PROBABILITY STATISTICS AND QUEUING THEORY
21MT2103RA

Topic:

Continuous probability distributions

Session - 8

AIM OF THE SESSION

To familiarize students with Uniform, Exponential distributions and its applications

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Demonstrate uniform and exponential distributions
2. Describe the applications of uniform and exponential distributions
3. List out the properties of uniform and discrete distribution
4. Solving problems of uniform and exponential distributions

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define uniform, exponential distributions
2. Describe the properties and applications of Uniform and Exponential
3. Summarize the concepts with their applications

CO2

- Continuous Random Variables: Uniform, Exponential and Normal

- Jointly Distributed Random Variables: Joint Distribution Functions, Independent Random Variables Statistics: Sample and population, Higher Order Moments, Variance, Standard Deviation, Measures of central tendency: Mean, Median, Mode, Measure of Dispersion: Variance, Standard deviation, coefficient of variation.

- Correlation and Linear regressionRandom Variables, Probability Distribution Function - Cumulative Distribution Function

Uniform Distribution

One of the simplest continuous distributions in all of statistics is the continuous uniform distribution.

This distribution is characterized by a density function that is "flat" and thus the probability is uniform in a closed interval, say [A, B].

Definition: The density function of the continuous uniform random variable X on the interval [A, B] is

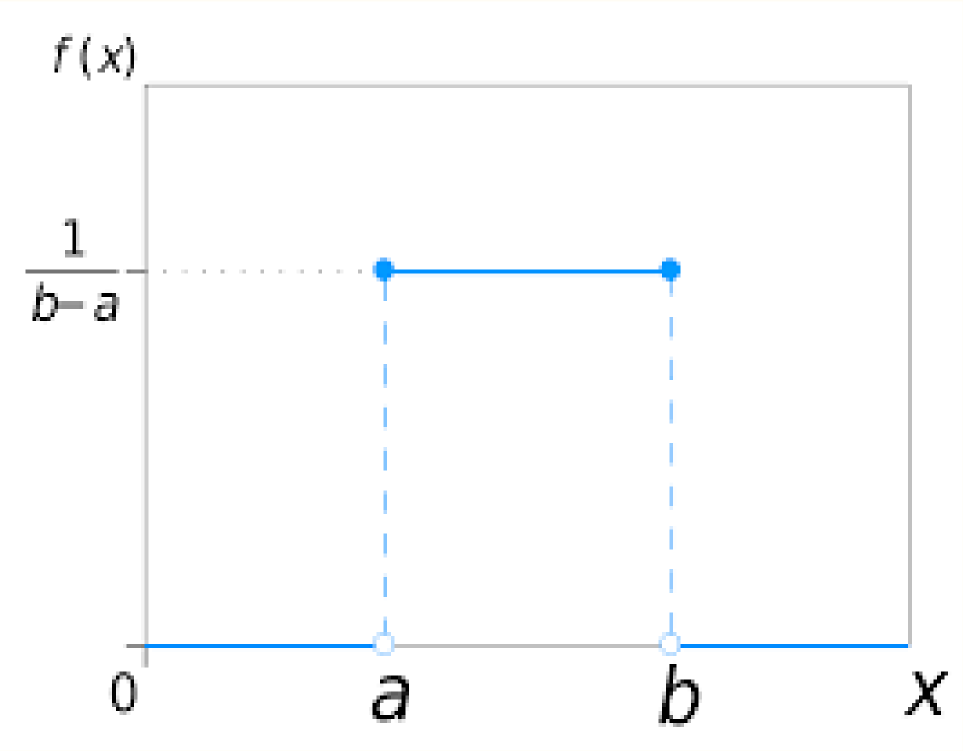
$$f(x; A, B) = \begin{cases} \dfrac{1}{B - A}, & A \leq x \leq B \\ 0 & \text{elsewhere} \end{cases}$$

Note: 1.

The density function forms a rectangle with base B-A and a constant height 1/ (B-A). As a result, the uniform distribution is often called the **rectangular distribution**.

. The probability is uniform in the interval [0,1] is said to be a standard uniform distribution.

Unifrom distribution



source: https://en.wikipedia.org/wiki/Continuous_uniform_distribution

The mean and variance of the uniform distribution are

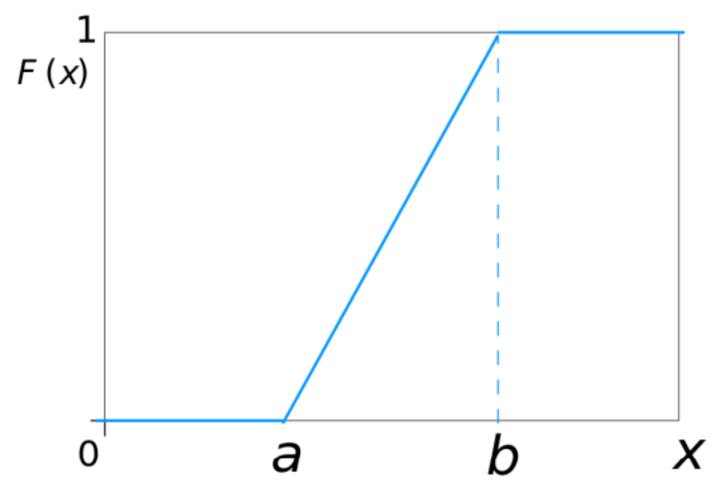
$$\mu = \frac{(A+B)}{2} \ and \ \sigma^2 = \frac{(B-A)^2}{12}$$

Unform distribution

The CDF, F(x), of a uniform random variable is shown below:

$$F(x) = \begin{cases} 0 & \text{for } x<a \\ \frac{x-a}{b-a} & \text{for } x\in [a,b] \\ 1 & \text{for } x>b \end{cases}$$

The graph of F(x) is shown below:



source: https://en.wikipedia.org/wiki/Continuous_uniform_distribution

Exponential distribution

Exponential distribution plays an important role in both queuing theory and reliability. Time between arrivals at service facilities, and time to failure of components and electrical systems, often is nicely modeled by the exponential distribution.

Definition: The continuous random variable X has an Exponential distribution, with parameter β (β>0), if its density function is given by

$$f(x, \beta) = \begin{cases} \beta\, e^{-\beta x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Note: 1) The mean and variance of the Exponential distribution are $\mu = \dfrac{1}{\beta}$ and $\sigma^2 = \dfrac{1}{\beta^2}$.

Cumulative distribution function F(x) of the Exponential distribution is $F(x) = \beta \int_0^x e^{-\beta t}\, dt = 1 - e^{-\beta x}$.

β – failure rate

Memory less property of the exponential distribution:

The types of applications of the exponential distribution in reliability, and component or machine life time problems is influenced by the memory less or lack of memory property of the exponential distribution.

For example, in the case of, say, an electronic component where distribution of life time has an exponential distribution, the probability that the component lasts, say t hours, that is

$$P(X \geq t_0 + t / x \geq t_0).$$

If X were the life of something, such as ‘how long this phone call to my mother will last’, the memoryless property says that after we have been talking for 5 minutes, the distribution of the remaining duration of the call is just the same as it was at the start. This is close to what happens in real life.

ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION

IMPORTANT FACTS

1. Rectangular distribution/ uniform distribution is constant function through the range of the random variable

2. Exponential distribution have the memory less property. It can uphold the present and future or past and present in its memory.

EXAMPLE-Uniform

Example: A bus arrives every 20 minutes at a specified stop beginning at 6:40 A. M. And continuing until 8:40 A. M. A certain passenger does not know the schedule, but arrives randomly (uniformly distributed) between 7:00 A. M. and 7:30 A. M. every morning. What is the probability that the passenger waits more than 5 minutes for a bus?

Solution: The passenger has to wait more than 5 minutes only if the arrival time is between 7:00 A. M. And 7: 15 A. M. or between 7:20 A. M. And 7:30 A.M. If X is a random variable that denotes the number of minutes past 7:00 A. M. That the passenger arrives, the desired probability is $P(0<X<15)+P(20<X<30)$

Now, X is a uniform random variable on (0,30). Therefore, the desired probability is given by

$F(15)+F(30)-F(20)=(15/30)+1-(20/30)=5/6$.

EXAMPLE-Exponential

At a receiving dock on an average 3 trucks arrive per hour to be unloaded at a warehouse. What are the probabilities that the time between the arrivals of successive trucks will be

a) less than 5 minutes b) at least 45 minutes c) is between 5 to 30 minutes

Solution: Assuming that the arrivals follow Poisson process, time interval between successive arrivals has an exponential distribution with mean 1/3 hours.

∴ Parameter of the exponential distribution β=3

The probability density is $f(x)=3e^{-3x}$

a) P(inter arrival time is less than 5 minutes)=P(inter arrival time is less than 1/12 hours)

$$= \int_0^{1/12} 3e^{-3x} = 1 - e^{-1/4} = 0.221$$

EXAMPLE-Exponential

b) P(inter arrival time is at least 45 minutes)=P(inter arrival time is at least ¾ hours)

$$= \int_{3/4}^{\infty} 3e^{-3x} = e^{-9/4} = 0.105$$

c) P(inter arrival time is between 5 to 30 minutes)

$$= \int_{1/12}^{1/2} 3e^{-3x} = \frac{3}{-3}[e^{-3x}]_{1/12}^{1/2} = -[e^{-\frac{3}{2}} - e^{-\frac{3}{12}}]$$

SUMMARY

In this session, the concepts of uniform and exponential distribution have described

1. Define Uniform distribution and its properties
2. Define Exponential distribution and its properties
3. Applications of Uniform
4. Applications of Exponential

SELF-ASSESSMENT QUESTIONS

A random variable has uniform distribution over the interval [-1,3]. This distribution has variance equal to:

(a) 8/5

(b) 4/3

(c) 13/4

(d) 9/2

For an exponential distribution with probability density function

$$f(x) = \frac{1}{2}e^{-\frac{x}{2}}; \qquad x \geq 0,$$

its mean and variance are:

(a) $\left[\frac{1}{2}, 2\right]$

(b) $\left[2, \frac{1}{4}\right]$

(c) $\left[\frac{1}{2}, \frac{1}{4}\right]$

(d) (2,4)…

TERMINAL QUESTIONS

1. Describe the memory less property of exponential distribution

2. Suppose that a large conference room for a certain company can be reserved for no more than 4 hours. However, the use of the conference room is such that both long and short conferences occur quite often. In fact, it can be assumed that length X of a conferences has a uniform distribution on the interval [0, 4].

a) What is the probability density function?

b) What is the probability that any given conference lasts at least 3 hours?

3. Suppose that a study of a certain computer system reveals that the response time, in seconds, has an exponential distribution with a mean of 3 seconds.$\frac{1}{\beta}$

a) What is the probability that response time exceeds 5 seconds?

b) What is the probability that response time is less than 10 seconds?

c) What are the mean and variance of response time?

4. The density function of the time Z in minutes between calls to an electrical supply store is given by

$$f(Z) = \begin{cases} \frac{1}{10} e^{-\frac{z}{10}}, & 0 < z < \infty \\ 0, & elsewhere \end{cases}$$

i) Estimate the mean and variance of time between calls?

ii) Estimate the probability that the time between calls exceeds the mean?

iii) Estimate the probability that the time between calls is less than 5 minutes?

5. The daily amount of coffee, in liters, is dispensed by a machine located in an airport lobby is a random variable ‘X’ having a continuous uniform distribution with A=7 and B=10. Obtain the probability that on a given day the amount of coffee dispensed by this machine will be

i) at most 8.8 liters
ii) more than 7.4 liters but less than 9.5 liters
iii) At least 8.5 liters

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. Chapter 1 of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons,Inc.

2. Richard A Johnson, Miller& Freund's Probability and statistics for Engineers, PHI, New Delhi, 11th Edition (2011).

Sites and Web links:

- Continuous Random Variables and their Distributions (probabilitycourse.com)

Notes: sections 1 to 1.3 of http://www.statslab.cam.ac.uk/~rrw1/prob/prob-weber.pdf

3. Section 3.1.1 of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

Video: https://www.youtube.com/watch?v=-5sOBWV0qH8&list=PLeB45KifGiuHesi4PALNZSYZFhViVGQJK&index=19

THANK YOU

Team – PSQT EVEN SEMESTER 2022-23