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Construction of a Haar measure on the projective limit group and random order values of non-atomic games

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Abstract

This paper constructs an increasing sequence of finite measurable subgroups of Lebesgue measure preserving (l.m.p.) automorphisms of the unit interval, and projective Borel homomorphisms of the subgroups. Then, by superimposing this group structure on the original Daniel–Kolmogorov model of stochastic processes it extends the Daniel–Kolmogorov consistency theorem which enables the construction of a separated measurable projective limit group and its representation as an uncountably large subgroup of l.m.p. automorphisms. Each such automorphism induces a distinct ordering of the players in the unit interval. With respect to this group of random orders, and again using the extended consistency theorem, a formula for the unique random order value operator, proposed by Raut, (Indian Statistical Institute discussion paper no. 8108, 1981) and Raut (Random order approach to Shapley-value of games and Haar measure (mimeo), 1993, University of California-San Diego), is derived for economically important classes of scalar and vector measure valued games in pNA. The formula is seen to be identical to the axiomatic value formula of Aumann and Shapley (values of non-atomic games, 1974, Princeton University Press).

JEL classification: C60; C71

Keywords: Non-atomic games; Random order; Shapley value; Haar measure; Extended Daniel–Kolmogorov consistency theorem

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1. Introduction

A cooperative game is a set function defined on an algebra of subsets of a fixed player set. A basic problem of cooperative game theory is to find a rule (i.e. a map) that assigns a finitely additive set function to each cooperative game such that the assignment rule has certain nice properties. For finite games, Shapley (1953) postulated three axioms for a map, namely linearity, efficiency and symmetry with respect to the group of automorphisms of the players (definitions are in the next section), and proved that such a map exists and is unique. A map that satisfies Shapley's three axioms is known as the axiomatic Shapley value operator, and the unique finitely additive set function that it assigns to a game is known as the axiomatic value of the game. He also proposed an alternative approach, later to be known as the random order approach, in which a player is given his expected marginal contribution in a random order chosen out of all equally likely orderings of the players. The corresponding map is known as the random order value operator and the finitely additive set function that is attached to a game by this rule is known as the random order value of the game. He showed that values from both approaches coincide for finite games. The value concept has been extended in many ways and applied to many interesting economic problems; see, for instance, the contributions in Roth (1988).

Aumann and Shapley (1974) extended the concept of an axiomatic value to games with a continuum of players and proved the existence and uniqueness of the value operator in large classes of games such as bv'NA and pNA, and they derived a formula for the axiomatic value for a class of scalar and vector measure valued games in pNA. [1] To prove the uniqueness of the axiomatic value operator they exploited the topological properties such as reproducibility and internality of the above spaces of games.

Aumann and Shapley (1974) also attempted to extend the random order approach to values of games with a continuum of players; however, they allowed too many orders and consequently arrived at the impossibility of finding a probability measure structure on their set of orders that could yield a random order value operator on the relevant spaces of games. Imposing a group structure on the set of orders, Raut (1981, 1993) reformulated and characterized the random order approach to values in a unified framework which encompasses both finite and infinite games, as follows.

I begin with a fixed subgroup of automorphisms of players, Θ, admitting a probability measure structure $(\Theta, \mathscr{A}_\Theta, \Gamma)$ such that it is also a 'separated measurable group'. A measurable group of automorphisms induces a set Ω of

[1] The Shapley value for measure valued games in pNA have many economic applications; see, for instance, Billera et al. (1978) and see Tauman (1988) for recent surveys of many other applications.

distinct orderings of players and a probability measure structure, $(\Omega, \mathscr{B}_\Omega, \mu)$, referred to as the set of random orders from which random coalitions may be generated. For each ordering of the players, I define the marginal contribution set function. The Θ-symmetric random order value of a game is the expected marginal contribution set function in a random order drawn according to $(\Omega, \mathscr{B}_\Omega, \mu)$. I have shown in Raut (1993) that on the linear space of games for which the Θ-symmetric random order value exists, the Θ-symmetric random order value operator is linear, efficient and symmetric with respect to the subgroup of automorphisms Θ; in particular, if Θ is the full group of automorphisms, then the random order value operator is also an axiomatic value operator on this linear space of games; furthermore, even when Θ is not the full subgroup, the Θ-symmetric random order value operator could be symmetric with respect to the full group of automorphisms on a linear subspace. It is clear that the larger is the group Θ, the larger is the linear subspace of games in which the Θ-symmetric random order value operator is also symmetric with respect to the full group of automorphisms. For the continuum of players case, I provided examples of the Θ-symmetric random order value operator for only finitely large Θ's. It was not clear, however, whether there exists a Θ-symmetric random order value operator with respect to uncountably large Θ's and what the properties are of such random order value operators.

In this paper I construct an increasing sequence of finite subgroups of Lebesgue measure preserving (l.m.p.) automorphisms and the projective homomorphisms on these subgroups. I then extend the Daniel–Kolmogorov consistency theorem of stochastic processes by superimposing this group structure on its original projective limit framework to construct a separated measurable projective limit group. The important feature of the construction of the finite subgroups and the projective homomorphisms on them is that we can imbed the projective limit group as an uncountably large subgroup of l.m.p. automorphisms of the unit interval. Thus the procedure of the paper provides a way to construct uncountably large separated measurable subgroups of l.m.p. automorphisms, which by itself has independent mathematical interest. However, as for the merits of the current paper, this construction helps to explicate interesting properties of the random order value operator; for instance, using the extended consistency theorem I derive a formula for the random order value with respect to this group of random orders for economically important classes of scalar and vector measure games in pNA and this value formula coincides with the Aumann–Shapley axiomatic value formula which they derived using different techniques.

In section 2 I provide the notation for games with a continuum of players, the concept of the Θ-symmetric random order value operator and conditions under which it exists (as in Raut, 1993). In Section 3 I construct the projective limit group by extending the Daniel–Kolmogorov consistency theorem. In Section 4 the random order value formula is derived for a class of scalar and vector measure games in pNA.

2. Notation and background

Let $I = [0, 1] \subset \Re$ be the set of players. Let $\mathscr{B}_I$ be the Borel σ-algebra of I, i.e. the sigma algebra generated by the set of open intervals in I. The elements of $\mathscr{B}_I$ are the set of all possible coalitions. A *game* is a set function $V : \mathscr{B}_I \to \Re$ such that $V(\emptyset) = 0$. Let G_I be the set of all games. Let FA be the set of finitely additive set functions on $(I, \mathscr{B}_I)$. A *measure* is a countably additive set function. We can easily check that G_I and FA are linear vector spaces. A *Borel automorphism* is a measurable map $\theta : (I, \mathscr{B}_I) \to (I, \mathscr{B}_I)$ such that it is one-one, onto and θ^{-1} is also measurable. Let $\mathscr{G}$ be the set of all Borel automorphisms on $(I, \mathscr{B}_I)$. We can check that $\mathscr{G}$ is a non-commutative (also known as non-Abelian) group, with the group multiplication taken as the composition of Borel automorphisms; and the group identity taken as the identity Borel automorphism.

For each $\theta \in \mathscr{G}$, we define the linear operator $\theta^* : G_I \to G_I$ by

$$(\theta^* V)(S) = V(\theta^{-1}(S)), \quad \forall S \in \mathscr{B}_I.$$

Given a subgroup of automorphisms, $\Theta \subset \mathscr{G}$, a linear subspace $Q \subset G_I$ is said to be *Θ-symmetric* if $\theta^* Q \subset Q$ for all $\theta \in \Theta$.

Let Q be a linear subspace of G_I. An operator $\Phi : Q \to FA$ is said to be *linear* if $\Phi(\alpha V_1 + V_2) = \alpha\Phi(V_1) + \Phi(V_2)$, $\forall V_1, V_2 \in Q$, $\alpha \in \Re$. Φ is said to be *efficient* if $\Phi V(I) = V(I)$, $\forall V \in Q$. For a Θ-symmetric space Q, the operator $\Phi : Q \to FA$ is said to be *Θ-symmetric* if $\Phi\theta^* V = \theta^*\Phi V$, $\forall \theta \in \Theta$, $V \in Q$.

Given an automorphism subgroup $\Theta \subset \mathscr{G}$, a *$\Theta$-symmetric axiomatic value operator* on a Θ-symmetric space of games Q is a linear, efficient, and Θ-symmetric operator $\Phi : Q \to FA$. The Θ-symmetry of $\Phi : Q \to FA$ has the following commutative diagram:

$$\begin{array}{ccc} Q \ni V & \xrightarrow{\Phi} & \Phi V \in FA \\ \theta^* \downarrow & & \downarrow \theta^* \\ Q \ni \theta^* V & \xrightarrow[\Phi]{} & \theta^*\Phi V \in FA \end{array}$$

A random order value operator for the continuum case, as proposed in Raut (1981, 1993) and followed in this paper, begins with a fixed symmetry group of automorphisms, $\Theta \subset \mathscr{G}$, with a measure structure $(\Theta, \mathscr{B}_\Theta, \Gamma)$ satisfying certain properties as specified below.

An *order generated by* $\theta \in \mathscr{G}$ is a binary relation, $\succ_\theta \subset I \times I$ defined by

$$\text{for any } s, t \in I, \quad s \succ_\theta t \quad \Leftrightarrow \theta(s) > \theta(t).$$

It is easy to see that $\succ_\theta$ is a transitive, irreflexive and complete order on I and that each $\theta \in \Theta$ generates an order. Let $\bar{I} = I \cup \{\infty\}$, and for all $\theta \in \mathscr{G}$ we assume that $\theta(\infty) = \infty$. For an order $\succ_\theta$, $\theta \in \mathscr{G}$, and $s \in \bar{I}$, we define *an initial segment* $I(s, \theta)$ by $I(s, \theta) = \{t \in I \mid \theta(s) > \theta(t)\}$. We view $I(s, \theta)$ as the set of players who are before player s in the order $\succ_\theta$.

Given a game V, and an order $\succ_\theta$, $\theta \in \mathscr{G}$, a *marginal contribution set function*, $(\phi[V])(\cdot, \theta)$ on $(I, \mathscr{B}_I)$ is a measure on $(I, \mathscr{B}_I)$ such that

$$(\phi[V])(I(s, \theta), \theta) = V(I(s, \theta)), \quad \forall s \in \bar{I}. \tag{1}$$

It can easily be shown that for a game V and an order $\succ_\theta$, $\theta \in \mathscr{G}$, if there exists a measure $\phi[V](\cdot, \theta)$ satisfying (1), then it is unique (Raut, 1993, proposition 2).

Let e be the identity map of I. It is easy to note that e is the identity element of $\mathscr{G}$ and it generates the standard order, $\succ_e$, of I. It is also easy to note that unlike in the finite player case, two Borel automorphisms of I may generate the same ordering of I. For instance, $\theta \in \mathscr{G}$ defined by $\theta(x) = x^2$, $x \in I$ and $e \in \mathscr{G}$ both generate the order $\succ_e$. Thus the set of orderings of players and the group of Borel automorphisms of players are not the same set. A probability model for the set of orders can be induced from $(\Theta, \mathscr{B}_\Theta, \Gamma)$ as follows.

Define an equivalence relation $\sim$ on $\Theta \times \Theta$ by

$$\theta_1 \sim \theta_2, \quad \text{for } \theta_1, \theta_2 \in \Theta \Leftrightarrow \theta_1, \theta_2 \text{ generate the same order on } I.$$

Let $\Theta_e = \{\theta \in \Theta \mid \theta \sim e\}$. It can be easily shown that Θ_e is a subgroup of Θ and the set of distinct orders, Ω, generated by the automorphisms in Θ is the set of right cosets given by

$$\Omega \equiv \Theta / \Theta_e \equiv \{\Theta_e \theta \mid \theta \in \Theta\}.$$

We use the *natural map* $\Pi : \Theta \to \Omega$ defined by $\Pi(\theta) = \Theta_e \theta$ to induce a probability measure structure $(\Omega, \mathscr{B}_\Omega, \mu)$ on the set of induced orders. $(\Omega, \mathscr{B}_\Omega, \mu)$ will be referred to as a set of random orders. In general, Θ_e is not a normal [2] subgroup of Θ and hence Ω is not necessarily a group. However, when Θ_e is a normal subgroup, as will be the case in our construction, the induced measure space $(\Omega, \mathscr{B}_\Omega, \mu)$ is also a measurable group. Notice that for any $\theta, \theta' \in \Theta$, such that $\theta \sim \theta'$, we have $I(s, \theta) = I(s, \theta')$; hence it follows from (1) that $\phi[V](S, \theta) = \phi[V](S, \theta')$ for all $S \in \mathscr{B}_I$. This allows us to unambiguously define $(\phi[V])(S, \omega) \equiv (\phi[V])(S, \theta)$, where θ is such that $\omega = \Theta_e \theta$.

The *expected marginal contribution set function for a game V* is a set function $\Phi_\Gamma V$ defined by

$$\begin{aligned}(\Phi_\Gamma V)(S) &= \int_\Omega (\phi[V])(S, \omega)\, \mathrm{d}\mu(\omega) \\ &= \int_\Theta (\phi[V])(S, \theta)\, \mathrm{d}\Gamma(\theta), \quad S \in \mathscr{B}_I. \end{aligned} \tag{2}$$

The second equality follows from the change of variable formula for Lebesgue integrals and the facts in the previous paragraph. Let us define the space of games:

$$LOR\Theta = \{V \in G_I \mid \phi[V](S, \theta) \text{ in (2) is integrable for all } S \in \mathscr{B}_I\}. \tag{3}$$

[2] N is a normal subgroup of G if for all $\theta \in G$, we have $\theta^{-1}\nu\theta \in N$ for all $\nu \in N$.

It can easily be shown that $LOR\Theta$ is a linear space and is symmetric with respect to Θ and that Φ_T is linear and efficient. In the finite player case, Shapley (1953) assumed a probability model for the set of orderings in which all orderings are equally likely and defined the random order value of a game as the finitely additive set function that yields the expected marginal contribution to a player. He observed that the random order value formula coincides with the unique axiomatic value formula. In Raut (1993), I have shown for the finite player case that when we treat the expected marginal contribution as a linear operator, Φ_T, on the space of games, $LOR\Theta$, the operator Φ_T is Θ symmetric *if and only if* the randomness of the orders is generated by a probability model that assigns equal probability to all automorphisms in Θ.

In the finite player case the set of automorphisms of players is finite, and for finite sets the concept of equal likelihood is intuitive. In the continuum case, however, the set of automorphisms of players is uncountable. The analogue of equal likelihood in the case of an infinite group is the following concept of a measurable group or invariant measure.

Definition 1. A measure space $(\Theta, \mathscr{A}_\Theta, \Gamma)$ is a *measurable group* if Θ is a group, the map $(\theta_1, \theta_2) \to \theta_1 \theta_2^{-1}$ from $(\Theta \times \Theta, \mathscr{A}_\Theta \times \mathscr{A}_\Theta)$ onto $(\Theta, \mathscr{A}_\Theta)$ is measurable, and Γ is σ-finite, not identically zero, and right invariant, i.e. $\Gamma(E\theta) = \Gamma(E)$, for all $E \in \mathscr{A}_\Theta$, and $\theta \in \Theta$, where $E\theta \equiv \{\sigma\theta \mid \sigma \in E\}$. Γ is known as the *right invariant measure.*

It is shown in Raut (1993) that if the integral in (2) is with respect to a right invariant probability measure on Θ, then the linear efficient operator Φ_T in (2) is Θ-symmetric. For any subgroup Θ, including the whole automorphism group $\Theta = \mathscr{G}$, there always exists a right invariant probability measure structure, for instance a trivial coarsest σ-algebra, $\mathscr{B} = \{\emptyset, \Theta\}$ with a trivial probability measure that assigns 0 to the empty set and 1 to the whole set. The coarser the σ-algebra, the more meager are the sets of measurable and integrable functions, and hence fewer games belong to $LOR\Theta$. With respect to coarser σ-algebras, $LOR\Theta$ may not contain any non-additive games, and hence such measurability structures are not interesting. We want the measurable group $(\Theta, \mathscr{B}_\Theta, \Gamma)$ to be separated in the following sense, so that it has enough measurable sets.

Definition 2. A measurable group $(\Theta, \mathscr{A}_\Theta, \Gamma)$ is *separated* [3] if $\forall \theta \in \Theta$, $\theta \neq e$, there exists $E \in \mathscr{A}_\Theta$ such that $0 < \Gamma(E) < \infty$ and $\Gamma(E\theta \Delta E) > 0$, where Δ is the symmetric difference operator between two sets.

[3] This separation notion for measurable groups is the analogue of the Hausdorff separation axiom for topological spaces; see Halmos (1950, p. 273).

We call a linear, efficient and Θ-symmetric operator, $\Phi_\Gamma : LOR\Theta \to FA$ defined in (2) and (3) with respect to a separated measurable group of automorphisms $(\Theta, \mathscr{B}_\Theta, \Gamma)$, a *$\Theta$-symmetric random order value operator*. It can be shown that given a fixed symmetry group of automorphisms, Θ, if we have two measurable structures $(\Theta, \mathscr{B}_\Theta, \Gamma)$ and $(\Theta, \mathscr{B}'_\Theta, \Gamma')$, then $\Phi_\Gamma = \Phi_{\Gamma'}$ on the common linear space of games. Thus once Θ is fixed, the Θ-symmetric random order value of a game $V \in G_I$ is unique. Furthermore, we will see later that if Θ contains a large enough number of 'important' automorphisms, and $\mathscr{B}_\Theta$ is 'fine enough', then for large classes of economically important games the Θ-symmetric random order value is also $\mathscr{G}$-symmetric and thus coincides with the axiomatic value studied in Aumann and Shapley (1974).

When we have an uncountably large subgroup $\Theta \subset \mathscr{G}$, it is not known if there exists a separated measurable group structure on it. In the next section we extend the Daniel–Kolmogorov consistency theorem of stochastic processes to construct an uncountably large separated measurable group of Lebesgue measure-preserving automorphisms $(\check{\Theta}, \check{\mathscr{B}}, \check{\Gamma})$, and in a later section we use the extended Daniel–Kolmogorov theorem to study the properties of the $\check{\Theta}$-symmetric random order value for economically important classes of games.

3. Construction of a measurable group of random orders and the extended Kolmogorov consistency theorem

We first construct an increasing sequence of finite groups of Lebesgue measure-preserving automorphisms, $\hat{\Theta}_0 \subset \hat{\Theta}_1 \subset \dots$. For each $n \geq 0$, we equip $\hat{\Theta}_n$ with the discrete σ-algebra [4] $\mathscr{B}_n$, and the counting probability measure Γ_n, so that $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$ is a measurable group; we define a suitable sequence of onto Borel homomorphisms, $f_n : \hat{\Theta}_n \to \hat{\Theta}_{n-1}$, $n \geq 1$. We then use the Daniel–Kolmogorov consistency theorem on $(\hat{\Theta}, \mathscr{B}_n, \Gamma_n)$ and f_{n+1}, $n \geq 0$ to construct the projective limit space $(\check{\Theta}, \check{\mathscr{B}}, \check{\Gamma})$ and show that it is a separated measurable group. The important feature of our construction of $\hat{\Theta}_n$'s and f_n's is that the projective limit space can be represented by an uncountably large subgroup of Lebesgue measure-preserving automorphisms of I.

The original formulation of the Daniel–Kolmogorov model does not require the $\hat{\Theta}_n$'s to be finite, and does not have any group structure to construct a probability model on the projective limit space. In our formulation, by imposing a group structure on it, we are able to construct a separated measurable group structure on the projective limit space. The finiteness of the $\hat{\Theta}_n$'s is not necessary in our formulation also, it is the outcome of our particular construction of the uncount-

[4] In compliance with the notion of a discrete topology, the powerset of a set X will be referred to here as the *discrete σ-algebra* of X.

ably large separated measurable group containing 'important' Lebesgue measure-preserving automorphisms. The importance of these automorphisms will become clear when we study the properties of the $\breve{\Theta}$-symmetric random order value operator in a later section.

Utilizing purely the measure-theoretic structure, a separated measurable group induces a topology, known as the Weil topology, such that with respect to this topology the group is a topological group (see Halmos, 1950, section 62). In our case, the projective limit space $\breve{\Theta}$ inherits a relative topology from the product topology of the product space $\mathscr{P} = \prod_{n=0}^{\infty} \hat{\Theta}_n$. We find that the Weil topology is equivalent to the relative product topology on $\breve{\Theta}$, and with respect to this topology the projective space $\breve{\Theta} \subset \mathscr{P}$ is a compact group, $\breve{\mathscr{B}}$ is the Borel σ-algebra, and $\breve{\Gamma}$ is the Haar probability measure. However, since we do not need these results to study the random order value, we will not prove these results in this paper.

In the rest of this section we provide details of the construction of the uncountably large separated projective limit group $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ and its imbedding in the group of l.m.p. automorphisms of I.

3.1. The projective limit group $\breve{\Theta}$

We recursively define the increasing sequence of finite groups $\hat{\Theta}_n$, $n \geq 0$, each containing l.m.p. automorphisms that are discontinuous at only a finite number of points of I. In the nth automorphism group $\hat{\Theta}_n$, the discontinuities of the automorphisms are at the points $k/2^n$, $k = 1, \ldots, 2^n - 1$. These $2^n - 1$ points in I determine 2^n dyadic subintervals of I: $I_k = k/2^n, (k+1)/2^n$, $k = 0, 1, \ldots, 2^n - 1$. In order for the automorphisms to be Lebesgue measure-preserving, we assume that in each subinterval I_k the automorphisms are linear with slope ± 1. For $n = 2$, such an automorphism is shown in part (a) of Fig. 1.

Let $N_n = \{0, 1, 2, \ldots, 2^n - 1\}$. We represent each automorphism in $\hat{\Theta}_n$ by a pair of functions π_n and $\mathscr{O}_n$ such that $\pi_n : N_n \to N_n$ is a permutation of N_n and $\mathscr{O}_n : N_n \to \{-1, 1\}$ is a map as follows. For each $k \in N_n$, $\pi_n(k)$ specifies which subinterval of the unit interval the image of the kth subinterval will be mapped to, and $\mathscr{O}_n \circ \pi_n(k)$ specifies the slope of the automorphism that the image subinterval will take. Sometimes we will refer to $\mathscr{O}_n$ as the *slope map*. We will denote such an automorphism, as described above, by

$$\theta_n = \left(\pi_n(k), \mathscr{O}_n \circ \pi_n(k)\right)_{k=0}^{2^n-1}. \tag{4}$$

An equivalent description of the above automorphism that we will often use is the following:

$$\theta_n(x) = \begin{cases} \dfrac{\pi_n(k)}{2^n} - \dfrac{k}{2n} + x, & \text{if } x \in I_k \text{ and } \mathscr{O}_n(\pi_n(k)) = +1, \\ \dfrac{\pi_n(k)+1}{2^n} - \dfrac{k}{2^n} - x, & \text{if } x \in I_k \text{ and } \mathscr{O}_n(\pi_n(k)) = -1, \\ & k = 0, 1, \ldots, 2^n - 1. \end{cases} \tag{5}$$

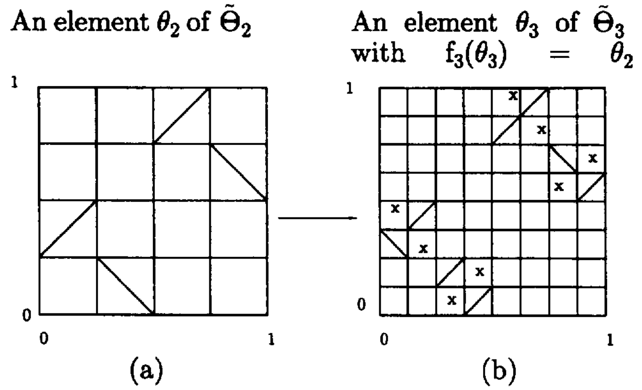

Fig. 1.

We will use θ_n to mean the representation (4), and $\theta_n(\cdot)$ or $\theta_n(x)$ to mean the representation (5) of an element in $\hat{\Theta}_n$.

Let $\mathfrak{F} = \{+1, -1\}$ denote the set of slopes. With the usual multiplication operation of real numbers and with $+1$ as the identity element, it is trivial to show that $\mathfrak{F}$ is a group.

For further illustration of these concepts, we have drawn a $\theta_2(x)$ in part (a) of Fig. 1, corresponding to the permutation $\pi_2(1) = 2$, $\pi_2(2) = 1$, $\pi_2(3) = 4$ and $\pi_2(4) = 3$, and the slope map $\mathscr{O}_2(1) = -1$, $\mathscr{O}_2(2) = +1$, $\mathscr{O}_2(3) = -1$, and $\mathscr{O}_2(4) = +1$.

For all $n \geq 0$, the finite subgroups $\hat{\Theta}_n$ of $\mathscr{G}$ are defined recursively as follows.

For $n = 0$, there is no subdivision of I, and we take

$$\hat{\Theta}_0 = \{\theta_0 = (\pi_0(0), \mathscr{O}_0(0)) \mid \mathscr{O}_0(0) \in \mathfrak{F}\}.$$

Note that $\hat{\Theta}_0$ has only two elements.

To define $\hat{\Theta}_1$, we take $n = 1$ and we have two dyadic subintervals of I denoted I_0 and I_1. Each $\theta_0 \in \hat{\Theta}_0$ induces a unique permutation π_{1,θ_0} of $N_1 = \{0, 1\}$ defined by

$$\pi_{1,\theta_0}(j) = i, \quad \text{if for all } x \in I_j,\ \theta_0(x) \in I_i,\ i, j \in N_1.$$

Given $\theta_0 \in \hat{\Theta}_0$, let us define

$$A_1(\theta_0) = \left\{\theta_1 = \left(\pi_{1,\theta_0}(k), \mathscr{O}_1(\pi_{1,\theta_0})(k)\right)_{k=0}^{1} \mid \mathscr{O}_1(\pi_{1,\theta_0})(j) \in \mathfrak{F},\ j = 0, 1\right\}.$$

We now define $\hat{\Theta}_1$ by

$$\hat{\Theta}_1 = \bigcup_{\theta_0 \in \hat{\Theta}_0} A_1(\theta_0).$$

Note that each $A_1(\theta_0)$ has $2\times 2=4$ elements, and hence $\hat{\Theta}_1$ has $2\times 4=8$ elements.

Let us suppose that we have already defined $\hat{\Theta}_{n-1}$. We now define $\hat{\Theta}_n$ from $\hat{\Theta}_{n-1}$.

Let the 2^n dyadic subintervals at stage n be denoted $I_0,\dots,I_{2^n-1}$. Each $\theta_{n-1}\in\hat{\Theta}_{n-1}$ induces a unique permutation $\pi_{n,\theta_{n-1}}$ of the set N_n defined by

$$\pi_{n,\theta_{n-1}}(j)=i,\quad \text{if for all } x\in I_j,\ \theta_{n-1}(x)\in I_i,\ i,\ j\in N_n. \tag{6}$$

For each $\theta_{n-1}\in\hat{\Theta}_{n-1}$, we define

$$\begin{aligned} &A_n(\theta_{n-1})\\ &\quad=\Big\{\theta_n=\big(\pi_{n,\theta_{n-1}}(k),\ \mathscr{O}_n(\pi_{n,\theta_{n-1}})(k)\big)_{k=0}^{2^n-1}\\ &\quad\times\mid \mathscr{O}_n(\pi_{n,\theta_{n-1}})(i)\in\mathfrak{F},\ \forall i\in N_n\Big\}\end{aligned}$$

and

$$\hat{\Theta}_n=\bigcup_{\theta_{n-1}\in\hat{\Theta}_{n-1}}A_n(\theta_{n-1}).$$

For each $n\geq 1$ we define the multiplication operation in $\hat{\Theta}_n$ as the composition of functions, namely, for $\hat{\theta}_n,\ \hat{\theta}'_n\in\hat{\Theta}_n$, we define $\hat{\theta}_n\hat{\theta}'_n=\hat{\theta}_n(\hat{\theta}'_n(x)),\ x\in I$. We will need the following proposition to establish that $\hat{\Theta}_n$ is a group whenever $\hat{\Theta}_{n-1}$ is.

Proposition 1. Suppose $\hat{\theta}_n=(\pi_{n,\hat{\theta}_{n-1}}(k),\ \mathscr{O}_n(\pi_{n,\hat{\theta}_{n-1}})(k))_{k=0}^{2^n-1}$, *and* $\hat{\theta}'_n=(\pi_{n,\hat{\theta}'_{n-1}}(k),\ \mathscr{O}'_n(\pi_{n,\hat{\theta}'_{n-1}})(k))_{k=0}^{2^n-1}$ *are two arbitrary members of* $\hat{\Theta}_n$. *Then*

$$\hat{\theta}_n\hat{\theta}'_n=\Big(\big(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}}\big)(k),\ \mathscr{O}^*_n\big(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}}\big)(k)\Big)_{k=0}^{2^n-1}, \tag{7}$$

where

$$\mathscr{O}^*_n\big(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}}\big)(k)=\mathscr{O}_n\big(\pi_{n,\hat{\theta}_{n-1}}\hat{\theta}'_{n-1}\big)(k)\cdot\mathscr{O}'_n\big(\pi_{n,\hat{\theta}'_{n-1}}\big)(k),\quad k\in N_n. \tag{8}$$

The following lemma will be useful in proving the above proposition. The proof of the lemma follows from construction, and hence is omitted.

Lemma 1. Let $\hat{\theta}_{n-1}$ *and* $\hat{\theta}'_{n-1}$ *be any two members of* $\hat{\Theta}_{n-1}$ *and let* $\pi_{n,\hat{\theta}_{n-1}}$ *and* $\pi_{n,\hat{\theta}'_{n-1}}$ *be the permutations on* N_n *induced by* $\hat{\theta}_{n-1}$ *and* $\hat{\theta}'_{n-1}$, *respectively. Then*

$$\big(\pi_{n,\theta_{n-1}}\circ\pi_{n,\theta'_{n-1}}\big)(k)=\pi_{n,\theta_{n-1}\theta'_{n-1}}(k),\quad \forall k\in N_n. \tag{9}$$

Proof of Proposition 1. Let $I_k,\ k\in N_n$, be an arbitrarily chosen nth order dyadic subinterval. We know that each member of $\hat{\Theta}_n$ transports I_k to one and only one of the nth order dyadic subintervals $I_0,\ I_1,\dots,I_{2^n-1}$. We want to trace the subinterval that the composite map $\hat{\theta}_n\hat{\theta}'_n$ transports I_k to. Note that under $\hat{\theta}'_n$, I_k

goes to $I_{\pi_{n,\hat{\theta}'_{n-1}}}(k)$, which goes to the subinterval $I_{(\pi_{n,\hat{\theta}_{n-1}} \circ \pi_{n,\hat{\theta}'_{n-1}})(k)}$, under the $\hat{\theta}_n$ map. Applying Lemma 1, we note that the composite map $\hat{\theta}_n \hat{\theta}'_n$ takes the kth dyadic subinterval to the subinterval $I_{(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}})(k)}$.

We now want to determine the slope of the subinterval $I_{(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}})(k)}$, which depends on the associated maps $\mathscr{O}_n$ and $\mathscr{O}'_n$. By drawing a suitable diagram if necessary, we can easily verify that the slope of the interval $I_{(\pi_{n,\hat{\theta}_{n-1}} \circ \pi_{n,\hat{\theta}'_{n-1}})(k)}$ under the composite map is given by the product of the slopes of $I_{(\pi_{n,\hat{\theta}'_{n-1}})(k)}$ under $\hat{\theta}'_n$ and that of $I_{(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}})(k)}$ under the map $\hat{\theta}_n$. □Q.E.D.

By construction, it is clear that $\hat{\Theta}_0$ is isomorphic to the group $\mathfrak{F}$; this can also be verified directly as follows: since $\hat{\Theta}_0$ has only two l.m.p. automorphisms, namely $\theta_0^0(x) = x$, $x \in I$, and $\theta_0^1(x) = 1 - x$, $x \in I$, with respect to the composition of the functions and with θ_0^0 as the identity element, $\hat{\Theta}_0$ is indeed a group. Since $\hat{\Theta}_0$ is a group, using Proposition 1 inductively on $n \geq 1$ we can easily establish the following proposition:

Proposition 2. *For $n \geq 0$, $\hat{\Theta}_n$ is a group with the identity map as the identity element of the group and with the multiplication operation as defined in* (7).

For each $n \geq 1$, we define the projection map $f_n : \hat{\Theta}_n \to \hat{\Theta}_{n-1}$ by $f_n(\hat{\theta}_n) = \hat{\theta}_{n-1}$, where $\hat{\theta}_{n-1}$ is the unique element of $\hat{\Theta}_{n-1}$ such that $\hat{\theta}_n \in A_n(\hat{\theta}_{n-1})$.

In part (b) of Fig. 1 we graphed a $\hat{\theta}_3 \in \hat{\Theta}_3$, which satisfies $f_3(\hat{\theta}_3) = \hat{\theta}_2$, where $\hat{\theta}_2 \in \hat{\Theta}_2$ has the graph as shown in part (a) of the same figure. As mentioned earlier, we equip each finite group $\hat{\Theta}_n$ with the discrete σ-algebra, $\mathscr{B}_n$, and the counting probability measure, Γ_n.

Proposition 3. *For each $n \geq 1$, f_n is a Borel homomorphism of the measurable group $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$ onto the measurable group $(\hat{\Theta}_{n-1}, \mathscr{B}_{n-1}, \Gamma_{n-1})$.*

Proof. Let $\hat{\theta}_n$ and $\hat{\theta}'_n$ be two arbitrary members of $\hat{\Theta}_n$. There exist unique $\hat{\theta}_{n-1}$ and $\hat{\theta}'_{n-1}$ in $\hat{\Theta}_{n-1}$ and slope maps $\mathscr{O}_n$ and $\mathscr{O}'_n$ of N_n such that $\hat{\theta}_n = (\pi_{n,\hat{\theta}_{n-1}}(k), \mathscr{O}_n(\pi_{n,\hat{\theta}_{n-1}})(k))_{k=0}^{2^n-1}$, $\hat{\theta}'_n = (\pi_{n,\hat{\theta}'_{n-1}}(k), \mathscr{O}'_n(\pi_{n,\hat{\theta}'_{n-1}})(k))_{k=0}^{2^n-1}$ and $f_n(\hat{\theta}_n) = \hat{\theta}_{n-1}$ and $f_n(\hat{\theta}'_n) = \hat{\theta}'_{n-1}$. To prove that f_n is a homomorphism, we need to show that $f_n(\hat{\theta}_n \hat{\theta}'_n) = f_n(\hat{\theta}_n) f_n(\hat{\theta}'_n)$. From (7) we have

$$\hat{\theta}_n \hat{\theta}'_n = \left(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}}(k), \mathscr{O}_n^* \left(\pi_{n,\hat{\theta}_{n-1}\hat{\theta}'_{n-1}}\right)(k)\right)_{k=0}^{2^n-1},$$

where $\mathscr{O}^*$ is as defined in (8). Hence,

$$f_n\left(\hat{\theta}_n \hat{\theta}'_n\right) = \hat{\theta}_{n-1} \hat{\theta}'_{n-1} = f_n\left(\hat{\theta}_n\right) f_n\left(\hat{\theta}'_{n-1}\right).$$

Since the $\mathscr{B}_n$'s are discrete σ-algebras, f_n obviously is a Borel map. □Q.E.D.

We define

$$\breve{\Theta} = \{\breve{\theta} = (\theta_0, \theta_1, \theta_2, \dots) \mid \theta_n \in \hat{\Theta}_n, \forall n \geq 0 \text{ and } f_n(\theta_n) = \theta_{n-1}, \forall n \geq 1\}.$$

For any two elements $\breve{\theta} = (\theta_0, \theta_1, \theta_2, \dots)$ and $\breve{\theta}' = (\theta'_0, \theta'_1, \theta'_2, \dots)$ from $\breve{\Theta}$, we define the multiplication operation $\breve{\theta} \circ \breve{\theta}'$ by

$$\breve{\theta} \circ \breve{\theta}' = (\theta_0\theta'_0, \theta_1\theta'_1, \theta_2\theta'_2 \dots).$$

By Proposition 3, $f_n(\theta_n\theta'_n) = f_n(\theta_n)f_n(\theta'_n) \in \hat{\Theta}_{n-1}$. Hence, $\breve{\theta} \circ \breve{\theta}' \in \breve{\Theta}$. With $\breve{\theta}^{-1} = (\theta_0^{-1}, \theta_1^{-1}, \theta_2^{-1}, \dots)$ as the inverse of $\breve{\theta} = (\theta_0, \theta_1, \theta_2, \dots)$, and with $\breve{e} = (e_0, e_1, \dots)$, where e_n is the identity element of $\hat{\Theta}_n$ as the unit element, we note that $\breve{\Theta}$ is a group.

We define for $n \geq 0$ the projection maps $\pi_n : \breve{\Theta} \to \hat{\Theta}_n$ by

$$\pi_n(\breve{\theta}) = \theta_n, \quad \text{where } \breve{\theta} = (\theta_0, \theta_1, \theta_2 \dots).$$

Let $\mathscr{F} = \bigcup_{n=1}^{\infty} \pi_n^{-1}(\mathscr{B}_n)$. It can easily be shown that $\mathscr{F}$ is a Boolean algebra. Let $\breve{\mathscr{B}}$ be the σ-algebra generated by $\mathscr{F}$. $(\breve{\Theta}, \breve{\mathscr{B}})$ is called the *projective limit* of the sequence of measure spaces $(\hat{\Theta}_n \mathscr{B}_n)$, $n \geq 0$, through the maps f_n, $n \geq 1$. The following is an extension of the Daniel–Kolmogorov consistency theorem of stochastic processes.

Theorem 1 (Generalized Daniel–Kolmogorov consistency theorem). *There exists a unique right invariant probability measure $\breve{\Gamma}$ on the projective limit $(\breve{\Theta}, \breve{\mathscr{B}})$ of the sequence of measurable groups $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)_0^{\infty}$ through the sequence of homomorphisms $\{f_n\}_0^{\infty}$ such that*

(i) $\breve{\Gamma}\pi_n^{-1} = \Gamma_n$, *and*

(ii) $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ *is a separated measurable group.*

Proof. It is clear from the construction of the $\hat{\Theta}_n$'s and f_n's that the sequence of counting probability measures $\{\Gamma_n\}_0^{\infty}$ on the measure spaces $(\hat{\Theta}_n, \mathscr{B}_n)_0^{\infty}$ satisfies the following (Kolmogorov) consistency condition:

$$\Gamma_n f_n^{-1} = \Gamma_{n-1}, \quad \text{for all } n \geq 1.$$

Hence by the Daniel–Kolmogorov consistency theorem (Parthasarathy 1977, proposition 27.4), there exists a probability measure $\breve{\Gamma}$ on the projective limit space $(\breve{\Theta}, \breve{\mathscr{B}})$ such that

$$\breve{\Gamma}\pi_n^{-1} = \Gamma_n, \quad \text{for all } n \geq 0.$$

We now show that the map $(\breve{\theta}, \breve{\theta}') \mapsto \breve{\theta}\breve{\theta}'^{-1}$ from $(\breve{\Theta} \times \breve{\Theta}, \breve{\mathscr{B}} \times \breve{\mathscr{B}})$ onto $(\breve{\Theta}, \breve{\mathscr{B}})$ is measurable, i.e. for any $E \in \breve{\mathscr{B}}$, $\{(\breve{\theta}, \breve{\theta}^*) \in \breve{\Theta} \times \breve{\Theta} \mid \breve{\theta}\breve{\theta}^{*-1} \in E\}$ is $\breve{\mathscr{B}} \times \breve{\mathscr{B}}$ measurable. Since $\breve{\mathscr{B}}$ is generated by the Boolean algebra $\mathscr{F} = \bigcup_{n=0}^{\infty} \pi_n^{-1}(\mathscr{B}_n)$, it is enough to show that for all $E \in \pi_n^{-1}(\mathscr{B}_n)$ the above is true. Since $\mathscr{B}_n$ has a finite number of elements, and is the power set of $\hat{\Theta}_n$, it is enough to show the measurability of $E = \pi_n^{-1}(\{\hat{\theta}_n\})$, for any $\hat{\theta}_n \in \hat{\Theta}_n$. It is easy to note that $\pi_n^{-1}(\{\hat{\theta}_n\})$ is equivalent to

$$C(\hat{\theta}_n) = \left\{(\hat{\theta}_0, \hat{\theta}_1, \dots, \hat{\theta}_n)\right\} \times f_{n+1}^{-1}(\hat{\theta}_n) \times f_{n+2}^{-1} f_{n+1}^{-1}(\hat{\theta}_n)$$
$$\times \dots, \quad \hat{\theta}_i \in \hat{\Theta}_i, \; 0 \le i \le n,$$

where $\hat{\theta}_i$, $i < n$, is uniquely determined by backward iterative projections of $\hat{\theta}_n$. We denote the above set by $C(\hat{\theta}_n)$ to denote its dependence on $\hat{\theta}_n \in \hat{\Theta}_n$. Let $\hat{\theta}_n \in \hat{\Theta}_n$ be arbitrarily fixed. We now note that

$$\left\{(\breve{\theta}', \breve{\theta}^*) \in \breve{\Theta} \times \breve{\Theta} \mid \breve{\theta}'\breve{\theta}^{*-1} \in \pi_n^{-1}\left(\{\hat{\theta}_n\}\right)\right\} = \bigcup_{\bar{\hat{\theta}}_n \in \hat{\Theta}_n} C\left(\hat{\theta}_n \bar{\hat{\theta}}_n\right) \times C\left(\bar{\hat{\theta}}_n\right).$$

Since each $C(\hat{\theta}_n \bar{\hat{\theta}}_n) \times C(\bar{\hat{\theta}}_n) \in \breve{\mathscr{B}} \times \breve{\mathscr{B}}$, their finite union also is in $\breve{\mathscr{B}} \times \breve{\mathscr{B}}$, and thus the map $(\breve{\theta}, \breve{\theta}') \mapsto \breve{\theta}\breve{\theta}'^{-1}$ from $(\breve{\Theta} \times \breve{\Theta}, \breve{\mathscr{B}} \times \breve{\mathscr{B}})$ onto $(\breve{\Theta}, \breve{\mathscr{B}})$ is measurable.

We now want to show that $\breve{\Gamma}$ is (right) invariant. Let $\breve{\theta} \in \breve{\Theta}$ be arbitrarily chosen. Let us consider the measure $\breve{\Gamma}_{\breve{\theta}}$ on $(\breve{\Theta}, \breve{\mathscr{B}})$ defined by

$$\breve{\Gamma}_{\breve{\theta}}(E) = \breve{\Gamma}(E\breve{\theta}), \quad E \in \breve{\mathscr{B}}.$$

We note that $\breve{\Gamma}_{\theta}(E) = \breve{\Gamma}(E)$ for all $E \in \pi_n^{-1}(\mathscr{B}_n)$, $n \ge 0$, and hence for all $E \in \mathscr{F}$, and since $\mathscr{F}$ generates $\breve{\mathscr{B}}$, hence for all $E \in \breve{\mathscr{B}}$. Thus, $\breve{\Gamma}$ is a (right) invariant probability measure. Hence $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ is a measurable group.

The uniqueness of $\breve{\Gamma}$ can be proved directly using the line of argument of the previous two paragraphs and noting that since the measurable group $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$ is finite, Γ_n is the unique (right) invariant probability measure. We then pass the argument to $\mathscr{F}$ and then to $\breve{\mathscr{B}}$.

We now prove the separability of the measurable group $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$. Let $\breve{\theta} \in \breve{\Theta}$, and $\breve{\theta} \neq \breve{e}$. Let $E_n = \{\breve{\theta} = (\theta_0, \theta_1, \dots, \theta_n, \dots) \in \breve{\Theta} \mid \theta_i = e_i, 0 \le i \le n\}$, where e_i is the identity element of $\hat{\Theta}_n$. Since $\breve{\theta} \neq \breve{e}$, there exists $n > 1$ such that $\breve{\theta} \notin E_n$. Let us fix such an n. We can easily show that $E_n \in \breve{\mathscr{B}}$, and is a normal subgroup of $\breve{\Theta}$ and hence $\breve{\Theta}/E_n$ is a group, and that $\hat{\Theta}_n \simeq \breve{\Theta}/E_n$, where $\simeq$ denotes group isomorphism. Since $\breve{\Gamma}$ is a right invariant probability measure, right cosets of E_n are equally likely with respect to $\breve{\Gamma}$. Since there are only a finite number of right cosets of E_n and they are disjoint subsets of $\breve{\Theta}$, it follows that $\breve{\Gamma}(E_n) > 0$ and $\breve{\Gamma}(E_n \breve{\theta} \Delta E_n) = 2\breve{\Gamma}(E_n) > 0$. □Q.E.D.

Remark 1. Since each $\hat{\Theta}_n$, $n \ge 0$, has at least two distinct elements, using the Cantor diagonalization argument we can show that the product space $\mathscr{P} = \prod_{n=0}^{\infty} \hat{\Theta}_n$ is uncountably large. Extending the same diagonalization argument to the projective limit $\breve{\Theta}$, which is a subset of $\mathscr{P}$, we can show that $\breve{\Theta}$ is also uncountably large as follows. Let us note that $\hat{\Theta}_0$ has two distinct elements, and for all $n \ge 1$

and $\hat{\theta}_{n-1} \in \hat{\Theta}_{n-1}$, the set $A_n(\hat{\theta}_{n-1})$ has at least two distinct elements. Let $E \subset \breve{\Theta}$ be a countable subset of $\breve{\Theta}$, and let $\breve{\theta}_0, \breve{\theta}_1, \ldots,$ be all the elements of E. Since $\breve{\theta}_n \in \breve{\Theta}$, we can write it as $\breve{\theta}_n = (\theta_{n,k})_{k=0}^{\infty}$, and $\theta_{n,k} \in \hat{\Theta}_n$ for all $n, k \geq 0$. We define $\breve{\theta}^* = (\theta_0^*, \theta_1^*, \ldots)$, where $\theta_n^* \in \hat{\Theta}_n$, $n \geq 0$, are defined recursively by $\theta_0^* \in \hat{\Theta}_0 - \{\theta_{0,0}\}$ and $\theta_n^* \in A_n(\theta_{n-1}^*) - \{\theta_{n,n}\}$. It is easy to note that $\breve{\theta}^* \notin E$. Thus every countable subset of $\breve{\Theta}$ is a proper subset of $\breve{\Theta}$. $\breve{\Theta}$ cannot be countable, otherwise it will mean that $\breve{\Theta}$ is a proper subset of $\breve{\Theta}$, which is absurd.

3.2. Embedding of the projective limit group $\breve{\Theta}$ as a subgroup of $\mathscr{G}$

For each $\breve{\theta} = (\theta_0, \theta_1, \ldots, \theta_n, \ldots) \in \breve{\Theta}$ we define a function $\breve{\theta}: I \to I$ by

$$\breve{\theta}(t) = \lim_{n \to \infty} \theta_n(t). \tag{10}$$

Proposition 4. *The limit in* (10) *exists for all $t \in I$, and the function $\breve{\theta}(\cdot)$ is a Lebesgue measure-preserving automorphism of $(I, \mathscr{B}_I)$.*

Proof. Note that from the definition of the θ_n's, it follows that $|\theta_n(t) - \theta_{n-1}(t)| \leq 1/2^n$, for all $n \geq 1$, and hence the limit in (10) exists for all $t \in I$.

To show that $\breve{\theta}(\cdot)$ is one-to-one, suppose $t, s \in I$, and $t \neq s$. Then there exists $n_0 \geq 0$ such that for all $n > n_0$, $\theta_n(t)$ and $\theta_n(s)$ are in separate nth order dyadic subintervals. Hence, $\breve{\theta}(t) \neq \breve{\theta}(s)$. The function corresponding to $\breve{\theta}^{-1}$ in $\breve{\Theta}$ defines the inverse function of $\breve{\theta}(\cdot)$. Since the function $\breve{\theta}(\cdot)$ on I corresponding to a $\breve{\theta} \in \breve{\Theta}$ is the point-wise limit of a sequence of measurable functions on I, $\breve{\theta}(\cdot)$ is a measurable function on $(I, \mathscr{B}_I)$. To show that $\lambda\breve{\theta}^{-1} = \lambda$, let $f: I \to \mathscr{R}$ be any (bounded) continuous function. Then

$$\begin{aligned}
\int f \, d\lambda\breve{\theta}^{-1} &= \int f(\breve{\theta}) \, d\lambda \\
&= \int f\Big(\lim_{n \to \infty} \theta_n(t)\Big) \, d\lambda(t) \\
&= \int \lim_{n \to \infty} f(\theta_n(t)) \, d\lambda(t) \\
&= \lim_{n \to \infty} \int f(\theta_n(t)) \, d\lambda(t) \qquad \text{(A)} \\
&= \lim_{n \to \infty} \int f \, d\lambda\theta_n^{-1} \\
&= \int f \, d\lambda. \qquad \text{(B)}
\end{aligned}$$

In deriving equality (A) above, we have made use of the Lebesgue dominated convergence theorem. Equality (B) above follows since θ_n is Lebesgue measure preserving for all $n \geq 0$. Since the above two integrals are identical for all bounded continuous functions, the corresponding measures are equal and we have $\lambda \breve{\theta}^{-1} = \lambda$. □Q.E.D.

Remark 2. From Proposition 4 it is clear that $\breve{\Theta}$ could be identified as a subgroup of the l.m.p. automorphisms of I. From last paragraph in the proof of Theorem 1, it is clear that $\breve{\Theta}$ contains all the component groups $\hat{\Theta}_n$, $n \geq 0$, with the identification $\hat{\Theta}_n \simeq \breve{\Theta}/E_n$. Alternatively, from (10) and Proposition 4 it is clear that we can also identify an element $\theta^* \in \hat{\Theta}_n$ with the element $\breve{\theta}^* = (\theta_0^*, \theta_1^*, \ldots, \theta_{n-1}^*, \theta^*, \theta^*, \ldots) \in \breve{\Theta}$, where $\theta_{k-1}^* = f_k(\theta_k^*)$, $1 \leq k \leq n$, and $\theta_n^* = \theta^*$. We will make use of these facts later.

4. Existence and characterization of the random order value operator on *LOR*$\breve{\Theta}$

Since $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ and $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$, $n \geq 0$, are all measurable groups, we know from Raut (1993, theorems 3 and 4) that there exist unique $\breve{\Theta}$-symmetric and $\hat{\Theta}_n$-symmetric random order value operators $\Phi_{\breve{\Gamma}}$ and Φ_{Γ_n} on $LOR\breve{\Theta}$ and $LOR\hat{\Theta}_n$, respectively. It can easily be shown that $LOR\breve{\Theta} \subset LOR\hat{\Theta}_n$ for all $n \geq 0$. A byproduct of our construction of $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ from the component measurable groups, $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$, is that on $LOR\breve{\Theta}$ the operator $\Phi_{\breve{\Gamma}}$ could be approximated by Φ_{Γ_n} in the sense of (12) below. We summarize these results in the following theorem.

Theorem 2. Let $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ be the projective limit of the measurable groups $(\hat{\Theta}_n, \mathscr{B}_n, \Gamma_n)$ and the projective homomorphisms f_{n+1}, $n \geq 0$. Then there exists a unique $\breve{\Theta}$-symmetric random order value operator $\Phi_{\breve{\Gamma}}$ on $LOR\breve{\Theta}$ given by

$$(\Phi_{\breve{\Gamma}} V)(S) = \int_{\breve{\Theta}} (\phi[V])(S, \breve{\theta}) \, \mathrm{d}\breve{\Gamma}(\breve{\theta}). \tag{11}$$

Furthermore, for all $S \in \mathscr{B}_I$ and $V \in LOR\breve{\Theta}$,

$$(\Phi_{\breve{\Gamma}} V)(S) = \lim_{n \to \infty} (\Phi_{\Gamma_n} V)(S), \tag{12}$$

where $(\Phi_{\Gamma_n} V)(S) = \int_{\Theta_n} \phi[V](S, \theta) \, \mathrm{d}\Gamma_n(\theta)$.

Proof. The first part follows from Raut (1993, theorems 3 and 4). Let us prove the limit in (12) for an arbitrary integrable function $h \in L_1(\breve{\Theta}, \breve{\Gamma})$. For each $\breve{\theta} = (\theta_0, \theta_1, \ldots, \theta_k, \ldots) \in \breve{\Theta}$, $h \in L_1(\breve{\Theta}, \breve{\Gamma})$, and for all $n \geq 0$, we define

$$\breve{\theta}_n \equiv (\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n, \theta_n \ldots)$$

$$h_n(\breve{\theta}) \equiv h(\breve{\theta}_n).$$

It is easy to note that each h_n, $n \geq 0$, is measurable. Note also that $h_n(\hat{\theta}) \to h(\breve{\theta})$ point-wise for all $\breve{\theta} \in \breve{\Theta}$ and that $|h_n(\breve{\theta})| \leq |h(\breve{\theta})|$ for all $\breve{\theta} \in \breve{\Theta}$. By the Lebesgue dominated convergence theorem, we have

$$\int_{\breve{\Theta}} h(\breve{\theta})\, \mathrm{d}\breve{\Gamma}(\breve{\theta}) = \lim_{n \to \infty} \int_{\breve{\Theta}} h_n(\breve{\theta})\, \mathrm{d}\breve{\Gamma}(\breve{\theta})$$

$$= \lim_{n \to \infty} \int_{\hat{\Theta}_n} h_n(\theta)\, \mathrm{d}\Gamma_n(\theta) \text{ by Theorem 1.}$$

In particular, for any $V \in LOR\breve{\Theta}$ and $S \in \mathscr{B}_I$, substituting $h(\breve{\theta}) \equiv \phi[V](S, \breve{\theta})$ in the above, Eq. (12) follows. □Q.E.D.

Remark 3. From (12) it is clear that the limit of a sequence of random order value operators with respect to a particular type of increasing sequence of finite subgroups of $\mathscr{G}$ is also a random order value operator with respect to the limit measurable subgroup which is uncountably large. This limiting result need not hold for every increasing sequence of measurable subgroups of $\mathscr{G}$.

4.1. Measure-valued games and the $\breve{\Theta}$-symmetric random order value

We first examine the kind of randomization of players that is performed by the random orders in $\hat{\Theta}_n$. This will guide us in finding large classes of economically important measure-valued games on which a $\breve{\Theta}$-symmetric random order value with respect to the measurable group $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ coincides with the axiomatic value for this class of games given by Aumann and Shapley (1974).

By way of illustration, let us consider the automorphism θ_2 that is depicted in part (a) of Fig. 1. Note that the set of players before player t, $t \in I$, in the random order $\theta_2 \in \hat{\Theta}_2$ is given by

$$I(t, \theta_2) = \begin{cases} [0, t) \cup I_2, & \text{if } t \in I_1, \\ (t, \frac{1}{2}), & \text{if } t \in I_2, \\ I_1 \cup I_2 \cup [\frac{1}{2}, t) \cup I_4, & \text{if } t \in I_3, \\ I_1 \cup I_2 \cup (t, 1), & \text{if } t \in I_4. \end{cases}$$

Note that the nature of the randomization produced by an element of $\hat{\Theta}_n$ depends on the associated permutation π_n and the slope map $\mathscr{O}_n$. Let us fix a $t \in I$ and suppose $t \in I_1$. Let us assume n is sufficiently large and fixed. Let us consider the random orders $\theta_n \in \hat{\Theta}_n$ that assign a constant value $t_0 \in I$ to t, i.e. $\theta_n(t) = t_0$ for all such n. Suppose, for simplicity, that $t_0 \neq 1/2^{n+1}$. [5] All these random orders will have either a positive slope or a negative slope at t. Without loss of generality

[5] To be more precise, t_0 is meant to be $t_0 \pm \mathrm{d}t_0$, where $\mathrm{d}t_0$ is infinitesimal, and similarly, t is meant to be $t \pm \mathrm{d}t$, where $\mathrm{d}t$ is infinitesimal. For our heuristic argument, however, we pretend t_0 and t are real numbers.

let us assume that all have a positive slope. From Fig. 1 it is clear that all these random orders place a particular type of sets of Lebesgue measure t_0 before player t; and the type of the sets depends on t_0, and the order of the dyadic subdivision, n. For example, suppose $t_0 < 1/2$, then all these random orders do not place any players from the interval $[1/2, 1]$. Thus for a given size t_0, $0 \leq t_0 \leq 1$, the random orders in $\breve{\Theta}$ allow t to form coalition only with certain sets of players of size t_0 but not every (Borel) set of players whose size is t_0. On the other hand, t gets to have any given player placed before him in a suitable random order and a suitable size t_0.

For games in which the worth of a coalition depends only on its size in the Lebesgue measure sense, i.e. if a set of players are effective [6] only through their (Lebesgue measure) size but not through any of their other identities, we expect and we will formally show that the expected marginal contribution of a player with respect to the group of random orders $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$ coincides with the axiomatic value for these games as characterized by Aumann and Shapley (1974).

More formally, let μ be a non-atomic probability measure on $(I, \mathscr{B}_I)$. Suppose, further, that $\mu \ll \lambda$. Let us consider games of the type $V(S) = (f \circ \mu)(S)$, $S \in \mathscr{B}_I$, $f \in C^1(I)$, the class of continuously differentiable functions on the unit interval. [7] Since $V \in pNA$, we know [8] that a marginal contribution function $\phi[V](\cdot, \breve{\theta})$ exists for each $\breve{\theta} \in \breve{\Theta}$. It follows that $\phi[V](\cdot, \breve{\theta}) \ll \mu \ll \lambda$ for all $\breve{\theta} \in \breve{\Theta}$. Thus by the Radon–Nikodyn theorem, there exists a unique function

$$h[f \circ \mu](t, \breve{\theta}) \equiv \frac{\mathrm{d}\phi[f \circ \mu](\cdot, \breve{\theta})}{\mathrm{d}\lambda}(t) \in L_1(\lambda).$$

We want to characterize $h[f \circ \mu](\cdot, \breve{\theta})$ in a suitable form. Let $m_{\breve{\theta}}(x) \equiv \mu\breve{\theta}^{-1}([0, x))$ for all $\breve{\theta} \in \breve{\Theta}$. We have the following result:

Proposition 5. Let $\breve{\theta}$ be any Lebesgue measure-preserving automorphism, $\mu \ll \lambda$ and f is continuously differentiable function on I, then we have

$$h[f \circ \mu](t, \breve{\theta}) = (f \circ m_{\breve{\theta}})'(\breve{\theta}(t)) \text{ a.e. } t(\lambda). \tag{13}$$

In particular, we have

$$h[f \circ \lambda](t, \breve{\theta}) = f'(\breve{\theta}(t)). \tag{14}$$

Proof. By Rudin (1966, theorem 8.6) we have

$$\lim_{n \to \infty} \left| \frac{\phi[V](E_n, \breve{\theta})}{\lambda(E_n)} - h[f \circ \mu](t, \breve{\theta}) \right| = 0 \text{ a.e. } t(\lambda),$$

[6] This concept is analogous to small group effectiveness, as in Wooders and Zame (1987).

[7] We assume that each function is continuously differentiable in the interior of I and has a continuous extension to the boundary of the interval.

[8] See Aumann and Shapley (1974).

where $\{E_n\}_0^\infty$ is any sequence of sets from $\mathscr{B}_I$ with the property that there exist $\alpha > 0$ and $r_n > 0$ such that each E_n lies in the open interval $(t - r_n/2,\ t + r_n/2)$, $\lambda(E_n) \geq \alpha r_n$, $n = 1, 2, \ldots$, and $r_n \to 0$ as $n \to \infty$.

Let

$$E_n = \begin{cases} I\left(t + \dfrac{1}{n},\ \breve{\theta}\right) - (t, \breve{\theta}), & \text{if } \breve{\theta}'(t) = +1, \\ I(t,\ \breve{\theta}) - I\left(t + \dfrac{1}{n},\ \breve{\theta}\right), & \text{if } \breve{\theta}'(t) = -1. \end{cases}$$

Note that $\lambda(E_n) = 1/n$ or $-1/n$ according as $\breve{\theta}'(t) = +1$ or -1. Using these facts, we have

$$\begin{aligned} h[f \circ \mu](t,\ \breve{\theta}) &= \lim_{n \to \infty} \frac{\phi[f \circ \mu]\big(E_n,\ \breve{\theta}\big)}{\lambda(E_n)} \\ &= \lim_{n \to \infty} \frac{\phi[f \circ \mu]\big(I(t + 1/n,\ \breve{\theta})\big) - \phi[f \circ \mu](t,\ \breve{\theta})}{\lambda(E_n)} \\ &= \lim_{n \to \infty} \frac{(f \circ \mu)\breve{\theta}^{-1}\big([0,\ \breve{\theta}(t + 1/n))\big) - (f \circ \mu)\breve{\theta}^{-1}\big([0,\ \breve{\theta}(t)\big)}{\lambda(E_n)} \\ &= \lim_{h \to 0} \frac{(f \circ m_{\breve{\theta}})\big(\breve{\theta}(t) + h\big) - (f \circ m_{\breve{\theta}})\big(\breve{\theta}(t)\big)}{h}, \quad h = \breve{\theta}'(t)\frac{1}{n} \\ &= (f \circ m_{\breve{\theta}})'\big(\breve{\theta}(t)\big), \text{ a.e. } t(\lambda). \end{aligned}$$

The second part of the proposition follows immediately from the first part since $m_{\breve{\theta}}(x) = x$ when $\mu = \lambda$. □Q.E.D.

We have the following analogue of Aumann and Shapley (1974, theorem A, p. 20).

Theorem 3. Let f: $I \to R$ be a continuously differentiable function such that $f(0) = 0$ and let μ be a non-atomic probability measure on $(I, \mathscr{B}_I)$. Then the $\breve{\Theta}$-symmetric random order value for the game $V(S) = f \circ \mu(S)$ exists and is given by

$$(\Phi_{\breve{\Gamma}}(f \circ \mu))(S) = \mu(S) \int_0^1 f'(x)\, \mathrm{d}x. \tag{15}$$

If, furthermore, $f(1) = 1$, then $\Phi_{\breve{\Gamma}}(f \circ \mu) = \mu$.

The following lemma will be used to prove the above theorem.

Lemma 2. Let $f\colon I \to \Re$ be continuous and let $t \in I$ be fixed. Then $f(\breve{\theta}(t))$ as a function of $\breve{\theta}$ is integrable with respect to $(\breve{\Theta}, \breve{\mathscr{B}}, \breve{\Gamma})$, and

$$\int_{\breve{\Theta}} f(\breve{\theta}(t))\,\mathrm{d}\breve{\Gamma} = \lim_{n\to\infty}\int_{\Theta_n} f(\hat{\theta}_n(t))\,\mathrm{d}\Gamma_n. \tag{16}$$

Proof of Lemma 2. Note that for any $\breve{\theta} = (\theta_0, \theta_1, \ldots, \theta_n, \ldots) \in \breve{\Theta}$, we have

$$\breve{\theta}(t) = \lim_{n\to\infty}\theta_n(t) = \lim_{n\to\infty}\pi_n(\breve{\theta}(t)).$$

Since f is continuous, we have

$$f(\breve{\theta}(t)) = f\Big(\lim_{n\to\infty}\pi_n\breve{\theta}(t)\Big) = \lim_{n\to\infty}(f\circ\pi_n)(\breve{\theta}(t)),\quad \forall\breve{\theta}\in\breve{\Theta}.$$

It is easy to see that each $(f\circ\pi_n)(\breve{\theta})$ is measurable and f, being the point-wise limit of a sequence of measurable functions, is also measurable. Since f is continuous with a compact domain, f is bounded by a constant $M > 0$, implying that $|(f\circ\pi_n)(\cdot)| \leq M$, and since $\breve{\Gamma}$ is a probability measure, by the Lebesgue dominated convergence theorem, f is integrable and we have

$$\begin{aligned}
\int_{\breve{\Theta}} f(\breve{\theta}(t))\,\mathrm{d}\breve{\Gamma} &= \lim_{n\to\infty}\int_{\breve{\Theta}}(f\circ\pi_n)(\breve{\theta}(t))\,\mathrm{d}\breve{\Gamma} \\
&= \lim_{n\to\infty}\int_{\Theta_n} f(\theta_n(t))\,\mathrm{d}\breve{\Gamma}\pi_n^{-1} \text{ by Theorem 1} \\
&= \lim_{n\to\infty}\int_{\Theta_n} f(\theta_n(t))\,\mathrm{d}\Gamma_n. \quad \square\text{Q.E.D.}
\end{aligned}$$

Proof of Theorem 3. Let us first prove (15), taking μ to be the Lebesgue measure λ. Note that

$$\begin{aligned}
\Phi_{\breve{\Gamma}}(f\circ\lambda)(S) &= \int_{\breve{\Theta}}\int_S h[V](t, \breve{\theta})\,\mathrm{d}\lambda(t)\,\mathrm{d}\breve{\Gamma}(\breve{\theta}) \\
&= \int_{\breve{\Theta}}\int_S f'(\breve{\theta}(t))\,\mathrm{d}\lambda(t)\,\mathrm{d}\breve{\Gamma}(\breve{\theta}), \text{ by (14)} \\
&= \int_S\left[\int_{\breve{\Theta}} f'(\breve{\theta}(t))\,\mathrm{d}\breve{\Gamma}(\breve{\theta})\right]\mathrm{d}\lambda(t), \text{ by Fubini's theorem} \\
&= \int_S\left[\lim_{n\to\infty}\int_{\Theta_n} f'(\hat{\theta}_n(t))\,\mathrm{d}\Gamma_n(\hat{\theta}_n)\right]\mathrm{d}\lambda(t), \text{ by Lemma 2} \\
&= \int_S \mathrm{d}\lambda(t)\int_0^1 f'(x)\,\mathrm{d}x, \text{ since } f' \text{ is Riemann integrable} \\
&= \lambda(S)\int_0^1 f'(x)\,\mathrm{d}x.
\end{aligned}$$

Thus (15) is true when μ is the Lebesgue measure. Note that for any $\xi \in \mathscr{G}$, $\Phi_{\check{\Gamma}}(\xi^*(f \circ \lambda))(S) = \Phi_{\check{\Gamma}}(f \circ \lambda)(\xi^{-1}(S)) = \lambda(\xi^{-1}(S))\int_0^1 f'(x)\,\mathrm{d}x = \xi^* \Phi_{\check{\Gamma}}(f \circ \lambda)(S)$. Thus the $\check{\Theta}$-symmetric random order value of a game of the form $(f \circ \lambda)(S)$ is symmetric with respect to the full group of automorphisms, $\mathscr{G}$.

For a general non-atomic measure μ, we know by the isomorphism theorem of measure theory (see Parthasarathy 1977, proposition 26.6) that there exists a $\xi \in \mathscr{G}$ such that $\mu\xi^{-1} = \lambda$. We now note that

$$\begin{aligned}
\Phi_{\check{\Gamma}}(f \circ \mu)(S) &= \Phi_{\check{\Gamma}}\xi^{*-1}\xi^*(f \circ \mu)(S)\\
&= \Phi_{\check{\Gamma}}\xi^{*-1}(f \circ \mu\xi^{-1})(S)\\
&= \Phi_{\check{\Gamma}}\xi^{*-1}(f \circ \lambda)(S)\\
&= \xi^{*-1}\Phi_{\check{\Gamma}}(f \circ \lambda)(S)\\
&= \xi^{*-1}\lambda(S)\int_0^1 f'(x)\,\mathrm{d}x\\
&= \mu(S)\int_0^1 f'(x)\,\mathrm{d}x. \quad \square \text{Q.E.D.}
\end{aligned}$$

Remark 4. Note that the above random order value is exactly the same as the axiomatic value given in Aumann and Shapley (1974). Thus for such games, both values coincide.

The following theorem is analogous to Aumann and Shapley (1974, theorem B, p. 23):

Theorem 4.[9] *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector of non-atomic positive measures on $(I, \mathscr{B}_I)$ such that for each i, $i = 1, \ldots, k, \exists \xi_i \in \mathscr{G}$ such that $\mu_i \xi_i = \mu_i(I)\lambda$, $\check{\theta}\xi_i \in \check{\Theta}$, and $\mathscr{R} = \{\mu(S) \in \Re_+^k \mid S \in \mathscr{B}_I\}$ has full rank. Let f: $\mathscr{R} \to \Re$ be a continuously differentiable function such that $f(0) = 0$. Then the random order value for the game $V(S) = (f \circ \mu)(S)$ exists and is given by*

$$(\Phi_{\check{\Gamma}}(f \circ \mu))(S) = \int_0^1 f_{\mu(S)}(\mu(I) \cdot x)\,\mathrm{d}x. \tag{17}$$

where $f_{\mu(S)}(x) \equiv \sum_{i=1}^k f_i(x)\mu_i(S)$.

Proof. It follows from (13) that

$$h[f \circ \mu](t, \check{\theta}) = \sum_{i=1}^k f_i\big(m_{\check{\theta}}(\check{\theta}(t))\big) \cdot \frac{\mathrm{d}\mu_i\check{\theta}^{-1}}{\mathrm{d}\lambda}\big(\check{\theta}(t)\big). \tag{18}$$

[9] The result seems to be true under more general conditions; however, I have not been able to establish it in this paper.

Note that

$$m_{\breve{\theta}}(\breve{\theta}(t)) = \begin{pmatrix} \mu_1(I)\cdot\lambda\xi_1^{-1}\breve{\theta}^{-1}([0,\breve{\theta}(t))) \\ \vdots \\ \mu_k(I)\cdot\lambda\xi_k^{-1}\breve{\theta}^{-1}([0,\breve{\theta}(t))) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1(I)\cdot\lambda(\breve{\theta}\xi_1)^{-1}([0,\breve{\theta}\xi_1\xi_1^{-1}(t))) \\ \vdots \\ \mu_k(I)\cdot\lambda(\breve{\theta}\xi_k)^{-1}([0,\breve{\theta}\xi_k\xi_k^{-1}(t))) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1(I)\cdot\lambda(I(\xi_1^{-1}(t),\breve{\theta}\xi_1) \\ \vdots \\ \mu_k(I)\cdot\lambda(\xi_k^{-1}(t),\breve{\theta}\xi_k) \end{pmatrix}, \text{ since } \breve{\theta}\xi_i \in \breve{\Theta},\ i=1,\ldots,k.$$

But $\lambda(I(\xi_i^{-1}(t),\breve{\theta}\xi_i)) = (\breve{\theta}\xi_i)(\xi_i^{-1}(t)) = \breve{\theta}(t)$ since $\breve{\theta}\xi_i \in \breve{\Theta}$. Hence, we have

$$m_{\breve{\theta}}(\breve{\theta}(t)) = (\mu(I)\cdot\breve{\theta}(t)), \tag{19}$$

where

$$\mu(I) = \begin{pmatrix} \mu_1(I) \\ \vdots \\ \mu_k(I) \end{pmatrix}.$$

Note also that

$$\frac{\mathrm{d}\mu_i\breve{\theta}^{-1}}{\mathrm{d}\lambda}(\breve{\theta}(t)) = \frac{\mathrm{d}\mu_i\breve{\theta}^{-1}}{\mathrm{d}\lambda\breve{\theta}^{-1}}(\breve{\theta}(t)), \text{ since } \breve{\theta} \text{ is l.m.p.}$$

$$= \frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(\breve{\theta}^{-1}(\breve{\theta}(t))),$$

$\times$ by Parthasarathy (1977, proposition 48.7)

$$= \frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(t), \text{ a.e. } t(\lambda).$$

Now note that

$$\Phi_{\breve{\Gamma}}(f\circ\mu)(S) = \int_{\breve{\Theta}}\int_S h[f\circ\mu](t,\breve{\theta})\,\mathrm{d}\lambda(t)\,\mathrm{d}\breve{\Gamma}(\breve{\theta})$$

$$= \int_S\int_{\breve{\Theta}} h[f\circ\mu](t,\breve{\theta})\,\mathrm{d}\breve{\Gamma}(\breve{\theta})\,\mathrm{d}\lambda(t), \text{ by Fubini's theorem}$$

$$= \int_S\left[\int_{\breve{\Theta}}\sum_{i=1}^k f_i(\mu(I)\cdot\breve{\theta}(t))\frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(t)\,\mathrm{d}\breve{\Gamma}\right]\mathrm{d}\lambda(t), \tag{A}$$

$$= \int_S \left[\sum_{i=1}^{k} \lim_{n \to \infty} \int_{\hat{\Theta}_n} f_i\left(\mu(I) \cdot \hat{\theta}_n \right) \mathrm{d}\Gamma_n\left(\hat{\theta}_n\right) \frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(t) \right] \mathrm{d}\lambda(t) \tag{B}$$

$$= \int_S \sum_{i=1}^{k} \left[\int_0^1 f_i(\mu(I) \cdot x) \, \mathrm{d}x \right] \frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(t) \, \mathrm{d}\lambda(t), \tag{C}$$

$$= \sum_{i=1}^{k} \int_0^1 f_i(\mu(I) \cdot x) \, \mathrm{d}x \int_S \frac{\mathrm{d}\mu_i}{\mathrm{d}\lambda}(t) \, \mathrm{d}\lambda(t), \tag{D}$$

$$= \int_0^1 \sum_{i=1}^{k} f_i(\mu(I) \cdot x) \mu_i(S) \, \mathrm{d}x$$

$$= \int_0^1 f_{\mu(S)}(\mu(I) \cdot x) \, \mathrm{d}x.$$

In the above derivation, equality (A) follows from (18) and (19); equality (B) follows from Lemma 2, equality (C) follows from a basic property of the Riemann integral of a continuous function on the closed interval I. □Q.E.D.

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