$$= E\left[e^{t x/h} e^{-at/h}\right]..$$

## Two dimensional Random Variable

Let X and Y be two random variables defined on the same sample space S, then the function (X,Y) that assigns a point in $R^2 = (R \times R)$ is called two-dimensional random variable.

### * joint probability Mass function ⇒

If (X,Y) is a two dimensional discrete random variable then the joint discrete function of X, Y also called the joint probability Mass function of X,Y denoted by $P_{XY}$ is defined as

$$P_{XY}(x_i, y_i) = \begin{cases} P(X = x_i, Y = y_i) & \text{for a value } (x_i, y_i) \text{ of } (X,Y) \\ 0 & \text{otherwise.} \end{cases}$$

### * Conditional probability function ⇒

Let (X,Y) be a discrete two-dimensional random variable, then the Conditional discrete density or conditional probability mass funⁿ of X given Y = y denoted by

$$f_{X|Y}(x|y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \text{ provided } P(Y = y) \neq 0$$

The Condition p.m.f of Y when X = x is given

$$f_{Y|X}(y|x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

Ex:-) for the joint probability distribution of two random variables X and Y given below.

| X \ Y | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| 1 | 4/36 | 3/36 | 2/36 | 1/36 | 10/36 |
| 2 | 1/36 | 3/36 | 3/36 | 2/36 | 9/36 |
| 3 | 5/36 | 1/36 | 1/36 | 1/36 | 8/36 |
| 4 | 1/36 | 2/36 | 1/36 | 5/36 | 9/36 |
| Total | 11/36 | 9/36 | 7/36 | 9/36 | $\boxed{1}$ |

find i) The Marginal distribution of X & Y

ii) Conditional distribution of X given the value Y=1

i) Marginal $\sum_y P(X=1, Y=y)$

$P(X=1) = P(X=1, \cancel{Y}=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$

$P(X=3) = \frac{8}{36}$, $P(X=4) = \frac{9}{36}$

Marginal Distribution of X

| Value of X, x | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P(X=x) | $\frac{10}{36}$ | $\frac{9}{36}$ | $\frac{8}{36}$ | $\frac{9}{36}$ |

nal distribution of Y

| ..lue of Y | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(Y=y)$ | 11/36 | 9/36 | 7/36 | 9/36 |

ii) Condition probability function

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)}$$

$$= \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = 5/11$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = 1/11$$

Hence Conditional distribution of X given Y=1 is

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x, Y=1)$ | 4/11 | 1/11 | 5/11 | 1/11 |

(*) Two dimensional distribution function ⇒

The distribution function of two-dimensional random variable $(X,Y)$ is a real valued function $F$ defined for all real $x$ & $y$ by relation

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

(*) Marginal probability function of $X$ and $Y$

$$f_X(x) = \begin{cases} \sum_y P_{XY}(x,y) & (\text{for discrete variable}) \\ \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy & (\text{for continous variable}) \end{cases}$$

$$f_Y(y) = \begin{cases} \sum_x P_{XY}(x,y) & (\text{for discrete variable}) \\ \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx & (\text{for continous variable}) \end{cases}$$

(*) Conditional density function ⇒

The Conditional density function of $Y$ when $X$ is given

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

The Conditional density function of $X$ when $Y$ is given

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

The joint pdf of a two dimensional random variable $(X,Y)$ is given by

$$f(x,y) = \begin{cases} 2 ; & 0<x<1 , 0<y<x \\ 0 ; & \text{elsewhere} \end{cases}$$

i) find Marginal density function of X and Y

ii) find the Conditional density function of Y given $X=x$ and conditional density function of X given $Y=y$

Sol$^n$ Marginal density function.

i) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^x 2\,dy = 2x$

$f_Y(y) = \int_y^1 2\,dx = 2(1-y)$

Conditional density function.

ii) $f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{2}{2x} = \dfrac{1}{x}$

$f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{1-y}$

(*) Expectation ⇒

The average value of a random phenomenon is termed as the Mathematical expectation or expected value.

* Expected value of a Random variable ⇒

The average value of discrete random variable X with pmf is given by

$$E(X) = \sum_x x f(x)$$

The average value of continous random variable X with pdf is given by

$$\cancel{E(y) = \sum_y y f(y)} \quad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Ex ⇒ Let X be a r.v with probability distribution.

| | | | |
|---|---|---|---|
| $x$ : | $-3$ | $6$ | $9$ |
| $P(X=x)$ : | $1/6$ | $1/2$ | $1/3$ |

find $E(X)$ & $E(X^2)$

$$E(X) = \sum x\,p(x) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$$

$$E(X^2) = \sum x^2 p(x) = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

Ex = find the expectation of the no. ~~of~~ a on a die when thrown.

Let X be the random variable representing the no. on a die when thrown. then X can take one of the value 1, 2, 3, ... 6 each with equal probability $\frac{1}{6}$

| $x$ | $P(x)$ |
|---|---|
| 1 | $\frac{1}{6}$ |
| 2 | $\frac{1}{6}$ |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

$$E(x) = \sum x p(x)$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$\Rightarrow \frac{1}{6}(1+2+3+4+5+6) = \frac{\cancel{21}\,7}{\cancel{6}\,2} = \frac{7}{2}$$

(4) properties of Expectation.

<u>1 property</u> $\Rightarrow$ Addition theorem of expectations.

<u>State</u> $\Rightarrow$ If X and Y are random variables, then

$E(X+Y) = E(X) + E(Y)$, provided all the expectations exist.

<u>proof</u> $\Rightarrow$ By defn

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx, \quad E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_{XY}(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\,dx\,dy$$

$$\Rightarrow \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\right]dx + \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\right]dy$$

Property - 5 Expectation

$$\int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$E(X+Y) = E(X) + E(Y)$

<u>Generalisation</u> ⇒ $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$

2) <u>property</u> Multiplication theorem of Expectation.

<u>Statement</u> ⇒ If X and Y are independent random variables then $E(XY) = E(X) \cdot E(Y)$

<u>Proof</u> ⇒ $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{XY}(x,y)\,dx\,dy$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_X(x)\, f_Y(y)\,dx\,dy$$

[ ∵ X & Y are independent so $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ ]

$$\Rightarrow \left[\int_{-\infty}^{\infty} x f_X(x)\,dx\right]\left[\int_{-\infty}^{\infty} y f_Y(y)\,dy\right]$$

$E(XY) \Rightarrow E(X)\,E(Y)$

<u>Generalisation</u> ⇒ $E(X_1, X_2 \dots X_n) = E(X_1)E(X_2) \dots E(X_n)$

<u>Property</u> (3) If X is a random variable and 'a' is constant, then

i) $E[a\psi(x)] = a\,E(\psi(x))$

ii) $E[\psi(x) + a] = E[\psi(x)] + a$

$$E[a\psi(x)] = \int_{-\infty}^{\infty} a\psi(x) f(x)\,dx$$

$$\Rightarrow a \int_{-\infty}^{\infty} \psi(x) f(x)\,dx = a\,E(\psi(x))$$

$$E[a\psi(x)] = a\,E(\psi(x))$$

(ii) $$E[\psi(x)+a] = \int_{-\infty}^{\infty} [\psi(x)+a] f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \psi(x) f(x)\,dx + \int_{-\infty}^{\infty} a\, f(x)\,dx$$

$$\Rightarrow E[\psi(x)] + a\int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow E[\psi(x)] + a$$

$$\Rightarrow E[\psi(x)+a] = E[\psi(x)] + a$$

<u>Property</u>-4 If $X$ is a random variable and $a$ & $b$ are constants then

$$E(ax+b) = aE(x)+b$$

<u>Proof</u> $$E(ax+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} ax f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx$$

$$\Rightarrow a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow aE(x)+b \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

property - 5 Expectation of a linear combination of Random variables

Let $x_1, x_2, \ldots, x_n$ be any $n$ random variables and if $a_1, a_2, \ldots, a_n$ are any $n$ constants, then

$$E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$$

provided all the expectation exist.

property - 6 If $X \geq 0$ then $E(X) \geq 0$

property - 7 If $X$ and $Y$ are two random variables such that $Y \leq X$, then $E(Y) \leq E(X)$, provided all the expectations exist.

property - 8 $\Rightarrow$ $|E(X)| \leq E|X|$, provided all the expectation exist.

## ...ent generating function :

The moment generating function (m.g.f) of a random variable X having the probability function $f(x)$ is given by

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum e^{tx} f(x) & [\text{for discrete}] \\ \int e^{tx} f(x)\,dx & (\text{for continous}] \end{cases}$$

### Properties of Moment generating function

1) $M_{cX}(t) = M_X(ct)$, c being a constant

Proof By def$^n$ $M_{cX}(t) = E(e^{tcX})$

$$M_X(ct) = E(e^{ctX})$$

from this we get $M_{cX}(t) = M_X(ct)$

2) The Moment generating function of the Sum of a number of independent random Variable is equal to the product of their respective mgf.

If $X_1, X_2, \ldots, X_n$ are independent random variables, then the M.g.f of their sum $X_1 + X_2 + \ldots + X_n$ is given by

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$$

Proof By def$^n$

$$M_{X_1+X_2+\ldots+X_n}(t) = E\left[e^{t(X_1+X_2+\ldots+X_n)}\right]$$

$$= E\left[e^{tX_1} . e^{tX_2} \ldots e^{tX_n}\right]$$

$$\Rightarrow E[e^{tX_1}] . E[e^{tX_2}] \ldots E[e^{tX_n}]$$

$\Rightarrow M_{x_1}(t) M_{x_2}(t) \ldots M_{x_n}(t)$ are independent

**property** → Effect of change of origin and scale.

Let us transform X to new variable U by changing both the origin and scale in X as follows:

$$U = \frac{X-a}{h}, \text{ where } a \;\&\; h \text{ are constants.}$$

$$M_U(t) = E(e^{tU}) = E\left[e^{t\left(\frac{x-a}{h}\right)}\right] = E\left[e^{tx/h} e^{-at/h}\right]$$

$$\Rightarrow e^{-at/h} E\left[e^{tx/h}\right]$$

$$\Rightarrow e^{-at/h} M_X\left(\frac{t}{h}\right)$$

**Ex** Let the r.v X assume the value 'r' with the probability law:

$$P(X=r) = q^{r-1} p \; ; \; r = 1, 2, \ldots$$

find M.g.f of X and hence its mean & Variance.

**Sol^n**
$$M_X(t) = E(e^{tX}) = \sum_{r=1}^{\infty} e^{tr} q^{r-1} p = \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$\Rightarrow \frac{p}{q} qe^t \sum_{r=1}^{\infty} (qe^t)^{r-1}$$

$$\Rightarrow pe^t \left[1 + qe^t + (qe^t)^2 + \ldots\right]$$

$$M_X(t) \Rightarrow \frac{pe^t}{1-qe^t}$$

$$M_X'(t) = \frac{pe^t}{(1-qe^t)^2} \qquad M_X''(t) = pe^t \frac{(1+qe^t)}{(1-qe^t)^3}$$

$$\mu_1' = M_X'(0) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$\mu_2' = M_x''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}$

$\therefore \mu_1' = \frac{1}{p}$

Variance $\mu_2 = \mu_2' - \mu_1'^2$

$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$

$$= P(a_1 < X \le b_1)\, P(a_2 < Y \le b_2), \text{ as desired.}$$

$\Rightarrow$ The events : $(a_1 < X \le b_1)$ and $(a_2 < Y \le b_2)$ are indepndent.

**Remark.** In case of discrete r.v.'s theorems 5·2 and 5·3 can be proved on replacing integration by summation over the given range of the variables.

**5·5·7. Generalisation to *n*-Dimensional Random Variable.** The concept of two-dimensional random variables and their joint and marginal distributions is § 5·5 to § 5·5·6 can be easily generalised to the case of $n$-dimensional random variable.

**Joint and Marginal Probability Mass Function.**

Let $(X_1, X_2, \dots, X_n)$ be a discrete $n$-dimensional *r.v.*, assuming discrete values, in some region, say, $R^n$ of the $n$-dimensional space. Then the joint *p.m.f.* of $(X_1, X_2, \dots, X_n)$ is defined as :

$$p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P\,[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]$$
$$= P\left[\bigcap_{i=1}^{n} (X_i = x_i)\right] \qquad \dots(5·21)$$

where,

(i) $p(x_1, x_2, \dots, x_n) \ \ge\ 0,\ \forall\, (x_1, x_2, \dots, x_n) \in R^n$, and

(ii) $\sum\limits_{x_1, x_2, \dots, x_n} p(x_1, x_2, \dots, x_n) = 1$

The marginal *p.m.f.* of any *r.v.*, say, $X_i$, is obtained on summing $p(x_1, x_2, \dots, x_n)$, over the values of all other variables except $X_i$. Thus,

$$P_{X_i}(x_i) = \sum_{\left(\substack{x_1, x_2, \dots, x_n \\ \text{except } x_i}\right)} p(x_1, x_2, \dots, x_n) \qquad \dots(5·21\,a)$$

In particular, if $p(x_1, x_2, x_3)$ is the joint *p.m.f.* of three *r.v.'s* $X_1$, $X_2$ and $X_3$, then the marginal *p.m.f.* of, say, $X_1$ is given by :

$$P_{X_1}(x) = \sum_{x_2, x_3} p(x_1, x_2, x_3), \qquad \dots(5·21\,b)$$

and so on.

As, in the case of two random variables, *the r.v.'s* $X_1, X_2, \dots, X_n$ *are independent if and only if their joint p.m.f. is equal to the product of their marginal p.m.f.'s, i.e., iff :*

$$p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \cdots p_{X_n}(x_n) \qquad \dots(5·21\,c)$$

**Joint and marginal Probability Density Function.**

Let $(X_1, X_2, \dots, X_n)$ be $n$-dimensional continuous *r.v.* assuming all the values in some region, say, $R_1^n$ of the $n$-dimensional space. Then the joint *p.d.f.* of $(X_1, X_2, \dots, X_n)$ is given by :

$$f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = \lim_{dx_1 \to 0,\, dx_2 \to 0, \dots\, dx_n \to 0} \frac{P\left[\bigcap_{i=1}^{n} (x_i < X_i < x_i + dx_i)\right]}{dx_1 \cdot dx_2 \cdot \dots \cdot dx_n} \qquad \dots(5·21d)$$

where :

(i) $f(x_1, x_2, \dots, x_n) \ge 0,\ \forall\, (x_1, x_2, \dots, x_n) \in R_1^n$, and

(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \dots dx_n = 1$

The marginal p.d.f. of any variable, say, $X_i$, is obtained on integrating the p.d.f. over the range of all the variables except $X_i$. Thus,

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n$$

In particular, for three r.v.'s $X_1, X_2, X_3$ with joint p.d.f. $f(x_1, x_2, x_3)$, the marginal p.d.f. of, say, $X_2$ is given by :

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3)\, dx_1\, dx_3$$

and so on.

The necessary and sufficient condition for the independence of r.v.'s $X_1, X_2, \ldots, X_n$ is that their joint p.d.f. is the product of their marginal p.d.f.'s i.e.,

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

**Example 5·30.** *In the random placement of three balls in three cells, describe the outcomes of the experiment. Let $X_i$ denote the number of balls in cell i; i = 1, 2, 3, and N the number of cells occupied. Obtain the joint distribution of : (a) $(X_1, N)$ and (b) ...*

**Solution.** (a) Let the three balls be denoted by $a$, $b$ and $c$. Then the outcomes of placing the three balls in three cells are as follows :

| | | |
|---|---|---|
| 1. {a \| b \| c} | 10. {ac \| b \| –} | 19. {b \| ca \| –} |
| 2. {a \| c \| b} | 11. {ac \| – \| b} | 20. {b \| – \| ca} |
| 3. {b \| a \| c} | 12. {– \| ac \| b} | 21. {– \| b \| ca} |
| 4. {b \| c \| a} | 13. {bc \| a \| –} | 22. {c \| ab \| –} |
| 5. {c \| a \| b} | 14. {bc \| – \| a} | 23. {c \| – \| ab} |
| 6. {c \| b \| a} | 15. {– \| bc \| a} | 24. {– \| c \| ab} |
| 7. {ab \| c \| –} | 16. {a \| bc \| –} | 25. {abc \| – \| –} |
| 8. {ab \| – \| c} | 17. {a \| – \| bc} | 26. {– \| abc \| –} |
| 9. {– \| ab \| c} | 18. {– \| a \| bc} | 27. {– \| – \| abc} |

Each of these arrangements represents a sample event, *i.e.*, a sample point, so the sample space contains 27 points.

Let $N$ denote the number of occupied cells. The favourable cases for N = 1 are at numbers 7 to 24, *i.e.*, 18; and for numbers 25, 26 and 27, *i.e.*, 3; for N = 2 are at numbers 7 to 24, *i.e.*, 18; and at numbers 1 to 6, *i.e.*, 6. Accordingly, the probability distribution of N is :

$$P(N=1) = \frac{3}{27}, \quad P(N=2) = \frac{18}{27}, \quad P(N=3) = \frac{6}{27}.$$

Let $X_1$ denote the number of balls placed in the first cell. Then from the table of sample points, we get

$$P(X_1=0) = \frac{8}{27}, \quad P(X_1=1) = \frac{12}{27}, \quad P(X_1=2) = \frac{6}{27} \quad \text{and } P(X_1=3) \ldots$$



The joint distribution ...

$P(N = 1, X_1 = 0)$ ...

$P(N = 1, X_1 = 3)$ ...

$P(N = 2, X_1 = 2)$ ...

$P(N = 3, X_1 = 1)$ ...

| $X_1$ \ $N$ | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| Distribution of N | |

(b) Proceeding on the same lines, the joint distribution of $X_1$ and $X_2$ can be obtained as shown in the adjoining table :

**Example 5·31.** *A bivariate population ... following pairs of ... noted against them :*

*Find the probability that ... if the two events X = 4 a...*

The joint distribution of $N$ and $X_1$ can be obtained as follows :

$$P(N=1, X_1=0)=\frac{2}{27}, \quad P(N=1, X_1=1)=0, \quad P(N=1, X_1=2)=0,$$
$$P(N=1, X_1=3)=\frac{1}{27}, \quad P(N=2, X_1=0)=\frac{6}{27}, \quad P(N=2, X_1=1)=\frac{6}{27},$$
$$P(N=2, X_1=2)=\frac{6}{27}, \quad P(N=2, X_1=3)=0, \quad P(N=3, X_1=0)=0,$$
$$P(N=3, X_1=1)=\frac{6}{27}, \text{ and } P(N=3, X_1=2)=0, \quad P(N=3, X_1=3)=0.$$

JOINT DISTRIBUTION OF $N$ AND $X_1$

| $X_1$ \ $N$ | 1 | 2 | 3 | Distribution of $X_1$ |
|---|---|---|---|---|
| 0 | $\frac{2}{27}$ | $\frac{6}{27}$ | 0 | $\frac{8}{27}$ |
| 1 | 0 | $\frac{6}{27}$ | $\frac{6}{27}$ | $\frac{12}{27}$ |
| 2 | 0 | $\frac{6}{27}$ | 0 | $\frac{6}{27}$ |
| 3 | $\frac{1}{27}$ | 0 | 0 | $\frac{1}{27}$ |
| Distribution of $N$ | $\frac{3}{27}$ | $\frac{18}{27}$ | $\frac{6}{27}$ | 1 |

(b) Proceeding on the same lines, the joint distribution of $X_1$ and $X_2$ can be obtained as shown in the adjoining table :

JOINT DISTRIBUTION OF $X_1$ and $X_2$

| $X_1$ \ $X_2$ | 0 | 1 | 2 | 3 | Distribution of $X_1$ |
|---|---|---|---|---|---|
| 0 | $\frac{1}{27}$ | $\frac{3}{27}$ | $\frac{3}{27}$ | $\frac{1}{27}$ | $\frac{8}{27}$ |
| 1 | $\frac{3}{27}$ | $\frac{6}{27}$ | $\frac{3}{27}$ | 0 | $\frac{12}{27}$ |
| 2 | $\frac{3}{27}$ | $\frac{3}{27}$ | 0 | 0 | $\frac{6}{27}$ |
| 3 | $\frac{1}{27}$ | 0 | 0 | 0 | $\frac{1}{27}$ |
| Distribution of $X_2$ | $\frac{8}{27}$ | $\frac{12}{27}$ | $\frac{6}{27}$ | $\frac{1}{27}$ | 1 |

**Example 5·31.** *A random observation on a bivariate population (X, Y) can yield one of the following pairs of values with probabilities noted against them :*

| For each observation pair | Probability |
|---|---|
| (1, 1) ; (2, 1) ; (3, 3) ; (4, 3) | $\frac{1}{20}$ |
| (3, 1) ; (4, 1) ; (1, 2) ; (2, 2) ; (3, 2) ; (4, 2); (1, 3) ; (2, 3) | $\frac{1}{10}$ |

*Find the probability that Y = 2 given that X = 4. Also find the probability that Y = 2. Examine if the two events X = 4 and Y = 2 are independent.*

**Solution.** $P(Y=2) = P\{(1,2)\cup(2,2)\cup(3,2)\cup(4,2)\} = \frac{4}{10} = \frac{2}{5}$.

$P(X=4) = P\{(4,1)\cup(4,2)\cup(4,3)\} = \frac{1}{10}+\frac{1}{10}+\frac{1}{20} = \frac{1}{4}$

$P(X=4, Y=2) = P\{(4,2)\} = \frac{1}{10}$

$P(Y=2 \mid X=4) = \frac{P(X=4\cap Y=2)}{P(X=4)} = \frac{1/10}{1/4} = \frac{2}{5}$

Now $P(X=4).P(Y=2) = \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10} = P(X=4\cap Y=2)$

Hence the events $X=4$ and $Y=2$ are independent.

**Problem 5·32.** *The joint probability distribution of two random variables X and Y given by :* $P(X=0, Y=1) = \frac{1}{3}$, $P(X=1, Y=-1) = \frac{1}{3}$, *and* $P(X=1, Y=1) = \frac{1}{3}$.

*Find (i) Marginal distributions of X and Y, and (ii) the conditional probability distribution of X given Y = 1.*

**Solution.** $P(X=-1)$

$= \sum_y P(X=-1, Y=y)$

$= P(X=-1, Y=-1)$

$+ P(X=-1, Y=0)$

$+ P(X=-1, Y=1) = 0$

Similarly $P(X=0) = \frac{1}{3}$

and $P(X=1) = \frac{2}{3}$

| Y \ X | −1 | 0 | 1 | Marginal (Y) |
|---|---|---|---|---|
| −1 | 0 | 0 | $\frac{1}{3}$ | |
| 0 | 0 | 0 | 0 | |
| 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | |
| Marginal (X) | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | |

Thus

Marginal distribution of X is :

| Values of X, $x$ : | −1 | 0 | 1 |
|---|---|---|---|
| $P(X=x)$ : | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |

Marginal distribution of Y is :

| Values of Y, $y$ : | −1 | 0 |
|---|---|---|
| $P(Y=y)$ : | $\frac{1}{3}$ | 0 |

(*ii*) The conditional probability distribution of X given Y is :

$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$. Now

$P(X=-1 \mid Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0$, $P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)}$

$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/3}{2/3} = \frac{1}{2}$

Thus the conditional distribution of X given Y = 1 is :

| Values of X = $x$ | −1 | 0 |
|---|---|---|
| $P(X=x \mid Y=1)$ | 0 | $\frac{1}{2}$ |

**Example 5·33.** For the following bivariate probability distribution of X and Y, find :

(i) $P(X \le 1, Y = 2)$,
(ii) $P(X \le 1)$,
(iii) $P(Y \le 3)$, and
(iv) $P(X < 3, Y \le 4)$.

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{32}$ | $\frac{2}{32}$ | $\frac{2}{32}$ | $\frac{3}{32}$ |
| 1 | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |
| 2 | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{64}$ | 0 | $\frac{2}{64}$ |

**Solution.** The marginal distributions are given below :

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | $p_X(x)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{32}$ | $\frac{2}{32}$ | $\frac{2}{32}$ | $\frac{3}{32}$ | $\frac{8}{32}$ |
| 1 | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{10}{16}$ |
| 2 | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{64}$ | 0 | $\frac{2}{64}$ | $\frac{8}{64}$ |
| $p_Y(y)$ | $\frac{3}{32}$ | $\frac{3}{32}$ | $\frac{11}{64}$ | $\frac{13}{64}$ | $\frac{6}{32}$ | $\frac{16}{64}$ | $\Sigma p(x) = 1$ <br> $\Sigma p(y) = 1$ |

(i) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{8}{32} + \frac{10}{16} = \frac{7}{8}$

(iii) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X < 3, Y \le 4) = P(X = 0, Y \le 4) + P(X = 1, Y \le 4) + P(X = 2, Y \le 4)$

$$= \left(\frac{1}{32} + \frac{2}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right) = \frac{9}{16}.$$

**Example 5·34.** For the joint probability distribution of two random variables X and Y given below :

| X \ Y | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| 1 | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{10}{36}$ |
| 2 | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{9}{36}$ |
| 3 | $\frac{5}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{8}{36}$ |
| 4 | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{5}{36}$ | $\frac{9}{36}$ |
| Total | $\frac{11}{36}$ | $\frac{9}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ | 1 |

*Find (i) the marginal distributions of X and Y, and*

*(ii) conditional distribution of X given the value of Y = 1 and that of Y given the value of X = 2.*

**Solution.** The marginal distribution of X is defined as :

$$P(X = x) = \sum_y P(X = x, Y = y)$$

$$\therefore \quad P(X=1) = \sum_y P(X=1, Y=y)$$
$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$$
$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}.$$

Similarly $P(X=2) = \sum_y P(X=2, Y=y) = \frac{9}{36}$ ; $P(X=3) = \sum_y P(X=3, Y=y) = \frac{8}{36}$

and $P(X=4) = \sum_y P(X=4, Y=y) = \frac{9}{36}$.

Similarly, we can obtain the marginal distribution of Y.

MARGINAL DISTRIBUTION OF X

| Values of X, $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{10}{36}$ | $\frac{9}{36}$ | $\frac{8}{36}$ | $\frac{9}{36}$ |

MARGINAL DISTRIBUTION OF Y

| Values of Y, $y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(Y=y)$ | $\frac{11}{36}$ | $\frac{9}{36}$ | $\frac{7}{36}$ | [illegible] |

(*ii*) The conditional probability function of X given Y is defined as follows :

$$P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}. \text{ Therefore}$$

$$\therefore \quad P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{4/36}{11/36} = \frac{4}{11}$$

$$P(X=2 \mid Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}.$$

$$P(X=3 \mid Y=1) = \frac{P(X=3, Y=1)}{P(Y=1)} = \frac{5/36}{11/36} = \frac{5}{11}$$

$$P(X=4 \mid Y=1) = \frac{P(X=4, Y=1)}{P(Y=1)} = \frac{1/36}{11/36} = \frac{1}{11}$$

Hence the conditional distribution of X given Y = 1 is :

| $x$ : | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x \mid Y=1)$ : | $\frac{4}{11}$ | $\frac{1}{11}$ | $\frac{5}{11}$ | $\frac{1}{11}$ |

Similarly, we can obtain the conditional distribution of Y for X = 2 as given below:

| $y$ : | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(Y=y \mid X=2)$ : | $\frac{1}{9}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{9}$ |

**Example 5·35.** *A two-dimensional r.v.* (X, Y) *have a bivariate distribution given by*

$$P(X=x, Y=y) = \frac{x^2+y}{32}, \text{ for } x = 0, 1, 2, 3 \text{ and } y = 0, 1.$$

*Find the marginal distributions of* X *and* Y.

(*b*) *A two-dimensional r.v.* (X, Y) *have a joint probability mass function* :

$$p(x, y) = \frac{1}{27}(2x+y), \text{ where } x \text{ and } y \text{ can assume only the integer values } 0, 1 \ldots$$

*Find the conditional distribution of* Y *for* X = x.

**Solution.** (*a*) We have

| Y \ X | 0 | 1 | 2 | 3 | Marginal distribution of Y, $P(Y = y)$ |
|---|---|---|---|---|---|
| 0 | 0 | $\frac{1}{32}$ | $\frac{4}{32}$ | $\frac{9}{32}$ | $\frac{14}{32}$ |
| 1 | $\frac{1}{32}$ | $\frac{2}{32}$ | $\frac{5}{32}$ | $\frac{10}{32}$ | $\frac{18}{32}$ |
| Marginal distribution of X, $P(X = x)$ | $\frac{1}{32}$ | $\frac{3}{32}$ | $\frac{9}{32}$ | $\frac{19}{32}$ | 1 |

The marginal probability distribution of X is given by :

$P(X = x) = \sum_y P(X = x, Y = y)$ and is tabulated in last row of above table.

The marginal probability distribution of Y is given by :

$P(Y = y) = \sum_x P(X = x, Y = y)$ and is tabulated in last column of above table.

(*b*) The joint probability function :

$$p_{XY}(x, y) = \frac{1}{27}(2x + y) \, ; x = 0, 1, 2; \; y = 0, 1, 2$$

gives the following table of joint probability distribution of X and Y.

JOINT PROBABILITY DISTRIBUTION $p(x, y)$ OF X AND Y

| X \ Y | 0 | 1 | 2 | $f_X(x)$ |
|---|---|---|---|---|
| 0 | 0 | $\frac{1}{27}$ | $\frac{2}{27}$ | $\frac{3}{27}$ |
| 1 | $\frac{2}{27}$ | $\frac{3}{27}$ | $\frac{4}{27}$ | $\frac{9}{27}$ |
| 2 | $\frac{4}{27}$ | $\frac{5}{27}$ | $\frac{6}{27}$ | $\frac{15}{27}$ |

For example, $p(0, 0) = \frac{1}{27}(0 + 2 \times 0) = 0$, $\quad p(1, 0) = \frac{1}{27}(0 + 2 \times 1) = \frac{2}{27}$;

$p(2, 0) = \frac{1}{27}(0 + 2 \times 2) = \frac{4}{27}$; and so on,

The conditional distribution of Y for $X = x$ is given by :

$$p_{Y \mid X}(Y = y \mid X = x) = \frac{p_{XY}(x, y)}{p_X(x)}$$ and is obtained in the adjoining table.

CONDITIONAL DISTRIBUTION OF Y FOR X = x

| Y \ X | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |
| 1 | $\frac{2}{9}$ | $\frac{3}{9}$ | $\frac{4}{9}$ |
| 2 | $\frac{4}{15}$ | $\frac{5}{15}$ | $\frac{6}{15}$ |

**Example 5·36.** *Two discrete random variables X and Y have the joint probability density function :*

$$p_{XY}(x, y) = \frac{\lambda^x e^{-\lambda} p^y (1 - p)^{x-y}}{y! \, (x - y)!}, \; y = 0, 1, 2, \ldots, x \; ; \; x = 0, 1, 2, \ldots$$

*where $\lambda$, $p$ are constants with $\lambda > 0$ and $0 < p < 1$.*

Find (i) The marginal probability density functions of X and Y.
(ii) The conditional distribution of Y for a given X and of X for a given Y.

**Solution.** (i) The marginal p.m.f. of X is given by :

$$p_X(x) = \sum_{y=0}^{x} p(x, y) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} \frac{x!\, p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} {}^xC_y\, p^y (1-p)^{x-y} = \frac{\lambda^x e^{-\lambda}}{x!} [p + (1-p)]^x = \frac{\lambda^x e^{-\lambda}}{x!}, \; x = 0, 1, \dots$$

which is the probability function of a Poisson distribution with parameter $\lambda$.

$$p_Y(y) = \sum_{x=y}^{\infty} p(x, y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!}$$

$[\because y = 0, 1, 2, \dots, x \Rightarrow y \le x \Rightarrow x \ge y]$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)} = \frac{e^{-\lambda p} (\lambda p)^y}{y!}; \; y = 0, 1, 2, \dots$$

which is the probability function of a Poisson distribution with parameter $\lambda p$.

(ii) The conditional distribution of Y for given X is :

$$p_{Y|X}(y \mid x) = \frac{p_{XY}(x, y)}{p_X(x)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y} x!}{y!(x-y)!\, \lambda^x e^{-\lambda}} = \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$

$$= {}^xC_y\, p^y (1-p)^{x-y}, \; x \ge y \text{ i.e., } y = 0, 1, 2, \dots x.$$

The conditional probability distribution of X for given Y is :

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!(x-y)!} \cdot \frac{y!}{e^{-\lambda p} (\lambda p)^y}$$

$$= \frac{e^{-\lambda q} (\lambda q)^{x-y}}{(x-y)!}; \; q = 1-p, \; x \ge y \text{ i.e., } x = y, y+1, y+2, \dots$$

**Example 5·37.** *If X and Y are two random variables having joint density function*

$$f(x, y) = \begin{cases} \frac{1}{8}(6-x-y); \; 0 \le x < 2, \; 2 \le y < 4 \\ 0, \text{ otherwise} \end{cases}$$

*Find* (i) $P(X < 1 \cap Y < 3)$, (ii) $P(X + Y < 3)$, *and* (iii) $P(X < 1 \mid Y < 3)$.

**Solution.** We have

(i) $P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x, y)\, dx\, dy = \int_{0}^{1} \int_{2}^{3} \frac{1}{8}(6-x-y)\, dx\, dy$ …

(ii) $P(X + Y < 3) = \int_{0}^{1} \int_{2}^{3-x} \frac{1}{8}(6-x-y)\, dx\, dy = \frac{5}{24}$

(iii) $P(X < 1 \mid Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} = \frac{3/8}{5/8} = \frac{3}{5}$

$\left[\text{From part (i) and } P(Y < 3) = \int_{0}^{2} \int_{2}^{3} \frac{1}{8}(6-x-y) \dots\right.$

**Example 5·38.** *Suppose that two-dimensional continuous random variable* … *joint p.d.f. given by :*

$$f(x, y) = \begin{cases} 6x^2 y, \; 0 < x < 1, \; 0 < y < 1 \\ 0, \text{ elsewhere} \end{cases}$$



(i) Verify that …

(ii) Find $P(0 < X < \frac{3}{4}$ …

**Solution.** (i) $\int_0^1 \int_0^1 f(x, y)\, dx\, dy =$ …

(ii) $P(0 < X < \frac{3}{4}, \frac{1}{3} <$ …

$P(X + Y < 1) =$ …

$=$ …

$P(X > Y) =$ …

$=$ …

$P(X < 1 \mid Y$ …

where $P(X < 1 \cap Y$ …

and $P(Y$ …

$\therefore$ $P(X < 1 \mid Y$ …

**Example 5·39.** *The* … *variable (X, Y) is given by* …

(i) Find the margina…

(ii) Find the conditi… function of X giv…

(iii) Check for indepe…

**Solution.** Evidently…

(i) The marginal p.d…

$f_X(x) = \begin{cases} \int \dots \\ 0, \dots \end{cases}$

0 < x < 1
0 < y < 1

(i) Verify that $\int_0^1 \int_0^1 f(x, y)\, dx dy = 1.$

(ii) Find $P(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2)$, $P(X + Y < 1)$, $P(X > Y)$ and $P(X < 1 \mid Y < 2)$.

**Solution.** (i)

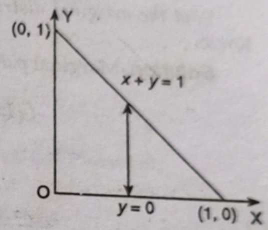
$$\int_0^1 \int_0^1 f(x, y)\, dx dy = \int_0^1 \int_0^1 6x^2 y\, dx dy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^1 dx = \int_0^1 3x^2 dx = \left|x^3\right|_0^1 = 1$$

(ii)
$$P\left(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2\right) = \int_0^{3/4} \int_{1/3}^{1} 6x^2 y\, dx dy + \int_0^{3/4} \int_1^2 0.\, dx dy$$

$$= \int_0^{3/4} 6x^2 \left|\frac{y^2}{2}\right|_{1/3}^1 dx = \frac{8}{9} \int_0^{3/4} 3x^2\, dx = \frac{8}{9}\left|x^3\right|_0^{3/4} = \frac{3}{8}.$$

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} 6x^2 y\, dx dy = \int_0^1 6x^2 \left|\frac{y^2}{2}\right|_0^{1-x} dx$$

$$= \int_0^1 3x^2(1-x)^2 dx = \frac{1}{10} \text{ [See Fig.]}$$




$$P(X > Y) = \int_0^1 \int_0^x 6x^2 y\, dx dy = \int_0^1 3x^2 \left|y^2\right|_0^x dx$$

$$= \int_0^1 3x^4 dx = \frac{3}{5}.$$

$$P(X < 1 \mid Y < 2) = \frac{P(X < 1 \cap Y < 2)}{P(Y < 2)}$$

where $P(X < 1 \cap Y < 2) = \int_0^1 \int_0^1 6x^2 y\, dx dy + \int_0^1 \int_1^2 0.\, dx dy = 1$

and $P(Y < 2) = \int_0^1 \int_0^2 f(x,y)\, dx dy = \int_0^1 \int_0^1 6x^2 y\, dx dy + \int_0^1 \int_1^2 0.\, dx dy = 1$

$\therefore$ $P(X < 1 \mid Y < 2) = \dfrac{P(X < 1 \cap Y < 2)}{P(Y < 2)} = 1.$

**Example 5·39.** *The joint probability density function of a two-dimensional random variable (X, Y) is given by :*

$$f(x, y) = \begin{cases} 2\,;\, 0 < x < 1, 0 < y < x\,; \\ 0, \text{ elsewhere} \end{cases}$$

*(i) Find the marginal density functions of X and Y.*

*(ii) Find the conditional density function of Y given X = x and conditional density function of X given Y = y.*

*(iii) Check for independence of X and Y.*

**Solution.** Evidently $f(x,y) \geq 0$ and $\int_0^1 \int_0^x 2\, dx dy = 2\int_0^1 x\, dx = 1.$

(i) The marginal p.d.f.'s of X and Y are given by :

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^x 2dy = 2x, 0 < x < 1 \\ 0, \text{ elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_y^1 2dx = 2(1-y),\ 0 < y < 1 \\ 0, \text{ elsewhere} \end{cases}$$

(ii) The conditional density function of Y given X, $(0 < x < 1)$ is :

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x},\ 0 < y < x.$$

The conditional density function of X given Y, $(0 < y < 1)$ is :

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)},\ y < x < 1$$

(iii) Since $f_X(x) f_Y(y) = 2(2x)(1-y) \neq f_{XY}(x, y)$, X and Y are not independent.

**Example 5.40.** *The joint p.d.f. of two random variables X and Y is given by :*

$$f(x, y) = \frac{9(1 + x + y)}{2(1 + x)^4 (1 + y)^4};\ 0 \le x < \infty,\ 0 \le y < \infty$$

*Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.*

**Solution.** Marginal p.d.f. of X is given by :

$$f_X(x) = \int_0^{\infty} f(x, y)\, dy = \frac{9}{2(1 + x)^4} \int_0^{\infty} \frac{(1 + y) + x}{(1 + y)^4}\, dy$$

$$= \frac{9}{2(1 + x)^4} \int_0^{\infty} \{(1 + y)^{-3} + x(1 + y)^{-4}\}\, dy$$

$$= \frac{9}{2(1 + x)^4} \left( \left| \frac{-1}{2(1 + y)^2} \right|_0^{\infty} + x \left| \frac{-1}{3(1 + y)^3} \right|_0^{\infty} \right)$$

$$= \frac{9}{2(1 + x)^4} \cdot \left( \frac{1}{2} + \frac{x}{3} \right) = \frac{3}{4} \cdot \frac{3 + 2x}{(1 + x)^4};\ 0 < x < \infty$$

Since $f(x, y)$ is symmetric in $x$ and $y$, the marginal p.d.f. of Y is given by :

$$f_Y(y) = \int_0^{\infty} f(x, y)\, dx = \frac{3}{4} \cdot \frac{3 + 2y}{(1 + y)^4};\ 0 < y < \infty$$

The conditional distribution of Y for $X = x$ is given by :

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{9(1 + x + y)}{2(1 + x)^4 (1 + y)^4} \times \frac{4(1 + x)^4}{3(3 + 2x)} = \frac{6(1 + x + y)}{(1 + y)^4 (3 + 2x)};\ 0 < y$$

**Example 5.41.** *Joint distribution of X and Y is given by :*

$$f(x,y) = 4xy\, e^{-(x^2 + y^2)};\ x \ge 0,\ y \ge 0.$$

*Test whether X and Y are independent. For the above joint distribution, find the conditional density of X given Y = y.*

**Solution.** Joint p.d.f. of X and Y is : $f_{XY}(x,y) = 4xy\ e^{-(x^2 + y^2)};\ x \ge 0,\ y \ge 0.$

Marginal density of X is given by :

$$f_X(x) = \int_0^{\infty} f_{XY}(x, y)\, dy = \int_0^{\infty} 4xy\, e^{-(x^2 + y^2)}\, dy = 4x\, e^{-x^2} \int_0^{\infty} y\, e^{-y^2}\, dy$$

$$= 4x\, e^{-x^2} \int_0^{\infty} e^{-t} \cdot \frac{dt}{2} = 2x \cdot e^{-x^2} \left| -e^{-t} \right|_0^{\infty}$$

$$\Rightarrow \quad V(U) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} a_i a_j \ldots$$

$$\Rightarrow \quad V\left( \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} a_i a_j \text{Cov}(X_i, X_j)$$

**Remarks 1.** If $a_i = 1$; $i = 1, 2, \ldots, n$, then

$$V(X_1 + X_2 + \ldots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n) + 2 \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, X_j) \quad \ldots(6\cdot32a)$$

2. If $X_1, X_2, \ldots X_n$ are independent (pairwise), then $\text{Cov}(X_i, X_j) = 0, (i \neq j)$.

Thus from (6·32) and (6·32a), we get

$$\left.\begin{aligned} &V(a_1X_1 + a_2X_2 + \ldots + a_n X_n) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n) \\ \text{and} \quad &V(X_1 + X_2 + \ldots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n), \end{aligned}\right\} \quad \ldots(6\cdot32b)$$

provided $X_1, X_2, \ldots, X_n$ are independent.

3. If $a_1 = 1 = a_2$ and $a_3 = a_4 = \ldots = a_n = 0$, then from (6·32), we get

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2 \text{Cov}(X_1, X_2)$$

Again if $a_1 = 1$, $a_2 = -1$ and $a_3 = a_4 = \ldots = a_n = 0$, then

$$V(X_1 - X_2) = V(X_1) + V(X_2) - 2 \text{Cov}(X_1, X_2)$$

Thus we have

$$V(X_1 \pm X_2) = V(X_1) + V(X_2) \pm 2 \text{Cov}(X_1, X_2) \quad \ldots(6\cdot32c)$$

If $X_1$ and $X_2$ are independent, then $\text{Cov}(X_1, X_2) = 0$ and we get

$$V(X_1 \pm X_2) = V(X_1) + V(X_2) \quad \ldots(6\cdot32d)$$

**Example 6·1.** *Let X be a random variable with the following probability distribution :*

| | | | | |
|---|---|---|---|---|
| $x$ | : | $-3$ | $6$ | $9$ |
| $P(X = x)$ | : | $1/6$ | $1/2$ | $1/3$ |

*Find E (X) and E (X²) and using the laws of expectation, evaluate E (2X + 1)².*

**Solution.** $E(X) = \Sigma x\, p(x) = (-3) \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$

$$E(X^2) = \Sigma x^2 p(x) = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

$$\therefore \quad E(2X+1)^2 = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + 1 = 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209.$$

**Example 6·2.** *(a) Find the expectation of the number on a die when thrown.*

*(b) Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them.*

**Solution.** (a) Let X be the random variable representing the number on a die when thrown. Then X can take any one of the values 1, 2, 3, ..., 6 each with equal probability $\frac{1}{6}$. Hence

$$E(X) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \ldots + \frac{1}{6} \times 6 = \frac{1}{6}(1 + 2 + 3 + \ldots + 6) = \frac{1}{6} \times \frac{6 \times 7}{2} = \frac{7}{2} \quad \ldots(*)$$

... [illegible] This does not mean that in a random throw of a dice, the player will get the ... [illegible] In fact, one can never get this (fractional) number in a throw of a dice. Rather, ... [illegible] that if the player tosses the dice for a "long" period, then on the average toss he will ...

(b) The probability function of X (the sum of numbers obtained on two dice), is

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Value of X : x | 2 | 3 | 4 | 5 | 6 | 7 | .... | 11 | 12 |
| Probability | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | .... | 2/36 | 1/36 |

$$E(X) = \sum_i p_i x_i$$

$$= 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12) = \frac{1}{36} \times 252 = 7$$

**Aliter.** Let $X_i$ be the number obtained on the $i$th dice $(i = 1, 2)$ when thrown. Then the sum of the number of points on two dice is given by :

$S = X_1 + X_2 \Rightarrow E(S) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7$ [On using (*)]

**Remark.** This result can be generalised to the sum of points obtained in a random throw of $n$ dice. Then

$$E(S) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{7}{2} = \frac{7n}{2}.$$

**Example 6·3.** *In four tosses of a coin, let X be the number of heads. Tabulate the 16 possible outcomes with the corresponding values of X. By simple counting, derive the probability distribution of X and hence calculate the expected value of X.*

**Solution.** Let $H$ represent a head, $T$ a tail and X, the random variable denoting the number of heads.

| S. No. | Outcomes | No. of Heads (X) | S. No. | Outcomes | No. of Heads (X) |
|---|---|---|---|---|---|
| 1 | HHHH | 4 | 9 | HTHT | 2 |
| 2 | HHHT | 3 | 10 | THTH | 2 |
| 3 | HHTH | 3 | 11 | THHT | 2 |
| 4 | HTHH | 3 | 12 | HTTT | 1 |
| 5 | THHH | 3 | 13 | THTT | 1 |
| 6 | HHTT | 2 | 14 | TTHT | 1 |
| 7 | HTTH | 2 | 15 | TTTH | 1 |
| 8 | TTHH | 2 | 16 | TTTT | 0 |

The random variable X takes the values 0, 1, 2, 3 and 4. Since, from the above table, we find that the number of cases favourable to the coming of 0, 1, 2, 3 and 4 heads are 1, 4, 6, 4 and 1 respectively, we have

$$P(X=0) = \frac{1}{16},\ P(X=1) = \frac{4}{16} = \frac{1}{4},\ P(X=2) = \frac{6}{16} = \frac{3}{8},\ P(X=3) = \frac{4}{16} = \frac{1}{4},\ P(X=4) = \frac{1}{16}$$

The probability distribution of X can be summarized as follows :

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 |
| $p(x)$ : | $\frac{1}{16}$ | $\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{1}{4}$ | $\frac{1}{16}$ |

$$E(X) = \sum_{x=0}^{4} x\,p(x) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16} = \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2.$$

**Example 6·4.** *An urn contains 7 white and 3 red balls. Two balls are drawn together, at random from this urn. Compute the probability that neither of them is white. Find also the probability of getting one white and one red ball. Hence compute the expected number of white balls drawn.*

**Solution.** Let X denote the number of white balls drawn. The probability distribution of X is obtained as follows :

| | | | |
|---|---|---|---|
| $x$ : | 0 | 1 | 2 |
| $p(x)$ : | $\frac{^3C_2}{^{10}C_2} = \frac{1}{15}$ | $\frac{^7C_1 \times {^3C_1}}{^{10}C_2} = \frac{7}{15}$ | $\frac{^7C_2}{^{10}C_2} = \frac{7}{15}$ |

Then expected number of white balls drawn is :

$$E(X) = 0 \times \frac{1}{15} + 1 \times \frac{7}{15} + 2 \times \frac{7}{15} = \frac{21}{15}.$$

**Example 6·5.** *A gamester has a disc with a freely revolving needle. The disc is divided into 20 equal sectors by thin lines and the sectors are marked 0, 1, 2, ..., 19. The gamester treats 5 or any multiple of 5 as lucky numbers and zero as a special lucky number. He allows a player to whirl the needle on a charge of 10 paise. When the needdle stops at the lucky number the gamester pays back the player twice the sum charged and at the special lucky number the gamester pays to the player 5 times of the sum charged. Is the game fair ? What is the expectation of the player ?*

**Solution.**

| Event | Favourable | $p(x)$ | Player's Gain $(x)$ |
|---|---|---|---|
| Lucky number | 5, 10, 15 | 3/20 | $20 - 10 = 10\,p$ |
| Special lucky No. | 0 | 1/20 | $50 - 10 = 40\,p$ |
| Other numbers | 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19 | 16/20 | $-10\,p$ |

$$\therefore \quad E(X) = \frac{3}{20} \times 10 + \frac{1}{20} \times 40 - \frac{16}{20} \times 10 = -\frac{9}{2} \neq 0,$$ *i.e.*, the game is *not fair*.

**Example 6·6.** *A box contains $2^n$ tickets among which $^nC_i$ tickets bear the number $i$; $i = 0, 1, 2, ..., n$. A group of m tickets is drawn. What is the expectation of the sum of their numbers ?*

**Solution.** Let $X_i$ ; $i = 1, 2, ..., m$ be the variable representing the number on the ith ticket drawn. Then the sum '$S$' of the numbers on the tickets drawn is given by :

$$S = X_1 + X_2 + ... + X_m = \sum_{i=1}^{m} X_i, \text{ so that } E(S) = \sum_{i=1}^{m} E(X_i)$$

$X_i$ is a random variable which can take any one of the possible values 0, 1, 2, ..., $n$ with respective probabilities : $^nC_0/2^n, {^nC_1}/2^n, {^nC_2}/2^n, ..., {^nC_n}/2^n$.

$$\therefore \quad E(X_i) = \frac{1}{2^n}\left(1.\,{^nC_1} + 2.\,{^nC_2} + 3.\,{^nC_3} + ... + n.\,{^nC_n}\right)$$

$$= \frac{1}{2^n}\left(1.n + 2.\,\frac{n(n-1)}{2!} + 3.\,\frac{n(n-1)(n-2)}{3!} + ... + n.\,1\right)$$

$$= \frac{n}{2^n}\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + ... + 1\right\}$$

$$= \frac{n}{2^n}\left({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1}\right) = \frac{n}{2^n}\cdot(1+1)^{n-1} = \frac{n}{2}$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i) = \sum_{i=1}^{m} (n/2) = \frac{mn}{2}.$$

**Example 6·7.** *A coin is tossed until a head appears. What is the expectation of the number of tosses required ?*

**Solution.** Let $X$ denote the number of tosses required to get the first head. Then $X$ can materialise in the following ways :

| Event | $x$ | Probability, $p(x)$ |
|---|---|---|
| $H$ | 1 | $\frac{1}{2}$ |
| $TH$ | 2 | $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ |
| $TTH$ | 3 | $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$ |
| ⋮ | ⋮ | ⋮ |

$$\therefore \quad E(X) = \sum_{x=1}^{\infty} x\,p(x) = 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16} + \dots \qquad \dots(*)$$

This is an arithmetic-geometric series with ratio of GP being $r = \frac{1}{2}$.

Let $\quad S = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + \dots$

Then $\quad \frac{1}{2}S = \quad \frac{1}{4} + 2\cdot\frac{1}{8} + 3\cdot\frac{1}{16} + \dots$

$$\therefore \quad \left(1-\tfrac{1}{2}\right)S = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots \quad\Rightarrow\quad \frac{1}{2}S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \quad \text{or} \quad S = 2.$$

[Since the sum of an infinite G. P. with first term $a$ and common ratio $r\,(<1)$ is $\frac{a}{1-r}$]

Hence, substituting in (*), we have $\qquad E(X) = 2.$

**Example 6·8.** *What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in each trial ?*

**Solution.** Let the random variable $X$ denote the number of failures preceding the first success. Then $X$ can take the values $0, 1, 2, \dots, \infty$. We have

$$P(X=x) = p(x) = P(x \text{ failures precede the first success}) = q^x p,$$

where $q = 1-p$, is the probability of failure in a trial. Then by def.,

$$E(X) = \sum_{x=0}^{\infty} x\,p(x) = \sum_{x=0}^{\infty} x\,q^x p = pq\sum_{x=1}^{\infty} x\,q^{x-1} = pq\,(1 + 2q + 3q^2 + 4q^3 + \dots)$$

Now $\quad 1 + 2q + 3q^2 + 4q^3 + \dots$ is an infinite arithmetic-geometric series.

Let $\qquad S = 1 + 2q + 3q^2 + 4q^3 + \dots$

$\qquad\quad qS = \qquad q + 2q^2 + 3q^3 + \dots$

$$\therefore \quad (1-q)S = 1 + q + q^2 + q^3 + \dots = \frac{1}{1-q} \quad\Rightarrow\quad S = \frac{1}{(1-q)^2}$$

$$\therefore \quad 1 + 2q + 3q^2 + 4q^3 \dots = \frac{1}{(1-q)^2}. \quad \text{Hence} \quad E(X) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}.$$

$(1+1)^{n-1} = \frac{n}{2}$

e expectation of the first head. Then X

| obability, p (x) |
|---|
| $\frac{1}{2}$ |
| $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ |
| $\times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ |
| ⋮ |

... (*)

$\frac{1}{2}$.

$S = 2$.

ratio $r (< 1)$ is $\frac{a}{1-r}$

es preceding the first ty p of success in each

ilures preceding the

$\cdot p,$

$q^2 + 4q^3 + \dots)$ ...(*)

ric series.

$\frac{1}{-q)^2}$

$= \frac{pq}{p^2} = \frac{q}{p}$. [From (*)]



**Example 6·9.** *A box contains 'a' white and 'b' black balls. 'c' balls are drawn at random. Find the expected value of the number of white balls drawn.*

**Solution.** Let a variable $X_i$, associated with $i$th draw, be defined as follows :

$$X_i = \begin{cases} 1, & \text{if } i\text{th ball drawn is white} \\ 0, & \text{if } i\text{th ball drawn is black} \end{cases}$$

Then the number 'S' of the white balls among 'c' balls drawn is given by :

$$S = X_1 + X_2 + \dots + X_c = \sum_{i=1}^{c} X_i \quad \Rightarrow \quad E(S) = \sum_{i=1}^{c} E(X_i) \qquad \dots(*)$$

Now $\quad P(X_i = 1) = P(\text{of drawing a white ball}) = \frac{a}{a+b}$

and $\quad P(X_i = 0) = P(\text{of drawing a black ball}) = \frac{b}{a+b}$

$\therefore \quad E(X_i) = 1 . P(X_i = 1) + 0 . P(X_i = 0) = \frac{a}{a+b}$

Hence $\quad E(S) = \sum_{i=1}^{c} \left(\frac{a}{a+b}\right) = \frac{ca}{a+b}$. [From (*)]

**Example 6·10.** *Let the r.v. X have the distribution :*

$$P(X=0) = P(X=2) = p \; ; \; P(X=1) = 1-2p, \text{ for } 0 \le p \le \tfrac{1}{2}.$$

*For what p is the Var (X) a maximum ?*

**Solution.** Here the r.v. X takes the values 0, 1, and 2 with respective probabilities $p, 1-2p$ and $p, 0 \le p \le \frac{1}{2}$. Thus

$$E(X) = 0 \times p + 1 \times (1-2p) + 2 \times p = 1, \; E(X^2) = 0 \times p + 1^2 \times (1-2p) + 2^2 \times p = 1 + 2p$$

$$\therefore \quad \text{Var}(X) = E(X^2) - \{E(X)\}^2 = 2p \; ; \; 0 \le p \le \tfrac{1}{2}$$

Obviously, for $0 \le p \le \frac{1}{2}$, Var (X) is maximum when $p = \frac{1}{2}$, and $[\text{Var}(X)]_{max} = 2 \times \frac{1}{2} = 1$.

**Example 6·11.** *(Random Walk Problem). Starting from the origin, unit steps are taken to the right with probability p and to the left with probability q $(= 1-p)$. Assuming independent movements, find the mean and variance of the distance moved from origin after n steps.*

**Solution.** Let us associate a variable $X_i$ with the $i$th step defined as follows :

$$X_i = \begin{cases} +1, & \text{if the } i\text{th step is towards the right,} \\ -1, & \text{if the } i\text{th step is towards the left.} \end{cases}$$

Then $S = X_1 + X_2 + \dots + X_n = \sum X_i$, represents the random distance moved from origin after $n$ steps.

$$E(X_i) = 1 \times p + (-1) \times q = p - q \quad \text{and} \quad E(X_i^2) = 1^2 \times p + (-1)^2 \times q = p + q = 1$$

$$\therefore \quad \text{Var}(X_i) = E(X_i^2) - \{E(X_i)\}^2 = (q+p)^2 - (p-q)^2 = 4pq$$

Hence $\quad E(S_n) = \sum_{i=1}^{n} E(X_i) = n(p-q) \quad \text{and} \quad V(S_n) = \sum_{i=1}^{n} V(X_i) = 4npq$

[$\because$ Movements of steps are independent.]