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Firm's R&D Behavior Under Rational Expectations

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ABSTRACT

This paper formulates the inter-temporal R&D investment decision problem of private firms using an optimal stochastic control framework. The paper explicitly derives the R&D investment decision rule and the cross equations parameter restrictions imposed by the hypothesis of rational expectations, using only the Riccati equation, and not requiring the Wiener-Kolmogorov prediction formula. Identification and estimation of the structural parameters are essential for evaluating policies to be free from Lucas critique. The paper finds conditions for identification of structural parameters, and discusses econometric procedures for estimation of structural parameters, and testing of the model.

Keywords: Research and development; Rational expectations; Stochastic control; Lucas critique

JEL Classifications: D21, L1

1. INTRODUCTION

Technological change is a major source of growth, and industrial R&D investment is a major determinant of technological change. Returns from industrial research and development (R&D) investments in the private sector depend on the evolution of market conditions and public policies over time. Thus the Lucas critique (1976) on policy evaluation applies in this context. The main point of the critique in the present context is that if a firm's R&D investment decision under uncertainty depends on its expectations about the future market conditions and policy changes, then, instead of estimating a R&D decision rule by throwing in arbitrarily some policy variables as regressors, one should model and estimate the parameters of the firm's objective function and the stochastic processes that govern the future environments in which the firm operates. To achieve this, one needs a tractable dynamic economic model of R&D investment which lends to estimation and testing using available econometric techniques.

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The empirical studies on R&D have not paid much attention to the above critique. One set of studies is concerned with testing the Schumpeterian hypothesis regarding the effects of firm size and intensity of rivalry on the pace of R&D investments within a static framework (see Levin and Reiss (1984), Kamien and Schwartz (1981) for an account of these studies). The other set of studies is concerned with the effect of R&D expenditures on productivity growth (Griliches (1984), Mohnen (1992), Mairesse and Sassenou (1991), for an account of studies on developed countries, and Raut (1995) for an account of studies on developing countries). Although many studies are directed toward policy analysis, these studies do not formulate R&D investment decisions using a dynamic economic model and then estimate the model parameters.[1]

In this paper I present a dynamic economic model of R&D investments and explore conditions under which the structural parameters can be identified and estimated. I explicitly model the process of knowledge creation, as in Griliches (1979), and Pakes and Griliches (1984). To impute a value to technological knowledge in each period, I assume that the timing of an innovation is unknown; however, the higher is the stock of knowledge, the higher is the probability of its taking place in any period. Applying techniques from the statistical decision theory, I impute a value to stock of technological knowledge. The problem of R&D investment decision is then cast into an econometrically tractable dynamic optimization problem of the type that Lucas suggested in his 1976 critique.

There are two approaches to solving a dynamic optimization problem: the dynamic programming approach and the variational approach. To find close form solution to their cost-of-adjustment model of labor supply, Hansen and Sargent (1981) gave up the dynamic programming approach, and followed the variational approach that uses Euler equation, Transversality condition and the Wiener-Kolmogorov prediction formula. In this paper, I show that when the control variable is one dimensional (which was also the case in their model) it is possible to derive a close form solution solely from the matrix Riccati equation, and thus the Wiener-Kolmogorov prediction formula is not required for this purpose. I then find conditions under which the structural parameters could be identified from the cross equations restrictions that are imposed by the hypothesis of rational expectations. I also discuss the econometric issues related to estimation and testing of the dynamic optimization model of R&D investment.

Section 2 sets up the dynamic optimization model of R&D and derives the close form solutions and cross equation restrictions that are imposed by the hypothesis of rational expectations. Section 3 deals with identification of structural parameters. Section 4 discusses econometric issues related to estimation and testing of the model.

1 Pakes (1984), however, goes a step closer in this direction: instead of deriving the reduced form solution with cross equations restrictions imposed by rational expectations, he, however, parameterizes directly the reduced form equations for estimation.

2. THE BASIC MODEL

The R&D investment of a private firm is a deliberate economic activity similar to investment in physical capital. A set of R&D inputs adds to stock of knowledge which might immediately be used or might be useful for further information production in later periods. Some accumulated knowledge may become obsolete in the light of new knowledge, and some of the previously acquired knowledge might be hard to retrieve. To take these aspects of knowledge creation process into account, I will assume that accumulated knowledge depreciates at a constant rate in each period. Let z_t be the stock of technological knowledge, net of depreciation at the beginning time t. The firm can increase this stock of knowledge by investing in R&D. Let R_t be the R&D investment in period t. The knowledge that R_t produces at time t also depends on random factors which I summarize in a random variable w_{1t}. I assume that w_{1t} is a white noise process with mean 0 and variance σ_1^2. I further assume that R&D investment has a gestation lag of one period, i.e., an investment R_t incurred at the beginning of period t adds to knowledge at the end of period t. The technology of knowledge accumulation is represented by the following production function.

$$z_{1t+1} = f(z_{1t}, R_t, w_{1t}) \tag{1}$$

This production process is general enough to incorporate many features of technology production. For instance, it allows the marginal product of R&D to vary from industry to industry depending on the R&D capability or strength of knowledge or the science base of that industry. It can also accommodate spillover effects of knowledge such as learning from others, public investment in basic research, and technological knowledge of other domestic and foreign firms by assuming that the firm acquires a constant flow of such spillover knowledge in each period.

2.1. Innovation Process

The knowledge is not a standard good for which there can be a market determining its value. The value of knowledge need be imputed from the value of the innovation that it may produce. Technological knowledge is an intangible, indivisible, inappropriable, i.e., difficult to institute a property right on, and involves externalities in its production and use. Patenting is a legal protection assuring only a partial appropriation (see Arrow (1962) for further discussions). Following the strategy used in statistical decision theory to evaluate information, I impute a private value to technological knowledge in the following way.

I assume that the timing of the innovation is not known with certainty, but the likelihood of its taking place in any period is higher, the greater is the stock of accumulated knowledge at the beginning of that period. Let $P(z_{1t})$ be the probability that the firm will reap the innovation in period t if its stock of knowledge is z_{1t}, given that it has not achieved it yet. Various forms for $P(.)$ are plausible. One would like it to satisfy the following: $dP(z)/dz > 0$ and $d^2P(z)/dz^2 < 0$. The reasonable forms for P(.) are as follows:

$$P(z) = 1 - e^{-\gamma z}, z \geq 0, \gamma > 0. \tag{2}$$

Another reasonable form is

$$P(z) = \vartheta(\mu z - \gamma z^2), 0 < z < \bar{z}, \mu, \gamma > 0. \tag{3}$$

where ϑ is a constant determined by μ,γ, and $\bar{z}$.

The value of the innovation in period t depends on a number of factors: It depends on the firm size z_{2t}, the intensity of rivalry z_{3t}, and the market condition, or the profitability from the current line of research ξ_t. The effect of the firm size on the value of technological knowledge may come through different channels. Following Nelson's (1959) interpretation, I argue that the larger firms having already established name and reputation in the market can more easily appropriate the benefits of an innovation, as for instance, by easier market penetration and product diversification. Therefore, a larger firm may envisage a bigger return from a given stock of technological knowledge than a smaller one[2]. The market concentration or the intensity of rivalry is an industry level attribute. More rivals in an industry means a higher chance of an innovated product or process being imitated, and also a higher chance of a similar innovation by another firm. Moreover, a higher number of rivals in an industry reduce the market share of a firm. All these mean a lower value for an innovation.[3]

Another important factor that influences the value of technological knowledge is the uncertainty about the demand for new products in the case of product innovation and the shift in the demand for an existing product in the case of process innovation. The higher is this uncertainty, the lower is the value of the stock of technological knowledge. This type of effect is sometimes referred to in the R&D literature as the Schmookler's hypothesis or the demand pull or the market opportunity hypothesis. I incorporate these factors in the valuation of knowledge by assuming that the present value of an innovation in period t is $\eta(z_{2t}, z_{3t}, \xi_t)$ which is a function of the firm size z_{2t}, the intensity of rivalry z_{3t} and the demand condition ξ_t in period t. This will be the case also if the innovation is patented and sold to another firm for a royalty payment, whose value is determined by the market conditions prevailing then.[4]

If the innovation is not reaped in period t, the firm will be left with a stock of accumulated knowledge, net of depreciation, which will increase the probability of an innovation in future. Thus the value of the end of period knowledge will depend on values of z_{2t}, z_{3t}, and ξ_t in future periods. The evolution of z_{2t}, z_{3t}, and ξ_t depend on many

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2 It should, however, be noted that the smaller firms are not necessarily restricted to use their knowledge only in their own production units as they can always sell it to another firm with licensing arrangements.

3 Also greater monopoly power reduces the incentive for innovation as the firm with monopoly power can continue to earn the monopoly rent without venturing into a new technological innovation. It is generally argued that an intermediate level of market concentration is most conducive to rapid technological innovation.

4 Kamien and Schwartz (1981), explicitly modeled rivalry using a subjective hazard function, and then derived a functional relationship between rivalry and the present value of an innovation.

factors. In a more general framework, market conditions should evolve endogenously as a result of strategic interactions among firms. A tractable general theory along this line does not yet exist. I simplify the present analysis by assuming that z_{2t}, z_{3t}, and ξ_t evolve over time according to an exogenously given Markov process. More specifically, I assume that the transition probability measure for the state variables $z_t \equiv (z_{1t}, z_{2t}, z_{3t}, \xi_t)$ is given by $q_v(dz_{t+1} \mid z_t, R_t)$, where v is a set of parameters characterizing the transition probability distribution.[5]

2.2. The decision problem of the firm

I assume that the cost of R&D is quadratic in input use. One period expected reward from a stock of knowledge z_{1t} in period t is then given by

$$u(z_t, R_t) \equiv \eta(z_{2t}, z_{3t}, \xi_t)P(z_{1t}) - 0.[1 - P(z_{1t})] - \theta R_t^2 \tag{4}$$

plus a stock of technological knowledge, z_{1t+1} as given by (1).

Assume that after reaping the targeted innovation, the firm will venture into another innovation that will use the knowledge of the previous pursuit. The firm then faces an infinite horizon for its R&D investment decisions.

Given the manger's visions on the sequences $\{z_{2t}\}$, $\{z_{3t}\}$, and $\{\xi_t\}$ that characterize the environment of the firm, the manager of the firm decides a sequence of R&D investments $\{R_t\}$ to maximize the following expected reward,

$$V_0 = E_0 \sum_{t=0}^{\infty} \beta^t u(z_t, R_t) \tag{5}$$

where the expectation operator $E_t(x) \equiv E(x \mid \Omega_t)$, and Ω_t is the information set of the firm at time t which includes past values of all the variables and the current values of the state variables. To solve this problem using the variational approach, the stochastic Euler equation of the above problem can be derived as follows,

$$2\theta R_t = \beta \int \left(\eta(z_{2t+1}, z_{3t+1}, \xi_{t+1}) P'(z_{1t+1}) \frac{\partial f(z_{1t}, R_t, w_{1t})}{\partial R_t} \right) q_v(dz_{t+1} \mid z_t, R_t),\ t = 0,1,2\ldots$$

which can be rewritten as

$$E_t\left(\phi(z_{t+1}, z_t, R_t; \delta) \mid z_t, R_t\right) = \int \phi(z_{t+1}, z_t, R_t; \delta) q_v(dz_{t+1} \mid z_t, R_t) = 0, \tag{6}$$

where, $\phi(z_{t+1}, z_t, R_t; \delta) \equiv \beta\eta(z_{2t+1}, z_{3t+1}, \xi_{t+1}) P'(z_{1t+1}) \dfrac{\partial f(z_{1t}, R_t, w_{1t})}{\partial R_t} - 2\theta R_t$

5 Notice that I am assuming here for simplicity that R&D activities affect neither the firm size nor the intensity of rivalry. While this assumption is innocuous in the short-run, for a medium to long-run analysis this may not be the case, see Landes (1969) for historical evidence and Levin [1981], and Levin and Reiss [1984] for studies on interaction of R&D expenditures by private firms and their effects on market concentration in a static framework.

and δ is a vector of parameters that specify the objective function in equation (5). The stochastic Euler equation (6) together with the associated Transversality condition can be used to solve the dynamic optimization problem.

For the dynamic programming approach to the above problem, the associated Bellman equation is given by

$$V(z) = \max_{R}\left(u(z,R) + \beta \int V(z')q_v\left(dz' \mid z,R\right)\right) \tag{7}$$

Under quite general assumptions[6] it is known that the above Bellman equation (7) (also known as functional equation) has a unique solution $V(.)$. Furthermore, there exists a stationary Markovian policy function $R = \rho\ (z)$ which is the argmax of the problem in equation (5) with z as the known initial state.

Notice that the optimal policy function $R = \rho\ (z)$ is a deterministic relation, and hence cannot be estimated econometrically. It needs an error term or stochastic component. The general practice in the literature is to assume that some of the components of the state variables in z are observed by the managers but not by the econometrician. I will assume that while there is a good estimate available for z_1, the market condition ξ is not observed by the econometrician. With a lot of pooled time series cross section data on R and the observed components of z, we can statistically estimate $\rho(.)$ very precisely. Denote the structural parameters corresponding to the objective function more generally by $\delta = (u\ (.), \beta)$. An important question related to the Lucas critique is then, could we identify the structural parameters δ and v non-parametrically from the estimated policy function $\rho(.)$? The answer is in general no (see Rust (1994) for details). Even if we jointly estimate the policy function $R=\rho\ (z)$ and the transition probability measure $q_v\left(dz' \mid z,R\right)$ from the observed data, the rational expectation hypothesis does not impose enough restrictions that could be used to identify all the structural parameters in general. For identification of parameters, one needs to impose further restrictions on specifications. I will illustrate this point in a linear-quadratic set-up in the next section.

2.3. Linear-Quadratic R&D Investment Problem

I assume that the technological knowledge is accumulated according to the following stochastic linear production function,

$$z_{1t+1} = a_1 z_{1t} + bR_t + c_1 + w_{1t},\ t \geq 0 \tag{8}$$

where $1-a_1$ is the depreciation rate for knowledge, b is a measure of the technological capability or the strength of knowledge, c_1 is the constant amount of spillover knowledge that accrues to the firm from all external sources. I assume the following linear specification

6 More specifically, the assumption are that $u(.)$ is bounded, R is non negative, bounded and the transition probability matrix satisfy the Feller property. For more precise statements of these assumptions, see Stokey and Lucas (1989, Theorems 9.6 and 9.8), see Bhattacharya and Majumdar (1989).

for η :

$$\eta(z_{2t}, z_{3t}, \xi_t) = r_0 + r_1\xi_t + r_2 z_{2t} + r_3 z_{3t} \tag{9}$$

where, r_0, r_1, $r_2 > 0$ and $r_3 < 0$. Substituting (3) in (9), one gets

$$\eta(z_{2t}, z_{3t}, \xi_t)\cdot P(z_{1t}) = (r_0 + r_1\xi_t + r_2 z_{2t} + r_3 z_{3t})\cdot\vartheta\left[\mu z_{1t} - \gamma z_{1t}^2\right]$$

Substituting the above in equation (4), regarding $r_0 + r_1\xi = \zeta$, and disregarding all third and higher order terms, equation (4) can be rewritten as,

$$u(z_t, R_t) = z_t' Q z_t + HR_t^2 \tag{10}$$

where, $Q = (q_{ij})_{i,j=1,\dots 4}$, $q_{11} = -r_0\gamma\vartheta$, $q_{12} = r_2\mu\vartheta$, $q_{13} = r_3\mu\vartheta$, $q_{14} = \mu\vartheta$, and other q_{ij} 's are all zero, H = $-\ \theta$ and $z_t = (z_{1t}, z_{2t}, z_{3t}, \zeta_t)'$. I assume that z_{2t} and z_{3t} follow third order autoregressive processes, and ζ_t follows a first order regressive process as follows:

$$\begin{cases} z_{2t+1} &= \delta_2(L)z_{2t} + c_2 + w_{2t} \\ z_{3t+1} &= \delta_3(L)z_{3t} + c_3 + w_{3t} \\ \zeta_{t+1} &= a_4\zeta_t + c_4 + w_{4t} \end{cases} \tag{11}$$

where $\qquad \delta_i(L) = a_{i1} + a_{i2}L + a_{i3}L^2,\ i = 2,\ 3$

and L is the lag operator, i.e., $LX_t = X_{t-1}$, and w_{it} is a white noise process with mean zero and variance σ_i^2, i = 2..4. Writing (8) and (11) together, and redefining the vector z to include the lag values of the state variables, we have the following vector autoregressive process of the state variables:

$$z_{t+1} = Az_t + BR_t + c + w_t \tag{12}$$

where

$$z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ z_{2t-1} \\ z_{2t-2} \\ z_{3t} \\ z_{3t-1} \\ z_{3t-2} \\ \xi_t \end{pmatrix} \quad A = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 \end{pmatrix} \quad B = \begin{pmatrix} b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$Q=\begin{pmatrix} q_{11} & q_{12} & 0 & 0 & q_{13} & 0 & 0 & q_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad c=\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \\ c_3 \\ 0 \\ 0 \\ c_4 \end{pmatrix} \quad w_t=\begin{pmatrix} w_{1t} \\ w_{2t} \\ 0 \\ 0 \\ w_{3t} \\ 0 \\ 0 \\ w_{4t} \end{pmatrix}$$

Notice that the functional equation (7) becomes

$$V(z_t)=\max_{R_t}\left(z_t'Qz_t+q_t'z_t+R_t'HR_t+h'R_t+\beta\int V(Az_t+BR_t+c+w_t)\mu(dw_t)\right). \quad (13)$$

While the above is written as a general quadratic problem, in the present context, however, q and h are zero vectors. Solution of (13) gives the optimal R_t as a function of z_t. Under certain conditions[7] on A, B, c, Q, and H, there exists an optimal stationary solution to functional equation (13) as stated in the following theorem:

Proposition 1: The optimal stationary solution of (13) is given by

$$R_t=-(Gz_t+g) \quad (14)$$

where

$$G=\beta(H+\beta B'KB)^{-1}B'KA \quad (15)$$

$$g=(H+\beta B'KB)^{-1}\left(\beta B'Kc+\frac{\beta}{2}B'k+\frac{h}{2}\right) \quad (16)$$

K is a positive definite solution of the following matrix Riccati equation:

$$K=Q+\beta A'[K-\beta KB(H+\beta B'KB)^{-1}B'K]A \quad (17)$$

and k is the solution to the following vector Riccati equation:

$$k'(I-\beta(A-BG))=q'+2g'(H+\beta B'KB)G+2\beta c'K(A-BG)-h'G-2\beta g'B'KA\,. \quad (18)$$

Proof. Guess a solution for V (.) of the form:

$$V(z)=z'Kz+k'z+\kappa \quad (19)$$

7 These conditions are controllability and observability as stated in Bertesekas (1976).

where K is a positive definite matrix, k is a vector of positive numbers and κ is a non-negative real number.

Substituting equation (19) in equation (13), one has

$$
\begin{aligned}
V(z_t) \;=\; & \max_{R_t}\Big\{z_t'Qz_t + q'z_t + R_t'HR_t + h'R_t \\
& +\beta\Big[z_t'A'KAz_t + 2z_t'A'KBR_t + 2z_t'A'Kc \\
& +R_t'B'KBR_t + 2R_t'B'Kc + c'Kc \\
& +E_t(w_t'Kw_t) \\
& +k'Az_t + k'BR_t + k'c + \kappa\Big]\Big\}
\end{aligned}
\tag{20}
$$

The first-order-condition of the above problem produces equation (14). To find the solution for K, k, and κ, substitute the optimal value of R_t from equation (14) in equation (20) and the presumed value of $V(z)$ from equation (19) on the left hand side of equation (20) and then collect the coefficients for the quadratic term $z_t'Kz_t$, the linear term $k'z_t$ and the constant term. After a few simplifications, one arrives at the expressions in equations (17) and (18).

Q.E.D

Using the formulas in Proposition 1, one can now derive the close form solution to the R&D investment decision problem as stated in Proposition 2.

Proposition 2: *A closed form solution of the optimization problem (13) is given by*

$$R_t = -g + \alpha_1 z_{1t} + \alpha_2(L)z_{2t} + \alpha_3(L)z_{3t} + \alpha_4\zeta_t \tag{21}$$

where
$$g = \frac{\lambda b}{a_1\theta(1-\lambda)}\left(\frac{q_{11}c_1}{1-\lambda a_1} + \frac{q_{12}c_2}{1-\lambda\delta_2(\lambda)} + \frac{q_{13}c_3}{1-\lambda\delta_3(\lambda)} + \frac{q_{14}c_4}{1-\lambda a_4}\right) \tag{22}$$

$$\alpha_1 = -\frac{b\lambda}{a_1\theta}a_1(1-a_1\lambda)^{-1}q_{11} \tag{23}$$

$$\alpha_j(L) = -\frac{b\lambda}{a_1\theta}q_{1j}\sum_{i=0}^{2}\frac{\left(\delta_j(\lambda)/\lambda^i\right)_+}{1-\lambda\delta_j(\lambda)}L^i,\quad j=2,\ 3 \tag{24}$$

$$\alpha_4 = -\frac{b\lambda}{a_1\theta} a_4(1-a_4\lambda)^{-1} q_{14} \tag{25}$$

and $\lambda = a_1\theta\beta/(b^2\beta k_{11}+\theta)$, and $()_+$ *denotes the* ***annihilation operator*** *that tells us to ignore negative powers of L.*

Following the variational approach, Hansen and Sargent (1981) used the stochastic Euler equation, the Wiener-Kolmogorov prediction formula, and the Transversality condition to derive their closed form solution to the labor supply problem. Following the dynamic programming approach, however, I have derived equations (21) and (22) directly from the matrix Riccati equation of the problem.

Proof. To find an explicit optimal decision rule R_t from equation (14), note that the vector Riccati equation (18) involves K, G and g, whereas the matrix Riccati equation K does not involve k and g. Note that

$$\beta B'KB + H = b^2\beta k_{11} + \theta$$

and

$$B'KA = b[a_1k_{11},\ a_{21}k_{12}+k_{13},\ a_{22}k_{12}+k_{14},\ a_{23}k_{12},\ a_{31}k_{15}+k_{16},\ a_{32}k_{15}+k_{17},\ a_{33}k_{15},\ a_3k_{15}]$$

Substituting these in equation (15), one derives the following expression for G :

$$G = \frac{-b\beta}{b^2\beta+\theta}[a_1k_{11},\ a_{21}k_{12}+k_{13},\ a_{22}k_{12}+k_{14},\ a_{23}k_{12},\ a_{31}k_{15}+k_{16},\ a_{32}k_{15}+k_{17},\ a_{33}k_{15},\ a_4k_{18}] \tag{26}$$

It is clear from the above that only the first row of the matrix K needs be computed. I compute k_{11}, k_{12},....k_{18} from the Riccati equation (17) as follows:

$$\begin{cases} K &= A'\left[\beta K - \beta^2 KBB'K/(\beta b^2 k_{11}+\theta)\right] A + Q \\ &= A'DA + Q \\ &= \begin{pmatrix} a_1^2 d_{11} & a_1d_{12}a_{21}+a_1d_{13} & a_1d_{12}a_{22}+a_1d_{14} & a_1d_{12}a_{23} & a_1d_{15}a_{31}+a_1d_{16} & \dots & a_1d_{18}a_4 \\ \dots & & \dots & & \dots & \dots & \\ a_4d_{81}a_1 & a_4d_{82}a_{21}+a_4d_{83} & a_4d_{82}a_{22}+a_4d_{84} & a_4d_{82}a_{23} & a_4d_{85}a_{31}+a_4d_{86} & \dots & a_4^2d_{88} \end{pmatrix} + Q \end{cases} \tag{27}$$

where

$$\begin{cases} D & = & (d_{ij})_{i,\, j=1,\,2,\,3,\,4} \\ & = & \beta K - \frac{\beta^2 KBB'K}{\beta b^2 k_{11} + \theta} \\ & = & \beta K - K \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} K \end{cases} \tag{28}$$

and $m = \beta^2 b^2 / (\beta b^2 k_{11} + \theta)$. Recall the notation: $\lambda = a_1 \theta \beta / (b^2 \beta k_{11} + \theta)$. It is now easy to compute d_{1j}'s as follows:

$$d_{11} = \beta k_{11} - m k_{11}^2 = \frac{\theta \beta k_{11}}{b^2 \beta k_{11} + \theta} = \lambda k_{11} / a_1$$

$$d_{12} = \beta k_{12} - m k_{11} k_{12} = \frac{\theta \beta k_{12}}{b^2 \beta k_{11} + \theta} = \lambda k_{12} / a_1$$

In general,

$$d_{1j} = \beta k_{1j} - m k_{11} k_{1j} = \frac{\theta \beta k_{1j}}{b^2 \beta k_{11} + \theta} = \lambda k_{1j} / a, \quad j = 1, \ 2, ..8$$

Substituting these on the right hand side of the last equality of the expressions in (27) and then equating the matrix elements of both sides, one gets the following:

Corresponding to the state variable z_1:

$$k_{11}(1 - a_1 \lambda) = q_{11} \tag{29}$$

Corresponding to the state variable z_2:

$$k_{12}(1 - \lambda a_{21}) - \lambda k_{13} = q_{12} \tag{30}$$

$$k_{13} - \lambda a_{22} k_{12} - \lambda k_{14} = 0 \tag{31}$$

$$k_{14} - \lambda a_{23} k_{12} = 0 \tag{32}$$

Corresponding to the state variable z_3:

$$k_{15}(1 - \lambda a_{21}) - \lambda k_{16} = q_{13} \tag{33}$$

$$k_{16} - \lambda a_{22} k_{15} - \lambda k_{17} = 0 \tag{34}$$

$$k_{17} - \lambda a_{23} k_{15} = 0 \tag{35}$$

Corresponding to the state variable ζ:

$$k_{18}(1 - a_4 \lambda) = q_{14} \tag{36}$$

Equations (30)-(32) yield,

$$k_{12} = \frac{q_{12}}{1-\lambda\delta_2(\lambda)}, \quad k_{13} = \frac{\left(\lambda a_{22} + \lambda^2 a_{23}\right) q_{12}}{1-\lambda\delta_2(\lambda)}, \quad k_{14} = \frac{\lambda a_{23} q_{12}}{1-\lambda\delta_2(\lambda)} \tag{37}$$

Denote the reduced form parameters corresponding to z_{2t} and its lag values on the right hand side of equation (26) by

$$\alpha_2(L) = \alpha_{21} + \alpha_{22} L + \alpha_{23} L^2.$$

Substituting equation (37) in equation (26) one gets,

$$\alpha_{21} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{\delta_2(\lambda)}{1-\lambda\delta_2(\lambda)} \tag{38}$$

$$\alpha_{22} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{a_{22} + \lambda a_{23}}{1-\lambda\delta_2(\lambda)} \tag{39}$$

$$\alpha_{23} = -\frac{b\lambda q_{12}}{a_1\theta} \cdot \frac{a_{23}}{1-\lambda\delta_2(\lambda)} \tag{40}$$

Similarly, denoting the reduced form parameters corresponding to z_{3t} and its lag values in equation (26) by $\alpha_3(L)=\alpha_{31}+\alpha_{32}L+\alpha_{33}L^2$ and proceeding exactly the same way with equations (33)-(34) as we did with equations(30)-(34), we derive the stated expressions for $\alpha_3(L)$ The solutions for k_{11} is obtained from equation (29) and solution for k_{18} is obtained from equation (36). Substituting these values of k_{11},k_{18} and the above solutions for α_2 (L) and α_3 (L) in equation (16), the close form solution for R_t as given in equation (21) is obtained.

To find an explicit solution for note that in the present case, q and h are 0. It follows from equation (16) that

$$g = \frac{b\beta}{b^2\beta+\theta}\left[k_{11}c_1 + k_{12}c_2 + k_{15}c_3 + \ k_{18}c_4 + k_1/2\right]$$

$$= \frac{b\beta}{b^2\beta+\theta}\left[\frac{q_{11}c_1}{1-\lambda a_1} + \frac{q_{12}c_2}{1-\lambda\delta_2(\lambda)} + \frac{q_{13}c_3}{1-\lambda\delta_3(\lambda)} + \frac{q_{14}c_4}{1-\lambda a_1} + \frac{k_1}{2}\right]$$

where k_1 is the first component of the vector k in the vector Ricaati equation (18) and and k_{11}, k_{12}, k_{15} and k_{18} are the components of the Matrix Riccati equation, which we have already solved above. To find k_1, substitute the above g in vector Ricaati equation (18), and then solve for k_1. After substituting this k_1 in the above expression, one gets the solution for g as in equation (22) of the theorem.

Q.E.D.

3. IDENTIFICATION OF STRUCTURAL PARAMETERS

In this section I show that even in the restrictive linear-quadratic case, while cross equation restrictions imposed by the rational expectations hypothesis are useful to identify some of the structural parameters, to identify all the structural parameters one needs to impose further structure. More specifically, I show that when the $\{z_{2t}, z_{3t}, \zeta_t\}$ process follows a first order autoregressive process, the cross equations restrictions do not identify all the structural parameters. When I assume a third or higher order autoregressive process, I show that a subset of the cross equations restrictions are enough to identify all the structural parameters, leaving the remaining restrictions for efficient estimation of the parameters and testing of the model.

3.1. First Order Autoregressive Process and the Identification Problem

Assume that z_{2t}, z_{3t}, and ζ_t follow first order autoregressive processes, i.e., assume that $a_{22} = a_{23} = a_{32} = a_{33} = 0$ in equation (21). Denote a_{21} as a_2 and a_{31} as a_3 to simplicity notation. The close form solution for R_t from the above theorem is then given by

$$R_t = -g + \alpha_1 z_{1t} + \alpha_2 z_{2t} + \alpha_3 z_{3t} + \alpha_4 \zeta_t \tag{41}$$

where,

$$g = \frac{b\lambda}{a_1\theta(1-\lambda)}\sum_{j=1}^{4} c_j(1-a_j\lambda)^{-1}q_{1j}$$

$$\alpha_i = -\frac{b\lambda}{a_1\theta}a_i(1-a_i\lambda)^{-1}q_{1i}, \ i=1,\dots 4$$

$$\lambda = \frac{a_1\theta\beta}{b^2\beta k_{11}+\theta}$$

and k_{11} is a positive solution of the quadratic equation:

$$k_{11}\left(1-\frac{\theta\beta a_1^2}{b^2\beta k_{11}+\theta}\right)=q_{11}.$$

Equation (41) and the system of equations (12) for the motion of the environment constitute the firm's decision rule. The assumption of rational expectations and a particular specification of the stochastic processes in equation (12) have generated cross equations restrictions.

The structural parameters are $\delta=(\theta,\beta,\ q_{1j},\ j=1,\ 2,\ldots 4)$ from the objective function; $v=\left(b,\ a_i,\ c_i,\ \sigma_i^2,\ i=1,\ 2,\ldots 4\right)$ from the stochastic processes (12) representing the environment. The second set of coefficients could be estimated from the system of equations (12). I will show that not all of the structural parameters from the objective function can be identified, and hence they could not be estimated from the observed data. To that end, from equation (41) it follows that

$$\sum_{j=1}^{4} c_j\alpha_j / a_j = (1-\lambda)g \tag{42}$$

$$k_{11}(1-a_1\lambda)=q_{11} \tag{43}$$

$$\lambda=\frac{a_1\theta\beta}{b^2\beta k_{11}+\theta} \tag{44}$$

$$\alpha_1=-\frac{\lambda b}{(1-a_1\lambda)\theta}q_{11} \tag{45}$$

Note that from equation (42), one can get an estimate of λ. Substituting the value of q_{11} from equation (45) in equation (43) and then substituting the value of k_{11} in equation (44) one gets

$$\lambda=\frac{a_1\beta}{1+b^2\beta\alpha_1/\lambda} \tag{46}$$

From equation (46) one can get an estimate of β. But since θ cancelled out, one cannot identify θ in this system. Therefore, the system of equations (12) and (41) is under identified.

3.2 Higher Order Autoregressive Process and the Identification Problem

Assume that z_{2t} and z_{3t} follow third order autoregressive processes as in equation (12). The close form solution for this specification is given by equations (21)-(25). The first set of structural parameters $\delta=(\theta,\beta,\ q_{1j},\ j = 1,\ 2,....4)$ from the objective function are same as in the previous subsection; the second set of structural parameters are now $v=\left(b,\ a_1,\ \delta_2(L),\ \delta_3(L),\ a_4,\ c_i,\ \sigma_i^2,\ i=1,..4\right)$ from the stochastic processes (11).

Equations (38)-(40) imply

$$\frac{\alpha_{21}}{\alpha_{23}}-\lambda\frac{\alpha_{22}}{\alpha_{23}}=\frac{a_{21}}{a_{23}} \tag{47}$$

From equation (47), one can estimate λ. It is now clear that given λ, one can estimate β from equations (43)-(46). (Note that equations (43)-(46) are valid in this case also).

Also note that substituting in equation (22) the values of q_{11} from equation (45), q_{12} from equation (39) and q_{13} and q_{14} from the equations that parallel equation (39), one can get an estimate of θ. From equation (39), one gets q_{12}, and from equations parallel to equation (39) for Z_{3t} and ζ_t, one can estimate q_{13} and q_{14}. Finally, from equation (45) one can estimate q_{11}. So, all the structural parameters could be recovered in this case. Note that in this identification strategy I have never used equations that are parallel of equations (38) and (40) corresponding to the variable z_{3t}. Therefore, the rational expectations hypothesis has imposed over identifying restrictions across equations. These over-identifying restrictions could be used for a more efficient estimation of the structural parameters and for a statistical testing of the model. I turn to these issues in the next section.

4. ECONOMETRIC ESTIMATION AND TESTING OF THE MODEL

Suppose we have data on z_{it}, R_{it} for a sample of firms $i = 1....n$, and $t = 1....,T$. The structural parameters to be estimated are the vectors δ and v as described in the previous section. There are broadly two approaches to the estimation of structural parameters of the dynamic optimization models: The variational approach which is also known as stochastic Euler equation approach, and the dynamic programming approach.

For the stochastic Euler equation approach, I rewrite Euler equations (6) as follows:

$$\varepsilon_{it}=\phi\left(z_{it+1},\ z_{it},\ R_{it};\ \delta,\ v\right),\ t=0,\ 1,...T$$

where ε_{it} is a random variable such that $E\ (\varepsilon_{it} \mid z_{it},\ R_{it}) = 0$. The structural parameters corresponding to the evolution of the state variables can be estimated from the pooled time series and cross section data on the state variables of firms. One can compute the sample moments $m(\delta,\ v)\equiv\frac{1}{nT}\Sigma\phi\left(z_{it+1},\ z_{it},\ R_{it};\ \delta,\ v\right)$ and use a simulated generalized method of moment (GMM) estimation procedure as proposed in Hansen and Singleton (1982).

For the dynamic programming approach, the general procedure is to use a "nested fixed point" algorithm as follows: As I noted earlier, the optimal solution $R_t = \rho(z_t)$ cannot be statistically estimated since it is a deterministic relationship, i.e. it does not have an error term. There are many, less convincing, ways to add an error term (see Rust(1986) for a discussion on this). A more meaningful way to introduce an error term in equation (21) is, however, to assume that while the decision makers observe all components of z_t, the econometrician does not observe some of the components and treat them as random variables. In this paper, I assume that ζ_t is not observed by the econometrician, and I assume that ζ_t is randomly distributed over firms. The nested fixed point approach involves two nested loops: The outer loop is over the structural parameters δ and v (note that v now includes the parameters of the probability distribution for ζ_t) and the inner loop is to solve the dynamic programming problem using a value iteration method or a policy iteration method for given values of the parameters in the outer loop and a randomly generated value of ζ_t. Given the optimal solution of the dynamic programming problem from the inner loop, one can define the likelihood function of the sample in the outer loop and maximize it with respect to the structural parameters δ and v to get the maximum likelihood estimate of the structural parameters. This procedure is, however, computationally unmanageable for most practical problems. The linear quadratic model produces a tractable procedure as described below.

For both the stochastic Euler equation approach and the dynamic programming approach, it is important that all the structural parameters are identified. This is impossible to achieve for general formulations. However, for the linear-quadratic case Hansen and Sargent has shown that variational approach can lead to identification and estimation of structural parameters. They used stochatic Euler equations, Transversality condition and Weiner-Kolmogorov prediction formula for that purpose. For the dynamic programming approach, however, I have shown above that using only the Riccati equation and assuming higher order processes for the environment variables, it is possible to derive a close form solution, and identify all the structural parameters from a subset of the cross equations restrictions implied by the hypothesis of rational expectations. I explain now how these restrictions could be used to estimate the model more efficiently and to test the model statistically.

Assume that $c_4 = 0$ and treat $\alpha_4\zeta_t$ as the disturbance term e_t in equation (21), i.e.,

$$\begin{aligned} e_t &= -\frac{b\lambda}{a_1\theta}(1-\lambda a_4)^{-1} q_{14}\zeta_t \\ &= -\frac{b\lambda}{a_1\theta}.q_{14}(1-\lambda a_4)^{-1}(1-a_4L)^{-1} w_{4t} \end{aligned}$$

i.e.,

$$(1-a_4L)e_t = -\frac{b\lambda}{a_1\theta}.q_{14}(1-\lambda a_4)w_{4t}$$

It is now clear that the error term $e_t \equiv \alpha_4 \zeta_t$ in equation (21) follows a first order auto-regressive process.[8] One can now use the method of maximum likelihood to estimate all the parameters subject to the over-identifying restrictions. Furthermore, using the maximized likelihood of the sample with and without the over identifying restrictions, one can construct the well-known likelihood ratio test to see if the restrictions are statistically rejected. If the restrictions are statistically rejected, we conclude that the particular linear quadratic model is not an appropriate model of R&D investment for the firms.

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8 For a higher order auto-regressive process it is straightforward to derive expressions similar to equation (48).

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