CE563 Computation Methods
The Runge-Kutta Methods

SECOND ORDER FORMULAS

Problem:
Given the differential equaition:

$$\begin{aligned} \frac{dy}{dx} &= f(x,y) && (1) \\ & && (2) \\ y(x_i) &= y_i && (3) \end{aligned}$$

and

$$y_{i+1} = y_i + [ak_1 + bk_2]\,h \tag{4}$$

$$\begin{aligned} k_1 &= f(x_i, y_i) && (5) \\ k_2 &= f(x_i + mh, y_i + k_1 mh) && (6) \end{aligned}$$

we seek to determine a, b, and m so that the expression for y_{i+1} agrees with the Taylor series expansion up to and including h^2 terms.

Solution:
Expand k_1 and k_2 in a Taylor series about x_i and y_i to give:

$$k_i = f(x_i, y_i) = f_i \tag{7}$$

and

$$\begin{aligned} k_2 &= f + \frac{\partial f}{\partial x} mh + \frac{\partial f}{\partial y} k_1 mh + O\left(h^2\right) \\ &= f + \frac{\partial f}{\partial x} mh + \frac{\partial f}{\partial y} f mh + O\left(h^2\right) \\ &= f + (f_x + f_y f)\, mh + O\left(h^2\right) && (8) \end{aligned}$$

where f and its derivatives are evaluated at (x_i, y_i).
Substitution now gives:

$$\begin{aligned} y_{i+1} &= y_i a f h + b f h + b\left(f_x + f_y f\right) m h^2 + O\left(h^3\right) \\ y_{i+1} &= y_i + (a+b) f h + m b\left(f_x + f_y f\right) h^2 + O\left(h^3\right) \qquad (9) \end{aligned}$$

which agrees with the Taylor series up to h^2 if:

$$\begin{aligned} (a+b) &= 1 \qquad (10) \\ mb &= 1/2 \qquad (11) \end{aligned}$$

Because the above two equations contain three parameters, there are an infinite number of ways to satify them, including

$$m = 1, a = b = 1/2 \quad \text{for modified Euler's method}$$

$$m = 1/2, a = 0, b = 1 \quad \text{for improved Euler's method}$$

FOURTH ORDER FORMULAS

Problem:
Given the following formula:

$$y_{i+1} = y_i + [ak_1 + bk_2 + ck_3 + dk_4]\,h \tag{12}$$

$$\begin{aligned}
k_1 &= f(x_i, y_i) && (13)\\
k_2 &= f(x_i + mh, y_i + k_1 mh) && (14)\\
k_3 &= f(x_i + nh, y_i + k_2 nh) && (15)\\
k_4 &= f(x_i + ph, y_i + k_3 ph) && (16)
\end{aligned}$$

we seek values for $a, b, c, d, m, n,$ and p so that y_{i+1} will agree with the Taylor series expansion up to and including terms of h^4.

After a great deal of algebra, the following equations can be found:

$$\begin{aligned}
a + b + c + d &= 1 && (17)\\
bm + cn + dp &= 1/2 && (18)\\
bm^2 + cn^2 + dp^2 &= 1/3 && (19)\\
bm^3 + cn^3 + dp^3 &= 1/4 && (20)\\
cmn + dnp &= 1/6 && (21)\\
cmn^2 + dnp^2 &= 1/8 && (22)\\
cm^2n + dn^2p &= 1/12 && (23)\\
dmnp &= 1/24 && (24)
\end{aligned}$$

There are seven equations and eight parameters. The traditional solution is:

$$\begin{aligned}
m &= 1/2 && (25)\\
n &= 1/2 && (26)\\
p &= 1 && (27)\\
a &= 1/6 && (28)\\
d &= 1/6 && (29)\\
b &= 1/3 && (30)\\
c &= 1/3 && (31)
\end{aligned}$$

which gives the following formula for the fourth order Runge-Kutta method:

$$
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\begin{aligned}
y_{i+1} &= y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]h \\
k_1 &= f\left(x_i, y_i\right) \\
k_2 &= f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \\
k_3 &= f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) \\
k_2 &= f\left(x_i + h, y_i + k_3 h\right)
\end{aligned}
}
$$

CE 563 COMPUTATIONAL METHODS

SECOND ORDER ODE'S

Problem:
Given the second order ordinary differential equation,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

determine $y(x)$ using a Runge-Kutta method.
Solution:
We begin by writing the equation as two, first-order equations as follows:

$$\begin{aligned}
\frac{dy'}{dx} &= f\{x,\, y,\, y'\} \\
\frac{dy}{dx} &= F\{x,\, y,\, y'\} \\
&= y'
\end{aligned}$$

Next, we apply our Runge-Kutta formulas to each of the two equations as follows:

Euler's Method (First-order R-K):

$\frac{dy'}{dx} = f\,(x,\, y,\, y')$	$\frac{dy}{dx} = y'$
$y'_{i+1} = y_i + f_1 h$	$y_{i+1} = y_1 + F_1 h$
$f_1 = f\,(x_i,\, y_i,\, y'_i)$	$F_1 = y'_i$
$y'_{i+1} = y'_i + f_1 h$	$y_{i+1} = y_i + y'_i h$
$f_1 = f\,(x_i,\, y_i,\, y'_i)$	

Modified Euler's Method (Second-order R-K):

$\dfrac{dy'}{dx} = f\left(x,\, y,\, y'\right)$	$\dfrac{dy}{dx} = y'$
$y'_{i+1} = y_i + \frac{1}{2}\left(f_1 + f_2\right)h$	$y_{i+1} = y_i + \frac{1}{2}\left(F_1 + F_2\right)h$
$f_1 = f\left(x_i,\, y_i,\, y'_i\right)$	$F_1 = y'_i$
$f_2 = f\left(x_i + h,\, y_i + F_1 h,\, y'_i + f_1 h\right)$	$F_2 = y'_i + f_1 h$
$y'_{i+1} = y'_i + \frac{1}{2}\left\{f_1 + f_2\right\}h$	$y_{i+1} = y_i + y'_i h + \frac{1}{2}\left\{f_1\right\}h^2$
$f_1 = f\left(x_i,\, y_i,\, y'_i\right)$	
$f_2 = f\left(x_i + h,\, y_i + y'_i h,\, y'_i + f_1 h\right)$	

Fourth-order Runge-Kutta Method:

$\dfrac{dy'}{dx} = f\left(x,\, y,\, y'\right)$	$\dfrac{dy}{dx} = y'$
$y'_{i+1} = y_i + \frac{1}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)h$	$y_{i+1} = y_1 + \frac{1}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)h$
$f_1 = f\left(x_i,\, y_i,\, y'_i\right)$	$F_1 = y'_i$
$f_2 = f\left(x_i + \frac{1}{2}h,\, y_i + F_1\frac{1}{2}h,\, y'_i + f_1\frac{1}{2}h\right)$	$F_2 = y'_i + f_1\frac{1}{2}h$
$f_3 = f\left(x_i + \frac{1}{2}h,\, y_i + F_2\frac{1}{2}h,\, y'_i + f_2\frac{1}{2}h\right)$	$F_3 = y'_i + f_2\frac{1}{2}h$
$f_4 = f\left(x_i + h,\, y_i + F_3h,\, y'_i + f_3h\right)$	$F_4 = y'_i + f_3h$
$y'_{i+1} = y_i + \frac{1}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)h$	$y_{i+1} = y_i + y'_ih + \frac{1}{6}\left\{f_1 + f_2 + f_3\right\}h^2$
$f_1 = f\left(x_i,\, y_i,\, y'_i\right)$	
$f_2 = f\left(x_i + \frac{1}{2}h,\, y_i + y'_i\frac{1}{2}h,\, y'_i + f_1\frac{1}{2}h\right)$	
$f_3 = f\left(x_i + \frac{1}{2}h,\, y_i + y'_i\frac{1}{2}h + f_1\frac{1}{4}h^2,\, y'_i + f_2\frac{1}{2}h\right)$	
$f_4 = f\left(x_i + h,\, y_i + y'_ih + f_2\frac{1}{2}h,\, y'_i + f_3h\right)$	