Lecture 7: Value Function Approximation

CS60077 : REINFORCEMENT LEARNING

Autumn 2023

Outline

Outline

Large-Scale Reinforcement Learning

Reinforcement learning can be used to solve *large* problems, e.g.

- Backgammon: 10^{20} states
- Computer Go: 10^{170} states
- Helicopter: continuous state space

How can we scale up the model-free methods for *prediction* and *control* from the last two lectures?

Tabular Methods and its Limitations

- State or action values are all represented by look-up tables
  - Either every state s or every state-action pair (s, a) has an entry in the form of $V(s)$ or $Q(s, a)$
- Unable to handle large (potentially infinite) state spaces
  - Huge memory need and also huge amount of time to update iteratively
  - Learned values at each state are decoupled – an update at one state affected no other state
  - Continuous states and actions *cannot* be handled
- States not encountered previously will not have a sensible policy update – *generalisation* issue

Learning using Function Approximators

- Parameterized value function representation means an update at one state affects many others helping *generalization*
- Challenge of learning function approximators using supervised learning mechanism:
  - State and update value pairs may act as the training data for this (non-conventional) supervised learning setup
  - Training set is not static unlike most traditional supervised learning setting
  - Control methods changes policy and thus the generated data
  - Target values of training examples are non-stationary during evaluation when bootstrapping applied
- Requires function approximation methods able to handle nonstationary target functions (that change over time)

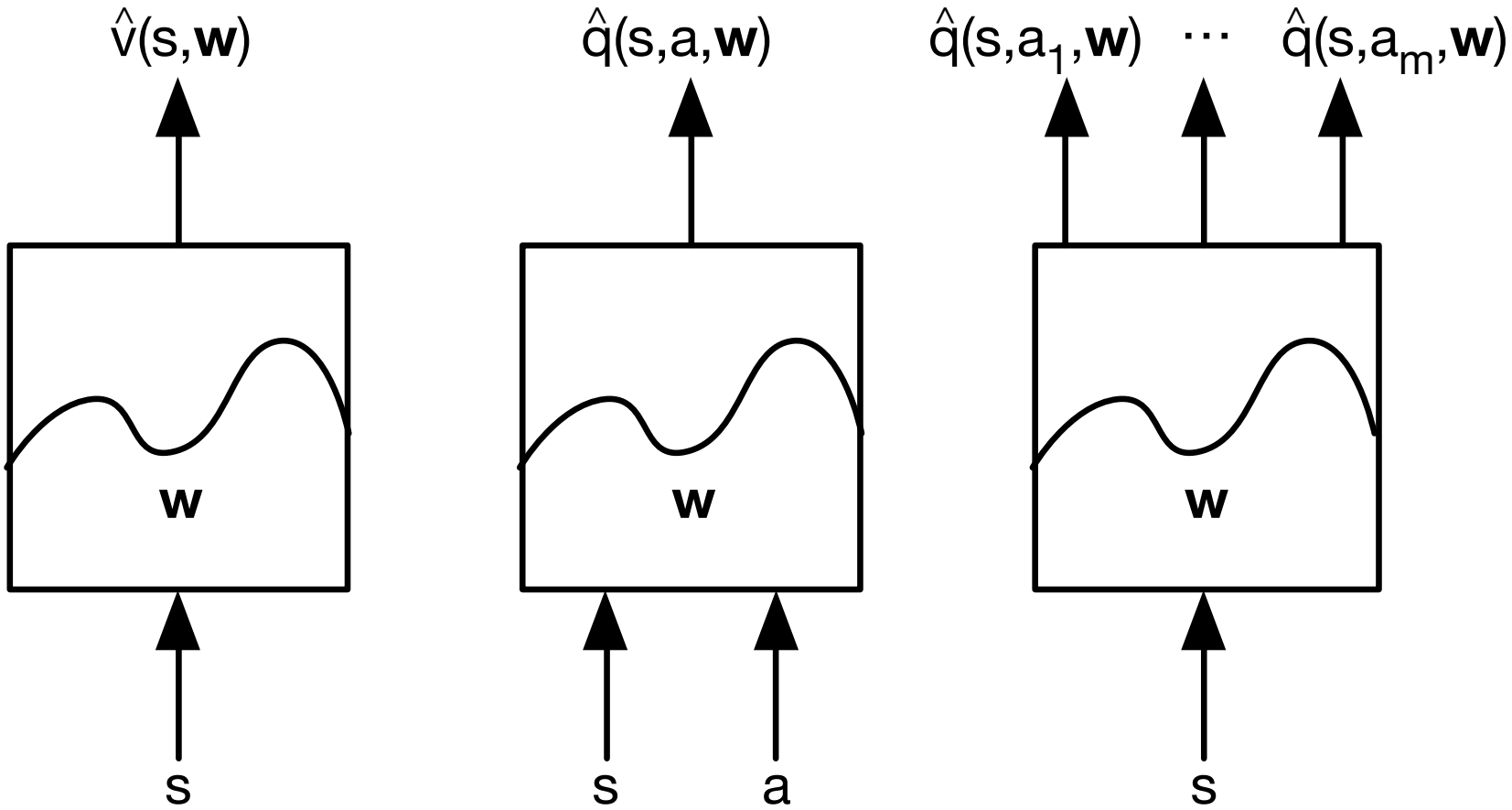
Value Function Approximation

- So far we have represented value function by a *lookup table*
  - Every state s has an entry $V(s)$
  - Or every state-action pair s, a has an entry $Q(s, a)$
- Problem with large MDPs:
  - There are too many states and/or actions to store in memory
  - It is too slow to learn the value of each state individually
- Solution for large MDPs:
  - Estimate value function with *function approximation*

$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$
$$\text{or } \hat{q}(s, a, \mathbf{w}) \approx q_\pi(s, a)$$

  - *Generalise* from seen states to unseen states
  - *Update* parameter $\mathbf{w}$ using MC or TD learning

Types of Value Function Approximation



Which Function Approximator?

There are many function approximators, e.g.

- Linear combinations of features
- Neural network
- Decision tree
- Nearest neighbour
- Fourier / wavelet bases
- ...

Which Function Approximator?

We consider differentiable function approximators, e.g.

- Linear combinations of features
- Neural network
- Decision tree
- Nearest neighbour
- Fourier / wavelet bases
- ...

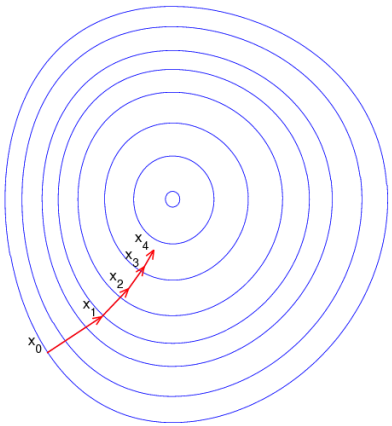
Furthermore, we require a training method that is suitable for non-stationary, non-iid data

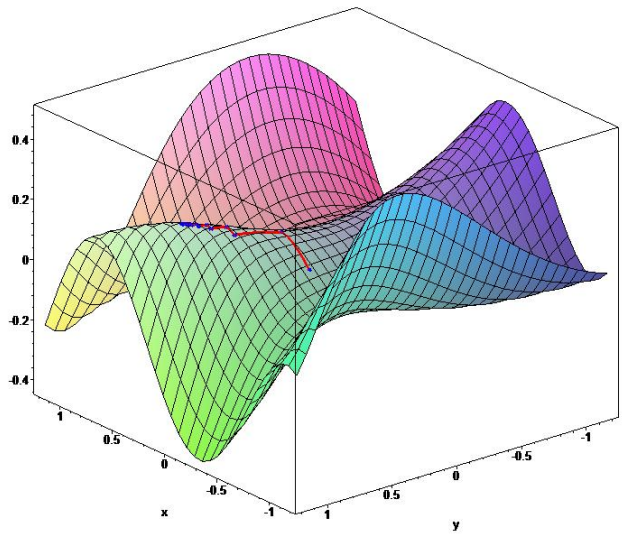
Outline

Gradient Descent

- Let $J(\mathbf{w})$ be a differentiable function of parameter vector $\mathbf{w}$
- Define the *gradient* of $J(\mathbf{w})$ to be

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \begin{pmatrix} \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_1} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}_n} \end{pmatrix}$$





- To find a local minimum of $J(\mathbf{w})$
- Adjust $\mathbf{w}$ in direction of -ve gradient

$$\Delta \mathbf{w} = -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w})$$

where α is a step-size parameter

Value Function Approx. By Stochastic Gradient Descent

- Goal: find parameter vector $\mathbf{w}$ minimising mean-squared error between approximate value fn $\hat{v}(s, \mathbf{w})$ and true value fn $v_\pi(s)$

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w}))^2 \right]$$

- Gradient descent finds a local minimum

$$\begin{aligned} \Delta \mathbf{w} &= -\frac{1}{2} \alpha \nabla_{\mathbf{w}} J(\mathbf{w}) \\ &= \alpha \mathbb{E}_\pi \left[(v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \right] \end{aligned}$$

- Stochastic gradient descent *samples* the gradient

$$\Delta \mathbf{w} = \alpha (v_\pi(S) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w})$$

- Expected update is equal to full gradient update

Feature Vectors

- Represent state by a *feature vector*

$$\mathbf{x}(S) = \begin{pmatrix} \mathbf{x}_1(S) \\ \vdots \\ \mathbf{x}_n(S) \end{pmatrix}$$

- For example:
  - Distance of robot from landmarks
  - Trends in the stock market
  - Piece and pawn configurations in chess

Linear Value Function Approximation

- Represent value function by a linear combination of features

$$\hat{v}(S, \mathbf{w}) = \mathbf{x}(S)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S) \mathbf{w}_j$$

- Objective function is quadratic in parameters $\mathbf{w}$

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(v_\pi(S) - \mathbf{x}(S)^\top \mathbf{w})^2 \right]$$

- Stochastic gradient descent converges on *global* optimum
- Update rule is particularly simple

$$\nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) = \mathbf{x}(S)$$
$$\Delta \mathbf{w} = \alpha (v_\pi(S) - \hat{v}(S, \mathbf{w})) \mathbf{x}(S)$$

Update = *step-size* $\times$ *prediction error* $\times$ *feature value*

Table Lookup Features

- Table lookup is a special case of linear value function approximation
- Using *table lookup features*

$$\mathbf{x}^{table}(S) = \begin{pmatrix} \mathbf{1}(S = s_1) \\ \vdots \\ \mathbf{1}(S = s_n) \end{pmatrix}$$

- Parameter vector $\mathbf{w}$ gives value of each individual state

$$\hat{v}(S, \mathbf{w}) = \begin{pmatrix} \mathbf{1}(S = s_1) \\ \vdots \\ \mathbf{1}(S = s_n) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}$$

Incremental Prediction Algorithms

- Have assumed true value function $v_\pi(s)$ given by supervisor
- But in RL there is no supervisor, only rewards
- In practice, we substitute a *target* for $v_\pi(s)$
  - For MC, the target is the return G_t

$$\Delta \mathbf{w} = \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

  - For TD(0), the target is the TD target $R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$

$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

  - For TD(λ), the target is the λ-return G_t^λ

$$\Delta \mathbf{w} = \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w})$$

Monte-Carlo with Value Function Approximation

- Return G_t is an unbiased, noisy sample of true value $v_\pi(S_t)$
- Can therefore apply supervised learning to "training data":

$$\langle S_1, G_1 \rangle, \langle S_2, G_2 \rangle, ..., \langle S_T, G_T \rangle$$

- For example, using *linear Monte-Carlo policy evaluation*

$$\begin{aligned} \Delta \mathbf{w} &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ &= \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t) \end{aligned}$$

- Monte-Carlo evaluation converges to a local optimum
- Even when using non-linear value function approximation

TD Learning with Value Function Approximation

- The TD-target $R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$ is a *biased* sample of true value $v_\pi(S_t)$
- Can still apply supervised learning to "training data":

$$\langle S_1, R_2 + \gamma \hat{v}(S_2, \mathbf{w}) \rangle, \langle S_2, R_3 + \gamma \hat{v}(S_3, \mathbf{w}) \rangle, ..., \langle S_{T-1}, R_T \rangle$$

- For example, using *linear TD(0)*

$$\begin{aligned} \Delta \mathbf{w} &= \alpha(R + \gamma \hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w})) \nabla_{\mathbf{w}} \hat{v}(S, \mathbf{w}) \\ &= \alpha \delta \mathbf{x}(S) \end{aligned}$$

- Linear TD(0) converges (close) to global optimum

TD(λ) with Value Function Approximation

- The λ-return G_t^λ is also a biased sample of true value $v_\pi(s)$
- Can again apply supervised learning to "training data":

$$\left\langle S_1, G_1^\lambda \right\rangle, \left\langle S_2, G_2^\lambda \right\rangle, ..., \left\langle S_{T-1}, G_{T-1}^\lambda \right\rangle$$

- Forward view linear TD(λ)

$$\begin{aligned}\Delta \mathbf{w} &= \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ &= \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)\end{aligned}$$

- Backward view linear TD(λ)

$$\begin{aligned}\delta_t &= R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}) \\ E_t &= \gamma\lambda E_{t-1} + \mathbf{x}(S_t) \\ \Delta \mathbf{w} &= \alpha \delta_t E_t\end{aligned}$$

Forward view and backward view linear TD(λ) are equivalent

Eligibility Trace for Function Approximation

- General TD learning associates eligibility values with states whose value functions get updated continuously
- Eligibility trace in Function Approximation methods associates eligibility to parameters which gets periodically updated
- Similarty of Eligibility value with Gradients:
  - Eligibility of a state signifies how much a state is responsible for the obtained final reward
  - Gradients (partial derivatives of predicted value with respect to parameters) indicate how much a particular parameter is responsible for the predicted output
  - This similarity is used by replacing the usual way of incrementing eligibility by 1, by an accumulation of gradients

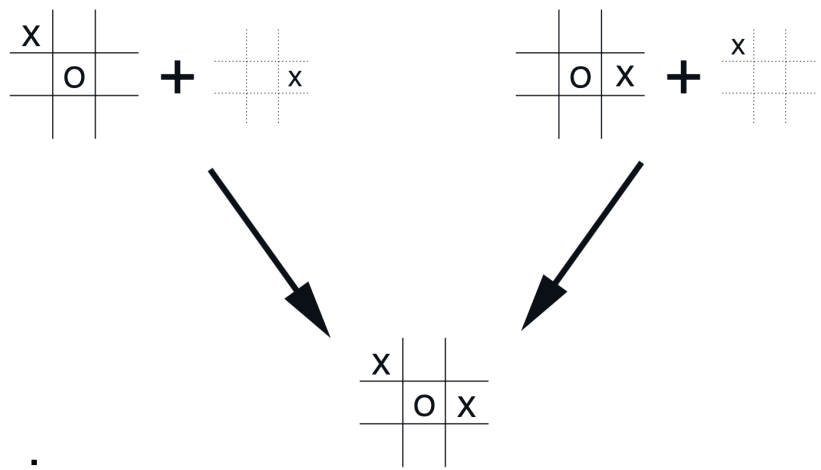
$$\begin{aligned} \delta_t &= R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}) \\ E_t &= \gamma \lambda E_{t-1} + \nabla_{\mathbf{w}} \hat{v}(S_t, \mathbf{w}) \\ \Delta \mathbf{w} &= \alpha \delta_t E_t \end{aligned}$$

Afterstates

Generalization with *large or continuous* action space

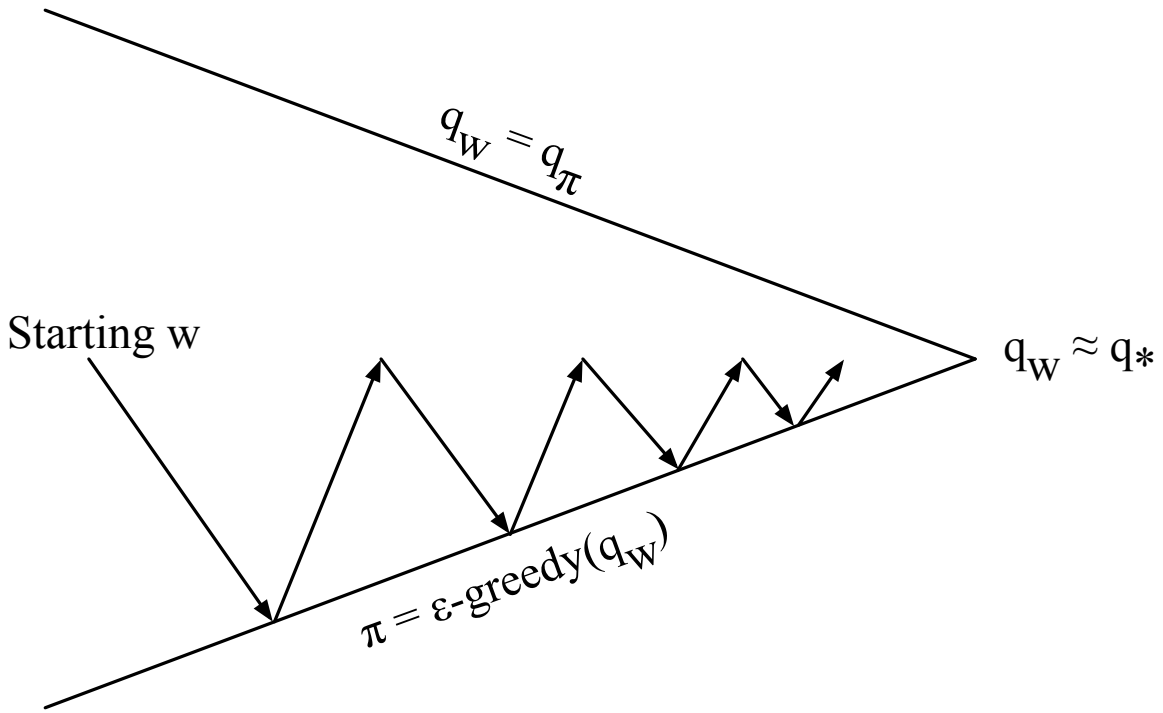
- Afterstate concept is based on separating stochasticity of states and deterministic nature of agent's moves.

- Example: The tic-tac-toe gameplay shown on the right indicates that position and move pairs are different but produce same "afterposition".



- Approximate value function learning:
  - A conventional action value function separately asses both pairs, whereas an afterstate value function asses both equally
  - Instead of learning action value over (s, a), state value function over afterstates are learned

Control with Value Function Approximation



Policy evaluation Approximate policy evaluation, $\hat{q}(\cdot, \cdot, \mathbf{w}) \approx q_\pi$

Policy improvement ϵ-greedy policy improvement

Action-Value Function Approximation

- Approximate the action-value function

$$\hat{q}(S, A, \mathbf{w}) \approx q_\pi(S, A)$$

- Minimise mean-squared error between approximate action-value fn $\hat{q}(S, A, \mathbf{w})$ and true action-value fn $q_\pi(S, A)$

$$J(\mathbf{w}) = \mathbb{E}_\pi \left[(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))^2 \right]$$

- Use stochastic gradient descent to find a local minimum

$$-\frac{1}{2} \nabla_{\mathbf{w}} J(\mathbf{w}) = (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})) \nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

$$\Delta \mathbf{w} = \alpha (q_\pi(S, A) - \hat{q}(S, A, \mathbf{w})) \nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w})$$

Linear Action-Value Function Approximation

- Represent state *and* action by a *feature vector*

$$\mathbf{x}(S, A) = \begin{pmatrix} \mathbf{x}_1(S, A) \\ \vdots \\ \mathbf{x}_n(S, A) \end{pmatrix}$$

- Represent action-value fn by linear combination of features

$$\hat{q}(S, A, \mathbf{w}) = \mathbf{x}(S, A)^\top \mathbf{w} = \sum_{j=1}^{n} \mathbf{x}_j(S, A)\mathbf{w}_j$$

- Stochastic gradient descent update

$$\nabla_{\mathbf{w}} \hat{q}(S, A, \mathbf{w}) = \mathbf{x}(S, A)$$
$$\Delta \mathbf{w} = \alpha(q_\pi(S, A) - \hat{q}(S, A, \mathbf{w}))\mathbf{x}(S, A)$$

Incremental Control Algorithms

- Like prediction, we must substitute a *target* for $q_\pi(S, A)$
  - For MC, the target is the return G_t

$$\Delta \mathbf{w} = \alpha(G_t - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{q}(S_t, A_t, \mathbf{w})$$

  - For TD(0), the target is the TD target $R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$

$$\Delta \mathbf{w} = \alpha(R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{q}(S_t, A_t, \mathbf{w})$$

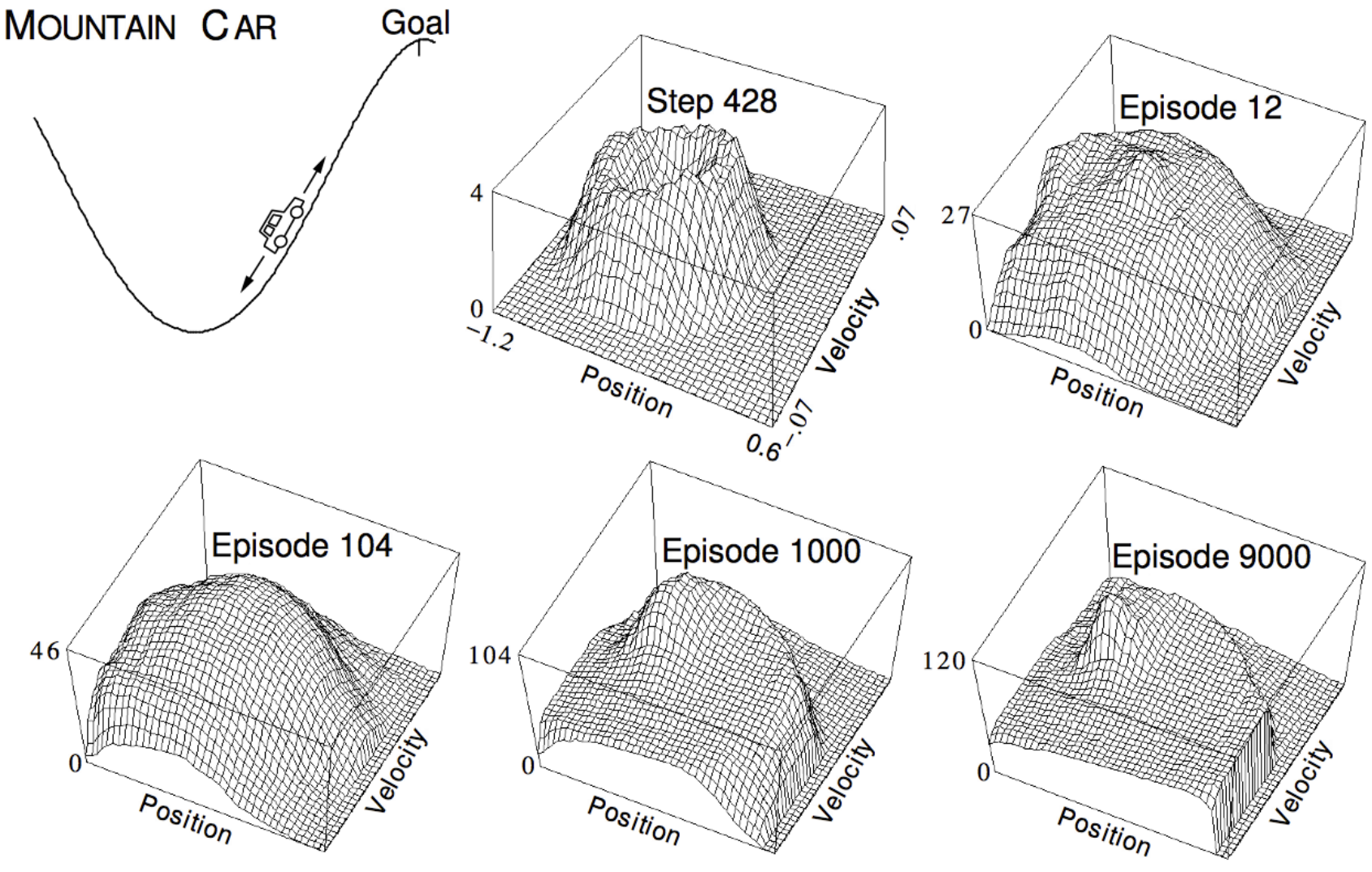
  - For forward-view TD(λ), target is the action-value λ-return

$$\Delta \mathbf{w} = \alpha(q_t^\lambda - \hat{q}(S_t, A_t, \mathbf{w}))\nabla_{\mathbf{w}} \hat{q}(S_t, A_t, \mathbf{w})$$

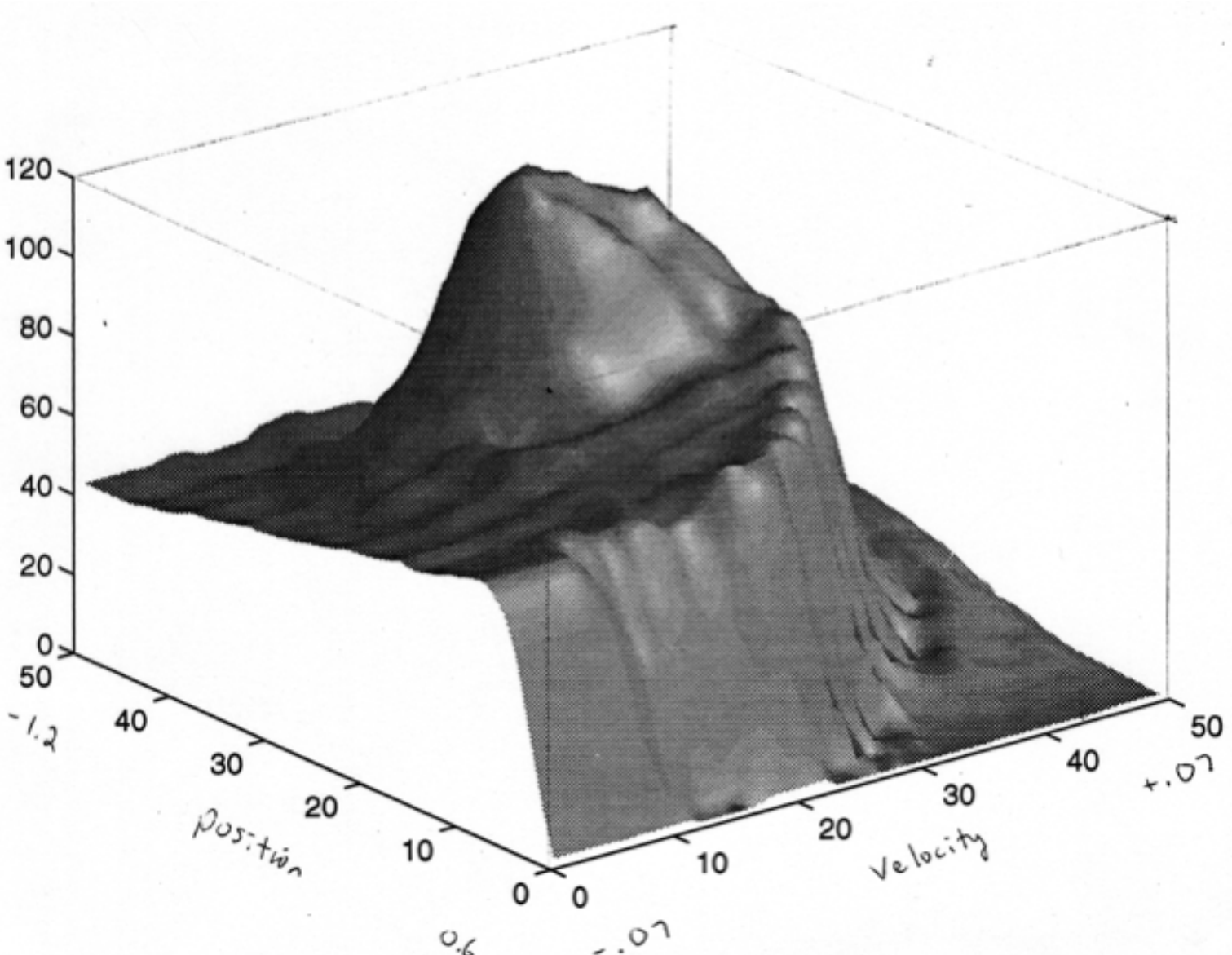
  - For backward-view TD(λ), equivalent update is

$$\begin{aligned}
\delta_t &= R_{t+1} + \gamma \hat{q}(S_{t+1}, A_{t+1}, \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}) \\
E_t &= \gamma \lambda E_{t-1} + \nabla_{\mathbf{w}} \hat{q}(S_t, A_t, \mathbf{w}) \\
\Delta \mathbf{w} &= \alpha \delta_t E_t
\end{aligned}$$

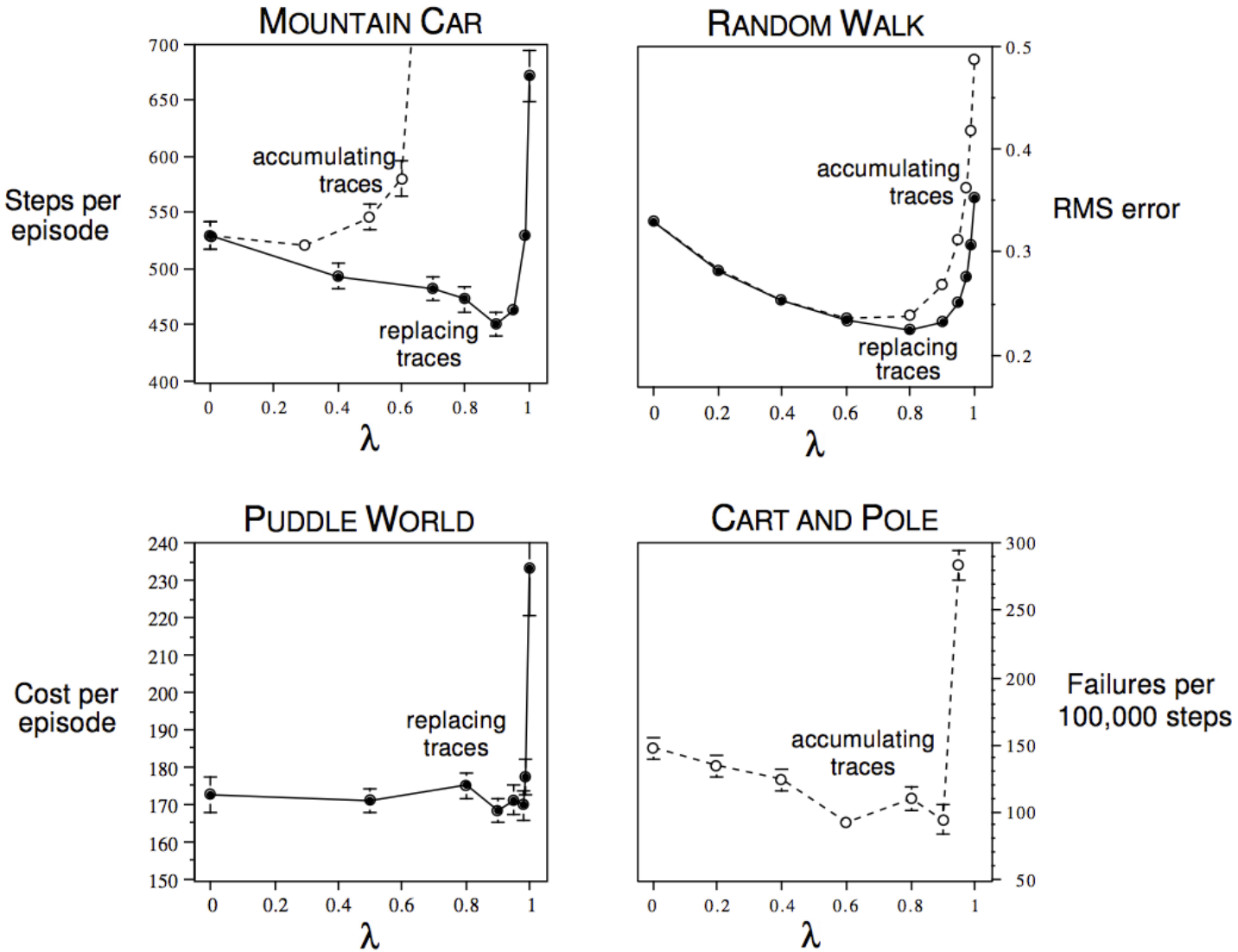
Linear Sarsa with Coarse Coding in Mountain Car



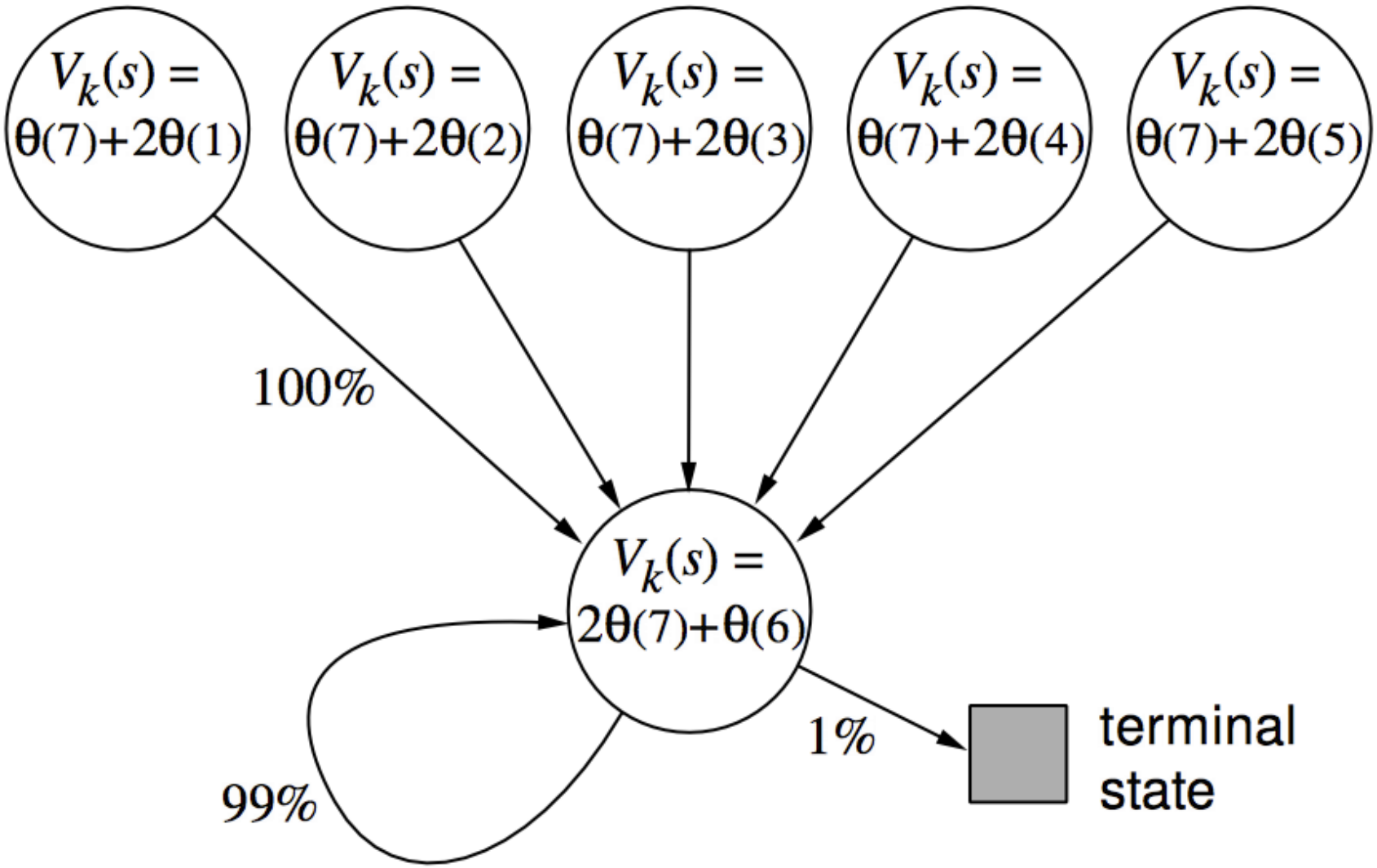
Linear Sarsa with Radial Basis Functions in Mountain Car



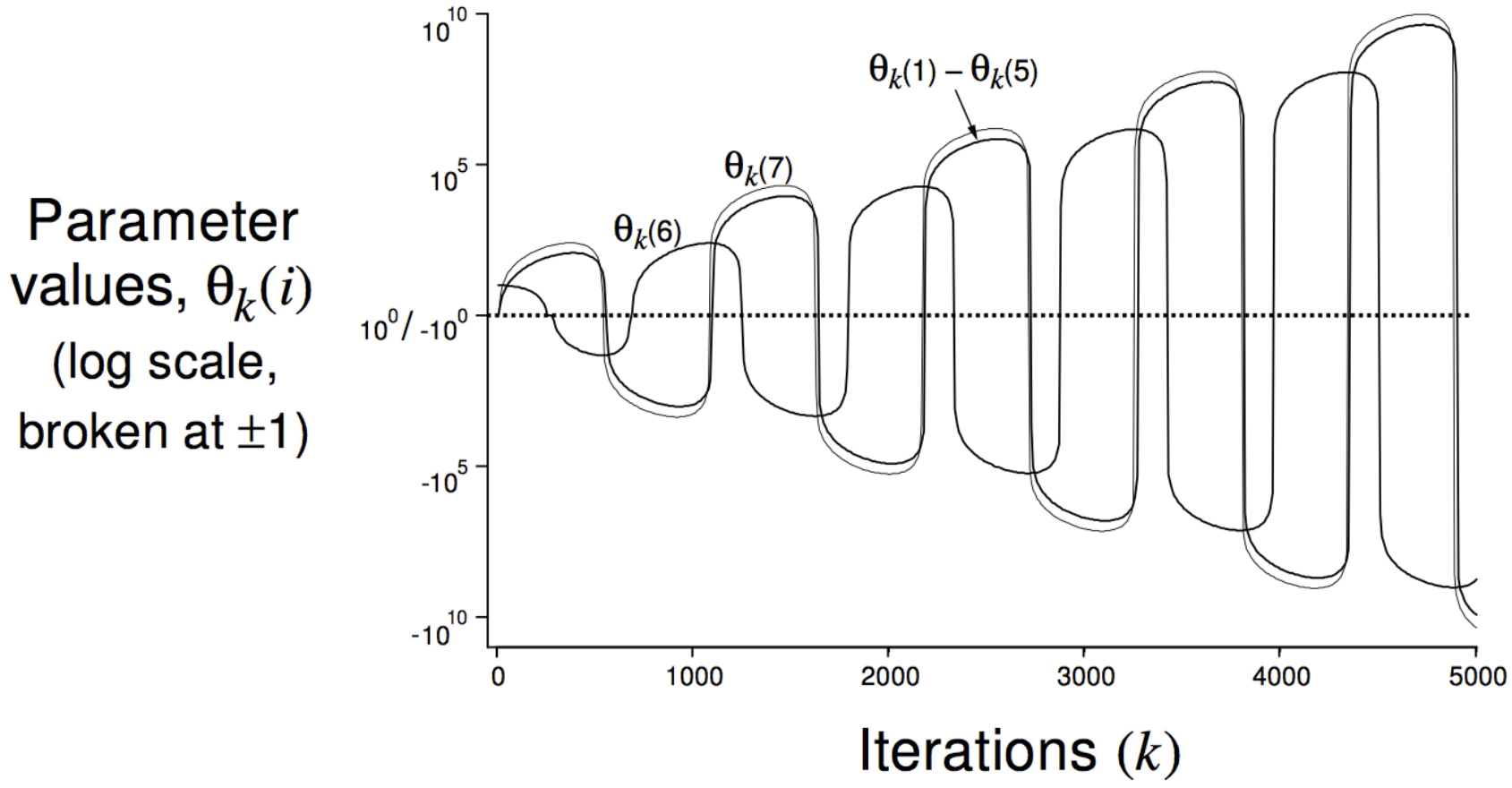
Study of λ: Should We Bootstrap?



Baird's Counterexample



Parameter Divergence in Baird's Counterexample



Convergence of Prediction Algorithms

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	TD(0)	✓	✓	✗
	TD(λ)	✓	✓	✗
Off-Policy	MC	✓	✓	✓
	TD(0)	✓	✗	✗
	TD(λ)	✓	✗	✗

Gradient Temporal-Difference Learning

- TD does not follow the gradient of *any* objective function
- This is why TD can diverge when off-policy or using non-linear function approximation
- Gradient TD follows true gradient of projected Bellman error

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	TD	✓	✓	✗
	Gradient TD	✓	✓	✓
Off-Policy	MC	✓	✓	✓
	TD	✓	✗	✗
	Gradient TD	✓	✓	✓

Convergence of Control Algorithms

Algorithm	Table Lookup	Linear	Non-Linear
Monte-Carlo Control	✓	(✓)	✗
Sarsa	✓	(✓)	✗
Q-learning	✓	✗	✗
Gradient Q-learning	✓	✓	✗

(✓) = chatters around near-optimal value function

Outline

Batch Reinforcement Learning

- Gradient descent is simple and appealing
- But it is *not* sample efficient
- Batch methods seek to find the best fitting value function
- Given the agent's experience ("training data")

Least Squares Prediction

- Given value function approximation $\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$
- And *experience* $\mathcal{D}$ consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

- Which parameters $\mathbf{w}$ give the *best fitting* value fn $\hat{v}(s, \mathbf{w})$?
- Least squares algorithms find parameter vector $\mathbf{w}$ minimising sum-squared error between $\hat{v}(s_t, \mathbf{w})$ and target values v_t^π,

$$\begin{aligned} LS(\mathbf{w}) &= \sum_{t=1}^{T} (v_t^\pi - \hat{v}(s_t, \mathbf{w}))^2 \\ &= \mathbb{E}_{\mathcal{D}} \left[(v^\pi - \hat{v}(s, \mathbf{w}))^2 \right] \end{aligned}$$

Stochastic Gradient Descent with Experience Replay

Given experience consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

Repeat:

1. Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

2. Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

Stochastic Gradient Descent with Experience Replay

Given experience consisting of $\langle state, value \rangle$ pairs

$$\mathcal{D} = \{\langle s_1, v_1^\pi \rangle, \langle s_2, v_2^\pi \rangle, ..., \langle s_T, v_T^\pi \rangle\}$$

Repeat:

1. Sample state, value from experience

$$\langle s, v^\pi \rangle \sim \mathcal{D}$$

2. Apply stochastic gradient descent update

$$\Delta \mathbf{w} = \alpha(v^\pi - \hat{v}(s, \mathbf{w}))\nabla_{\mathbf{w}} \hat{v}(s, \mathbf{w})$$

Converges to least squares solution

$$\mathbf{w}^\pi = \underset{\mathbf{w}}{\text{argmin}}\ LS(\mathbf{w})$$

Linear Least Squares Prediction

- Experience replay finds least squares solution
- But it may take many iterations
- Using *linear* value function approximation $\hat{v}(s, \mathbf{w}) = \mathbf{x}(s)^\top \mathbf{w}$
- We can solve the least squares solution directly

Linear Least Squares Prediction (2)

- At minimum of $LS(\mathbf{w})$, the expected update must be zero

$$\mathbb{E}_{\mathcal{D}}\left[\Delta\mathbf{w}\right] = 0$$

$$\alpha \sum_{t=1}^{T} \mathbf{x}(s_t)(v_t^{\pi} - \mathbf{x}(s_t)^{\top}\mathbf{w}) = 0$$

$$\sum_{t=1}^{T} \mathbf{x}(s_t) v_t^{\pi} = \sum_{t=1}^{T} \mathbf{x}(s_t)\mathbf{x}(s_t)^{\top}\mathbf{w}$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(s_t)\mathbf{x}(s_t)^{\top}\right)^{-1} \sum_{t=1}^{T} \mathbf{x}(s_t) v_t^{\pi}$$

- For N features, direct solution time is $O(N^3)$
- Incremental solution time is $O(N^2)$ using Shermann-Morrison

Linear Least Squares Prediction Algorithms

- We do not know true values v_t^π
- In practice, our "training data" must use noisy or biased samples of v_t^π

LSMC	Least Squares Monte-Carlo uses return $v_t^\pi \approx G_t$
LSTD	Least Squares Temporal-Difference uses TD target $v_t^\pi \approx R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w})$
LSTD(λ)	Least Squares TD(λ) uses λ-return $v_t^\pi \approx G_t^\lambda$

- In each case solve directly for fixed point of MC / TD / TD(λ)

Linear Least Squares Prediction Algorithms (2)

LSMC

$$0 = \sum_{t=1}^{T} \alpha(G_t - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(S_t)\mathbf{x}(S_t)^\top\right)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t)G_t$$

LSTD

$$0 = \sum_{t=1}^{T} \alpha(R_{t+1} + \gamma\hat{v}(S_{t+1}, \mathbf{w}) - \hat{v}(S_t, \mathbf{w}))\mathbf{x}(S_t)$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(S_t)(\mathbf{x}(S_t) - \gamma\mathbf{x}(S_{t+1}))^\top\right)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t)R_{t+1}$$

LSTD(λ)

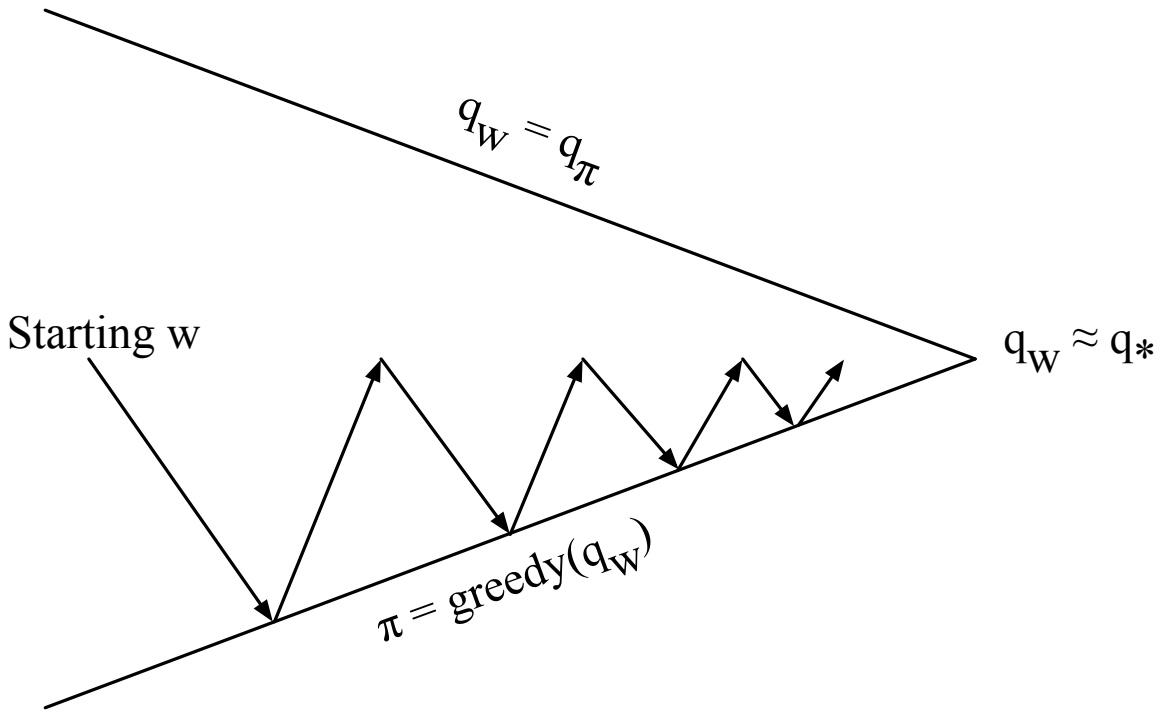
$$0 = \sum_{t=1}^{T} \alpha\delta_t E_t$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} E_t(\mathbf{x}(S_t) - \gamma\mathbf{x}(S_{t+1}))^\top\right)^{-1} \sum_{t=1}^{T} E_t R_{t+1}$$

Convergence of Linear Least Squares Prediction Algorithms

On/Off-Policy	Algorithm	Table Lookup	Linear	Non-Linear
On-Policy	MC	✓	✓	✓
	LSMC	✓	✓	-
	TD	✓	✓	✗
	LSTD	✓	✓	-
Off-Policy	MC	✓	✓	✓
	LSMC	✓	✓	-
	TD	✓	✗	✗
	LSTD	✓	✓	-

Least Squares Policy Iteration



Policy evaluation Policy evaluation by least squares Q-learning

Policy improvement Greedy policy improvement

Least Squares Action-Value Function Approximation

- Approximate action-value function $q_\pi(s, a)$
- using linear combination of features $\mathbf{x}(s, a)$

$$\hat{q}(s, a, \mathbf{w}) = \mathbf{x}(s, a)^\top \mathbf{w} \approx q_\pi(s, a)$$

- Minimise least squares error between $\hat{q}(s, a, \mathbf{w})$ and $q_\pi(s, a)$
- from experience generated using policy π
- consisting of $\langle (state, action), value \rangle$ pairs

$$\mathcal{D} = \{\langle (s_1, a_1), v_1^\pi \rangle, \langle (s_2, a_2), v_2^\pi \rangle, ..., \langle (s_T, a_T), v_T^\pi \rangle\}$$

Least Squares Control

- For policy evaluation, we want to efficiently use all experience
- For control, we also want to improve the policy
- This experience is generated from many policies
- So to evaluate $q_\pi(S, A)$ we must learn off-policy
- We use the same idea as Q-learning:
  - Use experience generated by old policy
    $S_t, A_t, R_{t+1}, S_{t+1} \sim \pi_{old}$
  - Consider alternative successor action $A' = \pi_{new}(S_{t+1})$
  - Update $\hat{q}(S_t, A_t, \mathbf{w})$ towards value of alternative action
    $R_{t+1} + \gamma \hat{q}(S_{t+1}, A', \mathbf{w}))$

Least Squares Q-Learning

- Consider the following linear Q-learning update

$$\delta = R_{t+1} + \gamma \hat{q}(S_{t+1}, \pi(S_{t+1}), \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w})$$
$$\Delta \mathbf{w} = \alpha \delta \mathbf{x}(S_t, A_t)$$

- LSTDQ algorithm: solve for total update = zero

$$0 = \sum_{t=1}^{T} \alpha(R_{t+1} + \gamma \hat{q}(S_{t+1}, \pi(S_{t+1}), \mathbf{w}) - \hat{q}(S_t, A_t, \mathbf{w}))\mathbf{x}(S_t, A_t)$$

$$\mathbf{w} = \left(\sum_{t=1}^{T} \mathbf{x}(S_t, A_t)(\mathbf{x}(S_t, A_t) - \gamma \mathbf{x}(S_{t+1}, \pi(S_{t+1})))^\top \right)^{-1} \sum_{t=1}^{T} \mathbf{x}(S_t, A_t) R_{t+1}$$

Least Squares Policy Iteration Algorithm

- The following pseudocode uses LSTDQ for policy evaluation
- It repeatedly re-evaluates experience $\mathcal{D}$ with different policies

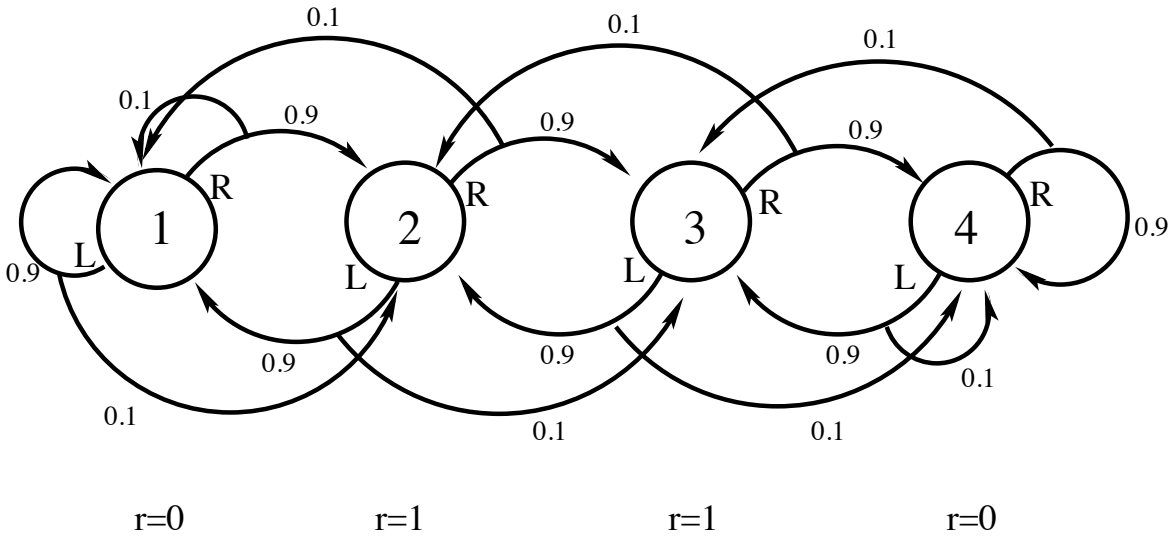
function LSPI-TD($\mathcal{D}, \pi_0$)
  $\pi' \leftarrow \pi_0$
  repeat
    $\pi \leftarrow \pi'$
    $Q \leftarrow$ **LSTDQ**($\pi, \mathcal{D}$)
    for all $s \in \mathcal{S}$ **do**
      $\pi'(s) \leftarrow \underset{a \in \mathcal{A}}{\text{argmax}}\ Q(s, a)$
    end for
  until $(\pi \approx \pi')$
  return π
end function

Convergence of Control Algorithms

Algorithm	Table Lookup	Linear	Non-Linear
Monte-Carlo Control	✓	(✓)	✗
Sarsa	✓	(✓)	✗
Q-learning	✓	✗	✗
LSPI	✓	(✓)	-

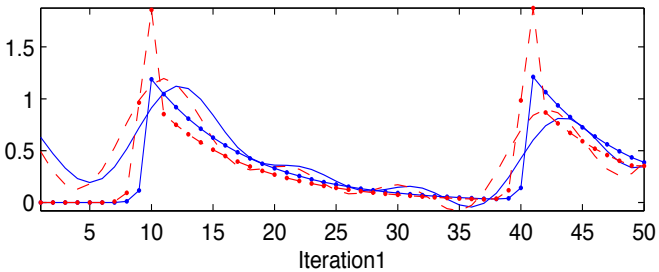
(✓) = chatters around near-optimal value function

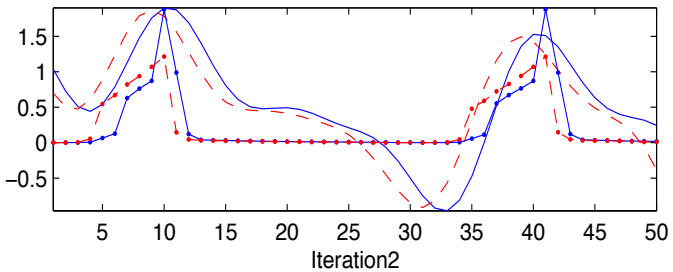
Chain Walk Example

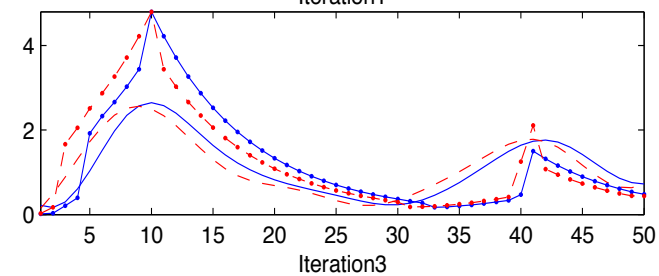


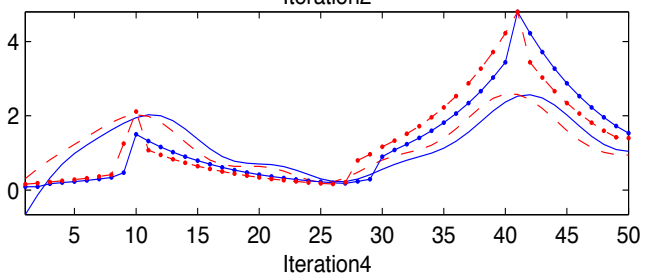
- Consider the 50 state version of this problem
- Reward $+1$ in states 10 and 41, 0 elsewhere
- Optimal policy: R (1-9), L (10-25), R (26-41), L (42, 50)
- Features: 10 evenly spaced Gaussians ($\sigma = 4$) for each action
- Experience: 10,000 steps from random walk policy

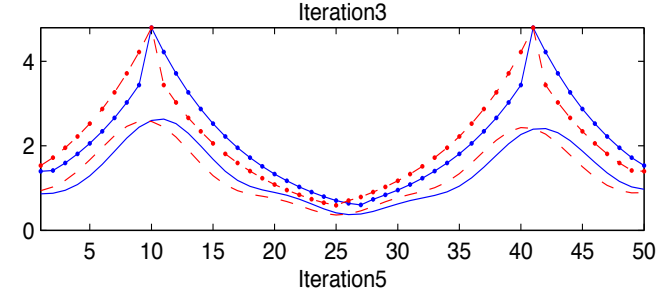
LSPI in Chain Walk: Action-Value Function

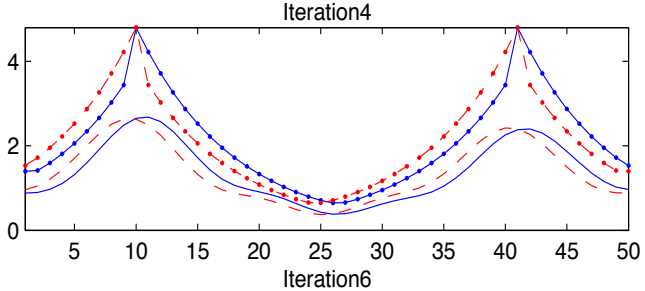


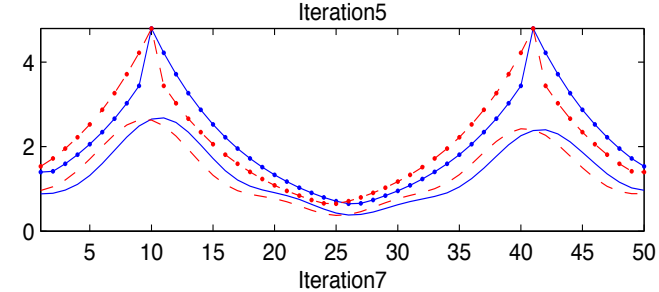




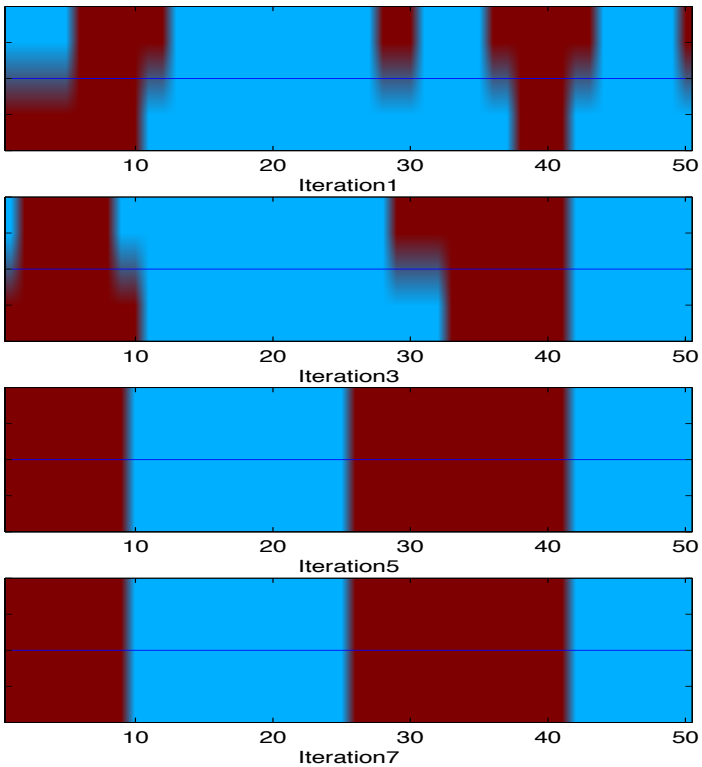








LSPI in Chain Walk: Policy



Thank You!

Questions?

The only stupid question is the one you were afraid to ask but never did!
– Rich Sutton