KTH ROYAL INSTITUTE
OF TECHNOLOGY

Degree Project in Technology

First cycle, 15 credits

Portfolio Optimization Problems with Transaction Costs

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Stockholm, Sweden 2023

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Abstract

Portfolio theory is a cornerstone of modern finance, and it is based on the idea that an investor can reduce risk by diversifying their investments across various assets. In practice, Harry Markowitz mean-variance optimization theory is expanded upon by taking into account variable and fixed transaction cost, making the model slightly more reliable. Estimation of parameters is done using historical data and the portfolios considered are those that would be of interest to Generation Z. Using transaction costs from some of Sweden's biggest and most popular banks, the impact of the transaction costs can be seen in the presented graphs. Though many more aspects could be considered to make the model even more realistic, the presented results give insight into how one might want to invest in the stock market to increase their chances of a good expected return given a minimal variance (risk).

Keywords

Portfolio optimization, variable transaction costs, fixed transaction costs

Acknowledgements

We want to thank our supervisor Dr. Shudian Zhao for her help and guidance in this project.

Acronyms

MVO Mean-variance optimization

MPT Modern Portfolio Theory

Contents

1. Introduction

Portfolio theory is a cornerstone of modern finance, and it is based on the idea that an investor can reduce risk by diversifying their investments across various assets. The goal of portfolio theory is to maximize the expected return for a given level of risk, or minimize the risk for a given level of return. The theory assumes that investors are rational and risk-averse, meaning that they will seek to minimize their exposure to risk.

1.1 Background

Harry Markowitz, an economist, is credited with creating the Modern Portfolio Theory (MPT) in 1952 [9]. He developed the theory to address the problem of how to allocate an individual's wealth among various assets with differing levels of risk. Markowitz's theory is based on two factors: risk and return (see section 2.2). In other words, the theory suggests that an investor should consider both the potential return of an investment and the risk involved in making that investment. Some assumptions are made however about the investor's behaviour, see section 2.1.

One of the key concepts of the MPT is the Efficient Frontier. The Efficient Frontier is a graph that shows the optimal mix of assets for a given level of risk. Points on the Efficient Frontier represent portfolios that provide the highest expected return for a given level of risk. The Efficient Frontier is determined by plotting the expected return and risk of various portfolios, and then connecting the dots to form a curve. The portfolios that lie on the Efficient Frontier are considered to be optimal because they provide the best return for a given level of risk. For a more thorough example, see section 2.1.

However, the original method created by Markowitz did not take into account transaction costs. Transaction costs refer to the expenses incurred when buying or selling an asset, such as brokerage fees, taxes, and bid-ask spreads. In this study, two types of transaction costs, fixed and proportional, are considered. The fixed transaction cost refers to a flat fee charged for each transaction, while the proportional transaction cost refers to a percentage of the value of the transaction.

1.2 Literature Review

Prior studies have identified a few problems with mean-variance optimization based on Markowitz's theory. One of the main criticisms of the MVO approach is the fact that the covariance matrix Σ and the expected return vector μ (see section 2.2) are not known and have to be estimated from historical data. This can lead to the so-called "Markowitz optimization enigma" which may lead to the portfolios performing poorly, having counter-intuitive asset allocation weights [7], and being sensitive to errors in these estimations [6].

Even though Markowitz's seminal paper on portfolio selection has made an impact in its field [9], another problem is the simplicity of the model. Thus, it take many years until portfolio managers actually started using portfolio optimization to manage real money. There are many different aspects to take into account for portfolio optimization, many still consider it impractical to apply [6]. In this study we take into account transactions costs, as they can have severe impact in realized risk-adjusted returns [6].

Even though a lot more factors could be considered in this paper to make the model more accurate, it is a good first look into the world of mean-variance optimization with more than just the basic theory that Markowitz proposed in 1952.

1.3 Purpose

This project is investigative in nature and aims to analyze the results of various portfolios and transaction costs. The goal is to determine the optimal asset weights that provide the highest expected return for a given level of risk, while also accounting for transaction costs. The results of this study ultimately do not contribute to novel research within the field, but makes it possible to get more insight into the subject in order to evolve it further in the near future.

2. Theoretical Background

2.1 Markowitz Modern Portfolio Theory

Founder of the Modern Portfolio Theory, Harry Markowitz, set the framework for the selection of a portfolio in 1952. Using statistical measurements of expectation as well as variance of return, Markowitz managed to describe the benefit and risk associated with an investment. The problem has two possible objectives, to either maximize the expected return or to minimize the portfolio variance. The solution for this one period static portfolio is then visualized through the efficient frontier. To find the optimal portfolio, Markowitz formulated a parametric quadratic program problem (QP). However, this method did not include transaction costs, see section 3.1.

Markowitz model is based on some assumptions of an investors behaviour. These are [19]:

- An investor views an investment as a particular expected return.
- An investor maximizes a one-period expected utility.
- An investor relates the risk of a certain portfolio to its variability of expected return.
- An investor bases their decisions solely on risk and expected return.
- An investor prefers higher expected returns to lower ones for a certain level of risk.

The solution to the Markowitz problem can be visualized by the efficient frontier. The curve is created through the two factors standard deviation (on the x-axis) and expected return (on the y-axis). The set of optimal portfolios creates a parabola called the minimum variance set.

When plotting the portfolio frontier one gets a concave curve with points representing the minimal variance. The frontier below in blue are not desirable since there exists another portfolio with the same risk but higher expected return.

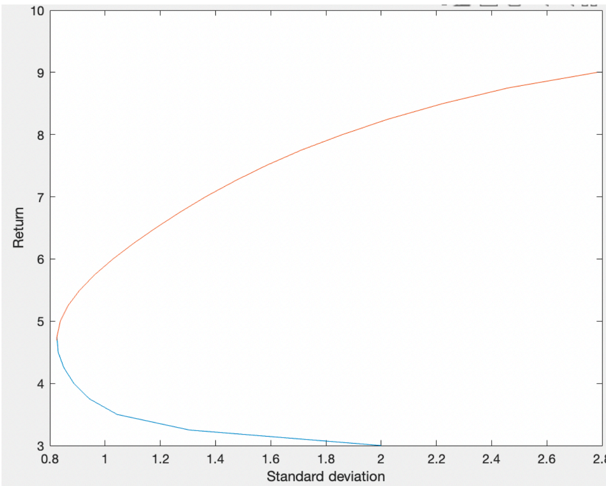

Figure 2.1.1: Red is the Efficient frontier, Blue is Standard MVO problem

The curve can in turn have different appearances depending on the problem formulation. For example one could consider shortselling or allowing to not invest the entire capital. To allow shortselling, one simply exclude the constraint $x \geq 0$ as can be seen in figure 2.1.2. Likewise, to illustrate that the investor does not have to invest the entire capital is represented by the constraint $\mathbf{1}^T x \leq 1$ [4], where $\mathbf{1}^T$ is the all-ones vector, as can be seen in figure 2.1.3.

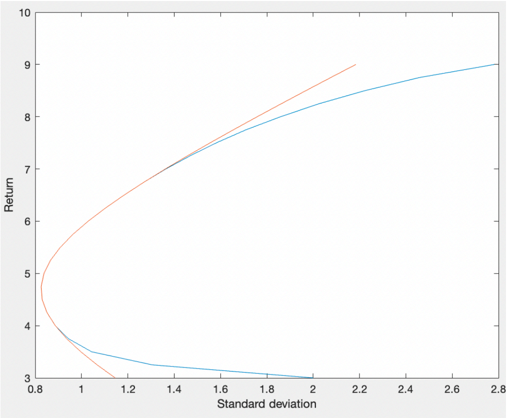

Figure 2.1.2: Red curve allows shortselling, Blue curve is standard MVO problem

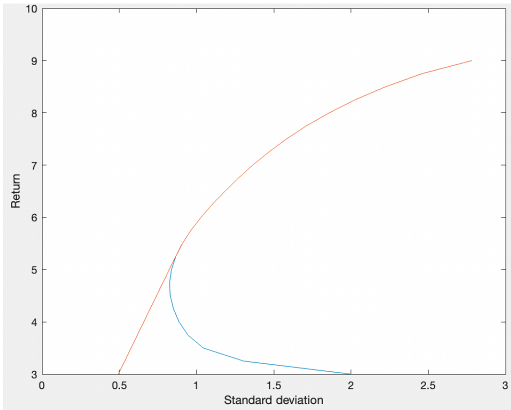

Figure 2.1.3: Red curve represents sum of assets do not have to equal one, Blue curve is standard MVO problem

2.2 Return and risk of a portfolio

When calculating the expected return of a portfolio according to MVO the following variables are essential:

$$\text{portfolio weight of asset } i = x_i = \frac{\text{value of investment}}{\text{total value of portfolio}}.$$

In addition, the variable r_i has to be introduced. r_i represents the expected return of asset i. However, it is not possible to obtain such a value since the market is unpredictable. Hence we approximate it using historical data. It can easily be done because of the following relation:

$$\mathbb{E}(r_p) = \sum \mathbb{E}(x_i r_i) = \sum x_i \, \mathbb{E}(r_i).$$

With these two concepts one can now calculate the expected return of the entire portfolio as:

$$\sum x_i r_i = x_1 r_1 + x_2 r_2 + ... + x_n r_n = r_p.$$

Furthermore, the above expression can be represented as:

$$Var(r_p) = x^T \Sigma x,$$

where x is a vector of all weights for the assets and Σ is the covariance matrix.

The variance of asset i is in the position (i, i) of the matrix and the covariance of asset i and j is at position (i, j) and (j, i) since the matrix is symmetrical.

Consequently, the risk is defined as $\sqrt{Var(r_p))}$ [1].

The risk is represented as the standard deviation or volatility of the portfolio. This is represented as a covariance matrix since there exists a correlation between the assets. One can also show this through the following derivation [1]:

$$Var(r_p) = Cov(r_p, r_p) = Cov(\sum x_i r_i, \sum x_i r_i) = x_i x_j \sum \sum Cov(r_i, r_j).$$

2.3 Transaction Costs

Transaction costs are associated with facilitating a transaction in the market. Examples are a broker's fee or an insurance company's premiums.

To buy or sell any asset in the market, one needs to find a dealer who can provide the desired asset. Then one needs to negotiate the price they are willing to buy or sell for, leading to a bargain. After both parties decides on a price a contract for the sale has to be made and for that legal help has to be ensured. This results in transaction costs, where the expense for availing of the service of the dealer and lawyer. It creates an additional charge the buyer has to pay besides the compromised price [17].

The three types of transaction costs are:

- Search and information costs
  These are the costs associated with looking for relevant information and meeting with agents with whom the transaction will take place. The stock exchange is one such example, as they bring the buyers and sellers of financial assets together. The stockbroker's fee is a type of information transaction costs.

- Bargaining cost
  These are the costs related to coming to an agreement that is agreeable to the parties involved in drawing up a contract. Bargaining costs can either be very

cheap, such as buying a newspaper, or can be very expensive, such as trading a basketball player from one team to another.

- Policing and enforcement costs
  These are the costs associated with making sure that the parties in the contract keep their word and do not default on the terms of the contract. In the real world, people often deviate from the contract, and thus, enforcement costs are incurred while governing contracts. Lawyer fees are an example of such a cost.

The transaction cost can be calculated as the difference between the cost of the particular purchase and the price the buyer has to pay [16].

2.4 Optimization problems

The central problem in optimization theory is typically expressed as a mathematical model as follows:

$$\begin{aligned} &\min f(x), \\ &\text{s.t. } x \in F. \end{aligned} \tag{P}$$

Here, f is a function defined over a set F of feasible solutions, and the set of values that f takes over F is denoted by $S := \{f(x) : x \in F\}$. The set S is a subset of the real numbers and has an infimum, which can be either $-\infty$ or $+\infty$.

The optimal value of problem (P) is defined to be the infimum of S, which is denoted as $\inf S = \inf_{x \in F} f(x)$. A vector $x \in F$ is called a feasible solution to problem (P) if it satisfies the constraints imposed by the problem, and a vector $\hat{x} \in \mathbb{R}^n$ is called an optimal solution to problem (P) if it is feasible and $f(\hat{x})$ is the minimum value of f over F.

The categorization of optimization problems depends on the form of the function $f(x)$ and the associated constraints. Linear programming, quadratic optimization, and nonlinear optimization are among the categories that may be applicable. In this context, we focus on quadratic optimization.

A quadratic function is a function of the form $f(x) = \frac{1}{2}x^T Hx + c^T x + c_0$, where $H \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $c \in \mathbb{R}^n$, and $c_0 \in \mathbb{R}$. In the presence of equality constraints, the quadratic optimization problem is expressed as:

$$\min f(x) = \frac{1}{2}x^T Hx + c^T x + c_0,$$
$$\text{s.t. } Ax = b.$$

If the system $Ax = b$ has no solution (i.e., $b \notin \text{rank}(A)$), then the feasible set F is empty. If $Ax = b$ has exactly one solution, then that solution is the unique optimal solution to the problem. However, if $Ax = b$ has multiple solutions, then any feasible solution can be expressed as a linear combination of a particular solution $\bar{x}$ and a vector in the null space of A, denoted by ker(A). Specifically, $x \in F$ if and only if $x = \bar{x} + Zv$ for some $v \in \mathbb{R}^k$, where k is the dimension of ker(A) and Z is an $n \times k$ matrix whose columns form a basis for ker(A).

An optimization problem is well-posed if the feasible set is convex and the objective function is convex. A convex optimization problem has the form:

$$\min f_0(x) = \frac{1}{2}x^T Hx + c^T x + c_0,$$
$$\text{s.t. } f_i(x) \leq b, i = 1, \ldots, m.$$

$f_0, f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$, where each function is convex, i.e., for any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, we have:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y).$$

Now, consider a quadratic function $f(x) = \frac{1}{2}x^T Hx + c^T x + c_0$, where H is a symmetric matrix in $\mathbb{R}^{n\times n}$, c is a vector in $\mathbb{R}^n$, and c_0 is a scalar in $\mathbb{R}$. The function f is convex if and only if H is positive semi-definite, and f is strictly convex if and only if H is positive definite [13].

In the context of optimization, the Markowitz problem involves continuous variables. However, the project being discussed here involves a mixed-integer problem. An integer linear program is a linear program with the additional constraint that some or all of the variables must be integers. If all variables must be integers, the problem is called a pure integer linear program. If only some variables are required to be integers, while others can be continuous, the problem is a mixed-integer linear program. In some cases, the integer variables are restricted to 0 or 1, and such problems are referred to as pure 0-1 linear programs or pure binary integer linear programs [10].

3. Method

We first start by formulating the problem we want to solve in this paper. We then consider all aspects that have to go in to solving the problem, such as data retrieval, calculating the covariance matrix and the expected return vector and then implementing it all in code.

3.1 Problem formulations

The purpose of the mean-variance optimization problem is to find the portfolio x containing n stocks that has the minimum risk of achieving a certain profit ρ,

$$\begin{aligned} &\min x^\top \Sigma x, \\ &\text{s.t. } \mathbf{1}^\top x = 1, \\ &x \geq 0, \\ &\mu x \geq \rho, \end{aligned} \tag{3.1}$$

where Σ is the covariance matrix, μ is the expected return vector, and $\mathbf{1}$ denotes the all-ones vector. However, what the problem (3.1) fails to take into consideration is that the changes of rebalancing a portfolio, i.e. buying and selling stocks, in reality incurs certain costs. Therefore, both fixed and variable transaction cost will be taken into account when formulating the problem further.

Transaction costs can be used to model a number of costs, such as brokerage fees, bid-ask spreads, taxes or fund loads. As stated previously, the simplest model would be to have no transaction costs, but a better model of realistic transaction costs is a linear one, where the transaction cost is proportional to the amount traded [8]:

$$C_i(\tilde{x}_i) = \begin{cases} v_i^+ \tilde{x}_i, & \tilde{x}_i \geq 0, \\ v_i^- \tilde{x}_i, & \tilde{x}_i < 0. \end{cases} \tag{3.2}$$

Here v_i^+ and v_i^- are the cost rates associated with buying and selling asset i respectively and the transaction volume is denoted as $\tilde{x} = x - x_0$, where $x_0 \in \mathbb{R}^n$ is the initial holding. One of the constraints in (3.1) will also be changed for the self-financing property of the portfolio, i.e. no external cash is put into or taken out of the portfolio, the costs are paid from the existing portfolio components:

$$\mathbf{1}^T x + \sum_i C_i(\tilde{x}_i) = \mathbf{1}^T x_0. \tag{3.3}$$

As the linear transaction cost functions C_i in (3.2) are clearly convex, the budget constraint (3.3) can also be handled by convex optimization. By introducing the variables $\tilde{x}^+ \geq 0$ and $\tilde{x}^- \geq 0$ and expressing the total transaction volume as $\tilde{x} = \tilde{x}^+ - \tilde{x}^-$, the transaction cost function (3.2) can be represented as $C_i = v_i^+ \tilde{x}_i^+ + v_i^- \tilde{x}_i^-$.

The fact is that convex optimization problems, even if non-linear or of large-scale, can be numerically solved with great efficiency. However, in practice, transaction costs are generally not convex functions of the amount traded and are more likely to rather be concave [8]. To include fixed transaction costs into the problem, a simple model with fixed and variable transaction costs will be considered. Let f_i^+ and f_i^- be the fixed costs associated with buying and selling asset i. The cost function (3.2) can be further modified as:

$$C_i(\tilde{x}_i) = \begin{cases} 0, & \tilde{x}_i = 0, \\ f_i^+ + v_i^+ \tilde{x}_i, & \tilde{x}_i > 0, \\ f_i^- - v_i^- \tilde{x}_i, & \tilde{x}_i < 0. \end{cases} \tag{3.4}$$

This is clearly no longer convex unless the fixed transaction costs are zero, meaning that the budget constraint (3.3) can no longer be handled by convex optimization.

The question is how we can model this so that the problem becomes solvable and how we can then use this model to investigate how the transaction costs affect how the optimal portfolio is chosen. This we will tackle by using the Python extension package `gurobipy`[5], see section 3.4.

3.2 Data retrieval

The youth demographic is often cited as a risk group in the realm of private economic matters due to their relative inexperience in financial management. However, according to data from Klarna's "Money Management Pulse Report", the Swedish youth exhibit a greater inclination to invest their financial resources in comparison to other age cohorts. Notably, a staggering 90% of individuals between the ages of 18 and 35 are actively saving, exceeding the national average of 84% [18]. These findings position Generation Z and Millennial populations as compelling groups warranting further analysis and scrutiny.

Thus, the selection of stocks will be grounded on statistical data from APEX Clearing, which highlights the 20 most alluring stocks based on preferences expressed by younger generations. The goal is that the resulting portfolio will be representable for this generation and hence used for our calculations and analysis.

In order to facilitate sound investment decisions based on the aforementioned 20 stocks preferred by younger generations, historical data for these stocks is extracted from Yahoo via the `yfinance` [20] Python package. The chosen data encompasses a daily time-series from 2017 to 2019, providing a comprehensive view of performance over a significant period. Moreover, alongside historical data, the beta value for each stock is also gathered, providing valuable insight into the relationship between each stock and the broader market. This information is indispensable for investors seeking to make informed decisions based on a nuanced understanding of the market dynamics underlying each stock.

To account for transaction costs associated with trading these selected stocks, we have examined the transaction costs charged by various banks commonly used for trading, including Avanza [12], Handelsbanken [11], and Nordea [2]. This data is found on the website of each bank and it is the costs for trading on the american market.

3.3 Estimating the covariance matrix and expected return vector

As stated previously, the covariance matrix Σ and the expected return vector μ have to be estimated from historical data. The historical data is gathered from Python's `yfinance` package. A reasonable approach for estimating Σ and μ is to use a time series of past returns. Unfortunately, it has been observed that small changes in the time series can lead to significant changes in the optimal portfolio [1]. See section 5.2 for discussion regarding this. For the simplicity of the estimation and to remain within the scopes of the project we use the time series estimation:

- r_{it} - return of asset i in period t
- r_{mt} - market return in period t
- r_{ft} - return of risk-free asset in period t
- ε_{it} - idiosyncratic risk
- β_i - beta value for asset i

Starting of with the covariance matrix, the following formulas are use to calculate each element:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2$$

$$\sigma_{ii} = \beta_i \beta_i \sigma_m^2 + \sigma_{\epsilon_i}^2.$$

To get σ_{ϵ_i}, ε_{it} has to be calculated from this relation:

$$r_{it} - r_{ft} = \beta_i (r_{mt} - r_{ft}) + \varepsilon_{it}.$$

In turn, the expected return vector can be calculated. The expected value is defined as the mean value. This is based upon the Central Limit Theorem.

$$\mu_i - \mathbb{E}(r_f) = \beta_i (\mathbb{E}(r_m) - \mathbb{E}(r_f)).$$

The calculated covariance matrix Σ and expected return vector μ used in this report can be seen in Appendix A.

3.4 Gurobi

The Branch-and-Bound algorithm is a widely used optimization technique for solving mixed-integer programming (MIP) problems. It is an iterative process that starts with the original MIP problem, denoted as P_0, and seeks to find the optimal solution by exploring a search tree of nodes.

At each step of the algorithm, an unexplored node, denoted as P, is selected from the search tree. The LP relaxation of P is then solved to obtain a fractional solution. If the LP solution satisfies all integer constraints, it is considered as a potential solution to the MIP problem. If not, the algorithm proceeds to choose a fractional variable, x, in the LP solution and creates two new MIPs.

As the algorithm explores the search tree, it maintains an upper bound on the objective function value, which is updated whenever an integer solution with a better objective function value is found. If a node's objective function value is worse than the current upper bound, it is fathomed, which means that the branch of the search tree rooted at that node is pruned.

The algorithm terminates when all nodes in the search tree have been explored or fathomed, and the incumbent node with the best integer solution is returned as the optimal solution to the MIP problem.

The Branch-and-Bound algorithm is a powerful optimization technique that can solve MIP problems to obtain the optimal solution. By exploring the search tree of the problem, it can find the optimal solution by iteratively generating child nodes, pruning the branches that do not lead to an optimal solution, and updating the upper bound of the objective function as it progresses through the tree [10].

3.5 Delimitations

3.5.1 Market behaviour

The data employed to approximate the anticipated return vector may not always be indicative of future performance. As widely acknowledged, the market is characterized by high levels of unpredictability, and unforeseeable events such as pandemics or conflicts could transpire, resulting in outcomes that diverge significantly from expectations [15].

3.5.2 Personal preferences

The Markowitz model relies on several assumptions regarding investor behavior (see section 2.1). Although many of these criteria may be satisfied, there will inevitably be additional biases present. For instance, many individuals may hold ethical values that preclude them from investing in the alcohol or weapons industries. Additionally, an investor's personal relationships with a particular company could motivate them to include it in their portfolio, despite unfavorable performance indicators.

3.5.3 Additional costs

Custody charges are fees for holding and safeguarding securities within an investor's portfolio [3]. These fees may vary based on factors such as the type and quantity of securities held, the size of the portfolio, and the level of service provided. Other costs include taxes on realized gains and management fees.

4. Result

4.1 Data

From APEX Clearing, the following statistical data was obtained, highlighting the 20 most alluring stocks based on preferences expressed by the younger generations. The stocks with the corresponding weights found in table 4.1.1 will be the basis for out calculations.

Ticker	TSLA	AAPL	AMZN	NIO	DIS	MSFT	NVDA	NFLX	FB	AMD	BA	BABA	BFKB	GOOGL	SQ	T	PLTR	CCL	AAL	GOOG
%	17.9	14.0	7.5	4.4	3.2	3.0	1.7	1.7	1.5	1.5	1.4	1.3	1.3	1.1	1.0	1.0	1.0	0.9	0.9	0.8

Table 4.1.1: Most popular stocks according to younger generation [14]

Through `yfinance` the data mentioned in section 3.2 was collected for the stocks in 4.1.1. A covariance matrix and expected return vector was calculated as defined in section 3.3. The covariance matrix Σ and expected return vector μ can be seen in Appendix A.

From the websites of the chosen banks (see section 3.2) the information found in table 4.1.2 about transaction costs was gathered.

Table 4.1.2: Courtage and minimum costs for the selected banks

Model	Courtage	Minimum cost
Handelsbanken [11]	0.09%	59 SEK
Nordea [2]	0.09%	29 SEK
Avanza [12]	0.25%	10 SEK

4.2 Different transaction costs

The presented results highlight the significant impact of transaction costs on investment portfolio performance, as evidenced by the comparison of various brokerage fees.

As can be seen in tables 4.2.1 and 4.2.2 below, the results further suggest that Nordea bank offers the most favorable risk-return portfolio among the evaluated institutions, followed by Avanza and Handelsbanken. However, it is important to note that these conclusions are based on the data and assumptions used in the comparison, and may not necessarily hold true for all investors or under different market conditions.

In addition, the objective of minimizing transaction costs when investing in order to achieve the most favorable risk-return profile is widely accepted in the field of finance, and the conclusion that having no transaction costs at all would enable the achievement of an optimal portfolio without additional costs is consistent with this principle.

Table 4.2.1: Comparing variance between banks for a set value of expected return

Model	Return $1.210 \cdot 10^{-2}$	Return $8.100 \cdot 10^{-3}$
No transaction cost	$2.563 \cdot 10^{-3}$	$1.558 \cdot 10^{-3}$
Handelsbanken	$2.583 \cdot 10^{-3}$	$1.568 \cdot 10^{-3}$
Nordea	$2.574 \cdot 10^{-3}$	$1.564 \cdot 10^{-3}$
Avanza	$2.579 \cdot 10^{-3}$	$1.567 \cdot 10^{-3}$

Table 4.2.2: Change of variance in percentage compared to model with no courtage for set values of expected return

Model	Return 0.001210	Return 0.000810	Courtage fee	Minimal cost
Handelsbanken	0.60%	0.7%	0.09%	59 SEK
Nordea	0.30%	0.4%	0.09%	29 SEK
Avanza	0.58%	0.6%	0.25%	10 SEK

The following figures 4.2.1, 4.2.2 and 4.2.3 display the efficient frontier for each of the models in comparison to the standard Mean-variance optimization (MVO) with no fees taken into account. As the variance increases, the red line progressively deviates from the blue line. In other words, as the variance becomes greater, the impact of transaction costs on returns becomes more pronounced.

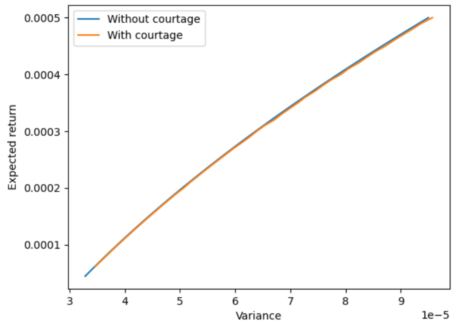

Figure 4.2.1: Handelsbanken

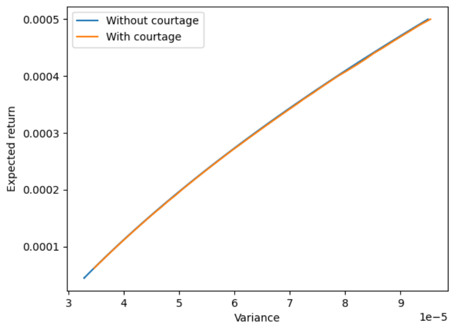

Figure 4.2.2: Nordea

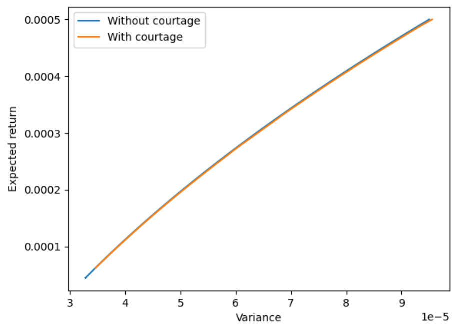

Figure 4.2.3: Avanza

An additional remark is that transaction costs have a significant impact on the buy and sell decisions, as illustrated by the fact that fewer assets are purchased when transaction costs are involved, particularly the fixed costs. This is because the model tends to avoid having to pay the fixed costs, resulting in lower overall asset purchases.

Regardless of the specific banks considered, the impact of different costs on the efficient frontier can be drawn. Specifically, the proportional costs tend to keep the efficient frontier looking smooth, but at the cost of pushing it down. On the other hand, the fixed costs tend to make the efficient frontier looking uneven and push it down. These observations are clearly illustrated by the presented graphs 4.2.4 and 4.2.5.

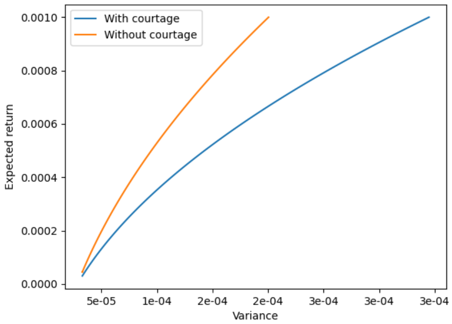

Figure 4.2.4: High proportional cost

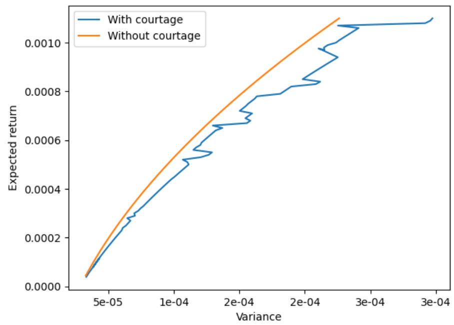

Figure 4.2.5: High fixed cost

The aforementioned findings underscore the crucial significance of meticulously contemplating the proportionate and fixed expenses while making investment decisions, given their momentous influence on the structure and placement of the efficient frontier. It is imperative to note that the term "high cost" pertains to a percentage of the entire portfolio capital, implying that transaction costs will exert a more substantial impact on a portfolio with greater monetary value.

4.3 Different initial portfolios

The study also examines the impact of different initial holdings on the resulting portfolios. Along with the previously investigated "Generation Z" portfolio an additional portfolio was considered. These portfolios provide a broader perspective on the effects of initial holdings on investment performance, and help illustrate the benefits of diversification and optimal portfolio selection.

The additional portfolio is an equal-weight portfolio, meaning that it has equal weights assigned to all assets, creating a well-diversified initial holding. This approach examines the impact of starting with a diversified portfolio, and how the model adjusts the asset allocation to achieve the desired risk-return profile.

Among the two different portfolios, the evenly spread one was found to be the most beneficial. The presented data points in table 4.3.1 and graphs 4.3.1 and 4.3.2 clearly illustrate the impact of the initial portfolio composition on the final result. Here, one can see that the equal-weights portfolio provides the lowest variance for the same expected return.

Table 4.3.1: Comparing variance between initial portfolios, with Avanza courtage, for some fixed values on expected return

Initial portfolio	Return $1.210 \cdot 10^{-2}$	Return $1.610 \cdot 10^{-2}$
Equal weights	$2.581 \cdot 10^{-3}$	$3.840 \cdot 10^{-3}$
Gen Z	$2.589 \cdot 10^{-3}$	$3.853 \cdot 10^{-3}$

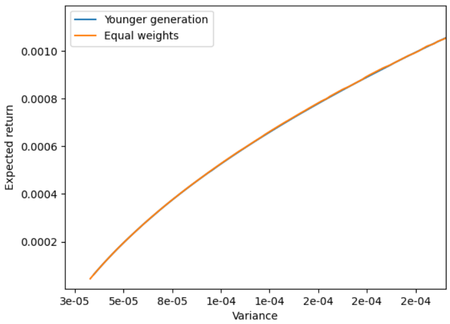

Figure 4.3.1: Efficient frontier for different initial portfolios

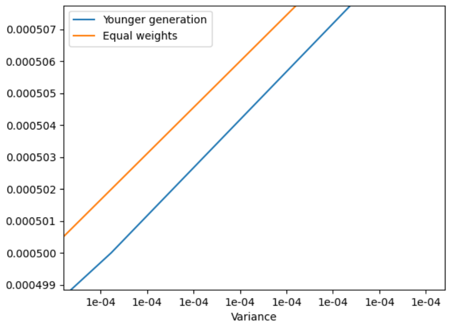

Figure 4.3.2: Close up on efficient frontier for different initial portfolios

Upon repeated execution of the code, wherein the resulting portfolio from each previous iteration is used as the initial portfolio for the subsequent iteration, an interesting observation can be made. Both portfolios, namely the "Generation Z" portfolio and the equal-weight portfolio, ultimately converge to a uniform portfolio. This convergence signifies that regardless of the initial holdings, the investment model identifies a common optimal allocation of assets that best suits the desired risk-return profile. This phenomenon demonstrates the power of the optimization process, as it consistently identifies a similar optimal portfolio regardless of the initial conditions. Furthermore, it is evident that the portfolio which is closest to the "limit portfolio" exhibits the most optimal performance. The "limit portfolio" can be understood as an idealized portfolio that represents the highest level of risk-adjusted return achievable under the given constraints and market conditions.

As a concrete result, a fixed value of return 0.0001, will after 10 iterations make both of the portfolios have the following proportions:

Ticker	TSLA	AAPL	AMZN	NIO	DIS	MSFT	NVDA	NFLX	FB	AMD	BA	BABA	GOOGL	SQ	T	CCL	AAL	GOOG
%	0	0	0	0	0	0	0	0	0	0	0	0.654	0	0	0.346	0	0	0

Table 4.3.2: Portfolio after 10 iterations with return constraint set to 0.0001

In addition, with the same execution but with a return of 0.0002 the following is obtained after 500 iterations:

Ticker	TSLA	AAPL	AMZN	NIO	DIS	MSFT	NVDA	NFLX	FB	AMD	BA	BABA	GOOGL	SQ	T	CCL	AAL	GOOG
%	0	0	0	0	0	0	0	0	0	0	0	0.0227	0	0	0.977	0	0	0

Table 4.3.3: Portfolio after 500 iterations with return constraint set to 0.0002

For higher values of returns the algorithm does not converge as easily. The general trend seen is that the convergence time increases rapidly with the return.

5. Conclusion

The results of this thesis provide valuable insights into the impact of transaction costs on investment portfolio performance. One of the key findings is that higher transaction costs, whether fixed or proportional, lead to greater variance for a given return. This highlights the importance of minimizing transaction costs in order to achieve the most favorable risk-return profile.

The work also compared the performance of different brokerage firms and found that Nordea bank offers the most favorable risk-return portfolio among the evaluated institutions, followed by Avanza and Handelsbanken. This also gives an insight of what combination of fixed and variable cost is most beneficial for an investor. However, it is important to note that these conclusions are based on the data and assumptions used in the comparison, and may not necessarily hold true for all investors or under different market conditions. Furthermore, these are only three of many different banks and there might be an even more beneficial courtage fees belonging to another bank.

Furthermore, the impact of different types of transaction costs on the efficient frontier was investigated. Proportional costs tend to keep the efficient frontier smooth, but at the cost of pushing it down. On the other hand, fixed costs tend to make the efficient frontier uneven and push it down. These observations highlight the need for careful consideration of proportionate and fixed expenses when making investment decisions. These findings are however impacted on the size of the portfolio. If the investor has a lot of money these findings will be more significant. Having a small capital and investing through a bank with high minimal cost creates great fluctuations in the efficient frontier. This makes the problem of finding the optimal portfolio even more difficult.

In addition, the thesis considered the impact of different initial portfolio compositions on the resulting portfolios. The evenly spread portfolio was found to be the most beneficial, while the portfolio composed solely of stocks based on what is popular with the younger generation performed the worst. These results emphasize the importance of diversification in portfolio management. It also implies that the preferences of the younger generation is not optimal from a MVO point of view. This is interesting, given that they tend to be exposed to a higher risk already having a greater proportion of their total assets and savings invested in the market. Therefore, it is of extra importance for the youths to make sure to have the optimal low-risk portfolio.

Another important insight provided by the work is the impact of transaction costs on buy and sell decisions. The model tends to avoid having to pay fixed costs, resulting in lower overall asset purchases. This underscores the need for careful consideration of transaction costs when making investment decisions, as these costs can have a significant impact on the structure and placement of the efficient frontier. Once again, the size of the portfolio will be significant for this result.

Overall, the results of this thesis highlight the crucial significance of meticulously contemplating transaction costs and portfolio composition when making investment decisions, given their momentous influence on portfolio performance. Depending on the size of the portfolio a standard MVO model can be a good approximation since the differences for the investigated 100,000 SEK portfolio was rather small. The study's findings provide valuable insights into the complex interplay between transaction costs, portfolio composition, and efficient frontier placement, and can help inform investment decisions in a variety of contexts.

5.1 Discussion

Beyond the key findings previously discussed, the thesis also delves into the impact of rebalancing frequency and the associated transaction costs on portfolio performance. Regular portfolio rebalancing is a strategy used by investors to maintain their desired risk-return profile. However, this work demonstrates that more frequent rebalancing can lead to higher transaction costs, which in turn negatively affects portfolio performance. Investors should, therefore, find a balance between maintaining their target asset allocation and minimizing transaction costs associated with rebalancing. This trade-off will depend on factors such as market volatility, investment horizon, and the investor's risk tolerance.

Another noteworthy insight from the work is the role of technology in reducing transaction costs. With the rise of fintech solutions and robo-advisors, investors have access to platforms that can help them optimize their investments with lower fees. Embracing technology and leveraging digital tools can lead to more efficient portfolio management, ultimately resulting in reduced transaction costs and improved portfolio performance.

The results and analysis also highlights the significance of investor education and financial literacy in making informed investment decisions. Investors who understand the importance of diversification, risk-return trade-offs, and the impact of transaction costs on their investments are more likely to make better decisions, leading to more favorable portfolio performance. Educating investors about the complex relationships among these factors can empower them to make more prudent choices in the pursuit of their financial goals.

5.2 Future Work

As briefly mentioned in section 3.3, the time series approach of estimating the covariance matrix Σ and the expected return vector μ is far from ideal. Small changes in the time series have been observed to make a significant difference in the optimal portfolio [1], which means that the reliability of the result is put in to question. This approach was considered and used for its simplicity and to keep within the scope of this project. However, there are other alternatives for estimating these parameters which one might take into consideration, for example Bayesian techniques or the Black-Litterman model [6]. The robust optimization methodology as well has emerged in the past few decades to answer some of the concerns regarding estimation errors and reliability of the decisions derived using portfolio optimization methods [6]. These methods could be considered for future projects or simply as an extension of this one.

In this project, we added the constraints of variable and fixed transaction cost in order to more accurately represent the real world. However, there are many more things that will have an effect on a portfolio and one's investments. In the paper *60 Years of portfolio optimization: Practical challenges and current trends* written by Kolm et al. [6], many more extensions of the original approach proposed by Markowitz are suggested. These include, but are not limited to, tax effects, various types of constraints that take specific investment guidelines and institutional features into account, and multi-period extensions of MVO to incorporate intertemporal effects such as hedging needs, changing market conditions, market impact costs, and alpha decay [6]. While some of these may be more complicated to implement, they will most likely contribute to a better representation of the real world in order to optimize our portfolio.

5.3 Final Words

We have come a long way since Markowitz published his seminal paper just over seventy years ago. Although the model in this paper could have been more extensive to try to achieve the best model currently possible, there still is no such thing as the perfect portfolio optimization methodology due to the uncertainty inherently present.

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Appendix - Contents

A. Covariance matrix and expected return vector

The covariance matrix Σ and the expected return vector μ have been reduced to only four decimals and the factor 10^{-4} has been factored out of the matrix and vector respectively for visual clarity. In the real calculations the values are not approximated in this way.

$$\Sigma = \begin{bmatrix}
3.9312 & 2.3848 & 2.2730 & 3.5027 & 2.4034 & 1.7140 & 3.3350 & 2.2544 & 2.2730 & 3.6890 & 2.6270 & 1.1365 & 2.0308 & 4.3411 & 1.4159 & 4.1548 & 2.8692 & 2.0308 \\
2.3848 & 1.4467 & 1.3789 & 2.1248 & 1.4580 & 1.0398 & 2.0231 & 1.3676 & 1.3789 & 2.2378 & 1.5936 & 0.6894 & 1.2319 & 2.6334 & 0.8589 & 2.5204 & 1.7405 & 1.2319 \\
2.2730 & 1.3789 & 1.3142 & 2.0252 & 1.3896 & 0.9910 & 1.9283 & 1.3034 & 1.3142 & 2.1329 & 1.5189 & 0.6571 & 1.1742 & 2.5100 & 0.8187 & 2.4023 & 1.6589 & 1.1742 \\
3.5027 & 2.1248 & 2.0252 & 3.1209 & 2.1414 & 1.5272 & 2.9715 & 2.0086 & 2.0252 & 3.2869 & 2.3406 & 1.0126 & 1.8094 & 3.8679 & 1.2616 & 3.7019 & 2.5564 & 1.8094 \\
2.4034 & 1.4580 & 1.3896 & 2.1414 & 1.4694 & 1.0479 & 2.0389 & 1.3782 & 1.3896 & 2.2553 & 1.6061 & 0.6948 & 1.2415 & 2.6540 & 0.8657 & 2.5401 & 1.7541 & 1.2415 \\
1.7140 & 1.0398 & 0.9910 & 1.5272 & 1.0479 & 0.7473 & 1.4541 & 0.9829 & 0.9910 & 1.6084 & 1.1454 & 0.4955 & 0.8854 & 1.8928 & 0.6173 & 1.8115 & 1.2510 & 0.8854 \\
3.3350 & 2.0231 & 1.9283 & 2.9715 & 2.0389 & 1.4541 & 2.8292 & 1.9125 & 1.9283 & 3.1295 & 2.2286 & 0.9641 & 1.7228 & 3.6827 & 1.2012 & 3.5247 & 2.4341 & 1.7228 \\
2.2544 & 1.3676 & 1.3034 & 2.0086 & 1.3782 & 0.9829 & 1.9125 & 1.2928 & 1.3034 & 2.1155 & 1.5065 & 0.6517 & 1.1645 & 2.4894 & 0.8120 & 2.3826 & 1.6453 & 1.1645 \\
2.2730 & 1.3789 & 1.3142 & 2.0252 & 1.3896 & 0.9910 & 1.9283 & 1.3034 & 1.3142 & 2.1329 & 1.5189 & 0.6571 & 1.1742 & 2.5100 & 0.8187 & 2.4023 & 1.6589 & 1.1742 \\
3.6890 & 2.2378 & 2.1329 & 3.2869 & 2.2553 & 1.6084 & 3.1295 & 2.1155 & 2.1329 & 3.4617 & 2.4651 & 1.0664 & 1.9057 & 4.0736 & 1.3287 & 3.8988 & 2.6924 & 1.9057 \\
2.6270 & 1.5936 & 1.5189 & 2.3406 & 1.6061 & 1.1454 & 2.2286 & 1.5065 & 1.5189 & 2.4651 & 1.7555 & 0.7594 & 1.3570 & 2.9009 & 0.9462 & 2.7764 & 1.9173 & 1.3570 \\
1.1365 & 0.6894 & 0.6571 & 1.0126 & 0.6948 & 0.4955 & 0.9641 & 0.6517 & 0.6571 & 1.0664 & 0.7594 & 0.3285 & 0.5871 & 1.2550 & 0.4093 & 1.2011 & 0.8294 & 0.5871 \\
2.0308 & 1.2319 & 1.1742 & 1.8094 & 1.2415 & 0.8854 & 1.7228 & 1.1645 & 1.1742 & 1.9057 & 1.3570 & 0.5871 & 1.0491 & 2.2425 & 0.7314 & 2.1463 & 1.4822 & 1.0491 \\
4.3411 & 2.6334 & 2.5100 & 3.8679 & 2.6540 & 1.8928 & 3.6827 & 2.4894 & 2.5100 & 4.0736 & 2.9009 & 1.2550 & 2.2425 & 4.7937 & 1.5636 & 4.5880 & 3.1684 & 2.2425 \\
1.4159 & 0.8589 & 0.8187 & 1.2616 & 0.8657 & 0.6173 & 1.2012 & 0.8120 & 0.8187 & 1.3287 & 0.9462 & 0.4093 & 0.7314 & 1.5636 & 0.5100 & 1.4965 & 1.0334 & 0.7314 \\
4.1548 & 2.5204 & 2.4023 & 3.7019 & 2.5401 & 1.8115 & 3.5247 & 2.3826 & 2.4023 & 3.8988 & 2.7764 & 1.2011 & 2.1463 & 4.5880 & 1.4965 & 4.3911 & 3.0324 & 2.1463 \\
2.8692 & 1.7405 & 1.6589 & 2.5564 & 1.7541 & 1.2510 & 2.4341 & 1.6453 & 1.6589 & 2.6924 & 1.9173 & 0.8294 & 1.4822 & 3.1684 & 1.0334 & 3.0324 & 2.0941 & 1.4822 \\
2.0308 & 1.2319 & 1.1742 & 1.8094 & 1.2415 & 0.8854 & 1.7228 & 1.1645 & 1.1742 & 1.9057 & 1.3570 & 0.5871 & 1.0491 & 2.2425 & 0.7314 & 2.1463 & 1.4822 & 1.0491
\end{bmatrix} \cdot 10^{-4}$$

$$\mu = \begin{bmatrix} 1.6427 & 0.7586 & 0.6946 & 1.3977 & 0.7692 & 0.3751 & 1.3018 & 0.6840 & 0.6946 & 1.5042 & 0.8970 & 0.0448 & 0.5562 & 1.8771 & 0.2046 & 1.7705 & 1.0355 & 0.5562 \end{bmatrix} \cdot 10^{-4}$$