Digital Communication

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CONTENTS

Chapter 1 Two Dice

1.1 Sum of Independant Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1.1.1 *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.1.1.2}$$

$$\implies X \in \{1,2,\ldots,12\} \tag{1.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.1.1.4}$$

1.1.2 *Convolution:* From (1.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.1.2.1}$$

$$= \sum_k \Pr(X_1 = n-k | X_2 = k)\, p_{X_2}(k) \tag{1.1.2.2}$$

after unconditioning. $\because X_1$ and X_2 are independent,

$$\Pr(X_1 = n-k | X_2 = k) = \Pr(X_1 = n-k) = p_{X_1}(n-k) \tag{1.1.2.3}$$

From (1.1.2.2) and (1.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (1.1.1.1) in (1.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.1.2.6}$$

From (1.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.1.2.7}$$

Substituting from (1.1.1.1) in (1.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.1.2.8}$$