Random variables and Distribution function

* <u>Random variable</u> ⇒ A random variable is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the events $\{\omega : X(\omega) \le a\} \in B$ where S = Sample space & B = The σ-field of subsets in S.

Ex 1. If a coin is tossed, then

$$S = \{\omega_1, \omega_2\} \quad \text{where} \quad \omega_1 = H, \ \omega_2 = T$$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = H \\ 0 & \text{if } \omega = T \end{cases}$$

$X(\omega)$ is random variable.

2. If a pair of fair dice is tossed then

$$S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$

$$n(S) = 36$$

Let X be a random variable with image set $X(S) = \{1,2,3,4,5,6\}$

$P(X=1) = P\{1,1\} = \frac{1}{36}$

$P(X=2) = P\{(2,1),(2,2),(1,2)\} = \frac{3}{36}$

$P(X=3)$ - - - -

* <u>Distribution function</u> ⇒ Let X be a random variable. The function F defined for all real x by

$$F(x) = P(X \le x) = P\{\omega : X(\omega) \le x\}, \quad -\infty < x < \infty$$

is called the distribution function of the random variable X.

properties of Distribution function

1. If F is the d.f of the r.v X and if $a < b$ then $P(a < x \le b) = F(b) - F(a)$

proof $\Rightarrow$ The events '$a < x \le b$' and '$x \le a$' are disjoint and their union is the event '$x \le b$' then by addition theorem of probability

$$P(a < x \le b) + P(x \le a) = P(x \le b)$$

$$\Rightarrow P(a < x \le b) = P(x \le b) - P(x \le a)$$

$$P(a < x \le b) = F(b) - F(a)$$

2. If F is d.f of one-dimensional r.v X then

i) $0 \le F(x) \le 1$

ii) $F(x) \le F(y)$ if $x < y$

[$\because$ A real valued function defined on S and taking values in $R(-\infty, \infty)$ is called one-dimensional r.v]

i.e, all distribution functions are monotonically non-decreasing and lie b/w 0 and 1.

3. If F is d.f of one-dimensional r.v X then

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad F(\infty) = \lim_{x \to \infty} F(x) = 1$$

proof $\Rightarrow$ Let us express the whole space S as a countable union of disjoint event as follows:

$$S = \left\{ \bigcup_{n=1}^{\infty} (-n < x \le -n+1) \right\} \cup \left\{ \bigcup_{n=0}^{\infty} (n < x \le n+1) \right\}$$

$$\Rightarrow P(S) = \sum_{n=1}^{\infty} P(-n < X \le -n+1) + \sum_{n=0}^{\infty} P(n < X \le n+1)$$

$$1 = \lim_{a \to \infty} \sum_{n=1}^{a} \{F(-n+1) - F(-n)\} + \lim_{b \to \infty} \sum_{n=0}^{b} \{F(n+1) - F(n)\}$$

$$= \lim_{a\to\infty} \{F(0) - F(-a)\} + \lim_{b\to\infty} \{F(b+1) - F(0)\}$$

$$= \{F(0) - F(-\infty)\} + \{F(\infty) - F(0)\}$$

$$1 = F(\infty) - F(-\infty) \quad \text{———} \quad (1)$$

Since $-\infty < \infty$, $F(-\infty) \le F(\infty)$ Also $F(-\infty) \geq 0$ and $F(\infty) \le 1$

$$0 \le F(-\infty) \le F(\infty) \le 1 \quad \text{———} \quad (2)$$

from (1) & (2) we get $F(-\infty) = 0$ and $F(\infty) = 1$

* Discrete Random variable ⇒ A real valued function defined on a discrete sample space is called a discrete random variable.

Ex ⇒ Marks obtained in a test, number of accidents per month no. of telephone calls per unit time.

* probability Mass function ⇒ If X is a discrete random variable with distinct values $x_1, x_2, \ldots, x_n$ then the function $p(x)$ defined as:

$$P_X(x) = \begin{cases} P(X = x_i) = P_i & \text{if } x = x_i \\ 0 & \text{if } x \neq x_i \; ; \; i = 1, 2 \ldots \end{cases}$$

is called the probability mass function of r.v X

Ex ⇒ A random variable X has the following probability function:

Value of X, x :	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) find k , ii) Evaluate $P(X<6)$, $P(X \geq 6)$ and $P(0<X<5)$

Solⁿ Since $\sum_{x=0}^{7} P(x) = 1$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$10k^2+9k-1=0 \Rightarrow (10k-1)(k+1)=0 \Rightarrow k=\frac{1}{10}$ or $k=-1$

But $p(x)$ can't be negative so $k=-1$, is rejected

Hence $k=\frac{1}{10}$

(ii) $P(X<6) = P(X=0)+P(X=1)+ \ldots + P(X=5)$

$= \frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100} = \frac{81}{100}$

$P(X\geq 6) = 1-P(X<6) = 1-\frac{81}{100} = \frac{19}{100}$

$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$8k = 8\times\frac{1}{10} = \frac{4}{5}$

Ex $\Rightarrow$ If $p(x) = \begin{cases} \frac{x}{15} ; & x=1,2,3,4,5 \\ 0 ; & \text{elsewhere} \end{cases}$

Find is $P\{X=1 \text{ or } 2\}$

$P(X=1 \text{ or } 2) = P(X=1)+P(X=2) = \frac{1}{15}+\frac{2}{15} = \frac{1}{5}$

* Continous Random variable $\Rightarrow$ A random variable X is said to be cts if it can take all possible values (integral as well as fractional)

Ex $\Rightarrow$ Age, height, weight etc.

Probability density function $\Rightarrow$ Consider the small interval $(x, x+dx)$ of length dx round the point x. Let $f(x)$ be any cts function of x so that $f(x)\,dx$ represents the probability that X falls in the infinitesimal interval $(x, x+dx)$. Symbolically,

$$P(x \leq X \leq x+dx) = f_X(x)\,dx$$