Trade, Growth, and Development
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CHAPTER 4

Endogenous Fertility and Growth Dynamics[1]

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4.1 Introduction

The main issue we address in this paper is what happens when we endogenize fertility in a neoclassical equilibrium growth model. It changes a wide spectrum of standard neoclassical growth results, ranging from the Ricardian equivalence theorem to the turnpike theorem. In this paper, however, we focus only on growth dynamics.

In recent years, there has been a lot of interest in endogenizing economic fluctuations of the long-waves type, as observed by Kondratieff [11], Kuznets [12] and Easterlin [8], and of the short-waves type, as observed in business cycles and stock market volatilities. Much of this research has been conducted in the neoclassical Ramsey growth framework, and in the overlapping generations general equilibrium framework. Two-period lived overlapping generations models generally exhibit very complex dynamics. When parental altruism is introduced in a two-period lived overlapping generations model (as in Barro [1]), the equilibrium dynamics of the model, under certain assumptions, can be characterized by an optimal growth path of a neoclassical one-sector Ramsey model, which does not exhibit dynamic complexities for its growth path. Subtle differences, however, occur in the equilibrium dynamics of these models when fertility is endogenized.

In Section 4.2, we first summarize the widely studied dynamics of the standard neoclassical growth model with exogenous fertility, and then report the striking differences in dynamics that arise when fertility is endogenized. We also point out the limitations of the literature in dealing with the dynamics of the Ramsey growth model when fertility is endogenized with two-period

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lived overlapping generations of agents who are altruistic towards their children (i.e., when the Barro model is extended to endogenize fertility). We then set out in Section 4.3 to formulate such a model and in Section 4.4, we contrast its dynamics with the dynamics of existing equilibrium growth models of endogenous fertility.

4.2 Backdrop

4.2.1 Growth dynamics in the Ramsey model with exogenous fertility

There are several reasons for the recent revival of interests in the Ramsey model. To endogenize the savings rate in the Solow growth model, it is often assumed that consumption and savings decisions are made by an infinitely lived agent or by one-period lived agents having an altruistic utility function. In either case, competitive equilibrium can be characterized as the solution of a Ramsey type growth model. More specifically, suppose agents are born as adults and live only for one period. We denote by c_t^t, the consumption in period t of an adult of generation t and by n_t, the number of children she has. Let the utility V_t of an adult of generation t, be given by

$$V_t = U(c_t^t) + \gamma(n_t) \cdot V_{t+1}, \quad t = 0, 1, \ldots \tag{4.1}$$

where $U(c_t^t)$ is the felicity index of one's life time consumption, and $\gamma(n_t)$ is the discount factor which may depend on the number of children. Suppose the child-rearing cost in period t is θ_t, which includes the opportunity cost of parents' child-rearing time cost. Then all the above problems can be cast in the form

$$\max_{\{k_t\}_1^\infty} V_0 = \sum_{t=0}^{\infty} \left[\prod_{\tau=o}^{t} \gamma(n_\tau) \right] U(c_t^t) \text{ subject to}$$
$$c_t^t + n_t[k_{t+1} + \theta_t] = f(k_t) + (1-\delta)k_t, k_0 \text{ given} \tag{4.2}$$

When fertility is exogenous, the general practice is to assume $\gamma(n_t) = \gamma$ a constant, and thus, $[\prod_{\tau=o}^{t} \gamma(n_\tau)] = \gamma^t$, and $\theta_t = 0$, for all $t \geq 0$.

In the one-sector Ramsey growth model, if the felicity index U is concave and the production function is concave, the optimal capital–labor ratio path, $\{k_t\}_1^\infty$, of the problem in (4.2) is monotonic in the sense that a capital–labor ratio path will be higher or lower in all periods if it starts from a higher level of k_0). Furthermore, the optimal capital-ratio path exhibits a turnpike property that as $t \to \infty$, k_t tends to a locally stable steady-state. Thus, it is apparent that the complex dynamics in optimal growth framework that are shown to occur in multi-sector growth framework by Benhabib and Nishimura [4] and

Boldrin and Montrucchio [6] cannot occur in standard neoclassical one-sector Ramsey growth framework.

In the above Ramsey framework, however, there is no overlap of generations, so the framework is not suitable for analyzing many economic issues that involve economic decisions over the life-cycle, such as the old-age pension motive for savings, pay-as-you-go social security transfers, or inter-vivos transfers. Two-period[2] Samuelson [20]–Diamond [7] overlapping generations (OLG) framework is more suitable for this. To describe it briefly, suppose that each agent lives for two periods: adult and old. Let us denote by c_t^t the adult age consumption of an agent of period t, and by c_{t+1}^t her consumption in period $t+1$ (she is old in period $t+1$). Suppose the felicity index of her consumption over the life-cycle is $U(c_t^t, c_{t+1}^t)$. Grandmont [9] and others have shown that the competitive equilibrium of this type of OLG economy produces very complex dynamics, including periodic fluctuations and chaos. Under the assumption that the life-cycle felicity index U is separable, it can be easily established (also see below for an exposition) that a social planner's problem of optimizing the (discounted) sum of utility functions of various generations turns out to be a one-sector Ramsey growth problem, and thus no new dynamic issues arise for this class of problems.

An important extension of this Samuelson–Diamond framework is by Barro [1], who assumes that parents have altruistic utility functions of the type (4.1) in which $U(c_t^t)$ is replaced by $U(c_t^t, c_{t+1}^t)$. This adds bequest motives to the life-cycle motives of the Samuelson–Diamond model. Even in this framework, under some additional assumptions (e.g., utility functions of children are consistent with their parents' utility functions, and $U(c_t^t, c_{t+1}^t)$ is separable, and the bequest motive is operative), Weil [21] has shown that the competitive equilibrium path of the economy is characterized by the optimal solution of an one-sector Ramsey growth model. Thus, at least for this class of OLG economies, parental altruism rules out complexities in the dynamics of competitive equilibrium path. Recently, however, Michel and Venditii [14] have shown that if the life-cycle felicity index, $U(c_t^t, c_{t+1}^t)$ is not separable, then the optimal growth path in the one-sector growth model can produce cycles.

4.2.2 Endogenous fertility and growth dynamics

Fertility has been endogenized in growth framework mainly along two lines:[3] in one strand, the motive for children is parental altruism or love, and in

[2] More than two but finite period lived OLG framework can be identified with a two-period lived agent OLG model. so it is general enough to restrict our discussions to two-period lived OLG models.

[3] There is, however, another class of models which endogenize population growth by postulating that population growth rate is a function of per capita income. See Nerlove and Raut [16] for an account of the dynamic properties of this kind of model. Our focus in this paper is, however, the models that endogenize fertility as pre-natal choice.

the second strand, the motive for children is old-age pension. These two frameworks of endogenous fertility are, respectively, the analogues of the one-sector Ramsey framework and the two-period lived OLG framework. More specifically, Barro and Becker [2] and Becker and Barro [3] consider an one-period lived OLG framework with dynastic utility function of the form (4.1), and the corresponding optimal growth problem is then characterized by (4.2), with n_t as a choice variable. Kemp and Kondo [10], Lapan and Enders [13] and Nishimura and Kunaponagkul [17] also make fertility endogenous in the Ramsey framework, but they assume that $\gamma(n_t)$ is constant. In such models, to incorporate a motive for children in the utility function, the tradition has been to assume that the felicity index, U, depends on c_t^t as well as on n_t. The two-period lived OLG growth models that incorporate the life-cycle motive for savings and the old-age pension motive for children, include Neher [15] and more recently Raut [18], and Raut and Srinivasan [19].

While there are subtle differences in the dynamics of the above two frameworks of endogenous fertility, they also share some common dynamics which are strikingly different from the dynamics of the neoclassical growth model with exogenous fertility. We point out a few important ones here.

- When fertility is endogenous, the nature of equilibrium dynamics in both types of models depends crucially on the form of the child-rearing cost. For instance, if the child-rearing cost involves resources other than parental time, the capital–labor ratio in both type of models converge to steady-state in two periods (see Barro and Becker [2] for this in their model, and see Raut [18] for the second type of model).
- As we mentioned earlier, within the precinct of the one-sector growth framework, when the fertility is exogenous, it is necessary to have two-period lived overlapping generations of agents with a nonseparable life-cycle felicity index in order to generate cycles in the optimal growth path (Michel and Venditti [14]). When the fertility is endogenous, however, the optimal growth path of the one-sector Ramsey growth model can generate cycles (see Barro and Becker [2] with Cobb–Douglas form for $\gamma(n_t)$, Benhabib and Nishimura [5] with any concave $\gamma(n_t)$, and also Lapan and Enders [13]).
- In a two-period OLG framework with an endogenous fertility, in which population density creates external effect on the total factor productivity of the economy, it is shown in Raut and Srinivasan [19] that depending on the nature of externality, the child-rearing cost function, the values of the parameters, and the initial conditions of the economy, the model can produce complex nonlinear dynamics, which "not only include neoclassical steady-state with exponential growth of population with constant per capita income and consumption, but also growth paths which do not converge to a steady-state and are even chaotic." There are also

equilibrium paths in which capital–labor ratio attains steady-state in finite period, but the fertility rate fluctuates over time.

It is important to integrate the old-age pension motive with an altruistic motive for childbearing and saving in the endogenous fertility case for the same reasons Barro introduced it in the exogenous fertility case and to examine its implications for equilibrium dynamics. Could it eliminate some of the complex dynamics of the two-period lived overlapping generations model with endogenous fertility as in Raut and Srinivasan [19]? Could it preserve the dynamics of the one-period lived Barro–Becker growth model with endogenous fertility? The existing results from the literature are not adequate to reflect on these issues. The reason is that, unlike in the exogenous fertility case mentioned above, even with separable life-cycle felicity index $U(c_t^t, c_{t+1}^t)$, we cannot convert this model into a Ramsey growth model of the type that has been studied so far. To see this, suppose that $U(c_t^t, c_{t+1}^t) = u(c_t^t) + v(c_{t+1}^t)$. Then the optimal growth problem of Barro model with endogenous fertility becomes

$$\max_{\{n_t, k_{t+1}, c_t^t, c_{t-1}^t\}_0^\infty} V_0 = \sum_{t=0}^{\infty} \left[\prod_{\tau=o}^{t} \gamma(n_\tau)\right] U(c_t^t, c_{t+1}^t) \text{ subject to}$$
$$c_t^t + c_t^{t-1}/n_{t-1} + n_t\left[\theta_t + k_{t+1}\right] = f(k_t) + (1-\delta)\, k_t, k_0 \text{ given} \quad (4.3)$$

Notice that we can rewrite V_0 as

$$V_0 = \gamma(n_0)u(c_0^0) + \sum_{t=1}^{\infty}\left[\prod_{\tau=o}^{t-1}\gamma(n_\tau)\right]\cdot\left[\gamma(n_t)u(c_t^t) + v(c_t^{t-1})\right]$$

For $t \geq 1$, let us denote by

$$\tilde{U}(c_t, n_t, n_{t-1}) \equiv \max_{c_t^t, c_t^{t-1}} \gamma(n_t)u(c_t^t) + v(c_t^{t-1}) \text{ subject to}$$
$$c_t^t + c_t^{t-1}/n_{t-1} = c_t$$

Then we have the following Ramsey model with endogenous fertility

$$\max_{\{n_t, k_{t+1}\}_0^\infty} V_0 = \gamma(n_0)u(c_0^0) + \sum_{t=1}^{\infty}\left[\prod_{\tau=o}^{t}\gamma(n_\tau)\right]\tilde{U}(c_t, n_t, n_{t-1}) \text{ subject to}$$
$$c_t + n_t\left[\theta_t + k_{t+1}\right] = f(k_t) + (1-\delta)\, k_t, k_0 \text{ given} \quad (4.4)$$

As pointed out earlier, the previous literature has either assumed $\tilde{U}$ to be independent of n_t's or $\gamma(n_t)$ to be independent of n_t, or both. In this paper, we study the dynamics of the above type of model. We do not impose separability of U from the beginning; we impose it only to derive some specific results. We do, however, impose some reasonable restrictions on the nature of the bequest and division of current consumption between living adult and old generations in each period to make our analysis possible within an one-sector optimal growth framework.

4.3 Basic Framework

4.3.1 Production sector

We assume that the productive sector has a constant returns to scale production function $Y_t = F(K_t, L_t)$, which uses capital K_t and labor L_t to produce output Y_t in each period t, $t \geq 0$. Capital takes one period to gestate. Old members of the households own capital. We adopt the convention that the producer borrows from the old members of the households the stock of capital K_t at the beginning of period t and pays them $(\partial F/\partial K)K_t$ amount of rental income during the period t and stock of depreciated capital $(1-\delta)K_t$. This depreciated capital, $(1-\delta)K_t$, is bequeathed to the L_t children by the L_{t-1} old parents at the end of period t before they die. Thus, at the beginning of period $t+1$ the stock of capital available for production is:

$$K_{t+1} = (1-\delta)K_t + L_t s_t \tag{4.5}$$

On the right-hand side of the above, the first term is the inherited capital and the second term is the new capital added by the adults of period t. We assume that $s_t \geq 0$, which is equivalent to the assumption that capital is irreversible.

$>$From (4.5) we have the following relationship:

$$k_{t+1} = \frac{(1-\delta)k_t + s_t}{n_t} \tag{4.6}$$

where n_t is the number of children chosen by an adult of period t.

4.3.2 Households

At the beginning of time, $t = 0$, assume that there is only one adult agent who has at her disposal an initial endowment of capital $k_0 > 0$. Each person lives for three periods: young, adult, and old. While young she is dependent on her parent for all decisions, including childhood consumption. As an adult, she earns income w_t in the labor market, out of which she decides the amount of savings s_t and the number of children $n_t \geq 0$. In the next period, she inherits $(1-\delta)k_t$ amount of physical capital assets from her deceased parents, and lives off the income $\rho_{t+1}[(1-\delta)k_t + s_t]$ from her assets, where ρ_{t+1} is the rental rate of capital in period $t+1$.

We assume that utility of agent t, V_t depends on her own life-cycle consumption and the discounted sum of the utilities of her identical children V_{t+1} as follows:

$$V_t = U(c_t^t, c_{t+1}^t) + \delta(n_t)n_t \cdot V_{t+1} \tag{4.7}$$

where $\delta(n_t)$ is the weight given to each child's utility. We assume that $\delta(n_t)$ is the decreasing function of the number of children, n_t. We denote by $\gamma(n_t) = \delta(n_t)n_t$.

The recursive equation (4.7) leads to the following welfare for agent $t = 0$ as a function of the stream of lifetime consumptions and fertility levels of her own and future generations:

$$V_0 = \sum_{t=0}^{\infty} \left(\prod_{\tau=0}^{t} \gamma(n_\tau) \right) U(c_t^t, c_{t+1}^t). \tag{4.8}$$

Assuming perfect foresight and complete enforceability of her decisions $\{n_t, k_{t+1}\}_0^\infty$ on subsequent generations, and for a given stream of future social security benefits $\{b_{t+1}\}_0^\infty$, the problem of the adult of generation $t = 0$ could be formally stated as follows:

$$\max_{\{(n_t,k_{t+1}\}_0^\infty} V_0 = \sum_{t=0}^{\infty} \left(\prod_{\tau=0}^{t} \gamma(n_\tau) \right) U(c_t^t, c_{t+1}^t) \text{ subject to} \tag{4.9}$$

$$\begin{aligned} &c_t^t = w_t - s_t - \theta_t n_t \\ &c_{t+1}^t = \rho_{t+1}[(1-\delta)k_t + s_t] \\ &t \geq 0, w_0 \text{ is given} \end{aligned} \tag{9}$$

where we have

$$\left.\begin{aligned} k_{t+1} &= \frac{(1-\delta)k_t + s_t}{n_t} \\ \rho_{t+1} &= f'(k_{t+1}) \\ w_{t+1} &= f(k_{t+1}) - k_{t+1}f'(k_{t+1}) \end{aligned}\right., \ t \geq 0 \tag{4.10}$$

Let us denote by $w(k) \equiv f(k) - kf'(k)$. Assume that the utility function, production function, and the degree of altruism are all concave and increasing; there exists a positive value $\bar{\gamma} < 1$, $\gamma(0) = 0$ and $\gamma(n) \leq \bar{\gamma}$ for all n. Under these conditions, the solution of the above problem in (4.9) is equivalent to the solution of the following Bellman equation of the dynamic programming problem:

$$\begin{aligned} &V(k_t) = \max_{n_t,k_{t+1}} U(c_t^t, c_{t+1}^t) + \gamma(n_t)V(k_{t+1}) \text{ subject to} \\ &c_t^t = w(k_t) + (1-\delta)k_t - (\theta + k_{t+1})n_t \\ &c_{t+1}^t = f'(k_{t+1})k_{t+1}n_t, \ t \geq 0, k_0 \text{ is given} \end{aligned} \tag{4.11}$$

4.4 Dynamic Properties of Competitive Equilibrium Paths

In order to study the dynamic properties of the competitive equilibrium path, we assume that the depreciation rate $\delta = 1$. We introduce the following notation. Define

$$W(k_t, k_{t+1}, n_t) \stackrel{\text{def}}{\equiv} U(w(k_t) - (\theta + k_{t+1})n_t, n_t R(k_{t+1}) + \gamma(n_t)V(k_{t+1}) \quad (4.12)$$

where, $R(k_{t+1}) = f'(k_{t+1})\, k_{t+1}$. Define

$$n(k_t, k_{t+1}) \stackrel{\text{def}}{\equiv} \arg\max_{n_t \geq 0} W(k_t, k_{t+1}, n_t). \quad (4.13)$$

Denote by

$$\bar{W}(k_t, k_{t+1}) \stackrel{\text{def}}{\equiv} W(k_t, k_{t+1}, n(k_t, k_{t+1})) \quad (4.14)$$

The original problem in (4.11) of finding $\{n_t, k_{t+1}\}_0^\infty$ is now equivalent to solving (4.13) and the following:

$$V(k_t) = \max_{k_{t+1}} \overline{W}(k_t, k_{t+1}). \quad (4.15)$$

Suppose the above problem has a unique solution, denoted as $k_{t+1} = P(k_t)$. This function is known as the *policy function.* The dynamic behavior of the optimal solution path is determined from the properties of the first order difference equation given by the policy function, which we study now.

Let the partial derivative of $W(x_1, x_2, x_3)$ with respect to x_i be denoted by $W_i(x_1, x_2, x_3)$, and the second order partial derivatives of W by W_{ij}. Then, we have

$$W_1 = -k_t f''(k_t) U_1 \quad (4.16)$$

$$W_2 = -n_t[U_1 - R'(k_{t+1})U_2] + \gamma(n_t)V'(k_{t+1}) \quad (4.17)$$

$$W_3 = -(\theta + k_{t+1})U_1 + R(k_{t+1})U_2 + \gamma'(n_t)V(k_{t+1}). \quad (4.18)$$

We assume that there exists an interior solution so that $W_2 = W_3 = 0$ are satisfied. We further assume that $W_{33} \neq 0$ and apply the implicit function theorem to $W_3(k_t, k_{t+1}, n_t) = 0$ to obtain:

$$\frac{\partial n(k_t, k_{t+1})}{\partial k_t} = -\frac{W_{31}}{W_{33}} \quad \text{and} \quad \frac{\partial n(k_t, k_{t+1})}{\partial k_{t+1}} = -\frac{W_{32}}{W_{33}}. \quad (4.19)$$

Using the above relationships, we can study the short- and long-run effects of an exogenous increase in capital–labor ratio on fertility. In particular, we can find conditions for the Easterlin hypothesis to hold. Notice that

$$\frac{\partial n_t}{\partial k_t} = \frac{\partial n(\cdot,\cdot)}{\partial k_t} + \frac{dk_{t+1}(\cdot)}{dk_t}\frac{\partial n(\cdot,\cdot)}{\partial k_{t+1}} \quad (4.20)$$

where $k_{t+1}(\cdot)$ is the solution of the problem in (4.14). To determine the sign of the partial defined in (4.20), and also for later use, we need to determine the signs of the partial derivatives in (4.19). For that we apply the implicit function theorem on equations (4.18) and (4.17) and obtain the following:

$$\mathrm{W}_{33} = (\theta+k_{t+1})^2U_{11} - 2(\theta+k_{t+1})R(k_{t+1})U_{21} + R(k_{t+1})^2U_{22} + \gamma''(n_t)\mathrm{V}(k_{t+1}) \tag{4.21}$$

$$\begin{aligned} W_{32} &= n_t[(\theta + k_{t+1})U_{11} - \{(\theta + k_{t+1})R'(k_{t+1}) + R_t\}U_{21} \\ &\quad + R(k_{t+1})R'(k_{t+1})U_{22}]R'(k_{t+1})U_2 - U_1 + \gamma'(n_t)\mathrm{V}'(k_{t+1}) \end{aligned} \tag{4.22}$$

$$W_{31} = f''(k_t)k_t[(\theta + k_{t+1})U_{11} - R(k_{t+1})U_{21}] \tag{4.23}$$

$$W_{21} = f''(k_t)k_t[n_t(U_{11} - R'(k_{t+1})U_{21})]. \tag{4.24}$$

If we assume that U is strictly concave, then it follows immediately that W_{33} in (4.21) is strictly negative. When we assume that U is separable, it easily follows that $W_{31} > 0$. However, even for nonseparable utility functions, $W_{31} > 0$, as we will see in the following three specific class of economies. We cannot, in general, determine the sign of W_{32} in equation (4.22). Assuming that $W_{32} < 0$, which is true for each of the following three examples, we note that dn_t/dk_t and dk_{t+1}/dk_t are inversely related. Thus in cases when $dk_{t+1}/dk_t < 0$, i.e., when the optimal $\{k_t\}_0^\infty$ is *oscillatory,* we have a theoretical basis for the well-known Easterlin [8] hypothesis, stated in the introduction.

In the following theorem we find conditions that characterize the dynamic properties of the optimal path. Let us assume that $\bar{W}$ in (4.15) is twice continuously differentiable at each point of an interior solution path $\{k_t\}_0^\infty$. By differentiating the first order condition of the problem (4.15), we note that dk_{t+1}/dk_t ($\equiv dP(k_t)/dk_t$) $= -\bar{W}_{21}/\bar{W}_{22}$. Since $\{k_t\}_0^\infty$ is optimal, $\bar{W}$ is locally concave with respect to the second argument at each k_t. Hence, we have $\bar{W}_{22} < 0$. Thus we note that the sign of $dk_{t+1}/dk_t >=$ or < 0 according to whether $\bar{W}_{21} >=$ or < 0. More precisely, we have the following theorem (also see Benhabib and Nishimura [5] for an alternative proof).

Theorem 1 *Let $\{k_t\}_0^\infty$ be an interior optimal solution of the problem* (4.8) *with $k_0 \neq k^*$, then the following are true:*
(i) $\bar{\mathrm{W}}_{21} < 0 \Rightarrow (k_t - k_{t+1})(k_{t+1} - k_{t+2}) < 0$,
(ii) $\bar{\mathrm{W}}_{21} > 0 \Rightarrow (k_t - k_{t+1})(k_{t+1} - k_{t+2}) > 0$,
(iii) $\bar{\mathrm{W}}_{21} = 0 \Rightarrow k_{t+2} = k^*$, *for all* $t \geq 0$.

Let us determine the sign of this crucial cross partial derivative.

The cross partial of $\bar{W}$ is related to the second derivatives of $W(k_t, k_{t+1}, n_t)$ as follows:

$$\bar{W}_{21} = W_{21} + W_{31}\frac{\partial n_t}{\partial k_{t+1}} + W_{23}\frac{\partial n_t}{\partial k_t} + W_{33}\frac{\partial n_t}{\partial k_t}\frac{\partial n_t}{\partial k_{t+1}}. \tag{4.25}$$

By substituting (4.19) into (4.25) we have

$$\bar{W}_{21} = \frac{W_{21}W_{33} - W_{31}W_{32}}{W_{33}} \tag{4.26}$$

Substituting (4.21)–(4.24) in (4.26) we have the following:

$$\begin{aligned}\bar{W}_{21} \;=\;& W_{33}^{-1} k_t f''(k_t)[\{n_t R(k_{t+1})[R(k_{t+1}) - (\theta + k_{t+1})R'(k_t)]\} \\ & (U_{22}U_{11} - U_{12}^2) - \{(\theta + k_{t+1})U_{11} - R(k_{t+1})U_{21}\}(R'(k_{t+1})U_2 - U_1) \\ & (1 - \gamma'(n_t)n_t/\gamma(n_t)) + ((U_{11} - R'(k_{t+1})U_{21}))\gamma''(n_t)\mathrm{V}(k_{t+1})n_t] \\ & W_{33}^{-1} \triangle \text{ , say.}\end{aligned} \tag{4.27}$$

It is not possible in general to determine the sign of the above cross partial derivative. We impose restrictions on the forms of the utility function $U(\cdot,\cdot)$ and the degree of altruism function $\gamma(n)$ along the lines of the available results in the literature to determine the sign of the above partial derivative and hence the dynamics of the equilibrium path.

4.4.1 Constant marginal utility of young age consumption

Let the utility function be given by $U(c_t^t, c_{t+1}^t) = c_t^t + v(c_{t+1}^t)$, that is, the marginal utility of the first period consumption is constant. In this case, $U_1 = 1$, $U_{11} = U_{12} = 0$ imply that $\bar{W}_{21} = 0$ and thus we have the following result:

Theorem 2 *In economies with special types of separable utility functions of the form $U(c_t^t, c_{t+1}^t) = c_t^t + v(c_{t+1}^t)$, the optimal sequence of capital-labor ratio, $\{k_t\}_0^\infty$ reaches steady-state at $t = 1$.*

It follows from the above theorem that the optimal fertility level, n_t, also reaches steady-state at $t = 1$. From equations (4.19) and (4.20), it follows that fertility and income are positively related in such economies. Since a steady-state is attained in finite time period, there is a unique steady-state.

In Barro and Becker [2] one-period lived agent framework, the above result is true for Cobb–Douglas $\gamma(n)$. In our two-period lived agent framework, the result is true for any general functional form for $\gamma(n)$, provided we restrict the felicity index to be separable and a linear function of the first period consumption.

4.4.2 Constant discount rate for progeny's welfare

In this section, we consider the case when $\gamma(n) = \gamma$, where $1 > \gamma > 0$, and characterize dynamic properties of optimal paths in terms of properties of a felicity index function, $U(c_t^t, c_{t+1}^t)$. We extend the one-period lived agent framework of Nishimura–Kunapongkul, Kemp–Kondo, and Lapan–Enders to two-period lived agents framework. By assuming $\gamma(n_t) = \gamma$, agents in these models are assumed to care about the welfare of a representative child; in one-period lived agent framework, previous models incorporate motives for children by assuming the utility function, U, to depend on c_t^t and n_t. We further assume that

$$U_{12} > 0 \tag{4.28}$$

$$f'(k) + f''(k)k > 0. \tag{4.29}$$

Notice that since $\gamma(n_t) =$ constant, we have

$$W_{33} = (\theta + k_{t+1})^2 U_{11} - 2(\theta + k_{t+1}) R(k_{t+1}) U_{21} + R(k_{t+1})^2 U_{22} \tag{4.30}$$

$\bar{W}_{21}$ from (4.27) can be expressed as

$$\begin{aligned}\triangle = \; & k_t f''(k_t) U_2^{-1} (\theta + k_{t+1}) \left[(\theta + k_{t+1}) n_t [U_1 - R'(k_{t+1}) U_2] (U_{22} U_{11} - U_{12}^2) \right. \\ & - \left. [U_2 U_{11} - U_1 U_{21}][R'(k_{t+1}) U_2 - U_1] \right] \\ & k_t f''(k_t) U_2^{-1} (\theta + k_{t+1})(U_1 - R'(k_{t+1}) U_2) \\ & \times \left[n_t (\theta + k_{t+1}) \left(U_{22} U_{11} - U_{12}^2 \right) + (U_2 U_{11} - U_1 U_{21}) \right].\end{aligned} \tag{4.31}$$

For further simplification of the above, let us consider the following life-cycle utility maximization problem by a representative agent who takes I, $(\theta + k)$, and μ as given to solve:

$$\begin{aligned}& \max_{\{n\}} U(c_1, c_2) \text{ subject to} \\ & c_1 + (\theta + k) n = I \\ & c_2 = n\mu\end{aligned}$$

Substituting the expression for c_2 in the utility function U, and denoting the Lagrange multiplier corresponding to the first constraint as λ, we have the following first order necessary conditions:

$$U_1 - \lambda = 0, \tag{4.32}$$

$$\mu U_2 - (\theta + k)\lambda = 0, \tag{4.33}$$

$$I - c_1 - n(\theta + k) = 0. \tag{4.34}$$

From (4.32)–(4.34), we get the following well known results from the demand theory:

$$\frac{dn}{dI} = \frac{(\theta + k) U_{11} - \mu U_{21}}{W_{33}}, \tag{4.35}$$

$$\frac{d\lambda}{dI} = \frac{\mu^2\left(U_{11}U_{22} - U_{12}^2\right)}{W_{33}}. \tag{4.36}$$

Let us denote the income elasticity of the demand for children, $e_n \equiv (I/n)(dn/dI)$, and the income elasticity of the marginal utility of income, $e_\lambda \equiv -(I/\lambda)(d\lambda/dI)$, for the above utility maximization problem. Using the facts that $\lambda = U_1$ from equation (4.32) and $\mu = (\theta + k_{t+1})U_1/U_2$ from equation (4.33), after simplifications we arrive at the following:

$$e_n - e_\lambda = \frac{I}{n}\frac{(\theta + k_{t+1})U_1}{W_{33}U_2^2}(n_t(\theta + k_{t+1})(U_{22}U_{11} - U_{12}^2) + (U_2U_{11} - U_1U_{21})). \tag{4.37}$$

Since $W_2 = 0$ is satisfied by the optimal path, we have

$$U_1 - R'(k_{t+1})\dot{U}_2 = \gamma(n_t)\mathrm{V}'(k_{t+1}) > 0 \tag{4.38}$$

Substituting equation (4.37) in equation (4.31) and using equation (4.38), we have the following theorem:

Theorem 3. *Let $k_0 \neq k^*$, then we have*
(i) $e_\lambda > e_n \Rightarrow \{k_t\}$ *is monotonic,*
(ii) $e_\lambda = e_n \Rightarrow k_t = k^*$ *for all* $t \geq 2$,
(iii) $e_\lambda < e_n \Rightarrow \{k_t\}$ *is oscillatory.*

Corollary: *The economies for which $e_\lambda = e_n$, we also have that $n_t = n^*$ for all $t \geq 1$, and when $e_\lambda < e_n$, we also have oscillatory $\{n_t\}$. We cannot tell how $\{n_t\}$ will behave if $e_\lambda > e_n$.*

The above theorem is an extension of the Lapan–Enders characterization of competitive equilibrium dynamics for the one-period lived Ramsey model. It should be noted that the results (i) and (iii) in the above theorem remain true even when γ depends on n, but it is close to a constant function.

4.4.3 Dynasty of one-period lived agents

We have pointed out earlier that most growth models of endogenous fertility and savings in the dynastic framework assume that agents live one period (Barro and Becker [2], Becker and Barro [3], Benhabib and Nishimura [5], and others). We can nest those models and derive their dynamic properties from our extended framework as follows: In the optimization problem (4.9) view saving s_t is for the purpose of bequest as opposed to old-age pension that we have maintained so far. In the notation of problem (4.9), these assumptions are equivalent to the following:

$$\mathrm{U}_2 = \mathrm{U}_{22} = \mathrm{U}_{21} = 0. \tag{4.39}$$

Using (4.39) in (4.27) and noting that $V'(k_{t+1}) = (\theta + k_{t+1})U_1/\gamma'(n_t)$ in this special case, we have the following:

$$\begin{aligned}\triangle &= k_t f''(k_t)(\theta + k_{t+1})U_{11}U_1\left[1 + n_t\left(\frac{\gamma''(n_t)\gamma(n_t) - [\gamma'(n_t)]^2}{\gamma'(n_t)\gamma(n_t)}\right)\right] \\ &= k_t f''(k_t)(\theta + k_{t+1})U_{11}U_1[1 - e(n_t)] \end{aligned} \tag{4.40}$$

where $e(n)$ is given by

$$e(n) \equiv \frac{-n}{\left(\frac{\gamma'(n)}{\gamma(n)}\right)} \frac{d\left(\frac{\gamma'(n)}{\gamma(n)}\right)}{dn} \tag{4.41}$$

Thus, the sign of $\overline{W}_{21}$ depends on the sign of $1 - e$, which depends only on the degree of altruism function $\gamma(n)$, but not on the utility function U or production function. Thus, we have proved the following result:

Theorem 4
(i) *if* $e < 1$, $\{k_t\}_0^\infty$ *is monotone,*
(ii) *if* $e = 1$, $\{k_t\}_0^\infty$ *reaches steady-state at* $t = 1$,
(iii) *if* $e > 1$, $\{k_t\}_0^\infty$ *oscillates.*

Corollary: *Let* $k_0 \neq k^*$. *If* $e = 1$, $\{n_t\}$ *reaches steady-state at* $t = 1$. *If* $e < 1$, $\{n_t\}$*is oscillatory.*

Barro and Becker [2] assumed Cobb–Douglas form for $\gamma(n)$ which forces $e = 1$.

It is important to note that Theorem 4(i) and (ii) are true even when U_2, U_{22}, and U_{21} are not zero, but are very close to zero. This extends Benhabib and Nishimura [5] to the dynastic model with two-period lived agents.

4.5 Conclusion

The one-sector neoclassical Ramsey growth model with exogenous fertility exhibits simple dynamics: The optimal path exhibits monotonicity and turnpike property. In this paper, we have briefly reviewed a host of one-sector equilibrium growth models with exogenous fertility, including the two-period lived overlapping generations model with parental altruism (i.e., the Barro model) which unifies the bequest motive and life-cycle motive for savings. Under general conditions, the dynamics of competitive equilibrium path and social optimal path in all these models share the above dynamic properties of the one-sector neoclassical Ramsey growth model with exogenous fertility.

The existing growth models with parental altruism and endogenous fertility study dynamics of the optimal path by restricting their analyses to one-period lived overlapping generations framework. Surveying the findings of various papers, we find strikingly different dynamics in one-sector equilibrium growth models when fertility is endogenous. We also show that unlike the exogenous fertility case, the existing Ramsey growth models of endogenous fertility are not able to embed the two-period lived overlapping generations model with parental altruism and endogenous fertility. We have formulated a two-period lived overlapping generations model of endogenous fertility with parental altruism and extended some of the existing results from the one-period lived framework to our general framework.

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