + m_2g(l_1 + l_2 - l_1 \cos \theta_1 - l_2 \cos \theta_2)$$

$$\therefore \text{Lagrangian} \quad L = T - V = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2$$

$$+ m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) - (m_1 + m_2)gl_1(1 - \cos \theta_1) - m_2gl_2(1 - \cos \theta_2) \quad \dots(ii)$$

Thus the Lagrange's equation corresponding to θ_1 , is

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 0, \text{ becomes}$$

$$-m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2)gl_1 \sin \theta_1 - \frac{d}{dt}[(m_1 + m_2)l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2 \cos(\theta_1 - \theta_2)] = 0$$

$$\text{or} \quad (m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2l_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)g \sin \theta_1 = 0 \quad \dots(iii)$$

Similarly from (ii), Lagrange's equation corresponding to θ_2 , i.e.,

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = 0$$

$$\text{becomes} \quad m_2l_1l_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2gl_2 \sin \theta_2 - \frac{d}{dt}[m_2l_2^2\dot{\theta}_2 + m_2l_1l_2\dot{\theta}_1 \cos(\theta_1 - \theta_2)] = 0$$

$$\text{or} \quad l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin \theta_2 = 0 \quad \dots(iv)$$

Now $\dot{\theta}_1$ and $\dot{\theta}_2$ being small, retaining first order terms only, (iii) and (iv) reduce to

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0 \quad \text{and} \quad l_1\ddot{\theta}_1 + l_2\ddot{\theta}_2 + g\theta_2 = 0.$$

PROBLEMS 35.5

1. In a single pendulum, a mass m is suspended by a light rod of length l and the system vibrates in a plane. Using Lagrange's equation, show that $\ddot{\theta} + (g/l) \sin \theta = 0$.

Show that if θ is small, the period of oscillation is $2\pi\sqrt{l/g}$.

2. Two masses m_1 and m_2 are connected by an inextensible string which passes over a fixed pulley. Using Lagrange's equations, show that the acceleration of either mass is numerically

$$= (m_1 - m_2) g / (m_1 + m_2).$$

3. A perfectly flexible rope of uniform density per unit length is suspended with its end points fixed. Show that it assumes the shape of a catenary.

4. A bead of mass m from rest slides without friction under gravity along a wire inclined at an angle α to the vertical and rotating with constant angular velocity ω . Show that in times t , the bead has滑 through a distance

$$\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \cosh(\omega t \sin \alpha - 1).$$

Integral Equations

1. Introduction.
2. Definition.
3. Conversion of a linear differential equation to an integral equation and vice versa.
4. Conversion of boundary value problems to integral equations using Green's function.
5. Solution of an integral equation.
6. Integral equations of the convolution type.
7. Abel's integral equation.
8. Integro-differential equations.
9. Integral equations with separable kernels.
10. Solution of Fredholm equations with separable kernels.
11. Solution of Fredholm and Volterra equations by the method of successive approximations.

36.1 INTRODUCTION

Integral equations play an effective role in the study of boundary value problems. Such equations also occur in many fields of mechanics and mathematical physics. Integral equations may be obtained directly from physical problems e.g., radiation transfer problem and neutron diffusion problem etc. They also arise as representation formulae for the solutions of differential equations. A differential equation can be replaced by an integral equation with the help of initial and boundary conditions.

Integral equations were first encountered in the theory of Fourier integrals. In 1826, another integral equation was obtained by Abel. Actual development of the theory of integral equations began with the works of the Italian mathematician V. Volterra (1896) and the Swedish mathematician I. Fredholm (1900).

36.2 DEFINITION

An *integral equation* is an equation in which an unknown function appears under the integral sign. We shall take up integral equations in which only linear functions of the unknown function are involved. The general type of *linear integral equation* is of the form

$$y(x) = F(x) + \lambda \int_a^b K(x, t) y(t) dt$$

where $F(x)$ and $K(x, t)$ are known functions while $y(x)$ is to be determined. The function $K(x, t)$ is called the *Kernel of the integral equation*.

If a and b are constants, the equation is known as *Fredholm integral equation*.

If a is a constant while b is a variable, it is called a *Volterra integral equation*.

36.3 CONVERSION OF A LINEAR DIFFERENTIAL EQUATION TO AN INTEGRAL EQUATION AND VICE VERSA

To make this transformation, the use of the following formula is necessary :

$$\int_a^x \int_a^x f(x) dx dx = \int_a^x (x-t) f(t) dt \quad \dots(I)$$

$$\text{In general, } \int_a^x \int_a^x \dots \int_a^x f(x) dx^n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt \quad \dots(II)$$

Proof. Let $I_n = \int_a^x (x-t)^{n-1} f(t) dt$... (1)

where n is a positive integer and a is a constant.

Differentiating both sides w.r.t. x , using Leibnitz's rule (p. 139)

$$\begin{aligned}\frac{dI_n}{dx} &= \int_a^x \frac{\partial}{\partial x} (x-t)^{n-1} f(t) dt + [(x-t)^{n-1} f(t)]_{t=x} \cdot 1 - [(x-t)^{n-1} f(t)]_{t=a} \cdot 0 \\ &= \int_a^x (n-1)(x-t)^{n-2} f(t) dt = (n-1) I_{n-1}(x)\end{aligned}\quad \dots(2)$$

Again differentiating (2) w.r.t. x

$$\frac{d^2 I_n}{dx^2} = (n-1) \frac{d}{dx} [I_{n-1}(x)] = (n-1)(n-2) I_{n-2}, \text{ using (1)}$$

Proceeding in this way, we get

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)(n-2) \dots 1 \cdot I_1(x) = (n-1)! I_1(x)$$

Now taking $n = 1$ in (1), we get

$$I_1 = \int_a^x f(t) dt = \int_a^x f(x_1) dx_1 \quad \dots(3)$$

Putting $x = a$ in (1), we obtain

$$I_n(a) = 0 \text{ for all } n$$

Taking $n = 2$ in (2), we get $\frac{dI_2}{dx} = I_1(x)$

$$\begin{aligned}\therefore I_2 &= \int_a^x I_1(x_2) dx_2 \quad [\because I_2(a) = 0] \\ &= \int_a^x \int_a^{x_2} f(x_1) dx_1 dx_2 \quad [\text{Using (3)}] \dots(4)\end{aligned}$$

Putting $n = 3$ in (2), we have $\frac{dI_3}{dx} = 2I_2(x)$

$$\begin{aligned}\therefore I_3 &= 2 \int_a^x I_2(x) dx \quad [\because I_3(a) = 0] \\ &= 2 \int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1) dx_1 dx_2 dx_3 \quad [\text{Using (4)}]\end{aligned}$$

Proceeding in this way, we get

$$I_n = (n-1)! \int_a^x \int_a^{x_n} \dots \int_a^{x_2} F(x_1) dx_1 dx_2 \dots dx_n$$

$$\text{i.e., } \int_a^x \int_a^{x_n} \dots \int_a^{x_2} f(x_1) dx_1 dx_2 \dots dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt$$

If x_2, x_3, \dots, x_n be the same as x , we get the formula (II) above.

Example 36.1. Convert the differential equation $y''(x) - 3y'(x) + 2y(x) = 5 \sin x$, $y(0) = 1$, $y'(0) = -2$ into an integral equation.

Solution. Integrating both sides of the given differential equation, we get

$$[y'(x) - y'(0)] - 3[y(x) - y(0)] + 2 \int_0^x y(t) dt = 5(1 - \cos x)$$

Since $y'(0) = -2$ and $y(0) = 1$, it becomes

$$y'(x) + 2 - 3y(x) + 3 + 2 \int_0^x y(t) dt = 5 - 5 \cos x$$

i.e.,

$$y'(x) - 3y(x) + 2 \int_0^x y(t) dt = -5 \cos x$$

Integrating again as before, we have

$$[y(x) - y(0)] - 3 \int_0^x y(t) dt + 2 \int_0^x \int_0^x y(t) dt = -5 \sin x$$

or

$$y(x) - 1 - 3 \int_0^x y(t) dt + 2 \int_0^x (x-t) y(t) dt = -5 \sin x \quad [\text{Using (I) above}]$$

or

$$y(x) + \int_0^x [2(x-t) - 3] y(t) dt = 1 - 5 \sin x$$

which is the desired integral equation.

Example 36.2. Show that the integral equation

$$y(x) = \int_0^x (x+t) y(t) dt + 1 \quad \dots(i)$$

is equivalent to the differential equation

$$y''(x) - 2x y'(x) - 3y(x) = 0, y(0) = 1, y'(0) = 0. \quad (\text{Kerala, M. Tech., 2005})$$

Solution. Differentiating (i) by Leibnitz's rule (p. 139), we have

$$\begin{aligned} \frac{dy}{dx} &= \int_0^x \frac{\partial}{\partial x} (x+t) y(t) dt + (x+x) y(x) \frac{d}{dx}(x) - (x+0) y(0) \frac{d}{dx}(0) \\ &= \int_0^x y(t) dt + 2xy(x) = \int_0^x y(x) dx + 2xy(x) \end{aligned} \quad \dots(ii)$$

Differentiating again w.r.t. x , we get

$$\frac{d^2y}{dx^2} = y(x) + 2[x y'(x) + 1 \cdot y(x)] = 2xy'(x) + 3y(x)$$

or

$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 3y(x) = 0 \quad \dots(iii)$$

Putting $x = 0$ in (i), we obtain

$$y(0) = \int_0^0 (x+t) y(t) dt + 1 \quad \text{i.e., } y(0) = 1$$

and putting $x = 0$ in (ii), we get $y'(0) = 0$.Hence (i) is equivalent to (iii) with initial conditions $y(0) = 1, y'(0) = 0$.**Example 36.3.** Show that the integral equation

$$y(x) = \int_0^x t(t-x) y(t) dt + \frac{1}{2} x^2, \quad \dots(iv)$$

is equivalent to the differential equation

$$\frac{d^2y}{dx^2} + xy = 1 \text{ and the conditions } y(0) = y'(0) = 0.$$

Solution. Differentiating (i) w.r.t. x by Leibnitz's rule (p. 139), we have

$$\begin{aligned} \frac{dy}{dx} &= \int_0^x \frac{\partial}{\partial x} [t(t-x)] y(t) dt + [t(t-x)y(t)]_{t=x} \cdot 1 + x - [t(t-x)y(t)]_{t=0} \cdot 0 \\ &= \int_0^x t(-1) y(t) dt + x = x - \int_0^x ty(t) dt \end{aligned} \quad \dots(v)$$

Differentiating (ii) w.r.t. x , we get

$$\frac{d^2y}{dx^2} = 1 - \left\{ \int_0^x \frac{\partial}{\partial x} [t y(t)] dt - [t y(t)]_{t=0} \cdot 0 + [ty(t)]_{t=x} \cdot 1 \right\} = 1 - xy(x)$$

or $\frac{d^2y}{dx^2} + xy = 1$ which is the differential equation corresponding to (i).

Also $y(0) = 0$ and $y'(0) = 0$.

Example 36.4. Find the integral equation corresponding to the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad y(0) = y(1) = 0.$$

Solution. Integrating both sides of the given differential equation w.r.t. x over $(0, x)$, we get

$$y'(x) - y'(0) + \lambda \int_0^x y(x) dx = 0$$

or $y'(x) = c - \lambda \int_0^x y(x) dx$, taking $y'(0) = c$... (i)

Again integrating (i) w.r.t. x in $(0, x)$, we obtain

$$\begin{aligned} y(x) - y(0) &= cx - \lambda \int_0^x \int_0^x y(x) dx \\ y(x) &= cx - \lambda \int_0^x (x-t) y(t) dt \end{aligned} \quad \dots (ii)$$

[Using (I) of § 36.3 and noting that $y(0) = 0$]

Putting $x = 1$ in (ii), we get

$$\begin{aligned} y(1) &= c - \lambda \int_0^1 (1-t) y(t) dt \quad [\because y(1) = 0] \\ \therefore c &= \lambda \int_0^1 (1-t) y(t) dt \end{aligned} \quad \dots (iii)$$

Substituting the value of c from (iii) in (ii), we get

$$\begin{aligned} y(x) &= \lambda x \int_0^1 (1-t) y(t) dt - \lambda \int_0^x (x-t) y(t) dt \\ &= \lambda x \left\{ \int_0^x (1-t) y(t) dt + \int_x^1 (1-t) y(t) dt \right\} - \lambda \int_0^x (x-t) y(t) dt \\ &= \lambda \int_0^x t(1-x) y(t) dt + \lambda \int_x^1 (1-t) y(t) dt \\ &= \lambda \left\{ \int_0^x K(x, t) y(t) dt + \int_x^1 K(x, t) y(t) dt \right\} \end{aligned}$$

where

$$K(x, t) = \begin{cases} t(1-x) & \text{when } t < x \\ x(1-t) & \text{when } t > x \end{cases}$$

Hence $y(x) = \lambda \int_0^1 K(x, t) y(t) dt$

which is a *Fredholm integral equation* with a symmetric kernel $K(x, t)$.

PROBLEMS 36.1

Transform each of the following boundary value problems into corresponding integral equations :

- $y'' + xy' + y = 0$, given that $y(0) = 1$, $y'(0) = 0$. (Madras, M.E., 2000)
- $y'' - \sin xy' + e^y = x$, given that $y = 1$, $\frac{dy}{dx} = -1$ when $x = 0$.
- $y'' + xy' = 1$, given that $y(0) = y'(0) = 0$. (Madras, 2000 S)
- $y'' + (1-x)y' + e^{-x}y = x^3 - 5x$, given that $y = -3$, $\frac{dy}{dx} = 4$ when $x = 0$.
- $\frac{d^3y}{dx^3} + y = \cos x$ given that $y = 0$, $y' = 1$, $y'' = 2$ at $x = 0$. (Kerala, M. Tech, 2005)

6. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - xy = \sin x$ given that $y = 1, y' = -1, y'' = \frac{1}{2}$ at $x = 0$.

7. $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 4 \cos 2x$ given that
 $y = -1, y' = 4, y'' = 0, y''' = 2$ when $x = 0$.

Convert each of the following integral equations into differential equations alongwith initial conditions :

8. $y(x) = \int_0^x (x+t) y(t) dt - 1.$

9. $y(x) = \int_0^x (x-t) y(t) dt + 3 \sin x.$

10. $y(x) + 3 \int_0^x (x-t)^2 y(t) dt = x^2 - 3x + 4.$

11. $y(x) + \int_0^x (x-t)^2 + 4(x-t) - 3 \} y(t) dt = e^{-x}.$

12. If $y''(x) = f(x); y(0) = y(l) = 0$, show that $y(x) = \int_0^l K(x,t) f(t) dt,$

where $K(x,t) = \begin{cases} \frac{t}{l}(x-l) & \text{when } t < x \\ \frac{x}{l}(t-l) & \text{when } t > x \end{cases}$

36.4 CONVERSION OF BOUNDARY VALUE PROBLEMS TO INTEGRAL EQUATIONS USING GREEN'S FUNCTION

Consider the linear homogeneous differential equation

$$L(y) + \phi(x) = 0 \quad \dots(1)$$

where $L(y) = \left[\frac{d}{dx} \left(p \frac{dy}{dx} \right) + q \right] y = py'' + p'y' + qy \quad \dots(2)$

together with the homogeneous boundary conditions of the form

$$\alpha y + \beta \frac{dy}{dx} = 0 \quad \dots(3)$$

Now let us find a function $G(x, t)$ which for a given number t , is given by $G_1(x)$ when $x < t$ and by $G_2(x)$ when $x > t$ and which has the following properties :

I. G_1 and G_2 satisfy the equation $L(G) = 0$ in their defined intervals i.e., $L(G_1) = 0$ when $x < t$; $L(G_2) = 0$ when $x > t$.

II. G_1 and G_2 satisfy the boundary conditions at the end points $x = a$ and $x = b$ respectively.

III. $G(x, t)$ is continuous at $x = t$ i.e., $G_1(t) = G_2(t)$.

IV. The derivative of G is continuous at every point within the range of x except at $x = t$ i.e., $G_2'(t) - G_1'(t) = -1/p(t)$

Def. $G(x, t)$ as defined above is called the **Green's function**.

If $G(x, t)$ exists, then the solution of the given problem can be transformed to the integral equation

$$y(x) = \int_a^b G(x, t) \phi(t) dt \quad \dots(4)$$

Let $y = y_1(x)$ and $y = y_2(x)$ be the non-trivial solutions of the equations $L(y) = 0$ which satisfy the homogeneous conditions at $x = a$ and $x = b$ respectively.

The above conditions I and II are satisfied if we write

$$G = \begin{cases} C_1 y_1(x), & \text{when } x < t \\ C_2 y_2(x), & \text{when } x > t \end{cases} \quad \dots(5)$$

Imposing the condition III on (5), we get

$$C_2 y_2(t) - C_1 y_1(t) = 0 \quad \dots(6)$$

Imposing the condition IV on (5), we have

$$C_2 y_2'(t) - C_1 y_1'(t) = -1/p(t) \quad \dots(7)$$

Equations (6) and (7) give a unique solution, if

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \neq 0$$

By Abel's formula*, we find that

$$y_1(t)y_2'(t) - y_2(t)y_1'(t) = C/p(t) \quad \dots(8)$$

Now [(7) $\times y_2(t)$ - (6) $\times y_1(t)$] gives on using (8),

$$C_1 = -\frac{1}{C} y_2(t) \text{ and } C_2 = -\frac{1}{C} y_1(t)$$

Thus (5) reduces to $G(x, t) = \begin{cases} -\frac{1}{C} y_1(x) y_2(t), & x < t \\ -\frac{1}{C} y_1(t) y_2(x), & x > t \end{cases} \quad \dots(9)$

Conversely it can be shown that the integral equation.

$$y(x) = \int_a^b G(x, t) \phi(t) dt$$

where $G(x, t)$ is as defined by (9), satisfies the differential equation $L(y) + \phi(x) = 0$ together with the prescribed boundary conditions.

Example 36.5. Transform the differential equation $y'' + y = x$, $y(0) = 1$, $y'(1) = 0$ to a Fredholm integral equation, finding the corresponding Green's function. (Madras M.E., 2000 S)

Solution. Given equation is $L(y) + \phi(x) = 0$...(i)

with the conditions $y(0) = 0$, $y'(1) = 0$, where $L(y) = y''$ and $\phi(x) = y - x$

The associated equation $L(y) = 0$ is $y''(y) = 0$...(ii)

Its solution is $y = C_1 x + C_2$...(iii)

Now $y_1(x)$ is a particular solution of (ii) satisfying the condition $y(0) = 0$.

\therefore Taking $C_2 = 0$ and $C_1 = 1$, we get $y_1(x) = x$...(iv)

$y_2(x)$ is another solution of (ii) satisfying the condition $y'(1) = 0$.

\therefore From (iii), $y'(1) = C_1 = 0$. Then $y(x) = C_2$

Taking $C_2 = 1$, a particular solution is $y_2(x) = 1$.

The constant C is found from $y_1(x)y_2'(x) - y_2(x)y_1'(x) = \frac{C}{p(x)}$

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\therefore p = 1$.

Thus $C = y_1(x)y_2'(x) - y_2(x)y_1'(x) = x \cdot 0 - 1 \cdot 1 = -1$.

\therefore Green's function is given by

$$G(x, t) = \begin{cases} \frac{-y_1(x)y_2(t)}{C}, & x < t \\ \frac{-y_1(t)y_2(x)}{C}, & x > t \end{cases} = \begin{cases} x, & x < t \\ t, & x > t \end{cases} \quad \dots(v)$$

Hence the equation (i) is equivalent to the integral equation

$$\begin{aligned} y(x) &= \int_0^1 G(x, t) \phi(t) dt = \int_0^1 G(x, t) \cdot (y - t) dt \\ &= \int_0^1 G(x, t) y(t) dt - \left\{ \int_0^x t \cdot dt + \int_x^1 t \cdot dt \right\} \end{aligned}$$

* The conditions that $y_1(x)$ and $y_2(x)$ satisfy the equation

$L(y) = 0$ are $(py_1)' + qy_1 = 0$...(i), $(py_2)' + qy_2 = 0$...(ii)

$[(i) \times y_2 - (ii) \times y_1]$ gives $y_2(py_1)' - y_1(py_2)' = 0$

or $[p(y_1y_2' - y_2y_1')]' = 0$ or $y_1y_2' - y_2y_1' = C/p$

$$\begin{aligned}
 &= \int_0^1 G(x, t) y(t) dt - \left\{ x \left| \frac{t^2}{2} \right|_0^x + \left| \frac{t^3}{3} \right|_x^1 \right\} \\
 &= \int_0^1 G(x, t) y(t) dt - \frac{1}{6}(x^3 + 2)
 \end{aligned}$$

where $G(x, t)$ is given by (v).

Example 36.6. Find the Green's function for the boundary value problem

$$d^2y/dx^2 + \mu^2 x = 0, \quad y(0) = 0 = y(1).$$

Solution. We observe that the solution $y_1 = \sin \mu x$ satisfies the boundary condition $y(0) = 0$ and the solution $y_2 = \sin \mu(x - 1)$ satisfies the second condition $y(1) = 0$. Also both these solutions are linearly independent.

The constant C is found from $y_1 y_2' - y_2 y_1' = C/p(x)$.

Since $L(y) = py'' + p'y' + qy = y''$ (given), $\therefore p = 1$

$$\therefore C = y_1 y_2' - y_2 y_1' = \mu \sin \mu x \cos \mu(x - 1) - \mu \sin \mu(x - 1) \cos \mu x = \mu \sin \mu$$

Hence the Green's function is given by

$$G(x, t) = \begin{cases} -\frac{y_1(x) y_2(t)}{C}, & x < t \\ -\frac{y_1(t) y_2(x)}{C}, & x > t \end{cases} = \begin{cases} -\frac{\sin \mu x \sin \mu(t-1)}{\mu \sin \mu}, & x < t \\ -\frac{\sin \mu t \sin \mu(x-1)}{\mu \sin \mu}, & x > t \end{cases}$$

PROBLEMS 36.2

- Transform the problem $d^2y/dx^2 + xy = 1 ; y(0) = 0 = y(1)$ to an integral equation, finding the corresponding Green's function.
- Transform the problem $y'' + y = x ; y(0) = 1, y'(1) = 0$ to an integral equation using Green's function.
- Construct an integral equation corresponding to the boundary value problem

$$\frac{d^2u}{dx^2} + e^u u = x ; u(0) = 0, u(1) = 1.$$

- Find the Green's function for the boundary value problem $d^2y/dx^2 - y = 0$ with $y(0) = y(1) = 0$.
- Transform the boundary value problem $x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} + (\lambda x^2 - 1) u = 0 ; u(0) = u(1) = 0$, to an integral equation.

[Hint. $u_1(x) = x, u_2(x) = \frac{1}{x} - x$ and $C = -2$]

36.5 SOLUTION OF AN INTEGRAL EQUATION

The solution of the integral equation $y(x) = F(x) + \lambda \int_a^b K(x, t) y(t) dt$ is a function $y(x)$ which when substituted in the equation reduces it to an identity w.r.t. x .

Example 36.7. Show that $y(x) = 2 - x$ is a solution of the integral equation

$$\int_0^x e^{x-t} y(t) dt = e^x + x - 1. \quad \dots(i)$$

Solution. Since

$$y(t) = 2 - t$$

$$\begin{aligned}
 \therefore \int_0^x e^{x-t} y(t) dt &= \int_0^x e^{x-t} (2-t) dt \\
 &= 2e^x \int_0^x e^{-t} dt - e^x \int_0^x t e^{-t} dt
 \end{aligned}$$

$$\begin{aligned}
 &= 2e^x \left| -e^{-t} \right|_0^x - e^x \left\{ \left| t \cdot (-e^{-t}) \right|_0^x - \int_0^x 1 \cdot (-e^{-t}) dt \right\} \\
 &= 2e^x (-e^{-x} + 1) + e^x (xe^{-x}) - e^x \left| -e^{-t} \right|_0^x \\
 &= -2 + 2e^x + x + e^x (e^{-x} - 1) = e^x + x - 1.
 \end{aligned}$$

Thus (i) is identically satisfied by $y(x) = 2 - x$. Hence $y(x) = 2 - x$ is a solution of (i).

Example 36.8. Show that the function $y(x) = (1+x^2)^{-3/2}$ is a solution of the Volterra integral equation :

$$y(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} y(t) dt. \quad \dots(i)$$

Solution. Substituting $y(x) = (1+x^2)^{-3/2}$ in the R.H.S. of (i), we have

$$\begin{aligned}
 &\frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \cdot \frac{1}{(1+t^2)^{3/2}} dt \\
 &= \frac{1}{1+x^2} + \frac{1}{1+x^2} \left\{ \frac{1}{(1+t^2)^{1/2}} \right\}_0^x \\
 &= \frac{1}{1+x^2} + \frac{1}{1+x^2} \cdot \frac{1}{(1+x^2)^{1/2}} - \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}} = y(x)
 \end{aligned}$$

Thus $y(x) = (1+x^2)^{-3/2}$ is a solution of the integral equation (i).

Example 36.9. Show that the function $y(x) = xe^x$ is a solution of the integral equation

$$y(x) = \sin x + 2 \int_0^x \cos(x-t) y(t) dt. \quad \dots(i)$$

Solution. Substituting $y(x) = xe^x$ in the R.H.S. of (i), we have

$$\begin{aligned}
 &\sin x + 2 \int_0^x \cos(x-t) \cdot te^t dt \\
 &= \sin x + 2 \left\{ \cos x \int_0^x t \cdot e^t \cos t dt + \sin x \int_0^x te^t \sin t dt \right\} \quad (\text{Integrating by parts}) \\
 &= \sin x + 2 \cos x \left\{ \frac{1}{2} \left| t \cdot e^t (\cos t + \sin t) \right|_0^x - \frac{1}{2} \int_0^x e^t (\cos t + \sin t) dt \right\} \\
 &\quad + 2 \sin x \left\{ \frac{1}{2} \left| t \cdot e^t (\sin t - \cos t) \right|_0^x - \frac{1}{2} \int_0^x e^t (\sin t - \cos t) dt \right\} \\
 &= \sin x + xe^x (\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x) \\
 &\quad - \cos x \int_0^x e^t (\cos t + \sin t) dt - \sin x \int_0^x t (\sin t - \cos t) dt \\
 &= \sin x + xe^x - \cos x \left| e^t \sin t \right|_0^x + \sin x \left| e^t \cos t \right|_0^x \\
 &= \sin x + xe^x - e^x \cos x \sin x + e^t \sin x \cos x - \sin x = xe^x
 \end{aligned}$$

Thus $y(x) = xe^x$ is a solution of the integral equation (i).

36.6 INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

$$y(x) = F(x) + \int_0^x K(x-t) y(t) dt$$

is an *integral equation of convolution type* and can be written as

$$y(x) = F(x) + K(x) * y(x)$$

[See p. 748]

It is a special integral equation of importance in applications.

Taking Laplace transform of both sides, assuming that $L[F(x)] = f(s)$ and $L[K(x)] = k(s)$ both exist, and using convolution theorem

$$\bar{y}(s) = f(s) + k(s) \cdot y(s) \quad \text{or} \quad \bar{y}(s) = \frac{f(s)}{1 - k(s)}$$

Now taking the inverse transform of both sides, we get the required solution.

Example 36.10. Solve the integral equation

$$y(x) = 3x^2 + \int_0^x y(t) \sin(x-t) dt.$$

Solution. Given integral equation can be written as

$$y(x) = 3x^2 + y(x) * \sin(x-t) \quad \dots(i)$$

Taking Laplace transform of both sides and using the convolution theorem (p. 748), we get

$$\bar{y} = \frac{6}{s^3} + y \cdot \frac{s}{s^2 + 1} \quad \text{or} \quad \bar{y} = \frac{6(s^2 + 1)}{s^5} = 6 \left(\frac{1}{s^3} + \frac{1}{s^5} \right)$$

$$\text{On inversion, we get } y = 6 \left(\frac{x^2}{2!} + \frac{x^4}{4!} \right) = 3x^2 + x^4/4$$

which is the required solution of (i).

Obs. The above solution can also be verified by direct substitution in the given integral equation.

Example 36.11. Solve $y(x) = x + 2 \int_0^x \cos(x-t) y(t) dt$.

Solution. The given equation can be written as

$$y(x) = x + 2 \cos(x) * y(x)$$

Taking Laplace transform of both sides and using convolution theorem, we get

$$\bar{y}(s) = \frac{1}{s^2} + 2 \frac{s}{s^2 + 1} \cdot \bar{y}(s) \quad \text{or} \quad \bar{y} \left(1 - \frac{2s}{s^2 + 1} \right) = \frac{1}{s^2}$$

or

$$\bar{y} = \frac{s^2 + 1}{s^2(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{2}{(s-1)^2}$$

$$\text{On inversion, we obtain } y = 2 + x - 2e^x + 2xe^x$$

Hence $y = x + 2 + 2(x-1)e^x$ is the desired solution.

Example 36.12. Solve the integral equation $\int_0^x y(t) y(x-t) dt = 4 \sin 9x$.

Solution. The given integral equation can be written as

$$y(x) * y(x) = 4 \sin 9x \quad \dots(i)$$

Taking Laplace transform of both sides and using the convolution theorem, we get

$$(\bar{y}(s))^2 = \frac{36}{s^2 + 81} \quad \text{or} \quad \bar{y} = \pm \frac{6}{\sqrt{s^2 + 81}}$$

On inversion and noting that $L^{-1} \frac{1}{\sqrt{(s^2 + a^2)}} = J_0(ax)$, we get

$$y = \pm 6 L^{-1} \left\{ \frac{1}{\sqrt{(s^2 + 9^2)}} \right\} = \pm 6 J_0(9x)$$

Thus $y = 6J_0(9x)$ and $y = -6J_0(9x)$ are both solutions of (i).

36.7 ABEL'S INTEGRAL EQUATION

The integral equation $\int_0^x \frac{y(t)}{(x-t)^\alpha} dx = G(x)$

such that $G(x)$ is given and α is a constant ($0 < \alpha < 1$), is known as *Abel's integral equation*. This is an important integral equation of the convolution type. An application of this equation is in finding the shape of a smooth wire lying in a vertical plane such that a bead placed anywhere on the wire slides to the lowest point in the same time. This is the well known *tautochrone problem* and the shape of the wire is a *cycloid*.

Example 36.13. Solve the Abel's integral equation $\int_0^x \frac{y(t)}{\sqrt{x-t}} dt = 1 + 2x - x^2$.

Solution. The given equation can be written as

$$y(x) * x^{-1/2} = 1 + 2x - x^2$$

Taking the Laplace transform of both sides and using convolution theorem, we get

$$\bar{y} \cdot L(x^{-1/2}) = L(1 + 2x - x^2)$$

or

$$\bar{y} \frac{\Gamma(1/2)}{s^{1/2}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3} \quad \text{or} \quad \bar{y} = \frac{1}{\Gamma(1/2)} \left(\frac{1}{s^{1/2}} + 2 \cdot \frac{1}{s^{3/2}} - 2 \cdot \frac{1}{s^{5/2}} \right)$$

On inversion, and noting that $L^{-1} \frac{1}{s^{n+1}} = \frac{x^n}{\Gamma(n+1)}$, we have

$$\begin{aligned} y &= \frac{1}{\Gamma(\frac{1}{2})} \frac{x^{-1/2}}{\Gamma(\frac{1}{2})} + 2 \cdot \frac{x^{1/2}}{\Gamma(\frac{3}{2})} - 2 \cdot \frac{x^{3/2}}{\Gamma(\frac{5}{2})} \\ &= \frac{1}{\pi} \left(x^{-1/2} + 4x^{1/2} - \frac{8}{3}x^{3/2} \right) \end{aligned}$$

Hence

$$y = \frac{1}{3\pi\sqrt{x}} (3 + 12x - 8x^3) \text{ is the desired solution.}$$

36.8 INTEGRO-DIFFERENTIAL EQUATIONS

An integral equation in which various derivatives of the unknown function $y(x)$ are also present, is called an *integro-differential equation*. An example of such an equation is

$$y'(x) = y(x) - \cos x + \int_0^x \sin(x-t) y(t) dt$$

The solution of integro-differential equation subject to given initial conditions can also be found by Laplace transforms as illustrated below :

Example 36.14. Solve $\frac{dy}{dx} + 3y + 2 \int_0^x y dx = x$, given $y(0) = 1$.

Solution. Given equation can be written as $y'(x) + 3y(x) + 2 \int_0^x y dx = x$

Taking Laplace transform of both sides, we get

$$L[y'(x)] + 3L[y(x)] + 2L \left\{ \int_0^x y(x) dx \right\} = L(x)$$

or

$$\{s \bar{y}(s) - y(0)\} + 3 \bar{y}(s) + 2 \frac{1}{s} \bar{y}(s) = \frac{1}{s^2} \quad [\text{Using } \S 21.6]$$

$$s \bar{y} + 3 \bar{y} + 2 \frac{1}{s} \bar{y} = 1 + \frac{1}{s^2} \quad [\because y(0) = 1]$$

or

$$\bar{y} = \frac{1+s^2}{s(s^2+3s+2)} = \frac{1+s^2}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{2}{s+1} + \frac{5}{2(s+2)}$$

On inversion, we obtain $y = \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s+1}\right) + \frac{5}{2} L^{-1}\left(\frac{1}{s+2}\right)$

Hence $y = \frac{1}{2} - 2e^{-x} + \frac{5}{2} e^{-2x}$ is the required solution.

Example 36.15. Solve $\frac{dy}{dx} = 3 \int_0^x \cos 2(x-t) y(t) dt + 2$ given $y(0) = 1$.

Solution. Given equation can be written as

$$y'(x) = 3 \cos 2x * y(x) + 2$$

Taking Laplace transform of both sides, we get

$$\begin{aligned} L[(y'(x))] &= 3L(\cos 2x) \cdot \bar{y}(s) + \frac{2}{s} \\ s\bar{y}(s) - y(0) &= \frac{3s\bar{y}(s)}{s^2+4} + \frac{2}{s} \quad \text{or} \quad \bar{y} = \frac{(s+2)(s^4+4)}{s^2(s^2+1)} \\ &= \frac{4}{s} + \frac{8}{s^2} - 3\frac{s}{s^2+1} - 6\frac{1}{s^2+1} \end{aligned} \quad [\because y(0) = 1]$$

On inversion, we obtain $y = 4 + 8x - 3 \cos x - 6 \sin x$

which is the required solution.

Obs. The given integro-differential equation can be converted into the following integral equation by integrating it from 0 to x and using $y(0) = 1$.

$$y(x) = 2x + 1 + 3 \int_0^x (x-t) \cos 2(x-t) y(t) dt.$$

PROBLEMS 36.3

1. Show that $y(x) = 1 - x$ is a solution of the integral equation $\int_0^x e^{x-t} y(t) dt = x$.

2. Show that $y(x) = 1$ is a solution of the Fredholm integral equation

$$y(x) + \int_0^1 x(e^{tx} - 1) y(t) dt = e^x - x.$$

3. Show that $y(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation $\int_0^x \frac{y(t)}{\sqrt{(x-t)}} dt = 1$.

4. Show that $y(x) = e^x (2x - 2/3)$ is a solution of the Fredholm integral equation

$$y(x) + \int_0^1 e^{x-t} y(t) dt = 2xe^x.$$

Solve each of the following integral equations :

5. $y(x) = x + \frac{1}{6} \int_0^x (x-t)^3 y(t) dt.$

6. $y(x) = x^2 + \int_0^x y(t) \sin(x-t) dt.$

7. $\int_0^x y(t) y(x-t) dt = 2y(x) + x - 2.$

8. $\int_0^x y(t) y(x-t) dt = 9 \sin 4x.$

9. Find a solution of the integral equation $y(x) = \frac{1}{2} \sin 2x + \int_0^x y(t) y(x-t) dt.$

Solve the following integral equations :

10. $\frac{dy}{dx} + 4y + 5 \int_0^x y(t) dt = e^{-x}, y(0) = 0.$

11. $\frac{dy}{dx} + 2y + \int_0^x y(t) dt = \sin x, y(0) = 1. \quad (\text{Mumbai, 2006})$

$$12. \quad y'(x) = x + \int_0^x y(x-t) \cos t dt, \quad y(0) = 1.$$

$$13. \quad y'(x) = \int_0^x y(t) \cos(x-t) dt, \quad y(0) = 1.$$

$$14. \quad \int_0^x \frac{y(t)}{\sqrt{(x-t)}} dt = \sqrt{x}.$$

$$15. \quad \int_0^x \frac{y(t)}{(x-t)^{1/3}} dt = x(x+1).$$

36.9 INTEGRAL EQUATIONS WITH SEPARABLE KERNELS

A kernel $K(x, t)$ of Fredholm integral equation is said to be *separable* (or *degenerate*) if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x alone and a function of t alone i.e., if it is of the form $K(x, t) = \sum_{n=1}^m f_n(x) g_n(t)$.

Also since $\cos(x+t) = \cos x \cos t - \sin x \sin t$, $\cos(x+t)$ is a separable kernel.

36.10 SOLUTION OF FREDHOLM EQUATIONS WITH SEPARABLE KERNELS

Consider the integral equation

$$y(x) = \lambda \int_a^b K(x, t) \cdot y(t) dt + F(x) \quad \dots(1)$$

where

$$K(x, t) = f_1(x) \cdot g_1(t) + f_2(x) \cdot g_2(t) + \dots + f_m(x) \cdot g_m(t) \quad \dots(2)$$

Substituting (2) in (1), we get

$$\begin{aligned} y(x) &= \lambda \int_a^b \left[\sum_{n=1}^m f_n(x) \cdot g_n(t) \right] y(t) dt + F(x) \\ &= \lambda \sum_{n=1}^m f_n(x) \left\{ \int_a^b g_n(t) y(t) dt \right\} + F(x) \end{aligned} \quad \dots(3)$$

Evidently $\int_a^b g_n(t) y(t) dt = C_n$ (say), is a constant and will be different for different values of n . Then (3) takes the form

$$y(x) = \lambda \sum_{n=1}^m C_n f_n(x) + F(x) \quad \dots(4)$$

This is a solution of (1) in which m constants C_1, C_2, \dots, C_m are to be determined.

Now multiplying both sides of (4) by $g_k(x)$ and integrating from a to b , we obtain

$$\int_a^b y(x) g_k(x) dx = \lambda \int_a^b \sum_{n=1}^m C_n f_n(x) g_k(x) dx + \int_a^b F(x) g_k(x) dx$$

Writing $\int_a^b g_k(x) f_n(x) dx = \alpha_{kn}$ and $\int_a^b g_k(x) F(x) dx = \beta_k$, the above equation becomes

$$\begin{aligned} C_k &= \lambda \sum_{n=1}^m C_n \alpha_{kn} + \beta_k \\ &= \lambda(C_1 \alpha_{k1} + C_2 \alpha_{k2} + \dots + C_m \alpha_{km}) + \beta_k \end{aligned}$$

Taking $k = 1, 2, \dots, m$, we get the following m equations which determine C_1, C_2, \dots, C_m :

$$\begin{aligned} (1 - \lambda \alpha_{11}) C_1 - \lambda \alpha_{12} C_2 - \dots - \lambda \alpha_{1m} C_m &= \beta_1 \\ - \lambda \alpha_{21} C_1 + (1 - \lambda \alpha_{22}) C_2 - \dots - \lambda \alpha_{2m} C_m &= \beta_2 \\ \dots & \\ - \lambda \alpha_{m1} C_1 - \lambda \alpha_{m2} C_2 - \dots + (1 - \lambda \alpha_{mm}) C_m &= \beta_m \end{aligned} \quad \dots(5)$$

Equations (5) will give a unique solution if the determinant Δ of the coefficients of C_1, C_2, \dots, C_m is not zero.

Now the following cases arise :

I. When $F(x) = 0$, (1) is said to be a *homogeneous integral equation* and all β 's are zero.

(i) If $\Delta \neq 0$, the only solution of (4) is the trivial solution $C_1 = C_2 = \dots = C_m = 0$. Then $y(x) = 0$ is the trivial solution of (1).

(ii) If $\Delta = 0$, at least one of the C 's can be assigned any value and the remaining C 's can be found accordingly. Then (4) gives infinitely many solutions. The values of λ for which $\Delta = 0$ are known as the *eigen values*. Any non-trivial solution of the homogeneous integral equation for a certain value of λ is called the corresponding *eigen function*. The solutions corresponding to eigen values of λ can be expressed as arbitrary multiples of eigen functions.

II. When $F(x) \neq 0$. Let us assume that $\int_a^b g_m(x) F(x) dx = 0$ so that $\beta_m = 0$

(i) If $\Delta \neq 0$, the only solution of (i) is the trivial solution $C_1 = C_2 = \dots = C_m = 0$.

Then $y = F(x)$ is the desired solution of (1).

(ii) If $\Delta = 0$, atleast one of the C 's can be given any value and the remaining C 's can be found.

Then (4) gives infinitely many solutions.

III. When atleast one of the β 's $\neq 0$

(i) If $\Delta \neq 0$, then equations (5) give a unique solution of the constants C .

Hence there is a unique solution of (1).

(ii) If $\Delta = 0$, then equations (5) will be inconsistent.

\therefore Either there is no solution or infinitely many solutions of (i) exist.

Example 36.16. Find the eigen values and eigen functions of the following homogeneous integral equations with degenerate kernels :

$$(i) y(x) = \lambda \int_0^1 (2xt - 4x^2) y(t) dt \quad (ii) y(x) = \frac{1}{e^2 - 1} \int_0^1 e^{x+t} y(t) dt.$$

Solution. (i) Given equation may be written as

$$y(x) = \lambda \left\{ 2x \int_0^1 ty(t) dt - 4x^2 \int_0^1 y(t) dt \right\}$$

or $y(x) = (2\lambda x) C_1 - (4\lambda x^2) C_2 \quad \dots(i)$

$$\text{where } C_1 = \int_0^1 ty(t) dt, C_2 = \int_0^1 y(t) dt$$

Substituting $y(x)$ in C_1, C_2 we get

$$C_1 = \int_0^1 t[(2\lambda t) C_1 - (4\lambda t^2) C_2] dt$$

$$C_2 = \int_0^1 [(2\lambda t) C_1 - (4\lambda t^2) C_2] dt$$

or

$$C_1 \left\{ 1 - 2\lambda \int_0^1 t^2 dt \right\} + C_2 \left\{ 4\lambda \int_0^1 t^3 dt \right\} = 0$$

$$C_1 \left\{ -2\lambda \int_0^1 t dt \right\} + C_2 \left\{ 1 + 4\lambda \int_0^1 t^2 dt \right\} = 0$$

or

$$C_1 \left(1 - \frac{2\lambda}{3} \right) + \lambda C_2 = 0 ; -\lambda C_1 + C_2 (1 + 4\lambda/3) = 0 \quad \dots(ii)$$

\therefore The determinant of eigen values will be

$$\begin{vmatrix} 1 - 2\lambda/3 & \lambda \\ -\lambda & 1 + 4\lambda/3 \end{vmatrix} = 0 \text{ or } (\lambda + 3)^2 = 0$$

\therefore The eigen values are $\lambda_1 = -3, \lambda_2 = -3$

For $\lambda_1 = \lambda_2 = -3$, the equations (ii) reduce to

$$3C_1 - 3C_2 = 0 ; 3C_1 - 3C_2 = 0 \text{ i.e., } C_1 = C_2$$

\therefore From (i) $y(x) = -6C_1(x - 2x^2) = x - 2x^2$ if $C_1 = -1/6$

Hence the eigen function corresponding to $\lambda_1 = \lambda_2 = -3$, is

$$y(x) = x - 2x^2$$

(ii) Given integral equation may be written as

$$y(x) = \frac{e^x}{e^2 - 1} \int_0^1 e^t e^t \cdot y(t) dt = \frac{e^x}{e^2 - 1} C, \quad \dots(i)$$

$$\text{where } C = \int_0^1 e^t y(t) dt$$

Substituting the value of $y(t)$ from (i) in C , we get

$$C = \int_0^1 e^t \left(\frac{e^t C}{e^2 - 1} \right) dt = \frac{C}{e^2 - 1} \int_0^1 e^{t^2} dt$$

or

$$C \left[1 - \frac{1}{e^2 - 1} \int_0^1 e^{t^2} dt \right] = 0 \quad \text{i.e. } C = 0$$

Hence from (i), $y(x) = 0$

which shows that the given integral equation has only trivial solution.

Thus it does not contain any eigen values or eigen functions.

Example 36.17. Solve the integral equation

$$y(x) = \cos x + \lambda \int_0^\pi \sin(x-t) y(t) dt.$$

Solution. Writing the given equation in the following form :

$$\begin{aligned} y(x) &= \cos x + \lambda \left\{ \sin x \int_0^\pi \cos t y(t) dt - \cos x \int_0^\pi \sin t y(t) dt \right\} \\ \text{or} \quad y(x) &= \cos x + (\lambda \sin x) C_1 - (\lambda \cos x) C_2 \end{aligned} \quad \dots(i)$$

$$\text{where } C_1 = \int_0^\pi \cos t \cdot y(t) dt, \quad C_2 = \int_0^\pi \sin t \cdot y(t) dt$$

Substituting $y(x)$ in C_1 and C_2 , we get

$$\begin{aligned} C_1 &= \int_0^\pi \cos t \{ \cos x + (\lambda \sin x) C_1 - (\lambda \cos x) C_2 \} dt \\ C_2 &= \int_0^\pi \sin t \{ \cos x + (\lambda \sin x) C_1 - (\lambda \cos x) C_2 \} dt \end{aligned}$$

$$\begin{aligned} \text{or} \quad C_1 \left\{ 1 - \lambda \int_0^\pi \cos t \sin t dt \right\} + C_2 \left\{ \lambda \int_0^\pi \cos^2 t dt \right\} &= \int_0^\pi \cos^2 t dt \\ C_1 \left\{ -\lambda \int_0^\pi \sin^2 t dt \right\} + C_2 \left\{ 1 + \lambda \int_0^\pi \sin t \cos t dt \right\} &= \int_0^\pi \sin t \cos t dt \end{aligned} \quad \dots(ii)$$

By evaluating each of the integrals in (ii), we get

$$C_1 + \frac{1}{2} C_2 \lambda \pi = \frac{1}{2} \pi; -\frac{1}{2} C_1 \lambda \pi + C_2 = 0 \quad \dots(iii)$$

The determinant of the equations (iii) is given by

$$\begin{vmatrix} 1 & \frac{1}{2} \lambda \pi \\ -\frac{1}{2} \lambda \pi & 1 \end{vmatrix} = 1 + \frac{1}{4} \lambda^2 \pi^2 \neq 0$$

Thus the equations (iii) have a unique solution

$$C_1 = \frac{2\pi}{4 + \lambda^2 \pi^2}; C_2 = \frac{\lambda \pi^2}{4 + \lambda^2 \pi^2}$$

Substituting these values of C_1 and C_2 in (i), we obtain the required solution

$$y(x) = \cos x + \frac{\lambda}{4 + \lambda^2 \pi^2} (2\pi \sin x - \lambda \pi^2 \cos x)$$

$$\text{or } y(x) = \frac{2}{4 + \lambda^2 \pi^2} (2 \cos x + \pi \lambda \sin x).$$

PROBLEMS 36.4

Determine the eigen values and eigen functions for the following homogeneous integral equations with degenerate kernels :

$$1. \quad y(x) = \lambda \int_0^1 (3x - 2) t \cdot y(t) dt.$$

$$2. \quad y(x) = \lambda \int_{-1}^1 (5x t^3 + 4x^2 t + 3xt) y(t) dt.$$

$$3. \quad y(x) = \lambda \int_0^{\pi/4} \sin^2 x \cdot y(t) dt.$$

$$4. \quad y(x) - \lambda \int_0^{2\pi} \sin x \sin t y(t) dt = 0. \quad (\text{Madras M.E., 2000 S})$$

$$5. \quad y(x) = \lambda \int_0^\pi \sin x \cos t \cdot y(t) dt.$$

$$6. \quad y(x) = \lambda \int_0^{2\pi} \sin(x+t) \cdot y(t) dt. \quad (\text{Madras M.E., 2000})$$

Solve the following integral equations :

$$7. \quad y(x) = x + \lambda \int_0^1 (x-t) y(t) dt.$$

$$8. \quad y(x) = x + \lambda \int_0^1 (1+x+t) y(t) dt.$$

$$9. \quad y(x) = x + \lambda \int_0^\pi (1 + \sin x \sin t) y(t) dt.$$

$$10. \quad y(x) = (2x - \pi) + 4 \int_0^{\pi/2} \sin^2 x \cdot y(t) dt.$$

$$11. \quad y(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \cdot y(t) dt.$$

$$12. \quad y(x) = x + \lambda \int_{-\pi}^{\pi} (x \cos t + t^2 \sin x + \cos x \sin t) y(t) dt.$$

$$13. \quad \text{Obtain the solution of } y(x) = 1 + \lambda \int_0^1 xt \cdot y(t) dt \text{ in the form } y(x) = 1 + \frac{3\lambda x}{2(3-\lambda)} \quad (\lambda \neq 3).$$

What happens when $\lambda = 3$?

14. For the integral equation

$$y(x) = F(x) + \lambda \int_0^1 (1 - 3xt) y(t) dt,$$

find the eigen values of λ and the corresponding eigen functions.

15. Obtain the most general solution of the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \sin(x+t) y(t) dt$$

where (i) $F(x) = x$ (ii) $F(x) = 1$, under the assumption that $\lambda \neq \pm 1/\pi$.

36.11 SOLUTION OF FREDHOLM INTEGRAL EQUATION BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the Fredholm equation $y(x) = F(x) + \lambda \int_a^b K(x, t) y(t) dt$... (1)

where $F(x)$ is continuous in $a \leq x \leq b$ and $K(x, t)$ is finite and continuous in the rectangle $a \leq x \leq b$ and $a \leq t \leq b$.

Replacing y under the integral sign by an initial approximation $y(0)$, we get the first approximation

$$y^{(1)}(x) = F(x) + \lambda \int_a^b K(x, t) y^{(0)}(t) dt \quad ... (2)$$

Replacing y under the integral sign in (1) by $y^{(1)}$, we get the next approximation

$$y^{(2)}(x) = F(x) + \lambda \int_a^b K(x, t) y^{(1)}(t) dt \quad ... (3)$$

Proceeding in this manner, we get the general formula for successive approximations as

$$y^{(n)}(x) = F(x) + \lambda \int_a^b K(x, t) y^{(n-1)}(t) dt \quad \dots(4)$$

We now, obtain the condition for the convergency of the sequence $y^{(n)}(x)$.

Replacing x by t and t by a dummy variable t_1 , (2) becomes

$$y^{(1)}(t) = F(t) + \lambda \int_a^b K(t, t_1) y^{(0)}(t_1) dt_1$$

Then (3) takes the form

$$\begin{aligned} y^{(2)}(x) &= F(x) + \lambda \int_a^b K(x, t) \left\{ F(t) + \lambda \int_a^b K(t, t_1) y^{(0)}(t_1) dt_1 \right\} dt \\ &= F(x) + \lambda \int_a^b K(x, t) F(t) dt + \lambda^2 \int_a^b K(x, t) \cdot \int_a^b K(t, t_1) y^{(0)}(t_1) dt_1 dt \end{aligned} \quad \dots(5)$$

Writing $K^* \phi(x) = \int_a^b K(x, t) \phi(t) dt$, the equations (1), (2) and (5) become

$$\begin{aligned} y(x) &= F(x) + \lambda K^* y(x) \\ y^{(1)}(x) &= F(x) + \lambda K^* y^{(0)}(x) \\ y^{(2)}(x) &= F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} y^{(0)}(x) \end{aligned}$$

Similarly $y^{(3)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} y^{(0)}(x)$

In general $y^{(n)}(x) = F(x) + \lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \lambda^3 K^{*3} F(x) + \dots + \lambda^n K^{*n} y^{(0)}(x)$

As $n \rightarrow \infty$, we get

$$y(x) = \text{Lt}_{n \rightarrow \infty} y^n(x) = F(x) + \text{Lt}_{n \rightarrow \infty} (\lambda K^* F(x) + \lambda^2 K^{*2} F(x) + \dots \infty) \quad \dots(6)$$

Now $F(x)$ and $K(x, t)$ being continuous for all values of x and t in (a, b) , $F(x) \leq M$ and $|K(x, t)| \leq m$ where M, m are their respective maximum values in (a, b) .

$$\begin{aligned} \therefore |K^* F(x)| &= \left| \int_a^b K(x, t) F(x) dt \right| \\ &\leq mM \left| \int_a^b dt \right| \leq mM(b-a) \end{aligned}$$

Similarly $K^{*r} F(x) \leq m^r M \cdot (b-a)^r$

$$\begin{aligned} \text{Then } |\lambda^r K^{*r} F(x)| &\leq |\lambda|^r \cdot m^r M(b-a)^r \\ &\leq M (|\lambda| m (b-a))^r \end{aligned}$$

$$\therefore \text{In (6), } \sum_1^\infty \lambda^r K^{*r} F(x) \leq M \sum_1^\infty (|\lambda| m (b-a))^r$$

Now the series on R.H.S. being a geometric series, converges for $|\lambda| m (b-a) < 1$.

Thus by comparison test, $\sum_1^\infty \lambda^r K^{*r} F(x)$ also converges for $|\lambda| m (b-a) < 1$

$$\text{i.e., for } |\lambda| < \frac{1}{m(b-a)} \quad \dots(7)$$

Hence the given integral equation (1) will have a continuous solution when the condition (7) is satisfied.

Obs. 1. To evaluate the successive terms in the series (6) conveniently, we choose $y^{(0)}(x) = F(x)$.

Obs. 2. Volterra integral equations can also be solved by following exactly similar procedure as above (See Example 36.19).

Example 36.18. Solve, by using the method of successive approximations, the integral equation

$$y(x) = 1 + \lambda \int_0^1 xt y(t) dt. \quad \dots(i)$$

Solution. Taking the initial approximation $y^{(0)}(x) = 1$ and substituting it in the R.H.S. of (i), we get

$$y^{(1)}(x) = 1 + \lambda \int_0^1 xt \cdot 1 dt = 1 + \lambda x \left| \frac{t^2}{2} \right|_0^1 = 1 + \frac{\lambda x}{2}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of (i), we have

$$\begin{aligned} y^{(2)}(x) &= 1 + \lambda \int_0^1 xt \left(1 + \lambda \frac{t}{2} \right) dt = 1 + \lambda x \int_0^1 \left(t + \frac{\lambda t^2}{2} \right) dt \\ &= 1 + \lambda x \left| \frac{t^2}{2} + \frac{\lambda}{2} \cdot \frac{t^3}{3} \right|_0^1 = 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6} \end{aligned}$$

Substituting $y^{(2)}(x)$ in the R.H.S. of (i), we get

$$\begin{aligned} y^{(3)}(x) &= 1 + \lambda \int_0^1 xt \left(1 + \frac{\lambda t}{2} + \frac{\lambda^2 t^2}{6} \right) dt = 1 + \lambda x \left| \frac{t^2}{2} + \left(\frac{\lambda}{2} + \frac{\lambda^2}{6} \right) \frac{t^3}{3} \right|_0^1 \\ &= 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6} + \frac{\lambda^3 x}{18} = 1 + \frac{\lambda x}{2} \left(1 + \frac{\lambda}{3} + \frac{\lambda^2}{3^2} + \dots \right) \end{aligned}$$

Hence the solution of (i) is

$$y(x) = 1 + \frac{\lambda x}{2} \left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3} \right)^2 + \left(\frac{\lambda}{3} \right)^3 + \dots \right)$$

As the number of terms tends to infinity, the exact solution is

$$\begin{aligned} y(x) &= 1 + \frac{\lambda x}{2} \left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3} \right)^2 + \left(\frac{\lambda}{3} \right)^3 + \dots \infty \right) \\ &= 1 + \frac{\lambda x}{2} \frac{1}{1 - \lambda/3} \quad [\text{Summing up the G.P. which converges for } \lambda/3 < 1] \end{aligned}$$

or

$$y(x) = 1 + \frac{3\lambda x}{2(3 - \lambda)} \quad \text{only if } \lambda < 3.$$

Example 36.19. Using the method of successive approximations, solve the Volterra integral equation

$$y(x) = 1 + x + \int_0^x (x-t) y(t) dt.$$

Solution. Taking the initial approximation $y^{(0)}(x) = 1 + x$ and substituting it in the R.H.S. of (i), we get

$$\begin{aligned} y^{(1)}(x) &= 1 + x + \int_0^x (x-t)(1+t) dt \\ &= 1 + x + x \left(x + \frac{x^2}{2} \right) - \left(\frac{x^2}{2} + \frac{x^3}{3} \right) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned}$$

Substituting $y^{(1)}(x)$ in the R.H.S. of (i), we obtain

$$\begin{aligned} y^{(2)}(x) &= 1 + x + \int_0^x (x-t) \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} \right) dt \\ &= 1 + x + x \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \right) - \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} \right) \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \end{aligned}$$

Proceeding in this manner, we get

$$y^{(n)}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!}$$

As $n \rightarrow \infty$, the exact solution of (i) is

$$y(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty = e^x.$$

PROBLEMS 36.5

Apply the method of successive approximations, to solve the following Fredholm integral equations : (1 to 3)

$$1. \quad y(x) = 1 - \lambda \int_0^1 xt \, y(t) \, dt.$$

$$2. \quad y(x) = \sin x + \lambda \int_0^{2\pi} \cos(x+t) \, y(t) \, dt.$$

$$3. \quad y(x) = 1 + \lambda \int_0^1 (1 - 3xt) \, y(t) \, dt.$$

Using the iterative method, solve the following Volterra integral equations : (4 to 6)

$$4. \quad y(x) = 1 + \int_0^x y(t) \, dt. \quad (\text{Madras M.E., 2000 S})$$

$$5. \quad y(x) = 1 + x - \int_0^x y(t) \, dt.$$

$$6. \quad y(x) = x + \int_0^x (t-x) \, y(t) \, dt.$$

$$7. \quad y(x) = 2(1+x^2) - \int_0^x xy(t) \, dt.$$

8. Choosing the initial approximation $y^{(0)}(x) = 0$, for the solution of the integral equation

$$y(x) = \int_0^x t(t-x) \, y(t) \, dt + \frac{x^2}{2}, \text{ show that } y^{(2)}(x) = \frac{x^2}{2} - \frac{x^5}{40}.$$

9. Starting with the initial approximation $y^{(0)}(x) = 1$, for the solution of the integral equation

$$y(x) = 1 + \int_0^x (x+t) \, y(t) \, dt,$$

$$\text{show that } y^{(3)}(x) = 1 + \frac{3x^2}{2} + \frac{7x^4}{8} + \frac{77x^6}{240}$$

(Madras M.E., 2000)

Discrete Mathematics

- I. Set Theory : 1. Sets. 2. Set operations. 3. Laws of set theory. 4. Principle of inclusion, Duality.
- II. Algebra of Logic : 5. Introduction; Propositions & statements; Compound statements. 6. Logical operations. 7. Statements generated by a set; Tautology; Contradiction. 8. Equivalence. 9. Duality law; Tautology implications. 10. Arguments. 11. Predicates. 12. Quantifiers. 13. Normal forms. 14. Inference theory.
- III. Boolean Algebra : 15. Introduction; Boolean function. 16. Duality. 17. Boolean identities. 18. Minimal Boolean function. 19. Disjunctive normal form. 20. Conjunctive normal form. 21. Switching circuits; Simplification of circuits.
- IV. Fuzzy Sets : 22. Fuzzy logic, Fuzzy set. 23. Fuzzy set operations. 24. Truth values. 25. Algebraic operations; Properties of fuzzy sets. 26. Generation of rules for fuzzy problems. 27. Classification of fuzzy propositions. 28. Applications of fuzzy sets.

I. SET THEORY

37.1 SETS

(1) The concept and the language of sets play a very important role in expressing mathematical ideas concisely and precisely. It was Cantor* who first introduced and developed the notion of sets in mathematical investigations. It is, therefore, essential for a student of engineering to grasp the basic ideas of *Set Theory*.

Def. A collection of objects defined by some property, is called a **set**. The objects belonging to a set are called its **elements or members**.

Examples of a set are (i) the set of positive integers less than 25, (ii) set of pages in a book, and (iii) set of women students in a college.

A set is denoted by a single capital letter e.g. A, B, \dots, S, X, Y and the elements of a set are generally denoted by small letters a, b, c, \dots, x, y, z .

When e is an element of a set S , we write $e \in S$ and read as 'e belongs to S '. When e is not an element of S , we write $e \notin S$.

If S be a set of odd integers, $3, 7, 11 \in S$ but $4, 6 \notin S$.

(2) Representation of a set

- (i) **Tabular form of a set.** In this, the elements are enclosed in curly brackets after separating them by commas, e.g., the set of positive even integers less than 9 is written as $S = \{2, 4, 6, 8\}$ and the set of prime numbers between 4 and 14 is $T = \{5, 7, 11, 13\}$.
- (ii) **Symbolic form of a set.** In this, the set is written as $\{x/P(x)\}$ where x is a typical element of the set and $P(x)$ is the property satisfied by this element. In symbolic form, the above two sets are

$$S = \{x/x = \text{a positive even number } < 9\}$$

$$T = \{x/x = \text{a prime number between 4 and 14}\}.$$

*The great German mathematician George Cantor (1845–1918), the creator of Set theory.

(3) Empty set or null set. A set which has no elements is called an **empty set or the null set** and is denoted by the symbol ϕ .

(4) Finite and infinite sets. A set is said to be **finite** if it has a finite number of elements. Otherwise a set is said to be **infinite**.

The number of distinct elements in a finite set A is called its **cardinality** and is denoted by $|A|$.

For instance, the set of days in a year is **finite**, the set of points in a line is an **infinite set**.

(5) Subset. If every element of a set A is also an element of set B , then A is called a **subset** of B and this relationship is denoted by $A \subset B$ or $B \supset A$; which is read as ' A is contained in B '.

Another definition : If A and B are two sets such that

$$x \in A \Rightarrow x \in B,$$

then A is called a subset of B .

The notation \Rightarrow stands for the word 'implies'.

For instance, the set V of vowels is a subset of the set A of the English alphabet and we write $V \subset A$.

(6) Power set. For a set A , collection of all subsets of A is called the **power set** of A and is denoted by $P(A)$.

If $A = [1, 2, 3]$ then $P(A)$ consists of 2^3 i.e. 8 elements $\phi, [1], [2], [3], [1, 2], [2, 3], [3, 1]$ and $[1, 2, 3]$.

In general, if A has n elements, then $P(A)$ has 2^n elements.

(7) Equality of sets. Two sets A and B are said to be equal if the elements of both are the same i.e., if each element of A is also an element of B and vice versa, and we write $A = B$.

In other words, if A and B are two sets such that

$$A \subset B \text{ and } B \subset A \Leftrightarrow A = B.$$

Here \Leftrightarrow stands for 'implies and is implied by' or 'if and only if'.

For instance, $\{2, 3, 5\} = \{3, 2, 5, 3\} = \{2, 5, 3, 2\}$, since the change in the order of elements or the repetition of an element is immaterial and all these contain the same elements 2, 3, 5.

(8) Proper and improper subsets. When the set B contains all the elements of A and some others, A is said to be a **proper subset** of B and is denoted by $A \subseteq B$.

i.e., if $A \subset B$ and $A \neq B$ then $A \subseteq B$.

If $A \subset B$ and every element of B is also an element of A i.e., $B \subset A$, then A is said to be an **improper subset** of B i.e., $A = B$.

For instance, the set of positive odd integers and the set of positive even integers are both proper subsets of the set of natural numbers.

(9) Universal set is that which has all the sets under investigation as its subsets. It is generally denoted by 'U'.

For instance the set of all letters of English alphabet is a universal set of the sets of the form $\{a, i, e, u\}, \{b, x, u, m\}$ etc.

Example 37.1. If A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$, then show that $A \subseteq C$.

Solution. Let x be any element of A .

Since $A \subseteq B$ i.e., all the elements of A belong to B ,

so $x \in A \Rightarrow x \in B$... (i)

Again as $B \subseteq C$ i.e., all elements of B belong to C ,

so $x \in B \Rightarrow x \in C$... (ii)

\therefore It follows from (i) and (ii) that $x \in A \Rightarrow x \in C$

i.e., $A \subseteq C$.

Example 37.2. Which of the following sets are equal?

$$S_1 = \{1, 2, 2, 3\}, S_2 = \{x : x^2 - 2x + 1 = 0\}$$

$$S_3 = \{3, 2, 1\} \text{ and } S_4 = \{x : x^3 - 6x^2 + 11x - 6 = 0\}.$$

Solution. Here $S_1 = \{1, 2, 2, 3\} = \{1, 2, 3\}$

$$S_2 = \{x : (x - 1)^2 = 0\} = \{1\}, S_3 = \{1, 2, 3\}$$

$$S_4 = \{x : (x - 1)(x - 2)(x - 3) = 0\} = \{1, 2, 3\}$$

From these we find that S_1, S_3, S_4 are equal.

37.2 SET OPERATIONS

(1) **Union** of two sets A and B is the set of all elements which belong to A or to B or to both. It is denoted by $A \cup B$ read as 'A union B' and is represented by the shaded portion in Fig. 37.1.

Symbolically $A \cup B = \{x/x \in A \text{ or } x \in B\}$.

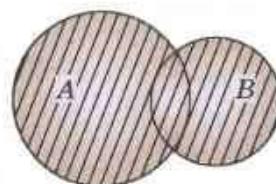


Fig. 37.1

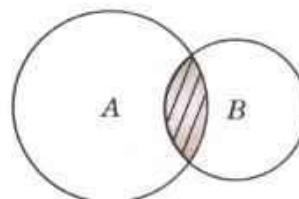


Fig. 37.2

(2) **Intersection** of two sets A and B is the set of elements which are common to both A and B . It is denoted by $A \cap B$ read as 'A intersection B' and is represented by the shaded portion in Fig. 37.2.

Symbolically $A \cap B = \{x/x \in A \text{ and } x \in B\}$

Such diagrams as Figs. 37.1 and 37.2 which exhibit the various relations between the sets are known as **Venn diagrams**.

(3) **Disjoint sets.** If the sets A and B have no common elements, they are called **disjoint sets**. Their intersection is an empty set.

For instance, if A be a set of boys in a college and B the set of girls in the same college, then A and B are disjoint sets i.e. $A \cap B = \emptyset$.

(4) **Complement of a set.** If $B \subset A$, the set of elements of A which are not in B is called the **complement of B in A** and is denoted by B^c in A . It is also known as the difference $A - B$ of sets A and B . Thus

$$B^c \text{ in } A = \{x/x \in A \text{ and } x \notin B\}$$

which is shown shaded in Fig. 37.3 (i).

If U be a universal set, then the set ' $U - A$ ' is called the complement of A and is denoted by A^c , which is shown shaded in Fig. 37.3 (ii).

For instance, if $U = \{1, 2, 3, 4, 5, \dots\}$ and $A = \{1, 3, 5, 7, \dots\}$, then $A^c = \{2, 4, 6, 8, \dots\}$.

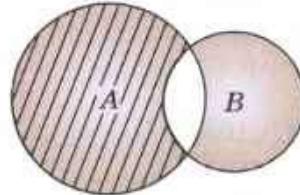


Fig. 37.3 (i)

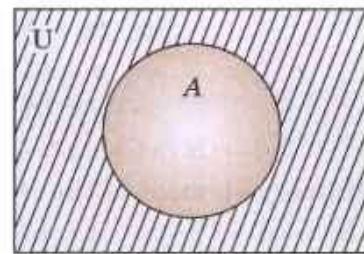


Fig. 37.3 (ii)

(5) **Cartesian product** of two sets A and B denoted by $A \times B$ is defined to be set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$ i.e.,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

For instance if $A = \{1, 2\}$, $B = \{1, 2, 3\}$, then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$, $B \times A = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$ $\therefore A \times B \neq B \times A$.

Example 37.3. If $A = \{2, 5, 6, 7\}$, $B = \{0, 2, 5, 7, 8\}$, $C = \{1, 2, 3, 5, 6\}$, show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution. Here

$$B \cap C = \{2, 5\}$$

\therefore

$$A \cup (B \cap C) = \{2, 5, 6, 7\} \quad \dots(i)$$

Again

$$A \cup B = \{0, 2, 5, 6, 7, 8\},$$

$$A \cup C = \{1, 2, 3, 5, 6, 7\}$$

\therefore

$$(A \cup B) \cap (A \cup C) = \{2, 5, 6, 7\} \quad \dots(ii)$$

Hence from (i) and (ii), we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example 37.4. With the help of Venn diagram, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

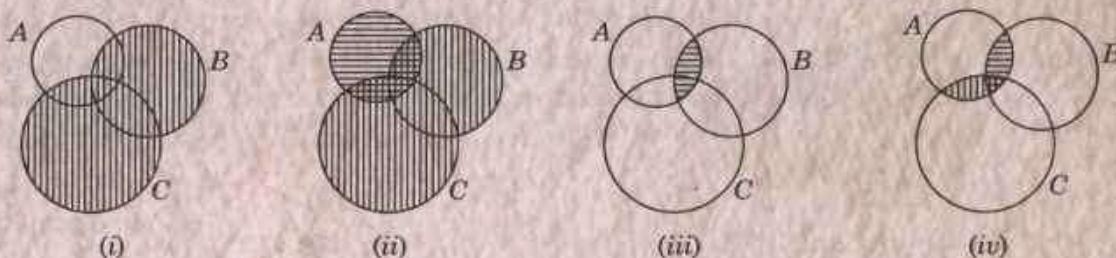


Fig. 37.4

Solution. First we draw vertical lines in the whole areas of B and C so as to represent $B \cup C$. [Fig. 37.4 (i)].

Now draw horizontal lines in the whole area A . Thus the double hatched area in [Fig. 37.4 (ii)] represents area common to A and $B \cup C$ i.e., $A \cap (B \cup C)$.

Again we draw horizontal lines in the area common to A and B so as to represent $A \cap B$ [Fig. 37.4 (iii)].

Now draw vertical lines in the area common to A and C , so as to represent $A \cap C$. Then the whole hatched area in [Fig. 37.4 (iv)] represents $(A \cap B) \cup (A \cap C)$.

Hence we observe that the double hatched area in Fig. 37.4 (ii) is equal to the total hatched area in Fig. 37.4 (iv).

Example 37.5. Prove that (i) $A - (B \cap C) = (A - B) \cup (A - C)$.

$$(ii) A \times (B \cap C) = (A \times B) \cap (A \times C).$$

(Tiruputi, 2001)

Solution. (i) Let x be an arbitrary element of the set $A - (B \cap C)$, then

$$\begin{aligned} x \in A - (B \cap C) &\Rightarrow x \in A \text{ and } x \notin (B \cap C) & [\because x \notin (A - B) \Rightarrow x \in A \text{ and } x \notin B] \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \\ &\Rightarrow x \in (A - B) \cup (A - C). \end{aligned}$$

$$\therefore A - (B \cap C) \subset (A - B) \cup (A - C) \quad \dots(i)$$

Again if x be an arbitrary element of the set $(A - B) \cup (A - C)$, then

$$\begin{aligned} x \in (A - B) \cup (A - C) &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin B \cap C \\ &\Rightarrow x \in A - (B \cap C) \end{aligned}$$

$$\therefore (A - B) \cup (A - C) \subset A - (B \cap C) \quad \dots(ii)$$

From (i) and (ii), we get $A - (B \cap C) = (A - B) \cup (A - C)$.

$$(ii) (x, y) \in A \times (B \cap C)$$

$$\begin{aligned} &\Rightarrow x \in A \text{ and } y \in (B \cap C) \\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C) \\ &\Rightarrow (x, y) \in (A \times B) \cap (A \times C) \end{aligned}$$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

37.3 LAWS OF SET THEORY

1. Commutative Law

$$A \cup B = B \cup A; A \cap B = B \cap A.$$

2. Associative Law

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributive Law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. Complement Law

$$A \cup A^c = U; A \cap A^c = \emptyset.$$

5. Identity Law

$$A \cup \emptyset = A = \emptyset \cup A$$

$$A \cap U = A = U \cap A.$$

6. Absorption Law

$$A \cup (A \cap B) = A; A \cap (A \cup B) = A$$

7. De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c; (A \cap B)^c = A^c \cup B^c$$

8. Involution Law

$$(A^c)^c = A.$$

37.4 PRINCIPLE OF INCLUSION

(1) If A and B be sets with cardinalities $|A|$ and $|B|$, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. The number of common elements in A and B is $|A \cap B|$. Each of these elements is counted twice in $|A| + |B|$, once in $|A|$ and once in $|B|$. This should be adjusted by subtracting the term $|A \cap B|$ from $|A| + |B|$.

Hence

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Obs. Using the distributive law, we can extend the above result for three sets A, B, C so that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

(V.T.U., 2002)

For

$$\begin{aligned} |A \cup B \cup C| &= |(A \cup B) \cup C| \\ &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| + |B \cap C| - |A \cap B \cap C| \end{aligned}$$

whence follows the result.

(2) **Duality.** If S be any identity involving sets and operations (e.g. complement, intersection \cap and union \cup etc.) and a new set S^* is obtained by replacing \cap by \cup , \cup by \cap , \emptyset by U and U by \emptyset in S , then the statement S^* is true and is called the **dual** of the statement S .

For instance, the dual of $A \cap (B \cup A) = A$ is $A \cup (B \cap A) = A$.

Example 37.6. In a survey conducted on 250 persons, it was found that 180 drink tea and 70 drink coffee and 50 take both tea and coffee. How many drink atleast one beverage and how many drink neither?

Solution. Let A be the set of tea drinkers and B the set of coffee drinkers. Then

$$|A \cup B| = |A| + |B| - |A \cap B| = 180 + 70 - 50 = 200$$

Hence 200 persons drink at least one beverage and $250 - 200 = 50$ persons drink neither tea nor coffee.

Example 37.7. How many integers between 1 and 468 are divisible by 3 but not by 5.

Solution. Number of integers between 1 and 468 which are divisible by 3 = $\left[\frac{468}{3} \right] = 156$

Number of integers between 1 and 468 which are divisible by 3 and 5 = $\left[\frac{468}{3 \times 5} \right] = 31$

Hence the number of integers between 1 and 468 divisible by 3 but not by 5 = $156 - 31 = 125$.

Example 37.8. How many integers are between 1 and 200 which are divisible by any one of the integers 2, 3 and 5?

Solution. Let A_1, A_2, A_3 denote the set of integers between 1 and 200 which are divisible by 2, 3, 5 respectively.

$$|A_1| = \left[\frac{200}{2} \right] = 100, |A_2| = \left[\frac{200}{3} \right] = 66, |A_3| = \left[\frac{200}{5} \right] = 40$$

$$|A_1 \cap A_2| = \left[\frac{200}{2 \times 3} \right] = 33, |A_1 \cap A_3| = \left[\frac{200}{2 \times 5} \right] = 20$$

$$|A_2 \cap A_3| = \left[\frac{200}{3 \times 5} \right] = 13, |A_1 \cap A_2 \cap A_3| = \left[\frac{200}{2 \times 3 \times 5} \right] = 6$$

$$\begin{aligned} \text{Hence } |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| \\ &\quad - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3| \\ &= 100 + 66 + 40 - 33 - 13 - 20 + 6 = 146. \end{aligned}$$

PROBLEMS 37.1

1. Show that the following sets are equal :

$$A = \{2, 1\}, B = \{1, 2, 1, 2, 1, 2\}, C = \{x : x^2 - 3x + 2 = 0\}.$$

2. Which of the following statements are true ? Give reason to support your answer.

(i) $\{a\} \subset \{a, b, c\}$	(ii) $a \subset \{a, b, c\}$	(iii) $a \subseteq \{a, b, c\}$
(iv) $\{a, b\} \subset \{a, b, c\}$	(v) $\{a, b\} \in \{a, b, c\}$	(vi) $\emptyset \subset \{a, b, c\}$.

3. Prove that

$$(i) B - A \text{ is a subset of } A^c. \quad (ii) B - A^c = B \cap A \quad (\text{Andhra, 2004})$$

$$(iii) \{A \subset B, B \subset C, C \subset A\} \Rightarrow A = C.$$

4. If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$, prove that

$$(i) (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \quad (ii) (A \cap B) \cup (A \cap C) = A \cap (B \cup C).$$

5. If $A = A \cup B$ then prove that $B = A \cap B$.

6. Prove that $A \cup B = B \Leftrightarrow A \subset B$.

7. With the help of the Venn-diagram, prove that

$$(i) (A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c. \quad (\text{Andhra, 2004})$$

$$(ii) A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \quad (\text{V.T.U., 2001 S})$$

8. If $B \subset A$, prove that

$$(i) B \cup C \subset A \cup C \quad (ii) B \cap C \subset A \cap C.$$

9. (i) If $A \cup B = A \cap B$, show that $A = B$.

$$(ii) \text{If } A \cup B = A \cup C \text{ and } A \cap B = A \cap C, \text{ show that } B = C.$$

10. (i) Prove that (i) $A - B = A - A \cap B$. (V.T.U., 2001 ; Madras, 2000)

$$(ii) A - (B \cup C) = (A - B) \cap (A - C).$$

11. Show that for any two sets A and B

$$(i) A - B = A \cap B^c \quad (ii) A \subseteq B \Leftrightarrow B^c \subseteq A^c$$

$$(iii) A \cup B = (A \cap \bar{B}) \cup (B \cap \bar{A}) \cup (A \cap B).$$

12. If A, B, C be sets such that $A \subset B$, $B \cap C = \emptyset$, show that $A \cap C = \emptyset$.

13. Show that $A \cup (B \cup C)^c = (A \cup B^c) \cap (A \cup C^c)$.

14. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (Andhra, 2001)
15. If S is any set and $P(s)$ is its power set and A and B belong to $P(s)$, prove that $B \cap (A - B) = \emptyset$.
16. If A and B are finite sets then prove that $A \cup B$ and $A \cap B$ are finite sets and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$
 (Andhra, 2004)
17. In survey conducted on 200 people, it was found that 140 are smokers while 80 are alcoholic and 40 are both smokers and alcoholics. Find how many are neither smokers nor alcoholics.
18. How many integers between 1 and 789 are divisible by 5 but not by 7.
19. How many integers are between 1 and 250 which are divisible by any of the integers 3, 5, and 7.
20. Out of a class of 153 students, 54 have taken History, 63 have taken Geography, 62 have taken Economics, and 43 have taken Geography and History, 45 have taken History and Economics, 46 have taken Geography and Economics and 37 have taken all the three subjects. How many of the students have not taken any of these three subjects? Use a Venn diagram.

II. ALGEBRA OF LOGIC

37.5 INTRODUCTION

(1) Logic is concerned with all types of reasoning such as valid statements, mathematical proofs, valid conclusions etc. Logical reasoning is used to prove theorems, to verify the correctness of computer programs and to draw conclusions from experiments. Later on, we shall observe that the algebra of sets and logic is analogous to the algebra of switching circuits which is similar to 'Boolean Algebra'.

(2) **Propositions and Statements.** A *proposition* is a declarative sentence which is either true (1) or false (0). Some authors use T and F respectively for 1 and 0. The truth or falsity of a proposition is defined as its *truth value*.

All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*.

Example 37.9. Which of the following are statements : (a) Agra is in India (b) $3 + 4 = 5$ (c) Where do you live ? (d) Do you speak Hindi ?

Solution. (a) and (b) are statements that happen (a) is true and (b) is false.

(c) and (d) are questions so they are not statements.

(3) **Compound statements.** The statement which is composed of sub-statements and logical connectives is called a *compound statement*.

e.g., 'It is raining and it is cold' is a compound statement as it is comprised of two sub-statements 'It is raining' and 'it is cold'.

(4) **Truth table.** The truth value of a compound statement is completely determined by the truth value of its substatements. A convenient way to represent a compound statement is by means of the **truth table** wherein the values of a compound statement are specified for all possible choices of the values of the sub statements.

We shall use the numbers 0 and 1 to denote the false and true statements. Also we use letters p, q, r, \dots to represent a proposition or logical variable.

37.6 LOGICAL OPERATORS

(1) **Conjunction.** If p and q are two statements then their conjunction p and q written as $p \wedge q$, is defined by the truth table 1.

Table 1

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Table 2

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For example, the conjunction of p : it is raining and q : I am cold is $p \wedge q$: It is raining and I am cold.

(2) Disjunction. If p and q are two statements, then their disjunction p or q written as $p \vee q$ is defined by the truth table 2.

For example, the disjunction of p and q for p : it is raining today ; q : 3 is an odd integer is

$p \vee q$: it is raining today or 3 is an odd integer.

(3) Negation. If p is a given statement and its negative 'not p ', written as $\sim p$ (or Np or $\neg p$) is defined by the following truth table :

p	$\sim p$
0	1
1	0

For example, the negation of the following statement

(a) p : $2 + 3 > 1$ is $\sim p$: $2 + 3 \leq 1$

(b) q : it is hot is $\sim q$: it is cold.

Example 37.10. If p be 'it is hot' and q be 'it is raining', describe each of the following statements by a sentence :

(a) $q \vee \sim p$ (b) $\sim p \wedge \sim q$ (c) $\sim (\sim p \vee q)$.

Solution. (a) It is raining or it is not hot. (b) It is not hot and it is not raining.

(c) It is hot but not raining.

(4) Conditional operator. The conditional statement 'if p then q ' written as $p \rightarrow q$ is defined by the truth table 4 :

Table 4

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Table 5

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Obs. The contrapositive of conditional statement $p \rightarrow q$ is the statement $\sim p \rightarrow \sim q$.

(5) Biconditional operator. If p and q be two statements, then the statement ' p if and only if q ' denoted by $p \leftrightarrow q$ and abbreviated as ' p if q ' is called a biconditional statement. The truth table for biconditional statement is table 5.

Example 37.11. Construct the truth tables for

- | | |
|--|--|
| (a) $p \wedge \sim q$ | (b) $(p \vee q) \vee \sim p$ |
| (c) $(p \rightarrow q) \wedge (q \rightarrow p)$ | (d) $(p \rightarrow q) \vee \sim (p \leftrightarrow \sim q)$. |

Solution. (a) The truth table is

p	q	$\sim q$	$p \wedge \sim q$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

(b) The truth table is

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
1	1	1	0	1
1	0	1	0	1
0	1	1	1	1
0	0	0	1	1

(c) The truth table is

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

(d) The truth table in this case is

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow \neg q)$	$p \rightarrow q \vee \neg(p \rightarrow \neg q)$
1	1	1	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	1
0	0	1	1	0	1	1

37.7 STATEMENTS GENERATED BY A SET

(1) If S be a set of statements, then any valid combination of statements in S with conjunction, disjunction or negation is a statement generated by S .

A statement generated by a set S need not include each element of S in its expression.

For example, if p, q, r are statements in S then

$$(a) (p \wedge q) \wedge r$$

$$(b) \neg q \wedge r$$

$$(c) (p \wedge q) \vee (\neg q \wedge r)$$

are statements generated by S . Their truth tables are :

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$ (a)	$\neg q$	$\neg q \wedge r$ (b)	(c)
1	1	1	1	1	0	0	1
1	1	0	1	0	0	0	1
1	0	1	0	0	1	1	1
0	1	1	0	0	0	0	0
1	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	1	1	1
0	0	0	0	0	1	0	0

(2) **Tautology** is an expression involving logical variables which is true for all cases in its truth table. It is also called a *logical truth*.

(3) **Contradiction** is an expression involving logical variables which is false for all cases in its truth table. Obviously, the negation of a contradiction is a tautology.

In other words, a statement formula which is a tautology is identically true, while a formula which is a contradiction is identically false.

Obs. The conjunction of two tautologies is also a tautology.

Example 37.12. Show that (a) $p \vee \neg p$ is a tautology (b) $p \rightarrow q \leftrightarrow (\neg p \vee q)$.

Solution. (a) The truth table is

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

Hence $p \vee \neg p$ is a tautology.

(b) The truth table is

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Hence $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ is true.

37.8 EQUIVALENCE

(1) If p and q be statements generated by the set of statements S , then p and q are equivalent if $p \leftrightarrow q$ is a tautology which is denoted by $p \Leftrightarrow q$.

If $p \rightarrow q$ is a tautology, then we say that p implies q and write it as $p \Rightarrow q$.

Obs. All tautologies are equivalent to each other and all contradictions are equivalent to each other.

(2) **Equivalent formulae.** Some basic equivalent formulae are given below which can be proved by using truth tables :

1. $p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$	<i>Idempotent laws</i>
2. $p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$	<i>Commutative laws</i>
3. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	<i>Associative laws</i>
4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	<i>Distributive laws</i>
5. $p \vee \sim p \Leftrightarrow 1$	$p \wedge \sim p \Leftrightarrow 0$	<i>Negation law</i>
6. $p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$	<i>Identity laws</i>
7. $p \vee 1 \Leftrightarrow 1$	$p \wedge 0 \Leftrightarrow 0$	<i>Null laws</i>
8. $p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$	<i>Absorption laws</i>
9. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$	$\sim(p \vee q) \Leftrightarrow \sim p \vee \sim q$	<i>De Morgan's laws</i>
10. $p \Rightarrow p \vee q$	$q \Rightarrow p \vee q$	<i>Disjunctive addition</i>
11. $p \wedge q \Rightarrow q$	$p \wedge q \Rightarrow q$	
12. $(p \vee q) \wedge \sim p \Rightarrow q$	$(p \vee q) \wedge \sim q \Rightarrow p$	
13. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$		<i>Chain rule</i>
14. $p \rightarrow q \Leftrightarrow \sim p \vee q$		<i>Conditional equivalence</i>
15. $p \Leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$		<i>Biconditional equivalence</i>

Example 37.13. Show that

$$(a) p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\sim q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$$

$$(b) [\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$$

Solution. (a) By conditional equivalence $q \rightarrow r \Leftrightarrow \sim q \vee r$

Replacing $q \rightarrow r$ by $\sim q \vee r$, we get $p \rightarrow (\sim q \vee r)$ which is equivalent to $\sim p \vee (\sim q \vee r)$ by the same rule.

Thus

$$\sim p \vee (\sim q \vee r) \Leftrightarrow (\sim p \vee \sim q) \vee r \quad [\text{By (3)}]$$

$$\Leftrightarrow \sim(p \wedge q) \vee r \quad [\text{By (9)}]$$

$$\Leftrightarrow (p \wedge q) \rightarrow r \quad [\text{By (14)}]$$

$$(b) [\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$$

$$\Leftrightarrow [\sim p \wedge (\sim q \wedge r)] \vee (q \vee p) \wedge r \quad [\text{By (4)}]$$

$$\Leftrightarrow [\sim p \wedge \sim q \wedge r] \vee (q \vee p) \wedge r \quad [\text{By (3)}]$$

$$\Leftrightarrow [\sim(p \vee q) \wedge r] \vee (q \vee p) \wedge r \quad [\text{By (9)}]$$

$$\Leftrightarrow [\sim(p \vee q) \vee (p \vee q)] \wedge r \quad [\text{By (4)}]$$

$$\Leftrightarrow 1 \wedge r \quad [\text{By (5)}]$$

$$\Leftrightarrow r \quad [\text{By (6)}]$$

37.9 DUALITY LAW

(1) Two formulae A and A^* are said to be duals of each other if either one can be obtained from the other by replacing \wedge by \vee and \vee by \wedge .

If the formula A contains special variables 1 or 0, then its dual A^* is obtained by replacing 1 by 0 and 0 by 1.

- e.g., (i) Dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$
(ii) Dual of $(p \wedge q) \vee 0$ is $(p \vee q) \wedge 1$.

(2) **Tautology implications.** A statement A is said to tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology which is read as "A implies B".

The implications listed below have important applications which can be proved by truth tables :

- | | |
|--|--|
| 1. $p \wedge q \Rightarrow p$ | $p \Rightarrow p \vee q$ |
| 2. $\sim p \Rightarrow p \rightarrow q$ | $q \Rightarrow p \rightarrow q$ |
| 3. $\sim(p \rightarrow q) \Rightarrow p$ | $\sim(p \rightarrow q) \Rightarrow \sim q$ |
| 4. $p \wedge(p \rightarrow q) \Rightarrow q$ | $\sim p \wedge(p \vee q) \Rightarrow q$ |
| 5. $(p \rightarrow q) \wedge(q \rightarrow r) \Rightarrow p \rightarrow r$ | |

37.10 ARGUMENTS

(1) An argument is an assertion that a given set of propositions p_1, p_2, \dots, p_n (called premises) yields another proposition q (called conclusion). The argument is symbolically written as " $p_1, p_2, \dots, p_n \vdash q$ ".

An argument $p_1, p_2, \dots, p_n \vdash q$ is true provided q is true whenever all the premises p_1, p_2, \dots, p_n are true. An argument which is true is said to be 'valid argument'. Otherwise it is called a fallacy.

Example 37.14. Show that

- (a) the argument $p \leftrightarrow q, q \vdash p$ is valid.
(b) the argument $p \rightarrow q, q \vdash p$ is a fallacy.

Solution. (a) Let us first prepare the truth table as follows :

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Since $p \leftrightarrow q$ is true in cases (rows) 1 and 4, and q is true in cases 1 and 3, therefore $p \leftrightarrow q$ and q both are true in case 1 only when p is also true. This shows that the given argument is valid.

(b) Let us first prepare the truth table below :

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

This table shows that $p \rightarrow q$ and q both are true in case 3 only while the conclusion p is false. Hence the given argument is a fallacy.

(2) **Theorem.** The argument $p_1, p_2, \dots, p_n \vdash q$ is valid if and only if the proposition $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.

The propositions p_1, p_2, \dots, p_n are simultaneously true if and only if the proposition $p_1 \wedge p_2 \wedge \dots \wedge p_n$ is true i.e., if the proposition $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.

Obs. The validity of an argument depends upon the particular form of the argument, not on the truth values of the statement appearing in the argument.

Example 37.15. Test the validity of the following argument :

S_1 : If 5 is less than 3, then 6 is not a prime number

S_2 : 5 is not less than 6

S_3 : 5 is a prime number.

Solution. Let '5 is less than 3' be p and '5 is a prime number' be q . Then the given argument is of the form $p \rightarrow \neg q, \neg p \vdash q$.

Since in the last line of the truth table, the premises $p \rightarrow \neg q$ and $\neg p$ are true but the conclusion q is false, therefore the argument is fallacy.

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg p$
1	1	0	0	0
1	0	1	1	0
0	1	0	1	1
0	0	1	1	1

PROBLEMS 37.2

- If p = Sam is a teacher, q = John is an honest boy, then translate the following into logical sentences :
 - $\neg(p \wedge q)$,
 - $p \vee \neg q$,
 - $\neg p \Leftrightarrow q$,
 - $p \Rightarrow \neg q$.
- Change the following sentence into symbols :
 - 'If I do not have car or I do not wear good dress then I am not a millionaire'.
 - Everyone who is healthy can do all kinds of work.(Anna, 2004 S)
- Prepare truth tables for the following statements (a) $(p \Rightarrow q) \wedge \neg q$, (b) $(p \Leftrightarrow q) \wedge (r \vee g)$.
- Write down the truth table of
 - $p \vee q$ (Madras, 1997)
 - $p \wedge (p \wedge q)$ (Madras, 2005 S)
- Verify that the following are tautologies :
 - $p \rightarrow [q \rightarrow (p \wedge q)]$ (Anna, 2005)
 - $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$
 - $(p \Rightarrow q \wedge r) \Rightarrow (\neg r \Rightarrow \neg p)$.
- Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology. (Anna, 2004 S ; Madras, 2003 S)
- Over the universe of positive integers

$p(n)$: n is prime and $n < 32$.

$q(n)$: n is a power of 3.

$r(n)$: n is a divisor of 27.

 - What are the truth sets of these propositions ?
 - Which of the three propositions implies one of the others ?
- Given the propositions over the natural numbers

p : $n < 4$, q : $2n > 17$ and r : n is a divisor of 18, what are the truth sets of

 - q ,
 - $p \wedge q$,
 - r ,
 - $q \rightarrow r$.(Madras, 1999)
- Show that (a) $\neg Q, P \rightarrow Q \Rightarrow \neg P$. (Madras, 2003)
- $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$. (Madras, 2001)
- Construct the truth table for (i) $(\neg p \rightarrow q) \wedge (q \Leftrightarrow p)$. (Bharthian, MSc. 2001)
- (ii) $\neg[P \vee (Q \wedge R)] \Leftrightarrow (P \vee Q) \wedge (P \vee R)$. (Andhra, 2004)
- Prove that the following statement is a contradiction :

$$S = [(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)].$$
- If p, q, r are three statements then prove that
 - $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
 - $(p \Rightarrow q) \vee r = (p \vee r) \Rightarrow (q \vee r)$
 - $\neg(p \vee q) = \neg p \wedge \neg q$.
- Define conjunction, conditional, biconditional and negation, with examples.

14. Show that (i) $p \wedge q$ logically implies $p \leftrightarrow q$.
(ii) $p \leftrightarrow \neg q$ does not logically imply $p \rightarrow q$. (U.P.T.U., 2001)
15. Write the duals of $(p \vee q) \wedge r$ and $(p \wedge q) \vee t$.
16. Show that $s \vee r$ is tautologically implied by $(p \vee q) \wedge (q \rightarrow r) \wedge (q \rightarrow s)$. (Andhra, 2004 ; Bharathian, 2001)
17. Let $P(n)$ be ' $8^n - 3^n$ is a multiple of 5'. Prove that $P(n)$ is a tautology on n .
18. Prove that $P \rightarrow \neg Q, R \rightarrow Q, R \vdash \neg P$ is a valid argument. (Madras, 2003)
19. Prove that $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge \neg q) \rightarrow r$. (Anna, 2005)
20. Show that $R \vee S$ follows logically from the premises $C \vee D, C \vee D \rightarrow \neg H, H \rightarrow A \wedge \neg B, A \wedge \neg B \rightarrow R \vee S$.

37.11 PREDICATES

Statements involving variables such as ' $x > 7$ ' and ' $x = y + 7$ ' are neither true nor false so long as the values of the variable x, y are not specified. We now, discuss the ways that propositions can be evolved from such statements. The statement ' $x > 7$ ' has two parts : First part—'the variable x ' is the *subject* of the statement ; Second part—'is greater than 7' is the *predicate* which refers to the property that the subject of statement can have. If $P(x)$ denotes the statement ' $x > 7$ ' then P is the *predicate* and x is the *variable*. The statement $P(x)$ is also known as the value of the propositional function P at x . The *predicates* are denoted by capital letters and the *objects* by the variables (denoted by small letters in brackets).

Thus *predicates are simple statements which turn out to be propositions involving variables whose values are not well specified*.

In other words, *predicate is a variable statement which becomes specific when particular values are assigned to the variables*.

There are statements which involves more than one variable consider the statement ' $x = y + 7$ ' which is denoted by $Q(x, y)$ where Q is the predicate and x, y are the variables. When values are assigned to the variables x, y , the statement $Q(x, y)$ has the truth value.

Similarly $R(x, y, z)$ denotes the statement of the type ' $x + y = z$ '. When values are assigned to x, y, z , this statement has a truth value.

For example, consider the statements (i) Ram is fair ; (ii) Sham is fair.

Here in (i) and (ii) 'is fair' is the predicate while Ram and Sham are the objects. If we denote the predicate by F and the objects by r and s , then the above statements can be symbolically expressed as (i) $F(r)$; (ii) $F(s)$.

Now consider the statement Ram is fair and the house is pink.

Writing 'the house is pink' as $P(h)$, the given statement can be expressed as $F(r) \wedge P(h)$.

37.12 QUANTIFIERS

(1) In a propositional function, when all the variables are assigned values, the resulting statement has a truth value. However, there is another method to create a proposition from a propositional function which is called *quantification*. It is of two types : *Universal quantification* and *Existential quantification*.

(2) **Universal quantification.** Many statements assert that a property is true for all values of a variable in a certain domain. This domain is termed as the universe of discourse and such a statement is expressed using universal quantification.

Thus the universal quantification of $P(x)$ is the proposition ' $P(x)$ is true for all values of x in the universe of discourse'.

The universal quantification of $P(x)$ is denoted by $\forall x P(x)$. The symbol \forall is called the *universal quantifier*.

Obs. When it is possible to list all the elements in the universe of discourse say : x_1, x_2, \dots, x_n , then the universal quantification $\forall x P(x)$ is same as the conjunction $P(x_1) \wedge P(x_2) \wedge \dots, P(x_n)$ for this conjunction is true if and only if $P(x_1), P(x_2), \dots, P(x_n)$ are all true.

Example 37.16. What is the truth value of the quantification $\forall x P(x)$ where

(a) $P(x)$ is the statement ' $x < 5$ ' and universe of discourse is the set of real numbers.

(b) $P(x)$ is the statement ' $x^2 < 18$ ' and the universe of discourse consists of positive integers not exceeding 5 ?

Solution. (a) For instance, $P(6)$ is false ; therefore $P(x)$ is not true for all real numbers x .

Thus $\forall x P(x)$ is false.

Example 37.22. Symbolise the expression 'All the world loves a lover.'

(Madras, 2001)

Solution. Let $p(x) : x \text{ is a person}$;
 $L(x) : x \text{ is a lover}$.

and $Q(x, y) : x \text{ loves } y$

Then the required expression is

$$(\forall x) [p(x) \rightarrow (\exists y) (p(y) \wedge L(y))] \rightarrow Q(x, y).$$

Summary. (i) $\forall Q(x)$ means that the predicate $Q(x)$ is true for all values in the universe of x .

(ii) $\exists Q(x)$ means that the predicate $Q(x)$ is satisfied if there is at least one value in the universe of x .

37.13 NORMAL FORMS

(1) For the given variables p_1, p_2, \dots, p_n , we may form a statement $S(p_1, p_2, \dots, p_n)$. The truth table for S will contain 2^n rows for all possible truth values of the n variables. The expression S may have the truth value 1 in all cases or may have the truth value 0 in all cases or have the truth value 1 for at least one combination of truth values assigned to the n variables. (Here S is said to be satisfiable). The problem of finding in a finite number of steps whether a given expression is a tautology or a contradiction or at least satisfiable is known as a *decision problem*. As the formation of a truth table is quite cumbersome, we go for an alternate approach called *normal form*.

In this approach, we use the word 'sum' in place of disjunction and 'product' in place of conjunction.

A sum of the variables and their negations is called an *elementary sum*. Similarly a product of the variables and their negation is called an *elementary product*.

(2) **Disjunctive normal form** of a given formula is the formula which is equivalent to the given formula and which contains the sum of the elementary products. The disjunctive normal form of a given form is not unique. In fact, several disjunctive normal forms can be obtained for a given formula by applying the distributive laws in different ways.

A given formula is however, identically false if every elementary product appearing in its disjunctive normal form is identically false.

Example 37.23. Obtain the disjunctive normal forms of

$$(i) p \wedge (p \rightarrow q) \quad (ii) \sim (p \vee q) \leftrightarrow (p \wedge q).$$

$$\text{Solution. } (i) p \wedge (p \rightarrow q) \Leftrightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q).$$

which is the desired disjunctive normal form.

$$\begin{aligned} (ii) \sim (p \vee q) &\Leftrightarrow (p \wedge q) \Leftrightarrow \sim (p \vee q) \wedge (p \wedge q) \vee (p \vee q) \wedge \sim (p \wedge q) \quad [\because E \Leftrightarrow F \Leftrightarrow (E \wedge F) \vee (\sim E \wedge \sim F)] \\ &\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge (\sim p \vee \sim q)] \\ &\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q] \\ &\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q) \end{aligned}$$

which is the desired disjunctive normal form.

(3) **Conjunctive normal form** of a given formula is that formula which is equivalent to the given formula and contains the product of elementary sums.

Example 37.24. Find a conjunctive normal form of $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

$$\begin{aligned} \text{Solution. } \sim (p \vee q) &\Leftrightarrow (p \wedge q) \Leftrightarrow [\sim (p \vee q) \rightarrow (p \wedge q)] \wedge [p \wedge q \rightarrow \sim (p \vee q)] \\ &\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \wedge [\sim (p \wedge q) \vee \sim (p \vee q)] \quad [\text{By conditional equivalence}] \\ &\Leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)] \\ &\Leftrightarrow p \vee q \vee p \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q) \end{aligned}$$

which is the required conjunctive normal form.

(4) **Principal disjunctive normal form.** Consider a formula for the propositions p and q using conjunction as $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$. These terms are called *minterms* or *Boolean conjunction* of p and q .

An equivalent formula for a given formula, consisting of disjunctions of minterms only is called the **principal disjunctive normal form (pdnf)** or **sum of products canonical form**.

Procedure to obtain the principle disjunctive normal form : (i) Replace the conditions and biconditions by their equivalent formulae containing \wedge , \vee , \neg only.

- (ii) Using DeMorgan's laws, apply negations to the variables.
- (iii) Apply the distribution laws.
- (iv) Introduce the missing factors to obtain minterms in the disjunctions.
- (v) Delete identical minterms appearing in the disjunctions.

Example 37.25. Obtain the pdnf for

$$(i) p \vee q \quad (ii) \neg(p \wedge q) \quad (iii) \neg p \vee q \text{ i.e., } p \rightarrow q.$$

$$\begin{aligned} \text{Solution. } (i) p \vee q &\Leftrightarrow [p \wedge (q \vee \neg q)] \vee [q \wedge (p \vee \neg p)] \\ &\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p) \\ &\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \end{aligned}$$

$$\begin{aligned} (ii) \neg(p \wedge q) &\Leftrightarrow (\neg p \vee \neg q) \Leftrightarrow [\neg p \wedge (\neg q \vee q)] \vee [\neg q \wedge (p \vee \neg p)] \\ &\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg q \wedge p) \vee (\neg q \wedge \neg p) \\ &\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg q \wedge p) \end{aligned}$$

$$\begin{aligned} (iii) \neg p \vee q &\Leftrightarrow \neg p \wedge (q \vee \neg q) \vee [q \wedge (p \vee \neg p)] \\ &\Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p) \\ &\Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p). \end{aligned}$$

(5) Principal conjunctive normal form. Consider a formula for the propositions p and q using disjunction as $p \vee q$, $\neg p \vee q$, $p \vee \neg q$, $\neg p \vee \neg q$. These terms are called *max terms*.

An equivalent formula for a given formula, consisting of conjunctions of the max terms only is called the **principal conjunctive normal form (pcnf)** or **product of sum canonical form**.

Procedure for obtaining pcnf for a given formula is similar to the one for pdnf as all assertions made for pdnf can be made for pcnf using duality principle.

Example 37.26. Obtain the principal disjunctive and conjunctive normal forms of

$$p \rightarrow [(p \rightarrow q) \wedge \neg(\neg q \vee \neg p)].$$

(Bharathiar, 2001)

$$\begin{aligned} \text{Solution. } (i) p \rightarrow [(p \rightarrow q) \wedge \neg(\neg q \vee \neg p)] &\Leftrightarrow \neg p \wedge [(\neg p \vee q) \wedge (q \wedge p)] \\ &\quad [\text{Using DeMorgan's law and equivalence } p \rightarrow q \Leftrightarrow \neg p \vee q.] \\ &\Leftrightarrow \neg p \vee [\neg p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)] \\ &\Leftrightarrow \neg p \vee (q \wedge p) \\ &\Leftrightarrow [\neg p \wedge (q \vee \neg q)] \vee (q \wedge p) \\ &\Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \end{aligned}$$

This is the desired pdnf.

$$\begin{aligned} (ii) p \rightarrow [(p \rightarrow q) \wedge \neg(\neg q \vee \neg p)] &\Leftrightarrow \neg p \vee [(\neg p \vee q) \wedge (q \wedge p)] \\ &\Leftrightarrow [\neg p \vee (\neg p \vee q)] \wedge [\neg p \vee (q \wedge p)] \\ &\Leftrightarrow (\neg p \vee q) \wedge (\neg p \vee q) \wedge (\neg p \vee p) \\ &\Leftrightarrow \neg p \vee q \end{aligned}$$

This is the desired pcnf.

Example 37.27. Obtain the principal conjunctive normal form for $(Q \rightarrow P) \wedge (\neg P \wedge Q)$. (Andhra, 2004)

$$\begin{aligned} \text{Solution. } (Q \rightarrow P) \wedge (\neg P \wedge Q) &\Leftrightarrow (\neg Q \vee P) \wedge (\neg P \wedge Q) \\ &\Leftrightarrow (\neg Q \vee P) \wedge [\neg P \vee (Q \wedge \neg Q) \wedge Q \vee (P \wedge \neg P)] \\ &\Leftrightarrow (\neg Q \vee P) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (Q \vee P) \wedge (Q \vee \neg P) \\ &\Leftrightarrow (\neg Q \vee P) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (Q \vee P). \end{aligned}$$

37.14 INFEERENCE THEORY

Inferring the conclusions from certain premises is known as the *inference theory*. When conclusion is reached from a set of premises by using the accepted rules of reasoning, then this process is called a *deduction* or a *formal proof*. A proof of a theorem is a valid argument.

The criteria for finding 'whether an argument is valid' are called *rules* which are expressed in terms of premises and conclusions or in terms of statement formulae.

The proofs are of two types : Direct or Indirect.

(i) If in a proof the truth of the premises directly shows the truth of the conclusions, then it is called a *direct proof*.

(ii) An *indirect proof* proceeds by assuming that p is true and also C is false, and then deduce a contradiction using p and $\neg C$, along with other premises.

The only difference between assumptions in a direct proof or an indirect proof is the negated conclusion.

Rules for deriving direct and indirect proofs :

- the proof should have finite steps only ;
- each step must be either a premise or a proposition which is implied from previous steps using valid equivalence or implication ;
- the last step for a direct proof must be conclusion while for an indirect proof it must be a contradiction.

Example 37.28. State whether the conclusion C follows logically from the premises R and S

- | | | | |
|-----|------------------------|----------------------------|------------------------|
| (a) | $R : p \rightarrow q,$ | $S : p,$ | $C : \neg q$ |
| (b) | $R : p,$ | $S : p \leftrightarrow q,$ | $C : \neg(p \wedge q)$ |
| (c) | $R : p \rightarrow q,$ | $S : p,$ | $C : q$ |

Solution. Let us first form the following truth table :

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg(p \wedge q)$	$p \leftrightarrow q$
1	1	0	0	1	0	1
1	0	0	1	0	1	0
0	1	1	0	1	0	0
0	0	1	1	1	1	1

(a) Here only the first row of premises R and S contains 1 but not the conclusion C .

Hence C is not valid.

(b) Only the first row of both R and S contains 1 but not the conclusion C .

Hence C is not valid.

(c) Here both R and S contain '1' only in the first row and the conclusion C also has '1' in that row. Hence our conclusion is valid.

Example 37.29. Find the direct and indirect proofs of $p \rightarrow (q \rightarrow r), \neg r \vee p, q \Rightarrow r \rightarrow s$.

Solution. Direct proof

- | | | | |
|-----------------------|---|--|-------------------|
| (i) $\neg r \vee p$ | (premise) | (ii) r | (another premise) |
| (iii) p | [By (i) and (ii)] | (iv) $p \rightarrow (q \rightarrow r)$ | (premise) |
| (v) $q \rightarrow r$ | [By (iii) and (iv)] | (vi) q | (premise) |
| (vii) r | [By (v) and (vi)], which is a conclusion. | | |

Indirect proof

- | | |
|---|------------------------------------|
| (i) $\neg(s \rightarrow r)$ | (negative of conclusion) |
| (ii) $s \wedge \neg r$ | (By conditional equivalence) |
| (iii) $\neg r \vee p$ | (premise) |
| (iv) $\neg r$ and (v) s | [By conjunctive simplification] |
| (vi) p | [By disjunction of (iii) and (iv)] |
| (vii) $p \rightarrow (q \rightarrow r)$ | (premise) |

- (viii) $q \rightarrow r$ [By (vi) and (vii)]
 (ix) q (premise)
 (x) r [By (viii) and (ix)]
 (xi) $r \wedge \neg r$ [By (x) and (iv)]

This is a contradiction.

PROBLEMS 37.3

- If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :
 (a) $(\forall x \in A) (x + 2 < 10)$ (b) $\exists x \in A (x + 2 = 10)$
- Negate each of the following statements :
 (a) $\forall x, x^3 = x$; (b) $\forall x, x + 5 > x$
 (c) Some students are 26 or older. (d) All students live in the hostels.
- What is the truth value of $\vee x P(x)$ where $P(x)$ is a statement ' $x^2 < 10$ ' and the universe of discourse consists of positive integers not exceeding 4.
- Use universal quantifier to state 'the sum of any two rational numbers is rational'.
- Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of a and b .
- Translate the following statements involving quantifiers, into formulae :
 (a) All rationals are reals. (b) No rationals are reals.
 (c) Some rationals are reals. (d) Some rationals are not reals.
- Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.
- Convert $\neg A \wedge (\neg B \rightarrow C) \Rightarrow 0$ into CNF. (V.T.U., MCA, 2001)
- Without constructing truth tables, obtain the product of sums canonical form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$. Hence find the sum of products canonicals form. (Anna, 2004 S)
- Find the direct proof of $p \rightarrow r, q \rightarrow s, p \vee q \Rightarrow s \vee r$.
- Prove that $P \rightarrow Q, Q \rightarrow R, P \vee R \Rightarrow R$ by using indirect method. (Anna, 2004 S)
- Using quantifier say $\forall z$ is not a real number ?
- State whether the conclusion C follows logically from the premises R and S
 (a) $R : p \rightarrow q, S : \neg q, C : q$ (b) $R : p \rightarrow q, S : q, C : \neg p$.

III. BOOLEAN ALGEBRA

37.15 INTRODUCTION

(1) The concept of Boolean algebra was first introduced by George Boole* in 1854 through his paper 'An investigation of the laws of thought'. It is basically two values i.e., (0, 1) set. Earlier it had applications to statements and sets which are either true or false. In 1938 Claude Shannon showed that basic rules given by Boole could be used to design circuits. These days however, Boolean algebra has wide applications to switching circuits, electrical networks and electronic computers.

Basically there are three operations in the Boolean algebra (i) AND, (ii) OR, and (iii) NOT, which are symbolically represented by \wedge , \vee and \neg respectively. Some authors, use the symbols $(+)$, $(.)$ and $(/)$ for the same operations.

Here ' \neg ' denotes the complement of an element and is defined by $0' = 1$ and $1' = 0$.

The operator \wedge (i.e., 'AND') has the following values $1 \wedge 1 = 1, 1 \wedge 0 = 0, 0 \wedge 1 = 0, 0 \wedge 0 = 0$; while the operator \vee (i.e., 'OR') has the values $1 \vee 1 = 1, 1 \vee 0 = 1, 0 \vee 1 = 1, 0 \vee 0 = 0$.

Def. Any non-empty set B with the binary operations ' \wedge ' and ' \vee ' and the unary operation ' \neg ' is called the Boolean algebra $[B, \wedge, \vee, \neg]$ if the following axioms hold where a, b, c are elements in B :

- Commutative law : $a \wedge b = b \wedge a ; a \vee b = b \vee a$
- Associative law : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
 $a \vee (b \vee c) = (a \vee b) \vee c$

*A British mathematician George Boole (1813–1864) who created Boolean algebra.

3. Distributive law : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
 $a \wedge b \vee c = (a \wedge b) \vee (a \wedge c)$

4. Complement law : $a \wedge a' = 0, a \vee a' = 1$

The operations \wedge , \vee and $/$ are called sum, product, and complement respectively. We shall follow the usual practice that $/$ has precedence over \vee , and \vee has precedence over \wedge , unless guided by brackets. e.g., $a \wedge b \vee c$ means $a \wedge (b \vee c)$ not $(a \wedge b) \vee c$ while $a \vee b'$ implies $a \vee b'$ not $(a \vee b)'$.

(2) Boolean function. The variable x is called a *Boolean variable* if it assumes values only from $B \{0, 1\}$.

Def. A function from the set $\{(x_1, x_2, \dots, x_n) : x_i \in B, 1 \leq i \leq n\}$ is called a *Boolean function of degree n*. Boolean functions can be represented by expressions comprised of variables and Boolean operations.

e.g., 0, 1, x_1, x_2, \dots, x_n are Boolean expressions in the variables x_i ($1 \leq i \leq n$). If p and q are Boolean expressions then $p \wedge q, p \vee q$ and p' are also Boolean expressions and each represents a Boolean function.

By substituting 0 and 1 for the variables in the expression, the values of this function can be found.

(3) If f and g be Boolean functions of degree n, then

(i) Complement of f is the function f' where

$$f'(x_1, x_2, \dots, x_n) = [f(x_1, x_2, \dots, x_n)]'$$

(ii) f and g are equal if $f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)$

(iii) Boolean sum $f \vee g$ is

$$(f \vee g)(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \vee g(x_1, x_2, \dots, x_n)$$

(iv) Boolean product $f \wedge g$ is

$$(f \wedge g)(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \wedge g(x_1, x_2, \dots, x_n).$$

(4) Power in a Boolean function : $x^2 = x \vee x = x$

$$x^3 = x^2 \vee x = x \vee x = x, \dots, x^n = x$$

Similarly, $2x = x, 3x = x$ etc.

Example 37.30. Find the values of the Boolean function $f = (x \wedge y') \vee z'$.

Solution. f being third degree Boolean function has 2^3 i.e. 8 values which are shown in the following table :

x	y	z	y'	z'	$x \wedge y'$	$(x \wedge y') \vee z'$
1	1	1	0	0	0	0
1	1	0	0	1	0	1
0	1	1	0	0	0	0
1	0	1	1	0	1	1
0	0	1	1	0	0	0
1	0	0	1	1	1	1
0	1	0	0	1	0	1
0	0	0	1	1	0	1

37.16 DUALITY

(1) The dual of any Boolean function is obtained by interchanging Boolean sums and Boolean products along with the interchange of zeros and ones.

For example the dual of $x \vee (y \wedge 0)$ is $x \wedge (y \vee 1)$.

The dual of any theorem of a Boolean algebra is also its theorem. This implies that the dual of any theorem in Boolean algebra is always true.

(2) Principle of duality. The dual of any theorem (or property) in Boolean algebra is also a theorem (or property).

37.17 BOOLEAN IDENTITIES

There are many identities in Boolean algebra which are quite useful in simplifying electrical circuits. Some of the important ones are given below :

1. <i>Identity law</i> :	$x \vee 0 = x ; x \wedge 1 = x$
2. <i>Dominance laws</i> :	$x \vee 1 = 1 ; x \wedge 0 = 0$
3. <i>Complement law</i> :	$x \vee x' = 1 ; x \wedge x' = 0$
4. <i>Idempotent law</i> :	$x \vee x = x ; x \wedge x = x$
5. <i>Double complement law</i> :	$(x')' = x$
6. <i>Commutative law</i> :	$x \vee y = y \vee x ; x \wedge y = y \wedge x$
7. <i>Associative law</i> :	$x \vee (y \vee z) = (x \vee y) \vee z$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
8. <i>Distributive law</i> :	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
9. <i>De-Morgan's law</i> :	$x \wedge y = x' \vee y' ; (x \vee y)' = x' \wedge y'$
10. <i>Absorption law</i> :	$x \wedge (x \vee y) = x$

(Bhopal, 2008)

Example 37.31. Let B be a Boolean algebra. Show that for all $a \in B$, there exists a unique complement a' .
(Andhra, 2004)

Solution. Let b and c be two complements of a .

$$\begin{aligned} \text{Then } b &= b \wedge 1 && [\because 0 \text{ is an additive identity}] \\ &= b \wedge (a \vee c) && [\because c \text{ is complement of } a] \\ &= (b \wedge a) \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c) \\ &= 0 \vee (b \wedge c) && [\because a \wedge b = a \wedge a' = 0] \\ &= b \wedge c && \dots(i) \end{aligned}$$

$$\begin{aligned} \text{Similarly } c &= c \wedge 1 = c \wedge (a \vee b) && [\because a \vee b = a \vee a' = 1] \\ &= (c \wedge a) \vee (c \wedge b) \\ &= (a \wedge c) \vee (b \wedge c) && [\because a \wedge c = a \wedge a' = 0] \\ &= 0 \vee (b \wedge c) \\ &= b \wedge c && \dots(ii) \end{aligned}$$

From (i) and (ii), we find that $b = c$.

Thus the complement of a is unique.

Example 37.32. In a Boolean algebra, show that

$$(i) x + (x \cdot y) = x \quad (ii) x \cdot (x + y) = x. \quad (\text{Bhopal, 2008})$$

$$\begin{aligned} \text{Solution. (i)} \quad x + (x \cdot y) &= x \wedge (x \vee y) = (x \vee 0) \wedge (x \vee y) && [\because x \vee 0 = x] \\ &= x \vee (0 \wedge y) && [\text{By distributive law}] \\ &= x \vee (y \wedge 0) && [\text{By commutative law}] \\ &= x \vee 0 && [\because y \wedge 0 = 0] \\ &= x. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad x \cdot (x + y) &= x \vee (x \wedge y) && [\because x \wedge 1 = x] \\ &= (x \wedge 1) \vee (x \wedge y) && [\text{By distributive law}] \\ &= x \wedge (1 \vee y) && [\text{By commutative law}] \\ &= x \wedge (y \vee 1) && [\because 1 \vee y = 1] \\ &= x \wedge 1 && [\because y \vee 1 = 1] \\ &= x. \end{aligned}$$

Example 37.33. Simplify the following :

$$(i) (x + y) \cdot x' \cdot y' \quad (ii) x \vee y \wedge y \vee z \wedge z \vee y \vee z' \quad (iii) x \vee y \wedge [(x \wedge y') \vee y].$$

Solution. (i)
$$\begin{aligned} (x \wedge y) \vee x' \vee y' &= (x \wedge y) \vee (x' \vee y') \\ &= (x \wedge y) \vee (x \wedge y)' \\ &= 1. \end{aligned}$$

[By De Morgan's law]

[$\because p \vee p' = 1$]

(ii)
$$\begin{aligned} x \vee y \wedge y \vee z \wedge y \vee z' &= (y \vee x) \wedge (y \vee z) \wedge (y \vee z') \\ &= [y \vee (x \wedge z)] \wedge (y \vee z') \\ &= y \vee [x \wedge z \wedge z'] \\ &= y \vee [x \wedge (z \wedge z')] \\ &= y \vee (x \wedge 0) \\ &= y \vee 0 \\ &= y. \end{aligned}$$

[By commutative law]
[By distributive law][By associative law]
[$\because z \wedge z' = 0$]
[$\because x \wedge 0 = 0$]

(iii)
$$\begin{aligned} x \vee y \wedge [(x \wedge y') \vee y]' &= x \vee y \wedge [y \vee (x \wedge y')]' \\ &= x \vee y \wedge [(y \vee x) \wedge (y \vee y')]' \\ &= (x \vee y) \wedge [(x \vee y) \wedge 1]' \\ &= (x \vee y) \wedge (x \vee y)' = 0. \end{aligned}$$

[By commutative law]
[By distributive law]
[$\because y \vee y' = 1$]**Example 37.34.** Show that

(i) $x \vee y \wedge y \vee z \wedge z \vee x = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

(ii) $(x \wedge y) \vee (x' \wedge z) = (x' \vee y) \wedge (x \vee z).$

(Bhopal, 2008)

Solution. (i) R.H.S.

$$\begin{aligned} &= (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \\ &= (x \wedge y) \vee (z \wedge y) \vee (z \wedge x) \\ &= (x \wedge y) \vee (z \wedge y \vee x) \\ &= (x \vee z) \wedge (y \vee z) \wedge [x \vee (y \vee x)] \wedge [y \vee (y \wedge x)] \\ &= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (y \vee x)] \\ &= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (x \vee y)] \\ &= (x \vee z) \wedge (y \vee z) \wedge (x \vee y) \\ &= (x \vee y) \wedge (y \vee z) \wedge (z \vee x) \\ &= \text{L.H.S.} \end{aligned}$$

[By commutative law]
[By distributive law][$\because x \vee x = x$ etc.][$\because p \wedge p = p$]
(By commutative law)

(ii) L.H.S.

$$\begin{aligned} &= (x \wedge y) \vee (x' \wedge z) \\ &= [x \vee (x' \wedge z)] \wedge [y \vee (x' \wedge z)] \\ &= [(x \vee x') \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)] \\ &= [1 \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)] \\ &= (x \vee z) \wedge (y \vee x') \wedge (y \vee z) \vee 1 \\ &= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee (x \wedge x')] \\ &= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee x] \vee (y \vee z) \vee x' \\ &= (x \vee z) \wedge [y \vee z \vee x] \wedge (x' \vee y) \vee [x' \wedge (y \vee z)] \\ &= (x \vee z) \vee (1 \wedge y) \wedge (x' \vee y) \vee (1 \wedge z) \\ &= [(x \vee z) \wedge 1] \wedge [(x' \vee y) \vee 1] \\ &= (x \vee z) \wedge (x' \vee y) = \text{R.H.S.} \end{aligned}$$

[By distributive law]

Example 37.35. Show that

(i) $x \vee y \wedge x' \vee y' = (x' \wedge y) \vee (x \wedge y')$

(ii) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)] = y.$

Solution. (i)
$$\begin{aligned} (x \vee y) \wedge (x' \vee y') &= [(x \vee y) \wedge x'] \vee [(x \vee y) \wedge y'] \\ &= [(x \wedge x') \vee (y \wedge x')] \vee [(x \wedge y') \vee (y \wedge y')] \\ &= \{0 \vee (x' \wedge y)\} \vee \{(x \wedge y') \vee 0\} \\ &= (x' \wedge y) \vee (x \wedge y'). \end{aligned}$$

[By distributive law]
[$\because x \wedge x' = 0$]

(ii)
$$\begin{aligned} [x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)] &= [(x \wedge x') \vee (x \wedge y)] \vee [(x' \wedge x) \vee (x' \wedge y)] \\ &= [0 \vee (x \wedge y)] \vee [0 \vee (x' \wedge y)] \\ &= (x \wedge y) \vee y \wedge 1 = y. \end{aligned}$$

[$\because x \wedge x' = 0$]

Example 37.36. If $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$, then prove that $a = b$.

Solution. Since $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$

$$\therefore (a \vee x) \wedge (a \vee x') = (b \vee x) \wedge (b \vee x')$$

i.e.,

$$a \vee (x \wedge x') = b \vee (x \wedge x')$$

or

$$a \vee 0 = b \vee 0$$

$$a = b.$$

[By distributive law]

$$[\because x \wedge x' = 0]$$

Example 37.37. In Boolean algebra $[B, +, \cdot, /]$, show that

$$(x \cdot y' + y \cdot z) \cdot (x \cdot z + y \cdot z') = x \cdot z$$

(Bhopal, 2008; M.P.T.U., 2001)

Solution. $\{(x \vee y') \wedge (y \vee z)\} \vee \{(x \vee z) \wedge (y \vee z')\}$

$$= \{(x \vee y') \wedge y\} \vee \{(x \vee y') \wedge z\} \vee \{(x \wedge z) \wedge y\} \vee \{(x \wedge z) \wedge z'\}$$

$$= \{(x \wedge y) \vee (y' \wedge y)\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z) \vee (z \wedge z')\}$$

$$= \{(x \wedge y) \vee 0\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z) \vee 0\}$$

$$= \{(x \wedge y) \vee (x \wedge y)\} \vee \{(x \wedge z) \vee (x \wedge z')\} \vee \{(z \wedge y) \vee (z \wedge y')\}$$

$$= (x \wedge y) \vee (x \wedge 1) \vee (z \wedge 1) = (x \wedge y) \vee x \vee z$$

$$= (x \vee x \vee z) \wedge (y \vee x \vee z) = (x \vee z) \wedge (y \vee x \vee z)$$

$$= (x \vee z) \wedge (1 \vee y) = (x \vee z) \wedge 1$$

$$= x \vee z.$$

PROBLEMS 37.4

1. Find the truth table for the Boolean function $f(x, y, z) = (x \wedge y') \vee (y \wedge z')$.

2. Write the dual of the Boolean expression $x + x' \cdot y = x' + y$.

(Andhra, 2004)

3. Simplify the following :

$$(i) (x \wedge y \wedge z)' \quad (ii) (x \vee y \vee z) \wedge (x' \wedge y' \wedge z').$$

4. In a Boolean algebra $[B, \wedge, \vee, /]$, prove that

$$(i) (x \wedge y) \vee (x \wedge y') = x. \quad (\text{Anna, 2005}) \quad (ii) x' \wedge (x \vee y) = x' \wedge y.$$

5. If $a \wedge x = b \wedge x$ and $a \wedge x' = b \wedge x'$, then show that $a = b$.

6. In a Boolean algebra $[B, \wedge, \vee, /]$, show that

$$(i) x \wedge (x \wedge y) = x \wedge y \quad (ii) x \vee (x \vee y) = x \vee y.$$

7. In Boolean algebra, prove that

$$(i) x \wedge (x' \vee y) = x \wedge y \quad (ii) x' \wedge y = x' \wedge (x \vee y).$$

8. Show that $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y'$.

9. If B be a Boolean algebra and $x, y, z \in B$, prove that

$$(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) = x \wedge y$$

10. Prove that $(x \vee y) \wedge [z \vee (x' \wedge y')] = (x \vee y) \wedge z$.

11. In a Boolean algebra B , prove that $(a + b)' = a' \cdot b' \forall a, b \in B$.

(Bhopal, 2009)

12. In any Boolean algebra, show that $a = b$ if and only if $a \cdot b' + a' \cdot b = 0$.

(Madras, 2001)

13. In Boolean algebra, show that

$$(i) (a + b) \cdot (a' + c) = a \cdot c + a' \cdot b + b \cdot c.$$

$$(ii) (a + b') \cdot (b + c') \cdot (c + a') = (a' + b) \cdot (b' + c) \cdot (c' + a).$$

(Andhra, 2004)

14. Give the truth table for the Boolean function

$f: B_3 \rightarrow B$ determined by the polynomial

$$P(x_1, x_2, x_3) = (x_1 \vee x_3) \wedge (x_1 \wedge (x_2 \vee x_3)).$$

(V.T.U., 2001)

37.18 MINIMAL BOOLEAN FUNCTION

Def. A minimal Boolean function in n variables is the product of x_1, x_2, \dots, x_n . It is also called minterm.

If x, y are two variables and x', y' are their complementary variables respectively, then $x \vee y, x' \vee y, x' \wedge y, x \wedge y'$, $x' \wedge y'$ are each a minimal Boolean function.

Similarly there are 2^3 i.e. 8 minimal Boolean functions in the three variables x, y, z i.e., $x \vee y \vee z, x' \vee y \vee z, x \vee y' \vee z, x \vee y \vee z', x \vee y' \vee z, x' \vee y \vee z', x' \vee y' \vee z, x' \vee y' \vee z'$.

In general, there are 2^n minimal Boolean functions (or minterms) in n variables.

Similarly the join of the variables x_1, x_2, \dots, x_n is called a **maxterm** and there will be 2^n maxterms.

37.19 DISJUNCTIVE NORMAL FORM

(1) **Def.** A Boolean function which can be expressed as sum of minimal Boolean functions is called a **Disjunctive normal form** or **minterm normal form** or **Canonical form**.

(2) If the number of distinct terms in a disjunctive normal form of Boolean function in n variables are 2^n , then it is called a **complete disjunctive normal form**.

(3) **Complement function of a disjunctive normal form** function f is the sum of all those terms of a complete disjunctive normal form which are not present in the disjunctive normal form of f . The complement of f is denoted by f' .

For example, if $f = (x \vee y) \wedge (x \vee y')$

then its complete disjunctive normal form in variables x and y is

$$(x \vee y) \wedge (x' \vee y) \wedge (x \wedge y') \wedge (x' \wedge y')$$

\therefore The complement function of this disjunctive normal form is $f'(x, y) = (x' \wedge y) \wedge (x' \wedge y')$.

Example 37.38. Find the value of the complete disjunctive normal form in three variables x, y, z .

Solution. The complete disjunctive normal form in three variables x, y, z is

$$\begin{aligned} f(x, y, z) &= (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \\ &\quad \wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z') \\ &= [(x \vee y) \vee (z \wedge z')] \wedge [(x \wedge y') \vee (z \wedge z')] \wedge [(x' \wedge y) \vee (z \wedge z')] \wedge [(x' \wedge y') \vee (z \wedge z')] \\ &= [(x \vee y) \vee 0] \wedge [(x \wedge y') \vee 0] \wedge [(x' \wedge y) \vee 0] \wedge [(x' \wedge y') \vee 0] \\ &= [x \vee (y \wedge y')] \wedge [x' \vee (y \wedge y')] \\ &= (x \vee 0) \wedge (x' \vee 0) = x \wedge x' = 0. \end{aligned}$$

37.20 CONJUNCTIVE NORMAL FORM

Def. If a Boolean function $f(x_1, x_2, \dots, x_n)$ is expressed in the form of factors and each factor is the sum of all the n -variables, then such a function is called a **conjunctive normal form** or **maxterm normal form** or **dual canonical form**.

(2) If a conjunctive normal of a function of n variables contains all the 2^n distinct factors, then such a function is called a **complete conjunctive normal form**.

(3) **Complement function of a conjunctive normal form** function f is a Boolean function which is the product of all those terms of complete conjunctive normal form which are not present in conjunctive normal form of f .

The complement of conjunctive normal form f is denoted by f' .

For example, if $f(x, y) = (x \wedge y) \vee (x \wedge y')$

then its complete conjunctive normal form in x and y is

$$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

\therefore The complement function of this conjunctive normal form is

$$f'(x, y) = (x' \wedge y) \vee (x' \wedge y')$$

Example 37.39. Given Boolean expression f , where $f(x_1, x_2, x_3) = (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3)$, simplify this expression stating the laws used and obtain the minterm normal form. (Bharathiar, 1997)

$$\begin{aligned} \text{Solution. Given } f(x_1, x_2, x_3) &= (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3) \\ &= (x_3' \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) \\ &= (x_2 \wedge x_3') \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3) \\ &= [x_2 \wedge (x_3' \vee x_3)] \vee (x_1' \wedge x_3) \end{aligned}$$

[By commutative law]

[By distributive law]

$$\begin{aligned}
 &= (x_2 \wedge 1) \vee (x_1' \wedge x_3) && [\because p \vee p' = 1] \\
 &= x_2 \vee x_1' \wedge x_3 && [\text{By identity law}]
 \end{aligned}$$

Minterm normal form of $f(x_1, x_2, x_3)$

$$\begin{aligned}
&= [(x_3' \wedge x_2) \wedge (x_1 \vee x_1')] \vee [(x_1' \wedge x_3) \wedge (x_2 \vee x_2')] \vee [(x_2 \wedge x_3) \wedge (x_1 \vee x_1')] \\
&= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1) \vee (x_2 \wedge x_3 \wedge x_1') \\
&= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1).
\end{aligned}$$

Example 37.40. Express the following functions into conjunctive normal forms :

$$(i) x' \wedge y \quad \quad \quad (ii) (x \wedge y) \vee (x' \wedge y').$$

Solution. (i) $x' \wedge y = x' \wedge y \wedge (z \vee z') = (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z')$.

$$(ii) \quad (x \wedge y) \vee (x' \wedge y') = [x \wedge y \wedge (z \vee z')] \vee [x' \wedge y' \wedge (z \vee z')] \\ = (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z').$$

Example 37.41. The function $f = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$ is in conjunctive normal form. Write its complement?

Solution. The complete conjunctive normal form in three variables x, y, z is $(x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \vee y \vee z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$.

\therefore The complement of the given function F is

$$F' = (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z).$$

PROBLEMS 37.5

- Find the value of a complete disjunctive normal form in
 - two variables x, y .
 - three variables x, y, z .
 - Express the following functions into disjunctive normal form :
 - $x \vee y$
 - $x \wedge (x' \vee y)$.
 - Express the Boolean function $F = A \vee (B' \wedge C)$ in a sum of minterms.
 - Convert the function $x \wedge y'$ to disjunctive normal form in three variables x, y, z .
 - Express the function $f = (x \vee y') \wedge (x \vee z) \wedge (x \vee y)$ into conjunctive normal form in which maximum number of variables are used.
 - Write the complement of the conjunctive normal form function $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$.

37.21 SWITCHING CIRCUITS

(1) A switching network is an arrangement of wires and switches (or gates) which connect two terminals. A switch can be either closed or open. A closed switch permits and an open switch stops flow of current

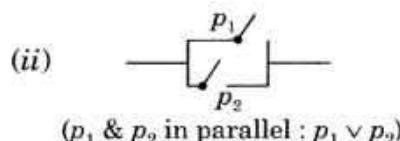
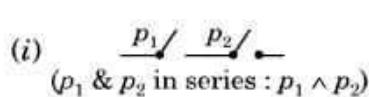
(2) If p denotes a switch, then p' denotes that switch which is open when p is closed and p' is closed when p is open.

If x denotes the state of the switch p , then x' represents the state of the switch p' . x is called the Boolean variable which is a binary variable.

If $x = 1$ denotes the switch is closed or current flows, then $x = 0$ denotes that the switch is open or current stops.

(3) Two switches p_1 and p_2 are either connected in series (represented by \wedge) or connected in parallel (represented by \vee).

These are shown as follows :



If $B [0, 1]$ is non-empty set and $\wedge, \vee, /$ are the operations on B , then the system $[(0, 1), \wedge, \vee, 1]$ is usually called *Boolean switching algebra*.

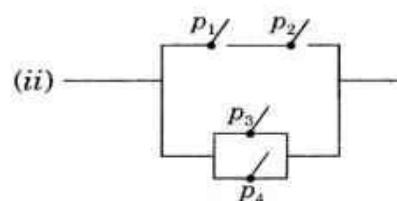
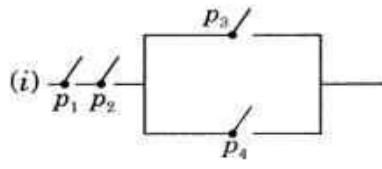
(4) Simplification of circuits. The simplification of a circuit means the least complicated circuit with minimum cost and best results. This depends on the cost of the equipment, number of switches and the type of the material used. Thus the simplification of circuits implies the use of lesser number of switches which can be achieved by using different properties of Boolean algebra. In other words, *the simplification of switching circuits is equivalent to simplification of the corresponding Boolean function*.

Example 37.42. Draw the circuit which represents the Boolean function :

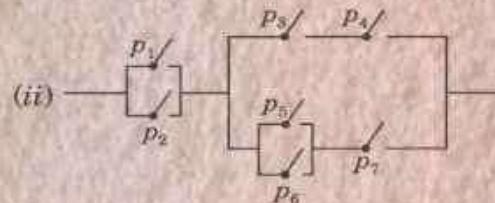
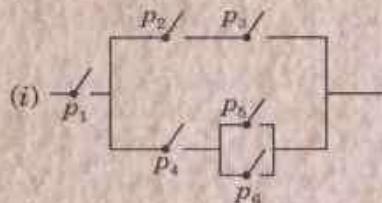
$$(i) (p_1 \wedge p_2) \wedge (p_3 \vee p_4) \quad (ii) (p_1 \wedge p_2) \vee (p_3 \vee p_4).$$

Solution. Here $p_1 \wedge p_2$ is a series circuit while $p_3 \vee p_4$ is a parallel circuit.

The required circuits are as follows :



Example 37.43. Write the Boolean functions representing the following circuits :



Also draw the circuit diagram which would be the complement of the circuit in (ii).

Solution. (i) The given circuit is represented by the Boolean function :

$$f = p_1 \wedge (p_2 \wedge p_3) \vee [p_4 \wedge (p_5 \vee p_6)]$$

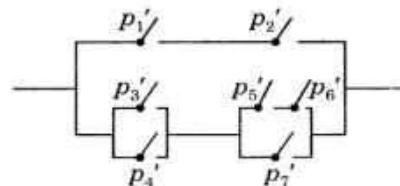
(ii) The Boolean function for the given circuit is

$$f = (p_1 \vee p_2) \wedge [(p_3 \wedge p_4) \vee ((p_5 \vee p_6) \wedge p_7)]$$

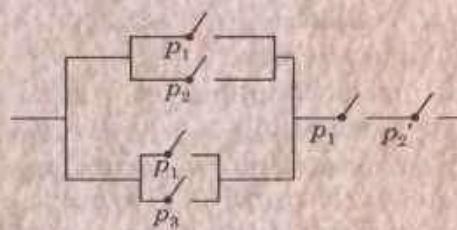
The complement of f i.e.,

$$\begin{aligned} f' &= (p_1 \vee p_2)' \vee [(p_3 \wedge p_4) \vee ((p_5 \vee p_6) \wedge p_7)]' \\ &= (p_1' \wedge p_2') \vee [(p_3 \wedge p_4)' \wedge ((p_5 \vee p_6) \wedge p_7)'] \\ &= (p_1' \wedge p_2') \vee [(p_3' \vee p_4') \wedge ((p_5' \wedge p_6') \vee p_7')] \end{aligned}$$

Its circuit diagram is as follows :



Example 37.44. Simplify the following circuit and draw the diagram of the resulting circuit :

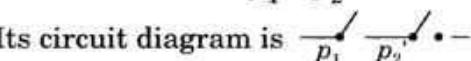


Solution. The given circuit is represented by the Boolean function f

$$\begin{aligned} &= [(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2') = (p_1 \vee p_2 \vee p_3) \wedge (p_1 \wedge p_2') \\ &= (p_1 \wedge p_1 \wedge p_2') \vee (p_2 \wedge p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') \\ &= (p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') = p_1 \wedge [p_2' \vee (p_3 \wedge p_2')] \\ &= p_1 \wedge p_2' \end{aligned}$$

(By distributive law)

[By absorption law]

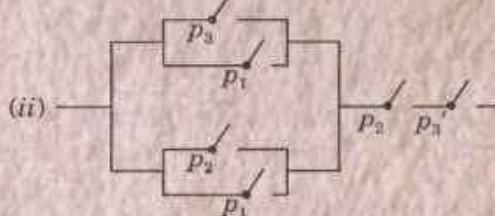
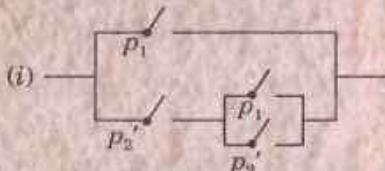
Its circuit diagram is 

PROBLEMS 37.6

1. Draw the circuit diagram represented by the Boolean functions :

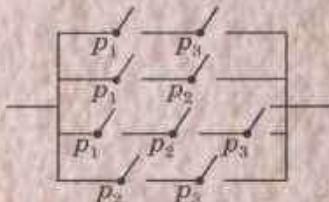
$$(i) [p_1 \wedge (p_1 \vee p_2)] \vee [p_2 \wedge (p_1' \vee p_2)] \quad (ii) p_1 \wedge [(p_2 \vee p_4') \vee (p_3' \wedge (p_1 \vee p_4 \vee p_3'))] \wedge p_2'$$

2. Write the Boolean functions representing the following circuits :



3. Simplify the Boolean functions, $p \vee (p' \wedge q) \vee (p \wedge q)$

4. Simplify the following circuit and draw the diagram of the resulting circuit :



5. Draw the simplified network of $f(x, y, z) = (x \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x' \vee y' \vee z)$.

(M.P.T.U., 2001)

6. Consider the function $f(x_1, x_2, x_3) = [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee (x_1 \vee x_2')$

(a) Simplify f algebraically. (b) Draw the switching circuit of f .

(c) Also find the minterm normal form of f .

(Madras, 1998)

IV. FUZZY SETS

37.22 FUZZY LOGIC

We have so far dealt with the fundamentals of classical logic. Besides this, we have *crisp logic* which deals with propositions that are required to be either true or false. There is however another type of logic which includes not only the crisp values but all the values between true (1) and false (0). But there is some degree of vagueness about the exact value between [0, 1]. *The logic to infer a definite outcome from such vague inputs is called fuzzy logic*.

(2) Fuzzy set. To provide a mathematical modelling to fuzzy logic, L.A. Zadeh introduced the concept of 'Fuzzy sets' in 1965 on the basis of a *membership function*. The theory of 'fuzzy sets' is now fully developed.

Def. A fuzzy set F of a non-zero set $X(x)$ is defined as $F = \{x, \mu_F(x)\} : x \in X$.

Here $\mu_F : X \rightarrow [0, 1]$ is a function called the *membership function* of F and $\mu_F(x)$ is the degree of membership of $x \in X$ in F .

In particular $\mu(x) = 1$ implies full membership

$\mu(x) = 0$ implies non-membership

and $0 < \mu(x) < 1$ means intermediate membership.

A fuzzy set F is, therefore, a set of pairs consisting of a particular element of the universe X and its degree of membership i.e., each x is assigned a value in the range (0, 1) indicating the extent to which x has the attribute F . It can also be represented as $F = \{[x_1, \mu_F(x_1)], [x_2, \mu_F(x_2)], \dots, [x_n, \mu_F(x_n)]\}$.

For example, if x is the number of cars in a lane, 'small' may be taken as a particular value of the fuzzy variable x and to each x is assigned a number in the range $(0, 1)$ then $\mu(x) \in (0, 1)$ is the *membership function*.
small

Example 37.45. In a car-race, all the cars complete the race in four time-groups : shortest time, moderate time, long time and longest time. If we note the time taken by each car in a group, it will give rise to a distribution of times. Now let us find the outcome of the race based on engine power, car speed and road conditions. Each of these variables may further be divided into :

- (i) low, medium and high for the variable engine power,
- (ii) slow, moderate and fast for the variable car speed,
- (iii) rough, bumpy and smooth for the variable road conditions.

Now we try to predict on some basis, in which of the four groups the car will finish, if it has low engine power, moderate speed and rough road.

Then the distribution for the engine power would correspond to the membership function for low, medium and high. Similarly the distribution for the speed would depend on the membership function for slow, moderate and fast, while the distribution for road conditions would depend on membership function for rough, bumpy and smooth.

37.23 FUZZY SET OPERATIONS

(1) A fuzzy set is said to be *normalised* when the largest element of the set (called *supremum*) is unity.

For instance, the set of members $\{5, 10, 15, 20, 25\}$ is normalised to $\{0.2, 0.4, 0.6, 0.8, 1\}$ by dividing each member by 25, the supremum in the set.

The normalization of a fuzzy set F is expressed as $\sup_{x \in X} F(x) = 1$.

(2) Complement. The complement of a fuzzy set F is the set F^c with degree of membership of an element in F^c equal to one minus degree of membership of this element in F . (Fig. 37.5)

For example, if $F = [0.4 \text{ Ram}, 0.6 \text{ Sham}, 0.8 \text{ Jyoti}, 0.9 \text{ Ritu}]$ be a set of intelligent students, then $F^c = [0.6 \text{ Ram}, 0.4 \text{ Sham}, 0.2 \text{ Jyoti}, 0.1 \text{ Ritu}]$ is a set of non-intelligent students.

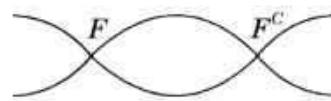


Fig. 37.5



Fig. 37.6

(3) Intersection. The intersection of two fuzzy sets F and G is the set $F \cap G$, where the degree of membership of an element in $F \cap G$ is the minimum of the degrees of membership of this element in F and G . (Fig. 37.6)

(4) Union. The union of two fuzzy sets F and G is the set $F \cup G$, where the degree of membership of an element in $F \cup G$, is the maximum of this element in F and G . (Fig. 37.7)

For example, if

$$F = [0.5 \text{ Rani}, 0.2 \text{ Suman}, 0.4 \text{ Anita}, 0.8 \text{ Sunita}]$$

be a set of fat girls, and

$$G = [0.1 \text{ Rani}, 0.6 \text{ Suman}, 0.9 \text{ Anita}, 0.5 \text{ Sunita}]$$

be a set of tall girls, then

$$F \cap G = [0.1 \text{ Rani}, 0.2 \text{ Suman}, 0.4 \text{ Anita}, 0.5 \text{ Sunita}]$$

and

$$F \cup G = [0.5 \text{ Rani}, 0.6 \text{ Suman}, 0.9 \text{ Anita}, 0.8 \text{ Sunita}]$$

(5) Equality. Two fuzzy sets F and G are said to be equal if and only if $F(x) = G(x)$ for all x in X .

(6) Subset. The fuzzy set F is said to be a subset of the fuzzy set G (i.e., $F \subseteq G$) if and only if $F(x) \leq G(x)$ for all $x \in X$.

(7) Double negation. If F is a fuzzy set, then $(F^c)^c = 1$.

(8) De Morgan's laws. If F and G are two fuzzy sets then

$$(F \cup G)^c = F^c \cap G^c ; (F \cap G)^c = F^c \cup G^c.$$

Example 37.46. Let the membership functions for the fuzzy sets F and G be as in the following table :

X	1	2	3	4	5	6	7	8	9	10
F	0	0	0.1	0.5	0.8	1	0.3	0.5	0	0
G	0	0	0	0	0.1	0.3	0.5	0.8	1	1
F^c	1	1	0.9	0.5	0.2	0	0.7	0.5	1	1

Then the corresponding

$$F \cap G = [0, 0.1, 0.3, 0.5]$$

$$F \cup G = [0.1, 0.5, 0.8, 1, 0.5, 0.8, 1, 1]$$

Clearly F is not a subset of G and G is not a subset of F .

37.24 TRUTH VALUE

(1) Truth value of the negation of a proposition in fuzzy logic is one minus the truth value of the proposition.

For example, if the truth value of the statement 'Ram is happy' is 0.8, then the truth value of 'Ram is not happy' is 0.2.

(2) Truth value of the conjunction of two propositions in the fuzzy logic is the minimum of the truth values of the two propositions.

(3) Truth value of the disjunction of two propositions in the fuzzy logic is the maximum of the truth values of two propositions.

For example, if the truth value of 'Ram is fat' is 0.6, and the truth value of 'John is fat' is 0.9, then the truth value of the statement

(a) 'Ram and John are fat' is 0.6.

(b) 'Ram or John is fat' is 0.9.

(c) 'neither Ram nor John is fat' is negation of minimum of negation of 'Ram is fat' (i.e., 0.4) and negation of 'John is fat' (i.e., 0.1) = 0.1.

(d) 'Ram is not fat or John is not fat' is maximum of 0.4 and 0.1 i.e., 0.4.

37.25 ALGEBRAIC OPERATIONS ON FUZZY SETS

(1) Algebraic sum of two fuzzy sets F and G is defined by the membership function

$$\mu_{F+G}(x) = \mu_F(x) + \mu_G(x) - \mu_F(x)\mu_G(x)$$

and is written as $F + G$.

Algebraic product of two fuzzy sets F and G is defined by the membership function.

$$\mu_{F.G.}(x) = \mu_F(x).\mu_G(x)$$

and is written as $F.G.$

(2) Properties of fuzzy set operations which are common to crisp set operations, are as under

1. *Idempotent* : $F \cup F = F$, $F \cap F = F$

2. *Identity* : $F \cup \phi = F$, $F \cap U = F$

3. *Commutative* : $F \cup G = G \cup F$, $F \cap G = G \cap F$

4. *Distributive* : $F \cap (G \cup H) = (F \cap G) \cup (F \cap H)$, $F \cup (G \cap H) = (F \cup G) \cap (F \cup H)$

5. *Associative* : $(F \cup G) \cup H = F \cup (G \cup H)$, $(F \cap G) \cap H = F \cap (G \cap H)$

6. *Absorption* : $F \cup (F \cap G) = F$, $F \cap (F \cup G) = F$.

37.26 GENERATION OF RULES FOR FUZZY PROBLEMS

We should know before hand all possible input-output relations while dealing with problems concerning fuzzy engines or fuzzy controls. These input-output rules are then expressed with 'if ... then' statements.

For instance, if F 's and G 's are inputs of fuzzy problems and R 's are the actions taken for each rule, then the set of 'if ... then' rules with two input variables F_1 and G_1 and the actions taken are shown in table 1.

i.e., if F_1 and or G_1 , then R_{11} , else
 if F_2 and or G_1 , then R_{21} , else
 if F_1 and or G_2 , then R_{12} , else
 if F_2 and or G_2 , then R_{22} .

Table 1

F_1	R_{11}	R_{12}
F_2	R_{21}	R_{22}
G_1	G_2	

In case, the fuzzy statements have more variables, then 'if ... then' rules becomes more complicated to tabulate.

However such a tabulation can be simplified by following a *decomposition process* as follows :

The decomposition process of three fuzzy variables F , G and H with actions R 's taken is shown below :

Table 2

F_1	R_{11}	R_{12}
F_2	R_{21}	R_{22}
G_1	G_2	

Table 3

H_1	R_{111}	R_{121}	R_{211}	R_{221}
H_2	R_{112}	R_{122}	R_{212}	R_{222}
	R_{11}	R_{12}	R_{21}	R_{22}

Here the statements F_2 and G_1 and H_1 then R_{211} is decomposed into 'if F_2 and G_1 then R_{21} ' and 'if R_{21} and H_1 , then R_{211} '.

Similarly R_{21} and H_2 then R_{212} .

This decomposition process can easily be extended to any number of input variables.

37.27 FUZZY PROPOSITIONS

(1) A *fuzzy number* is a fuzzy set $R \rightarrow [0, 1]$. We can easily extend classical two-valued logic to three-valued logic. Fuzzy logic, however is an extension of multi-valued logic. It provides foundations for approximate reasoning with imprecise fuzzy propositions using fuzzy set theory.

The classical propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees i.e., the truth and falsity are expressed by numbers in $[0, 1]$.

A variable whose values are 'words' or 'sentences' is called a *linguistic variable*. For example 'height' is a linguistic variable and its values are tall, very tall, quite tall, not tall, short, not very short, not quite tall etc.

(2) **Classification of fuzzy propositions.** The classification propositions are statements which are either true or false. In fuzzy logic, the truth or falsity of fuzzy propositions is assigned different degrees i.e., the truth and falsity are expressed by numbers in $[0, 1]$.

The fuzzy propositions of simple nature can be classified into the following four types. In each case, we introduce the relevant canonical form and then discuss its interpretation.

Type I. Unconditional and unqualified propositions.

The standard canonical form of this type of proposition is expressed as $p : u$ if F ... (1)

Here u is the variable that takes value u from some universal set U and F is a fuzzy set on U which represents a fuzzy predicate such as young, tall, low, high etc. Given a particular value u (say v), this value belongs to F with membership grade $F(v)$. This membership grade is then interpreted as the degree of truth $T(p)$ of proposition p

$$i.e., \quad T(p) = F(v) \quad \dots(2)$$

Here T is a fuzzy set on $[0, 1]$ which assigns the membership grade $F(v)$ to each value v of u .

In some fuzzy propositions, values of variable u in (1) are assigned to individuals in a given set I i.e., variable u becomes a function $u : I \rightarrow u$ where $u(i)$ is the value of v for individual i in U . Accordingly the canonical form (2) is modified to the form

$$p : u(i) \text{ is } F \text{ where } i \in I \quad \dots(3)$$

Example 37.47. Consider a set I of persons, each person is characterized by his 'age' and a fuzzy set expressing the predicate 'young' is given. Denoting our variable by 'age' and fuzzy set by 'young', the canonical form is

$$p : \text{age}(i) \text{ is young.}$$

Solution. The degree of truth of this proposition $T(p)$ is then determined for each person i in I by means of the equation $T(p) = \text{young}[\text{age}(i)]$.

Example 37.48. At a particular place on the earth, consider the air temperature u (in $^{\circ}\text{C}$). Let Fig. 37.8 represent the membership function as predicate 'high'. Assuming that all relevant temperature readings are given, the corresponding fuzzy proposition is expressed as,

$$p : \text{temp}(u) \text{ is high } (^{\circ}\text{C})$$

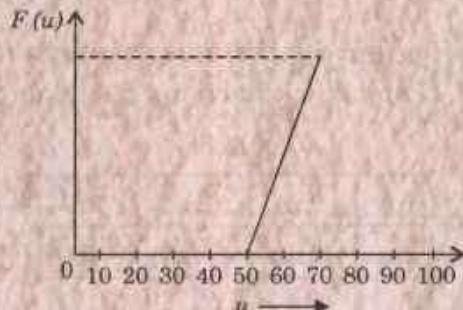


Fig. 37.8

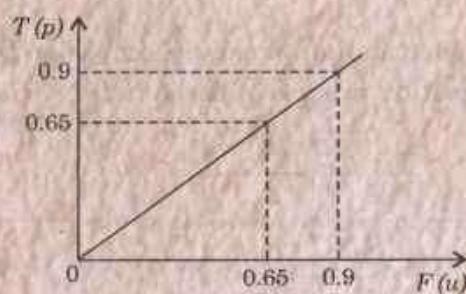


Fig. 37.9

Solution. The degree of truth $T(p)$ depends upon the actual value of the temperature and on the nature of predicate 'high' which is defined by the membership function F in Fig. 37.8.

e.g., if $u = 75$ then $F(75) = 0.65$ and $T(p) = 0.65$.

Type II. Conditional and unqualified propositions.

A proposition p of this type is expressed by the canonical form

$$p : \text{If } x \text{ is } F \text{ then } y \text{ is } G \quad \dots(4)$$

where x, y are variables whose values are in the sets X, Y and F, G are fuzzy sets on X and Y respectively. These propositions may also be viewed as propositions of the form

$$\{x, y\} \text{ is } R \quad \dots(5)$$

where R is a fuzzy set on $X \times Y$ which is determined for each $x \in X$ and each $y \in Y$ by the formula

$$R(x, y) = B[F(x), G(y)], \quad \dots(6)$$

where B is a binary operation as $[0, 1]$ representing a suitable *fuzzy implication*.

Type III. Unconditional and qualified propositions

A proposition of this type is expressed by either of the following canonical forms :

$$p : u \text{ is } F \text{ is } S \quad \dots(7)$$

or

$$p : \text{Prob}(U \text{ is } F) \text{ is } P \quad \dots(8)$$

where u is a variable that takes value v from some universal set U and F is a fuzzy set on U which represents a fuzzy predicate such as small, young, daughter etc.

$\text{Prob.}(U \text{ is } F)$ is the probability of fuzzy set event ' u is F ' ; S is the fuzzy truth qualifier and P is the fuzzy probability qualifier. S and P are both represented by fuzzy set on $[0, 1]$.

Example 37.49. Mary is 'young' is 'very true' where the predicate 'young' and the truth qualifier 'very true' are represented by the respective fuzzy sets shown in Fig. 37.10.

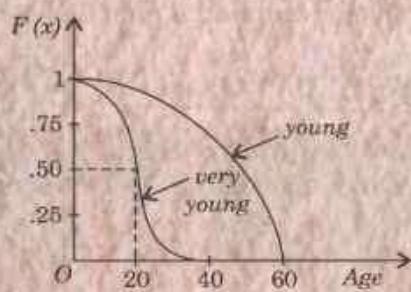


Fig. 37.10

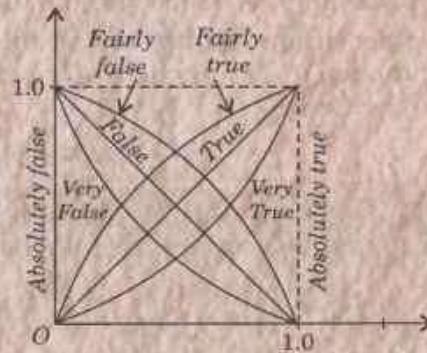


Fig. 37.11

Solution. The degree of truth $T(p)$ of any truth-qualified proposition p is given for each $u \in U$ by $T(p) = S\{F(u)\}$.

Assuming that Mary's age is 20, she belongs to the set representing the predicate 'very young' with membership grade 0.50, our proposition belongs to the set of propositions which are 'very true' with membership grade 0.50 as shown in Fig. 37.11. This implies that the degree of truth of our truth qualified proposition is 0.50.

If the proposition be modified by changing the predicate to 'young' or the truth qualifier to 'fairly true', we would obtain the corresponding degree of truth of such propositions by the same method.

Type IV. Conditional and qualified propositions

This type of propositions can either be expressed in the canonical form

$$p : \text{If } x \text{ is } F, \text{ then } y \text{ is } G \text{ in } S$$

or $p : \text{Prob}\{x \text{ is } F/y \text{ is } G\} \text{ in } P,$

where $\text{Prob}\{x \text{ is } F/y \text{ is } G\}$ is conditional probability.

37.28 APPLICATIONS OF FUZZY SETS

The concept of fuzzy sets has already influenced all engineering disciplines to various degrees.

Electrical engineering is the first such discipline where the utility of fuzzy logic and fuzzy sets has been recognized by developing controllers. Electronic circuits for fuzzy image processing have also been developed.

Some ideas regarding the application of fuzzy sets in *civil engineering* emerged around 1970. In the construction of bridges, dams, buildings etc. a designer has to take into account the safety factor for which the fuzzy theory has an effective role to play. Fuzzy set theory has also proved quite useful for assessing the life of existing constructions.

In *mechanical engineering* design problems, the utility of Fuzzy set theory was realised during mid 1980's. The membership function is expressed in terms of thermal expansion or corrosion or cost of different materials etc.

When the utility of fuzzy controllers was increasingly felt around mid 1980's, the need for computer hardware to implement the various operations involving fuzzy logic, had been recognized. In digital mode, fuzzy sets have been expressed as vectors of $(0, 1)$ members.

Fuzzy control and fuzzy decision making are two well-developed areas of fuzzy set theory. These are directly relevant to *industrial engineering problems*. The utility of fuzzy sets has also been recognized for estimating the service life of given equipment under various conditions.

Modern Reliability theory has also been developed on the assumption of fuzzy sets. At any given time, an engineering product may be in functioning state to some degree or in failed state to another degree. The behaviour of an engineering product with respect to its functioning state and failed state has been characterized as based on fuzzy set theory.

The use of fuzzy set theory in *Robotics* includes approximate reasoning, fuzzy controllers, fuzzy pattern recognition and fuzzy data bases.

PROBLEMS 37.7

- Given fuzzy sets $F_1 = [0.6 \text{ Sonu}, 0.9 \text{ Renu}, 0.7 \text{ Paul}, 0.3 \text{ Sham}]$
 $F_2 = [0.3 \text{ Sham}, 0.8 \text{ Paul}, 0.9 \text{ Renu}, 0.5 \text{ Sonu}]$
and $F_3 = [0.8 \text{ Paul}, 0.3 \text{ Sham}, 0.5 \text{ Sonu}, 0.9 \text{ Renu}]$
Which of the above two sets are equal?
- Write the complement set of the fuzzy set F , if $F = [0.8 \text{ Ram}, 0.3 \text{ Sham}, 0.6 \text{ John}, 0.7 \text{ Charu}]$.
- If $F = [0.3x_1, 0.7x_2, 0.5x_3, 0.8x_4]$ and $G = [0.4x_1, 0.6x_2, 0.1x_3, 0.9x_4]$ been two fuzzy sets, then write down $F \cup G$ and $F \cap G$.
- State the truth values of the negation of the following propositions :
 - Truth value of 'F is rich' is 0.8
 - Truth value of 'G is fat' is 0.6
 - Truth value of 'Mary is beautiful' is 0.7.
- Let the membership functions of fuzzy sets F and G be as follows :
 $X : [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$

$$F : [0, 0, 0, 0, 0.1, 0.3, 0.5, 0.9, 1, 1]$$

$$G : [0, 0, 0.1, 0.5, 0.9, 1, 0.9, 0.5, 0.0]$$

State whether (i) $F = G$ (ii) F is a subset of G .

Also write down F^c , $F \cap G$ and $F \cup G$.

6. The truth values of the statements

'Latif is a good player' is 0.7

and 'John is a good player' is 0.6.

What is the truth value of

(i) the conjunction of the above two prepositions.

(ii) the disjunction of the above prepositions.

7. Define a Fuzzy set and the standard operations on Fuzzy sets.

(Bhopal, 2009)

8. State the constituents of the pair in a fuzzy set.

9. Write a note on 'Fuzzy logic affects many disciplines'?

(Bhopal, 2001)

Tensor Analysis

1. Introduction. 2. Summation Convention. 3. Transformation of coordinates. Tensor of order zero. 4. Kronecker Delta. 5. Contravariant vectors, Covariant vectors. 6. Tensors of higher order. 7. Symmetric and skew-symmetric tensors. 8. Addition of tensors. 9. Outer product of two tensors. 10. Contraction of tensors. 11. Inner product of two tensors. 12. Quotient Law. 13. Riemannian space, Metric tensor. 14. Conjugate tensor. 15. Associated tensors. 16. Length of a vector, Angle between two vectors. 17. Christoffel symbols. 18. Transformation of Christoffel symbols. 19. Covariant differentiation of covariant vector ; Covariant differentiation of a contravariant vector. 20. Gradient, Divergence, Curl.

38.1 INTRODUCTION

Some physical quantities are specified by their magnitude only while others by their magnitude and direction. But certain quantities are associated with two or more directions. Such a quantity is called a *tensor*. The stress at a point of an elastic solid is an example of a tensor which depends on two directions—one normal to the area and other that of the force on it.

The properties of tensors are independent of the frames of reference used to describe them. That is why Einstein found tensors as a convenient tool for formulation of his Relativity theory. Since then, the subject of tensor analysis shot into prominence and is of great use in the study of Riemannian geometry, mechanics, elasticity, electro-magnet theory and numerous other fields of science and engineering. The emergence of tensor calculus as a symmetric subject is due to Ricci and his student Levi-Cita.

38.2 SUMMATION CONVENTION

Consider a sum of the type

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \quad i.e., \quad \sum_{i=1}^n a_i x_i \quad \dots(1)$$

In tensor analysis, the subscripts of the symbols x_1, x_2, \dots, x_n are replaced by superscripts and we write these as x^1, x^2, \dots, x^n . The superscripts do not stand for the various powers of x but act as labels to distinguish different symbols. The power of a symbol (say : x^i) will be indicated as $(x^i)^2, (x^i)^3$ etc. Hence (1) is written as

$$\sum_{i=1}^n a_i x^i. \quad \dots(2)$$

A still simpler notation is to drop the Σ sign and write (2) as $a_i x^i$...(3)

In this the repeated index i successively takes up the values 1, 2, ..., n and the expression (3) represents the sum of all such terms. The repeated index i over which the summation is to be done, is called a *dummy index* since it doesn't appear in the final result. This notation, known as *summation convention*, is due to Einstein. We shall adopt this convention throughout this chapter and take the sum whenever a letter appears in a term once as a subscript and once as superscript.

Example 38.1. Write the terms contained in $S = a_{ij}x^i x^j$ taking $n = 3$.

Solution. Since the index i occurs both as a subscript and as a superscript, we first sum on i from 1 to 3.

$$\therefore S = a_{1j}x^1 x^j + a_{2j}x^2 x^j + a_{3j}x^3 x^j$$

Now each term in S has to be summed up w.r.t. repeated index j from 1 to 3.

$$\begin{aligned}\therefore S &= a_{11}x^1 x^1 + a_{12}x^1 x^2 + a_{13}x^1 x^3 + a_{21}x^2 x^1 + a_{22}x^2 x^2 + a_{23}x^2 x^3 \\ &\quad + a_{31}x^3 x^1 + a_{32}x^3 x^2 + a_{33}x^3 x^3 \\ &= a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})x^1 x^2 + (a_{13} + a_{31})x^1 x^3 + (a_{23} + a_{32})x^2 x^3.\end{aligned}$$

Example 38.2. If f is a function of n variables x^i , write the differential of f .

Solution. Since $f = f(x^1, x^2, \dots, x^n)$

\therefore From Calculus, we have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \dots + \frac{\partial f}{\partial x^n} dx^n = \frac{\partial f}{\partial x^i} dx^i.$$

38.3 (1) TRANSFORMATION OF COORDINATES

In a 3-dimensional space, the coordinates of a point are (x^1, x^2, x^3) referred to a particular frame of reference. Similarly in an n -dimensional space, the coordinates of a point are n independent variables (x^1, x^2, \dots, x^n) with respect to a certain frame of reference. Let $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ be the coordinates of the same point referred to another frame of reference. Suppose, $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ are independent single-valued functions of x^1, x^2, \dots, x^n so that

$$\bar{x}^1 = \phi^1(x^1, x^2, \dots, x^n)$$

$$\bar{x}^2 = \phi^2(x^1, x^2, \dots, x^n)$$

.....

$$\bar{x}^n = \phi^n(x^1, x^2, \dots, x^n)$$

or more briefly

$$\bar{x}^i = \phi^i(x^1, x^2, \dots, x^n) \quad \dots(1)$$

We can solve the equations (1) and express x^i as functions of \bar{x}^i so that

$$x^i = \psi^i(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) \quad \dots(2)$$

The equations (1) and (2) are said to define a transformation of the coordinates from one frame of reference to another.

(2) Scalars or invariants. A function $\phi(x^1, x^2, x^3)$ is called a scalar or an invariant if its original value does not change upon transformation of coordinates from x^1, x^2, x^3 to $\bar{x}^1, \bar{x}^2, \bar{x}^3$.

i.e.,

$$\phi(x^1, x^2, x^3) = \psi(\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

A scalar or invariant is also called a tensor of order (or rank) zero.

38.4 KRONECKER DELTA*

The quantity δ_i^j defined by the relations

$$\delta_i^j = 0, \quad \text{when } j \neq i$$

and

$$\delta_i^j = 1, \quad \text{when } j = i, \text{ is called Kronecker delta.}$$

Evidently

$$\delta_1^1 = \delta_2^2 = \delta_3^3 = \dots = \delta_n^n = 1$$

while

$$\delta_1^2 = \delta_2^3 = \delta_3^2 = \dots = 0$$

*Called after the German mathematician Leopold Kronecker (1823–91) who made important contributions to number theory, algebra and group theory.

We note that by summing up w.r.t. the repeated index j ,

$$\begin{aligned}\delta_{3j}\delta_2^j &= a_{31}\delta_2^1 + a_{32}\delta_2^2 + a_{33}\delta_2^3 + a_{34}\delta_2^4 + \dots \\ &= 0 + a_{32} + 0 + 0 = a_{32}\end{aligned}$$

In general,

$$\begin{aligned}a_{ij}\delta_k^j &= a_{i1}\delta_k^1 + a_{i2}\delta_k^2 + \dots + a_{ik}\delta_k^k + \dots + a_{in}\delta_k^n \\ &= 0 + 0 + \dots + a_{ik} \cdot 1 + \dots + 0 = a_{ik}.\end{aligned}$$

Example 38.3. Show that $a_{ij}A^{kj} = \Delta\delta_i^k$, where Δ is a determinant of order three and A^{ij} are cofactors of a^{ij} .

(Delhi, 2002)

Solution. By expansion of determinants, we have

$$\begin{aligned}a_{11}A^{11} + a_{12}A^{12} + a_{13}A^{13} &= \Delta \\ a_{11}A^{21} + a_{12}A^{22} + a_{13}A^{23} &= 0 \\ a_{11}A^{31} + a_{12}A^{32} + a_{13}A^{33} &= 0\end{aligned}$$

which can be compactly written as

$$a_{1j}A^{1j} = \Delta, a_{1j}A^{2j} = 0, a_{1j}A^{3j} = 0$$

Using Kronecker delta notation, these can be combined into a single equation

$$a_{1j}A^{kj} = \Delta\delta_1^k$$

Similarly

$$a_{2j}A^{kj} = \Delta\delta_2^k, a_{3j}A^{kj} = \Delta\delta_3^k$$

All these nine equations are included in $a_{ij}A^{kj} = \Delta\delta_i^k$.

Example 38.4. If x^i and \bar{x}^i are independent coordinates of a point, show that

$$\frac{\partial x^j}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^i} = \delta_i^j.$$

Solution. The partial derivatives of ϕ in the two coordinate systems are different and are connected by the following formula of Differential Calculus :

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^1} \cdot \frac{\partial \bar{x}^1}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^2} \cdot \frac{\partial \bar{x}^2}{\partial x^i} + \frac{\partial \phi}{\partial \bar{x}^3} \cdot \frac{\partial \bar{x}^3}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^i}$$

$$\text{In particular, when } \phi = x^j, \text{ we have } \frac{\partial x^j}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^i} \quad \dots(i)$$

$$\left. \begin{array}{l} \text{Since } x^j \text{ is independent of } x^i, \frac{\partial x^j}{\partial x^i} = 0, \text{ when } j \neq i \\ = 1, \text{ when } j = i \end{array} \right\} \quad \dots(ii)$$

Hence the result follows from (i) and (ii).

38.5 (1) CONTRAVARIANT VECTORS

Let A^1, A^2, \dots, A^n (i.e., A^i) be a set of n functions of the coordinate system x^1, x^2, \dots, x^n (i.e., x^i). If these transform in another system of coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ (i.e., \bar{x}^i) according to the law

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \quad \dots(1)$$

then A^i are called components of a **contravariant vector or contravariant tensor of order one**.

An example of a contravariant vector. Let us transform the coordinates of a point x^i to \bar{x}^i in a n -dimensional space.

Since x is a function of x^i (i.e., x_1, x_2, \dots, x_n), therefore,

$$\begin{aligned}d\bar{x}^i &= \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \dots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n \\ &= \frac{\partial \bar{x}^i}{\partial x^j} dx^j, \text{ using the summation convention.}\end{aligned}$$

Comparing this with (1), it follows that the set of differentials dx^1, dx^2, \dots, dx^n is an example of a contravariant vector. That is why the coordinates of a point are numbered by superscripts and not by subscripts.

(2) Covariant vectors. Let A_i be a set of n functions of the coordinate system x^i . If these transform in another system of coordinates \bar{x}^i according to the law

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad \dots(2)$$

then A_i are called the components of a covariant vector or covariant tensor of order one.

An example of a covariant vector. Let ϕ be a function which has a fixed value at each point of space independent of the coordinate system employed. Therefore, ϕ is a function of the coordinates x^i in the first system and a function of the coordinates \bar{x}^i in the second system. By the chain rule

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^i} + \dots + \frac{\partial \phi}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^i} + \dots + \frac{\partial \phi}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}^i}$$

Comparing this equation with (2), it follows that the set of derivatives,

$$\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \dots, \frac{\partial \phi}{\partial x^n}$$

form a covariant vector.

Example 38.5. A covariant tensor has components $xy, 2y - z^2, xz$ in rectangular coordinates. Find its covariant components in spherical coordinates.

Solution. Here

$$\left. \begin{aligned} x^1 &= x, x^2 = y, x^3 = z \\ \bar{x}^1 &= r, \bar{x}^2 = \theta, \bar{x}^3 = \phi \end{aligned} \right\} \quad \dots(i)$$

and

Let

$$A_1 = xy, A_2 = 2y - z^2, A_3 = xz \quad \dots(ii)$$

According to the law of transformation, we have $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \quad (i = 1, 2, 3)$

and we wish to evaluate $\bar{A}_1, \bar{A}_2, \bar{A}_3$ where A_1, A_2, A_3 are known.

We know that $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$...(iii)

$$\begin{aligned} \text{Now } \bar{A}_1 &= \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3 = \frac{\partial x}{\partial r} xy + \frac{\partial y}{\partial r} (2y - z^2) + \frac{\partial z}{\partial r} xz \\ &= \sin \theta \cos \phi \cdot r \sin \theta \cos \phi \cdot r \sin \theta \sin \phi + \sin \theta \sin \phi (2r \sin \theta \sin \phi - r^2 \cos^2 \theta) \\ &\quad + \cos \theta \cdot r \sin \theta \cos \phi \cdot r \cos \theta \end{aligned} \quad \begin{matrix} \text{[From (i) and (ii)]} \\ \text{[From (iii)]} \end{matrix}$$

$$\text{Similarly } \bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \frac{\partial x^3}{\partial \bar{x}^2} A_3 = r \cos \theta \cos \phi \cdot r \sin \theta \cos \phi \cdot r \sin \theta \sin \phi$$

$$+ r \cos \theta \sin \phi (2r \sin \theta \sin \phi - r^2 \cos^2 \theta) + (-r \sin \theta) r \sin \theta \cos \phi \cdot r \cos \theta$$

and

$$\begin{aligned} \bar{A}_3 &= \frac{\partial x^1}{\partial \bar{x}^3} A_1 + \frac{\partial x^2}{\partial \bar{x}^3} A_2 + \frac{\partial x^3}{\partial \bar{x}^3} A_3 \\ &= -r \sin \theta \sin \phi \cdot r \sin \theta \cos \phi \cdot r \sin \theta \sin \phi \\ &\quad + r \sin \theta \cos \phi (2r \sin \theta \sin \phi - r^2 \cos^2 \theta) + 0 \end{aligned}$$

38.6 TENSORS OF HIGHER ORDER

Let i and j be each given values 1 to n , then the symbol A^{ij} will give rise to n^2 functions.

(1) If A^{ij} be a set of n^2 functions of the coordinates x^1, x^2, \dots, x^n which transforms to \bar{A}^{ij} in another system of coordinates $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$, according to the law

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} A^{kl} \quad \dots(1)$$

then the functions \bar{A}^{ij} are said to be components of a contravariant tensor of the second order.

(2) If A_{ij} be a set of n^2 functions of x^1, x^2, \dots, x^n which transform to \bar{A}_{ij} in another system of coordinates, $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ according to the law

$$\bar{A}_{ij} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^l}{\partial x^j} A_{kl}, \quad \dots(2)$$

then the functions \bar{A}_{ij} are said to be the components of a **covariant tensor of the second order**.

(3) If A^i_j be a set of n^2 functions of x^1, x^2, \dots, x^n which transform to \bar{A}_j^i in another system of coordinates, $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$, according to the law

$$\bar{A}_j^i = \frac{\partial \bar{x}^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial \bar{x}^j} A^k_l, \quad \dots(3)$$

then \bar{A}_j^i are said to be the components of a **mixed tensor of the second order**. It transforms like a contravariant vector with respect to the index i and like a covariant vector with regard to the index j . That is why i is placed as a superscript and j as subscript.

We can similarly define tensors of the orders higher than two.

Obs. Each of the above laws of transformation (1) to (3), give rise to n^2 equations as i and j are each given the value 1 to n .

Example 38.6. Show that the Kronecker delta is a mixed tensor of order two.

Solution. If δ_i^j transforms to $\bar{\delta}_i^j$ in the coordinate system $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ by the law for mixed tensors of order two, then

$$\bar{\delta}_i^j = \frac{\partial \bar{x}^l}{\partial x^i} \frac{\partial \bar{x}^j}{\partial x^m} \delta_l^m = \frac{\partial \bar{x}^j}{\partial x^m} \cdot \frac{\partial x^m}{\partial x^i} = \frac{\partial \bar{x}^j}{\partial x^i} = \delta_i^j \quad [\because \delta_l^m = 0 \text{ for } l \neq m]$$

Hence δ_i^j is a mixed tensor of order two, having the same components in every coordinate system.

Example 38.7. Show that the velocity of a fluid at any point is a contravariant tensor of rank one.

Solution. Let $dx^1/dt, dx^2/dt, dx^3/dt$ be the components of fluid velocity of the point (x^1, x^2, x^3) , i.e., dx^i/dt be the components of velocity in the coordinate system x^i . Suppose the corresponding components of velocity in the coordinate system \bar{x}^i are $d\bar{x}^i/dt$. Then $\bar{x}^1, \bar{x}^2, \bar{x}^3$ being the functions of x^1, x^2, x^3 which in turn are functions of t , we can write

$$\begin{aligned} \frac{d\bar{x}^j}{dt} &= \frac{\partial \bar{x}^j}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial \bar{x}^j}{\partial x^2} \frac{dx^2}{dt} + \frac{\partial \bar{x}^j}{\partial x^3} \frac{dx^3}{dt} \\ \text{or} \quad \frac{d\bar{x}^j}{dt} &= \frac{\partial \bar{x}^j}{\partial x^i} \frac{dx^i}{dt} \end{aligned} \quad \dots(i)$$

Now according to the law of tensor transformation, (i) shows that the velocity of a fluid is a contravariant tensor of rank one.

Example 38.8. Prove that there is no distribution between contravariant and covariant vectors if the transformation law is of the form $\bar{x}^i = a_m^i x^m + b^i$, where a 's and b 's are constants such that $a_r^i a_m^i = \delta_m^r$.

(Bhopal, 2003)

Solution. Given transformation $\bar{x}^i = a_m^i x^m + b^i$... (i)

$$\frac{\partial \bar{x}^i}{\partial x^m} = a_m^i \quad \dots(ii)$$

Also from (i),

$$\begin{aligned} a_r^i \bar{x}^i &= a_r^i a_m^i x^m + a_r^i b^i \\ &= \delta_m^r x^m + a_r^i b^i = x^r + a_r^i b^i. \end{aligned}$$

$$\therefore \frac{\partial x^r}{\partial \bar{x}^i} = a_r^i \quad \text{i.e.,} \quad \frac{\partial x^m}{\partial \bar{x}^i} = a_m^i \quad \dots(iii)$$

From (ii) and (iii), it is clear that any vector with components a, b, c will on transformation give the same components whether transformed as a contravariant vector or as a covariant vector. Thus in this case, there is no distinction between the two.

38.7 SYMMETRIC AND SKEW-SYMMETRIC TENSORS

(1) A tensor is said to be **symmetric** with respect to two contravariant (or two covariant) indices if its components remain unchanged on an interchange of the two indices.

Thus the tensor A^{ij} is symmetric if $A^{ij} = A^{ji}$, for every i and j .

(2) A tensor is said to be **skew-symmetric** with respect to two contravariant (or covariant) indices, if its components change sign on interchange of the two indices.

Thus the tensor A^{ij} is skew-symmetric if $A^{ji} = -A^{ij}$ for every i and j .

In general, the tensor A_{lm}^{ijk} is said to be symmetric or skew symmetric in i and j according as

$$A_{lm}^{ijk} = A_{lm}^{jik} \quad \text{or} \quad -A_{lm}^{jik}$$

Example 38.9. Show that (i) a symmetric tensor of the second order has only $\frac{1}{2}n(n + 1)$ different components.

(ii) A skew-symmetric tensor of the second order has only $\frac{1}{2}n(n - 1)$ different non-zero components.

Solution. (i) Let A^{ij} be a symmetric tensor of order two so that $A^{ij} = A^{ji}$.

If each of the indices i and j take the values 1 to n , then A^{ij} will have n^2 components. Out of these n^2 components, n components $A_{11}, A_{22}, \dots, A_{nn}$ are independent.

Thus the remaining components are $(n^2 - n)$ which can be taken in pairs ($\because A_{12} = A_{21}, A_{31} = A_{13}$ etc.)

Hence the total number of independent components

$$= n + \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n + 1)$$

(ii) Let A^{ij} be a skew-symmetric tensor of order two so that $A^{ij} = -A^{ji}$. As above, A^{ij} will have n^2 components. Out of these, n components $A^{11}, A^{22}, \dots, A^{nn}$ are all zero. [$\because A^{11} = -A^{11}$].

Omitting these, there are $(n^2 - n)$ components. Since $A^{12} = -A^{21}, A^{13} = -A^{31}$ etc., therefore ignoring the sign, $(n^2 - n)$ components can be taken in pairs.

Hence the total number of independent non-zero components

$$= \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n - 1).$$

PROBLEMS 38.1

1. Write the following using the summation convention :

$$(i) \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial\phi}{\partial x^2} \frac{dx^2}{dt} + \dots + \frac{\partial\phi}{\partial x^n} \frac{dx^n}{dt} \quad (ii) (x^1)^2 + (x^2)^2 + (x^3)^2 + \dots + (x^n)^2.$$

2. Write out in full the following :

$$(i) a_{ij}x^i x^j \quad (ii) g_{ij} dx^i dx^j \quad (iii) g_{lm} g^{mp}.$$

3. (a) Shows that δ_j^i is an invariant.

(Bhopal, 2003)

$$(b) \text{Evaluate } (i) \delta_j^i \delta_k^j \quad (ii) \delta_p^q \delta_q^r \delta_r^s.$$

$$4. \text{Show that } (i) \frac{\partial x^p}{\partial x^q} = \delta_q^p \quad (ii) \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^q}{\partial x^r} = \delta_r^p.$$

5. If the \bar{x} 's are n independent functions of x 's and i, j, k, l each take values from 1 to n , show that

$$\frac{\partial \bar{x}^k}{\partial x^l} \frac{\partial x^l}{\partial x^j} \cdot \delta_j^i = \delta_i^k.$$

6. Write down the law of transformation for the tensors

$$(i) A_l^k$$

$$(ii) C_{mn}.$$

7. A quantity $A(i, j, k, l, m)$ which is a function of the coordinates x^p transforms to another coordinate system \bar{x}^p according to the law :

$$\bar{A}(r, s, t, u, v) = \frac{\partial \bar{x}^r}{\partial x^i} \frac{\partial \bar{x}^s}{\partial x^j} \frac{\partial \bar{x}^t}{\partial x^k} \frac{\partial \bar{x}^u}{\partial x^l} \frac{\partial \bar{x}^v}{\partial x^m} A(i, j, k, l, m)$$

Is this quantity a tensor? If so express it suitably and state its nature and rank?

8. If the components of two tensors are equal in one coordinate system, show that they are equal in all coordinate systems.

9. A covariant tensor has components $2x - z, x^2y, yz$ in cartesian coordinate system. Find its components in
(a) cylindrical coordinates (Punjab, M.E., 1989) (b) spherical coordinates.

10. If g_{ij} denotes the components of a covariant tensor of rank two, show that the product $g_{ij} dx^i dx^j$ is an invariant scalar. (Delhi, 2002)

11. A contravariant tensor has components a, b, c in rectangular coordinates; find the components in spherical coordinates.

12. Prove that $A_{ij}B^iC^j$ is an invariant, if B^i and C^j are contravariant vectors and A_{ij} is a covariant tensor. (Madras, M.E., 2000)

13. Show that $\partial A_p / \partial x^q$ is not a tensor even though A_p is a covariant tensor of rank one. (Bhopal, 2003)

14. If a tensor A^{pqr}_{sym} is a skew-symmetric with respect to the indices p and q in one coordinate system, show that it remains skew-symmetric with respect to p and q in any coordinate system.

38.8 ADDITION OF TENSORS

The sum (or difference) of two tensors of the same order and type is another tensor of the same order and type.

Let A_{ij} and B_{ij} be two tensors of the same order and same type. Their components in the coordinates system $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ are \bar{A}_{ij} and \bar{B}_{ij} , such that

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl} \quad \text{and} \quad \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} B_{kl}$$

$$\therefore \bar{A}_{ij} \pm \bar{B}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} (A_{kl} \pm B_{kl}) \quad \text{i.e.,} \quad \bar{C}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} C_{kl}$$

Thus C_{ij} transforms in exactly the same manner as A_{ij} and B_{ij} and is, therefore, a tensor of the same order and same type.

38.9 OUTER PRODUCT OF TWO TENSORS

If A^i_j is a contravariant tensor of order two and B_{kl} is a covariant tensor of order two then their product is a mixed tensor C^i_{kl} of order four such that

$$C^i_{kl} = \bar{A}^i_j \bar{B}_{kl} = \left(\frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} A^{pq} \right) \left(\frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} B_{rs} \right) = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^{pq} B_{rs} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} C_{rs}^{pq}$$

But this is the law of transformation of a mixed tensor of order four. Therefore, C^i_{kl} is a mixed tensor of order four. Such products are called *outer products of two tensors*.

38.10 CONTRACTION OF A TENSOR

Consider a mixed tensor A_l^{ijk} of order four. By the law of transformation, we have

$$\bar{A}_l^{ijk} = \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^l} A_s^{pqr}$$

In this, put the covariant index $l = a$ contravariant index i , so that

$$\begin{aligned}\bar{A}_l^{ijk} &= \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial x^l} A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial x^p} A_s^{pqr} \\ &= \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} \delta_p^s A_s^{pqr} = \frac{\partial \bar{x}^j}{\partial x^q} \frac{\partial \bar{x}^k}{\partial x^r} A_p^{pqr}\end{aligned}$$

This shows that A_l^{ijk} is a contravariant tensor of order two.

The process of getting a tensor of lower order (reduced by 2) by putting a covariant index equal to a contravariant index and performing the summation indicated is known as **contraction**.

The tensors A_l^{ijk} and A_j^{ijk} obtained from contraction of the same tensor A_l^{ijk} are generally different from each other unless the tensor A_l^{ijk} is symmetric with respect to i and j (i.e., $A_j^{ijk} = A_i^{ijk}$).

38.11 INNER PRODUCT OF TWO TENSORS

Given the tensors A_k^i and B_{qr}^p , if we first form their outer product $A_k^i B_{qr}^p$ and contract this by putting $p = k$, then the result is $A_k^i B_{qr}^k$ which is also a tensor, called the *inner product of the given tensors*.

Hence the inner product of two tensors is obtained by first taking their outer product and then by contracting it. We can get several inner products for the same two tensors by contracting in different ways.

Example 38.10. Show that any inner product of the tensors A_r^p and B_t^{qs} is a tensor of rank three.

Solution. The transformation laws for A_r^p and B_t^{qs} are

$$\bar{A}_r^p = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^r} A_k^i \quad \dots(i) \quad \text{and} \quad \bar{B}_{t}^{qs} = \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial x^s}{\partial \bar{x}^t} \frac{\partial x^m}{\partial \bar{x}^l} B_{l m}^{jl} \quad \dots(ii)$$

∴ Inner product of \bar{A}_q^p and \bar{B}_t^{qs} is

$$\begin{aligned}\bar{A}_q^p \bar{B}_t^{qs} &= \left(\frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^q} \right) \left(\frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial x^s}{\partial \bar{x}^t} \frac{\partial x^m}{\partial \bar{x}^l} \right) A_k^i B_{l m}^{jl} \\ &= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^s}{\partial \bar{x}^t} \frac{\partial x^m}{\partial \bar{x}^l} \delta_j^k A_k^i B_{l m}^{jl} \\ &= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^s}{\partial \bar{x}^t} \frac{\partial x^m}{\partial \bar{x}^l} A_j^i B_{l m}^{jl} \quad \left[\because \frac{\partial x^k}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^j} = \frac{\partial x^k}{\partial x^j} = \delta_j^k \right]\end{aligned}$$

Hence the inner product of \bar{A}_q^p and \bar{B}_t^{qs} is a tensor of rank 3.

Similarly putting $p = t$ in the product of (i) and (ii) and noting that

$$\frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^m}{\partial \bar{x}^p} = \frac{\partial x^m}{\partial x^i} = \delta_i^m,$$

$A_r^p B_p^{qs}$ is found to be a tensor of rank 3.

Similarly, $A_r^p B_t^{qr}$ can also be shown to be a tensor of rank 3.

38.12 QUOTIENT LAW

To ascertain that a set of given functions forms the components of a tensor, we have to verify if the functions obey the tensor transformation laws. But this is a very tedious job. A simple test is provided by the quotient law which states that if the inner product of a set of functions with an arbitrary tensor is a tensor, then these set of functions are the components of a tensor.

The proof of this law is given below for a particular case.

Example 38.11. Show that the expression $A(i, j, k)$ is a tensor if its inner product with an arbitrary tensor B_k^l is a tensor.

Solution. Let $A(i, j, k) B_k^l = C_i^l$... (i)

where C_i^l is a tensor. In the coordinate system \bar{x}^i , let (i) transform to

$$\bar{A}(p, q, r) \bar{B}_r^{qs} = \bar{C}_p^s \quad \dots (ii)$$

where \bar{B}_r^{qs} and \bar{C}_p^s are the components of the tensors B_k^l and C_i^l . Expressing B_r^{qs} in terms of \bar{B}_k^l and \bar{C}_p^s in terms of \bar{C}_i^l , (ii) takes the form

$$\bar{A}(p, q, r) \frac{\partial \bar{x}^q}{\partial x^i} \frac{\partial \bar{x}^s}{\partial \bar{x}^l} \frac{\partial x^k}{\partial \bar{x}^r} B_k^l = \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^p} C_i^l \quad \dots (iii)$$

Multiplying (i) by $\frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^p}$ and subtracting from (iii), we get

$$\left\{ \bar{A}(p, q, r) \frac{\partial \bar{x}^q}{\partial x^i} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^k}{\partial \bar{x}^r} - A(i, j, k) \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^p} \right\} B_k^l = 0$$

Now B_k^l being an arbitrary tensor, the quantity within the brackets must be identically zero, i.e.,

$$\bar{A}(p, q, r) \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^k}{\partial \bar{x}^r} = A(i, j, k) \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial x^l}{\partial \bar{x}^p}$$

or $\bar{A}(p, q, r) = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k} = A(i, j, k)$

But this is the law of tensor transformation. Hence $A(i, j, k)$ is a tensor of order three, with i, j as covariant indices and k as contravariant index.

PROBLEMS 38.2

- Prove that if a tensor equation is true for one coordinate system, it is true for all coordinate systems.
- Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric.
- If A_{pq}^r and B_{pq}^r are tensors, prove that their sum and differences are also tensors.
- Show that A_{ij} is a tensor if its inner product with an arbitrary mixed tensor B_j^k is a tensor.
- Prove that (a) the contraction of the tensor A_p^q is an invariant.
(b) the contraction of the outer product of the tensors A_p^q and B_q^r is also an invariant.
- Let A_{rst}^{pq} be a tensor; choose $p = t$ and $q = s$ and show that A_{rs}^{pq} is also a tensor. What is its rank?

38.13 (1) RIEMANNIAN SPACE

The distance ds between two adjacent points whose rectangular Cartesian coordinates are (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by $ds^2 = dx^2 + dy^2 + dz^2$.

Riemann extended the concept of distance to a space of n dimensions and defined the distance ds between two adjacent points x^i and $x^i + dx^i$ ($i = 1, 2, \dots, n$) by the relation

$$ds^2 = a_{11}(dx^1)^2 + a_{22}(dx^2)^2 + \dots + a_{nn}(dx^n)^2 + a_{12}dx^1dx^2 + \dots + a_{lm}dx^l dx^m + \dots = a_{ij}dx^i dx^j, \text{ using summation convention.} \quad \dots (1)$$

The coefficients a_{ij} are the functions of the coordinates x^i . The quadratic form (1) is called a Riemannian metric and any space in which the distance is given by a such a metric is called a *Riemannian space*.*

If in a particular coordinate system X^i , the quadratic form (1) reduces to the form

$$ds^2 = (dX^1)^2 + (dX^2)^2 + \dots + (dX^n)^2,$$

then it is called a Euclidean metric and the corresponding space is called the *Euclidean space*.

* See footnote on p. 673

Obs. The geometry based on the Riemannian metric is called the *Riemannian geometry* and that based on the Euclidean metric is called the *Euclidean geometry*.

(2) Metric tensor. As in the physical space, the distance ds in the n -dimensional space is assumed to be independent of the coordinate system, i.e. a scalar invariant or a tensor of order zero. In the relation (1), dx^i and dx^j being displacements are components of a contravariant vector or a tensor of order one. Therefore, their outer product $dx^i dx^j$ is a contravariant tensor of order two. By the quotient law, the functions a_{ij} must be components of a covariant tensor of order two.

Let us write $a_{ij} = g_{ij} + h_{ij}$ where $g_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $h_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$.

Interchanging i and j , we have $g_{ji} = \frac{1}{2}(a_{ji} + a_{ij}) = g_{ij}$ and $h_{ji} = \frac{1}{2}(a_{ji} - a_{ij}) = -h_{ij}$

$\therefore g_{ij}$ is symmetric and h_{ij} is skew-symmetric. Thus (1) take the form

$$ds^2 = a_{ij} dx^i dx^j = (g_{ij} + h_{ij}) dx^i dx^j$$

Now $h_{ij} dx^i dx^j$ is zero, since h_{ij} is skew-symmetric. Hence $ds^2 = g_{ij} dx^i dx^j$ where g_{ij} is a covariant symmetric tensor of order two. It is called the *metric tensor* or the *first fundamental tensor*.

38.14 CONJUGATE TENSOR

Let g be the determinant $|g_{ij}|$ and G_{ij} denote the cofactors of g_{ij} in g . Define the function of g^{ij} by the relation $g^{ij} = G_{ij}/g$... (1)

Since the functions g_{ij} and G_{ij} are symmetric in the subscripts, the functions g^{ij} will be symmetric in the superscripts. Now

$$\begin{aligned} g_{ij} g^{ij} &= g_{ij} \frac{G_{ij}}{g} = \frac{g}{g} = 1 \quad \text{and} \quad g_{ij} g^{lj} = g_{ij} \frac{G_{lj}}{g} \\ &= 1, \text{ if } l = i, = 0, \text{ if } l \neq i \end{aligned}$$

Thus

$$g_{ij} g^{lj} = \delta_i^l \quad \dots (2)$$

If w^j be an arbitrary contravariant tensor, then its inner product with the tensor g_{ij} will be an arbitrary covariant tensor due to contraction, i.e.,

$$\begin{aligned} g_{ij} w^j &= v_i \\ \therefore g^{ij} v_i &= g^{ij} g_{lj} w^j = w^j \end{aligned} \quad \dots (3)$$

which is a contravariant tensor of order one. Therefore by quotient law, g^{ij} are the components of a contravariant tensor of order two. Hence g^{ij} is a symmetric contravariant tensor which is called the *conjugate tensor* or the *second fundamental tensor*.

Obs. In view of (2), the relation between g_{ij} and g^{ij} is reciprocal. As such the *first and second fundamental tensors are also called reciprocal tensors*.

Example 38.12. Find the components of the first and second fundamental tensors in spherical coordinates.

Solution. Let (x^1, x^2, x^3) be the rectangular cartesian coordinates and $\bar{x}^1, \bar{x}^2, \bar{x}^3$ be the spherical coordinates of a point so that

$$x^1 = r \sin \theta \cos \phi, x^2 = r \sin \theta \sin \phi, x^3 = r \cos \theta \quad \dots (i)$$

and

We know that $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$... (ii)

Let g_{pq} and \bar{g}_{ij} be the metric tensors in cartesian and spherical coordinates respectively.

Then $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq} dx^p dx^q$

$\therefore g_{11} = 1 = g_{22} = g_{33}$, and $g_{12} = 0 = g_{13} = g_{23}$ etc. ... (iii)

On transformation

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i} \frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i} \frac{\partial x^3}{\partial \bar{x}^j} g_{33} \quad \dots (iv)$$

Putting $i = j = 1$ in (iv), we have

$$\begin{aligned}\bar{g}_{11} &= \left(\frac{\partial x^1}{\partial \bar{x}^1} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1} \right)^2 g_{33} \\ &= \left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 \quad [\text{By (i) and (iii)}] \\ &= (\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2 = \sin^2 \theta + \cos^2 \theta = 1 \quad [\text{By (ii)}]\end{aligned}$$

Putting $i = j = 2$ in (iv), we have

$$\begin{aligned}\bar{g}_{22} &= \left(\frac{\partial x^1}{\partial \bar{x}^2} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2} \right)^2 g_{33} \\ &= \left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 \quad [\text{By (i) and (iii)}] \\ &= (r \cos \theta \cos \phi)^2 + (r \cos \theta \sin \phi)^2 + (-r \sin \theta)^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2\end{aligned}$$

Similarly $\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3} \right)^2 g_{33} = r^2 \sin^2 \theta$

and

$$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the first fundamental tensor, written in matrix form, is

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{vmatrix} = r^4 \sin^2 \theta$$

and the cofactors in g are given by

$$G_{11} = r^4 \sin^2 \theta, G_{22} = r^2 \sin^2 \theta, G_{33} = r^2; G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the second fundamental tensor are given by $g^{ij} = G_{ij}/g$. Hence the second fundamental

tensor in matrix form, is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2 \theta \end{vmatrix}$.

Example 38.13. Find the components of the metric tensor and the conjugate tensor in cylindrical coordinates.

Solution. Let (x^1, x^2, x^3) be the cartesian coordinates and $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ be the cylindrical coordinates of a point so that

$$x^1 = x, x^2 = y, x^3 = z \quad \text{and} \quad \bar{x}^1 = \rho, \bar{x}^2 = \phi, \bar{x}^3 = z \quad \dots(i)$$

We know that

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z \quad \dots(ii)$$

Let g_{pq} and \bar{g}_{ij} be the metric tensors in cartesian and cylindrical coordinates respectively.

Then $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{pq} dx^p$

$$\therefore g_{11} = 1 = g_{22} = g_{33} \quad \text{and} \quad g_{12} = 0 = g_{13} = g_{23} \text{ etc.} \quad \dots(iii)$$

On transformation,

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} g_{pq} = \frac{\partial x^1}{\partial \bar{x}^i} \frac{\partial x^1}{\partial \bar{x}^j} g_{11} + \frac{\partial x^2}{\partial \bar{x}^i} \frac{\partial x^2}{\partial \bar{x}^j} g_{22} + \frac{\partial x^3}{\partial \bar{x}^i} \frac{\partial x^3}{\partial \bar{x}^j} g_{33} \quad \dots(iv)$$

Putting $i = j = 1$ in (iv), we have

$$\begin{aligned}\bar{g}_{11} &= \left(\frac{\partial x^1}{\partial \bar{x}^1} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1} \right)^2 g_{33} = \left(\frac{\partial x}{\partial \rho} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2 \\ &= \cos^2 \phi + \sin^2 \phi + 0 = 1\end{aligned}\quad [\text{By (i) and (ii)}]$$

Putting $i = j = 2$ in (iv), we have

$$\begin{aligned}\bar{g}_{22} &= \left(\frac{\partial x^1}{\partial \bar{x}^2} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^2} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^2} \right)^2 g_{33} = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 \\ &= (-\rho \sin \phi)^2 + (\rho \cos \phi)^2 + 0 = \rho^2\end{aligned}\quad [\text{By (i) and (ii)}]$$

Similarly $\bar{g}_{33} = \left(\frac{\partial x^1}{\partial \bar{x}^3} \right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^3} \right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^3} \right)^2 g_{33} = 0 + 0 + 1 = 1$

and

$$\bar{g}_{12} = 0 = \bar{g}_{13} = \bar{g}_{21} = \bar{g}_{23} = \bar{g}_{31} = \bar{g}_{32}$$

Hence the metric tensor, written in matrix form, is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \therefore g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho^2$$

Also cofactors in g are given by

$$G_{11} = \rho^2, G_{22} = 1, G_{33} = \rho^2; G_{12} = 0 = G_{13} = G_{21} = G_{23} = G_{31} = G_{32}$$

The components of the conjugate tensor are given by $g^{ij} = G_{ij}/g$.

Hence the conjugate tensor in matrix form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

38.15 ASSOCIATED TENSORS

From (3) of § 38.14, we have $u^j \cdot g_{ij} = v_i$... (1)

i.e., the inner product of the tensor u^j with the fundamental tensor g_{ij} is another tensor v_i which is called the associated tensor of u^j .

Similarly, we have $v_l \cdot g^{lj} = u^j$... (2)

Hence u^j is the associated tensor of v_l .

Thus the indices of any tensor can be lowered or raised by forming its inner product with either of the fundamental tensors g_{ij} or g^{ij} as in (1) or (2) above.

38.16 (1) LENGTH OF A VECTOR

The vector \mathbf{A} is given by

$$\mathbf{A} = A^i g_i \quad \text{or} \quad \mathbf{A} = A_i g^i \quad \dots (1)$$

Also we have the associated vectors $A_i = g_{ij} A^j$... (2)

$$A^i = g^{ij} A_j \quad \dots (3)$$

\therefore Length of vector $\mathbf{A} = (\mathbf{A} \cdot \mathbf{A})^{1/2} = (A^i g_i \cdot A_j g^j)^{1/2}$ [By (1)]

$$= (g_{ij} A^i A^j)^{1/2} \quad [\because g_i \cdot g_k = g_{ij}]$$

$$= (A_i A^i)^{1/2} \quad [\text{By (2)}]$$

Also length of vector $\mathbf{A} = (\mathbf{A} \cdot \mathbf{A})^{1/2} = (A_i g^i \cdot A_j g^j)^{1/2} = (g^{ij} A_i A_j)^{1/2} = (A_i A^i)^{1/2}$ [By (3)]

Hence the magnitude or length of the vector $\mathbf{A} = \sqrt{(g_{ij} A^i A^j)} = \sqrt{(g^{ij} A_i A_j)} = \sqrt{(A_i A^i)}$... (4)

which is an invariant.

Obs. The length of a vector $A^1, 0, 0$ (in 3-dimensions) is $\sqrt{(g_{11}A^1A^1)}$, i.e. $\sqrt{g_{11}A^1}$. Similarly the length of the vector $0, A^2, 0$ is $\sqrt{g_{22}A^2}$ and the length of the vector $0, 0, A^3$ is $\sqrt{g_{33}A^3}$. Hence the physical components of a vector A^i are $\sqrt{g_{11}A^1}, \sqrt{g_{22}A^2}, \sqrt{g_{33}A^3}$.

(2) Angle between two vectors. Let \mathbf{A} and \mathbf{B} be the given vectors such that

$$\begin{aligned}\mathbf{A} &= A^i g_i \text{ and } \mathbf{B} = B^j g_j \\ \therefore \mathbf{A} \cdot \mathbf{B} &= |\mathbf{A}| |\mathbf{B}| \cos \theta\end{aligned}\quad \dots(5)$$

or

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \frac{A^i B^j g_{ij}}{\sqrt{(g_{ij} A^i A^j)} \sqrt{(g_{ij} B^i B^j)}} \quad [\text{Using (4) and (5)}]$$

In terms of associated vectors, we have

$$\cos \theta = \frac{A^i B_i}{\sqrt{(A^i A_i)} \sqrt{(B^i B_i)}}.$$

PROBLEMS 38.3

- If $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(x^3)^2 - 6dx^1 dx^2 + 4dx^2 dx^3$, find the values of g_{ij} and g^{ij} .
- Find g and g^{ij} corresponding to the metric $ds^2 = \frac{dr^2}{1-r^2/a^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$.
- The contravariant components of a vector \mathbf{A} in plane cartesian coordinates $x = x^1, y = x^2$ are (33, 56). Using the tensor law of transformation, obtain the new components in polar coordinates $r = \bar{x}^1$ and $\theta = \bar{x}^2$.
- Prove that the angles θ_{12}, θ_{23} and θ_{31} between the coordinate curves in a 3-dimensional coordinate system are given by $\cos \theta_{12} = \frac{g_{12}}{\sqrt{(g_{11}g_{22})}}, \cos \theta_{23} = \frac{g_{23}}{\sqrt{(g_{22}g_{33})}}, \cos \theta_{31} = \frac{g_{31}}{\sqrt{(g_{33}g_{11})}}$.
- Prove that for an orthogonal coordinate system
 - $g_{12} = g_{23} = g_{31} = 0$
 - $g^{11} = 1/g_{11}, g^{22} = 1/g_{22}, g^{33} = 1/g_{33}$.

38.17 CHRISTOFFEL SYMBOLS

Christoffel symbol of the first kind is denoted by $[ij, k]$ and is defined by

$$[ij, k] = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \quad \dots(1)$$

where g_{ij} are the components of the metric tensor.

Christoffel symbol of the second kind is denoted by $\{k\}_{ij}$ and is defined by

$$\{k\}_{ij} = g^{kl} [ij, l] \quad \dots(2)$$

Some authors write Christoffel symbol of the second kind as $\{ij, k\}$ or Γ_{ij}^k .

Obs. 1. No summation is indicated in the Christoffel symbol of the first kind, but summation is to be made over l in the Christoffel symbol of the second kind.

Obs. 2. It is evident from (1) and (2) that the Christoffel symbols of both kinds are symmetric in the indices i and j .

$$[ij, k] = [ji, k] \text{ and } \{k\}_{ij} = \{k\}_{ji}.$$

Example 38.14. If $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$, find the values of

- $[22, 1]$ and $[13, 3]$
- $\{1\}_{22}$ and $\{3\}_{13}$.

Solution. It is a 3-dimensional space in spherical coordinates such that

$$x^1 = r, x^2 = \theta \text{ and } x^3 = \phi$$

Clearly

$$g_{11} = 1, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta \text{ and } g_{ij} = 0 \text{ for } i \neq j. \quad \dots(i)$$

Also $g^{11} = 1, g^{22} = 1/r^2, g^{33} = 1/r^2 \sin^2 \theta$

(See Ex. 38.12) $\dots(ii)$

(a) Christoffel symbols of the first kind are given by

$$[ij, k] = \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right]_{i, j, k=1, 2, 3} \quad \dots(iii)$$

Taking $i = 2, j = 2$, and $k = 1$ in (iii), we get

$$[22, 1] = \frac{1}{2} \left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right] = \frac{1}{2} \left[\frac{\partial(0)}{\partial \theta} + \frac{\partial(0)}{\partial \theta} - \frac{\partial(r^2)}{\partial r} \right] = -r \quad \dots(iv)$$

Putting $i = 1, j = 3$ and $k = 3$ in (iii), we obtain

$$[13, 3] = \frac{1}{2} \left[\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3} \right] = \frac{1}{2} \left[\frac{\partial(r^2 \sin^2 \theta)}{\partial r} + \frac{\partial(0)}{\partial \phi} - \frac{\partial(0)}{\partial \phi} \right] = r \sin^2 \theta \quad \dots(v)$$

(b) Christoffel symbols of the second kind are defined by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl} [ij, l] = g^{k1} [ij, 1] + g^{k2} [ij, 2] + g^{k3} [ij, 3] \quad \dots(vi)$$

$$\begin{aligned} \therefore \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= g^{11}[22, 1] + g^{12}[22, 2] + g^{13}[22, 3] = [22, 1] + 0[22, 2] + 0[22, 3] \\ &= -r \end{aligned} \quad [\text{By (iv)}]$$

$$\begin{aligned} \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} &= g^{31}[13, 1] + g^{32}[13, 2] + g^{33}[13, 3] \\ &= 0[13, 1] + 0[13, 2] + \frac{1}{r^2 \sin^2 \theta} [13, 3] \end{aligned} \quad [\text{By (ii)}]$$

$$= \frac{1}{r^2 \sin^2 \theta} \cdot r \sin^2 \theta = \frac{1}{r}. \quad [\text{By (v)}]$$

Example 38.15. Prove that

$$(a) \frac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i] \quad (b) \frac{\partial g^{ij}}{\partial x^k} = -g^{jl} \begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{im} \begin{Bmatrix} j \\ mk \end{Bmatrix}.$$

Solution. (a) By definition of Christoffel symbol of the first kind, we have

$$[ik, j] = \frac{1}{2} \left[\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right] \quad \dots(i)$$

$$\text{and} \quad [jk, i] = \frac{1}{2} \left[\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right] \quad \dots(ii)$$

Since g_{ij} is a symmetric tensor, $\therefore g_{ij} = g_{ji}, g_{jk} = g_{kj}, g_{ki} = g_{ik}$

Adding (i) and (ii), we get the required result.

(b) We know that $g^{ij}g^{lj} = \delta_l^i$.

[Refer to § 38.14 (2)]

Differentiating w.r.t. x^k , we get

$$g^{ij} \frac{\partial g_{lj}}{\partial x^k} + g_{lj} \frac{\partial g^{ij}}{\partial x^k} = 0 \quad [\because \delta_l^i = 1 \text{ or } 0]$$

Multiplying by g^{lm} and transposing, we have

$$g^{lm} g_{lj} \frac{\partial g^{ij}}{\partial x^k} = -g^{lm} g^{ij} \frac{\partial g_{lj}}{\partial x^k} \quad \text{or} \quad \delta_j^m \frac{\partial g^{ij}}{\partial x^k} = -g^{lm} g^{ij} ([lk, j] + [jk, l]) \quad [\text{From (a)}]$$

or

$$\frac{\partial g^{im}}{\partial x^k} = -g^{lm} \left(g^{ij} [lk, j] \right) - g^{ij} \left(g^{lm} [jk, l] \right) = -g^{lm} \begin{Bmatrix} i \\ lk \end{Bmatrix} - g^{ij} \begin{Bmatrix} m \\ jk \end{Bmatrix}$$

Interchanging m and j , we obtain the desired result.

38.18 TRANSFORMATION OF CHRISTOFFEL SYMBOLS

The fundamental tensors g_{ij} , g^{ij} and also $[ij, k]$ are functions of the coordinates x^i . Let these transform to \bar{g}_{ij} , \bar{g}^{ij} and $[\bar{i}, \bar{j}, \bar{k}]$ in another coordinate system \bar{x}^i .

(1) *Law of transformation of Christoffel symbol of first kind.*

Let $[ij, k]$ which is a function of x^i , transform to $[\bar{i}, \bar{j}, \bar{k}]$ in another coordinate system \bar{x}^i . Then

$$[\bar{i}, \bar{j}, \bar{k}] = \frac{1}{2} \left(\frac{\partial \bar{g}_{jk}}{\partial \bar{x}^i} + \frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} - \frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} \right) \quad \dots(1)$$

Since \bar{g}_{ij} is a covariant tensor of order two, we have

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} g_{pq} \quad \dots(2)$$

Differentiating both sides w.r.t. \bar{x}^k , we get

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j} \right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial g_{pq}}{\partial x^r} \frac{\partial x^r}{\partial \bar{x}^k} \quad \dots(3)$$

[Note that g_{pq} is in terms of original coordinates x and to differentiate it w.r.t. \bar{x}^k , first we differentiate it w.r.t. x^r and then differentiate x^r w.r.t. \bar{x}^k .]

Interchanging i, k and also p, r in the last term of (3), we have

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^i} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^k} \frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial x^p}{\partial \bar{x}^k} \frac{\partial^2 x^q}{\partial \bar{x}^i \partial \bar{x}^j} \right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial g_{qr}}{\partial x^p} \quad \dots(4)$$

Similarly interchanging j, k and also q, r in the last term of (3), we get

$$\frac{\partial \bar{g}_{ik}}{\partial \bar{x}^j} = \left(\frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \right) g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial g_{pr}}{\partial x^q} \quad \dots(5)$$

Substituting the values from (3), (4) and (5) in (1), we obtain

$$[\bar{i}, \bar{j}, \bar{k}] = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} [pq, r] \quad \dots(6)$$

This is the desired law of transformation of Christoffel symbol of the first kind.

(2) *Law of transformation of Christoffel symbol of the second kind.*

Let $g^{kl} [ij, l]$ transform to $\bar{g}^{kl} [\bar{i}, \bar{j}, \bar{l}]$.

Since \bar{g}^{kl} is a contravariant tensor of order two.

$$\therefore \bar{g}^{kl} = \frac{\partial \bar{x}^k}{\partial x^s} \frac{\partial \bar{x}^l}{\partial x^t} g^{st} \quad \dots(7)$$

$$\text{From (6), we have } [\bar{i}, \bar{j}, \bar{l}] = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^l} g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^l} [pq, r] \quad \dots(8)$$

Multiplying the respective sides of (7) and (8), we get

$$\begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial \bar{x}^t}{\partial x^s} \delta_s^q \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} g^{st} g_{pq} + \frac{\partial \bar{x}^l}{\partial x^s} \delta_s^r \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} g^{st} [pq, r] \quad \dots(9)$$

Since $\delta_s^q g^{st} g_{pq} = g^{qt} g_{pq} = \delta_p^t$

and

$$\delta_s^r g^{st}[pq, r] = g^{rt}[pq, r] = \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^p}{\partial \bar{x}^i \partial \bar{x}^j} \delta_p^t \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix}$$

or

$$\begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} = \frac{\partial^2 x^t}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial \bar{x}^l}{\partial x^t} \begin{Bmatrix} t \\ pq \end{Bmatrix} \quad \dots(10)$$

This is the law of transformation of Christoffel symbol of the second kind.

Obs. 1. From (10), we obtain the following important relation :

$$\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial x^l}{\partial \bar{x}^i} \begin{Bmatrix} \bar{l} \\ ij \end{Bmatrix} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \begin{Bmatrix} t \\ pq \end{Bmatrix} \quad \dots(11)$$

Obs. 2. It is evident from (6) and (10) that the Christoffel 3-index symbols are not tensors. These symbols transform like tensors only for linear transformation of coordinates.

Example 38.16. Prove that $\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{\partial}{\partial x^i} (\log \sqrt{g})$.

Solution. Let G_{jk} be the co-factor of g_{ik} in g so that $g = g_{ik} G_{ik}$ (summation over k only)

$$\therefore \frac{\partial g}{\partial g_{ik}} = G_{ik} \quad [\because G_{ik} \text{ does not contain } g_{ik} \text{ implicitly}]$$

$$\text{Also } \frac{\partial g}{\partial g^j} = \frac{\partial g}{\partial g_{ik}} \frac{\partial g_{ik}}{\partial x^j} = G_{ik} \frac{\partial g_{ik}}{\partial x^j} \quad (\text{summation over } i \text{ and } k) \quad (i)$$

$$\text{We know that } g^{ik} = \frac{G_{ik}}{g} \quad \dots(ii)$$

Substituting the value of G_{ik} from (ii) in (i), we get

$$\frac{\partial g}{\partial x^j} = gg^{ik} \frac{\partial g_{ik}}{\partial x^j} \text{ or } \frac{1}{g} \frac{\partial g}{\partial x^j} = g^{ik} \frac{\partial g_{ik}}{\partial x^j}$$

$$\text{or } \frac{\partial}{\partial x^j} (\log g) = g^{ik} ([jk, i] + [ij, k]) \quad [\text{By Ex. 38.15 (a)}]$$

$$= g^{ik} [jk, i] + g^{ik} [ji, k] = \begin{Bmatrix} k \\ jk \end{Bmatrix} + \begin{Bmatrix} i \\ ji \end{Bmatrix} = 2 \begin{Bmatrix} i \\ ji \end{Bmatrix}$$

$$\text{Hence } \begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{1}{2} \frac{\partial}{\partial x^j} (\log g) = \frac{\partial}{\partial x^j} \log \sqrt{g}. \quad \dots(iii)$$

38.19 (1) COVARIANT DIFFERENTIATION OF A COVARIANT VECTOR

Let A_i and \bar{A}_i be the components of a covariant vector (*i.e.*, a tensor of first order) in the coordinate system x^i and \bar{x}^i respectively. Let us investigate the tensor character of the partial derivatives of A_i w.r.t. the variables x^j .

From the law of transformation, $\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p$

$$\text{Differentiating w.r.t. } \bar{x}^j, \text{ we have } \frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^i} A_p + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial A_p}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^j} \quad \dots(1)$$

(Note that A_p is not directly a function of \bar{x}^j).

Due to presence of the first term on the R.H.S. of (1), it is evident that $\frac{\partial A_p}{\partial x^q}$ is not a tensor.

On replacing this term by $\frac{\partial^2 x^s}{\partial \bar{x}^i \partial \bar{x}^j} A_s$ and substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial \bar{A}_l}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial \bar{x}^l} A_s \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} A_s \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} + \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A_p}{\partial \bar{x}^q}$$

or

$$\frac{\partial \bar{A}_l}{\partial \bar{x}^j} - \bar{A}_l \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\} = \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A_p}{\partial \bar{x}^q} - A_s \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \right]$$

This shows that the expression $\frac{\partial \bar{A}_l}{\partial \bar{x}^j} - \bar{A}_l \left\{ \begin{matrix} \bar{l} \\ i \ j \end{matrix} \right\}$

is a covariant tensor of the second order. This is called the covariant derivative of \bar{A}_l w.r.t. \bar{x}^j and is denoted by $\bar{A}_{l,j}$.

(2) Covariant differentiation of a contravariant vector. Let A^i and \bar{A}^i be the components of a contravariant vector in the coordinate systems x^i and \bar{x}^i . From the law of transformation $A^s = \frac{\partial x^s}{\partial \bar{x}^i} \bar{A}^i$.

Differentiating w.r.t. \bar{x}^j , we have $\frac{\partial A^s}{\partial \bar{x}^j} = \frac{\partial^2 x^s}{\partial \bar{x}^j \partial \bar{x}^i} \bar{A}^i + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$

Substituting for the second derivative from (11) of § 35.18, we get

$$\frac{\partial A^s}{\partial \bar{x}^q} \frac{\partial x^q}{\partial \bar{x}^j} = \frac{\partial x^s}{\partial \bar{x}^l} \left\{ \begin{matrix} l \\ i \ j \end{matrix} \right\} - \frac{\partial x^p}{\partial \bar{x}^i} \frac{\partial x^q}{\partial \bar{x}^j} \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \bar{A}^i + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Interchanging the dummy indices i, l in the first term on the R.H.S. and putting

$$\frac{\partial x^p}{\partial \bar{x}^i} \bar{A}^i = A^p \text{ in the second term, we obtain}$$

$$\frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial A^s}{\partial \bar{x}^q} = \frac{\partial x^s}{\partial \bar{x}^l} \left\{ \begin{matrix} \bar{l} \\ l \ j \end{matrix} \right\} \bar{A}^l - \frac{\partial x^q}{\partial \bar{x}^j} A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} + \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial \bar{A}^i}{\partial \bar{x}^j}$$

Transposing the second term on the R.H.S. to the L.H.S., we get

$$\frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A^s}{\partial \bar{x}^q} + A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \right] = \frac{\partial x^s}{\partial \bar{x}^i} \left[\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \left\{ \begin{matrix} \bar{l} \\ l \ j \end{matrix} \right\} \right]$$

or

$$\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \left\{ \begin{matrix} \bar{l} \\ l \ j \end{matrix} \right\} = \frac{\partial \bar{x}^i}{\partial x^s} \frac{\partial x^q}{\partial \bar{x}^j} \left[\frac{\partial A^s}{\partial \bar{x}^q} + A^p \left\{ \begin{matrix} s \\ p \ q \end{matrix} \right\} \right]$$

This shows that $\frac{\partial \bar{A}^i}{\partial \bar{x}^j} + \bar{A}^l \left\{ \begin{matrix} \bar{l} \\ l \ j \end{matrix} \right\}$ is a mixed tensor of the second order. This is called the *covariant derivative of A^i w.r.t. x^j* and is denoted by $A^i_{,j}$.

Obs. The following laws hold good for covariant differentiation :

(i) Covariant derivative of the sum (or difference) of two tensors = sum (or difference) of their covariant derivatives.

(ii) Covariant derivative of the product of two tensors = covariant derivative of first tensor \times second tensor + covariant derivative of second tensor \times first tensor.

Example 38.17. Prove that the covariant derivative of g^{ij} is zero.

Solution. Let A_i denote a covariant vector which moves parallel to itself so that

$$A_{i,k} \text{ or } \frac{\partial A_i}{\partial x^k} - A_l \left\{ \begin{matrix} l \\ i \ k \end{matrix} \right\} = 0 \quad \dots(i)$$

Let $\phi = g^{ij} A_i A_j$ so that ϕ is a scalar invariant. Differentiating it w.r.t. x^k , we have

$$\begin{aligned}\frac{\partial \phi}{\partial x^k} &= \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \frac{\partial A_i}{\partial x^k} A_j + g^{ij} A_i \frac{\partial A_j}{\partial x^k} \\ &= \frac{\partial g^{ij}}{\partial x^k} A_i A_j + g^{ij} \left\{ \begin{matrix} l \\ i \ k \end{matrix} \right\} A_l A_j + g^{ij} \left\{ \begin{matrix} l \\ j \ k \end{matrix} \right\} A_i A_l\end{aligned}\quad [\text{By (i)}]$$

Interchanging i and l in the second term and j and l in the last term on the right, we get

$$\frac{\partial \phi}{\partial x^k} = \left[\frac{\partial g^{ij}}{\partial x^k} + g^{lj} \left\{ \begin{matrix} i \\ l \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} \right] A_i A_j$$

Since $\partial \phi / \partial x^k$ is a covariant vector, the expression

$$\frac{\partial g^{ij}}{\partial x^k} + g^{lj} \left\{ \begin{matrix} i \\ l \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\}$$

is a tensor of the third order by quotient law. Thus it is the covariant derivative $g^{ij}_{,k}$.

$$\begin{aligned}\therefore g^{ij}_{,k} &= \frac{\partial g^{ij}}{\partial x^k} + g^{lj} \left\{ \begin{matrix} i \\ l \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} \\ &= -g^{jl} \left\{ \begin{matrix} i \\ l \ k \end{matrix} \right\} - g^{im} \left\{ \begin{matrix} j \\ m \ k \end{matrix} \right\} + g^{lj} \left\{ \begin{matrix} i \\ l \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} \\ &= -g^{im} \left\{ \begin{matrix} j \\ m \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} = -g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} + g^{il} \left\{ \begin{matrix} j \\ l \ k \end{matrix} \right\} = 0\end{aligned}\quad [\text{By Ex. 38.15 (b)}]$$

[Changing the dummy index m to l]

38.20 (1) GRADIENT

If ϕ be a scalar function of the coordinates, then the gradient of ϕ is denoted by $\text{grad } \phi = \frac{\partial \phi}{\partial x^i}$ which is a covariant vector.

(2) Divergence. The divergence of the contravariant vector A^i is defined by

$$\text{div } A^i = \frac{\partial A^i}{\partial x^i} + A^k \left\{ \begin{matrix} l \\ k \ i \end{matrix} \right\} \text{ which is sometimes written as } A^i_{,i}.$$

The divergence of the covariant vector A_i is defined by $\text{div } A_i = g^{ik} A_{ik}$.

(3) Curl. Let A_i be a covariant vector, then

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_k \left\{ \begin{matrix} k \\ i \ j \end{matrix} \right\} \text{ and } A_{j,i} = \frac{\partial A_j}{\partial x^i} - A_k \left\{ \begin{matrix} k \\ j \ i \end{matrix} \right\} \text{ are covariant tensors.}$$

$$\therefore A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

is a covariant tensor of second order, which is called curl of A_i .

$$\text{Thus } \text{curl } A_i = A_{i,j} - A_{j,i}.$$

Obs 1. $\text{curl } A_i$ is a skew-symmetric tensor.

$$\text{Since } A_{j,i} - A_{i,j} = -(A_{i,j} - A_{j,i}).$$

Obs. 2. Curl is a tensor and not a vector. In a 3-dimensional space, however, curl has only three independent non-zero components and it can, therefore, be taken as a vector.

Example 38.18. Prove that

$$(a) \text{div } A_i = \frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x^k} (\sqrt{g} A^k) \quad (b) \nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} g^{kr} \frac{\partial \phi}{\partial x^r} \right).$$

Solution. (a) Using

$$\begin{Bmatrix} i \\ k \ i \end{Bmatrix} = \frac{\partial}{\partial x^k} \log \sqrt{g} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k}$$

[By Ex. 38.16]

$$\begin{aligned} \operatorname{div} A^i &= \frac{\partial A^i}{\partial x^i} + A^k \begin{Bmatrix} i \\ k \ i \end{Bmatrix} = \frac{\partial A^i}{\partial x^i} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} = \frac{\partial A^i}{\partial x^i} + \frac{A^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k} \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g} A^k) \end{aligned} \quad \dots(i)$$

(ii) We have

$$\nabla^2 \phi = \operatorname{div} \operatorname{grad} \phi$$

... (ii)

and

$$\operatorname{grad} \phi = \frac{\partial \phi}{\partial x^r}, \text{ which is a covariant vector.}$$

The contravariant vector associated with $\partial \phi / \partial x^r$ is

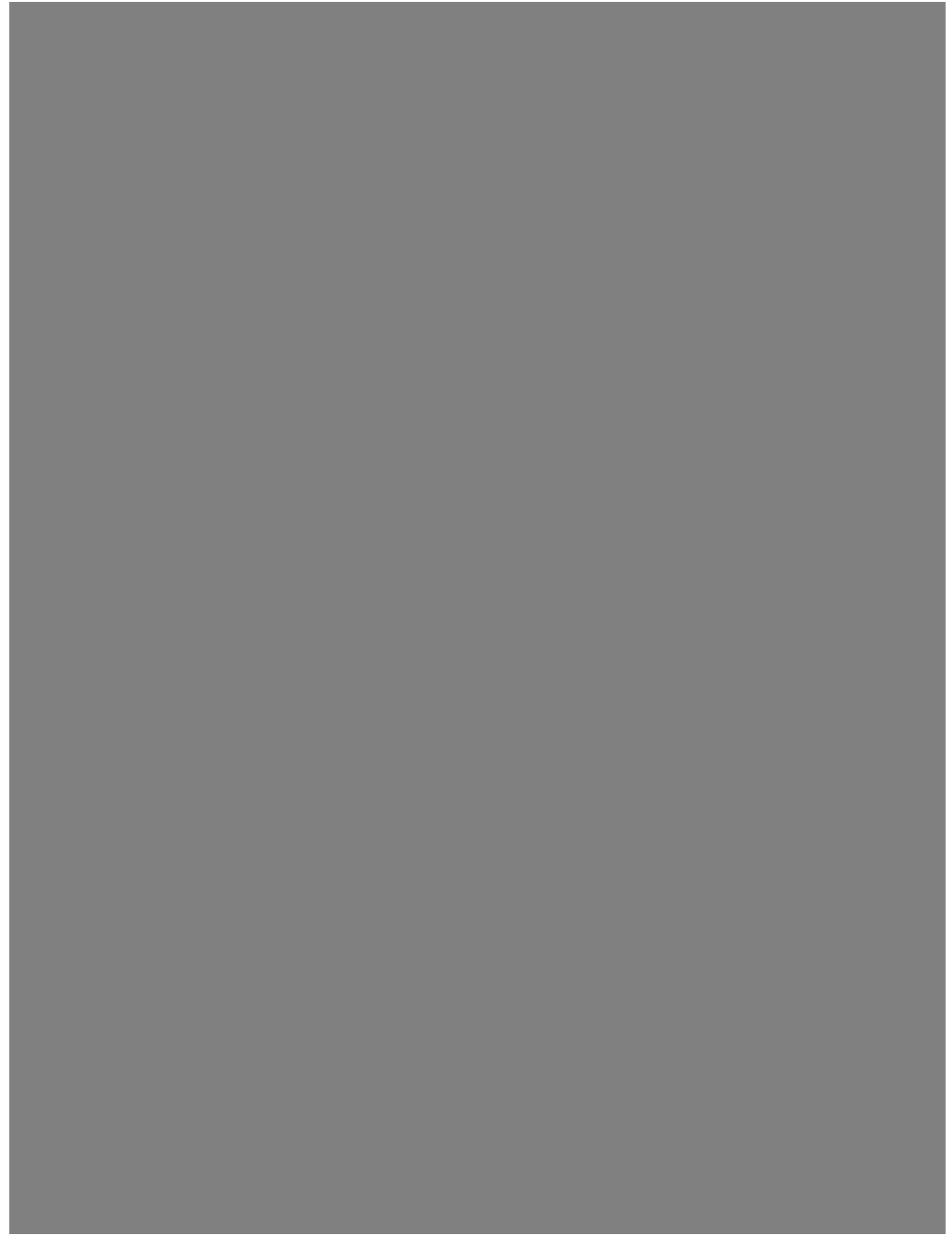
$$A^k = g^{kr} \frac{\partial \phi}{\partial x^r}.$$

$$\text{Then from (i) and (ii), } \nabla^2 \phi = \operatorname{div} \left(g^{kr} \frac{\partial \phi}{\partial x^r} \right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} g^{kr} \frac{\partial \phi}{\partial x^r} \right).$$

PROBLEMS 38.4

- Determine Christoffel symbols of the first and second kind for an orthogonal curvilinear coordinate system.
- Determine the Christoffel symbols of the first kind in (a) rectangular, (b) cylindrical and (c) spherical coordinates.
- Evaluate the Christoffel symbols of the second kind in (a) rectangular, (b) cylindrical and (c) spherical coordinates.
- Prove that (a) $[pq, r] = [qp, r]$, (b) $[pq, r] = g_{rs} \begin{Bmatrix} s \\ pq \end{Bmatrix}$
- If $(ds)^2 = r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$, find the values of
 - $[22, 1]$ and $[12, 2]$
 - $\begin{bmatrix} 1 \\ 22 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 12 \end{bmatrix}$.
- If $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$, find the values of
 - $[33, 1]$ and $[23, 3]$
 - $\begin{bmatrix} 1 \\ 33 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 23 \end{bmatrix}$.
- Show that the tensors g_{ij} , g^{ij} and δ^i_j are constants with respect to covariant differentiation.
- Write the covariant derivative w.r.t. x^k of the tensors u^{ij} and A^k_{ij} .
- Show that the covariant derivative of g_{ij} is zero.
- Find the covariant derivative $A^l_k B^{lm}_n$ with respect to x^q .
- Evaluate $\operatorname{div} A^i$ in (a) cylindrical, (b) spherical coordinates.
- Obtain the Laplace's equation in (a) cylindrical, (b) spherical coordinates.
- If $A_{ij,k}$ is the curl of a covariant vector, prove that

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0. \quad (\text{Madras M.E., 2000 S})$$
- Using tensor notation, show that
 - $\operatorname{div} \operatorname{curl} A^r = 0$
 - $\operatorname{curl} \operatorname{grad} \phi = 0$.



Useful Information

I. BASIC DATA

1. Basic Constants

$e = 2.7183$	$1/e = 0.3679$	$\log_e 2 = 0.6931$	$\log_e 3 = 1.0986$
$\pi = 3.1416$	$1/\pi = 0.3183$	$\log_e 10 = 2.3026$	$\log_{10} e = 0.4343$
$\sqrt{2} = 1.4142$	$\sqrt{3} = 1.732$	$1 \text{ rad.} = 57^\circ 17' 45''$	$1^\circ = 0.0174 \text{ rad.}$

2. Conversion Factors

1 ft. = 30.48 cm = 0.3048 m	1 m = 100 cm = 3.2804 ft.
1 ft ² = 0.0929 m ²	1 acre = 4840 yd ² = 4046.77 m ²
1 ft ³ = 0.0283 m ³	1 m ³ = 35.32 ft ³
1 m/sec = 3.2804 ft/sec.	1 mile/h = 1.609 km/h.

3. Systems of Units

Quantity	F.P.S. system	C.G.S. system	M.K.S. system
Length	foot (ft)	centimetre (cm)	metre (m)
Mass	pound (lb)	gram (gm)	kilogram (kg)
Time	second (sec)	second (sec)	second (sec)
Force	lb. wt.	dyne	newton (nt)

Note. The M.K.S. system is also known as the *International system of units (SI system)*.

4. Greek Letters Used

α	alpha	θ	theta	κ	kappa	τ	tau
β	beta	ϕ	phi	μ	mu	χ	chi
γ	gamma	ψ	psi	ν	nu	ω	omega
δ	delta	ξ	xi	π	pi	Γ	cap. gamma
ϵ	epsilon	η	eta	ρ	rho	Δ	cap. delta
ι	iota	ζ	zeta	σ	sigma	Σ	cap. sigma
		λ	lambda				

5. Some Notations

\in	belongs to	\cup	union
\notin	does not belong to	\cap	intersection
\Rightarrow	implies	\ni	such that
\Leftrightarrow	implies & implied by		

Factorial n i.e., $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$.

Double factorials : $(2n)!! = 2n(2n-2)(2n-4) \dots 6 \cdot 4 \cdot 2$.

$(2n-1)!! = (2n-1)(2n-3)(2n-5) \dots 5 \cdot 3 \cdot 1$.

Stirling's approximation. When n is large $n! \sim \sqrt{2\pi n} \cdot n^n e^{-n}$.

II. ALGEBRA

1. Quadratic equation : $ax^2 + bx + c = 0$ has roots

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}.$$

Roots are equal if $b^2 - 4ac = 0$

Roots are real and distinct if $b^2 - 4ac > 0$

Roots are imaginary if $b^2 - 4ac < 0$

2. Cubic equation : $x^3 + lx^2 + mx + n = 0$

Cardan's method :

(i) Remove x^2 term by putting $y = x - (-l/3)$

(ii) Equate coeffs. in the new equation and $y^3 - 3uvy - (u^3 + v^3) = 0$ $[\because y = u + v]$

(iii) Find u^3 and v^3 . Then find u and v .

(iv) Get $y = u + v$ and $x = y - l/3$.

3. Biquadratic equation : $x^4 + kx^3 + lx^2 + mx + n = 0$

I. *Ferrari's method :*

(i) Combine x^4 and x^3 terms into a perfect square by adding a term in λ .

(ii) Make R.H.S. a perfect square to find λ .

(iii) Solve resulting quadratic equations.

II. *Descarte's method :*

(i) Remove x^3 term by putting $y = x - (-k/4)$

(ii) Equate transformed expression to $(y^2 + py + q)(y^2 - py + q')$

(iii) Equate coeffs. of like powers from both sides.

(iv) Find p, q and q' and solve resulting quadratics.

4. Cross-multiplication : $a_1x + b_1y + c_1z = 0$

$$a_2x + b_2y + c_2z = 0$$

Then

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

5. Method of least squares :

(i) *Straight line of best fit* $y = a + bx$.

Normal equations : $\Sigma y = na + b\Sigma x, \Sigma xy = a\Sigma x + b\Sigma x^2$.

To find a, b, solve these equations.

(ii) *Parabola of best fit* $y = a + bx + cx^2$

Normal equations : $\Sigma y = na + b\Sigma x + c\Sigma x^2$,

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3, \Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

To find a, b, c, solve these equations.

6. Progressions :

(i) Numbers $a, a+d, a+2d, \dots$ are said to be in *Arithmetic progression (A.P.)*

Its n th term $T_n = a + \frac{n-1}{2}d$ and sum $S_n = \frac{n}{2}(2a + (n-1)d)$

(ii) Numbers a, ar, ar^2, \dots , are said to be in *Geometric progression (G.P.)*

Its n th term $T_n = ar^{n-1}$ and sum $S_n = \frac{a(1-r^n)}{1-r}, S_\infty = \frac{a}{1-r}$ ($r < 1$)

- (iii) Numbers $1/a, 1/(a+d), 1/(a+2d), \dots$ are said to be in *Harmonic progression (H.P.)* (i.e., a sequence is said to be in H.P. if its reciprocals are in A.P.)

Its n th term $T_n = 1/(a + (n-1)d)$

- (iv) If a and b be two numbers then their

$$\text{Arithmetic mean} = \frac{1}{2}(a+b), \text{ Geometric mean} = \sqrt{ab}, \text{ Harmonic mean} = 2ab/(a+b)$$

- (v) Natural numbers are 1, 2, 3, ..., n .

$$\Sigma n = \frac{n(n+1)}{2}, \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}, \Sigma n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2$$

7. Permutations and Combinations

$${}^nP_r = \frac{n!}{(n-r)!}; {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!}; {}^nC_{n-r} = {}^nC_r; {}^nC_0 = 1 = {}^nC_n.$$

8. Binomial theorem

- (i) When n is a positive integer

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n.$$

- (ii) When n is a negative integer or a fraction

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots \infty.$$

$$(iii) \text{ Binomial coefficients : } {}^nC_r = \frac{n!}{r!(n-r)!}$$

9. Logarithms

- (i) *Natural logarithm* $\log x$ has base e and is inverse of e^x .

Common logarithm $\log_{10} x = M \log x$ where $M = \log_{10} e = 0.4343$

- (ii) $\log_a 1 = 0; \log_a 0 = -\infty (a > 1); \log_a a = 1$.

- (iii) $\log(mn) = \log m + \log n; \log(m/n) = \log m - \log n; \log(m^n) = n \log m$.

10. Partial Fractions

A fraction of the form $\frac{a_0x^m + a_1x^{m-1} + \dots + a_m}{b_0x^n + b_1x^{n-1} + \dots + b_n}$

in which m and n are positive integers, is called a *rational algebraic fraction*. When the numerator is of a lower degree than the denominator, it is called a *proper fraction*.

To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. Then the proper fraction is resolved into a sum of partial fractions such that

- (i) to a non-repeated linear factor $x-a$ in the denominator corresponds a partial fraction of the form $A/(x-a)$;

- (ii) to a repeated linear factor $(x-a)^r$ in the denominator corresponds the sum of r partial fractions of the

$$\text{form } \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \dots + \frac{A_r}{(x-a)^r};$$

- (iii) to a non-repeated quadratic factor $(x^2 + ax + b)$ in the denominator, corresponds a partial fraction of the form $\frac{Ax+B}{x^2 + ax + b}$;

- (iv) to a repeated quadratic factor $(x^2 + ax + b)^r$ in the denominator, corresponds the sum of r partial fractions of the form $\frac{A_1x+B_1}{x^2 + ax + b} + \frac{A_2x+B_2}{(x^2 + ax + b)^2} + \dots + \frac{A_rx+B_r}{(x^2 + ax + b)^r}$.

Then we have to determine the unknown constants A, A_1, B_1 etc.

To obtain the partial fraction corresponding to the non-repeated linear factor $x-a$ in the denominator, put $x=a$ everywhere in the given fraction except in the factor $x-a$ itself.

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with (i) to (iv) above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of x or substitute convenient numerical values of x on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

11. Matrices

- (i) A system of mn numbers arranged in a rectangular array of m rows and n columns is called a **matrix** of order $m \times n$.
In particular if $m = n$, it is called a *square matrix* of order n .
- (ii) Two matrices of the same order can be added or subtracted by adding or subtracting the corresponding elements.
- (iii) Product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A .
- (iv) Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. If A is of order $m \times n$ and B is of order $n \times p$, then the product AB is a matrix of order $m \times p$, obtained by multiplying and adding the row elements of A with the corresponding column elements of B .
- (v) Transpose of a matrix A is the matrix obtained by interchanging its rows and columns and is denoted by A' .

A square matrix A is said to be *symmetric* if $A = A'$ and *skew symmetric* if $A = -A'$.

- (vi) If A and B are two square matrices such that $AB = I$ (i.e., a unit matrix), then B is called the *inverse* of A and is denoted by A^{-1} . Then $AA^{-1} = A^{-1}A = I$.
- (vii) *Rank of a matrix* is the largest order of any non-vanishing minor of the matrix.
- (viii) *Consistency of a system of equations in n unknowns*.

If the rank of the coefficient matrix A be r and that of the augmented matrix K be r' , then

(a) the equations are inconsistent (i.e. there is no solution) when $r \neq r'$,

(b) the equations are consistent when $r = r'$.

(c) the equations are consistent and there are infinite number of solutions when $r = r' < n$.

- (ix) *Eigen values*: If A is any square matrix of order n , then the determinant of the matrix $A - \lambda I_n$ equated to zero is called the *Characterist equation* of A and its roots are called the *eigen values* of A .
- (x) *Cayley Hamilton theorem*: Every square matrix satisfies its own characteristic equation.

12. Determinants

- (i) A **determinant** is defined for a square matrix A and is denoted by $| A |$. Unlike a matrix it has a single value e.g.,

$$\begin{aligned}| A | &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\end{aligned}$$

In this way, determinant can be expanded in terms of any row or column.

(ii) Properties :

- I. A determinant remains unaltered if its rows and columns are interchanged.
- II. A determinant vanishes if two of its rows (or columns) are identical or proportional.
- III. If each elements of a row (or column) consists of m terms, the determinant can be expressed as the sum of m determinants.
- IV. If to each elements of a row (or column) be added equi-multiples of the corresponding elements of two or more rows (or columns), the determinant remains unaltered.
- V. If the elements of a determinant Δ are functions of x and two parallel lines become identical when $x = a$, then $x - a$ is a factor of Δ .

III. GEOMETRY

1. Coordinates of a point : Cartesian (x, y) and polar (r, θ) .

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta \\ \text{or} \quad r &= \sqrt{(x^2 + y^2)}, & \theta &= \tan^{-1}(y/x). \text{ (Fig. 0.1).} \end{aligned}$$

Distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Point of division of the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$ is

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$$

In a triangle having vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3)

$$(i) \text{ Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(ii) Centroid (point of intersection of medians) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

(iii) Incentre (point of intersection of the internal bisectors of the angles) is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

where a, b, c are the lengths of the sides of the triangle.

(iv) Circumcentre is the point of intersection of the right bisectors of the sides of the triangle.

(v) Orthocentre is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides of the triangle.

2. Straight Line

$$(i) \text{ Slope of the line joining the points } (x_1, y_1) \text{ and } (x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the line $ax + by + c = 0$ is $-\frac{a}{b}$ i.e., $-\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Equation of a line

(a) having slope m and cutting an intercept c on y -axis is $y = mx + c$.

(b) cutting intercepts a and b from the axes is $\frac{x}{a} + \frac{y}{b} = 1$.

(c) passing through (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

(d) passing through (x_1, y_1) and making an $\angle \theta$ with the x -axis is $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$

(e) through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

$$(iii) \text{ Angle between two lines having slopes } m_1 \text{ and } m_2 \text{ is } \tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$$

Two lines are parallel if

$$m_1 = m_2$$

Two lines are perpendicular if

$$m_1 m_2 = -1$$

Any line parallel to the line

$$ax + by + c = 0 \text{ is } ax + by + k = 0$$

Any line perpendicular to

$$ax + by + c = 0 \text{ is } bx - ay + k = 0$$

$$(iv) \text{ Length of the perpendicular from } (x_1, y_1) \text{ to the line } ax + by + c = 0, \text{ is } \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

3. Circle

(i) Equation of the circle having centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

(ii) Equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle having centre $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

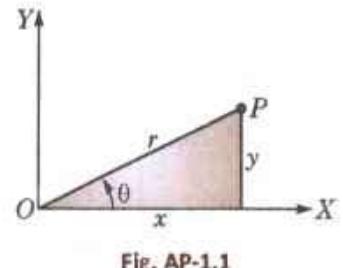


Fig. AP-1.1

- (iii) Equation of the tangent at the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
- (iv) Condition for the line $y = mx + c$ to touch the circle $x^2 + y^2 = a^2$ is $c = a\sqrt{1 + m^2}$.
- (v) Length of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

4. Parabola

- (i) Standard equation of the parabola $y^2 = 4ax$.

Its parametric equations are $x = at^2$, $y = 2at$.

Latus-rectum $LL' = 4a$, Focus is $S(a, 0)$

Directrix ZM is $x + a = 0$.

Focal distance of any point $P(x_1, y_1)$ on the parabola

$$y^2 = 4ax \text{ is } SP = x_1 + a$$

Equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

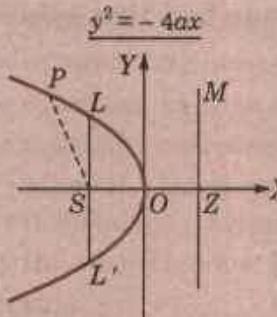
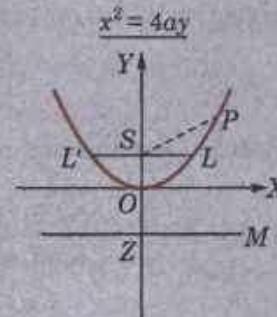
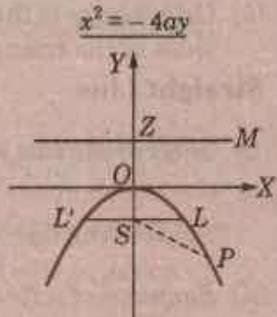
Condition for the line $y = mx + c$ to touch the parabola

$$y^2 = 4ax \text{ is } c = a/m.$$

Equation of the normal to the parabola $y^2 = 4ax$ in terms of its slope m is

$$y = mx - 2am - am^3.$$

- (ii) Other standard forms of parabola

$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
 <p>$y^2 = -4ax$</p> <p>Latus rectum : $LL' = 4a$ Focus : $S(-a, 0)$ Directrix ZM : $x = a$ Focal distance : $SP = a - x$</p>	 <p>$x^2 = 4ay$</p> <p>$LL' = 4a$ $S(0, a)$ $ZM : y = -a$ $SP = a + y$</p>	 <p>$x^2 = -4ay$</p> <p>$LL' = 4a$ $S(0, -a)$ $ZM : y = a$ $SP = a - y$</p>

5. Ellipse

- (i) Standard equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b > 0$).

Its parametric equations are $x = a \cos \theta$, $y = b \sin \theta$.

Eccentricity $e = \sqrt{1 - b^2/a^2}$,

Latus-rectum $LSL' = 2b^2/a$.

Foci $S(-ae, 0)$ and $S'(ae, 0)$

Directrices ZM ($x = -a/e$) and $Z'M'$ ($x = a/e$).

Sum of the focal distances of any point on the ellipse is equal to the major axis i.e.,

$$SP + S'P = 2a.$$

Equation of the tangent at the point (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

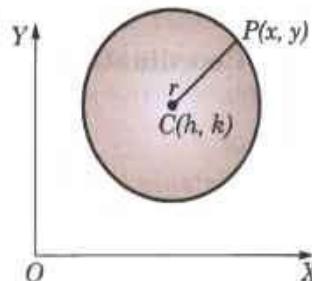


Fig. AP-1.2

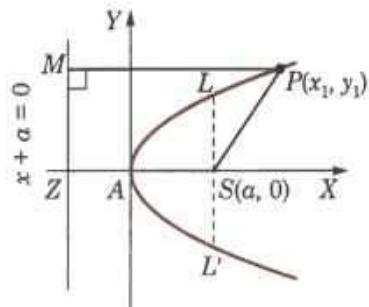


Fig. AP-1.3

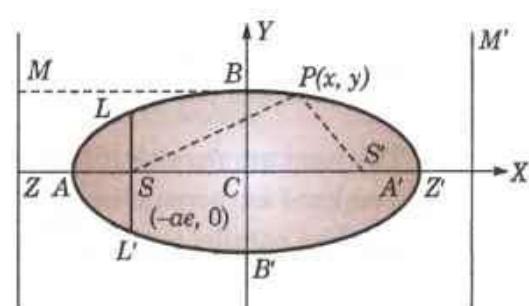


Fig. AP-1.4

Condition for the line $y = mx + c$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c = \sqrt{(a^2 m^2 + b^2)}$

(ii) Another standard form of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b > 0$)

Vertices: $A(0, a)$; $A'(0, -a)$

Foci: $S(0, ae)$; $S'(0, -ae)$

Directrices: $ZM: y = a/e$, $Z'M': y = -a/e$

Latus rectum: $LSL' = 2b^2/a$

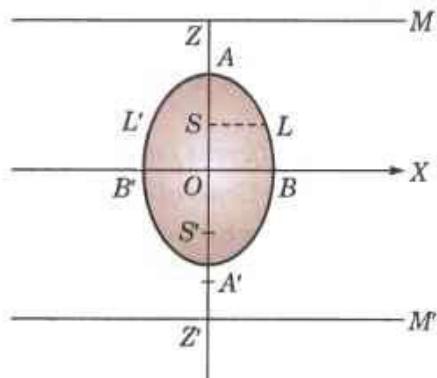


Fig. AP-1.4 (a)

6. Hyperbola

(i) Standard equations of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Its parametric equations are

$$x = a \sec \theta, \quad y = b \tan \theta.$$

Eccentricity $e = \sqrt{1 + b^2/a^2}$,

Latus-rectum $LSL' = 2b^2/a$.

Directrices

$$ZM (x = a/e) \text{ and } Z'M' (x = -a/e).$$

(ii) Equation of the tangent at the point (x_1, y_1) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(iii) Condition for the line $y = mx + c$ to touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } c = \sqrt{(a^2 m^2 - b^2)}$$

(iv) Asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

(v) Equation of the rectangular hyperbola with asymptotes as axes is $xy = c^2$.

Its parametric equations are $x = ct$, $y = c/t$.

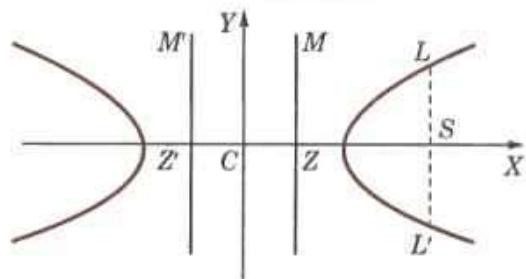


Fig. AP-1.5

7. Nature of a conic

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

(i) a pair of lines, if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} (= \Delta) = 0$

(ii) a circle, if $a = b$, $h = 0$, $\Delta \neq 0$

(iii) a parabola, if $ab - h^2 = 0$, $\Delta \neq 0$.

(iv) an ellipse, if $ab - h^2 > 0$, $\Delta \neq 0$.

(v) a hyperbola, if $ab - h^2 < 0$, $\Delta \neq 0$,

and a rectangular hyperbola if in addition, $a + b = 0$.

IV. SOLID GEOMETRY

1. (i) If l, m, n be the direction cosines of a line then $l^2 + m^2 + n^2 = 1$.

If a, b, c be the direction ratios of a line then $l = \frac{a}{\sqrt{\sum a^2}}$; $m = \frac{b}{\sqrt{\sum a^2}}$; $n = \frac{c}{\sqrt{\sum a^2}}$

(ii) If θ be the angle between the lines having d.c.'s l, m, n and l', m', n' , then

$$\cos \theta = ll' + mm' + nn'$$

Lines are perpendicular if, $ll' + mm' + nn' = 0$

Lines are parallel if $l = l'$, $m = m'$, $n = n'$

(iii) Projection of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line having d.c.'s $l, m, n = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

2. Plane

(i) Different forms of equation of a plane

— General form : $ax + by + cz = d$

where a, b, c are the d.r.s of a normal to the plane.

— Normal form : $lx + my + nz = p$

— Intercept form : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

— Any plane passing through the point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(ii) Angle θ between the planes $ax + by + cz = d$ and $a'x + b'y + c'z = d'$ is given by

$$\cos \theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)} \sqrt{(a'^2 + b'^2 + c'^2)}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$

Planes are parallel if $a/a' = b/b' = c/c'$

(iii) Any plane parallel to the plane $ax + by + cz = d$ is $ax + by + cz = k$.

3. Straight line

(i) Equation of the line through the point (x_1, y_1, z_1) having d.r.s a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad (\text{Symmetrical form})$$

(ii) Equation of the line through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (\text{Two point form})$$

(iii) Angle θ between the plane $ax + by + cz = d$ and the line

$$\frac{x - x_1}{a'} = \frac{y - y_1}{b'} = \frac{z - z_1}{c'} \\ \text{is } \sin \theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)} \sqrt{(a'^2 + b'^2 + c'^2)}}$$

Line is parallel to the plane if $aa' + bb' + cc' = 0$

Line is perpendicular to the plane if $a/a' = b/b' = c/c'$

(iv) Coplanar lines

Two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

and equation of the plane containing these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(v) Shortest distance between two skew lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

where l, m, n are given by $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

Equation of the line of S.D. is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

4. Sphere

(i) *Equation of the sphere having centre (a, b, c) and radius r is*

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

(ii) *Equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere having centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$*

(iii) *Equation of the sphere having the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$*

(iv) *Equation of a circle (i.e., section of a sphere $S = 0$ by the plane $U = 0$) is given by $S = 0$ and $U = 0$ taken together.*

(v) *Equation of any sphere through the circle of intersection of the sphere $S = 0$ and the plane $U = 0$ is $S + kU = 0$.*

(vi) *Tangent plane at any point (x_1, y_1, z_1) of the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

(vii) *Two spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and*

$$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \text{ cut orthogonally if } 2uu' + 2vv' + 2ww' = d + d'.$$

5. Cone

(i) *Equation of a cone with vertex at the origin is a homogeneous equation of the second degree in x, y, z .*

(ii) *Enveloping cone of the sphere $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$ where $S = x^2 + y^2 + z^2 - a^2$, $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$, $T = xx_1 + yy_1 + zz_1 - a^2$*

6. Quadric surfaces

(i) *Ellipsoid:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(ii) *Hyperboloid of one sheet:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Hyperboloid of two sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

(iii) *Cone:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

(iv) *Elliptic paraboloid:* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$

Hyperbolic paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$

7. Volumes and surface areas

Solid	Volume	Curved surface area	Total surface area
Cube (side a)	a^3	$4a^2$	$6a^2$
Cuboid (length l, breadth b, height h)	lbh	$2(l + b)h$	$2(lb + bh + hl)$
Sphere (radius r)	$\frac{4}{3}\pi r^3$	—	$4\pi r^2$
Cylinder (base radius r, height h)	$\pi r^2 h$	$2\pi rh$	$2\pi r(r + h)$
Cone (base radius r, height h)	$\frac{1}{3}\pi r^2 h$	$\pi r l$	$\pi r(r + l)$

where slant height l is given by $l = \sqrt{(r^2 + h^2)}$.

V. TRIGONOMETRY

1.

$\theta^\circ =$	0	30	45	60	90	180	270	360
$\sin \theta$	0	1/2	1/ $\sqrt{2}$	$\sqrt{3}/2$	1	0	-1	0
$\cos \theta$	1	$\sqrt{3}/2$	1/ $\sqrt{2}$	1/2	0	-1	0	1
$\tan \theta$	0	1/ $\sqrt{3}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

2. Signs and variations of t-ratios

Quadrant	$\sin \theta$	$\cos \theta$	$\tan \theta$
I	+	+	(0 to ∞)
II	+	-	($-\infty$ to 0)
III	-	-	(0 to ∞)
IV	-	+	($-\infty$ to 0)

3. Any t-ratio of $(n \cdot 90^\circ \pm \theta) = \pm$ same ratio of θ , when n is even. $= \pm$ co-ratio of θ , when n is odd.The sign + or - is to be decided from the quadrant in which $n \cdot 90^\circ \pm \theta$ lies.

e.g., $\sin 570^\circ = \sin (6 \times 90^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$,

$$\tan 315^\circ = \tan (3 \times 90^\circ + 45^\circ) = -\cot 45^\circ = -1$$

4. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\sin 2A = 2 \sin A \cos A = 2 \tan A / (1 + \tan^2 A)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}; \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

5. $\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$

$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$

$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

6. $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

$\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}$

7. $\sin 3A = 3 \sin A - 4 \sin^3 A$, $\cos 3A = 4 \cos^3 A - 3 \cos A$; $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

8. $a \sin x + b \cos x = r \sin(x + \theta)$
 $a \cos x + b \sin x = r \cos(x - \theta)$

where $a = r \cos \theta$, $b = r \sin \theta$ so that $r = \sqrt{(a^2 + b^2)}$, $\theta = \tan^{-1}(b/a)$

9. In any ΔABC :

(i) $a/\sin A = b/\sin B = c/\sin C$ (Sine formula)

(ii) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (Cosine formula)

(iii) $a = b \cos C + c \cos B$ (Projection formula)

(iv) Area of $\Delta ABC = \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

10. Series

(i) Exponential series : $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$

(ii) Sin, cos, sinh, cosh series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty, \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \infty$$

(iii) Log series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty, \quad \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right)$$

(iv) Gregory series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty, \quad \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty$$

11. (i) Complex number : $z = x + iy = r(\cos \theta + i \sin \theta) = r e^{i\theta}$ [see Fig. AP-1.1]

(ii) Euler's theorem : $\cos \theta + i \sin \theta = e^{i\theta}$

(iii) Demoivre's theorem : $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

12. (i) Hyperbolic functions : (i) $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$;

$$\tanh x = \frac{\sinh x}{\cosh x}; \coth x = \frac{\cosh x}{\sinh x}; \operatorname{sech} x = \frac{1}{\cosh x}; \operatorname{cosech} x = \frac{1}{\sinh x}$$

(ii) Relations between hyperbolic and circular functions :

$$\sin ix = i \sinh x; \cos ix = \cosh x; \tan ix = i \tanh x.$$

(iii) Inverse hyperbolic functions:

$$\sinh^{-1} x = \log[x + \sqrt{x^2 + 1}]; \cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]; \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

VI. DIFFERENTIAL CALCULUS

1. Standard limits :

(i) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$, n any rational number (ii) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$ (iv) $\lim_{x \rightarrow \infty} x^{1/x} = 1$

(v) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$

2. Differentiation

$$(i) \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \quad (\text{Chain Rule})$$

$$\frac{d}{dx}(ax + b)^n = n(ax + b)^{n-1} \cdot a$$

$$(ii) \frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

$$\frac{d}{dx}(\log_e x) = 1/x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$$

$$(iii) \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$(iv) \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$(v) \frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$(vi) D^n(ax + b)^m = m(m-1)(m-2) \dots (m-n+1)(ax + b)^{m-n} \cdot a^n$$

$$D^n \log(ax + b) = (-1)^{n-1} (n-1)! a^n / (ax + b)^n$$

$$D^n(e^{mx}) = m^n e^{mx}$$

$$D^n(a^{mx}) = m^n (\log a)^n \cdot a^{mx}$$

$$D^n \begin{bmatrix} \sin(ax+b) \\ \cos(ax+b) \end{bmatrix} = a^n \begin{bmatrix} \sin(ax+b+n\pi/2) \\ \cos(ax+b+n\pi/2) \end{bmatrix}$$

$$D^n e^{ax} \begin{bmatrix} \sin(bx+c) \\ \cos(bx+c) \end{bmatrix} = (a^2 + b^2)^{n/2} e^{ax} \begin{bmatrix} \sin(bx+c+n \tan^{-1} b/a) \\ \cos(bx+c+n \tan^{-1} b/a) \end{bmatrix}$$

$$(vii) \text{ Leibnitz theorem: } (uv)_n = u_n + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + {}^n C_r u_{n-r} v_r + \dots + {}^n C_n v_n$$

$$3. (i) \text{ Maclaurin's series: } f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$(ii) \text{ Taylor's series: } f(x+a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \dots$$

4. Curvature

$$(i) \text{ Radius of curvature } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}; \rho = r \frac{dr}{dp}.$$

$$(ii) \text{ Centre of curvature: } \bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \bar{y} = y + \frac{1}{y_2}(1+y_1^2).$$

(iii) **Evolute** is the locus of the centre of curvature of a curve. The curve is called the *involute* of the evolute.

(iv) **Envelope** of a curve $f(x, y, \alpha) = 0$ is the ' α ' eliminant from

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0.$$

The envelope of the normals to a curve is its **evolute**.

5. Asymptotes

(i) Asymptotes parallel to x -axis are obtained by equating to zero the coefficient of the highest power of x in the equation, provided this is not merely a constant.

Asymptotes parallel to y -axis are obtained by equating to zero the coefficient of highest power of y in the equation, provided this is not merely a constant.

(ii) Oblique asymptotes are obtained as follows:

Put $x = 1, y = m$ in the highest degrees terms getting $\phi_n(m)$

Put $\phi_n(m) = 0$ and find the values of m .

Find c from $c = -\phi_{n-1}(m)/\phi'_n(m)$

If two values of m are equal, then find c from

$$\frac{c^2}{2} \phi''_n(m) + c \phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

The asymptotes is $y = mx + c$.

(iii) Asymptotes of polar curve $1/r = f(\theta)$ is $r \sin(\theta - \alpha) = 1/f'(\alpha)$ where α is a root of $f(\theta) = 0$.

6. Curve tracing

(i) A curve is symmetrical about x -axis, if only even powers of y occur in its equation.

(ii) A curve is symmetrical about y -axis, if only even powers of x occur in its equation.

(iii) A curve is symmetrical about the line $y = x$, if on interchanging x and y , its equation remains unchanged.

(iv) A curve passes through the origin, if there is no constant term in its equation.

(v) Tangents to curve at the origin are found by equating to zero the lowest degree terms.

7. Partial Differentiation

(i) Euler's theorem. If u is a homogeneous function in x and y of degree n , then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu ; \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(ii) Chain rule : $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$, if $u = f(x, y), x = \phi(t), y = \psi(t)$.

(iii) $\frac{dy}{dx} = -\frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial y}$, if $\phi(x, y) = c$

(iv) Jacobian $J \left(\begin{matrix} u, v \\ x, y \end{matrix} \right) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$, then $JJ' = 1$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

(v) Taylor's series : $f(a+h, b+k) = f(a, b) + \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f + \frac{1}{2!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f + \dots$

(vi) Maxima Minima (a) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

(b) $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \frac{\partial^2 f}{\partial x \partial y}; \frac{\partial^2 f}{\partial x^2} < 0$ for maximum $\frac{\partial^2 f}{\partial x^2} > 0$ for minimum.

(vii) Leibnitz's Rule $\frac{d}{d\alpha} \left\{ \int_a^b f(x, \alpha) dx \right\} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx$

where $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions of x and α and a, b are constants.

VII. INTEGRAL CALCULUS

1. Integration

- (i) $\int x^n dx = \frac{x^{n+1}}{n+1}$ ($n \neq -1$) $\int \frac{1}{x} dx = \log_e x$
- $\int e^x dx = e^x$ $\int a^x dx = a^x / \log_e a$
- (ii) $\int \sin x dx = -\cos x$ $\int \cos x dx = \sin x$
- $\int \tan x dx = -\log \cos x$ $\int \cot x dx = \log \sin x$
- $\int \sec x dx = \log (\sec x + \tan x)$ $\int \operatorname{cosec} x dx = \log (\operatorname{cosec} x - \cot x)$
- $\int \sec^2 x dx = \tan x$ $\int \operatorname{cosec}^2 x dx = -\cot x$
- (iii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$ $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a}$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$ $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a}$
- (iv) $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$
- $\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$
- $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$
- (v) $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$
- (vi) $\int \sinh x dx = \cosh x$ $\int \cosh x dx = \sinh x$
- $\int \tanh x dx = \log \cosh x$ $\int \coth x dx = \log \sinh x$
- $\int \operatorname{sech}^2 x dx = \tanh x$ $\int \operatorname{cosech}^2 x dx = -\coth x$
- (vii) $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)\dots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even} \right)$
- $\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)\dots \times (n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} \times \left(\frac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even} \right)$
- (viii) $\int_0^a f(x) dx = \int_0^a f(a-x) dx$
- $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx,$ if $f(x)$ is an even function
 $= 0,$ if $f(x)$ is an odd function.
- $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx,$ if $f(2a-x) = f(x)$
 $= 0,$ if $f(2a-x) = -f(x).$

2. Lengths of curves

- (i) Length of curve $y = f(x)$ between $x = a, x = b$ is $\int_a^b \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx$
- (ii) Length of curve $x = f(y)$ between $y = a, y = b$ is $\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} dy$
- (iii) Length of curve $x = f(t), y = \phi(t)$ between $t = t_1, t = t_2$ is $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt$
- (iv) Length of curve $r = f(\theta)$ between $\theta = \alpha, \theta = \beta$ is $\int_{\alpha}^{\beta} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} d\theta$

3. Areas of curves

- (i) Area bounded by $y = f(x)$, x -axis and $x = a, x = b$ is $\int_a^b y dx$
- (ii) Area bounded by $x = f(y)$, y -axis and $y = a, y = b$ is $\int_a^b x dy$
- (iii) Area bounded by $r = f(\theta)$ and lines $\theta = \alpha, \theta = \beta$ is $\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

4. Volumes of revolution

- (i) Volume of revolution about x -axis of area bounded by $y = f(x)$, x -axis and $x = a, x = b$ is $\int_a^b \pi y^2 dx$
- (ii) Volume of revolution about y -axis of area bounded by $x = f(y)$, y -axis and $y = a, y = b$ is $\int_a^b \pi x^2 dy$
- (iii) Volume of revolution bounded by $r = f(\theta)$ and $\theta = \alpha, \theta = \beta$
 - (a) about $OX = \int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \sin \theta d\theta$
 - (b) about $OY = \int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \cos \theta d\theta$

5. Surface areas of revolution

- (i) Surface area of revolution about x -axis of curve $y = f(x)$ from $x = a$ to $x = b$ is $S = \int_{x=a}^{x=b} 2\pi y ds$
- Cartesian form : $S = \int_a^b 2\pi y \frac{ds}{dx} dx$ where $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$
- Parametric form : $S = \int 2\pi y \frac{ds}{dt} dt$ where $\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$
- Polar form : $S = \int 2\pi y \frac{ds}{d\theta} d\theta$ where $\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$

- (ii) Surface area of revolution about y -axis is $\int 2\pi x ds$.

6. Multiple integrals

- (i) Area = $\int_{x_1}^{x_2} \int_{y_1}^{y_2} dx dy$; Volume = $\int_{x_1}^{x_2} \int_{y_1}^{y_2} z dx dy$ or $\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx dy dz$
- (ii) C.G. of a plane lamina: $\bar{x} = \frac{\iint x \rho dx dy}{\iint \rho dx dy}$, $\bar{y} = \frac{\iint y \rho dx dy}{\iint \rho dx dy}$

$$C.G. \text{ of a solid } \bar{x} = \frac{\iiint_V x\rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}, \bar{y} = \frac{\iiint_V y\rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}, \bar{z} = \frac{\iiint_V z\rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}$$

$$(iii) \text{ Centre of pressure } \bar{x} = \frac{\iint_A px \, dx \, dy}{\iint_A p \, dx \, dy}, \bar{y} = \frac{\iint_A py \, dx \, dy}{\iint_A p \, dx \, dy}$$

$$(iv) M.I. \text{ about } x\text{-axis i.e., } I_x = \iiint_V \rho (y^2 + z^2) \, dx \, dy \, dz$$

$$M.I. \text{ about } y\text{-axis i.e., } I_y = \iiint_V \rho (z^2 + x^2) \, dx \, dy \, dz$$

$$M.I. \text{ about } z\text{-axis i.e., } I_z = \iiint_V \rho (x^2 + y^2) \, dx \, dy \, dz$$

$$7. \text{ Gamma function } \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx = (n-1)!, \Gamma(n+1) = n \Gamma(n) = n!, \Gamma(1/2) = \sqrt{\pi}$$

$$\text{Beta function } \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad (m > 0, n > 0)$$

VIII. VECTORS

1. (i) If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then $r = |\mathbf{R}| = \sqrt{(x^2 + y^2 + z^2)}$

(ii) \vec{PQ} = Position vector of Q – position vector of P .

2. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, then

(i) *Scalar product:* $\mathbf{A} \cdot \mathbf{B} = ab \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3$

(ii) *Vector product:* $\mathbf{A} \times \mathbf{B} = ab \sin \theta \hat{\mathbf{N}}$ = Area of the parallelogram having \mathbf{A} and \mathbf{B} as sides

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(iii) $\mathbf{B} \perp \mathbf{A}$ if $\mathbf{A} \cdot \mathbf{B} = 0$ and \mathbf{A} is parallel to \mathbf{B} if $\mathbf{A} \times \mathbf{B} = 0$

$$3. \text{ (i) Scalar triple product } [\mathbf{A} \mathbf{B} \mathbf{C}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{Volume of parallelopiped}$$

(ii) If $[\mathbf{A} \mathbf{B} \mathbf{C}] = 0$, then \mathbf{A} , \mathbf{B} , \mathbf{C} are coplanar.

$$\text{ (iii) Vector triple product } \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A}) \mathbf{B} - (\mathbf{C} \cdot \mathbf{B}) \mathbf{A}$$

$$4. \text{ (i) grad } f = \nabla f = \frac{\partial f}{\partial x} \mathbf{I} + \frac{\partial f}{\partial y} \mathbf{J} + \frac{\partial f}{\partial z} \mathbf{K}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} \quad \text{where } \mathbf{F} = f_1 \mathbf{I} + f_2 \mathbf{J} + f_3 \mathbf{K}$$

(ii) If $\text{div } \mathbf{F} = 0$, then \mathbf{F} is called a *solenoidal vector*.

(iii) If $\text{curl } \mathbf{F} = 0$ then \mathbf{F} is called an *irrotational vector*

5. *Velocity* = $d\mathbf{R}/dt$; *Acceleration* = $d^2\mathbf{R}/dt^2$; *Tangent vector* = $d\mathbf{R}/dt$; *Normal vector* = $\nabla \phi$

$$6. \text{ Green's theorem: } \int_C (\phi \, dx + \psi \, dy) = \iint_C \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx dy$$

Stoke's theorem: $\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{N} ds$

Gauss divergence theorem: $\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_V \operatorname{div} \mathbf{F} dv$

7. Coordinate systems

	Polar coordinates (r, θ)	Cylindrical coordinates (ρ, ϕ, z)	Spherical polar coordinates (r, θ, ϕ)
Coordinate transformations	$x = r \cos \theta$ $y = r \sin \theta$	$x = \rho \cos \phi$ $y = \rho \sin \phi$ $z = z$	$x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$
Jacobian	$\frac{\partial(x, y)}{\partial(r, \theta)} = r$	$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$	$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$
(Arc-length) ²	$(ds)^2 = (dr)^2 + r^2(d\theta)^2$ $dxdy = r d\theta dr$	$(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2$ $+ (dz)^2$	$(ds)^2 = (dr)^2 + r^2(d\theta)^2$ $+ (r \sin \theta)^2(d\phi)^2$
Volume-element		$dV = \rho d\rho d\phi dz$	$dV = r^2 \sin \theta dr d\theta d\phi$

IX. DIFFERENTIAL EQUATIONS

1. Equations of first order

(i) *Variables separable* : $f(y) dy/dx = \phi(x)$, $\int f(y) dy = \int \phi(x) dx + c$.

(ii) *Homogeneous equation* $dy/dx = f(x, y)/\phi(x, y)$ where $f(x, y)$ and $\phi(x, y)$ are of the same degree.

Put $y = vx$ so that $dy/dx = v + x dv/dx$.

(iii) *Equations reducible to homogenous form* : $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$

When $a/a' \neq b/b'$, put $x = X + h, y = Y + k$

When $a/a' = b/b'$, put $ax + by = t$.

(iv) *Leibnitz's linear equation* : $\frac{dy}{dx} + Py = Q$ where P, Q are functions of x .

I.F. = $e^{\int P dx}$, then solution is $y(\text{I.F.}) = \int Q(\text{I.F.}) dx + c$.

(v) *Bernoulli's equation* : $dy/dx + Py = Qy^n$, reducible to Leibnitz's equation by writing it as

$y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$ and putting $y^{1-n} = z$.

(vi) *Exact equation* : $M(x, y) dx + N(x, y) dy = 0$

Solution is $\int_{(y \text{ cons.})} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$, provided $\partial M/\partial y = \partial N/\partial x$.

(vii) *Clairaut's equation* : $y = px + f(p)$ where $p = dy/dx$.

Solution is obtained on replacing p by c .

2. Linear equations with constant coefficients

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

Symbolic form : $(D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = X$.

I. To find C.F.

Roots of A.E.	C.F.
(i) m_1, m_2, m_3, \dots	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots$
(ii) m_1, m_1, m_3, \dots	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots$
(iii) $\alpha + i\beta, \alpha - i\beta, m_3, \dots$	$e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots$
(iv) $\alpha \pm i\beta, \alpha \pm i\beta, m_5, \dots$	$e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \dots$

II. To find P.I.

$$(i) X = e^{ax}, \quad \text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \quad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \quad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \quad [f'(a) = 0, f''(a) \neq 0]$$

$$(ii) X = \sin(ax + b) \text{ or } \cos(ax + b)$$

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax + b) [\text{or } \cos(ax + b)], \text{ put } D^2 = -a^2, \quad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax + b) [\text{or } (\cos ax + b)], \text{ put } D^2 = -a^2, \quad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax + b) [\text{or } \cos(ax + b)], \text{ put } D^2 = -a^2, \quad [\phi'(-a^2) = 0, \phi''(-a^2) \neq 0]$$

$$(iii) X = x^m, \text{ P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m. \text{ Expand } [f(D)]^{-1} \text{ in ascending powers of } D \text{ as for as } D^m \text{ and}$$

operate on x^m term by term.

$$(iv) X = e^{ax} V, \text{ P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$$

III. Complete Solution : C.S. is $y = \text{C.F.} + \text{P.I.}$

$$3. \text{ Homogeneous linear equation : } x^3 \frac{d^3y}{dx^3} + k_1 x^2 \frac{d^2y}{dx^2} + k_2 x \frac{dy}{dx} + k_3 y = X$$

reduces to linear equation with constant coefficients by putting

$$x = e^t, x \frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y, x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

4. Lagrange's linear partial differential equation

$Pp + Qq = R, P, Q, R$ being functions of x, y, z .

To solve it (i) form the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) solve these equations giving $u = a, v = b$.

(iii) Complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

5. Homogeneous linear partial differential equations with constant coefficients

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Symbolic form : $(D^n + k_1 D^{n-1} D' + \dots + k_n D^n)z = F(x, y)$

To find C.F.

Roots of A.E.	C.F.
(i) m_1, m_2, m_3, \dots	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \dots$
(ii) m_1, m_1, m_3, \dots	$f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \dots$
(iii) m_1, m_1, m_1, \dots	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_3x) + \dots$

To find P.I.

(i) $F(x, y) = e^{ax+by}$, $P.I. = \frac{1}{f(D, D')} e^{ax+by}$, put $D = a$, $D' = b$.

(ii) $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos (mx + ny), \text{ put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2$$

(iii) $F(x, y) = x^m y^n$, $P.I. = [f(D, D')]^{-1} x^m y^n$. Expand $[f(D, D')]^{-1}$ and operate on $x^m y^n$.

(iv) $F(x, y)$ is any function of x and y , $P.I. = \frac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each

$$\text{partial fraction on } F(x, y) \text{ remembering that } \frac{1}{D - mD'} F(x, y) = \int F(x, c - mx) dx.$$

Complete solution: C.S. is $y = C.F. + P.I.$

X. INFINITE SERIES

- Basic test:** If $\lim_{n \rightarrow \infty} u_n \neq 0$ then the series $\sum u_n$ diverges.
- G.P. Series:** $1 + r + r^2 + r^3 + \dots \infty$ converge if $|r| < 1$; diverges if $r \geq 1$ and oscillates if $r \leq -1$.
- p-series:** $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots \infty$ converge for $p > 1$; diverges for $p \leq 1$.
- Comparison test:** If two positive term series $\sum u_n$ and $\sum v_n$ be such that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \text{finite quantity} (\neq 0)$, then $\sum u_n$ and $\sum v_n$ converge or diverge together.
- Ratio test:** In the positive term series $\sum u_n$, if $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
- Raabes' test:** In the positive term series $\sum u_n$, if $\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
- Logarithmic test:** In the positive term series $\sum u_n$, if $\lim_{n \rightarrow \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
- If u_n/u_{n+1} does not involve n as an exponent or a logarithm, then the series $\sum u_n$ diverges.
- Cauchy's root test:** In a positive term series $\sum u_n$ if $\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$, diverges for $\lambda > 1$ and fails for $\lambda = 1$.
- Integral test:** A positive term series $\sum f(n)$ converges or diverges according as $\int_1^\infty f(x) dx$ is finite or infinite where $f(n)$ is continuous in $1 < x < \infty$ and decreases as n increases.
- Leibnitz's test for alternating series:** An alternating series $u_1 - u_2 + u_3 - u_4 + \dots \infty$ converges if each term is numerically less than the previous term and $\lim_{n \rightarrow \infty} u_n = 0$.
if $\lim_{n \rightarrow \infty} u_n \neq 0$, then the given series is oscillatory.

12. *General Ratio test:* In an arbitrary term series $\sum u_n$ if $\lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = |k|$, then $\sum u_n$ is absolutely convergent if $|k| < 1$ and divergent if $|k| > 1$ and the test fails if $|k| = 1$.

XI. FOURIER SERIES

1. $f(x) = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots \text{ in } (0, 2\pi).$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx, a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx, b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$

2. $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ in any interval } (0, 2c),$

where $a_0 = \frac{1}{c} \int_0^{2c} f(x) dx, a_n = \frac{1}{c} \int_0^{2c} f(x) \cos \frac{n\pi x}{c} dx, b_n = \frac{1}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx$

3. For even function $f(x)$, Fourier expansion contains only cosine terms.

i.e., $a_0 = \frac{2}{c} \int_0^c f(x) dx, a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx, b_n = 0.$

For odd function $f(x)$, Fourier expansion contains only sine terms.

i.e., $a_0 = 0, a_n = 0, b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} dx.$

XII. TRANSFORMS

1. **Laplace Transforms.** $L(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$

(i) $L(1) = \frac{1}{s}$

(ii) $L(t^n) = \frac{n!}{s^{n+1}}$

(iii) $L(e^{at}) = \frac{1}{s-a}$

(iv) $L(\sin at) = \frac{a}{s^2 + a^2}$

(v) $L(\cos at) = \frac{s}{s^2 + a^2}$

(vi) $L(\sinh at) = \frac{a}{s^2 - a^2}$

(vii) $L(\cosh at) = \frac{s}{s^2 - a^2}$

(viii) $L(e^{at} f(t)) = F(s-a)$

(ix) $L f'(t) = sL f(t) - f(0)$

(x) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$

(xi) $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s) ds$

(xii) $u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t > a \end{cases}$

(xiii) $L[u(t-a)] = \frac{e^{-as}}{s}$

(xiv) $L\delta(t-a) = e^{-as}$

(xv) $L f(t) = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$ where $f(t)$ is a periodic function of period T .

2. Inverse Laplace Transforms

(i) $L^{-1}\left(\frac{1}{s}\right) = 1$

(ii) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

(iii) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(iv) $L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{1}{a} \sin at$

(v) $L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$

(vi) $L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \frac{1}{a} \sinh at$

$$(vii) L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at.$$

$$(viii) L^{-1}\left(\frac{s}{(s^2 + a^2)^2}\right) = \frac{1}{2a} t \sin at$$

$$(ix) L^{-1}\left[\frac{s^2 - a^2}{(s^2 + a^2)^2}\right] = t \cos at.$$

3. Fourier Transforms : $F(s) = \int_{-\infty}^{\infty} f(t) e^{ist} dt$

Fourier sine transform : $F_s(s) = \int_0^{\infty} f(t) \sin st dt$

Fourier cosine transform : $F_c(s) = \int_0^{\infty} f(t) \cos st dt$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F(u).$$

4. Z-Transforms : $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$(i) Z(1) = \frac{z}{z-1}$$

$$(ii) Z(n) = \frac{z}{(z-1)^2}$$

$$(iii) Z(n^2) = \frac{z^2 + z}{(z-1)^3}$$

$$(iv) Z(a^n) = \frac{z}{z-a}$$

$$(v) Z(na^n) = \frac{az}{(z-a)^2}$$

$$(vi) Z(\sin n\theta) = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$(vii) Z(\cos n\theta) = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$(viii) Z(\sinh n\theta) = \frac{z \sinh \theta}{z^2 - 2z \cosh \theta + 1}$$

$$(ix) Z(\cosh n\theta) = \frac{z(z - \cosh \theta)}{z^2 - 2z \cosh \theta + 1}$$

XIII. STATISTICS AND PROBABILITY

$$1. A.M. \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$2. S.D. \sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

$$3. Moments about the mean : \mu_0 = 1, \mu_1 = 0, \mu_2 = \sigma^2, \mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{\sum f_i}$$

4. Coeff. of skewness = (mean – mode)/σ which lies between – 1 and 1.

5. Kurtosis : $\beta_2 = \mu_4/\mu_2^2$.

$$6. \text{Coeff. of correlation } r = \frac{n \sum d_x d_y - \sum d_x \sum d_y}{\sqrt{[(n \sum d_x^2 - (\sum d_x)^2)(n \sum d_y^2 - (\sum d_y)^2)]}}; -1 < r < 1$$

$$7. \text{Line of regression of } y \text{ on } x : y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

$$\text{Line of regression of } x \text{ on } y : x - \bar{x} = \frac{r \sigma_x}{\sigma_y} (y - \bar{y})$$

$$8. \text{Probability } p(A) = \frac{\text{No. of ways favourable to } A}{\text{Total no. of equally likely ways}}, p + q = 1.$$

$$(i) p(A \text{ or } B) = p(A) + p(B), \quad (ii) p(A \text{ and } B) = p(A) \cdot p(B)$$

$$9. \text{Binomial distribution : } p(r) = {}^n C_r p^r q^{n-r}$$

$$\text{Mean} = np, \text{Variance} (\sigma^2) = npq$$

10. Poisson distribution : $p(r) = \frac{m^r}{r!} e^{-m}$

Mean = m , Variance (σ^2) = m .

11. Normal distribution : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, Standard variate = $\frac{x-\mu}{\sigma}$

(i) Probable error $\lambda = 0.6745 \sigma$.

(ii) 68% of values lie between $x = \mu - \sigma$ and $x = \mu + \sigma$.

95% of values lie between $x = \mu - 1.96 \sigma$ and $x = \mu + 1.96 \sigma$

99% of values lie between $x = \mu - 2.58 \sigma$ and $x = \mu + 2.58 \sigma$

XIV. NUMERICAL TECHNIQUES

1. Solution of equations

(i) Bisection method : $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

(ii) Method of False position : $x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$.

(iii) Newton-Raphson method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

(iv) Iterative formula to find $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

(v) Iterative formula to find \sqrt{N} is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

2. Solution of Linear Simultaneous equations

(i) Matrix inversion method. For the equations

$$a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$$

if $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

where A_1, B_1 , etc., are the co-factors of a_1, b_1 , etc., in the determinant $|A|$.

(ii) Gauss-elimination method. In this method the coefficient matrix is transformed to *upper triangular matrix*.

(iii) Gauss-Jordan method. In this method the coefficient matrix is transformed to *diagonal matrix*.

(iv) Gauss-Jordan method of finding the inverse of a matrix A . The matrices A and I are written side by side and the same row transformations are performed on both till A is reduced to I . Then the other matrix represents A^{-1} .

3. Finite differences and Interpolation

(i) Forward differences: $\Delta y_r = y_{r+1} - y_r$

Backward differences: $\nabla y_r = y_r - y_{r-1}$

Central differences: $\delta y_{n-1/2} = y_n - y_{n-1}$

(ii) Relations between operations :

$$\Delta = E - 1; \nabla = 1 - E^{-1}; \delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2}); \Delta = E\nabla = \nabla E = \delta E^{1/2}; E = e^{hD}$$

(iii) Factorial notation $|x|^r = x(x-1)(x-2)\dots(x-r+1)$.

Factorial polynomial $[x]^n = x(x-h)(x-2h)\dots(x-h-1h)$

(iv) *Newton's forward interpolation formula*

$$y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \text{ where } p = (x - x_0)/h.$$

(v) *Newton's backward interpolation formula:*

$$y_p = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots \text{ where } p = (x - x_n)/h.$$

(vi) *Stirling's formula:*

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

(vii) *Bessel's formula:*

$$\begin{aligned} y_p = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!} \Delta^3 y_{-1} \\ + \frac{(p+1)p(p-1)(p-2)}{4!} \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \dots \end{aligned}$$

(viii) *Lagrange's interpolation formula:*

$$\begin{aligned} y = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} y_1 + \dots + \\ \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} y_n \end{aligned}$$

(ix) *Newton's divided difference formula*

$$y = f(x) = y_0 + (x - x_0) [x_0, x_1] + (x - x_0)(x - x_1) [x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) [x_0, x_1, x_2, x_3] + \dots$$

$$\text{where } [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}, [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \text{ and so on.}$$

4. Numerical differentiation

(i) *Forward difference formulae:*

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right]$$

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \dots \right] \text{ and so on.}$$

(ii) *Backward difference formulae:*

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\nabla y_0 + \frac{1}{2} \nabla^2 y_0 + \frac{1}{3} \nabla^3 y_0 + \frac{1}{4} \nabla^4 y_0 + \dots \right]$$

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12} \nabla^4 y_0 + \dots \right] \text{ and so on.}$$

(iii) *Central difference formulae:*

$$\left(\frac{dy}{dx} \right)_{x_0} = \frac{1}{h} \left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6} \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30} \frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \dots \right]$$

$$\left(\frac{d^2y}{dx^2} \right)_{x_0} = \frac{1}{h^2} \left[\Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \frac{1}{90} \Delta^6 y_{-3} + \dots \right]$$

5. Numerical integration

(i) *Trapezoidal rule:*

$$\int_{x_0}^{x_0 + nh} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

(ii) *Simpson's 1/3 th rule:*

$$\int_{x_0}^{x_0 + nh} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

(Number of sub intervals should be taken as even)

(iii) *Simpson's 3/8 th rule:*

$$\int_{x_0}^{x_0 + nh} f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_5 + \dots + y_{n-2})]$$

(Number of sub-intervals should be taken as a multiple of 3)

(iv) *Weddle's rule:*

$$\int_{x_0}^{x_0 + nh} f(x) dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + \dots]$$

(Number of sub-intervals should be taken as multiple of 6)

6. Numerical solution of ordinary differential equations

(i) *Picard's method:* $y_1 = y_0 + \int_{x_0}^x (x, y_0) dx$

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx \text{ etc.}$$

(ii) *Taylor's method:*

$$y = y_0 + (x - x_0) (y)_0 + \frac{(x - x_0)^2}{2!} (y'')_0 + \frac{(x - x_0)^3}{3!} (y''')_0 + \dots$$

(iii) *Euler's method:* $y_2 = y_1 + h f(x_0 + h, y_1)$

Repeat this process till y_2 is stationary. Then calculate y_3 and so on.

(iv) *Modified Euler's method:* $y_2 = y_1 + \frac{1}{2} [f(x_0 + h, y_1) + f(x_0 + 2h, y_2)]$

Repeat this step till y_2 is stationary. Then calculate y_3 and so on.

(v) *Runge Kutta method:* $y_1 = y_0 + h$ where $h = \frac{1}{6}(k_1 + 2k_2, 2k_3 + k_4)$

such that $k_1 = h f(x_0, y_0)$; $k_2 = h f(x_0 + h/2, y_0 + k_1/2)$

$k_3 = h f(x_0 + h/2, y_0 + k_2/2)$; $k_4 = h f(x_0 + h, y_0 + k_3)$

(vi) *Milne's method*

Predictor formula: $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

Corrector formula: $y_4 = y_2 + \frac{h}{3}(f_2 - 4f_3 + f_4)$

(Four prior values are required to find the next values)

(vii) *Adams-Basforth method:*

Predictor formula: $y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

Corrector formula: $y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} - f_{-2})$

(Four prior values are required to find the next values)

7. Numerical solution of partial differential equations

(i) *Classification of a second order equations:*

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + \left(F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \right) = 0$$

is said to be

elliptic if $B^2 - 4AC < 0$

parabolic if $B^2 - 4AC = 0$

hyperbolic if $B^2 - 4AC > 0$

(ii) Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Standard 5-point formula: $u_{i,j} = \frac{1}{4} [u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}]$

Diagonal 5-point formula: $u_{i,j} = \frac{1}{4} [u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$

(iii) Poisson's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$.

Standard 5-point formula:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

(iv) One-dimensional heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Schmidt formula: $u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2d) u_{i,j} + \alpha u_{i+1,j}$ where $\alpha = kc^2/h^2$
when $\alpha = 1/2$, it reduces to

Bendre-Schmidt relation: $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

(v) Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Explicit formula for solution is

$$u_{i,j+1} = 2(1 - \alpha^2 c^2) u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \text{ where } \alpha = k/h$$

If α is so chosen that the coefficient of $u_{i,j}$ is zero i.e., $k = h/c$ then the above explicit formula takes the simplified form

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

Tables

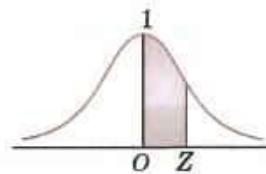
Table I : Gamma Function, $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$

α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$	α	$\Gamma(\alpha)$
1.00	1.000000	1.26	0.904397	1.52	0.887039	1.78	0.926227
1.01	0.994326	1.27	0.902503	1.53	0.887568	1.79	0.928767
1.02	0.988844	1.28	0.900719	1.54	0.888178	1.80	0.931384
1.03	0.983550	1.29	0.899042	1.55	0.888869	1.81	0.934076
1.04	0.978438	1.30	0.897471	1.56	0.889639	1.82	0.936845
1.05	0.973504	1.31	0.896004	1.57	0.890490	1.83	0.939690
1.06	0.968744	1.32	0.894640	1.58	0.891420	1.84	0.942612
1.07	0.964152	1.33	0.893378	1.59	0.892428	1.85	0.945611
1.08	0.959725	1.34	0.892215	1.60	0.893516	1.86	0.948687
1.09	0.955459	1.35	0.891151	1.61	0.894681	1.87	0.951840
1.10	0.951351	1.36	0.890184	1.62	0.895924	1.88	0.955071
1.11	0.947395	1.37	0.889313	1.63	0.897244	1.89	0.958380
1.12	0.943590	1.38	0.888537	1.64	0.898642	1.90	0.961766
1.13	0.939931	1.39	0.887854	1.65	0.900117	1.91	0.965231
1.14	0.936416	1.40	0.887264	1.66	0.901668	1.92	0.968774
1.15	0.933041	1.41	0.886764	1.67	0.903296	1.93	0.972397
1.16	0.929803	1.42	0.886356	1.68	0.905001	1.94	0.976099
1.17	0.926700	1.43	0.886036	1.69	0.906782	1.95	0.979881
1.18	0.923728	1.44	0.885805	1.70	0.908639	1.96	0.983742
1.19	0.920885	1.45	0.885661	1.71	0.910572	1.97	0.987685
1.20	0.918169	1.46	0.885604	1.72	0.912580	1.98	0.991708
1.21	0.915577	1.47	0.885633	1.73	0.914665	1.99	0.995813
1.22	0.913106	1.48	0.885747	1.74	0.916826	2.00	1.000000
1.23	0.910755	1.49	0.885945	1.75	0.919062		
1.24	0.908521	1.50	0.886227	1.76	0.921375		
1.25	0.906403	1.51	0.886592	1.77	0.923763		

Table II : Bessel Functions

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	-0.2601	0.3991
0.1	0.9975	0.0499	3.1	-0.2921	0.3009
0.2	0.9900	0.0995	3.2	-0.3202	0.2613
0.3	0.9776	0.1483	3.3	-0.3443	0.2207
0.4	0.9604	0.1960	3.4	-0.3643	0.1792
0.5	0.9385	0.2423	3.5	-0.3801	0.1374
0.6	0.9120	0.2867	3.6	-0.3918	0.0955
0.7	0.8812	0.3290	3.7	-0.3992	0.0538
0.8	0.8463	0.3688	3.8	-0.4026	0.0128
0.9	0.8075	0.4059	3.9	-0.4018	-0.0272
1.0	0.7652	0.4401	4.0	-0.3971	-0.0660
1.1	0.7196	0.4709	4.1	-0.3887	-0.1033
1.2	0.6711	0.4983	4.2	-0.3766	-0.1386
1.3	0.6201	0.5220	4.3	-0.3610	-0.1719
1.4	0.5669	0.5419	4.4	-0.3423	-0.2028
1.5	0.5118	0.5579	4.5	-0.3205	-0.2311
1.6	0.4554	0.5699	4.6	-0.2961	-0.2566
1.7	0.3980	0.5778	4.7	-0.2693	-0.2791
1.8	0.3400	0.5815	4.8	-0.2404	-0.2985
1.9	0.2818	0.5812	4.9	-0.2097	-0.3147
2.0	0.2239	0.5767	5.0	-0.1776	-0.3276
2.1	0.1666	0.5683	5.1	-0.1443	-0.3371
2.2	0.1104	0.5560	5.2	-0.1103	-0.3432
2.3	0.0555	0.5399	5.3	-0.0758	-0.3460
2.4	0.0025	0.5202	5.4	-0.0412	-0.3453
2.5	-0.0484	0.4971	5.5	-0.0068	-0.3414
2.6	-0.0968	0.4708	5.6	0.0270	-0.3343
2.7	-0.1424	0.4416	5.7	0.0599	-0.3241
2.8	-0.1850	0.4097	5.8	0.0917	-0.3110
2.9	-0.2243	0.3754	5.9	0.1220	-0.2951

Table III : Area under the Normal curve



x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993

Table IV : Values of $|t|$ with probability P and degrees of freedom v

$P \backslash v$	0.50	0.10	0.05	0.02	0.01
1	1.000	6.34	12.71	31.82	63.66
2	0.816	2.92	4.30	6.96	9.92
3	0.765	2.35	3.18	4.54	5.84
4	0.741	2.13	2.78	3.75	4.60
5	0.727	2.02	2.57	3.36	4.03
6	0.718	1.94	2.45	3.14	3.71
7	0.711	1.90	2.36	3.00	3.50
8	0.706	1.86	2.31	2.90	3.36
9	0.703	1.83	2.26	2.82	3.25
10	0.700	1.81	2.23	2.76	3.17
11	0.697	1.80	2.20	2.72	3.11
12	0.695	1.78	2.18	2.68	3.06
13	0.694	1.77	2.16	2.65	3.01
14	0.692	1.76	2.14	2.62	2.98
15	0.691	1.75	2.13	2.60	2.95
16	0.690	1.75	2.12	2.58	2.92
17	0.689	1.74	2.11	2.57	2.90
18	0.688	1.73	2.10	2.55	2.88
19	0.688	1.73	2.09	2.54	2.86
20	0.687	1.72	2.09	2.53	2.84
21	0.686	1.72	2.08	2.52	2.83
22	0.686	1.72	2.07	2.51	2.82
23	0.685	1.71	2.07	2.50	2.81
24	0.685	1.71	2.06	2.49	2.80
25	0.684	1.71	2.06	2.48	2.79
26	0.684	1.71	2.06	2.48	2.78
27	0.684	1.70	2.05	2.47	2.77
28	0.683	1.70	2.05	2.47	2.76
29	0.683	1.70	2.04	2.46	2.76
30	0.683	1.70	2.04	2.46	2.75

Table V : Values of χ^2 with probability P and df v

$v \backslash P$	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Table VI : 5% and 1% points of F

$v_1 \backslash v_2$	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60

Answers to Problems

Problems 1.1, page 5

1. $x^4 - 6x^3 + 3x^2 + 42x - 70 = 0$ 2. (i) $-2, 1 + 3i, 1 - 3i$ (ii) $2 \pm \sqrt{3}, 3, -5$
 5. Two roots between $(1, 2)$ and $(-3, -4)$
 6. $2, 2, -\frac{1}{3}$ 7. 3 8. $a = 2, b = 1$
 9. $6, 4, -1$ 10. $-4, 2, 6$ 11. $1, 1, 2, 2$
 12. $\frac{1}{2}(3 \pm \sqrt{5}); \frac{1}{2}(5 \pm \sqrt{5})$ 13. $1, 4, 7$ 14. $1, \frac{1}{2}, \frac{1}{4}$
 16. (i) $-5, -2, 1, 4$ (ii) $1, -2, 4, -8$ 17. (i) $m^2 - 2ln$, (ii) $lm - n$
 18. 36 19. (i) $4/3$, (ii) $16/9$.

Problems 1.2, page 8

1. $x^3 + 6x^2 - 36x + 27 = 0$ 2. $6x^5 - 7x^4 - 13x^3 + 4x^2 - 2 = 0$ 3. $10x^4 + 9x^3 + 8x^2 - 7x + 1 = 0$
 4. $-\frac{1}{2}, \frac{1}{3}, 2$ 5. (i) $x^3 - 9x^2 + 26x - 24 = 0$; (ii) $x^4 + 13x^3 + 60x^2 + 116x + 80 = 0$;
 (iii) $x^5 + 7 = 0$
 6. $x^3 + 15x^2 + 52x - 36 = 0$ 7. $y^3 + (p^3 + 3q)y^2 + 3q^2y + q^3 = 0$ 8. $3x^3 - 11x^2 + 9x - 2 = 0$
 9. $y^3 - qy^2 + pry - r^2 = 0$
 10. (a) $y^3 + 4my - 8n = 0$; (b) $nx^3 + m^2x^2 - 2mnx + n^2 = 0$; (c) $x(nx + m)^2 = n$
 11. $y^3 - 30y^2 + 225y - 68 = 0$
 12. (i) $\frac{-5 \pm \sqrt{21}}{2}, \frac{5 \pm \sqrt{91}i}{12}$ (ii) $2, 2, 1/2, 1/2$ (iii) $1, -2, 4, -1/2, 1/4$;
 (iv) $-1, -2, 3, -1/2, 1/3$; (v) $\pm 1, -3, -1/3, \frac{3 \pm \sqrt{5}}{2}$
 13. $\frac{1}{2}(5 \pm \sqrt{21})$; $\frac{1}{2}(-3 \pm \sqrt{5})$ 14. $-1, -2, -6, -7$.

Problems 1.3, page 11

1. $-6, 3, 3$ 2. $5, \frac{1}{2}(-5 \pm i\sqrt{3})$ 3. $6, -3 \pm 2\sqrt{(-3)}$
 4. $-1, -2, \frac{1}{2}$ 5. $\frac{1}{2}, -\frac{1}{6}(3 \pm i\sqrt{3})$ 6. $5, \frac{1}{2}(1 + i\sqrt{3})$
 7. $2 \cos \frac{2\pi}{9}, 2 \cos \frac{8\pi}{9}, 2 \cos \frac{14\pi}{9}$ 8. $\frac{1}{2}, \frac{-7 \pm 9i\sqrt{3}}{6}$.

Problems 1.4, page 12

- 1.** 1, 2, 3, 4
4. $-1, 3, 3 \pm \sqrt{30}$
7. $2 \pm \sqrt{3}, -2 \pm i\sqrt{3}$

- 2.** $-3, 1, \pm 2$
5. $1 \pm \sqrt{7}, 2 \pm \sqrt{3}$
8. $2, 4, 2 \pm 2i\sqrt{2}$.

- 3.** $4, -2, -1 \pm i$
6. $1 \pm 2i, -1 \pm \sqrt{2}$

Problems 1.5, page 15

- 1.** 1.32
4. (i) 0.71 rad

- 2.** 2.29
5. 1.81 rad

- 3.** 0.45
6. 0.26.

Problems 1.6, page 15

- | | | |
|--|--|---|
| 1. (d) | 2. (c) | 3. (c) |
| 4. (c) | 5. (c) | 6. (a) |
| 7. (d) | 8. (b) | 9. (c) |
| 10. (a) | 11. (c) | 12. minus |
| 13. § 15.1 (v) | 14. p/q | 15. 21 |
| 16. -3 and -2 | 17. Conjugate pairs | 18. $f(x)$ is continuous in (a, b) |
| 19. $x^3 - 9x^2 + 29x - 24 = 0$ | 20. $3, 6, -2$ | 21. $x^3 - 200x - 7000 = 0$ |
| 22. p/r | 23. $x^4 + 2x^3 - x^2 - 6x - 6 = 0$ | 24. 6 |
| 25. minus | 26. $pq = r$ | 27. $1, \frac{1}{2}(-1 \pm \sqrt{3}i)$ |
| 28. (iii) | 29. $1, 1, -2$ | 30. $x^3 - 7x^2 + 12x - 10 = 0$ |
| 31. Zero and 2 | 32. 21 | 33. True |
| 34. True. | | |

Problems 2.1 page 25

- | | |
|---|---|
| 5. (i) 1 (ii) 0 | 13. $(a-b)(b-c)(c-a)$ |
| 14. $(a-b)(b-c)(c-a)(ab+bc+ca)$ | 15. $(b-c)(c-a)(a-b)(a-1)(b-1)(c-1)$ |
| 16. $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ | 17. $x = 0, \pm \sqrt{(a^2 + b^2 + c^2 - ab - bc - ca)}$ |
| 18. $0, -\frac{1}{2}$ | |

Problems 2.2 page 31

- 1.** $x = 0, 3$

- 2.** $x = -3, y = -2, z = -4, a = 3$

3. $x = 3, y = 8$

4. $-2 \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$

5. $AB = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}$

7. (i) $[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$

(ii) $\begin{bmatrix} 8 & 7 \\ -122 & 104 \\ -365 & -131 \end{bmatrix}$ (iii) $\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}$

10. $3I$

11. $\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 327/19 \end{bmatrix}$

Problems 2.3, page 35

2. (i) $\begin{bmatrix} 3 & 0 & 5.5 \\ 0 & 7 & 1.5 \\ 5.5 & 1.5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -2.5 \\ -0.5 & 2.5 & 0 \end{bmatrix}$ (ii) $\begin{vmatrix} a & \frac{1}{2}(a+c) & \frac{1}{2}(b+c) \\ \frac{1}{2}(a+c) & b & \frac{1}{2}(a+b) \\ \frac{1}{2}(b+c) & \frac{1}{2}(a+b) & c \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{2}(a-c) & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-a) & 0 & \frac{1}{2}(b-a) \\ \frac{1}{2}(c-b) & \frac{1}{2}(a-b) & 0 \end{vmatrix}$

4. (i) $\begin{bmatrix} 2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1/33 & -4/33 & 2/11 \\ -4/33 & 14/33 & 13/33 \\ 2/11 & 13/33 & -1/33 \end{bmatrix}$

5. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. (i) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

Problems 2.4, page 40

1. 3 2. 2 3. 3 4. 2 5. 3
6. No value of p is possible.

7. (i) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (ii) $\frac{1}{21} \begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & 14 \\ -3 & 12 & 0 \end{bmatrix}$
(iii) $\begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

8. $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$, Rank (A) = 3

9. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ 10. (i) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;
(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. (i) 3 (ii) 3 (iii) 2 (iv) 3.

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Problems 2.5, Page 43

1. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ 2. $\frac{1}{8} \begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$ 4. $\begin{bmatrix} 7 & -3 & 0 & -5 \\ 8 & 1 & -2 & -11 \\ -5 & 0 & 1 & 6 \\ 19 & 5 & -6 & -28 \end{bmatrix}$

Problems 2.6, page 45

1. $x = 1, y = 2, z = 1$ 2. $x = 2, y = -1, z = 1/2$
3. $x = 1.2, y = 2.2, z = 3.2$ 4. $x = y = z = e^2$

5. $u = 1, v = 1/2, w = 1/3$
 7. $x = 2, y = 1, z = 0$
 9. $x = y = z = 2$
 11. $x = 1, y = -1, z = 1$
 13. $i_1 = 369/175, i_2 = 24/25, i_3 = 72/175$
6. $x_1 = 1, x_2 = -5, x_3 = 5$
 8. $x = 1/7, y = 10/7, z = 1/7$
 10. $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$
 12. $i_1 = 1.5, i_3 = 2.5$
 14. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5.$

Problems 2.7, page 50

- Consistent; $x = 1, y = 3k - 2, z = k$ for all k
- $k = 1, x = -3z, y = 2z + 1$; $k = 2, x = 1 - 3z, y = 2z$
- (i) $\lambda = 3, \mu \neq 10$; (ii) $\lambda \neq 3$; (iii) $\lambda = 3, \mu = 10$
- (i) Equations are inconsistent; (ii) consistent; $x = -1, y = 1, z = 2$; (iii) Equations are inconsistent; (iv) Consistent: $x = 2, y = 1, z = -4$
- If $a = -1, b = 6$, equations will be consistent and have infinite number of solutions
 If $a = -1, b \neq 6$, equations will be inconsistent;
 If $a \neq -1, b$ has any value, equations will be consistent and have a unique solution
- $\lambda \neq -5, x = 4/7, y = -9/7, z = 0$; $\lambda = -5, x = \frac{1}{7}(4 - 5k), y = \frac{1}{7}(13k - 9), z = k$ for all k
- $\lambda = -1, 1, 12$; $x = -1/11, y = -15/11$; $x = -5, y = 1$; $x = \frac{1}{2}, y = 1$
- $k = 3$ is the only real value for which $x = y = z$
- $\lambda = 1, x_1 = 2t - s, x_2 = t, x_3 = s$; $\lambda = -3, x_1 = -t, x_2 = -2t, x_3 = t$
- $\lambda = 1, -9$. For $\lambda = 1$, sol. is $x = k, y = -k, z = 2k$
 For $\lambda = -9$, sol. is $x = 3k, y = 9k, z = -2k$
- (i) Have infinite number of non-trivial solutions; $x = \lambda - 5\mu/3, y = \lambda - 4\mu/3, z = \lambda, w = \mu$ for all values of λ and μ . (ii) $x = 11k_2 + 6k_1, y = -8k_2 - 3k_1, z = k_2, w = k_1$ where k_1, k_2 are arbitrary constants.

Problems 2.8, page 54

- $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
- $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, A^{-1} = A' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$
- $Z = (BA)X$, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$
- $x_1 = 19y_1 - 9y_2 + 2y_3$; $x_2 = -4y_1 + 2y_2 - y_3$; $x_3 = -2y_1 + y_2$
- $a = \pm \frac{1}{\sqrt{2}}, b = \pm \frac{1}{\sqrt{6}}, c = \pm \frac{1}{\sqrt{3}}$
- (i) No. (ii) No. (iii) Yes, $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$.

Problems 2.9, page 60

- 10; 30
- (a) 10, 3; (1, 2), (3, -1); (b) -1, 6; (1, 1), (2, -5)
- (a) 0, 3, 5; (1, 2, 2), (2, 1, -2), (2, -2, 1) (b) 1, 2, 3; (1, 0, -1), (0, 1, 0), (1, 0, 1)
 (c) 5, -3, -3; (1, 2, -1), (2, -1, 0), (3, 0, 1)
 (d) 8, 2, 2; (2, -1, 1), (1, 0, -2), (1, 2, 0) (e) 2, 3, -1; (3, 1, 1), (-4, 1, -3), (0, 5, 5)
- (i) 8, 12, 6 (ii) 49, 121, 25
- 1, 1, 1/5
- (i) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$; (ii) $\frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$; (iii) $\frac{1}{4} \begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$; (iv) $\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$
- $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$, $\frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

11. (i) $\frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (ii) $\frac{1}{3} \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ (iii) $\frac{1}{27} \begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$

12. $625I$

13. $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0, \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$

14. $\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

15. $A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}, A^{-2} = \frac{1}{16} \begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$,

$A^{-3} = \frac{1}{64} \begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$.

Problems 2.10, page 67

3. $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

4. $A^n = \begin{bmatrix} 2^n + 3 \cdot 6^n & -3 \cdot 2^n + 3 \cdot 6^n \\ -2^n + 6^n & 3 \cdot 2^n + 6^n \end{bmatrix}; A^4 = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

5. $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

7. $(1, 1, -1), (1, 1, -1), (2, -1, 1); 4x^2 + y^2 + z^2$

8. $x^2 + y^2 - 2z^2$

10. (a) $1, 2, 4; (1, 0, 0), (0, 1, 1), (0, 1, -1)$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
(b) $x_1^2 + 2x_2^2 + 4x_3^2$

10. (i) $x_1^2 + 4x_2^2 + 4x_3^2, \begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, positive definite;

(ii) $3y^2 + 15z^2, \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, positive semidefinite

11. 2, 1

12. Indefinite.

Problems 2.11, Page 71

8. $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + i \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

Problems 2.12, page 72

1. (b)

2. (a)

3. (c)

4. (c)

5. (c)

6. (a)

7. (a)

8. (b)

9. (d)

10. (a)

11. (d)

12. (c)

13. (b)

14. (d)

15. (c)

16. 2

17. sum

18. 0, 8

19. 2

20. $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

21. 0

22. All the eigen values are ≥ 0 and at least one eigen value is zero.

23. (a) $n = p$, (b) $m = p, n = q$

24. 8

25. (b)

26. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 6 & 9 & 3 \end{bmatrix}$ 27. $x^2 + 4xy - 4y^2$ 28. $x = y = z = 0$ 29. 2, 2, 8
 30. $A^2 = A$ 31. 2 32. 1, 4, 9 33. (iv)
 34. 4 35. zero 36. Indefinite 37. 1–1
 38. The elements of its leading diagonal 39. 2 40. $\lambda_i, i = 1, 2, \dots, n$
 41. (c) 42. A or A^T 43. 1, 1/2, 1/3 44. $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$
 45. 1, 1/3 46. $x = 3 - 1$ 47. Symmetric ; skew-symmetric 48. 7 ; 5
 49. $\begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}$ 50. its determinant 51. $\lambda^2 - 6\lambda + 3 = 0$ 52. Augmented matrix
 53. $\begin{bmatrix} 4 & -1 & 3 \\ -1 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ 54. $\lambda_1^3, \lambda_2^3, \lambda_3^3$ 55. $1/\lambda$ 56. $\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$
 57. 38 58. 2 59. Index = 2, Signature = 1
 60. False 61. False 62. False 63. True
 64. False 65. True 66. False 67. True
 68. False 69. True 70. True 71. True.

Problems 3.1, page 80

1. (i) $\sqrt{159}; 6/\sqrt{159}, 1/\sqrt{159}, 11/\sqrt{159}$; (ii) $\sqrt{131}; 9/\sqrt{131}, -7/\sqrt{131}, 1/\sqrt{131}$
 3. 90° 5. $x = 1, y = -1$ 11. 2 : 1.

Problems 3.2, page 88

1. 5 2. (ii) A, B, C form a Δ , rt. \angle ed at C 4. $\cos^{-1}(.62)$
 6. 2.11 7. 13; 12/13, 4/13, 3/13 13. 60°
 14. $\cos^{-1}\left(-\frac{2}{3\sqrt{21}}\right).$

Problems 3.3, page 89

1. $\mathbf{I} - 10\mathbf{J} - 18\mathbf{K}, \frac{1}{5\sqrt{17}}\mathbf{I} - \frac{2}{\sqrt{17}}\mathbf{J} - \frac{18}{5\sqrt{17}}\mathbf{K}, \sin^{-1}\left(\frac{5\sqrt{17}}{21}\right)$ 3. $(2\mathbf{J} + \mathbf{K})/\sqrt{5}$
 5. $\frac{1}{2}\sqrt{(94)}$ 6. (b) $10\sqrt{3}$ 7. $-2/\sqrt{26}.$

Problems 3.4, page 92

1. 40 2. 17; $-24\mathbf{I} + 13\mathbf{J} + 4\mathbf{K}$ 3. 3.33
 4. 70.5 5. $2\mathbf{I} - 7\mathbf{J} - 2\mathbf{K}; \sqrt{(57)}$ 6. (1, 2, 2)
 7. 6 8. 8.25 9. $\frac{5}{6}(-3\mathbf{I} + 2\mathbf{J} + 10\mathbf{K}), \frac{5}{6}\sqrt{113}.$

Problems 3.5, page 96

1. 7 2. -4 3. (ii) Yes 4. Not linearly dependent
 5. 5/6 6. (i) 15; (ii) $1\frac{1}{3}$ 11. (i) $-7\mathbf{I} - 11\mathbf{J} + 5\mathbf{K}$; (ii) $-30\mathbf{I} - 15\mathbf{J} + 15\mathbf{K}$
 15. (b) $\frac{1}{6}abc \begin{vmatrix} 1 & \cos \psi & \cos \phi \\ \cos \psi & 1 & \cos \theta \\ \cos \phi & \cos \theta & 1 \end{vmatrix}.$

Problems 3.6, page 101

1. $2x - y + 3z = 9$
2. $\mathbf{R} \cdot (2\mathbf{I} + 2\mathbf{J} + \mathbf{K}) = 5$
3. $\frac{1}{3}(2\mathbf{I} + 2\mathbf{J} + \mathbf{K})$
4. 3
5. $4x - 3y + 2z = 3$
6. $x - 5y - 2z + 7 = 0$
8. $2; 2x + 2y - 3z = 6$
9. $\sum(x_1 - x_2) \left\{ x - \frac{1}{2}(x_1 + x_2) \right\} = 0$
10. $y = 2$
11. $3x + 4y - 5z = 9$
12. $k = 10.2; 5x - 15y - 21z = 34$
13. $1/6 \cdot 2x - 3y + 6z + 5 = 0$
14. $\cos^{-1}(\sqrt{2}/3)$
15. (i) $\frac{5}{\sqrt{83}}, \frac{-7}{\sqrt{83}}, \frac{-3}{\sqrt{83}}$ (ii) 83.7° (iii) $5x - 7y - 3z + 7 = 0$
17. $6x + 3y - 2z = 18; 2x - 3y - 6z = 6$
20. $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$
21. $xyz = 6k^3$
22. (i) $25x + 17y + 62z - 78 = 0$; (ii) $x + 35y - 10z - 156 = 0$; (iii) bisects the acute angle.

Problems 3.7, page 105

1. $\frac{x-3}{1} = \frac{y-2}{3} = \frac{z-4}{-2}$
2. $43^\circ 3'$
3. 90°
4. $x+2 = \frac{y-3}{2} = \frac{z-4}{2}$
5. 3
6. $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$
7. $\frac{x-1}{1} = \frac{y+2}{-19} = \frac{z-3}{3.5}; \frac{x-1}{11} = \frac{y+1}{13} = \frac{z-3}{-21.5}$
8. (3, 4, 5)
9. 4.1
10. 8.57
11. (3, 4, 5); (ii) $(26/7, -15/7, 17/7)$
12. $40^\circ 15'$
13. $29x - 27y - 22z = 85$
14. $2 - x = y + 1 = (z + 1)/3$.

Problems 3.8, page 107

1. $7x - 2y - 3z = 0$
2. $2x + 3y + 6z = 38$
3. $11x + 12y - 8z = 5$
4. $3y - z = 2$
5. $\frac{x-4}{7} = \frac{y-6}{-13} = \frac{z+2/3}{9}$
6. $x + y + 2z = 1, x + y + (2/5)z = 1$
7. $\frac{x+4/15}{-11} = \frac{y-2/5}{9} = \frac{z}{15}$
8. $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$.

Problems 3.9, page 110

1. $x - 2y + z = 0$
2. (5, -7, 6)
3. $\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \frac{a'\alpha + b'\beta + c'\gamma + d'}{a'l + b'm + c'n}$
4. (0, 1, 2); $4x + y - 2z + 3 = 0$
7. $-\frac{1}{6}(x-5) = y-3 = \frac{1}{2}(z-13)$
8. $\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}$
9. (2, 8, -3); (0, 1, 2); 8.83.

Problems 3.10, page 113

1. $1/\sqrt{6}; 11x + 2y - 7z + 0 = 0; 7x + y - 5z + 7 = 0$
2. $10.77; \frac{1}{2}(x-3) = \frac{1}{3}(y-5) = \frac{1}{4}(z-7); (3, 5, 7); (-1, -1, -1)$
3. $\frac{1}{\sqrt{5}}; 3x - 10y + 6z - 1 = 0 = x + 2z$.

Problems 3.11, page 115

4. First and second planes cut along $x - 36 = -\frac{1}{2}(y + 22) = z$.

Problems 3.12, page 118

1. $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0 ; (2, -3, -1) ; 3$
2. $x^2 + y^2 + z^2 - 2x + 2y - 4z = 0 ; (1, -1, 2) ; \sqrt{6}$
3. (a) $x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$ (b) $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
7. (i) $x^2 + y^2 + z^2 - ax - by - cz = 0$,

(ii) $x^2 + y^2 + z^2 - ax - by - cz = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 ; \left[\frac{a(b^{-2} + c^{-2})}{2\sum a^{-2}}, \frac{b(c^{-2} + a^{-2})}{2\sum a^{-2}}, \frac{c(a^{-2} + b^{-2})}{2\sum a^{-2}} \right]$

8. $(1, 3, 4) ; \sqrt{7}$
10. $x^2 + y^2 + z^2 + 2(x + y + z + 1) = 0$
11. $13(x^2 + y^2 + z^2) - 35x - 21y + 43z + 176 = 0$
12. $3(x^2 + y^2 + z^2) - 7x - 8y + z + 10 = 0$
13. $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$.

Problems 3.13, page 120

1. (i) $x + 3 = 0, x - 7 = 0$ (ii) $x + 2y + 2z = 9, 2x + y - 2z = 9$
2. $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$ and $5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 13 = 0$
3. (i) $x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$ 4. (ii) $x^2 + y^2 + z^2 - 14(x + y + z) + 98 = 0$
6. $\frac{x - 0.6}{-2} = \frac{y - 2.4}{7} = \frac{z}{5}$
8. $3\sqrt{6}, \sqrt{6}$
9. $3x + y + z + 6 = 0$
10. $(12/5, 4, 9/5)$.

Problems 3.14, page 124

1. $(\beta z - \gamma y)^2 = 4a(\alpha z - \gamma x)(z - \gamma)$
2. $528x^2 + 363y^2 + 76z^2 - 528xy - 264yz + 353zx + 704x + 1352z - 4436 = 0$
3. $5x^2 + 3y^2 + z^2 - 2xy - 6yz - 4zx + 6x + 8y + 10z - 26 = 0$
4. $x^2 + y^2 - 3z^2 - 2x - 2y + 6z - 1 = 0$
5. $x^2 + y^2 = z^2 \tan^2 \alpha$
6. $x^2 + 7y^2 + z^2 + 8xy + 8yz - 16zx = 0$
7. $4x^2 + 40y^2 + 19z^2 - 48xy - 72yz + 36zx = 0$
8. $x^2 - y^2 + z^2 + 4y - 4z = 0$
9. $yz \pm zx \pm xy = 0, \cos^{-1}(1/\sqrt{3}); x = y/\pm 1 = z/\pm 1$
10. $\cos^{-1} 4/\sqrt{41}; 25x^2 - 16y^2 - 16z^2 = 0$
11. $4x^2 + 4y^2 - z^2 + 20z - 100 = 0$
12. $x = y/2 = z/-1; x/-2 = y = z$
14. $-2x^2 + y^2 - 2z^2 + 4xy - 8xz + 4yz + 8x - 10y + 8z - 3 = 0$.

Problems 3.15, page 126

1. $5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy - 144 = 0$
2. $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$
3. $45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24zx - 42x - 112y - 126z - 392 = 0$
4. $x^2 + y^2 + z^2 - yz - zx - xy = a^2$
5. $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0; \pi$ units
6. $x^2 + y^2 - 2x - 4y - 11 = 0$
7. $a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0$.

Problems 3.16, page 131

1. Ellipsoid, 33.51
2. Hyperboloid of revolution of one sheet; Hyperbola $5x^2 - y^2 = 6$. No area

3. Right circular cylinder with axis along z -axis
 4. Hyperbolic paraboloid
 5. Hyperboloid of two sheets
 6. Parabolic cylinder
 7. Right circular cylinder
 8. Cone with vertex at the origin
 9. Hyperbolic paraboloid
 10. Hyperboloid of two sheets.

Problems 3.17, page 131

- | | | | | | |
|--|---------|---|---------|---|--------------------|
| 1. (b) | 2. (a) | 3. (c) | 4. (b) | 5. (b) | 6. (a) |
| 7. (c) | 8. (d) | 9. (c) | 10. (c) | 11. (d) | 12. (c) |
| 13. (b) | 14. (c) | 15. (c) | 16. (c) | 17. (c) | 18. (b) |
| 19. (c) | 20. (c) | 21. (b) | 22. (b) | 23. (c) | 24. (c) |
| 25. (b) | 26. (a) | 27. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ | | 28. 0, 0, 1 | 29. $x = 0, y = 0$ |
| 30. $(-3, 2, -1)$ | | 31. 8 or -10 | | 32. $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} = 0$ | |
| 33. $y^2 + z^2 = (bx/a)^2$ | | 34. $12x + 31y - 20z = 66$ | | 35. 523.6 | |
| 36. $\cos^{-1}(6/\sqrt{42})$ | | 37. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = (x + y + z)^2$ | | | |
| 38. $(3, 5, 7), (5, 8, 11)$ | | 38. $-(x - 1) = y - 2 = \frac{1}{6}(z - 3)$ | | 40. $x^2 + y^2 + z^2 + x - 6y - 7z + 9 = 0$ | |
| 41. $al + bm + cn = 0, ax_1 + by_1 + cz_1 + d = 0$ | | | | 42. $2/\sqrt{26}$ | |
| 43. $(3/2, -2, 2), 3\sqrt{5}/2$ | | 44. $\sqrt{44}/3$ | | 45. $\frac{x + 1/3}{1} = \frac{y + 2/3}{-2} = \frac{z}{1}$ | |
| 46. Parabolic cylinder | | 47. Hyperboloid of two sheets | | | |
| 48. $\cos^{-1}\left(\frac{7}{\sqrt{84}}\right)$ | | 49. 6, -4, 12 | | 50. 6 | |
| 51. True | | 52. True | | 53. True | |
| 54. Elliptic cylinder | | 55. $4(x^2 + y^2 + z^2) + 9(xy + yz + zx) = 0$ | | | |
| 56. $(\mathbf{I} - 2\mathbf{J} - 8\mathbf{K})/\sqrt{69}$ | | | | | |

Problems 4.1, page 135

6. $8t^3/(1-t^2)^3$ 7. $-\frac{3}{2}$ 8. $\sin t/a \cos^4 t$.

Problems 4.2, page 138

1. $(-1)^{n-1} (n-1)! 2^n [(2x+1)^{-n} + (2x-1)^{-n}]$ 2. $(-1)^n \left\{ \frac{n!}{(x+1)^{n+1}} - \frac{(n-1)!}{(x+2)^n} + \frac{(n-1)!}{(x+1)^n} \right\}$
 3. $\frac{1}{16} [2 \sin(x + n\pi/2) + 3^n \sin(3x + n\pi/2) - 5^n \sin(5x + n\pi/2)]$
 4. $\frac{1}{256} [9^n \cos(9\theta + n\pi/2) + 9.7^n \cos(7\theta + n\pi/2) + 36.5^n \cos(5\theta + n\pi/2) + 84.3^n \cos(3\theta + n\pi/2) + 126 \cos \theta]$
 5. $\frac{(20)^{n/2}}{2} [e^{2x} \sin(2x + n \tan^{-1} 2) - e^{-2x} \sin(4x - n \tan^{-1} 2)]$
 6. $\frac{1}{2} e^{5x} S(41)^{n/2} \cos[4x + n \tan^{-1}(0.8)] + (29)^{n/2} \cos[2x + n \tan^{-1}(0.4)]$

7. $\frac{(-1)^n n!}{3} \left\{ \frac{4}{(x-1)^{n+1}} \frac{1}{(x+2)^{n+1}} \right\}$

8. $(-1)^n n! \left\{ \frac{9(2)^{n-1}}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}} \right\}$

9. $\frac{(-1)^n n!}{3} \left\{ \frac{4}{(x+1)^{n+1}} + \frac{i-1}{4} \frac{1}{(x+i)^{n+1}} - \frac{i+1}{4} \frac{1}{(x-i)^{n+1}} \right\}$

10. $\frac{(-1)^n n! \cos(n+1)\theta}{(x^2+a^2)^{(n+1)/2}}$ where $\theta = \tan^{-1}(a/x)$

11. $2(-1)^{n-1} (n-1)! \sin n\alpha \sin^n \alpha$ where $\alpha = \cot^{-1} x$.

Problems 4.3, page 141

1. (i) $\frac{(-1)^{n-3} (n-3)!}{x^n} [(n-1)(n-2) + n(3-n)x^2];$

(ii) $\frac{1}{256} \{(\log 2)^n 2^x (\cos 9\theta + 9 \cos 7\theta + 3 \cos 5\theta + 84 \cos 3\theta + 126 \cos \theta) + {}^n C_1 (\log 2)^{n-1} 2^x [\cos 9\theta + \pi/2] + 9 \cos (7\theta + n\pi/2) + 36 \cos (5\theta + \pi/2) + 84 \cos (3\theta + \pi/2) + 126 \cos \theta + \pi/2] + \dots + 2^x [\cos (9\theta + n\pi/2) + 9 \cos (7\theta + n\pi/2) + 36 \cos (5\theta + n\pi/2) + 84 \cos (3\theta + n\pi/2) + 126 \cos (\theta + n\pi/2)]\}$

5. $y_{2m}(0) = 0, y_{2m+1}(0) = (-1)^m \cdot (2m)!$

7. $(y_n)_0 = 0$, if n is even
 $= m(1^2 - m^2)(3^2 - m^2) \dots [(2n-1)^2 - m^2]$, if n is odd

8. $(y_{2n})_0 = e^{m\pi/2} m^2 (2^2 + m^2)(4^2 + m^2) \dots [(2n-2)^2 + m^2]$

$(y_{2n+1})_0 = -e^{m\pi/2} m (1^2 + m^2)(3^2 + m^2) \dots [(2n-1)^2 + m^2]$

17. $\{m^2 - (n-2)^2\} \{m^2 - (n-4)^2\} \dots (m^2 - 2^2)m^2$, n even

$\{m^2 - (n-2)^2\} \{m^2 - (n-4)^2\} \dots (m^2 - 1^2)m$, n odd.

Problems 4.4, page 146

2. $x = (2m-1)a/(2m+2n-1)$

3. (i) $c = 3.154, 0.846$; (ii) $c = \pi/2$; (iii) $c = e-1$. (iv) $c = 0.5413$

6. 0.36

12. $\theta = 0.25$.

Problems 4.5, page 150

1. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

2. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \infty$

3. $1 + x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

4. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots \infty$

5. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \infty$

6. $\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$

24. $\frac{m \sin \theta}{1!} - \frac{m(m^2-1^2)}{3!} \sin^3 \theta + \frac{m(m^2-1^2)(m^2-3^2)}{5!} \sin^5 \theta - \dots$

25. $4 + 21(x-1) + 13(x-1)^2 + 2(x-1)^3$

26. (i) $e \left\{ 1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \right\}$

(ii) $\frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$

27. $1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \dots ; .9998$

29. $\log(0.5) = \sqrt{3}(x - \pi/3) - 2(x - \pi/3)^2 - \frac{4\sqrt{3}}{3}(x - \pi/3)^3 + \dots$

30. 0.8482

31. (i) 2.6121. (ii) 1.12.

Problems 4.6, page 154

- | | | | | | |
|------------------|---------------------|-----------------------------|---------------|------------|--------------|
| 1. $\log_e(a/b)$ | 2. $-1/3$ | 3. $1/3$ | 4. $a \log a$ | 5. 1 | 6. $1/18$ |
| 7. $1/2$ | 8. $1/12$ | 9. $3/2$ | 10. 0 | 11. $1/30$ | 12. 1 |
| 13. $1/3$ | 14. 2 | 15. 1 | 16. 1 | 17. 2 | 18. $11e/24$ |
| 19. $a = 2; 1$ | 20. $a = 5, b = -5$ | 21. $a = 1, b = 2, c = 1$. | | | |

Problems 4.7, page 156

- | | | | | | |
|--------------------|-------------------|----------------|------------------|-----------------|----------------|
| 1. $-1/3$ | 2. $1/2$ | 3. -2 | 4. $-1/3$ | 5. $2/3$ | 6. $1/e$ |
| 7. ae | 8. 1 | 9. e | 10. $1/\sqrt{e}$ | 11. $1/e$ | 12. 0 |
| 13. 0 | 14. 1 | 15. $e^{-1/6}$ | 16. e | 17. $e^{2/\pi}$ | 18. $e^{1/12}$ |
| 19. $-\frac{1}{2}$ | 20. $(6)^{1/3}$. | | | | |

Problems 4.8, page 160

- | | | |
|---|-------------|-------------|
| 1. $x - 20y = 7; 20x + y = 140$ | 2. (a, b) | 10. $\pi/4$ |
| 14. $T = 2a \sin t/2; N = 2a \tan t/2. \sin t/2; S.T. = a \sin t; S.N. = 2a \sin^2 t/2. \tan t/2$ | | |
| 15. $a \sin^3 \theta \tan \theta$. | | |

Problems 4.9, page 162

7. (i) $\pi/2$; (ii) $\pi/2$.

Problems 4.10, page 166

- | | | | |
|---|--|---------------------|-------------------------------|
| 4. $r^3 = 2ap^2$ | 5. $r^3 = a^2p$ | 6. $pa^m = r^{m+1}$ | 7. $r^{m+1} = \sqrt{2} a^m p$ |
| 8. $(1+m^2)p^2 = r^2$ | 9. (i) $\sqrt{1+9x/4a}$; (ii) $\cosh x/c$ | | 10. $a\theta$ |
| 11. (i) $2a \sin \theta/2$; (ii) $a \sqrt{\sec 2\theta}$; (iii) $r \sqrt{(8r-3)}$. | | | |

Problems 4.11, page 172

- | | | |
|---|--|---------------------------|
| 1. (i) $2a(1+t^2)^{3/2}$; (ii) y^2/c | | (iii) $(1+a^3)^{3b}/6a^2$ |
| 5. (i) $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}/ab$; (ii) $4a \sin \theta/2$; | | (iii) at |
| 11. (i) $3/2$; (ii) 1 ; | | (iii) $\sqrt{2a}$ |
| 12. (i) $\frac{4a}{3} \sin \frac{\theta}{2}$; (ii) $a^n/(n+1)r^{n-1}$ | | 14. $2\sqrt{(r^3/a)}$ |

Problems 4.12, page 176

- | | | |
|---|--|----------------------------|
| 1. $a(2+3t)t^2, -4\sqrt{2}at^{3/2}$ | 4. (i) $x = a(t - \sin t), y - 2a = a(1 + \cos t)$, (ii) $x = a \cos \theta, y = a \sin \theta$ | |
| 5. $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$ | | |
| 7. (i) $(x-3a/4)^2 + (y+3a/4)^2 = a^2/2$ (ii) $x^2 + y^2 - \frac{21}{8}(x+y) + \frac{432}{128} = 0$ | | |
| 11. $y^2 = 4ax$ | 12. $(x/a)^2 + (y/b)^2 = 1$ | 13. $27ay^2 = 4(x-2a)^3$. |

14. $(x/a)^2 + (y/b)^2 = 1$

15. $y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$

16. (i) $\sqrt{x} + \sqrt{y} = \sqrt{c}$; (ii) $4xy = c^2$; (iii) $x^{2/3} + y^{2/3} = c^{2/3}$

17. $x^{2/3} + y^{2/3} = c^{2/3}$

Problems 4.13, page 181

2. $a = 1, b = 1/4$, Point of minima 4. $x = 0.42l$

8. $\theta = \frac{\pi}{4} - \frac{\alpha}{2}; (1 - \sin \alpha)/(1 + \sin \alpha)$

13. $8 + 2\sqrt{7}, 2 + 2\sqrt{7}, 5 - \sqrt{7}$

16. $3\sqrt{3}a/4$

14. Depth is half the width

25. 2.5 km/hr.

5. $v = (aw^2/3b)^{1/4}$

10. Sq. with side $\sqrt{2}a$

15. $(a^{2/3} + b^{2/3})^{3/2}$

Problems 4.14, page 185

1. $x + y + a = 0$

2. $x = \pm a, y = \pm b$

3. $x = \pm a, y = \pm b$

4. $y = 0; x + 1 = 0; x + y = 0$

5. $y = x, y + 2x = 0, y + 2x + 1 = 0$

6. $x + a = 0; x - a = 0; x - y + \sqrt{2}a = 0; x - y - \sqrt{2}a = 0$

7. $x + 2y + 2 = 0, x + y = \pm 2\sqrt{2}$

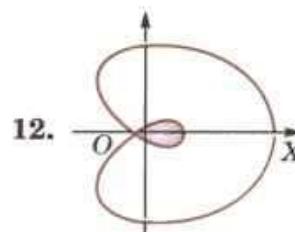
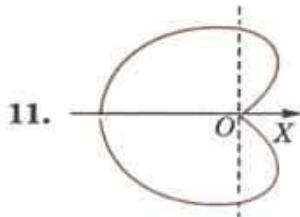
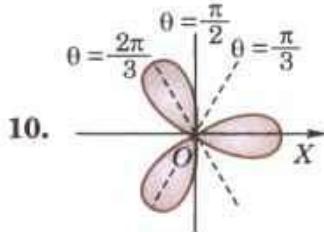
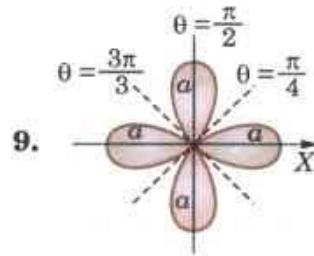
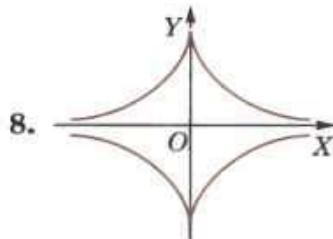
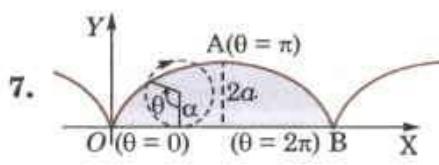
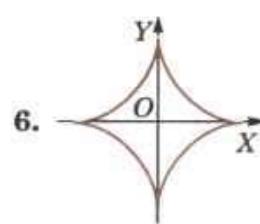
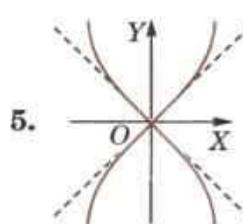
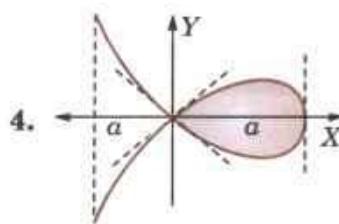
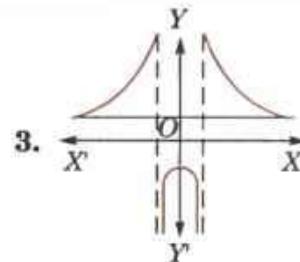
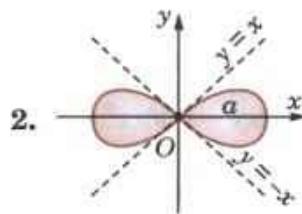
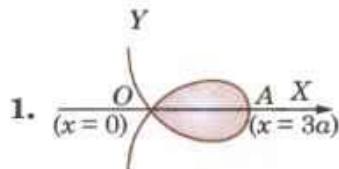
10. $r \cos \theta = a; r \cos \theta = -a$

11. $r \cos \theta = 0; r \cos \theta = 2a$

12. $r \sin \theta = 2$

13. $r \sin(\theta - m\pi/n) = a/n \cos m\pi$.

Problems 4.15, page 194



Problems 4.16, page 194

- | | | | | | |
|--|--|----------------|------------------------|-----------------------------------|---------|
| 1. c | 2. $x^2 + 4ay = 0$ | 3. $1/5$ | 4. (c) | 5. (a) | 6. (b) |
| 7. (c) | 8. (b) | 9. (b) | 10. (c) | 11. (b) | 12. (c) |
| 13. (c) | 14. (b) | 15. $x^2 = 4y$ | 16. of constant length | | |
| 17. $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ | 18. $\frac{1}{4} \left[2^n \cos\left(2x + n \frac{\pi}{2}\right) + 4^n \cos\left(4x + n \frac{\pi}{2}\right) + 6^n \cos\left(6x + n \frac{\pi}{2}\right) \right]$ | | | | |
| 19. $-32/3a$ | 20. True | | 21. $-a$ | 22. $2a(1 + t^2)^{3/2}$ | |
| 23. $(x - a)^2 + (y - b)^2 = k^{-2}$ | 24. envelope | | 25. $xy = c^2$ | 28. $e^x(x^3 - 12x^2 - 36x - 24)$ | |
| 26. α | 27. $2a$ | | 31. (B) | 34. node | |
| 29. (iii) | 30. $(x/a)^2 + (y/b)^2 = 1$ | | 38. $4b$ | | |
| 32. $c = 2.5$ | 33. $x = y$ | | 41. (D) | | |
| 35. Four loops of $r = a \sin 2\theta$ and three loops of $r = a \cos 3\theta$. | | | | | |
| 36. $y = \pm x$ | 37. $x = 4$ | | | | |
| 39. (A) | 40. $r > a$ | | | | |
| 42. (D) | 43. (C). | | | | |

Problems 5.1, page 198

1. $2/3$ 2. Does not exist 3. Zero 4. Does not exist
 7. Discontinuous.

Problems 5.2, page 202

1. (i) $xy(2 - \cos xy) - \sin xy ; x^2(1 - \cos xy) ;$
 (ii) $2x/(x^2 + y^2), 2y/(x^2 + y^2) ;$
 (iii) $(x^2 + 2xy - y^2)/[(x^2 + y^2)^2 + (x + y)^2] ; (y^2 + 2xy - x^2)/[(x^2 + y^2)^2 + (x + y)^2] ;$
 (iv) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{z}{1-z}$
11. $n = 2, -3$ 18. $e^{xyz}(x^2y^2z^2 + 3xyz + 1).$

Problems 5.4, page 208

13. $2u.$

Problems 5.5, page 211

2. $4a^2t(t^2 + 2)$ 2. $-2/(e^{2t} + e^{-2t})$ 3. zero
 4. 6.5 sq. ft./sec 6. $8e^{4t}.$

Problems 5.6, page 214

9. 0 ; 0.

Problems 5.7, page 218

6. zero 7. $x(yv + 1 - w) + z - 2uv$ 10. 0 ; $u = \tan v$
 11. $u^2 - v^2 = 8w$

Problems 5.8, page 220

1. $4X + Y + Z = 6$; $\frac{X - 2}{4} = Y - 1 = Z + 3$ 2. $3Y + 2Z - X - 3 = 0$, $1 - X = \frac{Y - 2}{3} = \frac{Z + 1}{2}$
3. $\frac{X}{x_1} + \frac{Y}{y_1} + \frac{Z}{z_1} = 3$; $x_1(X - x_1) = y_1(Y - y_1) = z_1(Z - z_1)$
4. $7X - 3Y + 8Z = 26$; $\frac{X - 1}{7} = \frac{Y + 1}{-3} = \frac{Z - 2}{8}$
5. $(-1, 2, 2/3)$ 7. $\frac{X - x}{x} = \frac{Y - y}{y} = \frac{Z - z}{z}$.

Problems 5.9, page 226

1. (i) $x - \frac{1}{6}(x^3 + 3xy^2)$
(ii) $\frac{1}{2\sqrt{2}} \left[1 + \{(x+1) + (y - \pi/4)\} + \frac{1}{2} \{(x+1)^2 - 2(x+1)(y - \pi/4) + (y - \pi/4)^2\} + \frac{1}{6} \{(x+1)^3 + 3(x+1)^2(y - \pi/4) - 3(x+1)(y - \pi/4)^2 - (y - \pi/4)^3\} + \dots \right]$
(iii) $1 + x + \frac{1}{2!}(x^2 - y^2) + \frac{1}{6}(y^3 - 3xy^2) + \dots$
2. $1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \dots$
3. -0.8232 4. -4500 units 5. 2% 7. 2%
11. $\frac{\alpha}{2} \cot \alpha + 2$ 12. Rs. 43.20 13. $(p - 3q - 4r)\%$ 14. $-1\frac{1}{3}\%$ 15. $5r$.

Problems 5.10, page 233

1. (i) (a, a) gives maximum if $a < 0$ and minimum if $a > 0$
(ii) Min. at (a, a) (iii) Max. at $(4, 0)$, Min. at $(6, 0)$
(iv) Max. at $(\pm 1, 0)$; Min. at $(0, \pm 1)$ (v) Max. at $(\pi/3, \pi/3)$; Min. at $(2\pi/3, 2\pi/3)$
2. $4, 2, 1$ 3. (i) $3a^2$; (ii) $p^2/(a^2 + b^2 + c^2)$; (iii) $3a^2$ 4. $12 \times 12 \times 6$ cm
6. $(0, 0, \pm 1)$ 8. 4, 1 9. 50
10. $4, 8, 12$ 11. Two stationary values of u are given by $\frac{l^2}{au-1} + \frac{m^2}{bu-1} + \frac{n^2}{cu-1} = 0$.

Problems 5.11, page 236

1. $\frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2 + a^2)}$ 2. $\frac{(-1)^n n!}{(m+1)^{n+1}}$
3. $\pi \log \left[\frac{1}{2} + \frac{1}{2} \sqrt{(1-a^2)} \right]$ 4. $-\pi/(a^2 - 1)^{3/2}$.

Problems 5.12, page 236

- | | | | | | |
|---------|---------|---------|---------|----------|---------------|
| 1. zero | 2. (a) | 3. 1 | 4. (b) | 5. (b) | 6. (b) |
| 7. (c) | 8. (c) | 9. (d) | 10. (d) | 11. (b) | 12. (d) |
| 13. (a) | 14. (d) | 15. (c) | 16. (b) | 17. zero | 18. $2/(x+y)$ |

19. $rt - s^2 < 0$ 20. (d) 21. $4u$ 22. $\partial(u, v)/\partial(x, y)$
 23. $f_x(a, b) = 0, f_y(a, b) = 0$ 24. (c) 25. $\frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$
 26. (c) 27. -1 28. (c) 29. equal
 30. False.

Problems 6.1, page 244

1. (i) $128/315$; (ii) $8/45$ 2. (i) $128/315$; (ii) $11\pi/192$
 3. (i) $\frac{(2n-3)(2n-5)}{(2n-2)(2n-4)} \dots \frac{3.1}{4.2} \frac{\pi}{2}$ (ii) $\frac{1}{8} \left(\frac{\pi}{8} + \frac{1}{6} \right)$ 4. $35\pi/10240$
 5. (i) $3\pi/512$ (ii) $1/144$ 6. (i) $5\pi/256$; (ii) $1/15$ 7. (i) $35\pi a^4/8$; (ii) $5\pi a^3/2$
 8. (i) $5\pi/8$; (ii) 28π .

Problems 6.2, page 247

1. (i) $\frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x$ (ii) $-\frac{1}{4} \cot^4 x + \frac{1}{2} \cot x + \log \sin x$
 3. $\frac{1}{2} \log 2 - \frac{1}{4}$ 4. $\frac{\pi}{4} - \frac{2}{3}$ 5. $I_n = \frac{(2)^{(n-2)/2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$
 6. (i) $\frac{1}{5} \sec^4 x \cdot \tan x + \frac{4}{15} (\sec^2 x + 2) \tan x$ (ii) $\frac{11\sqrt{3}}{4} + \frac{3}{8} \log(2 + \sqrt{3})$
 (iii) $-\frac{1}{4} \cot x \operatorname{cosec} x - \frac{3}{8} \cot x \operatorname{cosec} x + \frac{3}{8} \log(\operatorname{cosec} x - \cot x)$
 7. $\left\{ \frac{67\sqrt{2}}{48} + \frac{5}{16} \log(1 + \sqrt{2}) \right\} a^6$ 8. $\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t$.

Problems 6.3, page 250

1. $e^x (1 - x + x^2 - x^3 + x^4)$
 3. $\int x^m (\log x)^n dx = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$; $\int_0^1 x^5 (\log x)^3 dx = -1/216$
 5. $149/225$ 6. $3\pi^2/64 - 1/4$ 7. $\frac{5}{16}\pi^4 - 15\pi^2 + 120$
 11. $I_n = \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$; $\int_0^{\pi/2} e^{2x} \cos^3 x dx = \frac{2}{65} (3e^\pi - 11)$
 12. $24/85$.

Problems 6.4, page 254

7. (i) $3\pi/8$; (ii) $5\pi/8$; (iii) $3\pi/256$; (iv) $15\pi/640$ 8. (i) $16\pi/35$; (ii) $8\pi/315$.

Problems 6.5, page 256

1. $\log 2$ 2. $\frac{1}{3} \log 2$ 3. $1/3$ 4. $\pi/2$ 5. $\frac{1}{4} \log 2$ 6. $2e^{(\pi-4)/2}$.

Problems 6.6, page 260

1. (i) πab ; (ii) $8a^2/3$ 2. $21\frac{1}{12}$ 3. $2a^2/5$

4. (i) $8a^2/15\sqrt{3}$; (ii) $(2 - \pi/2)a^2$

8. $3\pi a^2$

11. $\frac{11\pi}{3} - 2\sqrt{3}$

15. $1\frac{1}{8}$.

5. (i) πa^2 ; (ii) $\frac{\sqrt{2}}{3}a^2$

9. $a^2/6$

13. $(3\pi - 8)a/6$

6. (i) πa^2 ; (ii) $4a^2$

10. $3\pi a^2$; πa^2

14. $64a^2/3$

Problems 6.7, page 262

1. (i) $3\pi a^2/2$; (ii) a^2

2. (i) $\pi a^2/8$; (ii) $\pi a^2/12$

5. $(1 - \pi/4)a^2$

6. $\pi a^2/2$.

Problems 6.8, page 265

1. $12\frac{11}{27}a$

2. (i) $\log(2 + \sqrt{3})$, (ii) $\log_e(e + 1/e)$

3. (i) $a[\sqrt{2} + \log(1 + \sqrt{2})]$; (ii) $(15/16 + \log 2)a$

4. (i) $4a/\sqrt{3}$; (ii) $4\sqrt{3}$

5. 37.85

7. (i) $8a$

8. $6a$

9. $4\sqrt{3}$

11. $2 + \frac{1}{2}\log 3$

12. $8a$

13. $\sqrt{2}\pi a \left\{ 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \cdot \left(\frac{1}{2}\right)^2 + \dots \right\}$

14. $2a[\sqrt{2} + \log(\sqrt{2} + 1)]$.

Problems 6.9, page 269

1. $\pi c^3(1 + \sinh 1 \cosh 1)$

2. $\pi h^2(a - h/3)$

3. $2\pi a^3$

4. $\pi a^3/12$

5. (i) $\frac{4}{3}\pi ab^2$; (ii) $\frac{4}{3}\pi a^2b$

6. $\frac{\pi h}{3}(r^2 + rR + R^2)$

7. $48\pi a^3$

8. (i) $2\pi a^3(\log 2 - 2/3)$; (ii) $\pi a^3/24$; (iii) $\pi/48$

9. (i) $5\pi^2 a^3$; (ii) $5\pi^2 a^3$

10. $32\pi a^3/105$

11. $4\pi^2 a^3$

13. (i) $\frac{4}{3}\pi a^3$; (ii) $\frac{8}{3}\pi a^3$

14. $\frac{4}{3}\pi a(a^2 + b^2)$

15. $\frac{\pi a^3}{4} \left\{ \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1) - \frac{1}{3} \right\}$

Problems 6.10, page 271

1. $\frac{\pi a^2}{2}(2 + \sinh 2)$

2. $\frac{8\pi a^2}{3}(2\sqrt{2} - 1)$

3. $2\pi ab \left\{ \frac{b}{a} + \frac{a}{\sqrt{(a^2 - b^2)}} \sin^{-1} [\sqrt{(a^2 - b^2)/a}] \right\}$

4. $\frac{1}{2}\pi r^2 h$; $\pi r\sqrt{r^2 + h^2}$, where r is the base-radius and h the height of the cone

5. $4\pi a^2$

8. $\frac{64}{3}\pi a^2$

9. $\frac{64}{3}\pi a^2$

12. $4\pi a^2$

13. $\frac{32}{5}\pi a^2$

14. $4\pi a^2(1 - 1/\sqrt{2})$

15. $\pi a^2[3\sqrt{2} - \log(\sqrt{2} + 1)]$.

Problems 6.11, page 271

1. (b)

2. (c)

3. (b)

4. (c)

5. (b)

6. (c)

7. (c)

8. (d)

9. (b)

10. (d)

11. (c)

12. (c)

13. (a)

14. $\frac{3\pi a^2}{2}$

15. (iii)

16. $\pi a^2/12$

17. 1

18. $7\pi/8$

19. (iii)

20. (iii)

21. $\pi h(r_1^2 + r_1 r_2 + r_2^2)$

22. (c)

23. (a)

24. (b) or (c)

25. (a).

Problems 7.1, page 280

1. 13

2. 3/35

3. $\frac{1}{2}(e - 1)$

4. $\frac{1}{4}\pi \log(1 + \sqrt{2})$

5. $a^4/8$

6. $\frac{\pi}{4}ab(a^2 + b^2)$

7. 3/56

8. $\pi a/4$

9. 241/60

10. $1 - 1/\sqrt{2}$

11. $\frac{\pi a^2}{4}(\log e - \frac{1}{2})$

12. 1/24

13. $\frac{2}{3}a^4$

14. $\pi a^2/b$

15. 1

16. (i) $\int_0^{2a} \int_{y\sqrt{2a}}^{2a} f(x, y) dx dy - \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x, y) dx dy;$ (ii) $\int_0^{\pi/2} \int_a^{ae^{\theta/2}} f(r, \theta) r dr d\theta$

18. $4a^2/3$

19. $45\pi/2.$

Problems 7.2, page 283

1. 4.5

2. 7/6

3. πa^2

4. $\frac{3}{2} \log_e 3 - \frac{2}{3}$

5. a^2

6. $a^2(1 - \pi/4)$

7. 4/3

8. $a^2/4.$

Problems 7.3, page 284

1. $\frac{abc}{3}(a^2 + b^2 + c^2)$

2. $\frac{8}{3}abc(a^2 + b^2 + c^2)$

3. 4/35

4. $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$

5. $\frac{8}{3} \log 2 - \frac{19}{9}$

6. $\frac{1}{4}(13 - 8e + e^2)$

7. $5\pi a^3/64.$

Problems 7.4, page 291

1. $\pi/8$

2. $\pi/2$

3. $8\left(\frac{\pi}{2} - \frac{5}{3}\right)a^2$

4. $\frac{2^{n+3}}{n+4}$

5. $\frac{\pi}{4} - \frac{1}{2}$

6. 0

7. $\frac{15\pi a^4}{64}$

9. π

10. $\pi^2/8$

11. $4\pi a$

12. $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$

14. $\pi a^8/12$

15. 3π

16. 4π

17. $\pi a^3(2 - \sqrt{2})/3$

18. $16a^3/3$

19. $\pi a^3/8$

20. $128a^3/15$

21. $3\pi a^3$

22. $4\sqrt{3}\pi$

23. $8a^4/3$

25. $\frac{1}{4}$

26. $\frac{1}{6}abc.$

Problems 7.5, page 293

2. 64

3. $2(\pi - 2)a^2$

4. $2\pi a^2$

5. $\frac{3\pi a^2}{4}.$

Problems 7.6, page 297

1. $182 \frac{7}{24} \lambda$

2. $21 \pi \mu a^4 / 32$

3. 30.375

4. $\left(\frac{3a}{20}, \frac{3a}{16} \right)$

5. $\left[\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+b)}{6(a+b)} \right]$

6. $\left(\frac{\pi a \sqrt{2}}{8}, 0 \right)$

8. $\bar{x} = 3a/5$, $\bar{y} = 9a/40$ where $a = OA$ 9. $(1/5, 1/5, 2/5)$.

10. $\bar{x} = 3/4$

11. $\left(\frac{16a}{35}, \frac{16b}{35}, \frac{16c}{35} \right)$

12. $\frac{27}{26}$ metres

13. $\left(\frac{3a}{8}, \frac{3\pi a}{16} \right)$

14. Divides the diagonal in the ratio 7 : 5

15. $\left(\frac{a}{2}, \frac{2}{3} h \right)$ where a is the base, h the depth

16. C.P. lies on the radius \perp to the bounding diameter at a depth $32a/(15\pi)$ from the centre.

Problems 7.7, page 301

1. $ab^3/12$

2. $5Ma^2/4$

3. $2M/9$

4. $\frac{1}{3}M(a^2 + b^2)$

5. $(21/32)\pi\rho a^4$

6. $\frac{2}{5}MR^2$

7. $\frac{1}{2}Mr^2; \frac{1}{12}M(3r^2 + 4h^2)$

8. (i) $\frac{3Mr^2}{10}$; (ii) $\frac{3M}{20}(r^2 + 4h^2)$; (iii) $\frac{M}{20}(3r^2 + 2h^2)$

9. 104803770 ρ

10. $\frac{1}{30}$

11. $\frac{\pi\rho abc(a^2 + b^2)}{30}$

12. $\frac{\rho a^2 b^2}{8}$.

Problems 7.8, page 309

1. (i) 3.323, (ii) 11.629; (iii) $\pi\sqrt{2}$; (iv) 0.1964; (v) 0.1227

2. (i) $\sqrt{\pi}/2$; (ii) $\Gamma(5/4)$; (iii) $\sqrt{\pi}/3$; (iv) $2^{p+q-1}\beta(p, q)$

4. $\pi/4\sqrt{2}$

7. $-3/8$

9. $\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{n\Gamma\left(\frac{m}{n}+p+1+\frac{1}{n}\right)}$, (i) $\frac{1}{396}$ (ii) $\frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

10. 16/35

15. $\frac{ka^2b^2c^2}{48}$.

Problems 7.9, page 312

2. $\frac{1}{2}K(\sqrt{3}/2)$

3. $\frac{2}{\sqrt{3}} \left\{ F\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\pi\right) - F\left(\sqrt{\frac{2}{3}}, \frac{1}{4}\pi\right) \right\}$

4. $2\sqrt{2}E(1/\sqrt{2}) - \sqrt{2}K(1/\sqrt{2})$

5. $erf(x) = \frac{2}{\pi} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \right)$; $erf(0) = 0$ 6. (i) 0.3248; (ii) 0.5204.

Problems 7.10, page 313

1. 4

2. Area of the triangle having vertices (0, 0), (0, 1), (1, 0)

3. $\sqrt{\pi}/2$

4. 3.1416

5. $15\sqrt{\pi}/8$

6. 92π

6. 26

8. $-1/3$

9. $1/2\beta(4, 3/4)$

10. 1

11. $3/4$ 12. $\frac{\pi a^3}{6}$ 13. (d) 14. $27/4$
 15. $\pi a^2/12$ 16. $-1/3$ 17. ∞ 18. $44/105$
 19. $r^2 \sin \theta dr d\theta d\phi$ 20. $e^2 - 1$ 21. $\frac{1}{4} \pi ab (a^2 + b^2)$. 22. $\frac{1}{4} \pi \log(1 + \sqrt{2})$
 23. $3/256$ 24. (c) 25. $\frac{6}{25} + \frac{1}{2} \sin \frac{3}{5}$ 26. $48/5$
 27. $1/6$ 28. $16/3$ 29. $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y) dy dx$
 30. $\sqrt{2} \pi$ 31. $3\pi/256$ 32. $1/2$ 33. $\int_0^{\pi/2} \int_0^{2a \cos \theta} r d\theta dr$
 34. $\frac{7h}{12}$ 35. $\left(\frac{3}{20}, \frac{3}{16}\right)$ 36. 16 37. $3Mr^2/10$
 38. 1 39. $\log 2$ 40. $\frac{1}{2} \sqrt{\pi}$ 41. (c) 42. (b).

Problems 8.1, page 318

3. (i) $t^3 \sin t + 7t^2 \cos t + 20t \sin t - 10t$; (ii) $(20t^3 + t \sin t - \cos t)\mathbf{I} - (2t \cos t + 2 \sin t + 75t^2)\mathbf{J} - t(t \sin t + 2t^2 \cos t + 10 \cos t)\mathbf{K}$
 5. $-4(\mathbf{I} + 2\mathbf{J})$ 6. (i) $(ua^2 \sec \alpha)$. (ii) $a^3 \tan \alpha$; $(\cos t \mathbf{J} - \sin t \mathbf{I}) \cos \alpha + \sin \alpha \mathbf{K}$
 7. $\{t\mathbf{I} + 2\mathbf{J} + (2t - 3)\mathbf{K}\}/\sqrt{(5t^2 - 12t + 13)}$; $\frac{1}{3}(2\mathbf{I} + 3\mathbf{J} + \mathbf{K})$
 8. $(x - a/\sqrt{2}) = y - a/\sqrt{2} = \left(z - \frac{a\pi}{4} \tan \alpha\right)/\sqrt{2} \tan \alpha$
 9. (i) $ab/(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}$; (ii) $1/4\sqrt{2}$
 10. (i) $\mathbf{R} = (p+q)\mathbf{I} + q\mathbf{J} + 2q\mathbf{K}; \frac{2\mathbf{J} - \mathbf{K}}{\sqrt{5}}$
 (ii) $\mathbf{R} = p\mathbf{I} + (p+2q)\mathbf{J} + (p+q)\mathbf{J} + (p+q)\mathbf{K}; (2\mathbf{K} - \mathbf{I} - \mathbf{J})/\sqrt{6}$.

Problems 8.2 page 321

1. $v(\text{at } t=0) = \sqrt{37}$, $a(\text{at } t=0) = \sqrt{325}$ 2. $\alpha = \pm 1/\sqrt{6}$
 3. $8\sqrt{14/7}; -\sqrt{14/7}$ 6. (a) d^2s/dt^2 ; v^2/p ; (b) 0; 3
 7. $\sqrt{17}$ m.p.h. in the direction $\tan^{-1}(0.25)$ North of East
 8. 21.29 knots/hr. in the direction $74^\circ 47'$ South of East.

Problems 8.3, page 325

1. (a) $2(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})/(x^2 + y^2 + z^2)$. (b) $\frac{6}{3} + c$ 2. $(-\mathbf{I} + 3\mathbf{J} + 2\mathbf{K})/\sqrt{14}$
 3. $12\frac{1}{3}$ 4. $15/\sqrt{17}$ 5. $a = 6, b = 24, c = -8$
 6. $-260/(69); \sqrt{1056}$ 7. $a = \pm \frac{20}{9}, b = \pm \frac{55}{9}, c = \pm \frac{55}{9}$ 8. $96(\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}); 96\sqrt{19}$
 9. 9 10. $\frac{1}{3}(2\mathbf{I} + 2\mathbf{J} - \mathbf{K})$ 11. $\cos^{-1}(1/\sqrt{22})$
 12. $\cos^{-1}(-1/\sqrt{30})$ 13. $a = -6, b = -10$.

Problems 8.4, page 333

1. (i) $12; 5\mathbf{I} - 16\mathbf{J} + 9\mathbf{K}$; (ii) $278; 5(27\mathbf{I} - 54\mathbf{J} + 8\mathbf{K})$; (iii) $-32; 0$
4. $a = -2; 4x(z-xy)\mathbf{I} + (y-2yz+4xy^2)\mathbf{J} + (2x^2+y^2-z^2-z)\mathbf{K}$
13. (i) 0 ; (ii) $2(x+z)\mathbf{J} + 2y\mathbf{K}$
14. (a) $2n(2n-1)/x^2+y^2+z^2)^{n+1}; n=1/2$
16. (i) $2(y^3+3x^2y-6xy^2)z\mathbf{I} + 2(3xy^2+x^3-6x^2y)z\mathbf{J} + 2(xy^2+x^3-3x^2y)y\mathbf{K}$; (ii) Zero
17. $1724/\sqrt{21}$.

Problems 8.5, page 335

1. $75\frac{1}{3}\mathbf{I} + 360\mathbf{J} - 42\mathbf{K}$
2. $(t^3-t+2)\mathbf{I} + (1-t^4)\mathbf{J} + (4-4\cos t-3t)\mathbf{K}$
3. $\mathbf{V} = 6 \sin 2t\mathbf{I} + 4(\cos 2t-1)\mathbf{J} + 8t^2\mathbf{K}; \mathbf{R} = 3(1-\cos 2t)\mathbf{I} + 2 \sin 2t\mathbf{J} + \frac{8t^3}{3}\mathbf{K}$.

Problems 8.6, page 336

1. 0
2. 35
3. $-2/3$
4. 5
5. $\frac{\pi^3\sqrt{2}}{3}$
6. zero
7. 303
8. $8\frac{8}{35}$
9. 9π
10. $\left(2-\frac{\pi}{4}\right)\mathbf{I} - \left(\pi-\frac{1}{2}\right)\mathbf{J}$.

Problems 8.7, page 339

2. $3\frac{1}{3}$
3. 8.

Problems 8.8, page 341

3. πab
4. πa^2
5. Zero
6. $128/5$
7. $35\pi a^4/16$.

Problems 8.9, page 345

3. $-2ab^2$
5. $\frac{19}{2}\pi$
6. Zero
10. 2
11. 0
12. π .

Problems 8.10, page 350

4. 108π
7. (i) $\frac{12}{5}\pi a^5$ (ii) $12(e-e^{-1})$
8. $4\pi a^3$
9. $\frac{\pi a^6}{12}$
10. $\frac{5}{4}\pi a^4 b$
11. -4π
12. $8/3$.

Problems 8.11, page 354

3. $14\frac{2}{3}$
4. $x^3y - y^2z^2 + z^3$
(ii) $x^2y + y^2z + z$;
5. (i) $\frac{1}{3}(x^3 + y^3 + z^3 - 3xyz)$;
(iii) $xz^3 - yz + 3x^2y$.
6. (i) Yes, $\frac{a}{2}(x^2 + y^2 - 2z^2)$;
(ii) Yes
7. $xy \sin z + \cos x + y^2 z + c$
8. $x^2y + xz^3; 202$
9. $a = 4, b = 2, c = -1$
10. $a = 4; 2x^2y - xz^3; 47$.

Problems 8.12, page 362

- (i) $(\rho \sin 2\phi - z \sin \phi) \mathbf{T}_\rho - (2\rho \sin^2 \phi + z \cos \phi) \mathbf{T}_\phi + 3\rho \cos \phi \mathbf{T}_z$
(ii) $(2\rho \cos^2 \phi - 3\rho^2 \sin^3 \phi) \mathbf{T}_\rho - (\rho \sin 2\phi + 3\rho^2 \sin^2 \phi \cos \phi) \mathbf{T}_\phi + \rho z \cos \phi \mathbf{T}_z$
- (i) $r \sin \theta [(\sin \theta (1 + \sin^2 \phi) + r \cos^2 \theta \sin \phi) \mathbf{T}_r$
 $+ (\cos \theta (1 + \sin^2 \phi) - r \sin \theta \cos \theta \sin \phi) \mathbf{T}_\theta + \sin \phi \cos \phi \mathbf{T}_\phi]$
(ii) $r^2 \sin \theta [(\sin^2 \theta \cos^2 \phi \sin \phi + \sin \theta \cos \theta \sin^2 \phi + \cos^2 \theta \cos \phi) \mathbf{T}_r$
 $+ (\sin \theta \cos \theta \cos^2 \phi \sin \phi + \cos^2 \theta \sin^2 \phi - \sin \theta \cos \theta \cos \phi) \mathbf{T}_\theta$
 $+ (\cos \theta \sin \phi \cos \phi - \sin \theta \sin^2 \phi \cos \phi) \mathbf{T}_\phi]$
- $\rho z \sin 2\phi \mathbf{T}_\rho + \rho z \cos 2\phi \mathbf{T}_\phi + \frac{1}{2} \rho^2 \sin 2\phi \mathbf{T}_z$.

Problems 8.13, page 363

- | | | | |
|--|---|--|---|
| 1. $1/\sqrt{14}$, $2/\sqrt{14}$, $3/14$ | 2. $\frac{1}{4}(x-2) = y-1 = z+3$ | 3. $dudv = \frac{1}{h_1 h_2} dxdy$ | |
| 4. $4x - 3z + 2xz$ | 5. zero | 6. $\frac{1}{2} \int_C (xdy - ydx)$ | |
| 7. 3V | 8. 3 ; 0 | 9. Irrotational | |
| 10. 4π | 11. solenoidal | 12. $-28/\sqrt{5}$ | 13. zero. |
| 14. $-(y\mathbf{I} + z\mathbf{J} + x\mathbf{K})$. | 15. $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. | 16. zero | 17. $-(12\mathbf{I} + 5\mathbf{J} + 8\mathbf{K})$ |
| 18. zero | 19. zero | 20. zero | 21. zero |
| 22. \mathbf{R}/r^2 ; $nr^{n-2} \mathbf{R}$ | 23. §8.5(2) | 24. $\frac{1}{\sqrt{21}} (2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$. | 25. 2, -2, 2 |
| 26. $6\frac{\sqrt{7}}{3}$ | 27. $2/r$ | 28. 7/3 | 29. zero |
| 30. (c) | 31. (c) | 32. (b) | 33. (c) |
| 34. (a) | 35. (a) | 36. $5u$ | 37. zero |
| 38. irrotational field | 39. (a) | 40. the rate at which fluid is originating at P per unit volume. | |
| 41. (a) | 42. it gives the maximum rate of change of ϕ . | 43. (iv) | |
| 44. (a) | 45. (a) | 46. (b) | 47. (a) |
| 48. (b) | 49. zero | 50. True | 51. True. |

Problems 9.1, page 366

- | | | | |
|---------------|---------------|---------------|---------------|
| 1. Convergent | 2. Convergent | 3. Convergent | 4. Divergent |
| 5. Convergent | 6. Convergent | 7. Convergent | 8. Divergent. |

Problems 9.2, page 367

- | | | | |
|---------------|---------------|----------------|---------------|
| 1. Convergent | 2. Convergent | 3. Oscillatory | 4. Convergent |
| 5. 15 m. | | | |

Problems 9.3, page 372

- | | | | |
|---------------|--|----------------|----------------|
| 1. Convergent | 2. Convergent | 3. Divergent | 4. Divergent |
| 5. Convergent | 6. Conv. for $p > 2$; Div. for $p \leq 2$. | | 7. Divergent |
| 8. Convergent | 9. Convergent | 10. Convergent | 11. Convergent |

12. Divergent

16. Convergent

13. Divergent

17. Divergent

14. Convergent

18. Convergent.

15. Convergent

Problems 9.4, page 3761. Conv. for $x < 1$; Div. for $x \geq 1$ 3. Conv. for $x \leq 1$; Div. for $x > 1$ 5. Convergent for all values of p

8. Convergent

11. Convergent

14. Convergent

16. Divergent

2. Conv. for $x < 1$; Div. for $x \geq 1$ 4. Conv. for $x \geq 1$; Div. for $x < 1$

6. Convergent

9. Conv. for $x < 1$; Div. for $x \geq 1$

12. Convergent

15. Conv. for $x < 1$, Div. for $x > 1$; Conv. for $p > 1$ and Div. for $p \leq 1$ 17. Conv. if $\beta > \alpha > 0$; Div. if $\alpha \geq \beta > 0$.

7. Convergent

10. Convergent

13. Divergent

Problems 9.5, page 3791. Conv. for $x \leq 1$; Div. for $x > 1$ 3. Conv. for $x < 1$; Div. for $x \geq 1$ 5. Conv. for $x < e$; Div. for $x \geq e$ 7. Conv. for $x \leq 1$; Div. for $x > 1$ 9. Conv. for $x^2 < 4$; Div. for $x^2 \geq 4$ 11. Conv. for $x < 1/e$; Div. for $x \geq 1/e$

13. Diverges

14. Conv. for $x < 1$; Div. for $x > 1$. When $x = 1$, Conv. for $b - a > 1$, Div. for $b - a \leq 1$.2. Conv. for $x \leq 1$; Div. for $x > 1$ 4. Conv. for $x < 2$; Div. for $x \geq 2$ 6. Conv. for $x \leq 1$; Div. for $x > 1$ 8. Conv. for $x^2 \leq 1$; Div. for $x^2 > 1$

10. Convergent

12. Conv. for $x < 1$; Div. for $x \geq 1$ **Problems 9.6, page 381**

1. Convergent

4. Convergent

6. Conv. for $x < \frac{1}{2}$; Div. for $x \geq \frac{1}{2}$

2. Convergent

5. Conv. for $x < 1$; Div. for $x \geq 1$

3. Convergent

1. Oscillatory

5. Convergent

9. Convergent

2. Convergent

6. Oscillatory

10. Oscillatory.

3. Convergent

7. Convergent

4. Convergent

8. Convergent

Problems 9.7, page 383

1. (i) and (ii) conditionally convergent

3. (i) Conditionally convgt. for $0 < p \leq 1$; (ii) Conditionally convgt4. Absolutely convergent for (i) $0 < x < 1$; (ii) $-1 < x \leq 1$; (iii) $|x| \leq 1$.5. Convergent for $x \leq 1$ and not convergent for $x > 1$ 6. (i) $-1 < x \leq 1$; (ii) $-1 < x \leq 1$;7. $-e < x \leq e$ 8. (i) Absolutely convergent (ii) convergent

9. Absolutely convergent.

Problems 9.9, page 3881. Conv. for $x < 1$; Div. for $x \geq 1$

4. Convergent

7. Conv. for $x < 1$; Div. for $x \geq 1$ 9. Conv. for $x < 1/4$; Div. for $x \geq 1/4$

2. Convergent

5. Divergent

8. Conv. for $x < 1$; Div. for $x \geq 1$

3. Divergent

6. Conv. for $x < 1$; Div. for $x \geq 1$ 10. Conv. for $x < 2$; Div. for $x \geq 2$

11. Convergent for all x 12. Conv. for $x < 1$; Div. for $x \geq 1$ 13. Convergent
 14. Absolutely convergent 15. Convergent
 16. Convergent for $p > 1$; divergent for $p \leq 1$.

Problems 9.10, page 391

1. Uniformly convergent for $0 \leq x \leq 1$. 2. to
 5. Uniformly convergent for all real values of x 6. Uniformly convergent for $0 \leq x \leq 1/a$
 10. (i) and (ii) Both converge uniformly for all real values of x .

Problems 9.11, page 392

- | | | | | | |
|-----------------------|-------------------|-----------------------|-----------------|----------------|-----------------|
| 1. (c) | 2. (d) | 3. (a) | 4. (b) | 5. (c) | 6. (d) |
| 7. (a) | 8. (b) | 9. (b) | 10. (d) | 11. (c) | |
| 12. (a) $(-1, 1)$ | (b) $(-1/2, 1/2)$ | 13. $-1 < x \leq 1$ | 14. $k > 1$ | 15. $a_n < k$ | 16. Oscillatory |
| 17. All values of x | | 18. $k < 1$ | 19. Convergent. | 20. Divergent. | 21. $q - p > 1$ |
| 22. Divergent | 23. Convergent. | 24. $0 < x < 4$ | 25. yes | 26. True | 27. Convergent |
| 28. Divergent | 29. $x > 1$ | 30. $0 \leq x \leq 1$ | 31. (b) | 32. (c) | 33. (d) |
| 34. (b) | 35. True. | | | | |

Problems 10.1, page 400

$$1. \frac{2 \sinh a\pi}{\pi} \left\{ \left(\frac{1}{2a} - \frac{a \cos x}{1^2 + a^2} + \frac{a \cos 2x}{2^2 + a^2} - \dots \right) + \left(\frac{\sin x}{1^2 + a^2} - \frac{2 \sin 2x}{2^2 + a^2} + \frac{3 \sin 3x}{3^2 + a^2} - \dots \right) \right\}$$

$$\frac{\pi}{\sinh \pi} = 2 \left[\frac{1}{2^2 + 1} - \frac{1}{3^2 + 1} + \frac{1}{4^2 + 1} - \dots \right].$$

Problems 10.2, page 401

1. No 2. No 3. Yes.

Problems 10.3, page 404

1. $\frac{1}{2}\pi - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$ 2. $\frac{I_0}{\pi} + \frac{1}{2} I_0 \sin x - \frac{2I_0}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}, \frac{1}{2}$
 3. $\frac{\pi^2}{6} - 2 \left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right) - \frac{1}{\pi} \left\{ \left(\frac{2}{1^3} - \frac{\pi^2}{1} \right) \sin x - \left(\frac{2}{2^3} - \frac{\pi^2}{2} \right) \sin 2x + \dots \right\}$
 4. $2 \left(\pi - \frac{4}{\pi} \right) \sin x - \pi \sin 2x + \frac{2}{3} \left(\pi - \frac{4}{9\pi} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \dots$
 5. $\frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right).$

Problems 10.4, page 408

$$1. -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n}$$

2. (i) $\frac{a^2}{3} \frac{a^2}{11^2} \left\{ \frac{1}{12} \cos \frac{2\pi x}{a} + \frac{1}{2^2} \cos 4\pi \frac{x}{a} + \dots \right\} \frac{a^2}{\pi} \left\{ \frac{1}{1} \sin \frac{2\pi x}{a} \frac{1}{2} \sin \frac{4\pi x}{a} + \dots \right\}$

(ii) $f(t) = \frac{2}{3} + \frac{4}{\pi^2} \left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \dots \right)$

4. $\frac{3}{2} - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left\{ \frac{\cos (2n-1)\pi x}{3} \right\}$

5. $\frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left[\frac{1}{1.3} \cos 2\omega t + \frac{1}{3.5} \cos 4\omega t + \frac{1}{5.7} \cos 6\omega t + \dots \right]$

6. $f(x) = \frac{\pi}{4} + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sin n\pi x$; put $x = 1/2$.

Problems 10.5, page 412

1. $\frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{4a^2}{n^2 \pi^2} (-1)^n \cos \frac{n\pi x}{a}$

3. $1 - \frac{1}{2} \cos x - \frac{2}{1.3} \cos 2x + \frac{2}{2.4} \cos 3x - \frac{2}{3.5} \cos 4x - \dots$

5. $\frac{1}{2}\pi - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$

6. (i) $\frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \dots + \frac{\cos 2nx}{4n^2 - 1} + \dots \right)$ (ii) $\frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{m+1}}{(4m^2 - 1)\pi} \cos \frac{2m\pi x}{l}$

7. $\frac{\pi}{2} + 1 - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right); \frac{\pi^2}{8}$

8. $\frac{4k}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right).$

Problems 10.6, page 416

2. $\frac{\pi}{2} - \frac{4}{\pi} \left[\cos x + \frac{1}{3^2} \cdot \cos 3x + \frac{1}{5^2} \cdot \cos 5x + \dots \right]; 2 [\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots]$

3. $\frac{\pi^2}{3} - 4 \left[\cos x - \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x - \frac{1}{4^2} \cos 4x + \dots \right]$

4. $\sum_{n=2}^{\infty} \frac{1}{n} \sin 2nx$

5. $\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x$

6. $\frac{8}{\pi^3} \left(\frac{\sin \pi t}{1^3} + \frac{\sin 3\pi t}{3^3} + \frac{\sin 5\pi t}{5^3} + \dots \right)$

7. $\frac{1}{n\pi} [1 - (-1)^n]$

8. $-\frac{1}{2} \sin x + \sum_{n=2}^{\infty} \frac{(-1)^n \cdot 2n}{n^2 - 1} \sin nx$

9. $\sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2} (1 - e \cos n\pi) \sin nx$

10. $\sum_{n=1}^{\infty} \frac{4}{\pi n^3} [1 - (-1)^n] \sin nx$

12. $\frac{l}{4} + \sum_{n=1}^{\infty} \frac{2l}{(n\pi)^2} \left\{ 2 \cos \frac{n\pi r}{2} - 1 - (-1)^n \right\} \cos \frac{n\pi x}{l}$

13. $\frac{8}{\pi} \cos \frac{\pi}{4} \left[\frac{\sin 2x}{1 \cdot 3} - \frac{\sin 6x}{5 \cdot 7} + \frac{\sin 10x}{9 \cdot 11} + \dots \right]$

14. $\frac{2l^2 h}{a(l-a)\pi^2} \left[\sin \frac{\pi a}{l} \sin \frac{\pi x}{l} + \frac{1}{2^2} \sin \frac{2\pi a}{l} \sin \frac{2\pi x}{l} + \frac{1}{3^2} \sin \frac{3\pi a}{l} \sin \frac{3\pi x}{l} + \dots \right].$

Problems 10.7, page 419

3. $\pi^4/96$.

Problems 10.8, page 420

1.
$$\sum_{n=-\infty}^{\infty} \frac{(\sinh al \cos n\pi l - i \cosh al \sin n\pi l)(a + in\pi)}{(a^2 + n^2\pi^2)} e^{inx/l}$$

2.
$$\frac{2}{\pi} \left\{ 1 - \frac{e^{2it} + e^{-2it}}{1 \cdot 3} - \frac{e^{4it} + e^{-4it}}{3 \cdot 5} - \frac{e^{6it} + e^{-6it}}{5 \cdot 7} - \dots \right\}$$

3.
$$\frac{a}{\pi} \sin a\pi \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2 - n^2}$$

4.
$$\sin h 9 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (9 + n\pi i)}{81 + (n\pi)^2} e^{n\pi ix/3}$$

5.
$$\frac{a}{2} - \frac{a}{\pi} \left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \frac{1}{5}(e^{5u} - e^{-5u}) + \dots \right] \text{ where } u = i\pi x/l.$$

Problems 10.9, page 423

1.
$$11.733 - 7.733 \cos 2x - 2.833 \cos 4x + \dots - 1.566 \sin 2x - 0.116 \sin 4x + \dots$$

2.
$$1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x)$$

3.
$$a_0 = 41.66, a_1 = -8.33, b_1 = -1.15 \quad 4. -0.0731$$

5.
$$y = 2.102 + 0.558 \cos x + 1.531 \sin x + 0.354 \cos 2x + 0.145 \sin 2x$$

6.
$$7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta - \dots$$

Problems 10.10, page 424

1. $2\pi/3$

2. $\frac{1}{2} \{f(c-0) + f(c+0)\}$

3. $(-1, 1)$ such that $f(x) = -f(-x)$

4. $f(x) = A$ when $0 < x < \pi$ and $f(x) = -A$ when $\pi < x < 2\pi$

5. Sine

6. § 10.11 (3)

7. Zero

8. not defined

9. odd

10. Cosine

11. even

12. $x = k/n$

13. Zero

14. Cosine

15. Zero

16. $\int_0^2 x^2 \cos \frac{n\pi x}{2} dx$

17. $\frac{1}{T} \int_a^{a+2T} f(x) \sin \frac{n\pi x}{T} dx$

18. § 10.3

19. Zero

20. $a_0 = \frac{2}{l} \int_0^l f(x) dx, a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$

21. $\frac{4}{\pi} \left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right\}$

22. π

23. Zero

24. $2l$

25. $\sum_{n=-\infty}^{\infty} (-1)^n \frac{(1-in\pi)}{1+n^2\pi^2} \sinh 1 e^{inx}$

26. False

27. $-\pi/2$

28. odd

29. Zero

30. 3.5355

31. zero

32. $-1/2$

33. $\frac{1}{2} a_n$

34. $\frac{\pi^2}{8}$

35. $\frac{\pi}{2} - \frac{4}{\pi} \left\{ \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \dots \right\}$

36. $f(x) = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos 4x$

37. $x^2 - x$

38. $x(l+x)$

39. True

40. False

41. False.

Problems 11.1, page 429

1. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0$

2. $\frac{d^2y}{dx^2} + 4y = 0$

3. $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy = 0$

4. $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$

5. $\frac{d^3y}{dx^3} - 7 \frac{dy}{dx} + 6y = 0$

6. $2xy \frac{dy}{dx} + x^2 - y^2 = 0$

7. $(x^2 - 25) \left(\frac{dy}{dx} \right)^2 + x^2 = 0$

8. $y \frac{dy}{dx} = 2a$

10. $y''' - 3y'' + 3y' - y = 0$

Problems 11.2, page 431

1. $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

2. $\log \frac{x}{y} - \frac{1}{x} - \frac{1}{y} = c$

3. $\tan x \tan y = c$

4. $(1+x^2)^{3/2} - 3\sqrt{1+y^2} = c$

5. $\tan y = c(1-e^x)$

6. $2e^{-y} = e^{-x^2} + 1$

7. $x = 2 \cos y$

8. $(x^2+1)(y^2+1) = c$

9. $3e^{2x} - 2e^{3y} + 8x^3 = c$

10. $(1-ay)(a+x) = cy$

11. $(x+1)(2-e^y) = c$

12. $a \log \left(\frac{x-y-a}{x-y+a} \right) = 2y + c$

13. $y = \tan^{-1}(x+y+1) + c$

14. $\tan(x+y) = \sec(x+y) + x + c$

15. $x = \operatorname{cosec}(x+y+1) - \cot(x+y+1) + c$

16. $\log \sin(y-x) = \frac{1}{2}x^2 + cz$

17. $\cos xy + \frac{1}{2x^2} = c$

Problems 11.3, page 432

1. $x(x^2 - 3y^2) = c$

2. $cy^3 = x^2 e^{-xy}$

3. $(x/y)^3 = 3 \log cy$

4. $y + \sqrt{x^2 + y^2} = c$

5. $y^2 = 2x[y + x \log(cx)]$

6. $x(c+y) = ay^2$

7. $y = 2x \tan^{-1}(cx)$

8. $e^{xy} = y + c$

9. $\log y - \frac{x^2}{4y^2} \left(z \log \frac{y}{x} + 1 \right) = c$

10. $\log x = \frac{1}{2} \left[\frac{y}{x} - \frac{1}{2} \sin \left(\frac{2y}{x} \right) \right] + c$

Problems 11.4, page 434

1. $(X^2 + 2Y^2)^2 = c \left(\frac{\sqrt{2}Y - X}{\sqrt{2}Y + X} \right)$ where $X = x + 1$, $Y = y - 1$

2. $(y-x)^3 = c(y+x-2)$

3. $(x+y)^7 = c(x-y-2/3)^3$

4. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

5. $3(2y-x) + \log(3x+3y+4) = c$

6. $x-y + \frac{3}{4} \log(8x-12y-5) = c$

7. $\log(x+y+\frac{1}{3}) + \frac{3}{2}(y-x) = c$

Problems 11.5, page 437

1. $y = ce^{-\tan x} + \tan x - 1$

2. $y = \log x + c/\log x$

3. $y \sec^2 x = \sec x - 2$

4. $y \cosh x = c + \frac{2}{3} \cosh^3 x$

5. $y \sqrt{(1-x^2)} = \sin^{-1} x + c$

6. $y = c(1-x)^2 + (1-x^2)$

7. $y(1+\sin x) = c - x^2/2$

8. $2r \sin^2 \theta + \sin^4 \theta = c$

9. $ye^{x^2} = 2x + c$

10. $x = y^3 + cy$

11. $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$

12. $xy^{-2} = c - e^{-y}$

13. $xe^{\tan^{-1} y} = \tan^{-1} y + c$

14. $xe^y = c + \tan y.$

Problems 11.6, page 439

1. $y^{-1} \sec x = \tan x + c$

2. $1/r = \sin \theta + c \cos \theta$

3. $x^2 + (4x^5 + c)y^4 = 0$

4. $1/y = x^2 - 2 + ce^{-x^2/2}$

5. $y^2 = x^2 + cx - 1$

6. $y/x = \log y + c$

7. $\sin y = (1+x)(e^x + c)$

8. $e^{x+y} = \frac{1}{2}e^{2x} + c$

9. $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$

10. $\cos y = \cos x (\sin x + c)$

11. $\sqrt{x} = \sqrt{y}(\log \sqrt{y} + c)$

12. $y^{-1} = \frac{1}{2} \log x + \frac{1}{4} + cx^2.$

Problems 11.7, page 442

1. $x^3 + y^3 - 3axy = c$

2. $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

3. $x^3 - 6x^2y - 6xy^2 + y^3 = c$

4. $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$

5. $e^{xy} + y^2 = c$

6. $x^5 + x^3y^2 - x^2y^3 - y^5 = c$

7. $x^3 + 3x^2y^2 + y^4 = c$

8. $x^2 - y^2 = cy^3$

9. $3y \cos 2x + 6y + 2y^3 = c$

10. $e^x = \sec x \tan y + c$

11. $x^2y + xy - x \tan y + \tan y = c.$

Problems 11.8, page 445

1. $ax + \tan^{-1} y/x = c$

2. $x^2 + y^2 - 2a^2 \tan^{-1}(y/x) = c$

3. $y + cx + \log x + 1 = 0$

4. $3 \log x - (y/x)^3 = c$

5. $\log(y/x) + \frac{1}{2}x^2y^2 = c$

6. $xy + \log(x/y) - (1/xy) = c$

7. $(y + 2/y^2)x + y^2 = c$

8. $4x^4y + 4x^3y^2 - x^4 = c$

9. $2 \cos(xy) + x^{-2} = c$

10. $\log(x/y) = c + xy$

11. $(x/y) + e^{x^3} = c$

12. $4(xy)^{1/3} - \frac{2}{3}(x/y)^{3/2} = c$

13. $4y \log x = y^2 + c.$

Problems 11.9, page 446

1. $(x - y + c)(x^2 + y^2 + c) = 0$

2. $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$

3. $x^2 + y^2 = cx$

4. $(y - cx)(y^2 - x^2 - c) = 0$

5. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0.$

Problems 11.10, page 448

1. $x + c = \frac{a}{2} \left[\log \frac{p-1}{\sqrt{(1+p^2)}} - \tan^{-1} p \right],$ with the given relation

2. $xy = c^2x + c$

3. $y = 2\sqrt{xc} + c^2$

4. $2cy = c^2x^2 + 1$

5. $x = (\log p - p + c)(p-1)^2,$ with the given relation

6. $x = \sin p + c,$ with the given relation.

Problems 11.11, page 449

1. $y = c(x - c)^2$

2. $y^2 = 2cx + c^3$

3. $(y + \alpha p)\sqrt{(p^2 - 1)} + a \cosh^{-1} p = c,$ with the given relation

4. $y + (1 + p^2)^{-1} = c,$ with the given relation.

Problems 11.12, page 450

1. (i) Gen. sol.: $y = cx + a/c^2$; Singular sol.: $2ax^2 = (2ac + x)^3$
(ii) Gen. sol.: $c = \log(cx - y)$; Singular sol.: $y = x(\log x - 1)$
(iii) Gen. sol. $y = cx + \sqrt{a^2c^2 + b^2}$; Singular sol. $y + \sqrt{1 - x^2} = 0$
(iv) Gen. sol. $y = cx - \sin^{-1} c$; Singular sol. $y = \sqrt{x^2 - 1} - \sin^{-1} \frac{\sqrt{x^2 - 1}}{x}$
2. $y = cx + (c - 2c^2)$ 3. $(y - cx)(c - 1) = c$
4. $(y - cx)(c + 1) + ac^2 = 0$ 5. $y^2 = cx^2 + c^2$ [Hint: Put $x^2 = u, y^2 = v$]
6. $xy = cy - c^2$ [Hint: Put $u = y, v = xy$] 7. $y^2 = cx^2 - \frac{2c}{1+c}$.

Problems 11.13, page 450

1. (i) 2. (ii) 3. (iii) 4. (i) 5. $\log y + c = x^2/2y^2$
6. $yx^2 = x^3 + c$ 7. $e^x + x^2y + cy = 0$. 8. (iii) 9. $x^2 + y^2 + 2 \tan^{-1} y/x = c$
10. $\log x + c = y^3/3x^3$ 11. (i) 12. $y^2 + 1/x + ce^{-y^2/2} = 2$
13. $y = cx + a/c^2$ 14. $c = \log(cx - y)$ 15. $xy = c$ or $x^2 - y^2 = c$
16. 2 17. $xy = c$ 18. (b) 19. (b)
20. $(1 + x^2)^{3/2} + (1 + y^2)^{3/2} = c$ 21. $y = 5e^{-x}$ 22. $x + y = u$
23. x^{-5} 24. § 11.11 (3) 25. $5x^4y^2 + 2(x^5 + y^5) = c$
26. $\sin(y/x) = cx$ 27. (a) 28. (c) 29. $x + y dy/dx = 0$
30. $e^{-x^2} + 2 \cos y = c$ 31. (c) 32. False 33. False.

Problems 12.1, page 454

1. (i) $9y + 4x^2 = 0$; (ii) $3(x + 3y) = 2(1 - e^{3x})$ 2. $y + 1 = 2e^{x^2/2}$
3. $x^2 + y^2 = cx$ 4. $y = \sqrt{a^2 - x^2} + a \log \left(\frac{\alpha - \sqrt{a^2 - x^2}}{x} \right) + c$
5. $y^2 = 4x$ 6. $y = ae^{cx}$ 7. $y = ax + b$
8. $x = 3y^2$ 9. (i) $r(\theta - \alpha) = c$; (ii) $r = a + b \cos \theta$
10. $r^2 = a^2 \sin 2\theta$ 11. $c^2x^2 = 2cy + 1$ 12. $r = ae^{\theta \cot \alpha}$.

Problems 12.2, page 457

1. $2x^2 + y^2 = c$ 2. $x^2 + 2y^2 = c^2$ 3. $3y^2 + 2x^2 = c^2$
4. $x^2 + y^2 + 2\mu y - c = 0$ 5. This system is self-orthogonal 6. $r = c(1 - \cos \theta)$
7. $r = b(\cos \theta - \sin \theta)$ 8. $r = 2b/(1 - \cos \theta)$ 9. $r^2 = c^2 \sin 2\theta$
10. $r^n n \sin \theta = b$ 13. $x^2 + y^2 + cx + 1 = 0$ 14. $y = cx$.

Problems 12.3, page 462

1. $V = \sqrt{\left(\frac{mg}{k}\right)} \tanh\left(\frac{9k}{m}t + c\right)$ 3. $\frac{1}{k} \log_e 2$
5. $2\sqrt{v_o/k}$ 6. $v^2 = 2gx - \frac{\lambda}{m} x^2$
10. $y = (\sqrt{150} - 0.001328t)^2$; $t_1 = 45$ min. 1 sec., $t_2 = 1$ hr. 16 min. 51 sec., $t_3 = 1$ hr. 38 min. 13 sec
11. 17 min. 4 sec.

Problems 12.4, page 465

1. 0.0006931 sec

2. $\frac{10}{L^2 + R^2} (R \sin t - L \cos t + Le^{-Rt/L})$

3. $i = \frac{1}{5} (1 - e^{-100})$

4. $i = ke^{-t/RC} + \frac{\omega CE_m}{\sqrt{(1 + R^2 C^2 \omega^2)}} \sin(\omega t + \theta)$ where $\theta = \cot^{-1}(RC\omega)$.

Problems 12.5, page 467

1. 52.5 mts

2. 48°C

3. *B* drinks hotter coffee

4. 490,000 cal

5. 2.16 cm.

Problems 12.6, page 469

1. 604.9

2. $2 \log 3 / \log 2$

3. $(1 - 1/p)^{21}$ times the original amount

4. 64.5 days

5. 21.5 gm

6. $t = 300 - 5 \log 2 + 5 \log \frac{0.7 - x}{0.5 - x}$

7. 3 hr. 50 min. 16 sec

8. $100(2 - e^{-t/20})$; 13.9 min.

Problems 12.7, page 469

1. $6(1 - e^{-3})$

2. 54 m

3. 90.25%

4. $r(\theta - \alpha) = c$

5. $y = ae^{cx}$

6. rectangular hyperbola

7. $x^2 - y^2 = c$

8. The system is self-orthogonal

9. $2\sqrt{v_0}/k$

10. $2 \log 3 / \log 2$

11. Sunil

12. (d)

13. (c)

14. (d)

15. 2.21

16. (c)

17. (c)

18. (a)

19. False

20. True.

Problems 13.1, page 474

1. $\frac{2}{3} e^{2t} \sin 3t$

2. $y = e^x (4 \cos 3x - \sin 3x)$

3. $y = c_1 + (c_2 + c_3 x) e^{-x/2}$

4. $y = c_1 e^{-x} + e^{x/2} \left(c_2 \cos \frac{\sqrt{3x}}{2} + c_3 \sin \frac{\sqrt{3x}}{2} \right)$

5. $y = (c_1 + c_2 x + c_3 x^2) e^x$

6. $y = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

7. $y = c_1 e^{-x} + c_2 e^{2x} + e^{x/2} (c_3 \cos x/\sqrt{2} + c_4 \sin x/\sqrt{2})$

8. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + c_5 e^x$.

Problems 13.2, page 486

1. $y = (c_1 + c_2 x) e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$

2. $y = \frac{3}{5} e^{-2x} (\cos x + 3 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$

3. $x = c_1 \cos nt + c_2 \sin nt + \frac{kx}{2n} \sin(nt + \alpha)$

4. $x = e^{-t} (c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t) + \frac{1}{4} (\sin t - \cos t)$

5. $y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{1}{10} (3 \sin 2x - \cos 2x)$

6. $y = c_1 e^x + c_2 e^{3x} + \frac{1}{884} (10 \cos 5x - 11 \sin 5x) + \frac{1}{20} (\sin x + 2 \cos x)$

7. $y = c_1 + (c_2 + c_3 x) e^{-x} - \frac{x^2}{2} e^{-x} + \frac{3}{50} \cos 2x - \frac{2}{25} \sin 2x$

8. $y = (c_1 + c_2 x) e^{-x} + \frac{1}{2} + \frac{1}{5} (2 \sin 2x + \cos 2x)$

9. $y = (c_1 + c_2 x) e^x + c_3 e^{3x} + \frac{1}{8} (xe^{3x} - x^2 e^x)$ 10. $y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{12} (2x^3 - 3x^2 + 9x)$

11. $y = c_1 + c_2 e^x + c_3 e^{-x} + xe^x - (x^2 + x) - 2 \sin x$

12. $y = e^{3x} (c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{17} e^{2x} + \frac{1}{565} (23 \sin x + 6 \cos x) + \frac{x}{25} + \frac{6}{625}$

13. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + x^4 - 24x^2 + 72 + \frac{1}{225} \sin 4x - \frac{1}{9} \sin 2x$

14. $y = c_1 e^{-2x} + c_2 e^{-3x} - \frac{e^{-2x}}{10} (\cos 2x + 2 \sin 2x)$

15. $y = e^{-x/2} \left\{ (c_1 + x/4) \cos (x\sqrt{3/2}) + (c_2 + x/4\sqrt{3}) \sin (x\sqrt{3/2}) \right\} + e^{x/2} \left\{ c_3 \cos \sqrt{3x/2} + c_4 \sin \sqrt{3x/2} \right\}$

16. $y = e^{-x} (c_1 \cos \sqrt{2x} + c_2 \sin \sqrt{2x}) + \frac{e^x}{41} (4 \sin x + 5 \cos x)$.

17. $y = c_1 e^{-x} + c_2 e^{-3x} - \frac{e^{-x}}{5} (\sin x + 2 \cos x) + \frac{e^{3x}}{22} \left(x - \frac{5}{11} \right)$

18. $y = c_1 \cos \sqrt{2x} + c_2 \sin \sqrt{2x} + \frac{e^{3x}}{11} \left(x^2 - \frac{12}{11}x + \frac{50}{121} \right) + \frac{e^x}{17} (4 \sin 2x - \cos 2x)$

19. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - (1/5) \cos x \cosh x$

20. $y = c_1 + (c_2 + c_3 x) e^{-x} + \frac{e^{2x}}{18} \left(x^2 - \frac{7x}{8} + \frac{11}{6} \right) + \frac{1}{100} (3 \sin 2x + 4 \cos 2x)$

21. $y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{7} \left(x \sin 3x - \frac{6}{7} \cos 3x \right)$

22. $y = (c_1 + c_2 x) e^{-x} + \frac{1}{2} \cos x + \frac{1}{2} (x - 1) \sin x$

23. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} (x \sin x + \cos x) + (xe^x/12) (2x^2 - 3x + 9)$

24. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + e^{-2x} \cdot e^{e^x}$

25. $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log (\sec ax + \tan ax)$.

Problems 13.3, page 490

1. $y = (c_1 - x/a) \cos ax + [c_2 + (1/a^2) \log \sin ax] \sin ax$

2. $y = c_1 \cos x + c_2 \sin x + \cos x \log (\cos x) + x \sin x$

3. $y = c_1 \cos x + c_2 \sin x - \cos x \log (\sec x + \tan x)$

4. $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x - \frac{x^2}{4} \cos x$ 5. $y = (c_1 + c_2 x) e^x + x e^x \log x$

6. $y = (e^x + e^{2x}) \log (1 + e^x) + (c_1 - 1 - x) e^x + (c_2 - x) e^{2x}$

7. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

8. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$

9. $y = c_1 \cos x + c_2 \sin x + \sin x \log (1 + \sin x) - x \cos x - 1$

10. $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} (x^2 + 3x + 3.5 - 2xe^x)$ 11. $y = c_1 \cos x + c_2 \sin x - x \sin x$

12. $y = c_1 e^{2x} + c_2 e^{3x} + x e^{3x} + \frac{1}{10} (\sin x + \cos x)$ 13. $y = c_1 e^x + c_2 e^{-2x} - \frac{1}{4} (2x + 1) - \frac{1}{10} (\cos x + 3 \sin x)$
14. $y = e^x (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{1}{27} (9x^3 + 18x^2 + 6x - 8) + \frac{1}{4} (\cos x - \sin x)$
15. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x.$

Problems 13.4, page 495

1. $y = c_1 x^2 + c_2 x^3 - x^2 \log x$
2. $y = c_1 x^4 + c_2 x^{-1} + \frac{x^4}{5} \log x$
3. $y = (c_1 + c_2 \log x) x^2 + \frac{1}{4} + 2x + \frac{1}{2} x^2 (\log x)^2$
4. $y = c_1 x^2 + c_2 x^{-1} + \frac{1}{3} (x^2 - 1/x) \log x$
5. $u = \frac{kr}{8} (a^2 - r^2)$
6. $c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2} \log x - \frac{3}{4}$
7. $y = c_1 x^{-1} + \sqrt{x} [c_2 \cos ((\sqrt{3}/2) \log x) + c_3 \sin ((\sqrt{3}/2) \log x)] + \frac{1}{2} x + \log x$
8. $y = c_1 x^{-2} + x [c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) - \sin (\log x)$
9. $y = c_1 x^{-1} + [c_2 \cos (\log x) + c_3 \sin (\log x)] x + 5x + 10 \log x/x$
10. $y = \frac{1}{x} (c_1 + c_2 \log x) + \frac{1}{x} \log \frac{x}{1-x}$
11. $y = x^{-2} (c_1 + c_2 \log x) + \frac{x}{9} \left(\log x - \frac{2}{3} \right)$
12. $y = c_1 x^3 + c_2 x^{-4} + \frac{x^3}{98} \log x (7 \log x - 2)$
13. $y = c_1 (2x+3)^a + c_2 (2x+3)^b - \frac{3}{14} (2x+3) + \frac{3}{4}$ where $a, b = \frac{3 \pm \sqrt{57}}{4}$
14. $y = c_1 (x-1) + c_2 (x-1)^2 + c_3 (x-1)^{-2} + \log (x+1) + 1$
15. $y = c_1 \cos \log (1+x) + c_2 \sin \log (1+x) - \frac{1}{3} \sin [2 \log (1+x)]$
16. $y = c_1 (3x+2)^{1/3} + c_2 (3x+2)^{-1} + \frac{1}{27} \left[\frac{1}{15} (3x+2)^2 + \frac{1}{4} (3x+2) - 7 \right]$

Problems 13.6, page 499

1. $x = (c_1 + c_2 x) e^{3x}; y = [(1-2x)(c_2 - 2c_1)] e^{3x}$
2. $x = e^t + e^{-t}, y = e^{-t} - e^t + \sin t$
3. $x = c_1 e^t + c_2 e^{-5t} + \frac{6}{7} e^{2t}; y = c_2 e^{-5t} - c_1 e^t + \frac{8}{7} e^{2t}$
4. $x = e^{6t} (c_1 \cos t + c_2 \sin t), y = e^{6t} [(c_1 - c_2) \cos t + (c_1 + c_2) \sin t]$
5. $x = \frac{1}{5} e^t + \frac{2}{5} e^{-t} - c_1 \sin 2t + c_2 \cos 2t, y = \frac{2}{5} e^t + \frac{1}{5} e^{-t} + c_1 \cos 2t + c_2 \sin 2t$
6. $x = c_1 e^t + c_2 e^{-5t} + \frac{3}{7} e^{2t} - \frac{2}{5} t - \frac{13}{25}, y = c_1 e^t - c_2 e^{-5t} - \frac{4}{7} e^{2t} - \frac{3t}{5} - \frac{12}{25}$
7. $x = -t - \frac{2}{3}, y = \frac{1}{2} t^2 + \frac{4}{3} t + c$
8. $y = c_1 e^x + c_2 e^{-2x} + 2e^{-x}, z = 3c_1 e^x + 2c_2 e^{-2x} + 3e^{-x}$
9. $x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5} (\cos t - 2 \sin t); y = 2c_1 e^{-t} - 2c_2 e^{3t} + \frac{1}{5} (\sin t + 2 \cos t)$
10. $x = \frac{1}{2} \left(t + \frac{1}{t} \right), y = \frac{1}{2} \left(-t + \frac{1}{t} \right)$
11. $x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5} (\cos t - 2 \sin t), y = 2c_1 e^{-t} - 2c_2 e^{3t} + \frac{2}{5} \cos t + \frac{1}{5} \sin t$
12. $x = (c_1 + c_2 t) e^{-t} + (c_3 + c_4 t) e^t, y = -\frac{1}{2} [c_1 + c_2 (1+t)] e^{-t} + \frac{1}{2} [c_4 (1-t) - c_3] e^t$

13. $x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4} \cos t + \frac{t}{4} \sin t$

$$y = -c_1 e^t - c_2 e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}(2+t)(\sin t - \cos t)$$

14. $x = \frac{8}{9}\left(1 - \cos \frac{3}{2}t\right), y = \frac{4}{3}t - \frac{8}{9} \sin \frac{3}{2}t.$

Problems 13.7, page 500

1. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

2. $-\frac{1}{25}(3 \sin 2x + 4 \cos 2x)$ **3.** $1/6$

5. (b)

8. $y = e^x(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$ **9.** $y = \cos x + 2 \sin x$

10. (ii)

11. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

12. $\frac{1}{10} \cosh 3x$

13. $y = a \log x + 6$

14. $y = (c_1 + c_2 x) e^{\sqrt{2}x} + (c_3 + c_4 x) e^{-\sqrt{2}x}$

15. $\frac{1}{6}x^3 e^{-x}$

16. $\frac{1}{2}e^{2x}$

17. $\sin 2x$

18. $\frac{1}{2}x^2 e^{-x}$

19. $y = (c_1 + c_2 x) e^{-x/2} + c_3$

20. (c)

21. (a)

22. $x e^{-t}$

23. $\frac{d^2y}{dt^2} + 7y = 2e^t.$

24. (c)

25. (a)

26. (b)

27. $y = (c_1 + c_2 \log x)x$

28. $x^2 y_2 + xy_1 - y = 0$

29. $\frac{1}{9} \log 2$

30. e^t

31. (d)

32. $y = c_1 e^{-x} + c_2 e^{2(1+\sqrt{2})x} + c_3 e^{2(1-\sqrt{2})x}$

33. $\frac{d^3y}{dx^3} + 2$

34. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

35. False

36. False.

Problems 14.1, page 506

1. 38 sec

4. $x = \frac{ue^{-\lambda nt}}{n\sqrt{1-\lambda^2}} \sin [nt\sqrt{(1-\lambda^2)}]$

6. It must be shortened by 1/8640 of its length

7. It must be increased by 0.0074 ft./sec²

8. 4321/4319

9. $k^2 > 4\mu, \theta = c_1 e^{\frac{-k+\lambda}{2}t} + c_2 e^{\frac{-k-\lambda}{2}t}$ where $\lambda = \sqrt{(k^2 - 4\mu)}$
 $k^2 = 4\mu, \theta = (c_1 + c_2 t) e^{-kt/2}$

$$k^2 < 4\mu, \theta = c_1 e^{-kt/2} \cos \left(\frac{\sqrt{4\mu - k^2}}{2}t + c_2 \right)$$

10. $x = \frac{F_0}{2n^2} (\sin nt - n \cos nt).$

Problems 14.2, page 513

2. 1 ft.; $\pi/2\sqrt{2}$ sec; $4\sqrt{2}$ ft/sec 4. $\pi/\sqrt{7}$
5. $x = e^{-5t} \left\{ \cos \sqrt{220}t + (5/\sqrt{220}) \sin \sqrt{220}t \right\}$. 6. 0.45 sec.; 1.15 sec
8. $x = \frac{10}{21}e^{-t} - \frac{17}{27}e^{-8t} - \frac{\sqrt{2}}{3} \sin \left(9t + \frac{\pi}{4} \right) \frac{\sqrt{2}}{3}, \frac{2\pi}{9}$ sec., $9/2\pi$ cycles/sec
9. $0.8(2 \sin 4t - \cos 4t)$
10. (i) $x = Ae^{-kt} \cos \left\{ t\sqrt{(b^2 - k^2)} + B \right\} + \{e^{-kt}/(b^2 + k^2 - n^2)\} \sin nt$
(ii) $x = Ae^{-kt} \cos \left\{ t\sqrt{(b^2 - k^2)} + B \right\} - (te^{-kt}/2n) \cos nt$

Problems 14.3, page 517

2. $i = I \sin (T/\sqrt{LC})$
3. $i = 2Eke^{-RT/2L} \sin (kt/2L)$, where $k = \sqrt{\left(\frac{4L - CR^2}{C} \right)}$
4. $R^2 > 4L/C$ for over damping; $R^2 = 4L/C$ for critical damping; $R^2 < 4L/C$ for under damping; critical resistance $= 2\sqrt{L/C}$
5. $q = e^{-500t}(0.002 \cos 1323t + 0.0008 \sin 1323t)$
8. (i) $i = Ae^{-\alpha t} \cosh(\beta t + \gamma)$; (ii) $i = Ae^{-\alpha t} \cos(\beta t + \gamma) + \frac{E}{R} \cos \phi \sin(pt + \phi)$
where $\alpha = -\frac{R}{2L}$, $\beta = \pm \sqrt{\left(\frac{R}{2L} \right)^2 - \frac{1}{CL}}$ and $\phi = \tan^{-1} \{(1 - CLp^2)/CRp\}$.

Problems 14.4, page 525

4. $y = \frac{wl}{2Pn} \operatorname{cosec} \frac{nl}{2} \cos \left(nx - \frac{nl}{2} \right) - \frac{wl}{2nP} \cot \frac{nl}{2} + \frac{w}{2P} (x^2 - lx)$
6. $y = \frac{F}{P} [n \sin nx - l \cos nx + l - x]$
7. $\pi^2 EI/4l^2$
8. $\frac{W}{2a^2} \left(\operatorname{sech} \frac{al}{2} - \sec al \right)$

Problems 14.5, page 528

1. $\frac{2u \sin \alpha}{g}; \frac{u^2 \sin 2\alpha}{g}$
2. (i) $\frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta}$; (iii) $\frac{u^2}{g(1 + \sin \beta)}$
4. $4x^2 + k^2y^2 = 4$
5. $i_1 = \frac{a}{p + \omega} \sin pt, i_2 = \frac{a}{p + \omega} \cos pt.$
6. $i_1 = \frac{E}{R} \left(\frac{2}{3} - \frac{1}{2} e^{-Rt/L} - \frac{1}{6} e^{-3Rt/L} \right); i_2 = \frac{E}{R} \left(\frac{1}{3} - \frac{1}{2} e^{-Rt/L} + \frac{1}{6} e^{-3Rt/L} \right)$
7. $x = a(nt - \sin nt), y = a(1 - \cos nt)$
8. $x = \frac{E}{H\omega} (1 - \cos \omega t), y = \frac{E}{H\omega} (\omega t - \sin \omega t)$, where $\omega = eH/m$.

Problems 14.6, page 529

1. (b)

5. (b)

2. (b)

6. (b)

3. (c)

7. (b)

4. (b)

8. 60 sec

9. $30/\pi\sqrt{LC}$

10. 0.0074 sec

11. resonance

12. $EI \frac{d^2y}{dx^2} - Py = \frac{w}{2}(x^2 - lx)$

13. $y = 0$ and $\frac{dy}{dx} = 0$

Problems 15.1, page 531

1. $y = -x^2 \sin x - 4x \cos x + c_1 x + c_2$

2. $y = \frac{x^4}{24} + \frac{x^3}{6} \log x - \frac{11}{35}x^3 + c_1 x^2 + c_2 x + c_3$

Problems 15.2, page 533

1. $2(y^{1/4} - 1) = x$

2. $\sqrt{(y^2 - 8y)} + 4 \cosh^{-1}(\frac{1}{4}y - 1) = 3x$

3. $r = \frac{\sqrt{(v^2 - a^2\omega^2)}}{\omega} \sinh \left[\omega t + \sinh^{-1} \frac{a\omega}{\sqrt{(v^2 - a^2\omega^2)}} \right]$

4. $t = \frac{h^{3/2}}{\sqrt{(2g)a}} \left[\cos^{-1} \sqrt{\frac{x}{h}} - \sqrt{\left(\frac{hx - x^2}{h} \right)} \right]$

Problems 15.3 page 534

1. $y = c_1 - x^2 - c_2/x$

2. $y = c_1 x + (c_1^2 + 1) \log(x - c_1) + c_2$

3. $15c_1^2 y = 4(c_1 x + a^2)^{5/2} + c_2 x + c_3$

4. $x^2 + y^2 = a^2$

5. $\theta = \frac{m}{\mu a} \log \left(1 + \frac{\mu a \omega t}{m} \right)$

6. $v = \frac{1}{r_1 - r_2} \left[v_1 r_1 - v_2 r_2 - \frac{(v_2 - v_1)r_1 r_2}{r} \right].$

Problems 15.4, page 536

1. $y = 2x - 2 \log(1 - c_1 e^{2x}) + c_2$

2. $y^2 = x^2 + c_1 x + c_2$

3. $\log y = c_1 e^x + c_2 e^{-x}$

4. $(\log y - 1)(c_1 x + c_2) = 1$

5. $(x - a)^2 + y^2 = c^2$, circles whose centres are on the x -axis.

Problems 15.5, page 537

1. $y = e^{x^2}(c_1 x + c_2)$

2. $y = (x^2 - x + c_1 x)e^x + c_2 x$

3. $y = e^x(c_1 \log x + x + c_2)$

4. $cy = 1 + (k - x) \cot x$

5. $y = \left[c_1 - \frac{1}{2} \cos x - \frac{1}{5} c_2 e^{-2x} (\cos x + 2 \sin x) \right] e^x.$

Problems 15.6, Page 539

1. $y = c_1 \cos(\sin x) + c_2 \sin(\sin x)$

2. $y = c_1 \cos(1/x) + c_2 \sin(1/x)$

3. $y = c_1 e^t + c_2 e^{-t} - t$ where $t = \cos x$

4. $y = c_1 \cos(2 \tan^{-1} x) + c_2 \sin(2 \tan^{-1} x)$

5. $y = c_1 e^{\sqrt{2} \sin x} + c_2 e^{-\sqrt{2} \sin x} + \sin^2 x.$

Problems 15.7, page 540

1. $nx - lz = c(mz - ny)$
4. $x^2 + y^2 - xz = cz$

2. $x^2 + y^2 + z^2 = cx$
5. $y(x+z) = c(x+y+z)$

3. $xy^2 = cz^3$
6. $x + y + z + \log(xyz) = c.$

Problems 15.8, page 541

1. $x^3 - y^3 = c_1, x^2 - z^2 = c_2$
2. $lx + my + nz = c_1, x^2 + y^2 + z^2 = c_2$
4. $x^2 - y - 2xy = c_1, x^2 - y^2 - z^2 = c_2$
5. $xyz = c_1, x^2 + y^2 + z^2 = c_2$
6. $y = c_1z, x^2 + y^2 + z^2 = c_2z.$

Problems 16.1, page 544

1. $y = a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$

2. $y = a_0 \left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \dots \right)$
 $+ a_1 \left(x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{x^{13}}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \dots \right)$

3. $y = a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \dots \right) + a_1 \left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \frac{x^7}{3 \cdot 5 \cdot 7} + \dots \right)$

4. $y = 4 + 5x - 4x^2 - \frac{5}{3}x^3 - \frac{x^5}{3} - \frac{x^7}{7} - \dots$
5. $y = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \dots \right) + a_1 x$

6. $y = a_0(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5}x^6 + \dots) + a_1 x.$

Problems 16.2, page 550

1. $y = c_1 \cos \sqrt{x} + c_2 \sin \sqrt{x}$

2. $y = a_0 \left(1 - x^2 + \frac{x^4}{4} \dots \right) + a_1 \left(x - \frac{x^3}{2} + \frac{3x^5}{10} \dots \right)$

3. $y = (c_1 + c_2 \log x) \left[1 + x + \frac{1}{(2!)^2} x^2 + \frac{1}{(3!)^3} x^3 + \dots \right] - 2c_2 \left[x + \frac{1}{(2!)^2} (1 + \frac{1}{2}) x^2 + \frac{1}{(3!)^2} (1 + \frac{1}{2} + \frac{1}{3}) x^3 + \dots \right]$

4. $y = c_1(1 + x + x^2/4 + x^3/4 \cdot 7 + \dots) + c_2 x^{2/3}(1 + \frac{1}{3}x + x^2/3 \cdot 6 + x^3/3 \cdot 6 \cdot 9 + \dots)$

5. $y = a_0 \left(1 - 2x + \frac{3}{2!} x^2 - \frac{4}{3!} x^3 + \dots \right) + a_1 \left[y_1 \log x + a_0 \left(3x - \frac{13}{4} x^2 + \dots \right) \right]$

6. $y = a_0 x \left(1 + \frac{x}{5} + \frac{x^2}{70} + \dots \right) + a_1 x^{-1/2} \left(1 - x - \frac{x^2}{2} + \dots \right)$

7. $y = c_1 x^{-\frac{1}{2}} \left(1 + \frac{x}{2} + \frac{x^2}{40} + \dots \right) + c_2 x^{1/4} \left(1 + \frac{x}{14} + \frac{x^2}{616} + \dots \right)$
 $+ c_2 \sqrt{x} (x + x^2/2 \cdot 3 + x^4/2 \cdot 4 \cdot 3 \cdot 7 + x^6/2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11 + \dots)$

8. $y = a_0 \sqrt{x}(1-x) + a_1 \left(1 - 3x + \frac{3x^2}{1 \cdot 3} + \frac{3x^3}{3 \cdot 5} + \frac{3x^4}{5 \cdot 7} + \dots \right)$

9. $y = a_0(1 - \frac{2}{3}x + \frac{1}{3}x^2 + \dots) + a_1x^4(1 - 2x + 3x^2 - 4x^3 + \dots)$

10. $y = c_1\left(1 + 3x^2 + \frac{3}{5}x^4 + \dots\right) + c_2x^{3/2}\left(1 + \frac{3}{8}x^2 - \frac{1.3}{8.16}x^4 + \frac{1.3.5}{8.16.24}x^6 + \dots\right).$

Problems 16.3, page 557

1. 0.224, 0.44.

Problems 16.4, page 562

1. $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

2. $y = c_1 J_{2/5}(x) + c_2 J_{-2/5}(x)$

3. $y = x^n [c_1 J_n(kx) + c_2 Y_n(kx)]$ where $n = \frac{1}{2}(1 - \alpha)$

4. $y = x[c_1 J_1(2x) + c_2 Y_1(2x)]$

5. $y = c_1 \sqrt{x} J_1(2\sqrt{x}) + c_2 \sqrt{x} Y_1(2\sqrt{x})$

7. $y = c_1 \sqrt{x} J_n(x) + c_2 \sqrt{x} J_{-n}(x)$

11. $x^2 = \sum_{n=1}^{\infty} \frac{2}{\alpha_n^2} \cdot \frac{1}{J_2^2(3\alpha_n)} (3\alpha_n J_1(3\alpha_n) - 2J_2(2\alpha_n))$

Problems 16.5, page 570

3. (i) $2P_3 + 4P_1$;

(ii) $\frac{2}{5}P_3 + \frac{4}{3}P_2 - \frac{2}{5}P_1 - \frac{7}{3}P_0$;

(iii) $\frac{8}{5}P_3 - 4P_2 + \frac{47}{5}P_1 + 4$

(iv) $\frac{8}{35}P_4 + \frac{6}{5}P_3 - \frac{2}{21}P_2 + \frac{34}{5}P_1 - \frac{224}{105}P_0$

9. (i) $f(x) = -\frac{7}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{5}P_3(x)$;

(ii) $f(x) = -\frac{32}{15}P_0(x) - \frac{4}{5}P_1(x) - \frac{40}{21}P_2(x) + \frac{2}{5}P_3(x) + \frac{8}{35}P_4(x)$.

Problems 16.6, page 572

1. $x^3 = \frac{1}{4}(3T_1 + T_3)$.

Problems 16.7, page 575

1. $y_n(x) = \sin nx$, $n = 1, 2, \dots$

2. $y_n(x) = \sin [(2n+1)\pi x/2l]$, $n = 0, 1, 2, \dots$

3. $y_n(x) = \cos nx$, $n = 0, 1, 2, \dots$

4. 1, $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ...

5. $y_n(x) = \sin \left[(2n+1)\frac{\pi}{2} \log |x|\right]$, $n = 0, 1, 2, \dots$

6. $[xe^{-x}y']' + ne^{-x^2}y = 0$, $p(x) = e^{-x}$

7. $[e^{-x^2}y']' + 2ne^{-x^2}y = 0$, $p(x) = e^{-x^2}$.

Problems 16.8, page 575

1. $\frac{1}{3}(10 - 9P_1 + 8P_2)$

2. $\sqrt{(2/\pi x)} \cos x$

3. $\frac{2}{(2n+1)}$

4. zero

5. zero

6. $J_n(x) = \frac{x}{2n} (J_{n+1}(x) + J_{n-1}(x))$

7. $\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = 0$

8. $x \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

9. $\sqrt{(2/\pi x)} \sin x$

- 10.** $x^n J_{n-1}(x)$ **11.** $\frac{1}{2}(3x^2 - 1)$ **12.** zero
13. True **14.** $P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2 - 1)^n$ **15.** $\alpha \neq \beta$
16. $2P_3 + 4P_1$ **17.** $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$
18. $-J_1(x)$ **19.** False **20.** True **21.** True **22.** True **23.** True
24. False **25.** (b) **26.** (c) **27.** (iv) **28.** (iii) **29.** (iii)
30. (iv) **31.** (iii) **32.** (iii) **33.** (iii) **34.** (iii) **35.** (iv)
36. 0, 1.

Problems 17.1, page 579

- 1.** $z = px + qy + p^2 + q^2$ **2.** $z^2(p^2 + q^2 + 1) = c^2$ **3.** $p^2 + q^2 = \tan^2 \alpha$
4. $p + q = px + qy$ **5.** $z^2(p^2 + q^2 + 1) = 9$ **6.** $py - qx = 0$
7. $py + qx = 0$ **8.** $qx - py = x + y$ **9.** $xyz = px + py - z$
10. $\frac{xyr}{y} = 2(px + qy - 2z)$ **11.** $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$ **12.** $x(y - z)p + y(z - x)q = z(x - y)$
13. $z \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$ **14.** $p + q = mz$ **15.** $px^2 + qy = 2y^2$
16. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = 0$ **17.** $\frac{\partial^2 v}{\partial t^2} = \frac{a^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right)$ **18.** $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$
19. $p(x - 2z) + q(2z - y) = y - x$ **20.** $(y - z)p + (z - x)q = x - y.$

Problems 17.2, page 581

- 1.** $z = \frac{x^2}{2} \log x + axy + \phi(x) + \psi(y)$ **2.** $z = \frac{1}{6} x^3 y + xf(y) + \phi(y)$
3. $u = -e^{-t} \sin x + \phi(x) + \psi(t)$ **4.** $z = f(x) + x\phi(y) + \psi(y) - \frac{1}{12} \sin(2x + 3y)$
5. $z = e^x \cosh y + e^{-x} \sinh x$ **6.** $z = \sin x + e^y \cos x.$

Problems 17.3, page 584

- 1.** $x = z^3 f(x/y)$ **2.** $\sqrt{x} - \sqrt{y} = f(\sqrt{x} - \sqrt{y})$ **3.** $x^2 + y^2 + z^2 = f(x + y + z)$
4. $[\cos(x + y) + \sin(x + y)]c^{y-x} = \phi \left[z^{\sqrt{2}} \tan \left(\frac{x+y}{2} + \frac{\pi}{8} \right) \right]$ **5.** $x^2 - y^2 = f(y^2 - z^2)$
6. $\phi \left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z} \right) = 0$ **7.** $x \log(x + y) - z = f(x + y)$ **8.** $x^2 + y^2 + 2z = [\log(xy)]$
9. $x^2 + y^2 - z^2 = f(x + y + z)$ **10.** $x + y + z = f(xyz)$ **11.** $\phi(x^2 + y^2 + z^2, xyz) = 0$
12. $x^2 + y^2 = f(y^2 - yz)$ **13.** $\phi(y/z, x^2 + y^2 + z^2) = 0$ **14.** $x^2 + y^2 + z^2 = f(y^2 - 2yz - z^2)$
15. $f \left(\frac{y}{z}, \frac{z}{x} - \frac{y}{x} + x^2 \right) = 0.$

Problems 17.4, page 587

1. $z = ax - ay/(1+a) + b$

4. $(1-a+az) = (x+ay+b)^2$

7. $\frac{8}{9}z = (x+a)^{3/2} + (y+a)^{3/2} + b$

9. $z = \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x_1 \sqrt{(x^2+a^2)}}{2} + \frac{y \sqrt{(y^2-a^2)}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{y}{a} + b$

10. $z = ax + by + \sin(a+b)$

11. $z = \frac{1}{6}(zx+a)^3 + a^2y + b$

13. $z = axy + a^2(x+y) + b.$

3. $4z(1+a^2) = (x+ay+b)^2$

6. $z = a(x-y) - (\cos x + \cos y) + b$

Problems 17.5, page 590

1. $z = [\sqrt{(ax)} + \sqrt{(b+y)}]^2 / (1+a)$

2. $z = ax^b y^{1/b}$

3. $\frac{z^2}{2} \pm \left\{ \frac{z}{2} \sqrt{z^2 - 4a^2} - 2a^2 \log \left(z + \sqrt{z^2 - 4a^2} \right) \right\} = 2ax + 2y + b$

4. $\log(z-ax) = y - a \log(a+y) + b$

5. $2\sqrt{(z-a-b)} = \sqrt{ax} + \frac{1}{\sqrt{a}}y + c$

6. $z = axe^{-y} - \frac{1}{2}a^2e^{-2y} + b.$

Problems 17.6, page 595

1. $z = f_1(y) + f_2(y+2x) + xf_3(y+2x)$

2. $z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \frac{e^{x+2y}}{27}$

3. $z = f_1(x+y) + xf_2(x+y) + \frac{x^2}{2} \times e^{x+y}$

4. $z = f_1(y+x) + zf_2(y+x) + f_3(y+2x) - e^{2x+y}$

5. $z = f_1(y+x) + xf_2(y+x) - \sin x$

6. $y = f_1(x-at) + f_2(x+at) - \frac{E}{p^2} \sin pt$

7. $z = f_1(y) + f_2(y+2x) + xf_3(y+2x) + 3x \cos(3x+2y)$

8. $f_1(yx) + f_2(y-2x) + f_3(y+3x) + \frac{1}{75} \sin(x+2y) + \frac{2}{3}x^3.$

9. $z = f_1(y+x) + f_2(y+2x) + \frac{1}{12}e^{2x-y} - xe^{x+y} - \frac{1}{3} \cos(x+2y)$

10. $z = f_1(y) + f_2(y+x) + \frac{1}{3}(\sin x \cos 2y + 2 \cos x \sin 2y)$

11. $z = f_1(y) + f_2(y+x) + \frac{1}{2}[\sin(x+2y) + \cos(x+2y)] - \frac{1}{6}[\sin(x-2y) + \cos(x-2y)]$

12. $z = f_1(y+x) + f_2(y-x) + \frac{3}{28}e^{x-y}[\sin(x+2y) - 2 \cos(x+2y)]$

13. $z = f_1(y-x) + f_2(y-2x) + 4x^3y - 3x^4$

14. $z = f_1(y-x) + xf_2(y-x) + \frac{1}{4}(x^4 - 2x^3y + 2x^2y^2)$

15. $z = f_1(y-x) + f_2(y+2x) + ye^x$

16. $z = f_1(y-x) + xf_2(y-x) + f_3(y+x) + \frac{e^x}{25}(\cos 2y + 2 \sin 2y)$

17. $z = f_1(y-x) + xf_2(y-x) + x \sin y.$

Problems 17.7, page 597

1. $z = e^{-x}\phi_1(y) + e^x\phi_2(y-x) - \frac{xe^{-x}}{2}$

2. $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{2x-y}$

3. $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + x + 2y + 6$

4. $z = f_1(y) + e^{-x}f_2(y+x) + \frac{1}{3}x^3 - x^2 + xy^2 + 6x$

5. $z = e^x\phi_1(y) + e^{-x}\phi_2(x+y) + \frac{1}{2}\cos(x+2y)$

6. $z = f_1(x) + e^{3y}f_2(2y-x) + \frac{3}{50}[4\cos(3x-2y) + 3\sin(3x-2y)]$

Problems 17.8, page 598

1. $z = \phi_1(x) + \phi_2(x+y+z)$

2. $z = \phi_1(y+\sin x) + \phi_2(y-\sin x)$

3. $z = \phi_1(xy^2) + \phi_2(x^2y)$

4. $z = \phi_1(y/x) + \phi_2(x^2+y^2) + xy$

5. $y = \phi_1(z) + e^x\phi_2(z)$

6. $y = \phi_1(x+y+z) + x\phi_2(x+y+z)$.

Problems 17.9, page 598

1. order two & degree two

2. $z = f_1(y+2x) + xf_2(y+2x)$

3. $z = -x^2 \sin xy + yf(x) + \phi(x)$

4. $x^2 + y^2 + z^2 = f(x+y+z)$

5. $-\frac{1}{2}\sin(x+y)$

6. $xp + yq = z$

7. $z = ax + (1 - \sqrt{a})^2y + c$

8. $\sqrt{x} - \sqrt{y} = f(6/x - \sqrt{z})$

9. $x \log(x+y) = z + f(x+y)$

10. First

11. $z = 2x + y \log x + f(xy)$

12. $\partial z / \partial x = \partial z / \partial y$

13. $z = f_1(y) + f_2(y+x) + f_3(y+2x)$

14. $4y^2p = q^2$

15. $u = \int f(y) dy + \phi(x)$

16. $c = 1$

17. $u = \frac{1}{6}x^3y + xf(y) + \phi(y)$

18. $f_1(y+x) + f_2(y+6x)$

19. (iv)

20. (iii)

21. (iii)

22. (iii)

23. (ii)

24. (iv)

25. (iv)

26. (i)

27. False

28. False

29. True

30. True

31. False.

Problems 18.1, page 601

1. $z = ce^{4ax^3} \cdot e^{-3ay^4}$

2. $u = ce^{k(1/y - 1/x)}$

3. $u = 8e^{-12x-3y}$

4. $u = 3e^{x-y} - e^{2x-5y}$

5. $u = 3e^{-5x-3y} + 2e^{-3x-2y}$

6. $u = \frac{1}{\sqrt{2}}\sinh \sqrt{2x} + e^{-3y} \sin x$.

Problems 18.2, page 610

1. $y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$, when $b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx$

2. $y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$ where

$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$

3. $y = \frac{8k}{\pi^2} \left(\sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{1}{3^2} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \dots \right)$

4. $y = \frac{8h}{\pi^2} \left(\sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{1}{3^2} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \dots \right)$

6. $y(x, t) = \frac{4l^2c}{a\pi^3} \left\{ \sin \frac{\pi x}{l} \sin \frac{\pi at}{l} - \frac{1}{33} \sin \frac{3\pi x}{l} \sin \frac{3\pi at}{l} \dots \right\}$

7. (i) $y(x, t) = a(x - x^2 - c^2t^2)$; (ii) $y(x, t) = \frac{a}{2}(1 - \cos 2\pi x \cos 2\pi ct)$.

Problems 18.3, page 617

1. $u(x, t) = \frac{400}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-(2n+1)c\pi/100 t} \sin \frac{(2n+1)\pi x}{100}$

2. $\sum_{n=odd} \frac{8a}{n^3\pi^3} \sin \frac{n\pi x}{l} e^{(n\pi x/l)^2 t}$

3. $u(x, t) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{[1-4(-1)^n]}{n} \sin \left(\frac{n\pi x}{30} \right) e^{\frac{(ann\pi)^2 t}{900}}$

4. $u(x, t) = -3x + 90 - \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{5} e^{-c^2 n^2 \pi^2 t / 25}$

5. $u(x, t) = \frac{5}{2} - \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-(2n-1)^2 c^2 \pi^2 t / 25}$

6. $u(x, t) = 50 - \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{50} e^{-c^2 \pi^2 (2n-1)^2 t / 2500}$

7. $\theta = \frac{4\theta_0}{\pi} \left[e^{-(\pi/2l)^2 kt} \cos \frac{\pi x}{2l} - \frac{1}{3} e^{-(3\pi/2l)^2 kt} \cos \frac{3\pi}{2l} x + \frac{1}{5} e^{-(5\pi/2l)^2 kt} \cos \frac{5\pi}{2l} x - \dots \right]$

8. $V = V_0 e^{-\sqrt{(n/2k)x}} \sin [nt - \sqrt{(n/2k)}x]$.

Problems 18.4, page 623

1. $u = -\frac{8}{\pi} \sum_{n=1, 3, 5, \dots}^{\infty} \frac{\sin nx \sinh n(\pi-y)}{n(n^2-4) \sinh nx}$

5. $u(x, y) = \frac{3200}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{20} \sinh \frac{(n-1)\pi y}{20}}{(2n-1)^2 \sinh (2n-1)\pi}$

8. $u(x, y) = u_0 \cosh \frac{\pi x}{a} \cosh \frac{\pi}{a}(b-y) \operatorname{sech} \frac{\pi b}{a}$.

Problems 18.5, page 626

1. $u(r, \theta) = \frac{8k}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a} \right)^{2n-1} \frac{\sin (2n-1)\theta}{(2n-1)^3}$

2. $u(r, \theta) = \frac{3200}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{r}{10} \right)^{2n-1} \frac{\sin (2n-1)\theta}{(2n-1)^3}$

3. $u(r, \pi) = \frac{2}{\pi} \sum_{n=1, 3, 5, \dots}^{\infty} \left(\frac{a}{r} \right)^{2n} \frac{r^{4n} - b^{4n}}{a^{4n} - b^{4n}} \cdot \frac{\sin 2n\theta}{n^3}$

4. $u(r, \theta) = \sum \frac{2k}{n\pi} \left(\frac{r}{a} \right)^{4n} (1 - \cos n\pi) \sin 4n\theta$

5. $u(r, \theta) = 50 - \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{r}{a} \right)^{2n-1} \sin (2n-1)\theta$

6. $u(r, \theta) = \cos \theta \left(\frac{200}{r} - \frac{r}{2} \right) + \sin \theta \left(2r - \frac{200}{r} \right)$. 7. $u(r, \theta) = 4 \cos \theta (r - 1/r) + 4 \sin \theta (r + 1/r)$.

Problems 18.6, page 630

1. $u = \sum_{m=1}^{\infty} \frac{J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)} \cos \alpha_m t J(\alpha_m r)$.

Problems 18.7, page 634

1. $e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}$; $i = i_0 - e_0 \sqrt{\left(\frac{C}{L}\right)} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{CL}}$

3. $v = \frac{20(l-x)}{l} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \exp(-n^2 \pi^2 t / RCl^2)$

$$i = \frac{20}{lR} + \frac{24}{lR} \sum_{n=1}^{\infty} (-1)^n \frac{n\pi x}{l} \exp(-n^2 \pi^2 t / RCl^2)$$

4. $v = V_0 \cos(pt - px\sqrt{LC})$.

Problems 18.9, page 638

1. $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 i}{\partial t^2} = LC \frac{\partial^2 i}{\partial x^2}$

2. $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$

3. If $u(x, t)$ is the temperature, then temperature gradient at a point is $\partial u / \partial x$ for all t .

4. elliptic

5. $\partial^2 y / \partial t^2 = c^2 \partial^2 y / \partial x^2$

6. $u = \frac{10}{l}x + 30$

7. parabolic partial differential equation

8. $r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$

9. $y = \frac{2h}{l}x, 0 < x < \frac{l}{2}; y = \frac{l}{2h}(2h - y), \frac{l}{2} < x < l$

10. $y(0, t) = 0, y(l, t) = 0, \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$

11. zero

12. $u(0, y) = 0, v(a, y) = 0, 0 < y < a; u_x(x, 0) = 0$ for all t and $u(x, a) = u$ for $0 < x < a$

13. $\frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(l, t) = 0$ for all t

14. $y(x, t) = f(x + ct) + f(x - ct)$

15. $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ where c^2 is the diffusivity

16. $u(x) = x^2 + 20$

17. § 18.8-(6), (7), (8)

18. $u = 8e^{-12x-3y}$

19. $z = 4e^{3x+t}$

20. $\alpha^2 (= k/s\rho)$ is called the diffusivity of the substance (cm^2/sec)

21. $\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t}, \frac{\partial^2 i}{\partial x^2} = RC \frac{\partial c}{\partial t}$

22. $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$

23. § 18.7 – (3), (4), (5)

25. False.

Problems 19.1, page 646

1. (i) $\sqrt{2} \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \right) \cos \left(\frac{\pi}{4} - \frac{\alpha}{2} \right)$. (ii) $-8i/25$

2. $\frac{-y}{x^2 + y^2 - 2x + 1}$

3. $x = \pm 1.5, y = \pm 2$

5. A circle: centre $(-1, 1)$ radius $\sqrt{2}$

8. $-1 + i\sqrt{3}, -1 - i\sqrt{3}, 1 - i\sqrt{3}, 4\sqrt{3}$

9. $-2 + 0.i, 1 - i\sqrt{3}$

10. $-1 - i, \sqrt{2}(\mp \sin 15^\circ \pm i \cos 15^\circ), \sqrt{2}(\mp \cos 15^\circ \pm i \sin 15^\circ)$

- 11.** (i) Annular region between the circles of radii 2 and 4 with centre $(-3, 0)$ including boundary of inner circle; (ii) Region of complex plane above the line $y = 2$; (iii) Infinite region bounded by the lines $\theta = \pi/3$ and $\theta = \pi/2$; (iv) Real axis and region above it between $x = \pm 2$
- 13.** (i) Ellipse with foci at $z = \pm 1$ and major axis = 3
(ii) (a) Right bisector of the line joining $z = 3$ and $z = -1$;
(b) Circle through the points $z = 3$ and $z = -1$;
- 14.** (i) Right bisector of the points 0 and 2; (ii) Circle through the points $\frac{1}{3}$ and 3
- 15.** (i) A straight line; (ii) Circle with centre $(1, 1/2)$ and radius $\sqrt{5}/2$.

Problems 19.2, page 650

5. $4m\pi/n(n+1)$

7. $2^{n+1} \sin^n \frac{\alpha-\beta}{2} \cos n \left(\frac{\pi+\alpha+\beta}{2} \right).$

Problems 19.3, page 653

- 1.** (i) $(2)^{1/8} [0.98 \pm i(0.195)]6$; $(2)^{1/2} [-0.195 \pm i(0.98)]$
(ii) $(2)^{1/5} \cos \frac{4n+3}{10}\pi$, where $n = 0, 1, 2, 3, 4$; (iii) $\pm 2\sqrt{2}$
(iv) $2^3\sqrt{2} \cos r\pi/9$, where $r = 1, 7$ or 13
- 3.** $\pm i, \frac{1}{2}(\sqrt{3} \pm i), \frac{1}{2}(-\sqrt{3} \pm i)$
- 5.** (i) $1, -1, \cos(\pm \pi/5), \cos(\pm 3\pi/5)$
(ii) $-1, \frac{1}{2}(1 \pm i\sqrt{3}), \pm(1+i)/\sqrt{2}, \pm(-1+i)/\sqrt{2}$
(iii) $\frac{1 \pm i}{\sqrt{2}}, \frac{-1 \pm i}{\sqrt{2}}; 1, \cos\left(\pm \frac{2\pi}{5}\right), \cos\left(\pm \frac{4\pi}{5}\right)$.
(iv) $\cos(2n+1)\pi/6$, where $n = 0, 1, 2, 3, 4, 5$; $2(2m+1)\pi/3$, where $m = 0, 1, 2$
- 6.** $(-1+i)/\sqrt{2}, (1-i)/\sqrt{2}$
- 7.** $\pm 1, \pm i, \pm \left(\cos \frac{\pi}{6} \pm i \sin \frac{\pi}{6} \right), \pm \left(\cos \frac{\pi}{3} \pm i \sin \frac{\pi}{3} \right)$; Last four values
- 9.** $x^3 + x^2 - 2x - 1 = 0$.

Problems 19.4, page 655

- 1.** $32 \cos^5 \theta - 24 \cos^3 \theta + 6 \cos \theta$
13. $-(2)^{-11} (\sin 12\theta - 2 \sin 10\theta - 4 \sin 8\theta + 10 \sin 6\theta + 5 \sin 4\theta - 20 \sin 2\theta)$
14. $\sin^5 \theta = A \sin \theta - B \sin 3\theta + C \sin 5\theta$.

Problems 19.5, page 661

- 1.** (i) $e^{x^2-y^2} \cos 2xy, e^{x^2-y^2} \sin 2xy$ (ii) $i e^5$. (iii) $e^{16} \cos 30, e^{16} \sin 30$
5. $[(pq' - p'q)(qr' - q'r)]^2 = [(pq' - p'q)^2 + (qr' - q'r)^2](pr' - p'r)^2$
- 10.** $\frac{1}{64} (\cosh 7\theta + 7 \cosh 5\theta + 21 \cosh 3\theta + 35 \cosh \theta)$
- 17.** $-\log^3$
- 18.** $-13/12$.

Problems 19.6, page 664

10. $\pm \frac{\pi}{4} + \frac{i}{4} \log \frac{1+\sin\theta}{1-\sin\theta}$ according as $\cos\theta$ is + ve or - ve

11. $\sin^{-1}(\sqrt{\sin\theta}) + i \log [\sqrt{(1+\sin\theta)} - \sqrt{\sin\theta}]$.

Problems 19.7, page 667

1. (i) $\log_e 10 + i [\tan^{-1}(4/3) \pm 2n\pi]$; (ii) $\log_e 1 + i (\pi + 2n\pi)$

4. (i) $\sqrt{2}e^{-(2n-\frac{1}{4})\pi}, (2n-\frac{1}{4})\pi + \log \sqrt{2}$; (ii) $e^{-\pi^2/8}, (\pi/4) \log_e 2$

9. $\sqrt{[\frac{1}{2}(\cos 2x + \cosh 2y)]} - i \tan^{-1}(\tan x \tanh y)$

10. (i) $2n\pi \pm i \log(2 + \sqrt{3})$; (ii) $-\frac{1}{2} \log 3 + (n + \frac{1}{2})i\pi$.

Problems 19.8, page 669

1. $e^{\sin\theta \cos\theta} \cos(\theta + \sin^2\theta)$

2. $\sin\alpha \cos(\cos\beta) \cosh(\sin\beta) - \cos\alpha \sin(\cos\beta) \sinh(\sin\beta)$

3. $\tan^{-1} \frac{x \sin\alpha}{1+x \cos\alpha}$, except when $x=1$ and $\alpha=(2n+1)\pi$ **4.** $\log(2 \cos\theta/2)$

5. $-\frac{1}{2} \tan^{-1}(\cos\beta \operatorname{cosech}\alpha)$

6. $\frac{1}{2} \tan^{-1} \frac{2c \sin\alpha}{1-c^2}$

7. $(2 \cos\theta)^{-1/2} \cos\theta/2$

8. $\sin \frac{n(\pi-\alpha)}{2} / (2 \sin\alpha/2)^n$

9. $\sin\left(\alpha + \frac{n-1}{2}\beta\right) \sin\frac{n\beta}{2} \operatorname{cosec}\frac{\beta}{2}$

10. $\frac{\cos\{\alpha + \frac{1}{2}(n-1)\beta\} \sin n\frac{\beta}{2}}{\sin\frac{1}{2}\beta}$

11. $\frac{\sin\alpha(\cos\alpha - \sin\alpha)}{1 - \sin 2\alpha + \sin^2\alpha}$

12. $\frac{1-x \cos\theta - x^n \cos n\theta + x^{n+1} \cos \overline{n-1}\theta}{1-2x \cos\theta + x^2}$.

Problems 19.9, page 670

2. 0.053 radians

3. $1^\circ 59'$

4. 39.7.

Problems 19.10, page 670

1. (b)

2. (c)

3. (b)

4. (c)

5. (b)

6. (d)

7. odd

8. $32 \cos^6\theta - 48 \cos^4\theta + 18 \cos^2\theta - 1$

9. $6(1-2i)$

10. $2i \sin n\theta$

11. $\frac{1}{25}(-6+17i)$

12. $\cosh x \cos y$

13. $\frac{1}{19}$ radians

14. 1

15. $-\cos x \sinh y$

16. $e^{-\pi/4\sqrt{2}}$

17. $\alpha + \beta + \gamma$

18. $2i n\pi$

19. real

20. $\sinh\phi$

21. $\sinh 2\phi / (\cosh 2\theta + \cosh 2\phi)$

22. $16 \cos^5\alpha - 20 \cos^3\alpha + 5 \cos\alpha$

23. an equilateral

24. $\pi/2$

25. $x = \pm 1, y = -4$.

26. a circle

27. True

28. True

29. True

30. True

31. False

32. True

33. False.

Problems 20.1, page 682

4. $a = 1, b = -6, c = 1, d = 2, e = 4$

6. (i) and (ii) Not analytic. (iii) Analytic

7. $p = -1$

11 & 12. $f(z)$ is not analytic at origin although C-R equations are satisfied there

14. (i) $z^3 + 3z^2 + 1 + ic$; (ii) $\cos z$; (iii) $\log z$; (iv) i/z ; (v) $e^z + i(c-z)$; (vi) $ze^{2z} + ic$; (vii) $z \sin z$; (viii) $x^2e^z + ic$

15. (i) $(1+i)/z + c$; (ii) $\cos z + c$; (iii) e^z ; (iv) $\bar{z}e^{-\bar{z}} + c$; (v) $1 + ize^{-z}$; (vi) $(2 \cos x \cosh y)/(\cos 2x + \cosh 2y)$

16. (i) $f(z) = c - iz^3$; (ii) $f(z) = \left(\frac{1}{2} - \cot \frac{z}{2}\right)$; (iii) $\frac{\cot z}{1+c} + c \left(\frac{1+3i}{5}\right)e^z + c$

17. $\psi = 3xy^2 - x^3 + c$

18. $2 \tan^{-1}(y/x)$; $2 \log z + c$

20. $v = C - e^{-2xy} \cos(x^2 - y^2)$; $f(z) = C - ie^{iz^2}$

22. (i) $x^4 - 6x^2y^2 + y^4 = c$; (ii) $x^2 - y^2 + 2e^x \sin y = c$; (iii) $r^2 \sin 2\theta = c'$

23. (i) $x/(x^2 + y^2)$; (ii) $\frac{1}{2} \log(x^2 + y^2)$

24. (i) $a(1 + \cos \theta + i \sin \theta \log r)$; (ii) $(r + 1/r) \cos \theta + (r - 1/r) \sin \theta + c$

27. $-2 \tan^{-1}[(y-2)/(x-1)]$, $2i \log(z-1-2i)$.

Problems 20.2, page 687

1. $z = \pm i$

2. $w = -1/z$

3. $w = (2i - 6z)/(iz - 3)$; fixed points $z = i, 2i$; no critical points

4. $w = \frac{(20 + 18i) - (32 + 12i)z}{(29 + 17i) - (11 - 3i)z}$

5. (i) $w = i(1-z)/(1+z)$; (ii) $w = \frac{(4i-2)+(5-3i)z}{2i+(1+i)z}$; (iii) $w = (1-z)/(1+z)$.

Problems 20.3, page 692

1. (i) $I(w) > 0$; (ii) Region bounded by the parabolas $v^2 = 4(1 \pm u)$; and $u^2 = 1 - 2v$; (iii) Region bounded by parabolas $v^2 = \frac{1}{4} \pm u$, $v^2 = 4(1 \pm u)$; (iv) Region boundary $p = 2\sqrt{p} \cos \frac{\phi}{2} + 3$.

2. $w = z^6$

3. Lines parallel to x and y axes map into two families of rectangular hyperbolas in the w -plane which cut each other at right angles. Lines parallel to u and v axes map into two families of parabolas in the z plane which cut each other orthogonally. It is conformal at $z = 0$

4. (a) Line $4v + 1 = 0$

5. (b) Every circle through the origin ($z = 0$) transforms into a st. line not passing through the origin ($w = 0$). If a line passes through $z = 0$, its image is a line through $w = 0$. (b) Circle with centre $(1/2, 1/2)$ and radius $(1/\sqrt{2})$ (c) Lemniscate $p^2 = \cos 2\phi$

10. $z = \pm a$

11. See § 20.10 (3)

14. See § 20.10(4).

Problems 20.4, page 694

1. $z = \cosh w$

3. $w = \sin z$

4. $w = \log z$.

Problems 20.5, page 696

1. (a) $(5-i)/6$ (b) $(5+i)/6$

2. (i) $4 + (25/3)i$; (ii) $4 + 8i$

3. $\frac{1}{6}(64i - 103)$

6. (i) i ; (ii) $2i$; (iii) 0

7. (a) $\frac{1}{3}(i-1)$, (b) $\frac{1}{6}(5i-3)$

9. (i) $\frac{2}{3}$; (ii) $-\frac{2}{3}$.

Problems 20.6, page 702

1. (i) 0 ; (ii) $2\pi i$
 4. (i) $5\pi i$; (ii) $\pi i/2$ (iii) $4\pi i$
 7. (i) $8\pi i$; (ii) 0

2. 0
 5. (a) $-10\pi i$ (b) $2\pi ie$
 8. (i) 0 ; (ii) $2\pi(6 + 13i)$; (iii) $12\pi i$

3. (a) zero ; (b) zero
 6. (i) $4\pi i$; (ii) $2\pi ie^{-1}$; (iii) $-\pi i$
 9. zero.

Problems 20.7, page 709

1. $\sum_{n=1}^{\infty} (-1)^{n+1} n(z-1)^n$; Convgt. in $|z-1| < 1$
2. $f(z) = \frac{1}{3} + \frac{u}{9}(z+i) - \frac{7}{27}(z+1)^2 + \dots$. Region of convergence is $|z+i| < 1$
3. (i) $\frac{1}{2}(z-1) - \frac{1}{2^2}(z-1)^2 + \frac{1}{2^3}(z-1)^3 - \dots$ (ii) $-(z-\pi/2) + (z-\pi/2)^3/3! - (z-\pi/2)^5/5! + \dots$
 (iii) (a) $f(z) = -\frac{1}{5} \sum_{n=0}^{\infty} (-1)^n (z+1)^n - \frac{1}{20} \sum_{n=0}^{\infty} \left(\frac{z+1}{4}\right)^n$ in the region $|z+1| < 1$. Also, $(z+1) < 4$
 (b) $f(z) = -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{z-1}{5}\right)^n \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$ in the region $|z-1| < 2$. Also $|z-1| < 3$
4. (i) $\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \dots + \frac{1}{2} \left[1 + \frac{z+1}{2} + \frac{(z+1)^2}{2^2} + \frac{(z+1)^3}{2^3} \right]$
 (ii) $\frac{1}{4z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right) - \frac{1}{12} \left(1 - \frac{z}{3} + \frac{z^2}{9} - \dots \right)$. (iii) $-\frac{1}{2(z-1)} - 3 \sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}}$
5. (i) $e \left[(z-1)^{-2} + (z-1)^{-1} + \frac{1}{2!} + \frac{1}{3!}(z-1) + \frac{1}{4!}(z-1)^2 + \dots \right]$
 (ii) $e^2(z-1)^{-3} + 2e^2(z-1)^{-2} + 2e^2(z-1)^{-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \dots$
6. (i) $\sum_{n=1}^{\infty} (-1)^{n-1} n \cdot (z-1)^{-n}$ for $|z-1| > 1$. (ii) $-\sum_{n=2}^{\infty} \frac{z^{2n-5}}{2(n-1)!}$
7. (i) $1 + \frac{3}{z} \left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots \right) - \frac{8}{3} \left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots \right)$
 (ii) $\frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{3^2}{(z+2)^3} + \dots + \frac{1}{5} \left(1 + \frac{z+2}{5} + \frac{(z+2)^2}{5^2} + \frac{(z+2)^3}{5^3} + \dots \right)$
 (iii) (a) $\frac{7}{z} - \frac{9}{z^2} - \frac{45}{z^3} - \frac{81}{z^4} + \dots$ (b) $\frac{5}{2(z-3)} + \frac{7}{12} - \frac{z-3}{24} - \frac{5(z-3)^2}{432} + \frac{7(z-3)^2}{864} + \dots$
8. (a) $\frac{z}{4} - \frac{5z^2}{16} + \frac{21}{64} z^5 - \dots$; (b) $\frac{1}{3} \left(\frac{1}{z^5} - \frac{1}{z^3} - \frac{z}{4} + \frac{z^3}{16} - \frac{z^5}{64} + \dots \right)$; (c) $\frac{1}{z^3} - \frac{3}{z^5} + \dots$
9. $z = 0, z = 2$ are the isolated singularities
 10. $z = 0$ is an isolated essential singularity
11. $z = 0$, is a non-isolated essential singularity
12. $z = 1$ is a pole of order 2
 13. $z = 1$ is a pole of order 4
14. $z = a$ is a double pole and $z = 0, \pm 1, \pm 2, \dots$ are simple poles.

Problems 20.8, page 715

1. $-\frac{1}{t} - 2i + 3t + 4i t^2 + \dots$ where $t = z - i ; -1$

2. (i) $3e/2$; (ii) $\operatorname{Res}(z = -ai/2)$
3. (i) $\operatorname{Res} f(0) = -1/2$, $\operatorname{Res} f(2) = 2\frac{1}{2}$;
(ii) $\operatorname{Res} f(-1) = 0$, $\operatorname{Res} f(i) = \frac{1+2i}{2(1-i)}$, $\operatorname{Res} f(-i) = \frac{2i-1}{2(i-1)}$
(iii) $\operatorname{Res} f(-1) = 1$, $\operatorname{Res} f(i) = \frac{2+i}{-1+i}$, $\operatorname{Res} f(-i) = \frac{-2+i}{1+i}$
4. (i) $\operatorname{Res} f(0) = -4/3$; (ii) $\operatorname{Res} f(i) = \frac{1}{2}e^{-1}$; $\operatorname{Res} f(-i) = \frac{1}{2}e$; (iii) $-i\pi/4$, $\operatorname{Res} f(n\pi) = 1$, n an integer.
5. (i) 0; (ii) $16\pi i/(2+3i)$
6. (i) $\frac{\pi ie^{-4}}{5}$ (ii) 0; (iii) $i\pi/4$ 7. (i) πi ; (ii) $\pi/2(3+2i)$, (iii) zero.
8. (i) $-2\pi i$; (ii) πi ; (iii) $\pi/16$
9. (i) $-2\pi i$; (ii) $8\pi i/3e^2$; (iii) $\frac{\pi e^{2i}}{2}$; (iv) $\frac{8\pi}{3}ie^{-2}$
10. (i) $\frac{21\pi i}{16}$; (ii) $2\pi i \sec 1(1 + \tan 1)$; (iii) $-2\pi i$
11. $2\pi i$ 12. $\frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} - \frac{31z^5}{15120} + \dots; -\frac{1}{3}\pi i$.

Problems 20.10, page 723

- | | | | | | |
|--|-----------|--------------------------------------|----------|---|---------|
| 1. (i) | 2. (iii) | 3. (a) | 4. (iii) | 5. (ii) | 6. (ii) |
| 7. (ii) | 8. (ii) | 9. (iii) | 10. (ii) | 11. (ii) | 12. (i) |
| 13. (ii) | 14. (iii) | 15. $v(x, y) = x^2 - y^2 + 2y + c$ | | | |
| 16. $3x^2y - y^3 + c$ | | 17. $u + iv$ is an analytic function | | 18. $2i/3$ | |
| 19. 1 | | 20. $z = \frac{1}{2}(a+b)$ | | 21. $2u+1=0$ | |
| 22. $z = 1, \frac{1}{2}(1 \pm \sqrt{3})$ | | 23. $z = 0$ | | 24. -1 | |
| 25. $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$ | | 26. zero | | 27. (iii) | |
| 28. $\frac{1}{2} \left[\frac{d^2}{dz^2} [(z-a)^3 f(z)] \right] z = a$ | | 29. zero | | 30. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ | |
| 31. zero | | 32. $e^x \sin y$ | | 33. zero | |
| 34. $z = 2$ | | 35. no point in the z -plane | | 36. $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$ | |
| 37. zero | | | | | |
| 38. A simply connected region is one in which any closed curve, lying entirely within it, can be shrunk to a point without going out of the region | | | | | |
| 39. which is analytic or regular | | 40. $z = 1, 2$ | | | |
| 41. $(z_1 - z_2)(z_3 - z_4)/(z_1 - z_4)(z_3 - z_2)$ | | 42. i | | | |
| 43. $\frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \dots$ | | 44. $\pm i$ | | | |
| 45. 1 | | 46. Magnification and rotation | | | |
| 47. The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity ($z=a$) is called the residue of $f(z)$ at that point. | | | | | |
| 48. $2\pi i$ | | 49. zero | | 50. $z = 1, 2$ | |
| 51. $-\frac{1}{2} \left\{ 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots \right\}$ | | 52. $i/2$ | | 53. $z = 0, 2$ | |
| 54. § 20.14 | | 55. $\pm i$ | | 56. zero | |
| 57. Zeroes are at $z = \pm 1$, singularity is at $z = 0$ | | | | 58. -1 | |

59. essential singularity

60. zero

$$61. \sin z = \frac{1}{\sqrt{2}} \left\{ 1 + (z - \pi/4) - \frac{(z - \pi/4)^2}{2!} - \frac{(z - \pi/4)^3}{3!} + \frac{(z - \pi/4)^4}{4!} + \dots \right\}$$

62. a circle with centre (3, 2) and radius 2 in w -plane.63. $n\pi$, n an integer

64. $\phi(a)/\psi'(a)$

65. circles

66. True

67. False

68. True

69. True

70. False

71. True

72. True

73. True

74. True

75. True

76. True

77. False

78. False

79. False

80. True

81. True

82. True.

Problems 21.1, page 732

1. $\frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^4+9}$

2. $\frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \sqrt[3]{\left(\frac{\pi}{s}\right)}$

3. $\frac{3s-20}{s^2-25}$

4. $\frac{s \cos b - a \sin b}{s^2 + a^2}$

5. $\frac{s^2 - 2s + 4}{s(s^2 + 4)}$

6. $\frac{2(s^2 - 5)}{(s^2 + 1)(s^2 + 25)}$

7. $\frac{\sqrt{\pi} - e^{1(1/4)s}}{2s^3/2}$

8. $\frac{5}{4} \left\{ \frac{1}{s^2 + 1} - \frac{3/2}{s^2 + 9} + \frac{1/2}{s^2 + 25} \right\}$

9. $\frac{s(s^2 + 28)}{(s^2 + 4)(s^2 + 36)}$

10. $\frac{b}{(s+a)^2 - b^2}$

11. $\frac{60}{s-2} - \frac{s-2}{s^2 - 4s + 20}$

12. $\frac{30(s+3)}{(s^2 + 6s + 13)(s^2 + 6s + 73)}$

13. $\frac{2}{(s+1)(s^2 + 2s + 5)}$

14. $\frac{1}{8} \left\{ \frac{3}{s-2} - \frac{4(s-2)}{s^2 - 4s + 8} + \frac{s-4}{s^2 - 8s + 32} \right\}$

15. $\frac{a(s^2 + 2a^2)}{s^4 + 4a^4}$

16. $\frac{3}{2} \left[\frac{1}{s^2 - 9} + \frac{s^2 - 13}{s^4 - 10s^2 + 169} \right]$

17. $\frac{2}{(s+2)^3}$

18. $\frac{1}{n} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$

19. $4 \frac{(4s^2 + 4s - 1)}{(4s^2 + 1)^2}$

20. $\frac{4}{s} - \frac{e^{-s}}{s}$

21. $\frac{1 + e^{-ns}}{s^2 + 1}$

22. $\frac{e^{-ns/3}}{s^2 + 1}$

23. $e^{-2\pi s/3} \frac{s}{s^2 + 1}$

24. $\frac{2}{s^3} - \frac{e^{-2s}}{s^3} (2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2} (5s - 1)$

25. $\frac{4}{(s-1)(s^2 - 2s + 5)}$

Problems 21.2, page 734

1. $(1/s^2 T) - e^{-sT}/s(1 - e^{sT})$

2. $[Ew/(s^2 + w^2)] \coth(\pi s/2w)$

3. $(a/s) \tanh(as/2)$

4. $(1/s^2) \tanh \frac{1}{2} as$

5. $\frac{1}{\sqrt{(s^2 + a^2)}}$

6. $(s^2 - 2as + a^2 + b^2)^{-1/2}$

7. $\frac{2}{(s-2)\sqrt{(s+1)}}$

Problems 21.3, page 740

1. $\frac{s+1}{s(s^2 + 2s + 2)}$

4. $\frac{2(3s^2 + 4)}{s^2(s^2 + 4)^2}$

5. $\frac{16}{(s^2 + 4)^2}$

6. $\frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$

7. $\frac{2as}{(s^2 - a^2)^2}$

8. $\frac{6(s-2)}{(s^2 - 4s + 13)^2}$

9. $\frac{8(s+2)}{s^2 + 4s + 20}$

10. $\frac{2(s^3 + 6s^2 + 9s + 2)}{(s^2 + 4s + 5)^3}$

11. $\log \{(s+b)/(s+a)\}$

12. $\cot^{-1}(s)$

13. $\frac{1}{2} \log \{(s^2 + 36)/(s^2 + 16)\}$

14. $\frac{1}{2} \log \left(\frac{s^2 + b^2}{(s-a)^2} \right)$

15. $\cot^{-1}(s+1)$

16. $\frac{1}{2} \log \left(\frac{s^2 + 9}{s^2} \right)$

17. $\cot^{-1}s - \frac{1}{2}s \log(1+s^{-2})$

18. $\frac{1}{s-\log 2} + \frac{2s}{(s^2+1)^2} + \frac{1}{2} \log \left(\frac{s^2+9}{s^2+4} \right)$

19. (i) $\log 2/3$; (ii) $\pi/8$; (iii) $12/169$; (iv) $\frac{8(s+1)}{s(s^2+2s+17)}$

21. (i) $\frac{1}{s} \cot^{-1}(s)$; (ii) $\frac{1}{s} \cdot \frac{s+1}{s^2+2s+2}$; (iii) $\frac{\cot^{-1}(s-1)}{s}$.

Problems 21.4, page 743

1. $\frac{1}{2} \left(\cos \frac{5t}{2} - \sin \frac{5t}{2} \right) - 4 \cosh 3t + 6 \sinh 3t$

2. $e^{3t} - e^{2t}$

3. $3e^{t/2} + 2e^{t/3}$

4. $e^{2t} + 2e^{-4t}$

5. $\frac{1}{3} (8e^{2t} - e^{-t})$

6. $\cosh t$

7. $e^t + e^{-2t} - 2e^{3t}$

8. $2e^{3t} - \frac{3}{5}e^{2t} - \frac{2}{5}e^{7t}$

9. $\frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$

10. $\frac{1}{2}t \sinh t$

11. $\frac{1}{3}t(e^t - e^{-2t})$

12. $\frac{1}{13} (3e^{3t} - 3 \cos 2t + 2 \sin 2t)$

13. $\frac{1}{2}(\sin t - te^{-t})$

14. $\frac{1}{2} [\cos at + \cosh at]$

15. $(\frac{1}{3}a^2) [e^{at} - e^{-at/2} (\cos(\sqrt{3}at/2) + \sqrt{3}\sin(\sqrt{3}at/2))]$

16. $\frac{1}{3} (5 \sin t - \sin 2t)$

17. $\frac{1}{3}e^{-2t} (6 \cos 3t - 7 \sin 3t)$

18. $\frac{1}{5} (1 + e^{-t}) \sin t + \frac{3}{5} (1 - e^{-t}) \cos t$

19. $\frac{1}{3}e^{-t} (\sin t + \sin 2t)$

20. $(2/\sqrt{3}) \sinh(\frac{1}{2}t) \sin(\frac{1}{2}\sqrt{3}t)$

21. $\cos at \sinh at$.

Problems 21.5, page 750

1. $\frac{1}{25} (e^{-5t} + 5t - 1)$

2. $\frac{1}{8} - \frac{1}{4} \left(t^2 + t + \frac{1}{2} \right) e^{-2t}$

3. $\frac{1}{a^2} \cos \left(\frac{bt}{a} \right)$

4. $\frac{1}{a^2} \left(t - \frac{1}{a} \sin at \right)$

5. $\frac{1}{2} t^2 + \cos t + 1$

6. $\frac{1}{2} t e^{-2t} \sin 2t$

7. $t \sin at$

8. $\frac{1}{2} (a^2 t^2 - 4at + 2) e^{-at}$

9. $\frac{1-e^t}{t}$

10. $\frac{1}{t} (e^{-bt} - e^{-at})$

11. $e^{-t} - e^{-2t} - e^{-3t}$

12. $\frac{1}{t} (\cos at - \cos bt)$

13. $\frac{2}{t} (1 - \cosh at)$

14. $\frac{2}{t} (e^t - \cos t)$

15. $\frac{\sin 2t}{t}$

16. $\frac{\sin t}{t}$

17. $\frac{2(\sinh t - t \cosh t)}{t^2}$

18. $\frac{e^{-bt} - e^{-at}}{a-b}$

19. $\frac{1}{2a^3} (\sin at - at \cos at)$

20. $\frac{1}{a^3} (at - \sin at)$

21. $t(e^{-t} + 1) + 2(e^{-t} - 1)$

22. $\frac{1}{16} (e^{2t} - e^{-2t} - 4te^{-2t})$

23. $\frac{1}{13} (3 \sin 3t + 2 \cos 2t - 2e^{-2t})$

24. $\frac{t^2}{2} + \cos t - 1$

25. $\frac{e^{-2t}}{54} (\sin 3t - 3t \cos 3t)$.

Problems 21.6, page 754

1. $y = \frac{7}{4} e^{-t} - \frac{3}{4} e^{-3t} - \frac{1}{2} te^{-t}$

2. $x = \frac{at}{2} \sinh t$

3. $y = t - 3 \sin t + \cos t$

4. $y = 2t + 3 + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$

5. $y = 4e^{2t} (1+t) - 7e^t$

6. $y = 2 \cos 5t + t \sin 5t$

7. $y = \frac{1}{2\omega} \sin \omega t$

8. $y = \frac{1}{8} e^t - \frac{1}{40} e^{-3t} - \frac{1}{10} (2 \sin t + \cos t)$

9. $y = \frac{1}{2} (\cos kt + \cosh kt)$

10. $y = \frac{1}{8} [(3-t^2) \sin t - 3t \cos t]$

11. $y = \frac{11}{3} e^{-t} (\sin t + \sin 2t)$

12. $y = \frac{-12}{5} + \frac{12}{5} e^{-t} \cos 2t + \frac{7}{10} e^{-t} \sin 2t$

13. $y = e^{2t} (x^2 - 6x + 12) - e^t (15x^2 + 7x + 11)$

14. $x = \frac{4}{9} \sin 2t - \frac{5}{9} \sin t - \frac{1}{3} t \cos 2t$

15. $y = \frac{1}{2} \left(\frac{3 \sin t}{t} - \cos t \right)$

16. $y = e^{2t}$

17. $y = t$

18. $y = 3J_0(2t)$

21. $(n \sin at - a \sin nt) F_0/mn(n^2 - a^2)$, where $n^2 = k/m$.

Problems 21.7, page 756

1. $x = \frac{1}{2} (e^t + \cos t + 2 \sin t - t \cos t)$, $y = \frac{1}{2} (t \sin t - e^t + \cos t - \sin t)$

2. $x = e^t + e^{-t}$, $y = e^{-t} - e^t + \sin t$

3. $x = 2 + t^2/2$, $y = -1 - t^2/2$

4. $x = \frac{1}{10} (5 - 2e^{-t} - 3e^{-6t/11})$, $y = \frac{1}{5} (e^{-t} - e^{-6t/11})$

5. $x = e^6 (1 + 2t) + 2e^{3t}$, $y = \sinh t + \cosh t - e^{-3t} - te^t$

6. $i_1 = \frac{a}{p+\omega} (\sin \omega t + \sin pt)$; $i_2 = \frac{a}{p-\omega} (\cos \omega t - \cos pt)$.

Problems 21.8, page 762

1. $\frac{2}{s^2 + 4} (e^{-2\pi s} - e^{-4\pi s})$

2. (i) $(1 - 2t) u(t - \pi) + 2tu(t)$, $\frac{2}{s^2} + \left(\frac{1 - 2\pi}{s} - \frac{2}{s^2} \right) e^{-as}$

(ii) $t^2 [u(t) - u(t-2)]$, $\frac{2(1 - e^{-2s})}{s^3} - \frac{4e^{-2s}(1+s)}{s^2}$

(iii) $[u(t) - u(t-T)] \cos(\omega t + \phi)$; $[(s \cos \phi - \omega \sin \phi) - e^{-sT} \times \{s \cos(\phi + \omega T) - \omega \sin(\phi + \omega T)\}] / (s^2 + \omega^2)$

3. (i) $\frac{s}{s^2 + 1} + \left(\frac{1}{s} + \frac{s}{s^2 + 1} \right) e^{-\pi s} - \left(\frac{1}{s} - \frac{1}{s^2 + 1} \right) e^{-2\pi s}$

(ii) $\frac{1}{s^2 + 1} + e^{-\pi s} \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 1} \right) + e^{-2\pi s} \left(\frac{s}{s^2 + 9} - \frac{s}{s^2 + 4} \right)$

(iii) $\frac{2}{s^3} \{1 + e^{-2s} (2s^2 - 1) - 2e^{-4s} (1 + 4s)\}$

4. (i) $e^{-s}/(s-1)$; (ii) $2e^{-s}/s^3$; (iii) $e^{-2s} \left\{ \frac{24}{s^4} + \frac{42}{s^3} + \frac{28}{s^2} + \frac{25}{s} \right\}$; (iv) $e^{-s}(2+2s+s^2+s^3)/s^3$

5. $20e^{-2}$

6. (i) $-\sin t \cdot u(t-\pi)$; (ii) $\frac{1}{3} e^{-4(t-2)} \sin 3(t-2) \cdot u(t-2)$;

(iii) $\frac{1}{2} e^{-(t-1)} (t-1)^2 u(t-1)$ (iv) $3 - 4(t-1)u(t-1) + 4(t-3)u(t-3)$

7. $y = \frac{1}{2} \sin 2t + \frac{1}{4} (1 - \cos 2t) - \left\{ \frac{1}{4} [1 - \cos(t-1)] u(t-1) \right\}$

8. $x = 3 - 2 \cos t + 2[t-4 - \sin(t-4)] u(t-4)$.

9. $y(x) = \begin{cases} \frac{2Wx^2(3l-5x)}{81EI}, & 0 < x < l/3 \\ \frac{2Wx^2(3l-5x)}{81EI} + \frac{W}{6EI} \left(x - \frac{l}{3} \right)^3, & \frac{l}{3} < x < l \end{cases}$

10. $y(x) = \frac{wl^2}{16EI} x^2 - \frac{wl}{12EI} x^3 + \frac{w}{24EI} x^4 - \frac{w}{24EI} (x-l/2)^4 u(x-l/2)$

11. $x = \frac{I}{mn} e^{-\mu t/2m} \sin nt$; $\frac{dx}{dt} = \frac{I}{m} e^{-\mu t/2m} \left(\cos nt - \frac{\mu}{2mn} \sin nt \right)$, where $n^2 = \frac{k}{m} - \frac{\mu^2}{4m^2}$.

Problems 21.9, page 764

1. $\frac{2s}{(s^2+1)^2}$

2. (d)

3. (b)

4. $\frac{1}{s^2 - 4s + 5}$

5. te^{-2t}

6. $\frac{1}{3} e^{-2t} \sin 3t$

7. $s \bar{f}(s) - f(0)$

8. $\frac{1}{8} (2-3t) e^{-3t/2}$

9. $\frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2-16} \right]$

10. $\frac{1}{s - \log 2}$

11. $\frac{k!}{(s+1)^{k+1}}$

12. $\frac{2}{13}$

13. $e^{-at/s}$

14. $\Gamma(3/2)/s^{3/2}$

15. $s^2 \bar{f}(s) - sf(0) - f'(0)$

16. $\frac{1}{4} \left[\frac{3s}{s^2+16} + \frac{s}{s^2+144} \right]$

17. $\cot^{-1}(s/a)$

18. $\frac{e^{-3t} t^4}{24}$

19. $\frac{s \cos 3 - 2 \sin 3}{s^2 + 4}$

20. (c)

21. $f(t-a) u(t-a)$

22. e^{-as}

23. $1 - 3t + 2t^2$

24. $\int_0^t f(t) dt$

25. $\int_0^T \frac{e^{-st} f(t) dt}{(1-e^{-st})}$

26. $\frac{1}{(s+1)^2(s+2)}$

27. $\frac{2(s-3)}{s^2-6s+34} + \frac{12}{s^2-6s+25}$

28. $1/(s - \log 4)$

29. $\frac{e^{-3t}}{\sqrt{\pi t}}$

30. (d)

31. $\frac{t}{2} \sin \frac{t}{2}$

32. (c)

33. (c)

34. (ii)

35. (iii)

36. (ii)

37. (iv)

38. (i)

39. $\int_a^b f(x)dx$

40. (ii)

41. (iii)

42. (d)

43. True

44. False

45. False.

Problems 22.1, page 776

1. $\frac{2}{\pi} \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda ; \frac{\pi}{2}$ for $|x| < 1, \frac{\pi}{4}$ for $|x| = 1, 0$ for $|x| > 1$

2. (i) $\frac{4}{\pi} \int_0^\infty \frac{\sin \omega - \omega \cos \omega}{\omega^3} \cos \omega x d\omega$ (ii) $\frac{2}{\pi} \int_0^\infty \cos \omega x \frac{a}{a + \omega^2} d\omega$

4. (i) $\frac{2\sin as}{s}, \pi$; (ii) $2 \{(a^2 s^2 - 2) \sin as + 2as \cos as\}/s^3$

5. $\frac{4}{s^3} (\sin sa - sa \cos sa)$

6. (i) $\sqrt{3\pi} e^{-3s^2/4}$ (ii) $\frac{\sqrt{\pi}}{2} e^{(3is - s^2/4)}$

7. $\frac{1 - \cos 2s}{s}, \frac{\sin 2s}{s}$

8. $\sqrt{\pi/2} e^{-as}$ 9. $\frac{a}{a^2 + s^2}; \frac{\pi}{2a} e^{-a\lambda}$

10. 1

13. (i) $\frac{\pi}{2a^2} (1 - e^{-as})$ (ii) $\tan^{-1}(s/a)$

11. $\frac{1}{a\sqrt{2}} e^{-s^2/4a^2}; \frac{1}{2a^3\sqrt{2}} e^{-s^2/4a^2}$

14. (i) $\frac{1}{2} \left\{ \frac{\sin [a(1-s)]}{1-s} - \frac{\sin [a(1+s)]}{1+s} \right\}$ (ii) $(2 \cos s - \cos 4s - 1)/s^2 - (2 \sin s)/s$

15. $2/(\pi s^2)$

16. $F_s(p) = -32(-1)^p/p\pi; F_c(p) = 32 \frac{(-1)^p - 1}{p^2\pi^2}$

17. $(\pi/s) \cos s/c$

19. $f(x) = (2 + 2 \cos x - 4 \cos 2x)/\pi x$ 20. $2/\pi(1+x^2)$.

Problems 22.2, page 780

2. $\frac{1}{4} \int_{-\infty}^{\infty} e^{-|x-t|+|t|} dt$

5. $2 \left(\frac{1 - \cos s}{s^2} \right).$

Problems 22.3, page 783

1. $\frac{1}{9} [e^{-t} + e^{2t} (3t - 1)]$

2. $\frac{1}{5} (e^{-2t} - 2 \sin t - \cos t)$

3. $(\sinh at - at)/a^2$

4. $\frac{1}{2} [e^x(x-1) + \cos x]$

5. $\frac{1}{2} (\sin t - t \cos t).$

Problems 22.4, page 791

1. $y = 30 e^{-75t} \cos 5x$

2. $\sum_{n=1}^{\infty} \frac{V_o}{n\pi} (1 - \cos n\pi) e^{-n^2 t \sin nx_0}$

3. $u(x, t) = \frac{2}{\pi} \int_0^{\infty} e^{-s^2 t} \left\{ \frac{\sin (1+x)s + \sin (1-x)s}{s} \right\} ds$

6. $\theta(x, t) = \theta_0 \operatorname{erf} \left(\frac{x}{2\sqrt{kt}} \right) + \theta_0 \sum_{n=1}^{\infty} (-1)^n \left\{ \operatorname{erf} \frac{nl-x}{2\sqrt{kt}} - \operatorname{erf} \frac{nl+x}{2\sqrt{kt}} \right\}.$

Problems 22.5, page 792

1. $F_c(s) = \int_0^\infty f(t) \cos st dt$

2. $s^2/2$

3. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

4. $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

5. $\int_{-\infty}^{\infty} t^n f(t) e^{ist} dt$

6. $e^{isa} F(s)$

7. $\frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t dt d\lambda$

8. $\frac{1}{a}$

9. $-s^2 [F(u)]$

10. $\frac{2}{\pi} \int_0^\infty \sin(\lambda x) d\lambda \int_0^\pi \sin(\lambda t) dt$ 11. $-\frac{n\pi}{l} \int_0^l f(x) \cdot \cos \frac{n\pi x}{l} dx$

12. $\frac{1}{a} F\left(\frac{\lambda}{a}\right)$

13. $f(x) = \frac{2}{\pi^3} \sum_{p=1}^{\infty} \left(\frac{1 - \cos p\pi}{p^2} \right) \sin px$

14. $\frac{1}{2} F(s/2)$

15. $1/(s^2 + 1)$

16. True

17. False

18. True

19. True

20. False

21. True.

Problems 23.1, page 800

1. (i) e^{az} ; (ii) $z(e^{-z} - 1)$; (iii) $\frac{z}{z - e^{i\theta}}$

2. (i) $\frac{2z}{(z-1)^2} + \frac{z/\sqrt{2}}{z^2 - \sqrt{2}z + 1} + \frac{z}{z-1}$; (ii) $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$; (iii) $\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$

6. (i) $\frac{z^2 \sin \theta}{z^2 - 2z \cos \theta + 1}$ (ii) $\cos \alpha \frac{z(z - \cos \pi/8)}{z^2 2z \cos \pi/8 + 1} - \sin \alpha \frac{z \sin \pi/8}{z^2 - 22 \cos \pi/8 + 1}$

7. $\frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}, |z| > 1; \frac{z(z^2 \cos \theta - 2z + \cos \theta)}{(z^2 - 2z \cos \theta + 1)^2}$ 8. $\frac{z^2}{z^2 + 1}, |z| > 1; \frac{z^2}{z^2 + a^2}$

9. (i) $\frac{z}{z - e^{-a}}$; (ii) $\frac{ze^{-a}}{(z - e^{-a})^2}$; (iii) $\frac{(z + e^{-a})e^{-a}}{(z - e^{-a})^3}$

12. $\frac{z^2(1 + 3z^2)}{(1-z)(1+z^2)}$ 13. $u_2 = 2, u_3 = 11$.

Problems 23.2 page 804

1. $\frac{z}{z-4}; |z| > 4$

2. $\frac{z}{2-z}; |z| < 2$

3. $\frac{3z}{(4-z)(z-3)}; 3 < |z| < 4$.

4. $\frac{5z}{(4-5z)^2}; |z| > 5$

5. $-\log(1-3/z); |z| > 3$

6. $e^{3/z}$, ROC is z -plane

7. $(1-e^a/z)^{-1}; |z| > |e^a|$.

Problems 23.3, page 807

1. $\frac{1}{2}(3^{n+1} - 1)$

2. $(n+1)a^n$

3. $\frac{1}{2}n(n-1)$

5. $4a^n$

6. $(1/3)^n - 2^n$

7. n

8. $\frac{3}{4} \left\{ \frac{1}{(2)^{n-1}} + \frac{1}{(-4)^n} \right\}$

9. $(n^2 + 7n + 4)(4)^{n-1}$

13. $u_n = (2)^{n-1} + (3)^{n-1} + (4)^{n-1}$ ($n > 0$)
12. $(-1)^{n+1} - 2n + \cos n\pi/2$
13. $1 - e^{-at}$
14. (i) $\left(-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} \dots\right) + \left(\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \dots\right)$; (ii) $(-2^{n-1})z^{-n}$, $n > 0$
 (iii) $(3^{n-1} - 2^{n-1})z^{-n}$, $n \geq 1, 0, n \leq 0$
15. $\frac{1}{2}(n-1)(n-2)5^{n-3}$, $n \geq 3$ and $= 0, n < 3$
16. $2(-i)^{n-1} - (-2)^{n-1}$
17. $\frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$
18. $\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$
19. $u_n = 1 + \frac{1}{2}[(i)^{n-2} + (-i)^{n-2}]$
20. $2n \sin(n\pi/2)$, $n = 0, 1, 2, \dots$

Problems 23.4, page 811

1. $y_k = \frac{8}{5}\left(\frac{1}{2}\right)^k - \frac{3}{5}\left(\frac{-1}{3}\right)^k$
2. $y(n) = (n-1)(-1)^{n-2}y(n-2) - 2^n$
3. $y_n = 2^{n-1} + (-2)^{n-1}$
4. $f(n) = 2 + (-4)^n$
5. $y_n = \frac{4}{3}\left[2(-1)^n + (2)^n\right]$
6. $36\left[\frac{1}{2} - (2)^n + \left(\frac{1}{2}\right)^n\right]$
7. $y_n = (c_1 + c_2n)3^n + \frac{1}{2}n(n-1)3^{n-2}$
8. $y_n = c_1 + c_2 \cdot 3^n + 5^n/8$
9. $y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$
10. 5.2^n
11. $y_n = \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n + \frac{1}{15}(2)^n$
12. $y_k = 1 - 2k + 2^k$
13. $y_n = c_1 4^n + \left(c_2 - \frac{n}{4}\right)2^n + 2n - \frac{8}{3}$
14. $y_k = \frac{1}{2}(k+2)\frac{1}{5^k} \cos \frac{k\pi}{2}$
15. $y_n = \left[\frac{1}{4} - \frac{9}{4}(-3)^n\right] \mu(n)$
16. $y_n = (-2)^{n-1}$, ($n \geq 1$).

Problems 23.5, page 811

1. $z/(z-1)$
2. $\sum_{n=0}^{\infty} u_n z^{-n}$
3. $z/(z-1)^2$
4. $az/(z-a)^2$
5. $\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$
6. $e^{1/z}$
7. $(z^2 + z)/(z-1)^3$
8. $Z(au_n + bv_n + cw_n) = aZ(u_n) + bZ(v_n) + cZ(w_n)$
9. 2^{n-1}
10. $(-1)^{n-1}n$
11. $u_0 = \lim_{z \rightarrow \infty} \{Z(u_n)\}$
12. False
13. False
14. True
15. True
16. False
17. False
18. False.

Problems 24.1, page 815

1. $a = 2.28, b = 6.1879, p = 30.46$
2. $a = 1120, b = 55.1$
3. $a = 0.2, b = 0.0044$
4. $a = 0.5012, n = 0.5$
5. $a = 0.115, b = 11.8$
6. $a = 4.1, b = 0.43$
7. $a = 0.0498, b = -0.02$

Problems 24.2, page 819

1. $y = 13.6x$
 2. 15.2 thousand tons
 3. $Y = 0.004P + 0.048$
 4. $R = 70.052 + 0.292t$
 5. $a = 0.545, b = 0.636$
 6. (a) $y = 4.193 + 1.117x$
 (b) $y = 8 - 0.5x$
 7. $y = 1.243 - 0.004x + 0.22x^2$
 8. $y = 0.34 - 0.78x + 0.99x^2$
 9. $y = 18.866 + 66.158x - 4.333x^2$
 10. $R = 3.48 - 0.002V + 0.0029V^2$ 11. $V = 2.593 - 0.326T + 0.023T^2$.

Problems 24.3, page 823

1. $6.32, b = 0.0095$
 2. $a = 1.52, b = 0.49$
 3. $a = 3, b = 2$
 4. $y = 7.187 - 5.16 \frac{1}{x}; 4.894$
 5. $a = 0.988, b = 3.275$
 6. $y = 2.978 x^{0.5143}; 5.8769$
 7. $a = 0.1839, b = 0.0221$
 8. $f(t) = 0.678 e^{-3t} + 0.312 e^{-2t}$
 9. $a = 146.3, k = -0.412$
 10. $a = 99.86, b = 1.2$.

Problems 24.4, page 826

1. $a = 11.1, b = 0.71$
 2. $y = 46.05 + 6.1x$
 3. $a = 0.0028, b = 0.01, c = 4.18$
 4. $a = 15.8, b = 2.1, c = -0.5$
 5. $a = 1.459, b = 0.062$.

Problems 24.5, page 828

1. $y = 0.12 + 0.47x$
 2. $y = 1.184 + .523x$
 3. $y = 1.53 + 0.063x + 0.074x^2$
 4. $y = 0.485 + 0.397x + 0.124x^2$.

Problems 24.6, page 829

1. $Y = aX + b$ where $X = x, Y = y/x$
 2. $Y = A + BX$, where $X = \log_{10} p, Y = \log_{10} v, A = \frac{1}{r} \log b, B = -1/r$
 3. § 24.4
 4. $y = aX + c$, where $X = x^b$
 5. (ii)
 6. $\Sigma y = nA + B\Sigma x, \Sigma xy = A\Sigma x + B\Sigma x^2$ where $y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$
 7. § 24.12
 8. Zero
 9. $y = aX + b$ where $X = x^2/\log_{10} z, Y = y/\log_{10} x$
 10. $a = 0.0167, b = 1.05$.
 11. The moments of the observed values of y are respectively equal to the moments of the calculated values of y .
 12. $a = 1.7, b = 1.26$
 13. $y = a + bx$ where $x = 1/x, y = 1/y$
 14. (r)
 15. (b)

Problems 25.1, page 837

1. 336.79
 2. 64% get more than 50 marks ; median 54.7, $Q_1 = 46, Q_3 = 61.5$
 3. Mean = 27.9 ; Median = 25.66 ; Mode = 21.85
 4. Mean = 32.58 ; Median = 32.6 ; Mode = 35.1
 5. 3.1%
 6. 1.3%
 7. 192 km/hr
 8. 60 km/hr
 9. 38.6 ; 36.2
 10. Median = 12.2 days ; Mode = 11.4 days

Problems 25.2, page 842

1. 4.45, 0.39
 3. 4, 7
 4. 10.04, 10.13, 11.69, 5.54, 2.35

5. 32, 32.6, 12.4 6. Q.D. = 10.9, S.D. = 15.26 7. 1.845 ; 1.8175
 8. 0.55, 1.24 ; first, yes 9. Height 10. A
11. *B* is a better player and more consistent
12. *A* is more efficient, *B* is more consistent 13. 161.3, 5.68.

Problems 25.3, page 845

1. $\mu_1 = \mu_3 = 0, \mu_2 = 2, \mu_4 = 11; \beta_1 = 0, \beta_2 = 2.75$ 2. 8.85 ; 5.25 ; 0.32 ; 1.09
 3. -0.2064 4. 0.22 ; 1.157 6. 0 ; 2.9
 7. $\beta_1 = 0.493; \beta_2 = 0.655$; platykurtic.

Problems 25.4, page 854

1. $r = 0.81; x = 0.5y + 0.5, y = 1.3x + 1.1$ 2. $r = 0.96$
 3. $r = 0.92$ 4. $r = -0.055$ 5. 0.7291
 6. $r = 0.4517$ 7. $r = 0.632; y = 0.467 + 0.8x, x = 0.167 + 0.5y$
 9. $m = (\beta - b)/(a - \alpha)$ 10. $\bar{x} = 4, \bar{y} = 7, r = -0.5$ 11. $\bar{x} = 9.06, \bar{y} = 5.52, r = 0.46$
 12. $r = 0.7395; \bar{x} = -0.1034; \bar{y} = 0.5172$ 13. 134.5
 14. 1.28 inch 15. 0.8545 16. 0.932.

Problems 25.5, page 855

1. (d) 2. (d) 3. (a) 4. (a)
 5. (b) 6. (a) 7. (b) 8. Coefficient of correlation
 9. No 10. 13.83 11. $\bar{x} = 2, \bar{y} = 3, r = \sqrt{3}$
 12. Zero 13. $\frac{1}{2}(Q_3 - Q_1)$ 14. -1
 15. § 25.9 16. $\frac{\Sigma XY}{n \sigma_x \sigma_y}$ 17. (\bar{x}, \bar{y})
 18. Reliability or consistency 19. $\sqrt{\beta_1}$ 20. degree of peakedness
 21. $y - \bar{y} = r \frac{\sigma_y}{\sigma_x} (x - \bar{x})$ 22. 100 σ/\bar{x} 23. $\tan^{-1} \left\{ \frac{1-r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right\}$
 24. 3 25. Two regression coefficients 26. perpendicular
 27. greater 28. ± 1 29. $1 - \frac{6 \sum d_i^2}{n^3 - n}$
 30. Coefficient of standard deviation 31. zero
 32. -1 and 1 33. -0.6 34. False
 35. True 36. False

Problems 26.1, page 858

1. 4096 2. 360 ; 120 3. 120 ; 115 4. (a) 676000 (b) 468000

Problems 26.2, page 868

1. (i) 7/12 ; (ii) $P(A/B) = 3/4, P(A \cup B) = 7/12, P(A' B') = 3/8$
 2. (a) 1/36 ; (b) 1/6, Yes 3. 36 : 30 : 25
 4. 1/7 5. (i) 1/4, (ii) 7/8, (iii) 11/16 6. 15/1024

- 7.** 3/28 **8.** 20/81 **9.** (i) 2816/4165 ; (ii) 2197/2025,
10. 0.11 **11.** (a) 6.739 ; (b) 0.024 **12.** (a) 1/114 ; (b) 685/1140
13. 2/801 **14.** 10/21 **15.** $1 - (1 - p_1)(1 - p_2) \dots (1 - p_n)$; 0.518
16. 15/17 **17.** 1/2 **18.** 5/12 **19.** (i) 83/110 (ii) 25/83
20. $1 - 2/(n - 1)$ **21.** 7/20 **22.** (a) 1/6 ; (b) 3/4 **23.** 61/90
24. 0.72 **25.** 0.2223 **26.** 0.88

Problems 26.3, page 871

- 1.** 3/11 **2.** 25/69, 28/69, 16/69 **3.** 0.3175, 0.254 **4.** 15/59.

Problems 26.4, page 878

- 1.** $k = 1$; $\mu = .8$, $\sigma^2 = 2.232$ **2.** $2\sqrt{5}$
3. $F(x) = 0$, $-\infty < x < 0$
 $= 1/8$, $0 \leq x \leq 1$
 $= 1/2$, $1 \leq x \leq 2$
 $= 7/8$, $2 \leq x \leq 3$
 $= 1$, $3 \leq x \leq \infty$
6. 2 **7.** $f(x)$ is a p.d.f. $\bar{x} = \frac{1}{2}$, $\sigma^2 = \frac{1}{20}$ **8.** (i) 9/16, 7/16 ; (ii) $k = 0.45$
9. (i) 0.37, (ii) 0.63 **10.** 4/9
12. $y_0 = 3/4$; Mean = 1 ; Variance = 1/5. **13.** 0.2
14. 1/3, 2/9 **15.** $F(x) = 0$, $x < x_1$; $(x - x_1)/(x_2 - x_1)$, $x_1 \leq x < x_2$; 1, $x \geq x_2$.

Problems 26.5, page 881

- 1.** $n = 4$, $p = q = \frac{1}{2}$; $\frac{15}{16}$ **2.** ${}^4C_r(1/6)^{4-r}(5/6)^r$; $r = 0, 1, 2, 3, 4$
3. (a) 0.02579 ; (b) 0.04571 ; (c) 1.024×10^{-7} **4.** 0.3456
5. 45927/50000 **6.** (i) 0.246 ; (ii) 0.345
7. (i) ${}^{20}C_1(1/20)(19/20)^{19}$; (ii) $\sum_{r=0}^5 {}^{20}C_r(1/20)^r(19/20)^{20-r}$; (iii) 19
8. (a) 0.08 ; (b) 0.26 ; (c) 0.92 **9.** (a) 250 ; (b) 25 ; (c) 500
10. (i) 0.59049 ; (ii) 0.32805 ; (iii) 0.08146 **11.** 11
12. 99.83 **13.** 0.3585, 0.3773, 0.1887 ; 0.0596
14. 600 **15.** 100 $(.432 + .568)^5$
16. 200 $(0.554 + 0.446)^6$.

Problems 26.6, page 884

- 1.** (i) 2 ; (ii) $\frac{2}{3} e^{-2}$ **2.** $P(0) = 0.2636$, $P(3) = 0.1041$, $P(> 3) = 0.1506$
4. $(10)^{15} e^{-10}/15! = 0.035$ **5.** 0.08
6. (i) 0.2231 (ii) 0.1913 **7.** 0.0008
8. $m = 0.51 = \sigma^2$; Poisson frequencies of 0, 1, 2, 3, 4 accidents are 180.1, 91.9, 23.4, 4, 0.6
9. 0.6 **11.** Theoretical frequencies are 44, 43, 21, 7, 1
12. Theoretical frequencies are 109, 142, 92, 40, 13, 3, 1, 0, 0, 0, 0.

Problems 26.7, page 890

2. (i) 0.1644 ; (ii) 0.7686 3. (i) 0.095 ; (ii) -0.995 4. 36.4
 5. (i) 16, (ii) 2 6. 294 7. 543
 8. (i) 79 ; (ii) 35% ; (iii) 11 10. 52 11. 67
 12. ₹ 866 13. $y = \frac{100}{\sqrt{(3.4\pi)}} e^{-\frac{(x-2)^2}{3.4}}$ 14. $\mu = 13.64$. $\sigma = 3.98$.

Problems 26.8, page 892

1. (a) 0.0287, (b) 0.9672, (c) 0.5111 2. (a) 0.7854, (b) 0.1815, (c) 0.1815
 3. (a) 0.97815, (b) 0.00595, (c) 0.01209.

Problems 26.9, page 893

3. $\frac{1}{2}(n+1), \frac{1}{12}(n^2 - 1)$ 5. Mean = $a + b$, variance = b^2
 6. $[(1 - e^t)/t]^2$ 8. $(1 - t)^{3/4}$.

Problems 26.10, page 894

- | | | | | | |
|---|--|--|--|---|--------|
| 1. (a) | 2. (b) | 3. (d) | 4. (b) | 5. 1/7 | 6. 1/2 |
| 7. (b) | 8. (b) | 9. (a) | 10. (c) | 11. 0.1288 | 12. 2 |
| 13. 0.21 | 14. 0.24 | | | | |
| 15. $X:$ 0 1 2 | | 16. § 25.5 | | 17. 0.7837 | |
| $p(x):$ 1/4 2/4 1/4 | | | | | |
| 18. zero | | 19. equal | | 20. $P(A) + P(B)$ | |
| 21. $\beta_1 = 0, \beta_2 = 3$ | | 22. 120 | | 23. 0.2646 | |
| 24. 1/9 | | 25. 0.2222 | | 26. 1/6 | |
| 27. e^{-3} | | 28. 5/36 | | 29. 2 | |
| 30. symmetrical | | 31. $1 - e^{-m}$ | | 32. six | |
| 33. 0.6915 | | 34. $(q + pe^t)^n$ | | 35. 4 : 5 | |
| 36. 1/6 | | 37. 3.5 | | 38. $\sqrt{2}$ | |
| 39. unity | | 40. $n \rightarrow \infty, p \rightarrow 0$ such that np is fixed | | | |
| 41. $P(A) + P(B)$ | | 42. ${}^yC_x / {}^{y+z}C_x$ | | 43. np | |
| 44. $P(A \cup B) = 0.72, P(A \cap B') = 0.1653$ | | | | 45. $\left(\frac{1}{3} + \frac{2}{3}\right)^{18}$ | |
| 46. 1 | 47. $\mu_r' = \left[\frac{d^r}{dt^r} (\sum p_i e^{tx_i}) \right]_{t=0}$ | | 48. 1/6 | | |
| 49. 3/4 | 50. 1/2 | 51. 0.2 | 52. 2/7 | | |
| 53. $(q + p)^n$ | 54. 2/3 | 55. Mean and S.D. | 56. 1/13 | | |
| 57. 2 | 58. § 26.6 | 59. $e^{-4/3}$ | 60. 1/3 | | |
| 61. True | 62. False | 63. True | 64. False | | |
| 65. False | 66. True | 67. False | 68. False | | |
| 69. $k = 2$ | 70. 8 | 71. $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, λ is a parameter | | | |
| 72. 25/12 | 73. 2 | 74. l | 75. 4 | | |
| 76. 8 | 77. n, m degrees of freedom | | 78. $\int_{-\infty}^{\infty} e^{t(x-a)} f(x) dx$ | | |

79. $-5/7$ 80. $\frac{e^{-x}x^{l-1}}{\Gamma(l)}, 0 < x < \infty$ 81. $\frac{1}{2}(b+a)$
 82. 50% 83. 3/8 84. 1 85. 3/4
 86. (iii) 87. $F(x) = \int_{-\infty}^x f(x) dx$ 88. $\frac{\lambda^{2r} e^{-\lambda}}{(2r)!}$ 89. $-\infty < t < \infty$
 90. 6 91. 1/9 92. False 93. 2
 94. $4(1-x)^3$ 95. 0.264

Problems 27.1, page 901

1. Die is biased 2. No 3. Yes
 4. 8.91% and 15.07% 5. Consistent 6. $p = 65/500$, S.E. = 0.015
 7. 37.5%; 30.3 and 44.7 respectively 8. No
 9. Difference is not significant
 10. $Z \sim 6.56$ so that the difference is significant 11. No.

Problems 27.2, page 904

1. No 2. Mean weight lies between 64.6 and 69.4 lbs.
 3. 0.0774 5. 62 6. 2.696 7. No
 8. No 9. (i) Yes, (ii) No 10. Yes.

Problems 27.3, page 910

1. 0.25 2. $t = 0.62$, Yes 3. 11.887 and 12.113 cm
 4. Refute the claim 5. Process is not under control 6. No
 7. Sample mean = 575.2 kg., S.E. = 2.75 kg 8. Accept null hypothesis
 9. Yes with 75% confidence. 10. No 11. No
 12. No

Problems 27.4, page 914

1. 0.41 2. Hypothesis is correct
 3. Significant at 5% level 4. Yes
 5. f_e : 33.82 161.78 315.98 308.48 150.54 29.4
 $\chi^2 = 7.97$. Binomial distribution gives a good fit at 5% level
 6. f_o : 305 365 210 80 18 12
 f_e : 301.2 361.4 216.8 86.7 26 7.9 ; $\chi^2 = 3.097$
 Poisson distribution gives a good fit at 5% level
 7. f_o : 314 355 204 85 29 12
 f_e : 301 362.2 217.3 86.9 26.1 6.5
 Poisson distribution can be fitted to the data
 8. $\chi^2 = 1.2$. The fit is quite good at 5% level.

Problems 27.5, page 917

1. First variance cannot be regarded as significantly greater than the second
 2. Not significant as $F = 2.1$ and $F_{0.05} = 4.15$ 4. Not significant as $F = 2.4$ and $F_{0.01} = 3.2$
 5. Product of firm B cannot be said to be of better quality than those of firm A.

Problems 27.6, page 917

1. § 27.3 (3) 2. § 27.15 3. § 27.3 (2)
 4. We are testing the hypothesis that one process is better than another
 5. § 27.11 6. 1 7. 50 8. $(0, \infty)$
 9. II 10. 8 ; 16 11. $r = n - 1$ 12. False
 13. Less than 30 14. § 27.17 15. $-\infty < t < \infty$ 16. (ii)
 17. True

Problems 28.1, page 926

Problems 28.2, page 929

Problems 28.3, page 936

1. $x = 7, y = -9, z = 5$ 2. $x = -51/4, y = 115/8, z = 35/4$ 3. $x = 1, y = 2, z = 3$
 4. $x_1 = 2, x_2 = -1, x_3 = 3$ 5. $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$ 6. $x = 1, y = 3, z = 5$
 7. $x = 8.7, y = 5.7, z = -1.3$ 8. $x = 1, y = 2, z = 3$ 9. $x = 7, y = -9, z = 5$
 10. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$ 11. $x = y = z = 1$ 12. $x = 1, y = 2, z = 3$

 13. $x = 35/18, y = 29/18, z = 5/18$ 14. $x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$ 15. $\begin{bmatrix} 1.2 & -0.4 & 0.2 \\ -0.2 & -0.1 & 0.3 \\ -0.4 & 0.3 & 0.1 \end{bmatrix}$

Problems 28.4, page 942

1. $x = 2.556, y = 1.722, z = -1.055$
 2. (a) $x = 2.426, y = 3.573, z = 1.926$ (b) $x = 2.426, y = 3.573, z = 1.926$
 3. $x = 1, y = 1, z = 1$
 4. $x = 0.998, y = 1.723, z = 2.024$
 6. $x = 1.052, y = 1.369, z = 1.962$
 8. $x = -13.223, y = 16.766, z = -2.306$
 9. $x = 1, y = 2, z = 3, u = 4$
 10. $x = 1.36, y = 2.103, z = 2.845$
 11. $x = y = z = 1$
 12. $x = 52.5, y = 44.5, z = 59.7$
 13. $x = 1.93, y = 3.57, z = 2.43$

Problems 28.5, page 943

1. $x = 2, y = 1$
 2. $x = -1.853, y = -1.927$

3. $x = 0.7974, y = 0.4006$
 5. $x = -3.131, y = 2.362$

4. $x = 3.162, y = 6.45$

Problems 28.6, page 945

1. (a) $5.38, \begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$; (b) $4.418, \begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$
 2. $3.41; [0.74, -1, 0.67]'$
 3. (a) $6, [1, 1, -1]'$ (b) $8, [1, -0.5, 0.5]'$
 4. (a) $7; [2.099/7, 0.467/7, 1]$ (b) $25.182, [1, 0.045, 0.068]'$ (d) $11.66 [0.025, 0.422, 1.000]$.

Problems 28.7, page 945

1. Newton-Raphson method
 2. $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$
 3. Chord AB
 4. $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$
 5. initial approximation x_0 is chosen sufficiently close to the root
 6. diagonal
 7. (c)
 8. (a)
 9. $x_{n+1} = \frac{1}{3}(2x_n + N/x_n^2)$
 10. $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$
 11. $x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$
 12. (a)
 13. $x_{n+1} = x_n(2 - Nx_n)$
 14. (b)
 15. Newton-Raphson method
 16. § 28.6
 17. Upper triangular matrix
 18. False
 19. True
 20. $x = 1, y = 1$.

Problems 29.1, page 952

1. 0.4
 2. -7459
 5. 239
 6. 4.68, 2.68, 55.8, 99.88
 8. (i) $1 - 2 \sin(x + 1/2) \sin 1/2$; (ii) $\tan^{-1}(1/2n^2)$;
 (iii) $192[x(x+4)(x+8)(x+12)(x+16)]$ (iv) $-2/[(x+2)(x+3)(x+4)]$
 9. (i) $e^{3x}[e^3 \log(1+1/x) + (e^3 - 1) \log 2x]$ (ii) $2^x(1-x)/(1+x)$
 (iii) $(a-1)^n ax^n$; (iv) $(-1)^n n!/[x(x+1)(x+2)\dots(x+n)]$.
 12. (i) -36 ; (ii) $24 \times 2^{10} \times 10!$
 14. $u = [x]^4 - 6[x]^3 + 13[x]^2 + x + 9$
 15. $4x^3 - 12x^2 + 8x + 1$; $12x(x-1)$
 16. $\frac{1}{2}[x]^4 + 3[x]^3 + 4[x] + c$
 17. $y(4) = 74, y(6) = 261$
 19. 15.

Problems 29.2, page 957

1. $\left(\frac{\Delta^2}{E}\right)u_x = u_{x+h} - 2u_x + u_{x-h}; \frac{\Delta^2 u_x}{Eu_x} = \frac{u_{x+2h} - 2u_{x+h} + u_x}{u_{x+h}}$
 2. (i) $2(\cos h - 1) \sin x$; (ii) $6x$; (iii) $2(\cos h - 1)[\sin(x+h) + 1]$; (iv) 8
 8. Error = 10
 9. 31
 11. $y(4) = 74, y(6) = 261$
 12. -99
 10. $f(1.5) = 0.222, f(5) = 22.022$
 13. $y_4 = 1$ approx
 15. (i) $n(3n^2 + 6n + 1)$; (ii) $\frac{n(n+1)(n+2)(n+3)}{4}$
 16. $2/(1-x)^3$.

Problems 29.3, page 961

- | | | | |
|--------------|------------------------|-------------------------------|--------------------------|
| 1. 5.54 | 2. 6.36 | 3. 1.1312 | 4. 0.788 |
| 5. ₹ 110.52 | 6. 8666 | 7. 352 ; 219 | 8. 0.9623, 0.2903 |
| 9. 24 approx | 10. $f(x) = 9x - 4x^2$ | 11. 1.625 | 12. 0.1955 |
| 13. 4.219 | 14. 2530 | 15. $y_1 = 0.1, y_{10} = 100$ | 16. $u_2 = 42, u_4 = 49$ |
| 17. 10, 22 | 18. 755. | | |

Problems 29.4, page 971

- | | | | |
|-------------|-----------------------------|----------|-----------|
| 1. 19.4 | 2. 12.826 | 3. 54000 | 4. 3.2219 |
| 5. 3.0375 | 6. 395 | 7. 3.347 | 8. 9 |
| 9. 3250.875 | 11. 2.5283 by all formulae. | | |

Problems 29.5, page 974

- | | | | |
|----------------------|--|--|--------|
| 1. 14.63 | 2. 2.8168 | 3. 0.89 | 4. 100 |
| 5. $648 + 30x - x^2$ | 6. $x^3 - 3x^2 + 5x - 6$ | 7. $x^5 - 9x^4 + 18x^3 - x^2 + 19x - 18$ | |
| 8. 3 | 9. $\frac{0.5}{x-1} - \frac{0.5}{x+1} + \frac{1}{x-2}$ | | |

Problems 29.6, page 977

- | | | | |
|--|-----------------------------|-----------------------------------|-----------|
| 1. 1 | 2. 3.09 | 3. 448, 3150 | 4. 133.19 |
| 5. $f(x) = \frac{1}{24}x^3 - 25x + 24 - \frac{7}{6}x^2 + \frac{557}{60}x - 25$ | 6. $f(x) = \frac{1}{20}x^3$ | 7. $f(x) = x^4 - 3x^3 + 5x^2 - 6$ | |
| 8. 31. | | | |

Problems 29.7, page 978

- | | | |
|---------|-----------|----------|
| 1. 11.5 | 2. 6.5928 | 3. 37.23 |
|---------|-----------|----------|

Problems 29.8, page 978

- | | |
|-----------|--------|
| 1. \$ 7.3 | 2. (b) |
|-----------|--------|

x	$f(x)$	I.D.D.	II.D.D.
5	7		
		2.9	
15	36		0.87
22	160	17.7	

- | | |
|---|-------------------------------------|
| 4. Intermediate value of the variables. | 5. \$ 7.8 |
| 6. $\frac{[x_1, x_2, x_3, x_4] - [x_0, x_1, x_2, x_3]}{x_4 - x_0}$ | 7. $\frac{-1}{4}$ and $\frac{1}{4}$ |
| 8. \$ 7.14 | |
| 9. $f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$ | |
| 10. $\frac{13}{5}$ | 11. (c) |
| 12. 1.857 | |
| 13. Extrapolation is the process of estimating the value of a function outside the given range of values. | |
| 14. $1/(abc)$ | 15. (a) |
| 16. $x^3 - 7x^2 - 18x - 12$ | 17. (b) |
| 18. $6h^2(x + h)$. | |

Problems 30.1, page 987

1. -27.9, 117.67 2. 4.75, 9 3. 0.63, 6.6
 4. (a) 0.493, -1.165 (b) 0.4473, -0.1583 ; (c) 0.4662, -0.2043 5. 2.8326
 6. -0.06; 0.5 7. 0.175 8. 13.13 m/sec
 9. (i) -52.4, (ii) -0.0191 10. 44.92 11. 0.085
 12. 3.82 rad/sec, 6.75 rad/sec² 13. 0.2561 15. 0.0186
 16. 135 17. $y_{\max}(1) = 0.25, y_{\min}(0) = 0$ 18. $\text{Max } f(10.04) = 1340.03$

Problems 30.2, page 995

1. (i) 0.695 (ii) 0.693 (iii) 0.693 2. (i) 0.7854 (ii) 0.7854, (iii) 0.78535, (iv) 0.7854
 3. 1.61 4. -6.436 5. (i) 70.16 (ii) 0.635
 6. 0.6305 7. (i) 2.0009 ; (ii) 1.1873 8. (i) 1.1249 (ii) 0.911
 9. (a) 1.8276551, .0001924 ; (b) 1.8278472, .0000003 ; (c) 1.8278470, 0000005 ; (d) 1.8278474, .00000001
 10. 1.3028 11. 403.67 12. 7.78
 13. 710 sq.ft 14. 3.032 15. 408.8 cub. cm.
 16. 1.063 sec ; 1.064 sec 17. 552 m ; 3 m/sec². 18. 30.87 m/sec.
 19. 29 min nearly.

Problems 30.3, page 996

1. (c) 2. $\frac{1}{h} \left[\Delta f(a) - \frac{1}{2} \Delta^2 f(a) + \frac{1}{3} \Delta^3 f(a) - \dots \right]$
 3. h should be small 4. 0.775 5. $2\frac{2}{3}$
 6. 30.8 7. (b)
 8. larger number of sub-intervals 9. 0.7854
 11. $y'_{x_n} \frac{1}{h} = \left[\nabla_{y_n} + \frac{1}{2} \nabla^2_{y_n} + \frac{1}{3} \nabla^3_{y_n} + \dots \right]$ 12. a multiple of 6
 13. (c) 14. 0.783 15. 0.69
 16. 1.36125 17. 1.36
 18. if the entire curve is itself a parabola
 19. even and multiples of 3 20. False

Problems 31.1, page 999

1. $y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$ 2. $\Delta y_n = (-1)^{n+1}/(n+1)$ 3. $u_{n+1} - 2u_n = 0$
 4. (i) $(x+2)y_{x+2} - 2(2x+1)y_{x+1} + xy_x = 0$; (ii) $(x^2+x)y_{x+2} - (2x^2+4x)y_{x+1} + (x^2+3x+2)y_x = 0$
 5. (ii) $y_{n+2} - 8y_{n+1} + 15y_n = 0$; (ii) $y_{n+2} - 6y_{n+1} + 4y_n = 0$
 6. (i) $(x-1)y_{x+2} - (3x-2)y_{x+1} + 2xy_x = 0$; (ii) $y_{x+2} - 4y_x = 0$;
 (iii) $y_{x+3} - 6y_{x+2} + 11y_{x+1} - 6y_x = 0$.

Problems 31.2, page 1002

1. $u_p = (c_1 + c_2 p) 3p$ 2. $y_n = c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3}$ 3. $u_n = c_1 \cos n\pi/2 + c_2 \sin n\pi/2$
 4. $y_n = c_1 \cdot 2^n + c_2 \cdot 3^n$. 5. $y_n = (2)^{n-1} + (-2)^{n-1}$ 6. $u_k = c_1 (-1)^k + (c_2 + c_3 k) 2^k$.
 7. $f(x) = (c_1 + c_2 x)(-1)^x + c_3 \cdot 2^x$ 8. $u_n = 2n + (-2)^n$. 9. $y_n = 6 + (n-3) 2^n$
 10. $u_n = 2^{n/2} [c_1 \cos n\pi/4 + c_2 \sin n\pi/4]$
 11. $y_m = 2^m \left\{ c_1 \cos \frac{m\pi}{4} + c_2 + \sin \frac{m\pi}{4} + c_3 \cos \frac{3m\pi}{4} + c_4 \sin \frac{3m\pi}{4} \right\}$ 14. $y_n = c_1 (-1)^n + c_2 (10)^n$

Problems 31.3, page 1005

1. $y_n = c_1(-1)^n + c_2(6)^n - 2^n/12$

2. $y_n = \left(\frac{n}{15} - \frac{1}{25}\right)(-3)^n + \frac{2^n}{25}$

3. $y_p = c_1 + c_2 p + c_3 p^2 + \frac{1}{6} p(p-1)(p-2)$

4. $y_n = 2^n \left(\frac{2}{\sqrt{3}} \sin \frac{n\pi}{3} - 2 \cos \frac{n\pi}{3}\right) + 2$

5. $y = c_1 + c_2 \cdot 3^x + \frac{1}{2} x \cdot 3^{x-1}$

6. $y_x = (c_1 + c_2 x) 2^x + 3x(x-1) 2^{x-3} + 5 \cdot 4^{x-1}$

7. $u_n = c_1 + c_2 (-1)^n + \frac{1}{2} \frac{\cos\left(\frac{n}{2}-1\right) - \cos\frac{n}{2}}{1-\cos 1}$

8. $y_p = c_1 \cos \frac{p}{2} + c_2 \sin \frac{p}{2} + \frac{p \cos\left(p-\frac{1}{2}\right)}{2 \sin \frac{1}{2}}$

9. $y_x = c_1 + 2^x + c_2 (-2)^x - \frac{1}{27} (9x^2 + 12x + 11)$

10. $y_n = c_1(-1)^n + c_2 \cos \frac{n\pi}{3} + c_3 + \sin \frac{n\pi}{3} + \frac{1}{2} n(n-3)$

11. $y_n = (c_1 + c_2 n)(3)^n + c_3(-1)^n + \frac{1}{3}(2)^n - \frac{3n}{4}$ 12. $y_n = (c_1 + c_2 n) 2^{-n} + \frac{2^n}{9} + n(n-1) \left(\frac{1}{2}\right)^{n-1}$

13. $y_n = c_1(-2)^n + c_2(-3)^n + \frac{n}{12} - \frac{7}{144}$ 14. $u_x + (c_1 + c_2 x)(-3)^x + \frac{2^x}{25}(5x-2) + \frac{2}{4} x^{x-2} + \frac{7}{16}$

15. $y_n = c_1(-2)^n + 2^n (c_2 \cos n\pi/3 + c_3 \sin n\pi/3) + \frac{3}{16} (2)^n + 2^{n-4} (2n+3)$

16. $u_n = \left\{c_1 + c_2 n + \frac{1}{48} n(n-1)^2(n-2)\right\} 2^n$. 17. $y_k = c_1 \cdot 2^k + c_2 \cdot 3^k + \frac{4^k}{2} (k^2 - 13k + 61)$

18. $y_n = 2^n \left\{(c_1 + n) \cos \frac{n\pi}{3} + c_2 + \sin \frac{n\pi}{3}\right\}$.

Problems 31.4, page 1006

1. $y_x = a + b(-1)^x + x, z_x = a + b(-1)^{x+1} - (x+1)$

2. $y_x = (a + bx)(-1)^x - \frac{1}{9} \cdot 2^{x+2}, z_x = \frac{2^x}{9} - (-1)^x [a + b(x - \frac{1}{2})]$

3. $u_n = 2 \cdot 4^n - 2 \frac{1}{2} n(n-1), v_n = 4^n + 2 + \frac{1}{n} + n(n+1)$

4. $u_x = -2a + b(-2)^x - c + \frac{1}{2x} (3-x), v_x = a + c + b(-2)^x + \frac{1}{2} x(x-1)$.

Problems 31.5, page 1007

1. $y_{i+1} - 2y_i + y_{i-1} = -\frac{l_m}{P} y_i$. Solve it for y_i .

Problems 31.6, page 1007

1. $y_{n+2} - 5y_{n+1} + 6y_n = 0$

2. $u_n = c_1 + c_2 n + c_3 n^2$

3. $u_n = c_1 + c_2 (-2)^n + c_3 (3)^n$

4. $y_n = c + 2^n$.

5. $y_n = c(2)^n - (n+1)$

7. $y_n = c(2)^k + 1$

6. $(x^2 + x)y_{x+2} - (2x^2 + 4x)y_{x+1} - (x^2 + 3x + 2)y_x = 0$

9. Third

8. $y_n = (2)^{n+2} + (-2)^{n-1}$

10. $(x+2)y_{n+2} - 2(n+1)y_{n+1} + ny_n = 0$

12. $y_n = (C_1 + C_2 n)2^n$

15. True.

11. Second

14. $y_{n+2} - 6y_{n+1} + 9y_n = 0$

Problems 32.1, page 1012

1. $y = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$

2. 0.0214

3. $y = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \dots$

4. (a) 0.9138, (b) 0.1938

5. $y(1.1) = 0.1103, y(1.2) = 0.2428$. Exact $y(1.1) = 0.1103$ and $y(1.2) = 0.2428$

6. 1.1053425

7. 1.1272

8. 1.005.

Problems 32.2, page 1017

1. 1.1831808

2. 1.1448

3. 4.5559

4. $y(0.1) = 0.095, y(0.2) = 0.181, y(0.3) = 0.259$

5. $y(0.2) = 1.2046, y(0.4) = 1.4644$

6. 2.2352

7. 1.0928

8. 5.051.

Problems 32.3, page 1021

1. 1.7278

2. 1.1749

3. 1.0207, 1.0438

4. 2.5005

5. $y(0.1) = 0.9052, y(0.2) = 0.8213$

6. $y(0.1) = 2.9919, y(0.2) = 2.9627$

7. 0.3487

8. 0.8489

9. 1.1678

10. 1.0911, 1.1677, 1.2352, 1.2902, 1.338.

Problems 32.4, page 1026

1. 3.795

2. 1.2797

3. $y(1.4) = 3.0794$

4. $y(4.5) = 1.023$

5. $y(0.4) = 2.162$

6. $y(0.4) = 0.441$.

Problems 32.5, page 1030

1. 0.2416

2. 1.0408

3. 0.6897

4. $y(4.4) = 1.019$

5. 2.5751

6. $y(1.4) = 0.949$.

Problems 32.6, page 1034

1. $y_3 = 1 + \frac{x}{2} + \frac{3}{40}x^5 + \frac{1}{40}x^6 + \frac{1}{192}x^9, z_3 = \frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{34}x^8 + \frac{7}{340}x^9 + \frac{1}{256}x^{12}$

2. $y(0.1) = 0.105, z(0.1) = 0.999; y(0.2) = 0.22, Z(0.2) = 0.997$

3. $y(0.1) = 2.0845, z(0.1) = 0.5867$

4. $y_2 = 1 + \frac{1}{2}x + \frac{3}{40}x^5$

5. $y(0.1) = 0.5075$

6. $y(0.2) = 0.9802, y(0.2) = -0.196$

7. -0.5159

8. $\theta(0.2) = 0.8367, (\frac{d\theta}{dt})_{0.2} = 3.6545$.

Problems 32.7, page 1038

1. 0.14031

2. $y(.25) = y(.75) = 2.4, y(.5) = 3.2$

3. $y(1.25) = 1.3513, y(1.5) = 1.635, y(1.75) = 1.8505$

4. $y(.25) = -0.3473, y(.5) = -0.9508, y(.75) = -1.7257$

5. $n = 2, y(0.5) = 0.1389$, true value = 0.1505 ; $n = 4, y(0.5) = 0.147$

6. $y(0.25) = 0.062, y(0.5) = 0.25, y(0.75) = 0.562$

7. $y(1) = 1.0171, y(2) = 1.094$

8. $y(1/3) = 1.1539, y(2/3) = 3.9231; y(1) = 7.4615.$

Problems 32.8, page 1038

1. (b)

4. § 31.4

7. $y_4 = y_2 + \frac{h}{3} (f_2 + 4f_3 + f_4)$

9. $y_1 = y_0 + \frac{h}{24} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

10. Milne's method and Adam-Bashforth method

11. four

12. $y = 1 + \frac{x^2}{2} + \frac{x^4}{8}$

13. (b)

14. $y_4 = y_0 + \frac{4h}{3} (2f_1 - f_2 + 2f_3)$

15. $y_1 = y_0 + \frac{h}{24} (9f_1 + 19f_0 - 5f_{-1} + f_{-2})$

16. 1.1818

17. $\frac{dy}{dx} = z, \frac{dz}{dx} + y(1 + yz) = 0$

18. § 31.7

19. starting values

20. Picard's and Runge-Kutta methods

21. It agrees with Taylor's series solution upto the term in h^4

22. (d)

23. $y_{i+1} + 2(h^2 - 1)y_i + y_{i-1} = 0$

24. (a)

25. True

26. False

27. False.

Problems 33.1, page 1041

1. Parabolic

2. Hyperbolic

3. (i) Parabolic (ii) Elliptic (iii) Elliptic

4. Outside the ellipse $(x/0.5)^2 + (y/0.25)^2 = 1$.**Problems 33.2, page 1050**

1. $u_1 = 1.999, u_2 = 2.999, u_3 = 3.999$

2. 2.37, 5.6, 9.87, 2.89, 6.14, 9.89, 3.02, 6.17, 9.51

3. $u_1 = 10.188, u_2 = 0.5, u_3 = 1.188, u_4 = 0.25, u_5 = 0.625, u_6 = 1.25$

4. $u_1 = 26.66, u_2 = 33.33, u_3 = 43.33, u_4 = 46.66$

5. $u_1 = 0.99, u_2 = 1.49, u_3 = 0.49$

6. $u_1 = 1.57, u_2 = 3.71, u_3 = 6.57, u_4 = 2.06, u_5 = 4.69, u_6 = 8.06, u_7 = 2, u_8 = 4.92, u_9 = 9$

7. $u_1 = -3, u_2 = -2, u_3 = -2$.

Problems 33.3, page 1054

1.

<i>i</i>	0	1	2	3	4
<i>j</i>	0	3	4	3	0
	0	2	3	2	0
	0	1.5	2	1.5	0
	0	1	1.5	1	0
	0	0.75	1	0.75	0
	0	0.5	0.75	0.5	0

2.

<i>j</i>	<i>i</i>	0	1	2	3	4	5	6	7	8	9	10
<i>j</i>	0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
<i>j</i>	1	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0
<i>j</i>	2	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
<i>j</i>	3	0	0.07	0.133	0.18	0.21	0.22	0.21	0.18	0.133	0.07	0

3.

<i>j</i>	<i>i</i>	0	1	2	3	4	5
<i>j</i>	0	0	24	84	144	144	0
<i>j</i>	1	0	42	84	114	72	0
<i>j</i>	2	0	42	78	78	57	0
<i>j</i>	3	0	39	60	67.5	39	0
<i>j</i>	4	0	30	53.25	49.5	33.75	0
<i>j</i>	5	0	26.6	39.75	43.5	24.75	0
<i>j</i>	6	0	19.88	35.06	32.25	21.75	0

4.

<i>j</i>	<i>i</i>	0	1	2	3	4
<i>j</i>	0	0	0.5	1	0.5	0
<i>j</i>	1	0	0.5	0.5	0.5	0
<i>j</i>	2	0	0.25	0.5	0.25	0
<i>j</i>	3	0	0.25	0.25	0.25	0

Problems 33.4, page 1060

1.

$t = 0.3 ; x =$	0.1	0.2	0.3	0.4	0.5
Numerical sol. $u =$	0.02	0.04	0.06	0.075	0.08
Exact sol. $u =$	0.02	0.04	0.06	0.075	0.08

2.

<i>j</i>	<i>i</i>	0	1	2	3	4	5
<i>j</i>	0	0	20	15	10	5	0
<i>j</i>	1	0	7.5	15	10	5	0
<i>j</i>	2	0	-5	2.5	10	5	0
<i>j</i>	3	0	-5	-10	-2.5	5	0
<i>j</i>	4	0	-5	-10	-15	-7.5	0
<i>j</i>	5	0	-5	-10	-15	-20	0

3.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	0	10.19	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	5.08	0
2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0
3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0
4	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0.02	0
5	0	0	0	0	0	0	0	0	0	-0.02	0

4.

$i \backslash j$	0	0.1	0.2	0.3	0.4	0.5
0.1	0	0.037	0.07	0.096	0.113	0.119
0.2	0	0.031	0.059	0.082	0.096	0.101
0.3	0	0.023	0.043	0.059	0.07	0.074
0.4	0	0.012	0.023	0.031	0.037	0.039
0.5	0	0	0	0	0	0

Problems 33.5, page 1060

1. (b)

4. Poisson's equation

7. § 32.8 (2)

9. hyperbolic

11. $u_{i,j+1} = 2(1 - 4\alpha^2)u_{i,j} + 4\alpha^2(u_{i-1,j} + u_{i+1,j} - u_{i,j-1})$ where $\alpha = k/h$.12. $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

15. Elliptic

16. $u(0, t) = 0 = u(1, t)$, ($t > 0$) ; $u(x, 0) = f(x)$, $0 < x < l$; $\delta x / \delta t(x, 0) = 0$, $0 < x < l$ 17. $u_{i,j+1} = u_{i+1,j} - u_{i-1,j} - u_{i,j-1}$ 18. $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = f(x, y)$ 19. 100 20. $\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$ 21. $y < 0$ 22. $k = 1/4$.**Problems 34.1, page 1063**1. Max. $Z = 1.2x_1 + 1.4x_2$; subject to $40x_1 + 25x_2 \leq 1000$, $35x_1 + 28x_2 \leq 980$, $25x_1 + 35x_2 \leq 875$ and $x_1, x_2 \geq 0$ 2. Max. $Z = 3x_1 + 2x_2 + 4x_3$; subject to $4x_1 + 3x_2 + 5x_3 \leq 2000$, $2x_1 + 2x_2 + 4x_3 \leq 2500$, $100 \leq x_1 \leq 150$, $200 \leq x_2$ and $50 \leq x_3$ 3. Max. $Z = 3x_1 + 2x_2 + x_3$; subject to $3x_1 + 4x_2 + 3x_3 \leq 42$, $5x_1 + 3x_3 \leq 45$, $3x_1 + 6x_2 + 2x_3 \leq 41$ and $x_1, x_2, x_3 \geq 0$.4. Max. $Z = 400x + 300y$; subject to $x + y \leq 200$, $x \geq 20$, $y \geq 4x$, $x \geq 0$, $y \geq 0$ 5. Min. $Z = x_1 + x_2$; subject to $2x_1 + x_2 \geq 12$, $5x_1 + 8x_2 \geq 74$, $x_1 + 6x_2 \geq 24$ and $x_1, x_2, x_3 \geq 0$.

6. Max. $Z = 0.15x_1 + 0.25x_2$; subject to
 $2x_1 + 5x_2 \leq 480,000, 5x_1 + 4x_2 \leq 720,000,$
 $8x_1 + 16x_2 \leq 300,000, 0 \leq x_1 \leq 25,000 \text{ and } 0 \leq x_2 \leq 7,000$
7. Min. $Z = 41x_1 + 35x_2 + 96x_3$; subject to
 $2x_1 + 3x_2 + 7x_3 \geq 1250, x_1 + x_2 \geq 250, 5x_1 + 3x_2 \geq 900,$
 $6x_1 + 25x_2 + x_3 \geq 232.5 \text{ and } x_1, x_2, x_3 \geq 0$
8. Min. $Z = 100x_1 + 250x_2 + 160x_3$; subject to
 $0.94x_1 + x_2 + 1.04x_3 \leq 0.98, 10x_1 + 15x_2 + 17x_3 \geq 14,$
 $470x_1 + 500x_2 + 520x_3 \geq 495, x_1 + x_2 + x_3 = 1 \text{ and } x_1, x_2, x_3 \geq 0.$

Problems 34.2, page 1069

1. $x_1 = 100, x_2 = 50$; max. $Z = 550$ 2. $x_1 = 8/15, x_2 = 12/5$, max. $Z = 24.8$
 3. $x_1 = 15, x_2 = 0$; max. $Z = 300$ 4. $x_1 = 1000, x_2 = 500$; max. $Z = 5500$
 5. 450 units of product B only; max. profit = ₹ 1800
 6. $X = 2, Y = 4.5$; max. profit = ₹ 37
 7. $A = 1.18$ units, $B = 0.53$ units; max. profit = ₹ 14.50 approx.
 8. 2000/11 units of product A and 1000/11 units of B ; max. profit = ₹ 10,000
 9. $x_1 = 4, x_2 = 0$; max. $z = 8$ 10. Unbounded solution
 11. $x_1 = 2, x_2 = 4$; min. $Z = 64$
 12. Production cost will be min. if G and J run for 12 and 4 days respectively.

Problems 34.3, page 1074

1. Max. $Z = 3x_1 + 5x_2 + 8x_3$; subject to $2x_1 - 5x_2 + s_1 = 6,$
 $3x_1 + 2x_2 + x_3 - s_2 = 5, 3x_1 + 4x_3 + s_3 = 3;$
 $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$
2. Min. $Z = 3x_1 + 2x_2 + 5x_3$, subject to $-5x_1 + 2x_2 + s_1 = 5,$
 $2x_1 + 3x_2 + 4x_3 - s_2 = 7, 2x_1 + 5x_3 + s_3 \geq 3$
 $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$
3. Max. $Z = 3x_1 - 2x_2 + 4x_4 - 4x_5$; subject to
 $x_1 + 2x_2 + x_4 - x_5 + s_1 = 8, 2x_1 - x_2 + x_4 - x_5 - s_2 = 2,$
 $-4x_1 + 2x_2 + 3x_4 - 3x_5 = 6; x_1, x_2, x_4, x_5, s_1, s_2 \geq 0$
4. (i) $x_1 = 2, x_3 = 1$ (Basic); $x_3 = 0$ (Non-basic). (ii) $x_1 = 5,$
 $x_3 = -1$ (Basic); $x_2 = 0$ (Non-basic); (iii) $x_2 = 5/3, x_3 = 2/3$ (Basic); $x_1 = 0$ (Non-basic). All the three basic solutions are non-degenerate
6. Basic solutions are (i) $x_1 = 2, x_2 = 1$ (Basic) and $x_3 = 0$;
(ii) $x_1 = x_3 = 1$ (Basic) and $x_2 = 0$; (iii) $x_2 = -1, x_3 = 2$ (Basic) and $x_1 = 0$
(a) First two solutions are non-degenerate basic feasible solutions
(b) First solution is optimal and $Z_{\max} = 5.$

Problems 34.4, page 1081

1. $x_1 = 2, x_2 = 4$, max. $Z = 14$ 2. $x_1 = 0, x_2 = 20$; max. $Z = 200$
 3. $x_1 = 7/3, x_2 = 4/3$; max. $Z = 16$ 4. $x_1 = 5, x_2 = x_3 = 0$; max. $Z = 50$
 5. $x_1 = 0, x_2 = 100, x_3 = 230$; max. $Z = 1350$ 6. $x_1 = 89/41, x_2 = 50/41, x_3 = 62/41$; max. $Z = ₹ 765/41$
 7. $x_1 = 4, x_2 = 5, x_3 = 0$; min. $Z = -11$
 8. $x_1 = 280/13, x_2 = 0, x_3 = 20/13, x_4 = 180/13$; max. $Z = 2280/13$
 9. $x_1 = 0, x_2 = 400$ units; max. profit = ₹ 1200
 10. $x_1 = 125, x_2 = 250$ units; max. profit = ₹ 2250
 11. $x_1 = 400$ gms, $x_2 = 0$; min. cost = ₹ 2 12. $x_1 = 0, x_2 = x_3 = 50$; max. profit = ₹ 700

13. $x_1 = 0.5, x_2 = x_3 = 0.04$ units ; min. cost = ₹ 5.80

14. Averages for corn, wheat, soyabean are 250, 625, zero respectively to achieve a max. profit of ₹ 32,000.

Problems 34.5, page 1088

1. $x_1 = 0, x_2 = 2, x_3 = 0$; max. $Z = 4$
2. $x_1 = 3, x_2 = 2, x_3 = 0$; max. $Z = 8$
3. $x_1 = x_2 = -6/15$; max. $Z = -48/5$
4. $x_1 = 23/3, x_2 = 5, x_3 = 0$; max. $Z = 85/3$
5. $x_1 = x_2 = x_3 = 5/2, x_4 = 0$; max. $Z = 15$
6. $x_1 = 21/13, x_2 = 10/13$; max. $Z = 31/13$
7. Infeasible
8. $x_1 = 23/3, x_2 = 5, x_3 = 0$; max. $Z = 85/3$
9. $x_1 = 55/7, x_2 = 30/7, x_3 = 0$; max. $Z = 155/7$
10. $x_1 = 2, x_2 = 0$; max. $Z = 18$
11. Degenerate solution : $x_1 = 0$ (non-basic); $x_2 = 1, x_3 = 0$ (basic); max. $Z = 3$.

Problems 34.6, page 1091

1. Min. $W = 26y_1 + 7y_2$; subject to $6y_1 + 4y_2 \geq 10$
 $5y_1 + 2y_2 \geq 13, 3y_1 + 5y_2 \geq 19; y_1, y_2, y_3 \geq 0$
2. Max. $W = 11y_1 + 7y_2 + y_3 + 5y_4$; subject to $3y_1 + 2y_2 - y_3 + 3y_4 \leq 2$,
 $4y_1 + 3y_2 + 2y_3 + 2y_4 \leq 4, y_1 - 2y_2 + 3y_3 + 2y_4 \leq 3$;
 $y_1, y_2, y_3, y_4 \geq 0$
3. Min. $W = -3y_1 + y_2 + 4y_3$; subject to $y_1 + 3y_2 - 2y_3 \leq -3$,
 $y_1 + y_3 \geq 16, y_1 - 2y_2 + y_3 \leq -7$;
 $y_1, y_2 \geq 0, y_3$, unrestricted in sign
4. Max. $W = -5y_1 + 9y_2 + 8y_3$; subject to $-2y_1 + 4y_2 - 8y_3 \leq 3$,
 $3y_1 - 2y_2 + 4y_3 \leq -2, -y_1 + 3y_3 = 1$;
 $y_1, y_2 \geq 0, y_3$ unrestricted
5. Min. $y = 3y_1 + 4y_2 + y_3 + 6y_4$; subject to $5y_1 - 2y_2 + y_3 - 3y_4 \geq 2$,
 $6y_1 + y_2 - 5y_3 - 3y_4 \geq 5, -y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6, y_1, y_2, y_3, y_4 \geq 0$.

Problems 34.7, page 1094

1. $x_1 = x_2 = 0, x_3 = 5/2$; min. $Z = 2.5$
2. $x_1 = 4, x_2 = 2$; max. $Z = 10$.
3. $x_1 = 7, x_2 = 0$, max. $Z = 21$
4. $x_1 = 0, x_2 = 100, x_3 = 230$; max. $Z = 1350$.

Problems 34.8, page 1097

1. $x_1 = 0, x_2 = 1$; max. $Z = -1$
2. $x_1 = 3/5, x_2 = 6/5$; min. $Z = 12/5$
3. $x_1 = 6, x_2 = 2, x_3 = 0$; min. $Z = 10$
4. $x_1 = 65/23, x_2 = 0, x_3 = 20/23, x_4 = 0$; min. $Z = 215/23$.

Problems 34.9, page 1104

1. $x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{24} = 125, x_{33} = 275, x_{34} = 125$; min. cost = ₹ 12075
2. $x_{13} = 14, x_{21} = 6, x_{22} = 5, x_{23} = 1, x_{32} = 5$; min. cost = 143
3. $x_{11} = 50, x_{12} = 100, x_{21} = 150, x_{33} = 150, x_{42} = 100, x_{43} = 50$; min. tonnage = 3300
4. $x_{11} = 140, x_{13} = 60, x_{21} = 40, x_{22} = 120, x_{33} = 90$; min. cost = ₹ 5920
5. $x_{11} = 5, x_{14} = 2, x_{22} = 3, x_{23} = 7, x_{32} = 5, x_{34} = 13$;
min. cost = ₹ 799 and maximum saving = ₹ 201
6. $x_{11} = 150, x_{13} = 20, x_{22} = 160, x_{24} = 40, x_{33} = 90, x_{34} = 90$; max. profit = 4920
7. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$; min. cost = 33
8. $x_{13} = 0, x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200$,
 $x_{34} = 300, x_{43} = 300$; min. cost = 9200.

Problems 34.10, page 1109

1. $x_{11} = x_{22} = x_{33} = 1$; min. cost = ₹ 18
2. $A \rightarrow 2, B \rightarrow 3, C \rightarrow 4, D \rightarrow 1$; min. $Z = 38$
3. $I \rightarrow B, II \rightarrow A, III \rightarrow D, IV \rightarrow C$; min. cost = ₹ 49
4. $A \rightarrow$ Dyn. Prog., $B \rightarrow$ Queuing Th., $C \rightarrow$ Reg. Analysis, $D \rightarrow$ L.P.; min. time = 28 hrs
5. (i) $A \rightarrow I, B \rightarrow II, C \rightarrow III, D \rightarrow IV$; (ii) $A \rightarrow I, B \rightarrow III, C \rightarrow II, D \rightarrow IV$
6. $1 \rightarrow IV, 2 \rightarrow II, 3 \rightarrow VI, 4 \rightarrow I, 5 \rightarrow III, 6 \rightarrow V$; max. profit = ₹ 270

Problems 34.11, page 1110

- | | | |
|---|-----------------------------------|---------------------------------|
| 1. § 34.5 Def. 2 | 2. it provides an optimality test | 3. § 34.11 |
| 4. § 34.16 (1) | 5. § 34.13 | 6. § 34.6 (1) |
| 7. Min. $W = 7y_1 + 5y_2$, subject to $2y_1 + 3y_2 \leq 4$,
$3y_1 - 2y_2 \leq 9$, $2y_1 + 4y_2 \leq 2$, $y_1 \geq 0$, y_2 is unrestricted in sign | | |
| 8. § 34.12 (2) | 9. § 34.14 | |
| 10. Minimize $Z = (2x_{11} + 3x_{12} + 11x_{13} + 4x_{14}) + (5x_{21} + 6x_{22} + 8x_{23} + 7x_{24})$,
subject to $x_{11} + x_{12} + x_{13} + x_{14} = a_1 (= 15)$, $x_{21} + x_{22} + x_{23} + x_{24} = a_2 (= 20)$,
$x_{11} + x_{21} = b_1 (= 10)$, $x_{12} + x_{22} = b_2 (= 5)$; $x_{13} + x_{23} = b_3 (= 12)$; $x_{14} + x_{24} = b_4 (= 8)$ and $x_{ij} \geq 0$.
[$\because \Sigma a_i = \Sigma b_j = 35$] | | |
| 11. (i) $x_1 = 3, x_2 = 5, x_3 = 0$; (ii) $x_1 = 0.5, x_2 = 0, x_3 = 2.5$ | | 12. § 34.5 (Def. 4) |
| 13. § 34.15 | 14. balanced | 15. § 34.9 |
| 16. § 34.7 (3) | 17. optimal | |
| 18. Minimize $y = 5y_4 - 3y_3$, subject to $y_4 + y_3 = 5$, $2y_4 - 5y_3 \geq 6$, $y_3 \geq 0$ and y_4 unrestricted | | |
| 19. 5 | 20. Max. $Z = 5/19$ | 21. § 34.7 |
| 22. § 34.16 | 23. § 34.7 [2 (ii)] | |
| 24. Min. $W = 2y_1 + 4y_2 + 3y_3$, subject to $-y_1 + y_2 + y_3 \geq 2$, $2y_1 + y_2 \geq 1$, $y_1, y_2 \geq 0$ | | |
| 25. North west corner rule and Vogeli approximation method | | 26. Slash or surplus variables. |

Problems 35.1, page 1118

- | | |
|---|--|
| 1. (i) $y = \frac{1}{4}x^2 + c_1x + c_2$; (ii) $y = c_1x^{-1} + c_2$ | 3. $y = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{2}$ |
| 4. $y = -x \cos x/2$ | 6. $y = \sinh(c_1x + c_2)$ |
| 7. $y = x^2 - 1$ | 9. $y = 2 \sin x$ |
| 12. The spirals of the family $r = a \sec(\phi \sin \alpha + b)$. | |

Problems 35.2, page 1120

1. $y = \pm 2 \sin mx, \text{ where } m \text{ is an integer}$
2. $y = \lambda x^2 + ax + b$, where λ, a, b are determined from the isoperimetric and boundary conditions.
4. $y(x) = \frac{1}{2}(1 - \cos x) + \frac{1}{4}(2 - \pi) \sin x$.

Problems 35.3, page 1124

- | | |
|---|---|
| 2. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$,
$z = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x - 2c_2 \sin x + 2c_4 \cos x$ | 5. $y = -\frac{\lambda}{24\mu} (x^2 - a^2)^2$ |
| 3. $y = a_n \sin nx, n = 1, 2, 3 \dots$ | 4. $y = \cos x$ |

6. (i) $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$;
(ii) $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 + x^7/7$!.

Problems 35.4, page 1126

1. (i) and (ii) $\bar{y} = \frac{5}{18} x(1-x)$ 2. $\bar{y} = \frac{x}{4} (5x-1)$ 4. $\bar{y} = 0.58 + 0.27x$
5. (i) and (ii) $c_1 = 0.93, c_2 = -0.05$ 6. 0.05 7. $c_1 = 3.27, c_2 = -2.69$
8. $y = \frac{1}{2} (5x^2 - 3x)$.

Problems 36.1, page 1134

1. $y(x) = 6x - 5 + \int_0^x (5 - 6x + 6t) y(t) dt$.
2. $y(x) = x - \sin x + e^x (x-1) + \int_0^x [\sin x - e^x (x-t)] y(t) dt$
3. $y(x) = \int_0^x t(t-x) y(t) dt + \frac{1}{2} x^2$
4. $y(x) + \int_0^x [1 + x - 2t + (x-t) e^{-t}] y(t) dt = \frac{x^5}{20} - \frac{5x^3}{6} + x - 3$
5. $y(x) = \cos x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \frac{1}{2} \int_0^x (x-t)^2 y(t) dt$
6. $y(x) = \sin x - \frac{1}{2} + \int_0^x \left\{ \frac{1}{2} x(x-t)^2 - 1 \right\} y(t) dt$
7. $y(x) - \int_0^x [4 - 6(x-t) + 2(x-t)^2 - \frac{1}{6}(x-t)^3] y(t) dt = \frac{1}{4} \cos 2x - \frac{19}{12} + \frac{32}{3} x - \frac{85}{6} x^2 + \frac{20}{3} x^3$
8. $y''(x) - 2xy'(x) - 3y(x) = 0$; $y(0) = 1, y'(0) = 0$
9. $y''(x) - y(x) + 3 \sin x = 0$; $y(0) = 3, y'(0) = 0$
10. $y'''(x) + 6y(x) = 0$; $y(0) = 4, y'(0) = -3, y''(0) = 2$
11. $y'''(x) - 3y''(x) + 4y'(x) + 2y(x) + e^{-x} = 0$; $y(0) = 1, y'(0) = 2, y''(0) = 3$.

Problems 36.2, page 1137

1. $y(x) = \int_0^1 G(x, t) ty(t) dt + \frac{1}{2} x(x-1)$, where $G(x, t) = x(1-t)$, $(x < t)$ and $= t(1-x)$, $x > t$
2. $y(x) = \int_0^1 G(x, t) y(t) dt + \frac{1}{6} (x^3 - 3x + 6)$, where $G(x, t) = x$, $x < t$ and $= t$, $x > t$
3. $u(x) = \int_0^1 G(x, t) e^t u(t) dt + \frac{1}{6} (x^3 + x)$, where $G(x, t) = x(1-2t)$, $x < t$ and $= t(1-2x)$, $x > t$
4. $G(x, t) = \frac{\sinh x \sinh (t-1)}{\sinh 1}$, $x < t$ and $= \frac{\sinh t \sinh (x-1)}{\sinh 1}$, $x > t$
5. $u(x) = \lambda \int_0^1 G(x, t) t \cdot u(t) dt$, where $G(x, t) = \frac{1}{2} x \left(\frac{1}{t-t} \right)$, $x < t$ and $= \frac{1}{2} \cdot t \left(\frac{1}{x-x} \right)$, $x > t$

Problems 36.3, page 1141

5. $y(x) = \frac{1}{2} (\sin x + \sinh x)$ 6. $y(x) = x^2 + \frac{1}{12} x^4$ 7. $y(x) = 1$
8. $y(x) = \pm 6J_0(4x)$ 9. $y(x) = J_1(2x)$ ($x > 0$)

10. $y(x) = \frac{1}{2} e^{-2x} (\cos x + 3 \sin x) - \frac{1}{2} e^{-x}$

12. $y(x) = 1 + x^2 + x^4/24.$

15. $y(x) = \frac{3\sqrt{3}}{4\pi} x^{1/3} (3x + 2).$

11. $y(x) = (1-x)e^{-x} + 1/2 \sin x$

14. $y(x) = 1/2$

Problems 36.4, page 1145

1. Has no eigen values and eigen functions

2. Eigen value $\lambda = 1/4$; eigen function is $y(x) = x^2 + 3x/2$

3. Eigen values $\lambda = 8/(\pi - 2)$; eigen function is $y(x) = \sin^2(x)$

4. Eigen value $\lambda = 1/\pi$, $y = \sin x$

5. Has no eigen values or eigen functions

6. Eigen values are $\lambda = \pm 1/\pi$; eigen functions are $y(x) = \cos x + \sin x$, $y(x) = \cos x - \sin x$

7. $y(x) = x + \frac{\lambda [(6-\lambda)x - 4]}{12 + \lambda^2}$

8. $y(x) = x + \frac{\lambda}{12(1-2\lambda)-\lambda^2} [10 + (6+\lambda)x]$

9. $y(x) = x + \frac{\lambda}{(1-\lambda\pi)\left(1-\frac{1}{2}\lambda\pi\right)+4\lambda^2} \left\{ 2\lambda\pi + \frac{1}{2}\pi^2\left(1-\frac{1}{2}\lambda\pi\right) + \pi(1-2\lambda\pi)\sin x \right\}$

10. $y(x) = 2x - \pi + \frac{\pi^2 \sin^2 x}{\pi - 1}$

11. $y(x) = \frac{2}{2-\lambda} \sin x, \lambda \neq 2$

12. $y(x) = x + \frac{2\lambda\pi}{1+2\lambda^2\pi^2} (\lambda\pi x - 4\lambda\pi \sin x + \cos x)$

13. There is no solution to the integral equation when $\lambda = 3$

14. $\lambda_1 = 2, \lambda_2 = -2$; $y_1(x) = 1 - x, y_2(x) = 1 - 3x$

15. (i) When $F(x) = x$, solution is $y(x) = x + \lambda \left\{ \frac{2\lambda\pi^2}{\lambda^2\pi^2 - 1} \sin x + \frac{2\pi}{\lambda^2\pi^2 - 1} \cos x \right\}$

(ii) When $F(x) = 1$, solution is $y(x) = 1$.

Problems 36.5, page 1148

1. $y(x) = 1 - \frac{3\lambda x}{2(3+\lambda)} (\lambda \neq -3)$

2. $y(x) = \frac{\sin x}{1+\lambda\pi}$ only if $|\lambda| < \frac{1}{\pi}$

3. $y(x) = \frac{4 + 2\lambda(2-3x)}{4-\lambda^2} (\lambda \neq 2)$

4. $y = e^x$

5. $y(x) = 1$

6. $y = \sin x$

7. $y(x) = 2$.

Problems 37.1, page 1154

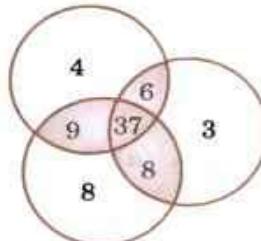
2. (i) True, since $\{a\}$ is a subset of the set $\{a, b, c\}$; (ii) and (iii) False, since the element a cannot be a subset of the set $\{a, b, c\}$; (iv) True, since the set $\{a, b\}$ is a subset of the set $\{a, b, c\}$; (v) False, since the set $\{a, b\}$ is not an element of the set $\{a, b, c\}$; (vi) True, since the null set \emptyset is a subset of every set.

17. 20

18. 105

19. 136

20. Number of students not taking any of these courses is 71.



Problems 37.2, page 1160

- (a) It is not true that Sam is a teacher and John is an honest boy ; (b) Sam is a teacher and John is not an honest boy ; (c) Sam is not a teacher iff John is an honest boy ; (d) If Sam is a teacher then John is not an honest boy.
- (a) $(p \vee q) \Rightarrow r$ where $p = I$ have no car, $q = I$ do not wear good dress, $r = I$ am not, a millionaire.

3.

p	q	$\neg q$	$p \Rightarrow q$	$p \Rightarrow q \wedge \neg q$
1	1	0	1	0
1	0	1	0	0
0	1	0	1	0
0	0	1	1	1

(b)

p	q	r	$p \Leftrightarrow q$	$r \vee q$	$(p \Leftrightarrow q) \wedge (r \vee q)$
1	1	1	1	1	1
1	1	0	1	1	0
1	0	1	0	1	0
1	0	0	0	0	0
0	1	1	0	1	0
0	1	0	0	1	0
0	0	1	1	1	1
0	0	0	1	0	0

7. (i) $T_p = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31]$

$T_q = [1, 3, 9, 27, \dots], T_r = [1, 3, 9, 7]$

(ii) $T_r \leq T_q$

10. (i)

p	q	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

15. (i) Dual of $(p \wedge q) \vee r$ is $(p \wedge q) \vee r$ (ii) Dual of $(p \wedge q) \vee t$ is $(p \vee q) \wedge t$

Problems 37.3, page 1166

- (a) $(\forall x \in A) (x + 2 < 10)$ (b) $(\exists x \in A) (x + 2 = 10)$
- (a) $\forall x, (x^3 \neq x)$ (b) $\neg \forall x, (x + 5 \leq x)$
- (c) None of the students are 26 or older (d) Some students do not live in the hostels.
- $\forall x P(x)$ is false
- $\forall (a, b) R(a + x = b)$
- (a) $\forall x [Q(x) \rightarrow R(x)]$ (b) $\neg \forall x [Q(x) \rightarrow R(x)]$, (c) $\exists x [Q(x) \wedge R(x)]$, (d) $\exists x [Q(x) \wedge \neg R(x)]$
- $(\neg A \vee \neg A) \wedge (B \vee \neg B) \wedge (\neg A \vee C) \wedge (B \vee C)$
10. 1. $p \vee q$ (Premise), 2. $\neg p \rightarrow q$ (conditional equivalence)
3. $q \rightarrow s$ (Premise) 4. $\neg p \rightarrow s$ (2, 3 chain rule)
5. $p \rightarrow r$ (Premise) 6. $\neg s \rightarrow p$ (4, conditional equivalence)
7. $\neg s \rightarrow r$ (5, 6 chain rule) 8. $s \vee r$ (7, conditional equivalence)
- (b) $R(x = \sqrt{Z})$
- (a) Conclusion is not valid (b) Conclusion is not valid

Problems 37.4, page 1170

1.

x	y	z	$x \wedge y$	z'	$y \wedge z'$	$(x \wedge y) \vee (y \wedge z')$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	0	1	1	1
1	0	0	0	1	0	0
1	1	0	1	1	1	1
1	0	1	0	0	0	0
0	1	1	0	0	0	0
1	1	1	1	0	0	1

2. $x \vee z' \wedge y = x \wedge y$

3. (i) $x' \vee y' \vee z'$ (ii) 0

14.

x_1	x_2	x_3	$x_1 \vee x_3$	x_3'	$x_2 \vee x_3'$	$x_1 \wedge (x_2 \vee x_3')$	P
0	0	1	1	0	0	0	0
0	1	0	0	1	1	0	0
0	1	0	0	1	1	0	0
1	0	0	1	1	1	1	1
1	1	0	1	1	1	1	1
0	1	1	1	0	1	0	0
1	0	1	1	0	0	0	0
1	1	1	1	0	1	1	1

Problems 37.5, page 1172

1. (i) 0 (ii) 0

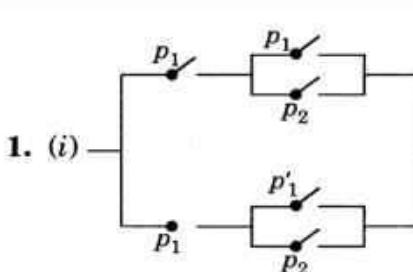
2. (i) $(x \vee y \vee z) \wedge (x \vee y \vee z')$ (ii) $x \vee y \wedge (x \vee y') \wedge (x' \vee y)$

4. $(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z')$

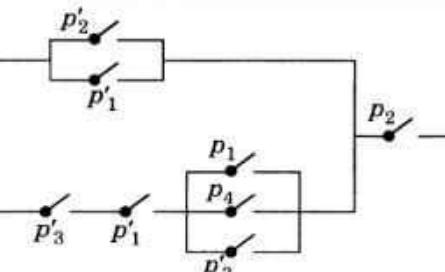
5. $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$

6. $F' = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$

Problems 37.6, page 1174



(ii)

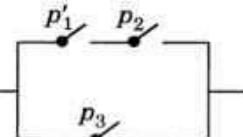


2. (i) $p_1 \vee [p_2' \wedge (p_1 \vee p_2')]$

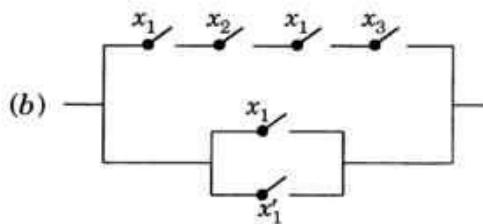
(ii) $[(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2')$

3. $x \wedge y$

4. $p_1' \wedge p_2 \vee p_3$;



6. (a) $x_1 \wedge x_2'$



(c) $(x_1 \vee x_2' \vee x_3) \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_3 \vee x_2' \vee x_1) \wedge (x_3 \vee x_2' \vee x_1')$

Problems 37.7, page 1179

- $F_2 = F_3$
- $F \cup G = [0.4 x_1, 0.7 x_2, 0.5 x_3, 0.9 x_4]$
 $F \cap G = [0.3 x_1, 0.6 x_2, 0.1 x_3, 0.8 x_4]$
- (i) Truth value of 'F is not rich' is 0.2
(ii) Truth value of 'G is not fat' is 0.4
(iii) Truth value of 'Mary is not beautiful' is 0.3
- (i) $F \neq G$
(ii) F is not a subset of G ; G is not a subset of F .
(iii) $F^c = [1, 1, 1, 1, 0.9, 0.7, 0.5, 0.1, 0, 0]$
 $F \cap G = [0, 0.1, 0.3, 0.5]$
 $F \cup G = [0.1, 0.5, 0.9, 1, 0.9, 0.9, 1, 1]$
- (i) Truth value of the conjunction of 'Latif and John are good players' is 0.6.
(ii) Truth value of the disjunction of 'Latif and John are good players' is 0.7.
- Members and its degree of membership.

Problems 38.1, page 1186

- (i) $\frac{d\phi}{dt} = \frac{d\phi}{dx^i} \cdot \frac{dx^i}{dt}$; (ii) $x^i x^i$
- (i) $a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})x^1 x^2 + (a_{13} + a_{31})x^1 x^3 + (a_{23} + a_{32})x^2 x^3$
(ii) $g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}dx^1 dx^2 + 2g_{23}dx^2 dx^3 + 2g_{31}dx^3 dx^1$.
(iii) $g_{11} = g_{12} g^{2p} + \dots + g_{ln} g^{np}$
- (i) δ_k^i ; (ii) δ_s^p .
- (i) $\bar{A}_p^{qr} = \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial x^i}{\partial \bar{x}^p} A_i^{jk}$; (ii) $\bar{C}_{pq} = \frac{\partial x^m}{\partial \bar{x}^p} \frac{\partial x^n}{\partial \bar{x}^q} C_{mn}$
- Yes, $A_{kl}^{ij} m$, contravariant order 3, covariant order 2, Rank 5
- (a) $2\rho \cos^2 \phi - z \cos \phi + \rho^3 \sin^2 \phi \cos^2 \phi, -\rho^2 \sin 2\phi + \rho z \sin \phi + \rho^4 \sin \phi \cos^3 \phi, \rho z \sin \phi$;
(b) $2r \sin^2 \theta \cos^2 \phi - r \sin \theta \cos \theta \cos \phi + r^3 \sin^4 \theta \sin^2 \phi \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \sin \phi$;
 $2r^2 \sin \theta \cos \theta \cos^2 \phi - r^2 \cos^2 \theta \cos \phi + r^4 \sin^3 \theta \cos \theta \sin^2 \phi \cos^2 \phi - r^3 \sin^2 \theta \cos \theta \sin \phi$;
 $-2r^2 \sin^2 \theta \sin \phi \cos \phi + r^2 \sin \theta \cos \theta \sin \phi + r^4 \sin^4 \theta \sin \phi \cos^3 \phi$
- (a $\cos \phi + b \sin \phi$) $\sin \theta + c \cos \theta$; $((a \cos \phi + b \sin \phi) \cos \theta - c \sin \theta)/r$; $(b \cos \phi - a \sin \phi)/r \sin \theta$.

Problems 38.2, page 1189

6. Rank = 1

Problems 36.3, page 1193

- $g = 4, g^{11} = 2, g^{22} = 5, g^{33} = 1.5, g^{12} = 3, g^{23} = -2.5, g^{13} = -1.5$,
- $g_{11} = 1, g_{22} = \rho^2, g_{33} = 1, g_{ij} = 0 (i \neq j)$; $g^{11} = 1, g^{22} = \rho^{-2}, g^{33} = 1, g^{ij} = 0 (i \neq j)$
- $g = r^4 \sin^2 \theta / (1 - r^2/R^2)$; $g^{11} = 1 - r^2/R^2, g^{22} = 1/r^2, g^{33} = (r \sin \theta)^{-2}, g^{ij} = 0 (i \neq j)$.

Problems 38.4, page 1199

1. (a) $[ii, i] = \frac{1}{2} \partial g^{ij} / \partial x^i$, $[ii, k] = -\frac{1}{2} \partial g_{ik} / \partial x^k$,

$[ik, k] = [ki, k] = \frac{1}{2} \partial g_{kk} / \partial x^i$, $[ij, k] = 0$, when i, j, k are all different

(b) $\begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{1}{2} g^{ij} \frac{\partial g_{ii}}{\partial x^i}$, $\begin{Bmatrix} k \\ ii \end{Bmatrix} = -\frac{1}{2} g^{kk} \frac{\partial g_{ii}}{\partial x^k}$

$\begin{Bmatrix} k \\ ik \end{Bmatrix} = \begin{Bmatrix} k \\ ki \end{Bmatrix} = \frac{1}{2} g^{kk} \frac{\partial g_{kk}}{\partial x^i}$ (no summation over i or k)

$\begin{Bmatrix} k \\ ij \end{Bmatrix} = 0$, when i, j, k are all different

2. (a) All are zero

(b) $[21, 2] = \rho = [12, 2]$; $[22, 1] = \rho$, all others are zero

(c) $[21, 2] = r = [12, 2]$; $[31, 3] = r \sin^2 \theta = [13, 3]$; $[32, 3]$

$= r^2 \sin \theta \cos \phi = [23, 3]$; $[22, 1] = -r$; $[33, 1] = -r \sin^2 \theta$;

$[33, 2] = -r^2 \sin \theta \cos \phi$; all others are zero

3. (a) All are zero

(b) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\rho$, $\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{\rho}$, all others are zero

(c) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r$, $\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -r \sin^2 \theta$, $\begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\sin \theta \cos \theta$, $\begin{Bmatrix} 3 \\ 31 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \cot \theta$, all others are zero

5. (a) $r^2 \sin \theta \cos \theta$; $r^2 \sin \theta \cos \theta$

(b) $-\sin \theta \cos \theta$; $\cot \theta$

6. (a) $-r \sin^2 \theta$; $r^2 \sin \theta \cos \theta$

(b) $-r \sin^2 \theta$; $\cot \theta$

8. (a) $u^{ij}_{,k} = \frac{\partial u^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{sj} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{is}$;

(b) $A^h_{ij,k} = \frac{\partial A^h_{ij}}{\partial x^k} - \begin{Bmatrix} s \\ ik \end{Bmatrix} A^h_{sj} + \begin{Bmatrix} h \\ ks \end{Bmatrix} A^s_{ij}$

10. $A^j_{k,q} B_n^{lm} + A^j_{k,l} B_{n,q}^{lm}$

11. (a) $\frac{1}{\rho} \left[\frac{\partial}{\partial p} (\rho A_p) + \frac{\partial}{\partial \phi} (A_\phi) + \frac{\partial}{\partial z} (\rho A_z) \right]$; (b) $\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial r} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$

12. (a) $\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 v}{\partial \phi^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} = 0$ (b) $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v}{\partial \phi^2} + \frac{2}{r} \frac{\partial v}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial v}{\partial \theta} = 0$.

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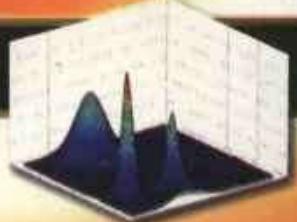
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Higher Engineering Mathematics

Dr. B.S. Grewal

The book provides a clear exposition of essential tools of applied mathematics from a modern point of view and meets complete requirements of engineering and computer science students. Every effort has been made to keep the presentation at once simple and lucid. It is written with the firm conviction that a good book is one that can be read with minimum guidance from the instructor. To achieve this, more than the usual number of solved examples, followed by properly graded problems have been given. Many of the examples and problems have been selected from recent papers of various university and other engineering examinations. Basic Concepts and Useful Information has been given in an Appendix. However, the subject matter has been set in eight main units:

- **Algebra & Geometry** : Solution of equations, Linear algebra: Determinants, Matrices, Vector algebra and Solid geometry.
- **Calculus** : Differential calculus, Partial differentiation, Integral calculus, Multiple integrals, Vector calculus.
- **Series** : Infinite series and Fourier series.
- **Differential Equations** : Differential equations of first order and their applications, Linear differential equations and their applications, Differential equations of different types, Series solution of differential equations and special functions, Partial differential equations and their applications.
- **Complex Analysis** : Complex numbers and functions, Calculus of complex functions.
- **Transforms** : Laplace transforms, Fourier transforms and Z-transforms.
- **Numerical Techniques** : Empirical Laws and Curve fitting, Statistical methods, Probability and Distributions, Sampling and Inference, Numerical methods, Finite differences and Interpolation, Difference equations, Numerical solution of Ordinary and Partial differential equations, Linear programming.
- **Special Topics** : Calculus of variations, Integral equations, Discrete mathematics, Tensors.

An exhaustive list of '*Objective Type of Questions*' has been given at the end of each chapter. *Standard Tables, Answers to Problems*, and a fairly comprehensive *Index* is given at the end.



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