UNIT - 4

(Permutations and Combinations)

Topics.

1. Permutations < with out Repetitions / with Repetitions.
2. Combinations.
3. Applications.

Note(1) Combinatorics is a branch of mathematics that deals with counting, arrangement and selection of objects. ⇒ Algorithm Analysis, Graph Theory, Cryptography, Data structure - BST, optimization, etc.

(A) (Permutations without Repetitions)

Permutations :- The arrangements of objects by taking some or all out of n. different objects (of a set), are called permutations.

(i) The number of all permutations of n-different objects taking all at a time are $n!$

(ii) The number of all permutations of n-different objects taking some ie $(1 \leq r \leq n)$ or r-objects at a time, are ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

Ex Consider digits 1, 2 & 3. The arrangements are

123 , 231 , 312
132 , 213 , 321

Here we see number change when order is changed. Total number of permutations are 6.

(B) Permutations with Repetitions

- If we have n-objects (Not all different) but p objects are of Ist kind, q-objects are second type & r-objects alike Third and all remaining objects are different. then number of Permutations of such n-objects taking all at a time is

$$= \frac{n!}{p! \times q! \times r!}$$

Ex How many ways are there to arrange the nine letters in word ALLAHABAD.

Sol:- $n = 9$ & 4 A's, 2 L's.

Total number of arrangements $= \frac{9!}{4! \times 2!}$

$= 7,560$ ways.

Note (1)

$$^nP_1 = \frac{n!}{(n-1)!} = n$$

$$^nP_2 = \frac{n!}{(n-2)!} = n.(n-1)$$

(increasing)

$$^nP_3 = \frac{n!}{(n-3)!} = n.(n-1)(n-2).$$

$\vdots$

$$^nP_n = n!$$

Factorial function :-

Let n be a positive Integer. Then, the continued product of first n natural number is called factorial n & denoted by $n!$ or $\lfloor n$.

Therefore ; $$n! = n.(n-1).(n-2) \ldots 3.2.1$$

Note :- ① $\lfloor n = n . \lfloor n-1$

② $0! = 1 = 1!$

Ex Compute

(i) $\dfrac{8!}{4! \times 3!}$ (ii) $\dfrac{30!}{28!}$ (iii) LCM. $(4!, 5!, 6$

(iv) $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!}$

Ex. Convert into factorials

(i) $6.7.8.9 = \dfrac{9!}{5!}$

(ii) $2.4.6.8.12 = 2^5 . 5!$

(iii) $1.3.5.7.9.11 = \dfrac{11!}{2^5 . 5!}$ etc.

Note: $^4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$ ABCD

ABCD	BCDA	CDAB	DABC
ABDC	BCAD	CDBA	DACB
ACBD	~~CADB~~	CABD	DBAC
ACDB	~~CABD~~	CADB	DBCA
ADBC	BDAC	CBAD	DCAB
ADCB	BDCA	CBDA	DCBA
	BACD		
	BADC		

<u>Find</u> x or n. If

(i) $\frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = \frac{x}{7!}$ $x = 259$

(ii) $(n+1)! = 12 \times (n-1)!$ $n = 3$

(iii) $\frac{1}{9!} + \frac{1}{10!} = \frac{n}{11!}$ $n = 121$

(iv) $(n+2)! = 1{,}560 \cdot \underline{\lfloor n}$ $n = 38, -41.$

Problems based on $({}^nP_r)$

<u>Ex9</u> (i) Evaluate ${}^{12}P_4$ 11880

(ii) ${}^6P_4$ 360

(iii) ${}^8P_8$ $8! = 40{,}320$

<u>Ex10</u> If ${}^nP_4 = 20 \times {}^nP_2$ find n.

$$\frac{n!}{(n-4)!} = 20 \times \frac{n!}{(n-2)!}$$

$$n(n-1)(n-2)(n-3) = 20 \times n(n-1)$$

$$(n-2)(n-3) = 20$$

$$n^2 - 5n + 6 = 20$$

$$n^2 - 5n - 14 = 0 \quad \Rightarrow \quad n = 7, -2 \quad \text{ANS}$$

<u>Ex11</u> find r, If ${}^{20}P_r = 6{,}840$.

$$\frac{20!}{(20-r)!} = 6840 = 20 \times 19 \times 18$$

$$\frac{20!}{(20-r)!} = \frac{20!}{17!} \Rightarrow r = 3 \quad \text{ANS}$$

Ex $\quad {}^{n+5}P_{n+1} = \dfrac{11 \cdot (n-1)}{2} \times {}^{n+3}P_n$, find n.

$$\frac{(n+5)!}{((n+5)-(n+1))!} = \frac{11 \cdot (n-1)}{2} \times \frac{(n+3)!}{(n+3-n)!}$$

$$\frac{(n+5)(n+4)\cancel{(n+3)!}}{4!} = \frac{11 \cdot (n-1)}{2} \times \frac{\cancel{(n+3)!}}{3!}$$

$$\frac{(n+5)\cdot(n+4)}{4} = \frac{11\,(n-1)}{2}$$

$$n^2+9n+20 = 22\,(n-1)$$

$$n^2+9n+20 = 22n-22$$

$$n^2-13n+42=0 \Rightarrow \qquad n=6,7 \qquad \text{ANS}$$

Ex 14 $\quad$ If ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_n$ find n.

$$\frac{9!}{4!} + 5 \cdot \frac{9!}{5!} = \frac{10!}{(10-n)!}$$

$$2 \cdot \frac{9!}{4!} = \frac{10 \cdot 9!}{(10-n)!}$$

$$\frac{1}{4!} = \frac{5}{(10-n)!}$$

$$(10-n)! = 5 \times 4! \qquad \Rightarrow n=5$$

Permutation $\qquad$ n. object $\{$Set$\}$

All . different $\qquad$ Not . All different (Some Repeat or identical)

Taking r no $\qquad$ Taking all no $\qquad$ $n!$

(5)

Ex 22 Each of 2 women & 3 men can occupy one chair
230 out of 8 chairs, numbered 1 to 8.
The women can occupy any two chairs 1 to 4, then 3 men can occupy 3 chair from Rest of 6.
How many ways it can be done.

Sol:-

1 2 3 4 5 6 7 8

women

$$^4P_2 \times {^6P_3} = 12 \times 120 = 1440. \quad \text{ANS}$$

Ex 24
231 ~~A Person has four notes of rupees 1, 2, 5 & 10. denomination. The numbers of different sum he can form?~~

Ex 25
231. In How many ways four students can stand in a line such ~~that~~ two of them are always togather.

Sol:-

1	2
A	B
B	A

3 4

$$= {^2P_2} \times {^3P_3} = 2! \times 3! = 12. \quad \text{ANS}$$

Ex 31
234 How many four digit number can be formed by using digits 1 to 5. Also out of these How many will be even?

Sol:- $^5P_4 = 120$ (All ~~dig~~ numbers).

for Even, last place will be fixed for 2 or 4.

$$^2P_1 \times {^4P_3} = 2 \times 24 = 48.$$

			2 or 4
			$^2P_1$

Ex 29
233 The letters of word TUESDAY are arranged in a line that each arrangement ends with letters. Also How many start with ~~the~~ letter. T.

						S

$$^6P_6 = 720.$$

T						S

$$^5P_5 = 120 \text{ ways.} \quad \text{ANS}$$

Combinations :-

Each of different group which can be formed by taking some or all objects (irrespective of order of arrangement) is called called Combinations

The total numbers of combinations of n-distinct object by taking r at a time is denoted by ^nC_r (where $1 < r \leq n$).

Also, $^nC_r = \frac{n!}{r! \cdot (n-r)!}$

Note ① $^nC_0 = {^nC_n}$

② $^nC_r = {^nC_{n-r}}$

Ex $^nP_r = 720$ & $^nC_r = 120$ find r ; $r = 3$

Ex $^{107}C_{87} = {^{107}C_x}$ find x. (since $x = n - r$) $x = 20$

Ex $x \cdot {^{77}C_{55}} = 23 \cdot {^{78}C_{55}}$ find x.

Ex 71 / 253 How many ways a committee of 5 members can be selected from 6 men & 5 women, consisting 3 m & 2 ladies? $({^6C_3} \times {^5C_2} = 200)$

Ex 72 / 253 A committee of 5 is to be formed out of 6 men & 4 ladies. In how many ways it can be done. when it consist.

(1) atleast 2 ladies included ⇒

2 ladies	3 men	$\rightarrow {^4C_2} \times {^6C_3}$	$= 120$
3 ladies	2 men	$\rightarrow {^4C_3} \times {^6C_2}$	$= 60$
4 ladies	1 men	$\rightarrow {^4C_4} \times {^6C_1}$	$= 20$
			Total $= 200$

(2) atmost 2 ladies include ⇒

1 ladies 4 men $\rightarrow {^4C_1} \times {^6C_4} = 60$

Total = 10.

Repeatable / Non-Repeatable [R] ^{NR}.

If there are n- different objects and we need to select r- at timed. Repeatition allowed then number of ways it can be done are

$$(n)^r$$

Ex. How many different PIN 4-digit can be generated by number 0 to 9.

$$\underset{10}{\overset{1}{\square}}\ \underset{10}{\overset{2}{\square}}\ \underset{10}{\overset{3}{\square}}\ \underset{10}{\overset{4}{\square}} = 10^4$$

Ex. 23 / 230 A question paper had 10-questions. Each question has two choice true or false. How many different ways are there to solved QP.

Sol:- $[R]^{NR} = [2]^{10} = 1024.$

Ex 53 / 248 A man has 6 friends to invite. In How many ways he can send invitation to them if he has 3 servents to carry cards?

Sol:- $3^6 = 729.$ Ans

Permutations (with Repeatition) $\frac{n!}{r!\,s!\,t!}$

Ex 26
232

How many ways are there to arrange nine letter of word ALLAHABAD.

Sol:-
$$\frac{9!}{4!\cdot 2!} = 7{,}560.$$

Ex 27
233

How many different messages can be formed by sequences of four dashes and three dots.

$$\frac{7!}{4!\,3!} = 35.$$

Ex. 28
233

Find how many ways to point 12 officer. so that 3 of them are green, 2 are pink, 2 are yellow & rest are white.

$$\frac{12!}{3!\cdot 2!\cdot 2!\cdot 5!} = 1{,}66{,}320.$$

Ex 30
233

How many signals can be made by using 6 flags of different colours when any numbers of them can be hoisted at a time?

Sol:-

One-at time	$6P_1$	$= 6$
Two at time	$6P_2$	$= 30$
THREE ———	$6P_3$	$= 120$
4 ———	$6P_4$	$= 360$
5 ———	$6P_5$	$= 720$
6 ———	$6P_6$	$= 720$
	Total	$= 1956$
