

Digital Modulation Techniques

P. N. V. S. S. K. HAVISH, S. S. Ashish and G V V Sharma

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Abstract—The manual frames the problems of receiver design and performance analysis in digital communication as applications of probability theory.

1 BPSK

Problem 1. The *signal constellation diagram* for BPSK is given by Fig. 1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

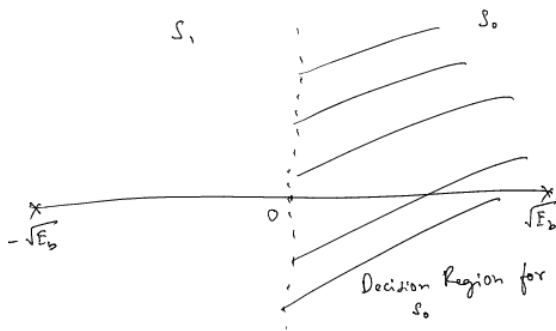


Fig. 1

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (2)$$

The authors are with the Department of Electrical Engineering, IIT, Hyderabad 502285 India e-mail: {ee16btech11023, ee16btech11043, jbala, gadepall}@iith.ac.in. All material in the manuscript is released under GNU GPL. Free to use for all.

where the AWGN $n \sim \mathcal{N}(0, \frac{N_0}{2})$.

Problem 2. From Fig. 1 obtain a decision rule for BPSK

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\geq}} 0 \quad (3)$$

Problem 3. Repeat the previous exercise using the MAP criterion.

Problem 4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0|s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (4)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (5)$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in (5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \quad (6)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \quad (7)$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

Problem 5. The PDF of $w \sim \mathcal{N}(0, 1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \quad (8)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (9)$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad (10)$$

Problem 6. Verify the bit error rate (BER) plots for

BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
import numpy as np
import mpmath as mp
import matplotlib.pyplot as plt

def qfunc(x):
    return 0.5*mp.erfc(x/mp.
        sqrt(2))

#Number of SNR samples
snrlen = 10
#SNR values in dB
snrdb = np.linspace(0,9,10)
#Number of samples
simlen = int(1e5)
#Simulated BER declaration
err = []
#Analytical BER declaration
ber = []

#for SNR 0 to 10 dB
for i in range(0,snrlen):
    #Generating AWGN, 0 mean
    #unit variance
    noise = np.random.normal
        (0,1,simlen)
    #from dB to actual SNR
    snr = 10**((0.1*snrdb[i]))
    #Received symbol in
    #baseband
    rx = mp.sqrt(snr) + noise
    #storing the index for the
    #received symbol
    #in error
    err_ind = np.nonzero(rx <
        0)
    #calculating the total
    #number of errors
    err_n = np.size(err_ind)
    #calculating the simulated
    #BER
    err.append(err_n/simlen)
    #calculating the
    #analytical BER
    ber.append(qfunc(mp.sqrt(
        snr)))
```

```
plt.semilogy(snrdb.T,ber,label='
    Analysis')
plt.semilogy(snrdb.T,err,'o',label
    ='Sim')
plt.xlabel('SNR$\\left(\\frac{E_b
    }{N_0}\\right)$')
plt.ylabel('$P_e$')
plt.legend()
plt.grid()
#plt.savefig('../figs/bpsk_ber.eps
    ')
plt.show()
```

yields Fig. 2

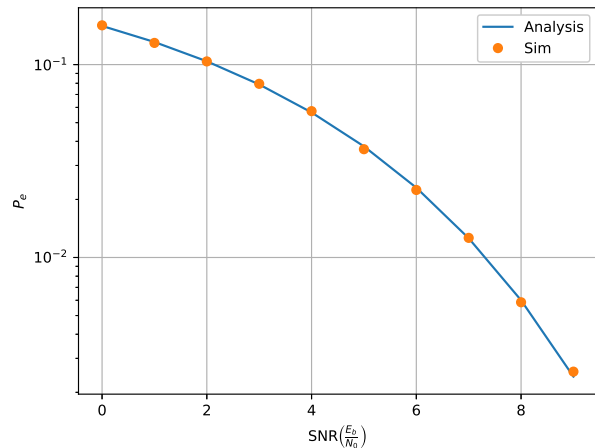


Fig. 2

Problem 7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (11)$$

2 COHERENT BFSK

Problem 8. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 3. Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|_{s_0} = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (12)$$

and

$$\mathbf{y}|_{s_1} = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (13)$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

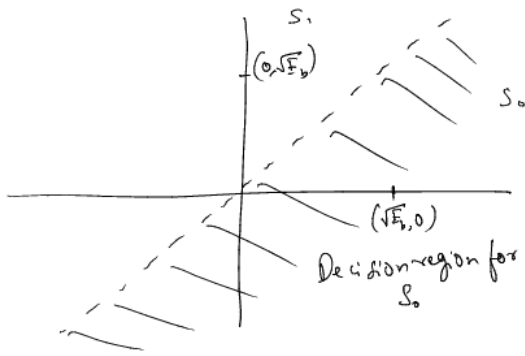


Fig. 3

Problem 9. Obtain a decision rule for BFSK from Fig. 3.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\geq}} y_2 \quad (14)$$

Problem 10. Repeat the previous exercise using the MAP criterion.

Problem 11. Derive and plot the probability of error. Verify through simulation.

3 QPSK

Problem 12. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \quad (15)$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \quad (16)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \quad (17)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \sigma^2 \mathbf{I} \quad (18)$$

1) Show that the MAP decision for detecting s_0 results in

$$|r|_2 < r_1 \quad (19)$$

2) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2 . Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (20)$$

X and Y are said to be uncorrelated if $\rho = 0$

- 3) Show that if X and Y are uncorrelated Verify this numerically.
- 4) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.
- 5) Show that $X, Y \sim \mathcal{N}(0, 2\sigma^2)$.
- 6) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
- 7) Find $\Pr(X < A, Y < A)$.
- 8) Verify the above through simulation.

4 M-PSK

Problem 13. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \sin\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots, M-1$. Let

$$\mathbf{r} | s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (21)$$

where $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$.

1) Substituting

$$r_1 = R \cos \theta \quad (22)$$

$$r_2 = R \sin \theta \quad (23)$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \quad (24)$$

2) Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (25)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^\infty e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (26)$$

3) Using the above, evaluate

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \quad (27)$$

for large values of γ .

4) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos \theta d\theta \quad (28)$$

5) Find $P_{e|s_0}$.