A Non-standard Analysis of Aumann-Shapley Random Order Values of Non-atomic Games*

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Abstract

Using techniques from the non-standard analysis, a non-standard analogue of the Aumann-Shapley random order value of non-atomic games is provided. The paper introduces the notion of effectively ergodic family of automorphism groups. It is shown that for a wide class of games, the non-standard random order value with respect to an effectively ergodic family of automorphism groups coincides with the standard Aumann-Shapley value.

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1 Introduction

A basic problem in cooperative game theory is to find rules for dividing the worth of the grand coalition among the players so that certain fairness is achieved. Mathematically, the problem is to find a mapping or an operator satisfying pre-specified conditions from the space of all set functions to the space of additive set functions. Using the linear vector space structure of the space of games with finite number of players, Shapley, 1953 proved the existence and uniqueness of the operator that satisfies certain axioms characterizing a notion of fair division. The solution thus obtained is known as *axiomatic value*. Shapley also provided an alternative set of fairness properties which come to be known as the *random order value*. In this approach, a player is given his expected marginal contribution in a random ordering of players, each ordering being equally likely among all possible orderings of the players. showed that the formulas for value from both approaches coincide. Aumann and Shapley, 1974 extended the concept of axiomatic value to games with a continuum of players and proved the existence and uniqueness of an axiomatic value operator on economically important spaces of games, including the space pNA. Their attempt to extend the random order value to the continuum case led to proving their well-known impossibility principle: *There does not exist a measure structure on the set of orders that they considered with respect to which a random order value could be assigned to games in pNA.*

The most important fairness properties of the Shapley value are derived from the symmetry axiom. This axiom was originally specified with respect to the whole group $\mathcal{G}$ of automorphisms of the players set. Many economic applications dictate the symmetry to be restricted to subgroups, see for instance, Monderer, 1989; Monderer, 1986. It is important to note that the main fairness property of the random order value arises from the fact that each player has an equal chance of forming a coalition with a set of players of any size and any name. The random order value assigns to each player the average of his marginal contributions over all coalitions which he may join. Raut, 1997; 2003 proved that when the measure structure on the set of orders are induced from the Haar measure structure of an automorphism subgroup Θ, there exists a Θ-symmetric random order value operator on large spaces of games. 1997 constructed a Haar measure structure on an uncountably large group of automorphisms $\breve{\Theta}$, and showed that $\breve{\Theta}$-symmetric random order value coincides with fully symmetric value for a large class of economically important games. In this paper I extend the random order approach to non-atomic games using the non-standard analysis, and thus avoid many measure theoretic complications of the approach in ; 2003.

In section 2, I first describe the standard concepts on values of non-atomic games, and then define the non-standard analogues of these concepts. In section 3, I state and prove the main results. I relegate all the remarks to section 4.

2 Basic Notation and Concepts

I use two dots over a symbol to denote a linear operator, a * prescript before a symbol to denote a non-standard element, and a ~ over a symbol to denote the non-standard extension of a standard element represented by the symbol. The abbreviation l.m.p. will mean Lebesgue measure preserving.

2.1 The Standard Framework

Let $I = [0,1] \subset \Re$ be the set of players. Let $\mathcal{B}_I$ be the Borel σ-algebra of I, i.e., the sigma algebra generated by the set of open intervals in I. The elements of $\mathcal{B}_I$ are the set of all possible coalitions. A *game* is a set function $V : \mathcal{B}_I \rightarrow \Re$ such that $V(\emptyset) = 0$. Let G_I be the set of all games. Let FA be the set of finitely additive set functions on $(I, \mathcal{B}_I)$. A *measure* is a countably additive set function. One can check easily that G_I and FA are linear vector spaces. A *Borel automorphism* is a measurable map $\theta : (I, \mathcal{B}_I) \rightarrow (I, \mathcal{B}_I)$ such that it is one-one, onto and θ^{-1} is also measurable. Let $\mathcal{G}$ be the set of all Borel automorphisms on $(I, \mathcal{B}_I)$. One can check that with composition of functions as group multiplication operation and identity function as the group identity, the set $\mathcal{G}$ is a non-commutative (also known as non-abelian) group. For each $\theta \in \mathcal{G}$, define the linear operator $\ddot{\theta} : G_I \rightarrow G_I$ by $(\ddot{\theta}V)(S) = V(\theta^{-1}(S)), \forall S \in \mathcal{B}_I$. Given a subgroup of automorphisms, $\Theta \subset \mathcal{G}$, a linear subspace $Q \subset G_I$ is said to be Θ-symmetric if $\ddot{\theta}Q \subset Q$ for all $\theta \in \Theta$.

Let Q be a linear subspace of G_I. An operator $\ddot{\Phi} : Q \rightarrow FA$ is said to be *linear* if $\ddot{\Phi}(\alpha V_1 + V_2) = \alpha\ddot{\Phi}(V_1) + \ddot{\Phi}(V_2) \forall V_1, V_2 \in Q, \alpha \in \Re$. The operator $\ddot{\Phi}$ is said to be *efficient* if $\ddot{\Phi}V(I) = V(I) \forall V \in Q$. For a Θ-symmetric space Q, the operator $\ddot{\Phi} : Q \rightarrow FA$ is said to be Θ-symmetric if $\ddot{\Phi}\ddot{\theta}V = \ddot{\theta}\ddot{\Phi}V, \forall \theta \in \Theta, V \in Q$.

Given an automorphism subgroup $\Theta \subset \mathcal{G}$, a Θ-symmetric *axiomatic value operator* on a Θ-symmetric space of games Q is a linear, efficient, and Θ-symmetric operator $\ddot{\Phi} : Q \rightarrow FA$.

Each $\theta \in \mathcal{G}$ generates a linear order $\succ_\theta$ on I as follows: for any $s, t \in I$, define $s \succ_\theta t$ if and only if $\theta(s) > \theta(t)$. Denote by $\bar{I} = I \cup \{\infty\}$ and for any automorphism θ, define $\theta(\infty) = \infty$. For each $\theta \in \mathcal{G}$, and $s \in \bar{I}$, define an initial segment $I(s,\theta) = \{t \in I | \theta(t) < \theta(s)\}$. Given a game $V \in G_I$, and an automorphism $\theta \in \mathcal{G}$, define a *marginal contribution measure* $(\phi^\theta V)$ by

$$\left(\phi^\theta V\right)(I(s,\theta)) = V(I(s,\theta)), \forall s \in \bar{I} \tag{1}$$

A set function $V \in G_I$ is said to be *a normalized set function* if (i) $V(A_n) \to 0$ as $n \to \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \downarrow \emptyset$ as $n \to \infty$, and (ii) $V(A_n) \to V(A)$ as $n \to \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \uparrow A$ as $n \to \infty$, where $A \in \mathcal{B}_I$. Denote by NBV the set of normalized set functions of bounded variations. It is known that $\phi^\theta V$ exists for any Borel automorphism θ and for each game $V \in NBV$. Let Θ_n be a finite group of l.m.p. automorphisms. A finite group of automorphisms generates a finite set of orders on which it is easy to define a measure that assigns equal likelihood to each order and thus it is easy to define the random order Shapley value restricting the set of orders to any finite set. Denote by $\ddot{\Phi}_n$ the linear operator which associates each game $V \in NBV$ its random order Shapley value with respect to the Haar measure on Θ_n equipped with the discrete σ-algebra, i.e., $(\ddot{\Phi}_n V)(S) = \frac{1}{N_n} \sum \left(\phi^{\theta_n} V\right)(S)$, where N_n is the number of elements in Θ_n.

One way to extend the above finite sum or the concept of equal likelihood to uncountably large set of orders is to use a measurable group structure (1997; 2003). Another way to extend the above finite sum to the case of uncountably large index set is to use the concept of hyper-finite sum from the non-standard analysis. I follow the latter approach in this paper as described in the next section.

2.2 The Non-Standard Framework

I now extend the above concepts to the non-standard framework. I closely follow the exposition of the non-standard analysis by Lindstrom, 1988. Denote by $\mathcal{N}$ the set of positive natural numbers. It is known that there exists a finitely additive probability measure m on $\mathcal{N}$ such that (i) for all $A \subset \mathcal{N}$, $m(A)$ is either 0 or 1, and $m(A) = 0$ for all finite A. Fix such a finitely additive measure m on $\mathcal{N}$. Define an equivalence relation $\sim$ on $\mathcal{G}^{\mathcal{N}}$ as follows:

$$\begin{aligned} \text{for } \theta &= (\theta_1, \theta_2, ...) \text{ and } \theta' = (\theta'_1, \theta'_2, ...) \text{ from } \mathcal{G}^{\mathcal{N}}, \\ \text{define } \theta &\sim \theta' \Longleftrightarrow m\left\{n | \theta_n = \theta'_n\right\} = 1 \end{aligned} \tag{2}$$

Define the *non-standard automorphism group* ${}^*\mathcal{G} \equiv \mathcal{G}^{\mathcal{N}} / \sim$. For an element $\theta = (\theta_1, \theta_2, ...)$ from $\mathcal{G}^{\mathcal{N}}$, denote the corresponding equivalence class from ${}^*\mathcal{G}$ by ${}^*\theta$. When I need to refer to a particular element in the equivalence class θ^*, I will refer the element by $\langle \theta_n \rangle$. Non-standard real number system ${}^*\Re$ is similarly defined by, ${}^*\Re = \Re^{\mathcal{N}} / \sim$, where the $\sim$ on $\Re^{\mathcal{N}}$ is defined by Eq. (2) for sequences of real numbers in place of automorphisms. In denoting the non-standard real numbers I follow the same notational convention as for the automorphisms above. Most operations on $\mathcal{G}$ and $\Re$ can be lifted to the corresponding non-standard spaces through component wise operations.

Let $\Theta_1, \Theta_2, ..$ be a sequence of finite subgroups of l.m.p. automorphisms. Denote the internal set ${}^*\Theta = \langle \Theta_n \rangle \subset {}^*\mathcal{G}$, defined by ${}^*\theta = \langle \theta_n \rangle \in {}^*\Theta \Longleftrightarrow m\left\{n | \theta_n \in \Theta_n\right\} = 1$. Identify any element θ of the finite subgroups Θ_n in the internal set ${}^*\Theta$ as the equivalence class $\langle (\theta, \theta, \cdots) \rangle$, and denote it by $\tilde{\theta}$. With this convention, note that the hyper finite internal set ${}^*\Theta$ contains all its components, Θ_n, $n \geq 1$ and it is a group with the group operation between two elements being lifted component-wise. I will refer to ${}^*\Theta$ as the *internal group of l.m.p. automorphisms* corresponding to the sequence of l.m.p. automorphisms $\{\Theta_n\}$.

A *non-standard game* is an equivalence class of games ${}^*V = \langle V_n \rangle$, where $V_n \in G_I$. Given a space of standard games Q, denote by ${}^*Q = \{{}^*V = \langle V_n \rangle | V_n \in Q \text{ for all } n \geq 1\}$ the induced set of non-standard set functions. Given a non-standard automorphism ${}^*\theta = \langle \theta_n \rangle$, define a linear operator ${}^*\ddot{\theta}$ which takes a non-standard game ${}^*V = \langle V_n \rangle$ to another non-standard game as follows

$${}^*\ddot{\theta} \ \left({}^*V(S)\right) = \left\langle \left(\ddot{\theta}_n V_n\right)(S) \equiv V_n\left(\theta_n^{-1}(S)\right) \right\rangle, \tag{3}$$

A space of non-standard games ${}^*Q$ is said to be ${}^*\Theta$-*symmetric* if for all ${}^*\theta \in^* \Theta$ and for all ${}^*V \in {}^*Q$, the non-standard game ${}^*\ddot{\theta}\,{}^*V \in {}^*Q$. A linear operator $\ddot{\Phi} : {}^*Q \longrightarrow {}^*FA$ defined on a ${}^*\Theta$-symmetric space of games ${}^*Q$ is said to be a${}^*\Theta$-*symmetric operator* if for all ${}^*V = \langle V_n \rangle \in {}^*Q$ and for all ${}^*\theta = \langle \theta_n \rangle \in {}^*\Theta$, we have that $\ddot{\Phi}({}^*\ddot{\theta}\ {}^*V) = {}^*\ddot{\theta}\left(\ddot{\Phi}\ {}^*V\right)$, and it is said to be an *efficient operator* if $\left(\ddot{\Phi}\,{}^*V\right)(I) = \langle V_n(I) \rangle$. A *non-standard* ${}^*\Theta$-*symmetric value operator* on a ${}^*\Theta$-symmetric space of non-standard games ${}^*Q$ is a linear, efficient and ${}^*\Theta$-symmetric operator.

Given a ${}^*\theta = \langle\theta_n\rangle \in {}^*\Theta$, define a linear order $\succ_{^*\theta}$ on the unit interval I as follows: for any two $t, s \in I$, define $t \succ_{^*\theta} s \Longleftrightarrow m\{n|\theta_n(t) > \theta_n(s)\} = 1$. I refer to $\succ_{^*\theta}$ as a *non-standard order*. For the above to be an well-defined and useful concept of order for our analysis, the non-standard order $\succ_{\tilde{\theta}}$ and the standard order $\succ_\theta$ defined earlier should coincide for each $\theta \in \mathcal{G}$, and every member of the equivalence class $\langle\theta_n\rangle$ should generate the same linear order on I. Both facts are true, and can be easily verified.

Given a non-standard game ${}^*V = \langle V_n\rangle$, and a non-standard automorphism ${}^*\theta = \langle\theta_n\rangle$, define a *non-standard marginal contribution measure* $\phi^{^*\theta}\ {}^*V : \mathcal{B}_I \longrightarrow {}^*\Re$ by $\left(\phi^{^*\theta}\ {}^*V\right)(S) = \langle(\phi^{\theta_n} V_n)(S)\rangle, S \in \mathcal{B}_I$. Here again the non-standard marginal contribution measure $\phi^{^*\theta}\ {}^*V$ is well defined in the sense that it is independent of which representative $(\theta_1, \theta_2, ...)$ is used for ${}^*\theta$, and also for any $\theta \in \Theta_n, n \geq 1$, $\left(\phi^{\tilde{\theta}}\tilde{V}\right)(S) = (\phi^\theta V)(S)$.

Denote the infinite non-standard integer ${}^*N \equiv < 2, 2^2, ..., 2^n, ... >$ and its standard components by $N_n = 2^n$, $n \geq 1$. For a non-standard game ${}^*V = \langle V_n\rangle$, and for an internal group of automorphisms ${}^*\Theta$ with an associated sequence of l.m.p. automorphisms $\{\Theta_n\}$, define the *hyper-finite sum,*

$$\begin{aligned} \sum_{^*\theta \in {}^*\Theta} \frac{1}{^*N}\left(\phi^{^*\theta}\,{}^*V\right)(S) &\equiv \left\langle \frac{1}{N_n}\sum_{\theta_n\in\Theta_n}\left(\phi^{\theta_n}V\right)(S)\right\rangle, S \in \mathcal{B}_I \qquad (4) \\ &= \left\langle\left(\ddot{\Phi}_n V_n\right)(S)\right\rangle, S \in \mathcal{B}_I \end{aligned}$$

I refer to the above non-standard finitely additive measure as the *non-standard random order value* of the non-standard game ${}^*V$ with respect to the non-standard automorphism group ${}^*\Theta = \langle\Theta_n\rangle$.

The operator ${}^*\ddot{\Phi} : {}^*Q \longrightarrow {}^*FA$ that associates to each non-standard game ${}^*V \in {}^*Q$ the non-standard random order value defined in Eq. (4) is said to be a *non-standard random order value operator with respect to a non-standard automorphism group* ${}^*\Theta$. In Theorem 1 in the next section I prove that on the ${}^*\Theta$-symmetric space of non-standard games ${}^*NBV$, the non-standard random order value operator ${}^*\ddot{\Phi}$ with respect to a non-standard automorphism group ${}^*\Theta$ is linear, efficient and ${}^*\Theta$-symmetric.

The natural question is: When does a non-standard random order value coincide with the standard Aumann-Shapley axiomatic value? This is answered in Theorem 2. For this theorem, I need the following concepts and facts: A group of Lebesgue measure preserving automorphisms Θ is **ergodic** if for each $\theta \in \Theta$, and for any $E \in \mathcal{B}_I$ with $\lambda\left(\theta^{-1}E \triangle E\right) = 0$ (i.e., for any θ-invariant set E) implies $\lambda(E) = 0$ or 1. An **effectively ergodic family** of automorphisms is an increasing sequence of finite[1] groups of automorphisms $\Theta_1 \subset \Theta_2 \subset ...$ such that for any $E \in \mathcal{B}_I$ with $\lambda(E) \neq 0$, we have $\lim_{n\longrightarrow\infty} \frac{\sum_{\theta_n\in\Theta_n} \chi_E(\theta_n(t))}{\#\Theta_n} = \lambda(E)$ for almost all $t(\lambda)$. Note that since this equality will also hold for simple functions, using the usual limiting arguments one can show that for any $g \in L_1(I, \mathcal{B}_I, \lambda)$, $\lim_{n\longrightarrow\infty} \frac{\sum_{\theta_n\in\Theta_n} g(\theta_n(t))}{\#\Theta_n} = \int g(x)\, d\lambda(x)$ a.e.$t(\lambda)$. This equality is generally stated to hold in individual ergodic theorem when we replace $\theta_n(t)$ with $\theta^n(t)$ (i.e., n compositions of θ) and $\#\Theta_n$ by n. In the present context, I am creating an effectively ergodic orbit with the help of an effectively ergodic family of recurrent automorphisms to achieve a thorough mixing of players in the random ordering.

3 The Main Results

For the main results I will need the following lemma and proposition.

Lemma 1. : Let $S \subset \Re$, and $\theta : S \to S$, and $\pi : S \to S$ be two automorphisms of S. Denote by $I(s,\theta) = \{t \in S | \theta(t) < \theta(s)\}$ for an automorphism, θ. Then, $\pi^{-1}(I(s,\theta)) = I\left(\pi^{-1}(s), \theta\pi\right)$.

Proof. The result follows from the following equivalent statements:

$$\begin{aligned} x \in \pi^{-1}(I(s,\theta)) &\Leftrightarrow \pi(x) \in I(s,\theta) \\ &\Leftrightarrow \theta(\pi(x)) < \theta(s) \\ &\Leftrightarrow (\theta\pi)(x) < (\theta\pi)\pi^{-1}(s) \\ &\Leftrightarrow x \in I\left(\pi^{-1}(s), \theta\pi\right) \end{aligned}$$

Q.E.D.

Proposition 1. : Let Θ be any fixed subgroup of automorphisms in $\mathcal{G}$. Suppose for a game $V \in G_I$, the marginal contribution measure $\phi^\theta V$ exists for all $\theta \in \Theta$. Then for any $\pi \in \Theta$, the marginal contribution measure $\phi^\theta(\ddot{\pi}V)$ for the game $\ddot{\pi}V$ also exists for all $\theta \in \Theta$, and it is related to the marginal contribution measure of V by,

$$\phi^\theta(\ddot{\pi}V)(S) = (\phi^{\theta\pi}V)\left(\pi^{-1}(S)\right), \forall S \in \mathcal{B}_I \tag{5}$$

[1] Note that a finite automorphism group can contain only recurrent automorphisms but not ergodic automorphisms.

Proof. Note that for any $s \in \bar{I}$,

$$\begin{aligned}
(\phi^{\theta}(\ddot{\pi}V))(I(s,\theta)) &= (\ddot{\pi}V)(I(s,\theta)) \text{ by definition of } \phi^{\theta} \\
&= V\left(\pi^{-1}I(s,\theta)\right) \text{ by definition of } \ddot{\pi} \\
&= V\left(I\left(\pi^{-1}(s),\theta\pi\right)\right) \text{ by lemma } \mathbf{1} \\
&= (\phi^{\theta\pi}V)\left(I(\pi^{-1}(s),\theta\pi)\right) \text{ by definition of } \phi^{\theta\pi} \\
&= ((\phi^{\theta\pi}V)\pi^{-1})\,(I(s,\theta)) \text{ by lemma } \mathbf{1}
\end{aligned}$$

Since they agree on the initial segments in $I(s,\theta), \forall s \in \bar{I}$, they agree on $\mathcal{B}_I$. Thus the measure $(\phi^{\theta}\ddot{\pi}V)$ exists whenever the measure $(\phi^{\theta\pi}V\pi^{-1})$ exists. Since $\theta\pi \in \Theta$, by the hypothesis of the Proposition, $(\phi^{\theta\pi}V)$ exists, and since $(\phi^{\theta\pi}V)\pi^{-1}$ is a measure whenever $(\phi^{\theta\pi}V)$ is a measure. Hence I conclude that $\phi^{\theta}(\ddot{\pi}V)$ exists for all $\theta \in \Theta$ and is given by the right hand side of Eq. (5).

Q.E.D.

The following theorem is on the existence of non-standard random order value operator.

Theorem 1. Let $\{\Theta_n\}$ be a sequence of finite groups of l.m.p. automorphisms. Let ${}^*\Theta$ be the non-standard internal group of l.m.p. automorphisms corresponding to the sequence $\{\Theta_n\}$. The non-standard random order value operator ${}^*\ddot{\Phi}$ on the ${}^*\Theta$-symmetric space of games ${}^*NBV$ is linear, efficient and *Θ-symmetric.

Proof. Linearity and efficiency follows trivially. To show that ${}^*\ddot{\Phi}$ is ${}^*\Theta$-symmetric, let ${}^*\pi = \langle \pi_n \rangle$, $\pi_n \in \Theta_n$. Let ${}^*V \in {}^*NBV$ be an arbitrary game, then

$$\begin{aligned}
{}^*\ddot{\Phi}\left({}^*\ddot{\pi}\left({}^*V(S)\right)\right) &= \left\langle \frac{1}{N_n}\sum_{\theta_n\in\Theta_n}\phi^{\theta_n}\left(\ddot{\pi}_n V_n\right)(S)\right\rangle \\
&= \left\langle \frac{1}{N_n}\sum_{\theta_n\in\Theta_n}\phi^{\theta_n\pi_n}\left(V_n\right)\left(\pi_n^{-1}(S)\right)\right\rangle \text{ by Proposition 1} \\
&= \left\langle \frac{1}{N_n}\sum_{\theta_n\pi_n\in\Theta_n}\phi^{\theta_n\pi_n}\left(V_n\right)\left(\pi_n^{-1}(S)\right)\right\rangle \text{ relabeling the summation index} \\
&= \left\langle \left(\ddot{\Phi}_n V_n\right)\left(\pi_n^{-1}(S)\right)\right\rangle \\
&= \left\langle \ddot{\pi}_n\left(\ddot{\Phi}_n V_n\right)(S)\right\rangle \\
&= {}^*\ddot{\pi}{}^*\ddot{\Phi}{}^*V(S)
\end{aligned}$$

Hence ${}^*\ddot{\Phi} \circ {}^*\ddot{\pi} = {}^*\ddot{\pi} \circ {}^*\ddot{\Phi}$ on ${}^*NVB$.

Q.E.D.

Analogue of Aumann and Shapley, 1974, Theorem A and Raut, 1997, Theorem 3 is the following:

Theorem 2. Let λ be the Lebesgue measure on $(I, \mathcal{B}_I)$. Let $f : I \to \Re$ be a real valued function which is differentiable a.e.(λ), and $f' \in L_1(I, \mathcal{B}_I, \lambda)$. The non-standard random order value with respect to an effectively ergodic family of l.m.p. automorphisms$\{\Theta_n\}$ yields the following diagonal formula for the scalar measure game $f \circ \lambda$:

$$\ddot{\Phi}[f \circ \lambda](S) = \lambda(S) \int_0^1 f'(x) d\lambda(x) \tag{6}$$

Proof. Denote by $N_n = \#\Theta_n$, and by ${}^*N = \langle N_n \rangle$. I will establish that the non-standard random order value of $(f \circ \lambda)(S)$ coincides with its axiomatic Aumann-Shapley value. To that end, notice that

$$\begin{aligned}
\sum_{\breve{\theta} \in {}^*\Theta} \frac{1}{{}^*N} \left({}^*\phi^{\tilde{\theta}} \left(\widetilde{f \circ \lambda} \right) \right)(S) &= \left\langle \frac{1}{N_n} \sum_{\theta_n \in \Theta_n} \left(\phi^{\theta_n} (f \circ \lambda) \right)(S) \right\rangle \\
&= \left\langle \frac{1}{N_n} \sum_{\theta_n \in \Theta_n} \int_S f'(\theta_n(t))\, d\lambda(t) \right\rangle \text{ by Raut [1997, Proposition 5]} \\
&= \left\langle \int_S \left[\frac{1}{N_n} \sum_{\theta_n \in \Theta_n} f'(\theta_n(\xi(t)) \right] d\lambda(t) \right\rangle \text{ by Fubini's theorem} \\
&= \int_S d\lambda(t) \int_0^1 f'(x) d\lambda(x) \text{ ...(A)}
\end{aligned}$$

The step (A) follows since $\{\Theta_n\}$ is an effectively ergodic family of l.m.p. automorphisms..

Q.E.D.

4 Remarks

In the definition of the operator ${}^*\ddot{\Phi}$, $\frac{1}{N}$ is the non-standard analogue of Haar measure or equal likelihood.

If each Θ_n contains at least two distinct elements from its predecessors, ${}^*\Theta$ is then an uncountably large group. An example of effectively ergodic family of l.m.p. automorphisms is a projective limit group constructed in . It is shown in 2003 that the projective limit group is isomorphic to the unit interval.

In this approach, the component groups Θ_n could be taken to be uncountably large compact groups and the approach could be further extended in which indexing parameter n could run over real numbers instead of integers.

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