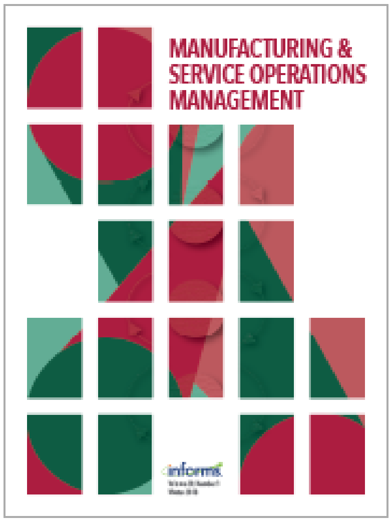


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# Data-Driven Distributionally Robust CVaR Portfolio Optimization Under A Regime-Switching Ambiguity Set

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**Abstract.** ***Problem definition*:** Nonstationarity of the random environment is a critical yet challenging concern in decision-making under uncertainty. We illustrate the challenge from the nonstationarity and the solution framework using the portfolio selection problem, a typical decision problem in a time-varying financial market. ***Methodology/Results*:** This paper models the nonstationarity by a regime-switching ambiguity set. In particular, we incorporate the time-varying feature of the stochastic environment into the traditional Wasserstein ambiguity set to build our regime-switching ambiguity set. This modeling framework has strong financial interpretations because the financial market is exposed to different economic cycles. We show that the proposed distributional optimization framework is computationally tractable. We further provide a general data-driven portfolio allocation framework based on a covariate-based estimation and a hidden Markov model. We prove that the approach can include the underlying distribution with a high probability when the sample size is larger than a quantitative bound, from which we further analyze the quality of the obtained portfolio. Extensive empirical studies are conducted to show that the proposed portfolio consistently outperforms the equally weighted portfolio (the $1/N$ strategy) and other benchmarks across both time and data sets. In particular, we show that the proposed portfolio exhibited a prompt response to the regime change in the 2008 financial crisis by reallocating the wealth into appropriate asset classes on account of the time-varying feature of our proposed model. ***Managerial implications*:** The proposed framework helps decision-makers hedge against time-varying uncertainties. Specifically, applying the proposed framework to portfolio selection problems helps investors respond promptly to the regime change in financial markets and adjust their portfolio allocation accordingly.

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**Keywords:** time-varying uncertainty • regime-switching ambiguity • hidden Markov model • portfolio selection

## 1. Introduction

A typical challenge in decision-making with uncertainty is the changing random environment. For instance, in e-commerce, demands can be highly nonstationary, changing with seasons or even the time of day. In supply chains, the high volatility of disruption results in a significant nonstationarity of the market during a disruption. The time-varying stochastic environment is especially relevant in the application of portfolio optimization because economic conditions change from time to time. Bridgewater Associates (2012) revealed that different asset classes do well in certain regimes and poorly in others. Before that, they created the All-Weather strategy in 1996, which allocated portfolios via the risk of each asset class, known as risk parity. Various funds, such as the Canadian Pension Fund, gradually recognized the benefits of asset allocation via risky parity and have adopted this strategy since then. In this paper, we focus on the portfolio selection problem to illustrate the challenge of the time-varying uncertainties and our solution framework.

Modern portfolio theory, pioneered by Markowitz (1952), is central to finance and is the building block of many economic models. Portfolio optimization is the process of looking for the optimal balance between the reward and risk of the portfolio, which is traditionally measured by the mean and the variance of the portfolio's return. The inferior out-of-sample performance of the mean-variance portfolio (see Michaud 1989, DeMiguel et al. 2009) attracts many proposals of remedies to the estimation error, such as the robust control approach (see Hansen et al. 2006, Fouque et al. 2016, Pun 2021) and the use of alternative risk measures. Value-at-risk (VaR) and conditional value-at-risk (CVaR) are two popular quantile-based risk measures in the finance industry, and they are widely studied in many other fields, such as engineering and economics. However, CVaR is considered a better risk measure than VaR for portfolio optimization because of its coherent property. In addition, CVaR can be expressed as a minimization formula while preserving convexity (see Rockafellar and Uryasev 2002). However, the inaccurate estimations of the returns' distribution would produce a poor out-of-sample performance of the constructed portfolio (Litterman 2003).

To overcome the risk of inaccurate estimations because of limited historical data, distributionally robust optimization (DRO) approaches are deployed. DRO allows the decision maker to optimize against the worst-case distribution over a set of distributions with only partial statistical information inferred from the available data. We name the set of distributions an ambiguity set. Specifically, a statistical-distance-based ambiguity set has been widely studied recently. This type of ambiguity set is constructed to contain all the distributions centering around a nominal distribution (center) within a certain distance threshold, where the distance can be measured by various metrics, such as $\phi-$divergence and the Wasserstein metric, that are of interest in this paper. In the application of portfolio allocation, this center is often chosen as the empirical distribution (see Pflug et al. 2012, Blanchet et al. 2022), which converges to the true distribution as the sample goes to infinity if the returns of assets are assumed to be i.i.d. across time. However, this assumption is often violated in real applications because of the nonstationary nature of the decision-making environment.

Various data-driven optimization approaches are proposed to cope with the nonstationary nature of the decision-making environment. One stream of literature considers a set of distributions across time to capture the possible evolution of uncertainties. Among them, Markov-chain-based models are favored in capturing the evolution of uncertainties (see Zhou et al. 2021, Li et al. 2021). Our paper models the time-varying uncertainties using a regime-switching ambiguity set. In particular, the distribution of the uncertain return depends on different regimes, which correspond to different market conditions. Regimes evolve over time following a Markov chain. Under each regime, a Wasserstein-distance-based ambiguity set is constructed to capture a family of distributions that are close to the empirical estimates of returns. We propose two approaches to estimate the ambiguity set: (1) predetermining the regimes using covariates (such as macroeconomic variables) and using frequency to estimate the transition probabilities and (2) applying a hidden Markov model (HMM) to automatically calibrate regimes and transition probabilities from data. To the best of our knowledge, this paper is the first to derive a general data-driven framework for regime-switching models in a DRO setting.

We summarize the contributions of this paper as follows. First, we generalize the white paper of Bridgewater Associates (2012) to divide the financial market into four different economic conditions and then estimate regimes and the corresponding transition probabilities matrix using two methods: (1) predetermining regimes from covariates and estimating transition probabilities using the frequency of occurrence from one regime to another and (2) HMM. Subsequently, we incorporate the regime-switching nature of asset returns into the distributionally robust portfolio optimization problem. We further provide a tractable reformulation for the DRO problem. Second, we prove that the proposed ambiguity set can include the underlying distribution with a high probability when the sample size and ambiguity aversion level (which determines the size of the ambiguity set) are larger than a quantitative bound. We further analyze the quality of the obtained portfolio from our data-driven approach. Finally, by conducting extensive empirical studies with the data sets in DeMiguel et al. (2009) and a data set with various asset classes, we show that the portfolio from our approach performs stably and consistently beats the equally weighted portfolio in terms of Sharpe ratio. In addition, we numerically show that our proposed formulation can guide investors to shift their portfolio weightings to less risky assets to optimize their risk profiles, particularly in years such as the 2008–2009 great recession.

The remainder of this paper is organized as follows. Section 2 reviews the related works. Section 3 introduces our distributionally robust portfolio optimization model. Specifically, we formulate the distributionally robust portfolio optimization model under a regime-switching ambiguity set and present its tractable reformulations. In Section 4, we introduce our data-driven portfolio allocation approach. In particular, two methods are proposed to determine the financial regimes and estimate the corresponding transition probabilities. We further analyze the statistical performance guarantees of the proposed data-driven methods. Section 5 conducts numerical studies with extensive empirical data sets to compare the out-of-sample performance of the proposed portfolio with other benchmarks. Finally, Section 6 concludes the paper.

Throughout this paper, we use the following notations. For a vector $\mathbf{a}$ or a matrix $A$, its transpose is denoted by $\mathbf{a}'$ (or $A'$). We denote by $\mathbf{1}$ the vector of ones with appropriate dimensions. For $\boldsymbol{a} = (a_1, \ldots, a_p)' \in \mathbb{R}^p$ and a positive integer $m$, we denote by $\ell_m$-norm $\|\boldsymbol{a}\|_m = \left(\sum_{j=1}^{p} |a_j|^m\right)^{1/m}$. For an integer $M$, $[M] := \{1, \ldots, M\}$ denotes the index set from 1 to $M$. Let $\mathcal{P}(\mathcal{H})$ be the space of Borel probability measures supported on $\mathcal{H}$, where $\mathcal{H}$ could be a product space. We denote the $(a, \boldsymbol{b}) \in \Omega_{soc}$ as the second-order conic constraint, that is, $\|\boldsymbol{b}\|_2^2 \le a^2$. We use $\delta_x$ to represent Dirac measure on the single point $x$ and $\mathbb{I}_{\{\cdot\}}$ to denote the indicator function. For two distributions $\mu$ and $\nu$, we denote $\mu \otimes \nu$ as their product measure. We denote $f(n) = \Theta(g(n))$ such that $\exists k_1, k_2 > 0$ and $\exists n_0$ such that $\forall n > n_0$, $k_1 g(n) \le f(n) \le k_2 g(n)$. Besides, $X_n = O_p(a_n)$ as $n \to \infty$ means that $\forall \varepsilon > 0$, $\exists M, N > 0$ such that $\Pr(|X_n/a_n| > M) \le \varepsilon$, $\forall n > N$.

## 2. Related Works

In this section, we briefly review the works related to our study from three aspects, namely distributionally robust optimization, optimization under time-varying uncertainties, and regime-switching models.

### 2.1. Distributionally Robust Optimization

Distributionally robust optimization has received wide attention over the past decades. Different forms of ambiguity sets are developed. Delage and Ye (2010) and Wiesemann et al. (2014) examined a moment-based ambiguity set. They provided tractable reformulations for various DRO problems and showed that the model structure of the portfolio optimization admits an analytical solution for its optimal decision. Liu et al. (2019) and Kang et al. (2019) investigated tractable reformulations for portfolio optimization under various moment-based ambiguity sets using the worst-case CVaR as a risk measure. However, using a moment-based ambiguity set may lead to a too-conservative portfolio allocation decision.

Recently, many researchers have focused on studying a statistical-distance-based ambiguity set. Some commonly used distance measures include the $\phi$-divergence (Ben-Tal et al. 2013) and Wasserstein distance (Esfahani and Kuhn 2018). We refer readers to a more comprehensive discussion in Rahimian and Mehrotra (2019). Wasserstein ambiguity sets are known to yield a higher chance of nesting the true probability distribution and thus achieve a better out-of-sample performance guarantee. As a result, they are widely used in real applications of portfolio optimization models (see Esfahani and Kuhn 2018, Blanchet et al. 2022). Our paper uses the Wasserstein ambiguity set in designing the portfolio allocation policy.

A recent inspiring work by Chen et al. (2020) provided a generic framework for a large family of ambiguity sets. They propose a *robust stochastic optimization* framework based on a scenario-wise ambiguity set, which is flexible in recreating various types of ambiguity sets in the literature, including the Wasserstein ambiguity set that is of interest in this paper. Specifically, their scenario-wise ambiguity set is formulated as follows:

$$\bar{\mathcal{U}}(\theta) := \left\{ \mathbb{P} \in \mathcal{P}(\mathbb{R}^I \times [N]) \;\middle|\; \begin{array}{l} (\tilde{\boldsymbol{r}}, \tilde{n}) \sim \mathbb{P} \\ \mathbb{E}^{\mathbb{P}}[\tilde{\boldsymbol{r}} \mid \tilde{n} = n] \in \Omega_n, \quad \forall n \in [N] \\ \mathbb{P}[\tilde{\boldsymbol{r}} \in \mathcal{Z}_n \mid \tilde{n} = n] = 1, \quad \forall n \in [N] \\ \mathbb{P}[\tilde{n} = n] = p_n, \quad \forall n \in [N] \end{array} \right\} \tag{1}$$

where $\tilde{\boldsymbol{r}}$ represents a random vector, $\tilde{n}$ indicates a set of random scenarios, and $p_n$ denotes the probability of each scenario $n$. $\Omega_n$ is defined as a closed and convex set that can contain some moment conditions, and $\mathcal{Z}_n$ specifies the support of $\tilde{\boldsymbol{r}}$. They provide several applications of the ambiguity set to show how it recreates the existing ambiguity sets in the literature. In particular, by defining each scenario as an observation, denoted by $\boldsymbol{r}_n$, and the support of the random vector under each scenario as $\mathcal{Z}_n = \{\tilde{\boldsymbol{r}} \mid \rho(\tilde{\boldsymbol{r}}, \boldsymbol{r}_n) \le \theta\}$, where $\rho(\cdot,\cdot)$ denotes the cost function in the Wasserstein metric, the ambiguity set is reduced to a Wasserstein ambiguity set. This paper builds on their framework and incorporates the regime-switching pattern in the ambiguity set to cope with the nonstationarity.

### 2.2. Optimization Under Time-Varying Uncertainty

Traditional DRO models often assume that the distribution of uncertainties is time invariant. However, time-varying uncertainties are a common feature in practice, especially in financial problems. Ignoring the time-varying feature of uncertainties in a decision-making model may deteriorate the solution's (out-of-sample) performance.

Recently, some work has started modeling the uncertainty across time as a dependent structure, specifically through some Markov chains. Dou and Anitescu (2019) studied an optimization model where the uncertain parameters are sampled from a special type of time series, vector autoregression process (VAR). They obtained a tractable DRO reformulation for the problem and provided an out-of-sample guarantee for their method. In contrast, our paper models the time-varying uncertainties from a Markov chain perspective. Sutter et al. (2020) and Li et al. (2021) studied an optimization model where the uncertain parameters are sampled from a finite-state Markov chain to model the time-varying behavior. They prove that their data-driven DRO model at any given decision converges almost surely to its true expected loss as the sample size goes to infinity. Moreover, they also showed that the solution from their DRO model can generate the least possible out-of-sample risk among the vast class of approaches. In contrast, the random samples (returns) in our model are governed by a Markov chain (for regimes) rather than directly sampled from a Markov chain; that is, the random return's distribution is determined by regimes that evolve across time according to a Markov chain. In addition, we can establish a finite-sample bound for the ambiguity level to cover the true distribution with a high probability under mild assumptions. We also theoretically analyze the quality of the obtained solution from our data-driven DRO model. Yang (2017) and Ramani and Ghate (2022) modeled the time-varying uncertainties in a Markov decision process with state-dependent rewards. States evolve over time; hence, the (uncertain) rewards' distributions vary across time. This problem setting is similar to ours but in a multistage setting. However, their estimation of the ambiguity set requires multiple i.i.d. sample trajectories. Nevertheless, in many data sets, we can observe only one trajectory of historical data. In summary, we model the time-varying uncertainties by regime-dependent returns, where the regime evolves over time according to a Markov chain. We provide a tractable reformulation for our DRO model with time-varying uncertainties and establish a nonasymptotic performance guarantee for the proposed data-driven approach.

Another related stream of literature is to study the estimation of uncertainties from exogenous covariates (see Bertsimas and Van Parys 2022, and Esteban-Pérez and Morales 2022). They assume that the covariates are exogenous and i.i.d. across time. The uncertainties' distribution could vary, depending on different realizations of the covariates, which leads to time-varying uncertainties. In contrast, we assume that the covariates are sampled from a Markov chain rather than i.i.d across time. In particular, we use the market state and macroeconomic variables as covariates in predicting regimes that determine the returns' distribution.

In summary, this paper contributes to the optimization under time-varying uncertainties from two perspectives. From the modeling perspective, we exploit one specific type of nonstationarity of the time series data by assuming that the uncertainties (e.g., uncertain returns) generating the data depend on each regime, which may switch from one to another across time according to a Markov chain. We then incorporate the evolution of latent uncertainty sources (such as macroeconomics variables) when using them as covariates to predict the distribution of the uncertain returns in different regimes. From the theoretical perspective, we show that our proposed ambiguity sets can include the underlying distribution of interest with high probability and enjoy tractable reformulations and practical interpretations.

### 2.3. Regime Switching Models

In the finance literature, regime-switching models have been studied in modeling stock returns in the context of economic modeling (Hamilton 1989, 2010). Evidenced by the economic observations (see Tu 2010 and Chang et al. 2017), the parameters of the return process are regime switching. Tu (2010) investigated the economic value of accounting for regimes in making portfolio allocation decisions and pointed out that ignoring the regime-switching feature will lead to significant certainty-equivalent losses. Subsequent studies examined how to optimize the portfolio allocation decision in a regime-switching market and demonstrated the improvement made by incorporating the regime-switching behavior (Bae et al. 2014). Many variants of portfolio models have also been investigated under the regime-switching pattern by either directly observing the history states or inferring states through HMM (Guidolin and Timmermann 2007, Fu et al. 2014, Nystrup et al. 2017, Costa et al. 2019, Costa and Kwon 2020, Oprisor and Kwon 2021). However, most of them directly use the estimated parameters from the regime-switching model as input in their portfolio optimization model, which does not consider the estimation error of these parameters with finite samples. In contrast, we introduce a variant of Wasserstein ambiguity sets to capture the regime-switching pattern and theoretically analyze the effect of estimation errors on the portfolio's performance. In particular, we show that our constructed ambiguity set with a certain ambiguity aversion level can contain the true distribution of interest with a high probability under mild assumptions. Under the same ambiguity aversion level, we can bound the risk of the obtained portfolio. To the best of our knowledge, this paper is the first to examine portfolio optimization in a regime-switching market from the DRO perspective.

## 3. Distributionally Robust Portfolio Optimization with Regime-Switching Ambiguity

In this section, we first model the distributionally robust portfolio optimization problem and introduce our regime-switching ambiguity set. Next, we provide a tractable reformulation of the DRO model and discuss the relation between our optimal solution and the equally weighted portfolio, a widely studied portfolio in the literature.

### 3.1. Preliminaries

**3.1.1. Wasserstein Ambiguity Set.** The Wasserstein ambiguity set is representable by restricting the probability distributions to a certain statistical distance of a reference distribution, such as the empirical probability distribution (denoted by $\hat{\mathbb{P}}_e$). To define a classical Wasserstein ambiguity set, we first introduce the following definitions. For a lower-semicontinuous cost function $\rho: \mathbb{R}^I \times \mathbb{R}^I \longmapsto [0, \infty)$ with $\rho(\boldsymbol{u}, \boldsymbol{u}) = 0$ for any $\boldsymbol{u} \in \mathbb{R}^I$, the Wasserstein distance between two probability distributions

$\mathbb{P}$ and $\mathbb{Q}$ is defined as

$$W(\mathbb{P},\mathbb{Q}) := \inf_{\nu}\{\mathbb{E}^{\nu}[\rho(\boldsymbol{p},\boldsymbol{q})] \mid \nu \in \mathcal{P}(\mathbb{R}^I \times \mathbb{R}^I), \Pi_{\boldsymbol{p}}\nu = \mathbb{P},\ \Pi_{\boldsymbol{q}}\nu = \mathbb{Q}\}, \quad (2)$$

where $\nu$ is the joint distribution of $\mathbb{P}$ and $\mathbb{Q}$, and $\Pi_{\boldsymbol{p}}\nu$ and $\Pi_{\boldsymbol{q}}\nu$ are the projections of $\nu$ over $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. The metric (2) refers to the optimal transportation cost of moving the mass from $\mathbb{P}$ into the mass of $\mathbb{Q}$ under a cost $\rho(\boldsymbol{p},\boldsymbol{q})$ per unit of mass transported from location $\boldsymbol{p}$ to location $\boldsymbol{q}$. This paper considers the cost function as $\ell_m$-norm: $\rho(\boldsymbol{p},\boldsymbol{q}) = \|\boldsymbol{p}-\boldsymbol{q}\|_m$, and in this case, (2) is known as the Wasserstein distance with $\ell_m$-norm, and we denote it by $W_m$. Define the ambiguity set as

$$\mathcal{U}_w(\theta) = \{\mathbb{P} \in \mathcal{P}(\mathbb{R}^I) \mid W_m(\mathbb{P},\hat{\mathbb{P}}) \le \theta\}, \quad (3)$$

which is a Wasserstein ambiguity set of radius $\theta$ centered at the reference distribution $\hat{\mathbb{P}}$. Along this line of work, the distinct feature of our paper is the incorporation of regime-switching behavior arising from the investing environment.

**3.1.2. CVaR Minimization Formula.** Consider a market with $I$ risky assets whose random return vector is represented by $\boldsymbol{r} = (r_1,\ldots,r_I)'$ for the specified investment horizon. The portfolio of the investor is characterized by the portfolio weight vector $\boldsymbol{x} = (x_1,\ldots,x_I)'$ with $\mathbf{1}'\boldsymbol{x} = 1$, where $x_i$ is the proportion of the investor wealth that is invested in the $i$-th asset at the beginning of the investment for $i = 1,\ldots,I$. In our framework, we consider the feasible portfolios that satisfy the general linear equality and inequality constraints, that is,

$$\boldsymbol{x} \in \mathcal{X} := \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ C\boldsymbol{x} \ge \boldsymbol{d}\},$$

where $A \in \mathbb{R}^{m\times I}$, $\boldsymbol{b} \in \mathbb{R}^m$, $C \in \mathbb{R}^{p\times I}$, and $\boldsymbol{d} \in \mathbb{R}^p$. Some realistic portfolio constraints, such as leverage or no-shorting constraints, can be incorporated into our framework. For notation simplicity, we embed the budget constraint $\mathbf{1}'\boldsymbol{x} = 1$ into $\mathcal{X}$ as a necessary constraint.

Given the portfolio $\boldsymbol{x}$, the portfolio's return is equal to $\boldsymbol{x}'\boldsymbol{r}$. Thus, we define the loss (function) of the investor as $f(\boldsymbol{x},\boldsymbol{r}) := -\boldsymbol{x}'\boldsymbol{r}$ and name it as $\mathbb{P}$. Assume that the random return vector $\boldsymbol{r}$ has a probability density function, $p(\boldsymbol{r})$. For a portfolio weight vector $\boldsymbol{x}$ and a constant $v$, the cumulative distribution function of the loss $f(\boldsymbol{x},\boldsymbol{r})$ is denoted by $\psi(\boldsymbol{x},v) := \int_{f(\boldsymbol{x},\boldsymbol{r})\le v} p(\boldsymbol{r})d\boldsymbol{r}$. For a confidence level $\eta \in (0,1)$, VaR is defined as $\mathrm{VaR}_\eta(\boldsymbol{x}) = \inf\{v \in \mathbb{R} \mid \psi(\boldsymbol{x},v) \ge \eta\}$, and the corresponding CVaR is defined as the conditional expectation of the loss $f(\boldsymbol{x},\boldsymbol{r})$ greater than $\mathrm{VaR}_\eta$:

$$\mathrm{CVaR}_\eta^{\mathbb{P}}(\boldsymbol{x}) = \mathbb{E}^{\mathbb{P}}[f(\boldsymbol{x},\boldsymbol{r}) \mid f(\boldsymbol{x},\boldsymbol{r}) \ge \mathrm{VaR}_\eta(\boldsymbol{x})] = \frac{1}{(1-\eta)}\int_{f(\boldsymbol{x},\boldsymbol{r})\ge \mathrm{VaR}_\eta(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{r})p(\boldsymbol{r})d\boldsymbol{r}. \quad (4)$$

As shown in Rockafellar and Uryasev (2002), the $\mathrm{VaR}_\eta(\boldsymbol{x})$ and $\mathrm{CVaR}_\eta(\boldsymbol{x})$ are the minimizer and the minimum value of a function $F_\eta(\boldsymbol{x},v)$ with respect to $v$ under $\boldsymbol{r} \sim \mathbb{P}$, respectively, where

$$F_\eta^{\mathbb{P}}(\boldsymbol{x},v) := v + \frac{1}{1-\eta}\mathbb{E}^{\mathbb{P}}[(f(\boldsymbol{x},\boldsymbol{r})-v)^+],$$

in which $g^+ := \max\{g,0\}$ is the positive part of $g$. It is shown that $F_\eta(\boldsymbol{x},v)$ is convex in $\boldsymbol{x}$ and $v$.

The stochastic CVaR portfolio optimization can be formulated as

$$\min_{\boldsymbol{x}\in\mathcal{X}} \mathrm{CVaR}_\eta^{\mathbb{P}}(\boldsymbol{x}) = \min_{\boldsymbol{x}\in\mathcal{X},\, v\in\mathbb{R}} F_\eta^{\mathbb{P}}(\boldsymbol{x},v) = \min_{\boldsymbol{x}\in\mathcal{X},\, v\in\mathbb{R}} \left\{v + \frac{1}{1-\eta}\mathbb{E}^{\mathbb{P}}[G(\boldsymbol{x},\boldsymbol{r})]\right\}, \quad (5)$$

where

$$G(\boldsymbol{x},\boldsymbol{r}) = \min_{y\in\mathcal{Y}} y \quad \text{with } \mathcal{Y} = \{y \mid y \ge -\boldsymbol{x}'\boldsymbol{r} - v,\ y \ge 0\}. \quad (6)$$

Given that the feasible set $\mathcal{X}$ is convex, the solution to (5) can be obtained by standard convex optimization algorithms (see Krokhmal et al. 2001).

## 3.2. Regime-Switching Ambiguity Set

In practice, the computation of CVaR or the expectation in (5) requires the knowledge of the return distribution. Traditionally, by using the historical return data of the length $N$, $\{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\}$, one may adopt the following empirical version of (5) as its proxy:

$$\min_{\boldsymbol{x}\in\mathcal{X},\, v\in\mathbb{R}} \left\{v + \frac{1}{N(1-\eta)}\sum_{n=1}^{N}[G(\boldsymbol{x},\boldsymbol{r}_n)]\right\}, \quad (7)$$

Essentially, in (7), the return distribution is estimated by the empirical probability distribution, denoted by $\hat{\mathbb{P}}_e := \sum_{n\in[N]}\delta_{\boldsymbol{r}_n}/N$, where $\delta_{\boldsymbol{r}_n}$ denotes a Dirac delta function placed at a location $\boldsymbol{r}_n \in \mathbb{R}^I$. However, this estimation procedure suffers from considerable errors, leading to the resulting portfolio's inferior empirical performance.

A popular remedy is to apply a DRO approach. Zhu and Fukushima (2009) studied the CVaR under the worst-case distribution among a family of distributions named ambiguity set. They showed that worst-case expected CVaR inherits the properties of CVaR, including subadditivity, positive homogeneity, monotonicity, and translation invariance. The corresponding robust CVaR portfolio optimization is formulated as

$$\min_{\boldsymbol{x}\in\mathcal{X},\, v\in\mathbb{R}} \left\{v + \frac{1}{1-\eta}\sup_{\mathbb{P}\in\mathcal{U}}\mathbb{E}^{\mathbb{P}}[G(\boldsymbol{x},\boldsymbol{r})]\right\}, \quad (8)$$

where $\mathcal{U}$ is the ambiguity set of return distributions. Note that problem (8) is a two-stage optimization problem, which is generally difficult to solve, but the CVaR metric here allows tractable reformulations (Zhu and Fukushima 2009, Hanasusanto et al. 2015, Long et al. 2023).

**Remark 1.** If we define $\mathcal{U} = \mathcal{U}_w(\theta)$ in (3), then problem (8) is a distributionally robust CVaR portfolio optimization under a Wasserstein ambiguity set, where the radius $\theta$ represents the level of ambiguity aversion of the investor. It is easy to see that when $\theta = 0$, this robust CVaR portfolio optimization is reduced to (7) because $\mathbb{P} \equiv \hat{\mathbb{P}}_e$ in this case.

Because of the existence of time-varying economic conditions in the financial market, the ambiguity of asset returns is regime dependent, where each regime refers to an economic cycle. It is crucial to incorporate the prediction of economic conditions in constructing an ambiguity set. In our ambiguity set, we define each economic condition as a regime. Conditioning on a regime, we model the random return in a Wasserstein ambiguity set defined under Chen et al. (2020). Specifically, the regime-switching Wasserstein ambiguity set is defined as

$$\mathcal{U}_{rs}(\boldsymbol{\theta}) = \left\{ \begin{array}{l|l} & (\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{u}}, \tilde{n}, \tilde{k}) \sim \mathbb{P} \\ & \mathbb{E}^{\mathbb{P}}[\tilde{u}_k] \le \theta_k, \ \ \forall k \in [K] \\ & \mathbb{P}[\tilde{\boldsymbol{r}} \in \mathbb{R}^I, \rho(\tilde{\boldsymbol{r}}, \boldsymbol{r}_{nk}) \le \tilde{u}_k \mid n \in [N_k], k \in [K]] = 1 \\ \mathbb{P} \in \mathcal{P}(\mathbb{R}^I \times \mathbb{R}^K \times [N_k]_{k=1}^K \times [K]) & \mathbb{P}[\tilde{n} = n \mid \tilde{k} = k] = \dfrac{1}{N_k}, \\ & \qquad \forall n \in [N_k], k \in [K] \\ & \mathbb{P}[\tilde{k} = k] = w_k, \ \ \forall k \in [K] \end{array} \right\}, \tag{9}$$

Where there are $K$ different regimes, $\tilde{k}$ denotes the regime (e.g., the economic condition) from which the uncertain return is drawn and takes value in $[K]$. In each regime $k$, there are $N_k$ return samples ($\boldsymbol{r}_{nk}$ for $k \in [N_k]$), which provide an empirical distribution of returns $\hat{\mathbb{P}}_k := \sum_{n\in[N_k]} \delta_{\boldsymbol{r}_{nk}}/N_k$. We use $\tilde{n}$ to represent a discrete random scenario of returns that takes values in $[N_k]$ under each regime $k$. The sample of return in scenario $n$ under regime $k$ is $\boldsymbol{r}_{nk}$. $\rho(\cdot,\cdot)$ denotes the cost function in the Wasserstein metric (2). $\theta_k$ measures the ambiguity aversion level under each regime $k$. Following a traditional approach (see Wiesemann et al. 2014, Chen et al. 2020), we introduce a vector of $K$ auxiliary random variables $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \ldots, \tilde{u}_K)'$ in the ambiguity set for the sake of modeling tractability. $w_k$ denotes the probability that a random return is from regime $k$. We will provide different ways to determine these parameters in Section 4.

The essence of the proposed ambiguity set is to restrict the distribution within a prespecified threshold $\theta_k$ from the empirical distribution in each regime. A natural relaxation is that we can relax the constraint of the distribution to a set of distributions located within an average threshold from the average empirical distribution, where the average is taken across different regimes. This relaxation helps apply the existing methods in the literature of Wasserstein distance to optimize (8). However, we show in Proposition 1 that the proposed ambiguity set (projected to the random returns) is a strict subset of the traditional Wasserstein ambiguity set that bounds the distance to the average empirical distribution by the average threshold. In other words, it is not straightforward to apply the existing methods in the literature of the Wasserstein ambiguity set to reformulate (8) with the ambiguity set $\mathcal{U}_{rs}(\boldsymbol{\theta})$ in (9).

**Proposition 1.** *The regime-switching ambiguity set is a strict subset of the Wasserstein ambiguity set with the distance upper bounded by the weighted average ambiguity aversion level* $\sum_{k\in[K]} w_k\theta_k$, *that is,*

$$\Pi_{\tilde{\boldsymbol{r}}}\mathcal{U}_{rs}(\boldsymbol{\theta}) \subsetneq \mathcal{U}_w\left(\sum_{k\in[K]} w_k\theta_k\right) = \left\{\mathbb{P} \in \mathcal{P}(\mathbb{R}^I) \mid W_m(\mathbb{P}, \hat{\mathbb{P}}) \le \sum_{k\in[K]} w_k\theta_k\right\}, \tag{10}$$

*where* $\hat{\mathbb{P}} = \sum_{k\in[K]} w_k \sum_{n\in[N_k]} \delta_{\boldsymbol{r}_{nk}}/N_k$, *and* $\Pi_{\tilde{\boldsymbol{r}}}\mathcal{U}_{rs}(\boldsymbol{\theta})$ *is the projection on the first coordinate that corresponds to* $\tilde{\boldsymbol{r}}$. *Specifically, it represents a set of* $\tilde{\boldsymbol{r}}$ *where there exists some* $(\tilde{\boldsymbol{u}}, \tilde{n}, \tilde{k})$ *such that* $(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{u}}, \tilde{n}, \tilde{k}) \in \mathcal{U}_{rs}(\boldsymbol{\theta})$.

**Remark 2.** If there is only one regime ($K = 1$, $k = \tilde{k} = w_k \equiv 1$, and $N_k = N$), (9) is reduced to (1) in the application of the Wasserstein ambiguity set according to Chen et al. (2020). The essential difference between (9) and (1) is that we introduce a regime random variable $\tilde{k}$ that transits from one to another according to a Markov chain in (9) to characterize the regime-switching ambiguity. (9) represents $K$ different Wasserstein ambiguity sets mixed with the time-varying (state-dependent) transition probabilities.

In summary, the key contribution of this part is to refine the representation of the ambiguity set initiated by Chen et al. (2020). The refined ambiguity set helps incorporate the financial intuitions into the modeling framework. But it also brings challenges in computing the optimal portfolio allocation. In the next subsection, we will provide a reformulation of the DRO problem (8) under a regime-switching ambiguity set and conditions of when the problem remains tractable.

### 3.3. Reformulation of Distributionally Robust Portfolio Optimization with Regime-Switching Ambiguity

The proposed portfolio is found by solving the problem (8) with $\mathcal{U} = \mathcal{U}_{rs}(\boldsymbol{\theta})$,

$$\min_{\boldsymbol{x}\in\mathcal{X}, v\in\mathbb{R}} \left\{ v + \frac{1}{1-\eta} \sup_{\mathbb{P}\in\mathcal{U}_{rs}(\boldsymbol{\theta})} \mathbb{E}^{\mathbb{P}}[G(\boldsymbol{x}, \boldsymbol{r})] \right\}, \tag{11}$$

where $G$ is given in (6). The following proposition shows that the distributionally robust CVaR portfolio optimization with the regime-switching ambiguity set where the Wasserstein distance is defined in the $\ell_m$-norm $(m \geq 1)$ can be reformulated as a convex optimization problem.

**Theorem 1.** *$\forall m \in [1, +\infty]$, the problem (11) under the ambiguity set $\mathcal{U}_{rs}(\boldsymbol{\theta})$ in (9) with $\rho(\cdot,\cdot)$ defined by $\ell_m$-norm, can be reformulated as*

$$
\begin{aligned}
\min_{v,\boldsymbol{x},\boldsymbol{\alpha},\beta} \quad & v + \frac{1}{1-\eta}\left[\sum_{k\in[K]}\sum_{n\in[N_k]}\frac{w_k}{N_k}\alpha_{nk} + \beta\sum_{k\in[K]} w_k\theta_k\right] \\
\text{s.t.} \quad & \boldsymbol{r}'_{nk}\boldsymbol{x} \geq -\alpha_{nk} - v, \ \forall n \in [N_k], k \in [K], \\
& \beta \geq \|\boldsymbol{x}\|_{m^*}, \\
& \alpha_{nk} \geq 0, \ \forall n \in [N_k], k \in [K], \\
& \boldsymbol{x} \in \mathcal{X}. 
\end{aligned} \tag{12}
$$

*In particular, when $m = 1$ and $m = 2$, (12) is a linear program and second-order conic program, respectively, which are both tractable.*

In practice, we can incorporate the target on the empirical mean return in the constraint set $\mathcal{X}$ without affecting the tractability. This is because the formula of the empirical mean return is a linear function of the portfolio allocation decision $\boldsymbol{x}$. Specifically, the constraint on the target portfolio's return is $\boldsymbol{\mu}'\boldsymbol{x} \geq R$, where $\boldsymbol{\mu}$ is the anticipated mean return vector, which can be estimated from data, and $R$ is the targeted return (see Yao et al. 2013). An alternative way to consider the target of mean returns in the literature is the robust Mean-CVaR model, where the worst-case expected return is upper bounded by a target. We provide a tractable DRO formulation for the robust Mean-CVaR model in Appendix C.

Finally, we would like to comment on the solution from our model. Note that a popular portfolio allocation policy studied in the literature is to equally allocate portfolios. We show in Proposition 2 that our model would converge to this classical equally weighted portfolio when the radius of our ambiguity set is large enough.

**Proposition 2.** *When $\sum_k w_k\theta_k \to \infty$, the solution of the DRO problem (8) with ambiguity set $\mathcal{U}_{rs}(\boldsymbol{\theta})$ $\forall m \in [1, +\infty]$ converges to the equally weighted portfolio.*

## 4. Data-Driven Portfolio Allocation Approach: Algorithms and Theoretical Analysis

The key to determining our ambiguity set is to determine the regimes and estimate the transition probabilities between every pair of them. In this section, we will provide two algorithms to estimate regimes and transition probabilities. This section is organized as follows. First, we will provide an overall procedure to construct the optimal portfolio from historical data, where two different approaches are introduced to determine regimes and estimate the associated transition probabilities. Then, we theoretically analyze the performance of the constructed portfolios. Before proceeding, we first list the following assumption on the regimes.

**Assumption 1** (Markov Chain). *The sequences of regimes $\{s_i\}_{i\in\mathbb{Z}}$ form a time-homogeneous ergodic Markov chain in $[K]$ with a unique stationary distribution $(\pi_1, \ldots, \pi_K)$. The conditional distribution of the true return $\tilde{\boldsymbol{r}} \mid s = i$, $\forall i \in [K]$ exhibits variations across different regimes.*

### 4.1. Algorithms

We present our data-driven portfolio allocation algorithm in Algorithm 1, with specific blocks in Algorithms 2–4 in Appendix D. Here, the index $n$ of $\boldsymbol{r}, s$, and $o$ represents the return, regime, and output observation at time $n$, respectively. We denote the constructed portfolio weight from LP (12) as $\boldsymbol{x}^*(\boldsymbol{\theta}; \mathcal{R}_t, s)$ given the regime $s$ under the training sample $\mathcal{R}_t$ with the ambiguity aversion level $\boldsymbol{\theta}$.

**Algorithm 1** (Data-Driven Regime-Switching Portfolio Selection Procedure)

**Input:** Window size $N$, whole time period with returns $M(\leq T)$, sequences of asset returns $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M\}$, where $\boldsymbol{r}_i \in \mathbb{R}^I$, $\forall i \in [M]$; estimated regimes (input of Algorithm 2) or observational sequences (input of Algorithm 3), set of candidate aversion level $\Theta$ (input of Algorithm 4 in Appendix D).

1: **for** $t = 1, \ldots, M - N$ **do**
2: Obtain the training sample of asset returns $\mathcal{R}_t := \{\boldsymbol{r}_t, \ldots, \boldsymbol{r}_{N+t-1}\}$ by the "rolling-sample" approach;
3: Determine the $\{w_k, \{\boldsymbol{r}_{nk}\}_{n\in[N_k]}\}_{k\in[K]}$ in the ambiguity set (9) from $\mathcal{R}_t$ using the frequency-based approach in Algorithm 2 or the hidden Markov approach in Algorithm 3.
4: Cross-validation to determine the best ambiguity aversion level $\boldsymbol{\theta}^*$ from $\Theta$ (see Algorithm 4 in Appendix D).
5: Solve the optimization problem in (12) to obtain the optimal portfolio weight vector $\boldsymbol{x}^*(\boldsymbol{\theta}^*; \mathcal{R}_t, s_{N+t-1})$, denoted by $\boldsymbol{x}^*_{N+t}$ for simplicity. Then, the realized total return denoted by $\hat{g}_{N+t}$ is computed by $\hat{g}_{N+t} = \boldsymbol{r}'_{N+t}\boldsymbol{x}^*_{N+t} \in \mathbb{R}$.
6: **end for**

**Output:** constructed portfolio weight $\{\boldsymbol{x}^*_{N+1}, \ldots, \boldsymbol{x}^*_M\}$; the associated total return sequence $\{\hat{g}_{N+1}, \ldots, \hat{g}_M\}$.

This algorithm applies the "rolling-sample" approach proposed by DeMiguel et al. (2009). Specifically, given a $T$-month-long data, in each month $t$, we use the data set from months $t$ to $t + N - 1$ to estimate the ambiguity set $\mathcal{U}_{rs}(\boldsymbol{\theta})$ (see (9)) and solve the portfolio optimization problem (see (12)), which generates the portfolio weight and the total return in month $t + N$. We repeat the process by adding the returns for the next period and dropping the earliest one until we reach the end of the data set.

In estimating the ambiguity set $\mathcal{U}_{rs}(\boldsymbol{\theta})$, several parameters need to be determined, which include each regime, its associated weight, and the ambiguity aversion level $\boldsymbol{\theta}$. We apply two different approaches, named a frequency-based approach (see Section 4.1.1) and a hidden-Markov approach (see Section 4.1.2), to estimate regimes and their weights and use cross-validation to tune $\boldsymbol{\theta}$. The detailed procedure for cross-validation is shown in Algorithm 4 in Appendix D. Different from the standard $m$-fold cross-validation procedure (i.e., Ban et al. 2018), we cannot split samples randomly because the distribution in the subsequent periods depends on the regime at the current time. We also cannot validate $\mathcal{D}_{t,1}$ using the training data $\mathcal{D}_t \backslash \mathcal{D}_{t,1}$ because the validating step requires $s_{t-1}$, which is beyond the input in our implementation. To resolve these issues, we split samples in the temporal order and validate from the second fold (start from $i=2$ instead of $i=1$ in Step 2 of Algorithm 4 in Appendix D).

**4.1.1. A Frequency-Based Approach.** This section introduces several common financial practices in determining regimes and provides estimation methods for transition probabilities.

A simple yet widely used definition of the financial market regimes is bull and bear markets (see Tu 2010). These two regimes are characterized by the market's trend. If the market return tends to be positive, it is defined as a bull market; otherwise, it is a bear market. The market return can be measured by the market portfolio (capitalized weighted index) or equally weighted portfolio of all the assets in the market. In our experiments, we stick with the latter because some assets of our choice are sector portfolios. We can compute the market return for bull and bear markets and determine the corresponding regime at each time point.

When there are different asset classes in the market, the bull and bear markets appear to be insufficient in describing the market regimes. Inspired by the success of the All-Weather fund launched by Bridgewater Associates in 1996, one can determine the financial regimes using some macroeconomic variables. For example, bonds perform best during times of deflationary recession; stocks do well during periods of growth. Holding cash becomes the most attractive option when the money supply is tight. By investigating market dynamics and potential return scenarios over a long period, Bridgewater Associates (2012) pointed out that economic drivers, such as inflation and growth rates, can determine which asset class an investor should invest in. Therefore, it is potentially interesting to incorporate this regime-switching nature of assets in estimating the regimes. Figure 1 presents the categorization of the regimes by Bridgewater Associates (2012) and the asset classes that perform well in each regime. For the four regimes defined in Bridgewater Associates (2012), we can classify the data into four different economic

**Figure 1.** (Color online) The Weathers (regimes) Defined in Bridgewater Associates (2012) with Asset Classes Rising over Time in the Boxes

| | | Growth | Inflation |
|---|---|---|---|
| Market Expectations | Rising | Equities<br>Commodities<br>Corporate Credit<br>EM Credit | IL Bonds<br>Commodities<br>EM Credit |
| | Falling | Nominal Bonds<br>IL Bonds | Equities<br>Nominal Bonds |

*Note.* IL, inflation-linked; EM, emerging market.

scenarios using the $2 \times 2$ interactions of U.S. CPI and GDP data that represent the levels of inflation and growth, respectively. Specifically, we determine whether the inflation is rising above or falling below the market's expectations by subtracting the moving average of the recent one-year U.S. CPI from its current value. Similarly, we do the same for the growth using the U.S. GDP data.

**Algorithm 2** (Frequency-Based Approach)

**Input:** Return sequences $\mathcal{R}_t$, associated regime sequences $\{s_t, \ldots, s_{t+N-1}\}$ determined in Section 4.1.1.

1: Obtain the parameters of transition probability matrix by

$$a_{j,k} = \frac{\sum_{i=t}^{t+N-2} \mathbb{I}_{\{s_i=j,\, s_{i+1}=k\}}}{\sum_{i=t}^{t+N-2} \mathbb{I}_{\{s_i=j\}}}, \quad \forall j,k \in [K].$$

2: From the last **observed** regime $j^* := s_{N+t-1}$, determine the regime frequency by $w_k = a_{j^*,k}$ and associated return sequences by $\{\boldsymbol{r}_{nk}\}_{\{n\in[N_k]\}} = \{\boldsymbol{r}_i \mid s_i = k, t \le i \le N+t-1\}$, $\forall k \in [K]$ for (9).

**Output:** Transition probability matrix $A = \{a_{j,k}\}_{j,k\in[K]}$; $\{w_k, \{\boldsymbol{r}_{nk}\}_{n\in[N_k]}\}_{k\in[K]}$ in the ambiguity set (9).

After determining the regimes, we can estimate the transition probability matrix $A = \{a_{j,k}\}_{j,k\in[K]}$ for the Markov chain by the frequency of occurrence from one regime to another. Note that $\sum_{k\in[K]} a_{j,k} = 1$ for all $j \in [K]$. The general procedure of the frequency-based approach is shown in Algorithm 2. We also illustrate how our algorithm determines the parameters in (9) in each period using Example 1. Here, we denote the transition probability matrix of the true regime sequence to be $A^* = \{a^*_{j,k}\}_{j,k\in[K]}$.

**Example 1.** Suppose we have 10 sequential history samples of asset returns $\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{10}\}$, and there are two underlying regimes 1 and 2. $\tilde{\boldsymbol{r}} \mid s = 1 \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma), \tilde{\boldsymbol{r}} \mid s = 2 \sim \mathcal{N}(-\boldsymbol{\mu}, \Sigma)$. The underlying true regime is $\boldsymbol{s}^* = \{1,2,2,1,1,2,1,2,2,1\}$ with regime transition matrix $A^* = \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$. We would like to model the ambiguity set of $\tilde{\boldsymbol{r}}_{11}$ from data. Because $s_{10} = 1$, the underlying true distribution of our interest is $\tilde{\boldsymbol{r}}_{11} \sim 0.2\mathcal{N}(\boldsymbol{\mu}, \Sigma) + 0.8\mathcal{N}(-\boldsymbol{\mu}, \Sigma)$.

Using the frequency-based approach, given the input $\{\boldsymbol{r}_i\}_{i\in[10]}$ and one observed regime sequence, $\boldsymbol{s} = \{1, 2, 1, 1, 1, 2, 2, 1, 2, 1\}$, we apply Step 1 of Algorithm 2 to estimate the transition matrix $A = \begin{pmatrix} 0.4 & 0.6 \\ 0.75 & 0.25 \end{pmatrix}$. Because the last observed state is $s_{10} = 1$, we output $w_1 = a_{1,1} = 0.4, w_2 = a_{1,2} = 0.6,\quad N_1 = 6, N_2 = 4, \{\boldsymbol{r}_{n1}\}_{n\in[N_1]} = \{\boldsymbol{r}_i \mid s_i = 1\} = \{\boldsymbol{r}_1, \boldsymbol{r}_3, \boldsymbol{r}_4, \boldsymbol{r}_5, \boldsymbol{r}_8, \boldsymbol{r}_{10}\}, \{\boldsymbol{r}_{n2}\}_{n\in[N_2]} = \{\boldsymbol{r}_i \mid s_i = 2\} = \{\boldsymbol{r}_2, \boldsymbol{r}_6, \boldsymbol{r}_7, \boldsymbol{r}_9\}$, according to Step 2 of Algorithm 2, which determines the parameters of the ambiguity set in (9).

#### 4.1.2. A Hidden-Markov Approach.

The frequency-based method requires us to know the exact regime of the economy at any point in time, which may not be directly observable and representative of the asset return regime. Therefore, we propose a hidden-Markov model (HMM) to determine the regimes and the corresponding transition probability matrix simultaneously; see Ghahramani (2001) for an introduction to HMM and Stratonovich (1960) and Baum and Petrie (1966) for the mathematics behind HMM. Note that even though the regimes can be hidden, we still need to specify the number of the regimes using this approach.

Following Assumption 1, we assume that there are $K$ regimes in the financial market. We use $\boldsymbol{o} = \{o_1, \ldots, o_T\}$ to denote a sequence of $T$ observations. The definition of each observation is based on the available information in the data set. In Section 5.2, Appendix F.1, when dealing with the classical DeMiguel et al. (2009)'s data sets, the historical return data are available, and we use $o_t \in \{+, -\}$ to denote whether the market returns are above the risk-free rate (given by the 90-day Treasury-bill yield) at time $t$. In Section 5.3, there are five different asset classes in the data: equities, treasury bonds, commodities, credit, and treasury inflation-protected securities (TIPS), indexed from $i = 1$ to $i = 5$. Then we define $o_t = i$ if the asset class $i$ enjoys the highest return among all five asset classes at time $t$.

We use $\mathbb{P}^{HMM}$ to denote the probability distribution of the sequence of observations and the corresponding regimes. An HMM instantiates two simplifying assumptions:

***Markov Assumption.*** The probability of a particular regime depends only on the previous regime: $\mathbb{P}^{HMM}(s_t \mid s_{t-1}, s_{t-2}, \ldots, s_1) = \mathbb{P}^{HMM}(s_t \mid s_{t-1})$.

***Output Independence.*** The probability of an output observation $o_t$ depends only on the regime $s_t$ that produces the observation and not on any other regimes or observations: $\mathbb{P}^{HMM}(o_t \mid s_1, \ldots, s_T, o_1, \ldots, o_T) = \mathbb{P}^{HMM}(o_t \mid s_t)$.

Given a sequence of observations $\boldsymbol{o}$, the HMM can tackle the following two tasks:

1. Provided the inputs of transition probability matrix $A = \{a_{j,k}\}_{j,k\in[K]}$ and a sequence of observation likelihoods (emission probabilities) denoted by $B = \{b_{i,j}\}_{i\in[K]}^{j\in\mathcal{O}}$, where $b_{i,j} = \Pr(o = j \mid s = i)$, determine $\mathbb{P}^{HMM}(\boldsymbol{o} \mid A, B)$ and the most likely hidden regime sequence $(s_1, \ldots, s_T)$ corresponding to the observations.
2. Provided a given regime sequence, learn the HMM parameters $A$ and $B$.

The standard algorithm for training HMM is the Viterbi algorithm for the first task (Viterbi 1967, Forney 1973) and the forward-backward algorithm (Baum 1972), which is a special case of expectation-maximization (EM) algorithm (Dempster et al. 1977), for the second task. The combined algorithm first computes an initial estimate for the regimes and the transition probabilities and iteratively improves $\mathbb{P}^{HMM}(\boldsymbol{o} \mid A, B)$. Readers are referred to the appendix of Jurafsky and Martin (2008) for a tutorial on training HMM. We denote the combined algorithm as one `HMM training oracle` with only a sequence of observations $\boldsymbol{o}$ as input. The output of the `HMM training oracle` includes its most likely regime assignment for the observations and the transition probability matrix. The specific procedure of HMM is shown in Algorithm 3. Following the earlier Example 1, suppose the output emission probability matrix $B = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$. Using the HMM approach, given the input $\{\boldsymbol{r}_i\}_{i\in[10]}$ and one observation sequence $\boldsymbol{o} = \{-, +, +, -, +, +, -, -, +, -\}$ following the first definition above, we output the estimated transition matrix $A = \begin{pmatrix} 0.25 & 0.75 \\ 0.6 & 0.4 \end{pmatrix}$ and the most likely regimes $\boldsymbol{s} = \{1, 2, 2, 1, 2, 2, 1, 1, 2, 1\}$ by calling the HMM training oracle in Python (the training oracle is implemented using the code `hmm.GaussianHMM(2, 'full')` under Python 3) according to Step 1 of Algorithm 3. Because the last predicted regime $s_{10} = 1$, we output $w_1 = a_{1,1} = 0.25, w_2 = a_{1,2} = 0.75,\quad N_1 = 5, N_2 = 5, \{\boldsymbol{r}_{n1}\}_{n\in[N_1]} = \{\boldsymbol{r}_i \mid s_i = 1\} = \{\boldsymbol{r}_1, \boldsymbol{r}_4, \boldsymbol{r}_7, \boldsymbol{r}_8, \boldsymbol{r}_{10}\}, \{\boldsymbol{r}_{n2}\}_{n\in[N_2]} = \{\boldsymbol{r}_i \mid s_i = 2\} = \{\boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_5, \boldsymbol{r}_6, \boldsymbol{r}_9\}$, for (9) according to Step 2 of Algorithm 3.

**Algorithm 3** (HMM Approach)

**Input:** Return sequences $\mathcal{R}_t$, associated observation sequences $\{o_t, \ldots, o_{t+N-1}\}$ determined in Section 4.1.2

1: Obtain the parameters of transition probability matrix $A = \{a_{j,k}\}_{j,k\in[K]}$ and the most likely regimes $\{s_t, \ldots, s_{t+N-1}\}$ from one `HMM training oracle`.

2: From the last **predicted** regime $j^* := s_{N+t-1}$ the step above, determine the regime frequency by $w_k = a_{j^*,k}$ and associated return sequences by $\{\boldsymbol{r}_{nk}\}_{\{n\in[N_k]\}} = \{\boldsymbol{r}_i \mid s_i = k, t \le i \le N + t - 1\}$, $\forall k \in [K]$ for (9).

**Output:** Transition probability matrix $A = \{a_{j,k}\}_{j,k\in[K]}$, most likely regimes $\{s_t, \ldots, s_{t+N-1}\}$, $\{w_k, \{\boldsymbol{r}_{nk}\}_{n\in[N_k]}\}_{k\in[K]}$ in the ambiguity set (9).

Note that no matter which approach is used, the transition probability matrix is dynamic and will change across time based on the historical data, following the initiative of Bazzi et al. (2016). The length of observations can be different from that of the return data because the specification of the regimes is more of a long-term feature, whereas the historical return data are used to infer the returns in the near future. In our case study, we use the 10-year data to determine the regimes.

### 4.2. Theoretical Analysis of Regime Switching Models

In this section, we analyze the theoretical performance of the regime-switching model when the true underlying regime sequences of history returns are known; that is, the estimation of the regime using covariates (such as U.S. CPI and GDP data) is accurate. For the sake of analysis, we need to make additional assumptions besides Assumption 1.

**Assumption 2** (K-Regime). *For the underlying true distribution of interest in (5),* $\mathbb{P}^* := \sum_{i\in[K]} w_k^* \mathbb{P}_k^*$ *with* $w_k^* > 0$, $\forall k \in [K]$; $\sum_{k\in[K]} w_k^* = 1$. $\{\mathbb{P}_k^*\}_{k\in[K]}$ *are K different distributions.*

Here, $\{w_k^*\}_{k\in[K]}$ is different across time because the last regime $j^*$ varies across time in Algorithm 2, and we have $w_k^* = a_{j^*,k}^*$, where $A^* = \{a_{j,k}^*\}_{j,k\in[K]}$ represents the underlying true transition probability matrix in Algorithm 2. At the $t$-th period in Steps $2-5$ in Algorithm 1, the underlying true distribution $\mathbb{P}^*$ in Assumption 3 refers to the distribution of the random return at the time $N+t$.

**Assumption 3** (Light Tail). *The underlying true distribution* $\mathbb{P}^*$ *is a light tail distribution, that is,* $\mathbb{E}^{\tilde{\boldsymbol{r}}\sim\mathbb{P}_k^*}[\exp(\|\tilde{\boldsymbol{r}}\|^a)] < \infty$, $\forall k \in [K]$ *for some* $a > 0$.

This assumption is necessary to guarantee the traditional concentration result for the empirical distribution (see Fournier and Guillin 2015, Esfahani and Kuhn 2018).

The main results are presented here.

**Theorem 2** (Coverage of True Distribution). *Under Assumptions 1, 2, and 3, if the sample size* $N \geq c_0^2 \log(2K/\delta\sqrt{\pi_*})/(\pi_*\gamma_{\mathrm{ps}})$ *(where* $\gamma_{\mathrm{ps}}$ *is its pseudo-spectral gap with the definition deferred in Appendix B.1), then* $\mathbb{P}^* \in \Pi_{\tilde{\boldsymbol{r}}}\mathcal{U}_{rs}(\boldsymbol{\theta})$ *with a probability of at least* $1-\delta$ *if the ball size*

$$\theta_k \geq \left(\frac{c_{k,1}\log(K/\delta) + c_{k,2}}{N_k}\right)^{\frac{1}{l}} + \frac{c_0}{\min_{i\in[K],w_i\neq 0} w_i} \sqrt{\frac{K\log(2K/\delta)}{N\pi_*}} \max_{i\neq k} W_m(\mathbb{P}_k^*, \mathbb{P}_i^*),\ \forall k \in [K], \tag{13}$$

*where* $c_{k,1}, c_{k,2}$ *are constants only depending on* $\mathbb{P}_k^*$, $c_0$ *is a constant only depending on the Markov chain, and* $\pi_* = \min_{i\in[K]}\pi_i$.

We briefly discuss some of the key ideas in proving this result. We estimate $\{w_k^*\}_{k\in[K]}$ and $\{\mathbb{P}_k^*\}_{k\in[K]}$ by their empirical estimators, denoted by $w_k$ and $\hat{\mathbb{P}}_k = \sum_{n\in[N_k]} \delta_{\boldsymbol{r}_{nk}}/N_k$, respectively. To ease the analysis, we first reconstruct $\mathbb{P}^*$ as a convex combination of $K$ different new regimes $\hat{\mathbb{P}}_k^*$ with the coefficient of each $\hat{\mathbb{P}}_k^*$ being the estimated weight $w_k$. Specially, we show that $\mathbb{P}^* := \sum_{k\in[K]} w_k^*\mathbb{P}_k^*$ can be represented by $\sum_{k\in[K]} w_k \hat{\mathbb{P}}_k^*$, where $\hat{\mathbb{P}}_k^* = \sum_{i\in[K]}\alpha_{ik}\mathbb{P}_i^*$, $\forall k\in[K]$ with $\sum_{i\in[K]}\alpha_{ik} = 1, \alpha_{kk} = \min\{1, w_k^*/w_k\}$, $\forall k\in[K]$ and $\alpha_{ik}\geq 0, i,k\in[K]$. By doing so, we can build a connection between $\mathbb{P}^*$ and the empirical estimators ($w_k$ and $\hat{\mathbb{P}}_k$) and reduce the analysis of the difference between $\mathbb{P}^* = \sum_{k\in[K]} w_k\hat{\mathbb{P}}_k^*$ and $\hat{\mathbb{P}} = \sum_{k\in[K]} w_k\hat{\mathbb{P}}_k$ to bounding the difference between $\hat{\mathbb{P}}_k^*$ and $\hat{\mathbb{P}}_k$ under each regime $k$, that is, $W_m(\hat{\mathbb{P}}_k^*, \hat{\mathbb{P}}_k)$. By triangle inequalities $W_m(\hat{\mathbb{P}}_k^*, \hat{\mathbb{P}}_k) \leq W_m(\mathbb{P}_k^*, \hat{\mathbb{P}}_k) + W_m(\hat{\mathbb{P}}_k^*, \mathbb{P}_k^*)$, we turn our attention to bounding $W_m(\mathbb{P}_k^*, \hat{\mathbb{P}}_k)$ and $W_m(\hat{\mathbb{P}}_k^*, \mathbb{P}_k^*)$, respectively. The bound of $W_m(\mathbb{P}_k^*, \hat{\mathbb{P}}_k)$ is a straightforward application of the measure concentration result in Esfahani and Kuhn (2018). This term characterizes the estimation error of $\mathbb{P}_k^*$ from its empirical distribution in each regime $k$. The second term ($W_m(\hat{\mathbb{P}}_k^*, \mathbb{P}_k^*)$) arises from the error in estimating the regime-switching behavior, that is, $|w_k - w_k^*|, k\in[K]$ and its magnified effects on the difference of each $\mathbb{P}_i \neq \mathbb{P}_k$ from the Wasserstein distance, that is, $W_m(\mathbb{P}_k^*, \mathbb{P}_i^*)$. The bound of $W_m(\hat{\mathbb{P}}_k^*, \mathbb{P}_k^*)$ leverages the estimation error rate of the transition matrix of the Markov chain $|w_k - w_k^*|$ in Wolfer and Kontorovich (2021).

**Remark 3.** When $K=1$, there is only one regime. The second term in the bound disappears. The result is consistent with the standard Wasserstein measure concentration results established in Esfahani and Kuhn (2018), which confirms the validity of our results to some extent.

**Remark 4.** We claim that the bound in Theorem 2 cannot be further improved using other estimators rather than the empirical ones as we choose in this paper. Note that the bound is built on two key concentration results: the upper bound of the Wasserstein distance between the empirical distribution $\hat{\mathbb{P}}_k$ and the true distribution $\mathbb{P}_k^*$ under each regime, which is $O_p(N_k^{-1/l})$ (see Lemma 5 in Appendix B.2), and the estimation error bound of the Markov chain's transition matrix $\sum_{k\in[K]}|w_k - w_k^*|$, which is $O_p(N^{-1/2})$ (see Lemma 3 in Appendix B.2). These two bounds can be shown to be tight in the order of $N$. Specifically, according to Theorem 3.2 in Wolfer and Kontorovich (2021), for the set of Markov chains satisfying Assumptions 1 and 2, there exists one instance of Markov chain $S$ such that for any estimator $\{\hat{w}_k\}_{k\in[K]}$ (as a function of history states $\{s_i\}_{i\in[N]}$), with probability at least $0.1, \sum_{k\in[K]}|\hat{w}_k - w_k^*(S)|$ is lower bounded by $C_1\sqrt{K/N\pi_k}$ for some constant $C_1$. Similarly, according to Singh and Póczos (2018), one can show that for the set of distributions $\mathcal{P}$ satisfying Assumption 3 and any estimator $\hat{\mathbb{P}}$, we have $\inf_{\hat{\mathbb{P}}}\sup_{\mathbb{P}_k^*\in\mathcal{P}}\mathbb{E}_{\{\boldsymbol{r}_{nk}\}_{n\in[N_k]}\sim\mathbb{P}_k^*}[W_m(\mathbb{P}_k^*,\hat{\mathbb{P}})]\geq C_2/N_k^{1/I}$ for some constants $C_2$.

**4.2.1. Data-Independent Measure Concentration.** The lower bound of the choice of the ball size in (13) is a function of $N_k$, $w_k$, and thus depends on the observational regimes of history samples. To remove this dependence, we further take advantage of the concentration tools in bounding the empirical frequency of states in the Markov chain, that is, $|N_k/N-\pi_k|$ (shown in Lemma 3 in Appendix B.2), to revise the concentration guarantee. Specifically, we can get a minimum required ball size independent of $N_k$, $w_k$ when the sample size $N$ is large enough. Proposition 3 summarizes the results.

**Proposition 3** (Data-Independent Measure Concentration). *Under Assumptions 1, 2 and 3, $\forall\delta>0$, and the Markov chain is reversible so that spectral gap $\gamma_s$ (definition in Appendix B.1) is meaningful if the sample size satisfies* $N\geq\max\{(c_0^2)/(\pi_*\gamma_s)\log((2K)/\delta\sqrt{1/\pi_*}),(\log(4K/\delta))/(2c_0'\pi_*^2),(c_0^2K\log(2K/\delta))/(\pi_*(\min_{i\in[K]}w_i^*)^2)\}$, *and then* $\mathbb{P}^*\in\Pi_{\tilde{r}}\mathcal{U}_{rs}(\theta)$ *with a probability of at least $1-\delta$ if the ball size*

$$\theta_k\geq\left(\frac{c_{k,1}\log(2K/\delta)+c_{k,2}}{2N\pi_k}\right)^{\frac{1}{I}}+\frac{c_0}{2\min_{i\in[K]}w_i^*}\sqrt{\frac{K\log(4K/\delta)}{N\pi_*}}\max_{i\neq k}W_m(\mathbb{P}_k^*,\mathbb{P}_i^*),\ \forall k\in[K],\tag{14}$$

*where $c_0'$ is a positive constant and depends on the spectral gap $\lambda_s$.*

**Remark 5.** Note that when the sample size $N$ is large enough, the first term will dominate the minimum required ball size in (13) because $\Theta((N_k)^{-1/I})=\Theta((N\pi_k)^{-1/I})>\Theta((N\pi_k)^{-1/2})$, since $\pi_k>0$ from Assumption 1. Hence, we use the first term, which is in the order of $N^{-1/I}$, and choose a more conservative $\theta_k=\gamma N^{-1/I}$ with a tuning hyperparameter $\gamma$ in our numerical studies.

Suppose the underlying true distribution is a mixture of $K=2$ Gaussian distributions, that is, $\mathbb{P}^*=w_1^*\mathcal{N}(\boldsymbol{\mu},\Sigma)+w_2^*\mathcal{N}(-\boldsymbol{\mu},\Sigma)$. If the norm $m=1$, from Theorem 2, the minimum required ball size reduces to $\theta_k\geq((c_{k,1}\log(2/\delta)+c_{k,2})/(N_k))^{1/I}+(2c_0\|\boldsymbol{\mu}\|_2^2)/(\min\{w_1,w_2\})\sqrt{(2\log(2/\delta))/(N\pi_*)},k\in\{1,2\}$ because $W_1(\mathcal{N}(\boldsymbol{\mu}_1,\Sigma),\mathcal{N}(\boldsymbol{\mu}_2,\Sigma))\leq\sqrt{\|\boldsymbol{\mu}_1-\boldsymbol{\mu}_2\|_2^2}$ by Dowson and Landau (1982). From the bound, we can see that if $\boldsymbol{\mu}$ is larger (implying that the distribution in each regime $k$ ($\mathbb{P}_k^*$) is very different from each other) or the frequency difference between $w_1^*,w_2^*$ is larger (implying that $w_1^*,w_2^*$ significantly differs), then a larger ball size $\theta_k$ is required. On the other hand, if $\pi_*$ is very small, which implies that one regime happens with a very small probability, then we cannot obtain enough samples in this regime to accurately estimate its corresponding $w_k$. Therefore, it also requires a large ball size according to the bound.

**4.2.2. Performance of the Obtained Portfolio.** Now, we are ready to analyze the performance of the obtained portfolio using our data-driven approach. Theorem 2 shows that when the ambiguity aversion level $\boldsymbol{\theta}$ is large enough, the ambiguity set will include the true distribution with a high probability. Based on it, we further show that under the same ambiguity aversion level, the obtained portfolio using our data-driven approach can generate the risk measured by CVaR no larger than a certain threshold from the true risk level. Specifically, we list our results below in Theorem 3.

**Theorem 3.** *Denote the solution $\hat{\boldsymbol{x}},\boldsymbol{x}^*$ to each optimization problem*:

$$\hat{\boldsymbol{x}}\in\arg\min_{\boldsymbol{x}\in\mathcal{X}}\sup_{\mathbb{P}\in\mathcal{U}_{rs}(\boldsymbol{\theta})}CVaR_\eta^{\mathbb{P}}(\boldsymbol{x})$$
$$\boldsymbol{x}^*\in\arg\min_{\boldsymbol{x}\in\mathcal{X}}CVaR_\eta^{\mathbb{P}^*}(\boldsymbol{x}).$$

*Then, with probability $1-\delta$, if $\theta_k$ satisfies (13), we would have*

$$CVaR_\eta^{\mathbb{P}^*}(\hat{\boldsymbol{x}})-CVaR_\eta^{\mathbb{P}^*}(\boldsymbol{x}^*)\leq\frac{2\|\boldsymbol{x}^*\|_{m^*}}{1-\eta}\sum_k w_k\theta_k,$$

*where m means that we take $\ell_m$-norm as the cost function in the Wasserstein distance.*

In summary, Theorem 2 establishes the concentration result of the true distribution for the frequency-based approach, based on which we can analyze the performance of the obtained portfolio using this frequency-based data-driven optimization approach (see Theorem 3). Note that in this analysis, we assume that the true regimes of history returns are observed. However, in practice, the true underlying regimes of history returns are unknown but inferred from algorithms. Therefore, we can no longer ensure that the output state sequence is "almost" the same as the true regime sequence such that $W_m(\mathbb{P}_k^*,\hat{\mathbb{P}}_k),\sum_{k\in[K]}|w_k-w_k^*|\to 0$ as $N\to\infty$. In this case, we use the HMM to dynamically determine the regime and estimate the transition probability matrix. Although HMM has shown its outstanding performance numerically, to the best of our knowledge, it is still an open problem to analyze its concentration performance theoretically. We leave it for future research.

## 5. Case Studies

In this section, we examine the performance of the proposed regime-switching distributionally robust CVaR portfolio using the data sets used in DeMiguel et al. (2009) with an extended test period and a data set with different asset classes. All the data are monthly data. We compare our proposed portfolio with the benchmark portfolios: equally weighted (EW) portfolio, which is a robust choice that is hard to beat (see DeMiguel et al. 2009); minimum-variance portfolio, which outperforms classical mean-variance portfolio (see Jagannathan and Ma 2003); and the robust mean-CVaR portfolio with a moment-based ambiguity set where the first two moments are specified (see Kang et al. 2019). We use the shorthand, "EW," "MVP," and "DR Mean-CVaR (Moments)" for the aforementioned portfolios, respectively.

### 5.1. Data Sets and Numerical Settings

We include the six data sets considered in DeMiguel et al. (2009). They are S&P sectors, industry portfolios, international portfolios, Mkt/SMB/HML, FF 1-factor, and FF 4-factor; see DeMiguel et al. (2009) for the detailed descriptions of the data sets. The time periods of the data sets are up to November 2004 (from July 1963 at the earliest). We extend the data sets to April 2019 by downloading the same classes of data from Ken French's website and MSCI (Morgan Stanley Capital International). Each asset's returns can be directly input to Algorithm 2 (frequency-based approach). To get the input of Algorithm 3 (HMM), we need to generate observational sequences $o_t \in \{+,-\}$ based on the asset's returns. Specifically, if the return is above the risk-free rate (given by the 90-day Treasury-bill yield) at time $t$, then $o_t = +$; otherwise, $o_t = -$. The empirical results for the six types of data sets over different periods (up to 2004 and up to 2019) are presented in Section 5.2 and Appendix F.1, respectively.

In Section 5.3, we include another data set comprising five different asset classes to explore the regime-switching property of the proposed portfolio. The five asset classes are (i) equities, (ii) 10-year treasury note, (iii) treasury inflation-protected securities (TIPS), (iv) commodities, and (v) credit. The asset prices are downloaded from various sources, including BlackRock and Barclays Indices. The time period spans from January 2004 to April 2019. The class of equities is formed using the Global Industry Classification Standard (GICS), and the 10 industries considered are Energy, Materials, Industrials, Consumer Discretionary, Consumer Staples, Healthcare, Financials, Information-Technology, Telecommunications, and Utilities. Gold, Silver, and West Texas Intermediate (WTI) are selected to represent the commodity class. PIMCO Investment Grade Corporate Fund is selected to represent the credit class. All of them are selected mainly because they are liquid and tradable in the market. It will be interesting to see whether our portfolio can invest in the appropriate asset classes over a long time period. Besides using the assets' returns as input of Algorithm 2 (frequency-based approach), we also need to estimate regimes using either the bull and bear markets or the "weather" conditions by Bridgewater Associates (2012) based on some macroeconomic variables (see construction details in Section 4.1.1) and input them to Algorithm 2. To get the input of Algorithm 3 (HMM), we need to generate new observational sequences $o_t \in \{1,2,\dots,5\}$ (see construction details in Section 4.1.2). Specifically, $o_t = i$ if the asset class $i$ enjoys the highest return among all five assets at time $t$. In addition to the aforementioned benchmarks, we also introduce a stochastic regime-switching model and a DRO portfolio optimization model with the traditional Wasserstein ambiguity set to examine the value of considering a robust model for the regime-switching model and the value of adding the regime-switching feature in a DRO model, respectively. The shorthand and the corresponding specifications are presented in Table 1. We add a nonnegative constraint, $x_i \geq 0$ for $i \in [I]$, into $\mathcal{X}$ for all the CVaR and Mean-CVaR portfolios. The Mean-CVaR portfolios are obtained by adding first-moment constraint $\boldsymbol{\mu}'\boldsymbol{x} \geq R$ to the formulation (11), where $\boldsymbol{\mu}$ is estimated by the sample mean vector and $R$ is a target. We set $R$ to the 40% lower quantile of the historical returns of all assets for the respective datasets. We choose the set of candidate ambiguity aversion level $\Theta = \{\boldsymbol{\theta} \in \mathbb{R}^K : \theta_k = \gamma N^{-1/I},\ \forall k \in [K]; \gamma \in \{0.02, 0.04, \dots, 0.1\}\}$ and adopt cross-validation shown in Appendix D to tune $\gamma$ for all DRO models in our experiments. For DeMiguel et al. (2009)'s data sets across different time periods, we compare only EW, RS CVaR (HMM), DR CVaR (Wasserstein), and RSDR CVaR (HMM) because of the space limit, whereas for our own data set we compare all the portfolios.

We measure the performance of each model using four commonly used performance metrics in the finance literature: the out-of-sample Sharpe ratio, certainty equivalent (CEQ) return, maximum drawdown, and turnover. We refer interested readers to Appendix E for their definitions. Generally, a good portfolio should have a high Sharpe ratio and CEQ and a low maximum drawdown and turnover.

### 5.2. Empirical Results of DeMiguel et al. (2009)'s Data Sets During the Same Time

We compare the performance of our proposed portfolios with the same considerations of DeMiguel et al. (2009). We report the 1/N strategy (EW) from their paper only because it is the best among the rest in their paper. Moreover, EW portfolio is an extreme case of the (RS)DR CVaR portfolios, as suggested in Proposition 2. Our empirical results are presented in Table 2.

**Table 1.** CVaR Portfolios for Comparison

| Shorthand of strategy | Mean constraint | Robustness | No. of regimes ($K$) | How the regimes are determined |
|---|---|---|---|---|
| RSDR Mean-CVaR (HMM) | Yes | Yes | 4 | HMM with observations about five asset classes |
| RSDR Mean-CVaR (Weathers) | Yes | Yes | 4 | Frequency-based approach with Bridgewater's regimes |
| RSDR Mean-CVaR (bull and bear) | Yes | Yes | 2 | Frequency-based approach for bull and bear markets |
| RS Mean-CVaR (HMM) | Yes | No | 4 | HMM with observations about five asset classes |
| RS Mean-CVaR (Weathers) | Yes | No | 4 | Frequency-based approach with Bridgewater's regimes |
| RS Mean-CVaR (bull and bear) | Yes | No | 2 | Frequency-based approach for bull and bear markets |
| DR Mean-CVaR (Wasserstein) | Yes | Yes | 1 | NA |
| Mean-CVaR | Yes | No | 1 | NA |
| RSDR CVaR (HMM) | No | Yes | 4 | HMM with observations about five asset classes |
| RSDR CVaR (Weathers) | No | Yes | 4 | Frequency-based approach with Bridgewater's regimes |
| RSDR CVaR (bull and bear) | No | Yes | 2 | Frequency-based approach for bull and bear markets |
| RS CVaR (HMM) | No | No | 4 | HMM with observations about five asset classes |
| RS CVaR (Weathers) | No | No | 4 | Frequency-based approach with Bridgewater's regimes |
| RS CVaR (bull and bear) | No | No | 2 | Frequency-based approach for bull and bear markets |
| DR CVaR (Wasserstein) | No | Yes | 1 | NA |
| CVaR | No | No | 1 | NA |

*Note.* NA means that no regimes are determined in the corresponding approach.

We first evaluate the marginal benefits of regime-switching and DRO models by comparing our approach with the RS CVaR (HMM) and DR CVaR (Wasserstein) portfolios. Without robustness, RS CVaR (HMM) appears to be more sensitive than RSDR CVaR (HMM) because one can see that the performance of RS CVaR (HMM) is volatile compared with EW and RSDR CVaR (HMM) across different data sets. It demonstrates the efficiency of incorporating robustness into the regime-switching model to mitigate the errors from estimating nonstationary parameters. DR CVaR (Wasserstein) performs analogous to RSDR CVaR (HMM) in the tested data. We can even see an evident decrease in the turnover from DR CVaR (Wasserstein) to RSDR CVaR (HMM), which implies that adding robustness for nonstationary optimization models can stabilize the positions of portfolios. The good performance of DR CVaR (Wasserstein) is due mainly to the asset structure. Each data set in DeMiguel et al. (2009) is from only one class, and the return changes of each asset across the data set are highly similar. Hence, the nonstationarity behavior incorporating the regime behavior does not bring significant gains to the original model. In contrast, when there are multiple assets in the data set in Section 5.3, RSDR CVaR (HMM) shows

**Table 2.** Empirical Results of DeMiguel et al. (2009)'s Data Sets During The Same Time

| Strategy | S&P sectors $I = 11$ | Industry sectors $I = 11$ | International portfolios $I = 9$ | Mkt/SMB/HML $I = 3$ | FF 1-factor $I = 21$ | FF 4-factor $I = 24$ |
|---|---|---|---|---|---|---|
| 1. Sharpe ratio | | | | | | |
| EW | 0.1876 | 0.1353 | 0.1277 | **0.2240** | 0.1623 | 0.1753 |
| RS CVaR (HMM) | 0.0811 | **0.1387** | 0.0947 | 0.2073 | 0.1716 | **0.2554** |
| DR CVaR (Wasserstein) | 0.1876 | 0.1368 | 0.1258 | **0.2240** | **0.1844** | 0.2334 |
| RSDR CVaR (HMM) | **0.1879** | 0.1369 | **0.1304** | **0.2240** | 0.1831 | 0.2185 |
| 2. CEQ | | | | | | |
| EW | **0.0069** | 0.0050 | 0.0046 | **0.0039** | 0.0073 | **0.0072** |
| RS CVaR (HMM) | 0.0025 | 0.0047 | 0.0032 | 0.0035 | 0.0069 | 0.0046 |
| DR CVaR (Wasserstein) | **0.0069** | **0.0051** | 0.0045 | **0.0039** | **0.0080** | 0.0055 |
| RSDR CVaR (HMM) | **0.0069** | 0.0050 | **0.0047** | **0.0039** | **0.0080** | 0.0063 |
| 3. Maximum drawdown | | | | | | |
| EW | 0.3147 | 0.4488 | 0.3941 | 0.1732 | 0.4466 | 0.3896 |
| RS CVaR (HMM) | 0.3860 | **0.4234** | **0.3903** | 0.1912 | **0.3969** | **0.1302** |
| DR CVaR (Wasserstein) | 0.3473 | 0.4495 | 0.3966 | **0.1729** | 0.4172 | 0.2653 |
| RSDR CVaR (HMM) | **0.3056** | 0.4485 | 0.3958 | 0.1732 | 0.4228 | 0.2941 |
| 4. Turnover | | | | | | |
| EW | **0.0376** | **0.0244** | **0.0334** | **0.0266** | **0.0190** | **0.0226** |
| RS CVaR (HMM) | 0.4266 | 0.3481 | 0.4010 | 0.1010 | 0.5461 | 0.2081 |
| DR CVaR (Wasserstein) | 0.0376 | 0.0286 | 0.0381 | **0.0266** | 0.0556 | 0.1729 |
| RSDR CVaR (HMM) | 0.0790 | 0.0289 | 0.0510 | **0.0266** | 0.1253 | 0.1964 |

*Note.* Boldfaced values mean that the corresponding approach is the best by this metric at this given data set.

its significant performance improvement over DR CVaR (Wasserstein) because different asset classes can behave quite differently under different regimes characterized by Bridgewater Associates (2012) (see Figure 1). In summary, RS CVaR (HMM) and DR CVaR (Wasserstein) perform quite differently in different data sets, with no clear winner. RSDR CVaR (HMM) appears to be a robust choice combining the merits of those two.

It can also be seen that the RSDR CVaR (HMM) uniformly outperforms the EW portfolio in terms of Sharpe ratio over all data sets. It is worthwhile to note that DeMiguel et al. (2009) has shown that none of the model-based strategies, including those incorporating the Bayesian framework, portfolio constraints, and various combinations, can uniformly outperform EW portfolio. Therefore, it implies that our proposed RSDR CVaR (HMM) outperforms all the portfolios considered in DeMiguel et al. (2009) in terms of Sharpe ratio. However, the turnovers of the proposed CVaR models compared with the EW portfolio are generally higher. It is reasonable because the proposed models allocate portfolio weights via risk parity, and each asset class has a different risk based on the regime, which implies that there are more occasions to rebalance the portfolio for these models compared with the EW portfolio because the regime changes across time.

We further extend the studies to a data set that is extended from DeMiguel et al. (2009)'s data to the first quarter of 2019, covering the 2008 financial crisis. In general, we observe similar results. We defer the details to Appendix F.1.

### 5.3. Empirical Results for Tradable Assets of Different Classes from 2004 to 2019

Finally, we investigate whether the RSDR Mean-CVaR (HMM) is robust across different data sets and how the regime-switching ambiguity model leads to a smart portfolio allocation under different market conditions. To this end, we propose a data set with multiple asset classes constructed in Section 5.1. We also include other models for comparison and observe that other variations of the model are dominated by RSDR Mean-CVaR (HMM). Note that in this study, we take $M = 36$ because the earliest data we can get starts from January 2004, and we want to analyze our model during the 2008–2009 great recession period.

We present the empirical results for model comparison in Table 3. From the table, we first observe that the RSDR Mean-CVaR (HMM) performs the best among all the models in terms of Sharpe ratio and CEQ. Second, all regime-switching models based on the market and macroeconomic states outperform other benchmarks. Moreover, by comparing our RSDR CVaR models with DR CVaR models, one can see a significant performance improvement by taking regime-switching into account in the DR model. Third, we note that portfolios with RS or DR alone can hardly outperform any benchmark, but RSDR portfolios, especially with HMM, promise a huge advantage over all three benchmarks in terms of Sharpe ratio. It is also noteworthy that when we incorporate the distributional robustness (see the comparison between RS and RSDR models) for each scenario determination, the turnover rate drops by a large margin, whereas the out-of-sample portfolio return and Sharpe ratio are often higher. Finally, our proposed HMM approach is generally better than the frequency-based approach.

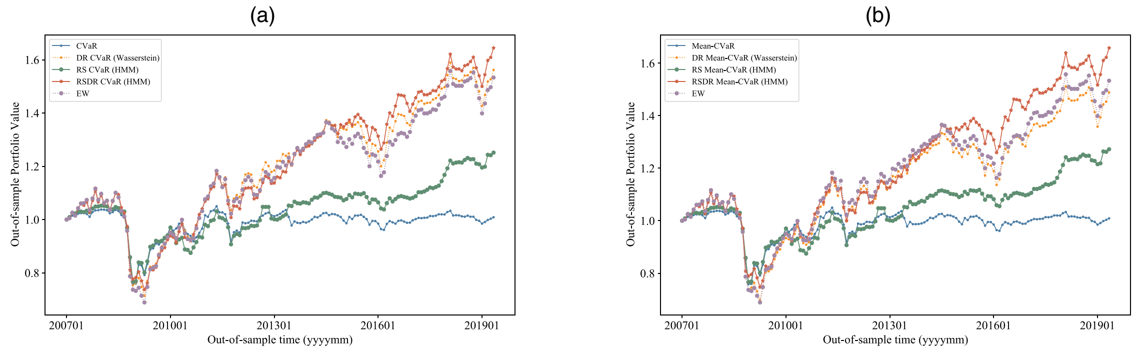
To better illustrate our point of view, we further visualize the portfolio wealth changes over time with an initial wealth of unit 1. We compare two clusters of Mean-CVaR models and CVaR models using different approaches to determining the regimes. The results of our proposed

**Table 3.** Empirical Results for Tradable Assets of Different Classes from 2004 to 2019 Q1

| Strategy | Sharpe ratio | CEQ | Maximum drawdown | Turnover |
|---|---|---|---|---|
| RSDR Mean-CVaR (HMM) | **0.1406** | **0.0036** | 0.3252 | 0.2452 |
| RSDR Mean-CVaR (Weathers) | 0.1197 | 0.0031 | 0.3463 | 0.1844 |
| RSDR Mean-CVaR (bull and bear) | 0.1293 | 0.0033 | 0.2931 | 0.1695 |
| RS Mean-CVaR (HMM) | 0.0818 | 0.0016 | 0.2714 | 0.5866 |
| RS Mean-CVaR (Weathers) | 0.0329 | 0.0005 | 0.2644 | 0.5712 |
| RS Mean-CVaR (bull and bear) | 0.0232 | 0.0003 | 0.2503 | 0.2043 |
| DR Mean-CVaR (Wasserstein) | 0.0788 | 0.0018 | 0.3369 | 0.0811 |
| Mean-CVaR | 0.0140 | 0.0001 | 0.2890 | 0.1854 |
| RSDR CVaR (HMM) | 0.1373 | 0.0035 | 0.3348 | 0.2490 |
| RSDR CVaR (Weathers) | 0.1177 | 0.0031 | 0.3540 | 0.1831 |
| RSDR CVaR (bull and bear) | 0.1272 | 0.0032 | 0.2930 | 0.1887 |
| RS CVaR (HMM) | 0.0770 | 0.0015 | 0.2714 | 0.5812 |
| RS CVaR (Weathers) | 0.0366 | 0.0006 | 0.2644 | 0.5614 |
| RS CVaR (bull and bear) | 0.0110 | (0.0000) | 0.2796 | 0.1934 |
| DR CVaR (Wasserstein) | 0.1081 | 0.0026 | 0.3197 | 0.1924 |
| CVaR | 0.0140 | 0.0001 | 0.2694 | 0.1854 |
| EW | 0.1090 | 0.0029 | 0.3830 | **0.0376** |
| MVP | 0.0297 | 0.0003 | **0.2020** | 0.0995 |
| DR Mean-CVaR (moments) | 0.1185 | 0.0023 | 0.2890 | 0.0515 |

*Note.* Boldfaced values mean that the corresponding approach is the best by this metric.

**Figure 2.** (Color online) Comparison of Different HMM Models

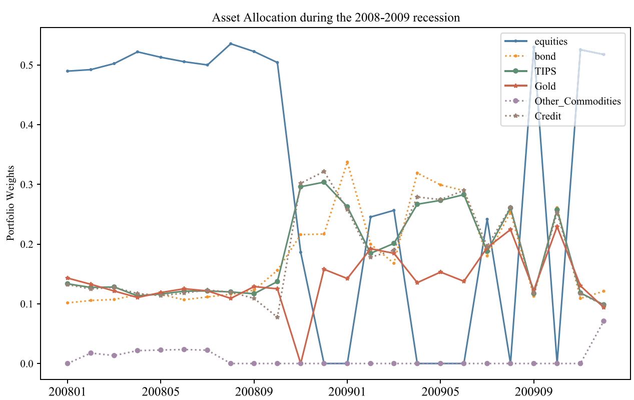



*Note.* (a) CVaR (HMM); (b) Mean-CVaR (HMM).

HMM approaches are shown in Figure 2. The frequency-based approaches present an analogous pattern, and we report the results in Appendix F.3. One can see that the proposed regime-switching ambiguity set outperforms the conventional ambiguity set. Here, the noticeable superiority of regime-switching models is ascribed to the multiclass feature of the data set. Note that in such data, different asset classes such as equities, commodities, and credit may behave differently under different regimes characterized by Bridgewater Associates (2012) (see Figure 1). The superior performance of our proposed approach in this multiclass data set indicates that our method's advantage becomes more significant if the data contain multiple sources of time-varying uncertainties.

Finally, note that the financial crisis in 2008 brought an apparent change in the financial regime. It is interesting to investigate the portfolio's performance during 2008–2009 only. Figure 3 shows how the portfolio weights of different asset classes varied during 2008 for the RSDR CVaR (Weathers). According to the figure, our model can identify the regime change starting around November 2008 and adjust the portfolio weight such that the allocation to asset classes with lower risks, like gold and bond, rises over time, and asset classes with higher risks like equities fall over time. (Note that during the recessionary environment, equities yield negative returns but gold and bond yield positive returns in Figure 1.) Therefore, this allows our model to outperform the EW portfolio and S&P 500 Index in terms of annual returns, which are −33.25% and −38.49%, respectively, whereas RSDR Mean-CVaR (HMM, Weathers, Bull & Bear) yielded "only" −25.85%, −30.03%, and −25.14% annual return in 2008, respectively. Our modeling framework of the regime-switching ambiguity set helps investors allocate their portfolio such that the weightage of highly risky asset classes falls over time to maximize risk-weighted returns in distressed years. We also test different parameter values under the framework of RSDR CVaR models. The obtained insights are similar. We refer interested readers to Appendix F.4 for the detailed results.

**Figure 3.** (Color online) Asset Allocation for RSDR CVaR (Weathers) During the 2008–2009 Great Recession ($\gamma = 0.01$)




In summary, from the performance comparison across different data, we can see that RSDR CVaR (HMM) is very robust in achieving good performance in various data sets. We believe that the superior performance of our algorithm can be observed under other data sets or parameter settings. This is because our intervention in portfolio construction is minimal. We only tune the hyperparameter $\gamma$ (related to robustness) via cross-validation (see Appendix D). From the performance comparison in single-class and multiclass data, we can see that the constructed ambiguity set is well designed for nonstationary financial time series data. Its credibility is linked mainly to the persistence of multiple economic regimes and model uncertainty in each regime. It is one of the insights we bring to the literature.

## 6. Conclusions

We formulate a general framework to model the regime-switching nature of uncertainties in a decision-making environment from a distributionally robust perspective. In particular, we construct a regime-switching ambiguity set and propose two methods to estimate the transition probability and determine the portfolio allocation. We prove that the proposed DRO problem can be reformulated as a linear program (or second-order cone program) that can be solved efficiently. Moreover, we prove that our proposed ambiguity set can include the underlying distribution of random returns with high probability under mild assumptions. To further validate our method, we also demonstrate numerically that our proposed portfolios significantly outperform the benchmark portfolios across various data sets. The improvement is attributed to the regime-switching feature of the model to allocate portfolio weightings based on the risk, which depends on the current environment. In addition, we examine the marginal contributions of regime-switching and DRO models, which validate the necessity of each of them as well as the benefits of merging them, as in our proposal. In summary, our proposed model, RSDR Mean-CVaR (HMM), helps investors allocate their portfolio to achieve a better Sharpe ratio and CEQ than other models.

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