Bayesian Estimation

Machine Learning Techniques

Karthik Thiagarajan

Bayes' Theorem

Bayes' Theorem

Bayes' Theorem

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

Bayes' Theorem

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

Bayes' Theorem

Belief

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

1) Prior

Bayes' Theorem

Belief

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

1) Prior
2) Evidence (Data)

Bayes' Theorem

Belief

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

1) Prior
2) Evidence (Data)
3) Likelihood

Bayes' Theorem

Belief

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

Distribution

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

Distribution

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

Distribution Function

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

Distribution

Function

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Bayes' Theorem

Belief

Distribution

Function

$$P(\theta \mid D) = \frac{P(\theta) \cdot P(D \mid \theta)}{P(D)}$$

Scalar

$$\text{Posterior} = \frac{\text{Prior} \cdot \text{Likelihood}}{\text{Evidence}}$$

1) Prior
2) Evidence (Data)
3) Likelihood
4) Posterior

Beta Distribution

$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$

Beta Distribution

$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$

$p \in (0, 1)$

Beta Distribution

$$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

$$p \in (0, 1)$$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}$$

Beta Distribution

$$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

$$p \in (0, 1)$$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}$$

$$B(\alpha, \beta) = \int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-1} du$$

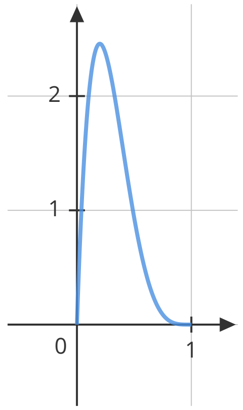
Beta Distribution

$$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

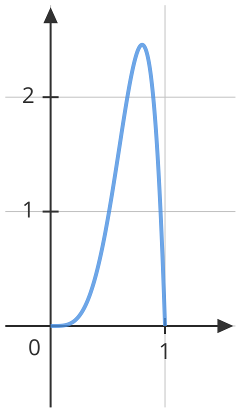
$$p \in (0, 1)$$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha - 1} \cdot (1 - p)^{\beta - 1}$$

$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du$$



$\texttt{Beta}(2, 5)$



$\texttt{Beta}(5, 2)$

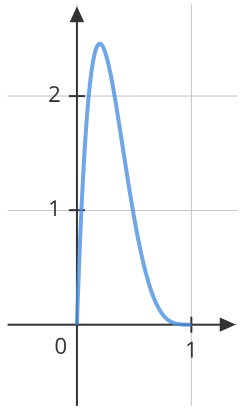
Beta Distribution

$$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

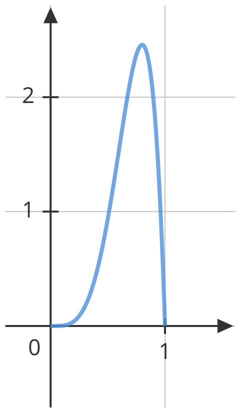
$$p \in (0, 1)$$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}$$

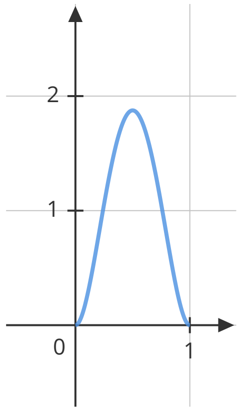
$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du$$



$\texttt{Beta}(2, 5)$



$\texttt{Beta}(5, 2)$



$\texttt{Beta}(3, 3)$

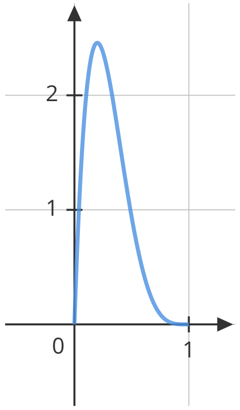
Beta Distribution

$\texttt{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$

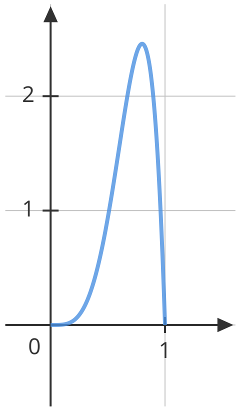
$p \in (0, 1)$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}$$

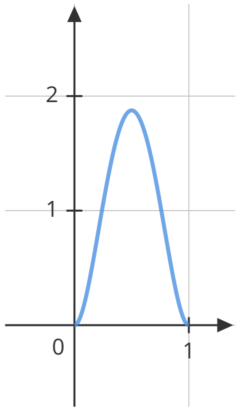
$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du$$



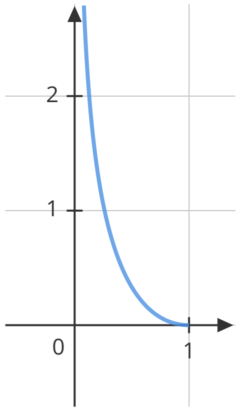
$\texttt{Beta}(2, 5)$



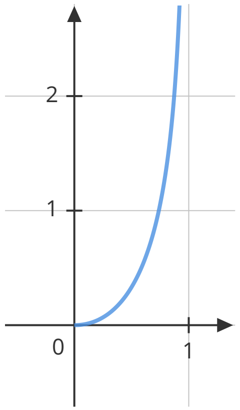
$\texttt{Beta}(5, 2)$



$\texttt{Beta}(3, 3)$



$\texttt{Beta}(0.5, 3)$



$\texttt{Beta}(3, 0.5)$

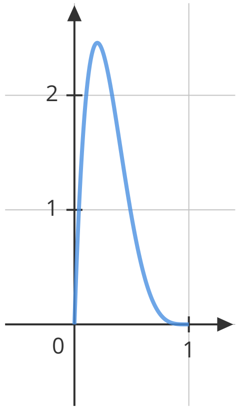
Beta Distribution

$$\text{Beta}(\alpha, \beta) \qquad \alpha > 0, \beta > 0$$

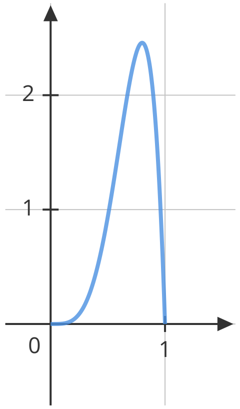
$$p \in (0, 1)$$

$$f(p; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}$$

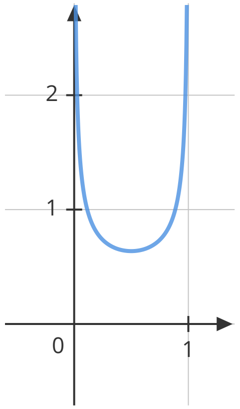
$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du$$



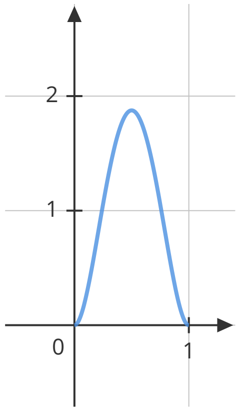
$\text{Beta}(2, 5)$



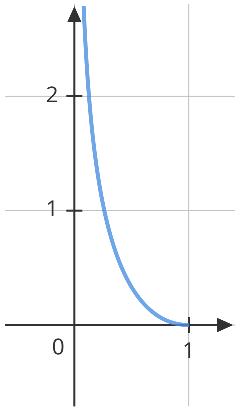
$\text{Beta}(5, 2)$



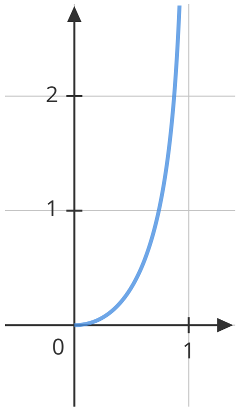
$\text{Beta}(0.5, 0.5)$



$\text{Beta}(3, 3)$



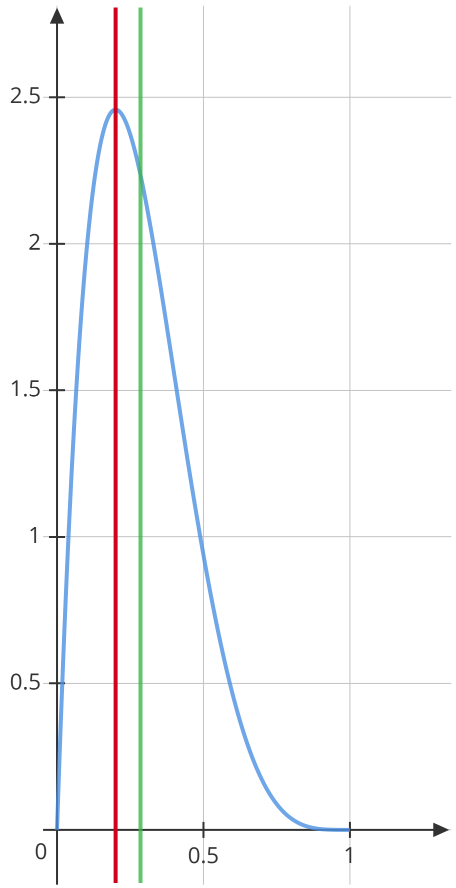
$\text{Beta}(0.5, 3)$



$\text{Beta}(3, 0.5)$

Beta Distribution

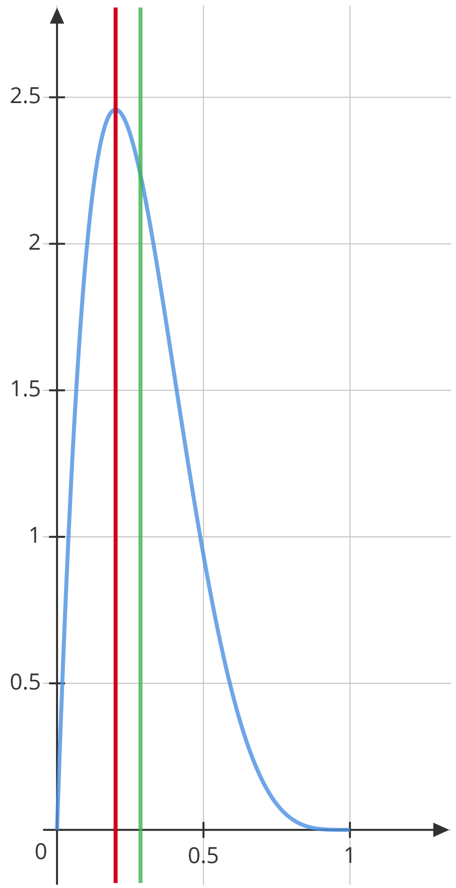
Beta$(2, 5)$



Beta Distribution

$$\text{Mean} = \frac{\alpha}{\alpha + \beta}$$

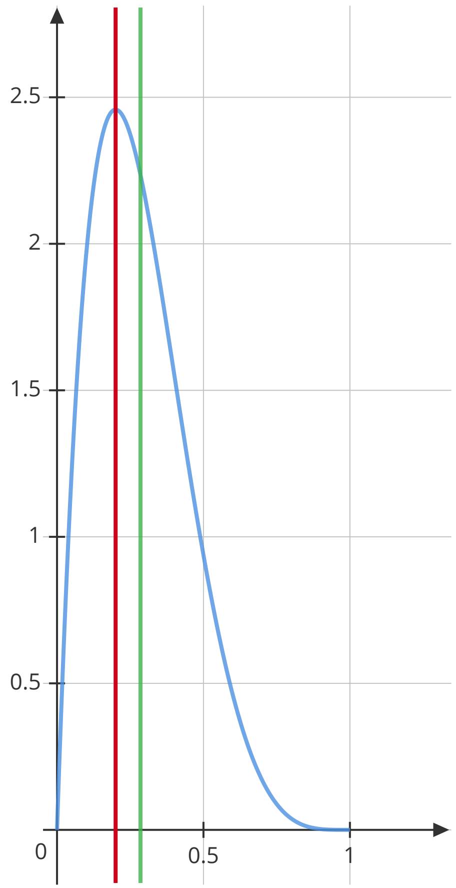
$\text{Beta}(2, 5)$



Beta Distribution

$$\texttt{Mean} = \frac{\alpha}{\alpha+\beta}$$

$\texttt{Beta}(2,5)$

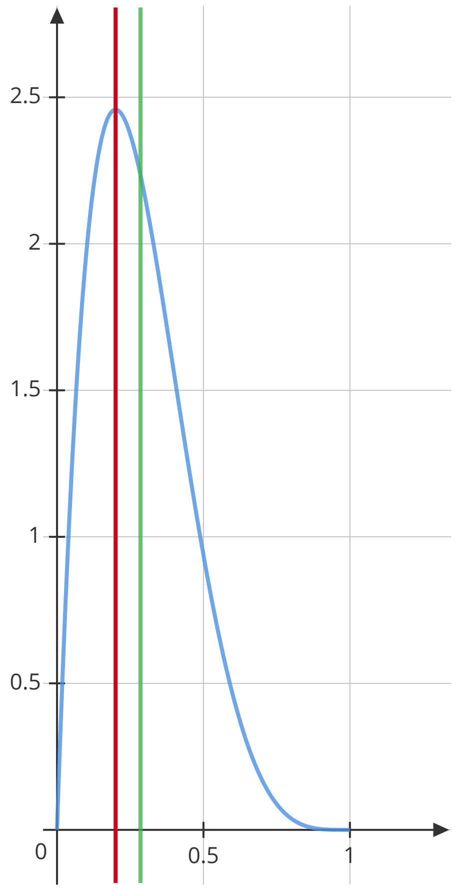


$$\texttt{Mode} = \begin{cases} \dfrac{\alpha-1}{\alpha+\beta-2} & \alpha,\beta>1 \\ 0 & \alpha\leqslant 1, \beta>1 \\ 1 & \alpha>1, \beta\leqslant 1 \\ (0,1) & \alpha=\beta=1 \\ \{0,1\} & \alpha,\beta<1 \end{cases}$$

Beta Distribution

$$\texttt{Mean} = \frac{\alpha}{\alpha + \beta}$$

$\texttt{Beta}(2, 5)$



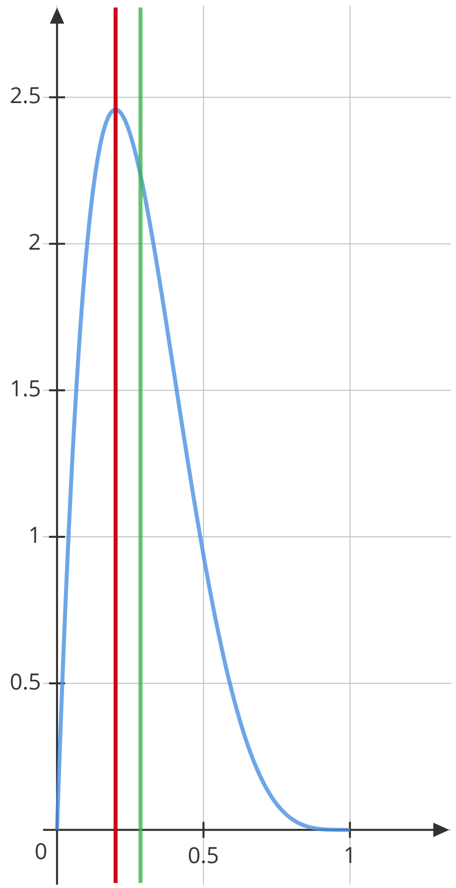
$$\texttt{Mode} = \begin{cases} \dfrac{\alpha - 1}{\alpha + \beta - 2} & \alpha, \beta > 1 \\ 0 & \alpha \leqslant 1, \beta > 1 \\ 1 & \alpha > 1, \beta \leqslant 1 \\ (0, 1) & \alpha = \beta = 1 \\ \{0, 1\} & \alpha, \beta < 1 \end{cases}$$

$$\frac{d \log(f(p))}{dp} = 0$$

Beta Distribution

$$\texttt{Mean} = \frac{\alpha}{\alpha+\beta}$$

$\texttt{Beta}(2,5)$



$$\texttt{Mode} = \begin{cases} \dfrac{\alpha-1}{\alpha+\beta-2} & \alpha, \beta > 1 \\ 0 & \alpha \leqslant 1, \beta > 1 \\ 1 & \alpha > 1, \beta \leqslant 1 \\ (0,1) & \alpha = \beta = 1 \\ \{0,1\} & \alpha, \beta < 1 \end{cases}$$

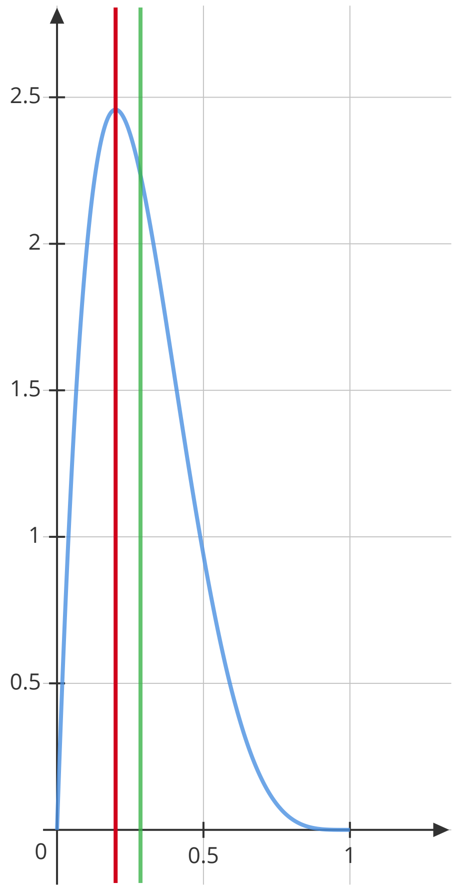
$$\frac{d \log(f(p))}{dp} = 0$$

$$\frac{(\alpha-1)}{p} - \frac{(\beta-1)}{1-p} = 0$$

Beta Distribution

$$\texttt{Mean} = \frac{\alpha}{\alpha+\beta}$$

$\texttt{Beta}(2,5)$



$$\texttt{Mode} = \begin{cases} \dfrac{\alpha-1}{\alpha+\beta-2} & \alpha,\beta > 1 \\ 0 & \alpha \leqslant 1, \beta > 1 \\ 1 & \alpha > 1, \beta \leqslant 1 \\ (0,1) & \alpha = \beta = 1 \\ \{0,1\} & \alpha, \beta < 1 \end{cases}$$

$$\frac{d \log(f(p))}{dp} = 0$$

$$\frac{(\alpha-1)}{p} - \frac{(\beta-1)}{1-p} = 0$$

$$p = \frac{\alpha-1}{\alpha+\beta-2}$$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

$p \to$ parameter

$p \sim \texttt{Beta}(\alpha, \beta)$

$\alpha, \beta \to$ hyperparameters

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

p → parameter

$p \sim \text{Beta}(\alpha, \beta)$

α, β → hyperparameters

Prior: $f(p) = \dfrac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

p → parameter

$p \sim \text{Beta}(\alpha, \beta)$

α, β → hyperparameters

Prior: $f(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$

Likelihood: $p^{n_h}(1-p)^{n_t}$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

p → parameter

$p \sim \text{Beta}(\alpha, \beta)$

α, β → hyperparameters

Prior: $f(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$

Posterior $\propto$

Likelihood: $p^{n_h}(1-p)^{n_t}$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

$p \to$ parameter

$p \sim \texttt{Beta}(\alpha, \beta)$

$\alpha, \beta \to$ hyperparameters

Prior: $f(p) = \dfrac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$

Posterior $\propto$ Prior $\times$ Likelihood

Likelihood: $p^{n_h}(1-p)^{n_t}$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

$p \to$ parameter

$p \sim \texttt{Beta}(\alpha, \beta)$

$\alpha, \beta \to$ hyperparameters

Prior: $f(p) = \dfrac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$

Likelihood: $p^{n_h}(1-p)^{n_t}$

$$\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$$

$$\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1}$$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

$p \rightarrow$ parameter

$p \sim \text{Beta}(\alpha, \beta)$

$\alpha, \beta \rightarrow$ hyperparameters

Prior: $f(p) = \dfrac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$

Likelihood: $p^{n_h}(1-p)^{n_t}$

$$\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$$

$$\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1}$$

$$\propto \text{Beta}(n_h+\alpha, n_t+\beta)$$

$$\texttt{Prior} \xrightarrow{\texttt{Likelihood}} \texttt{Posterior}$$

$$X_i \sim Br(p) \qquad p \sim \texttt{Beta}(\alpha, \beta)$$

$$p \to \texttt{parameter} \qquad \alpha, \beta \to \texttt{hyperparameters}$$

$$\texttt{Prior}: \; f(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\texttt{Likelihood}: \; p^{n_h}(1-p)^{n_t}$$

$$\texttt{Posterior} = \texttt{Beta}(n_h+\alpha, n_t+\beta)$$

$$\begin{aligned}\texttt{Posterior} &\propto \texttt{Prior} \times \texttt{Likelihood} \\ &\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1} \\ &\propto \texttt{Beta}(n_h+\alpha, n_t+\beta)\end{aligned}$$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$

$p \sim \text{Beta}(\alpha, \beta)$

p → parameter

α, β → hyperparameters

$$\text{Prior}: \ f(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\text{Likelihood}: \ p^{n_h}(1-p)^{n_t}$$

$$\text{Posterior} = \text{Beta}(n_h + \alpha, n_t + \beta)$$

$$\begin{aligned} \text{Posterior} &\propto \text{Prior} \times \text{Likelihood} \\ &\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1} \\ &\propto \text{Beta}(n_h + \alpha, n_t + \beta) \end{aligned}$$

Beta distribution is a conjugate prior for the Bernoulli distribution

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$ $\quad$ $p \sim \text{Beta}(\alpha, \beta)$

$p \rightarrow$ parameter $\quad$ $\alpha, \beta \rightarrow$ hyperparameters

$$\text{Prior}: \; f(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\text{Likelihood}: \; p^{n_h}(1-p)^{n_t}$$

$$\text{Posterior} = \text{Beta}(n_h + \alpha, n_t + \beta)$$

$$\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$$

$$\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1}$$

$$\propto \text{Beta}(n_h + \alpha, n_t + \beta)$$

Beta distribution is a conjugate prior for the Bernoulli distribution

$$\text{Hard to compute} \longleftarrow P(D) = \int_\theta P(\theta) \cdot P(D \mid \theta) d\theta$$

$$\text{Prior} \xrightarrow{\text{Likelihood}} \text{Posterior}$$

$X_i \sim Br(p)$ $\qquad$ $p \sim \text{Beta}(\alpha, \beta)$

p → parameter $\qquad$ α, β → hyperparameters

$$\text{Prior}: \; f(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\text{Likelihood}: \; p^{n_h}(1-p)^{n_t}$$

$$\text{Posterior} = \text{Beta}(n_h + \alpha, n_t + \beta)$$

α, β: Pseudo-observations

$$\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$$

$$\propto p^{n_h+\alpha-1} \cdot (1-p)^{n_t+\beta-1}$$

$$\propto \text{Beta}(n_h + \alpha, n_t + \beta)$$

Beta distribution is a conjugate prior for the Bernoulli distribution

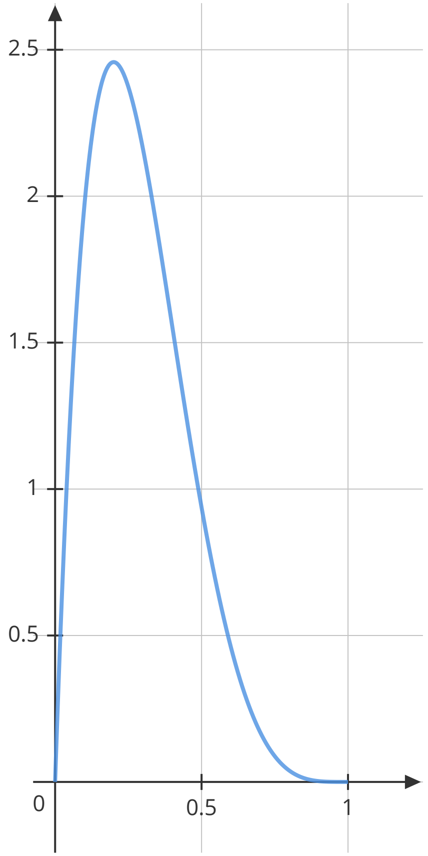
$$\text{Hard to compute} \longleftarrow P(D) = \int_\theta P(\theta) \cdot P(D \mid \theta) d\theta$$

Example-1

p is closer to 0 than it is to 1

Example-1

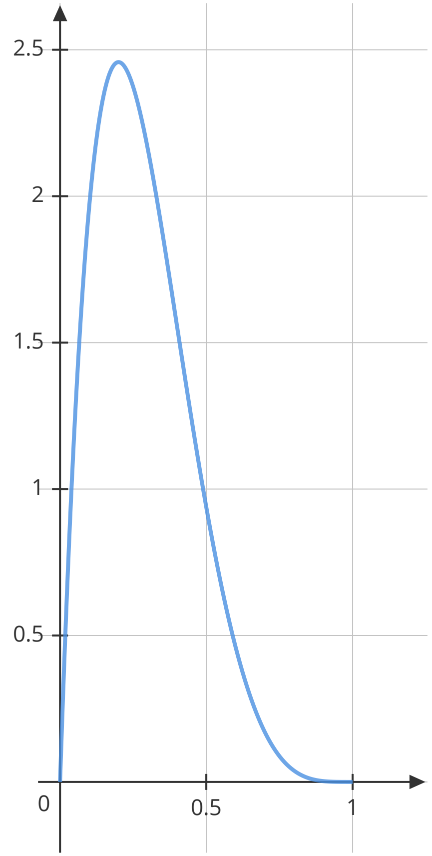
p is closer to 0 than it is to 1



Example-1

p is closer to 0 than it is to 1

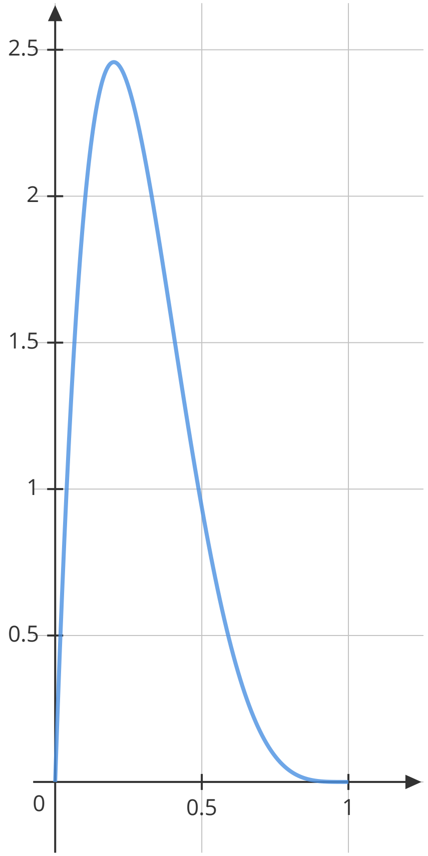
Beta$(2, 5)$



Example-1

p is closer to 0 than it is to 1

Beta$(2, 5)$



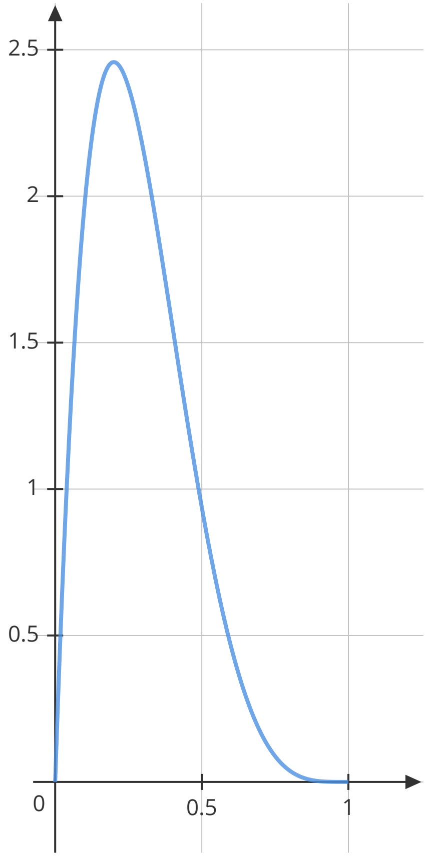
$$D = \{1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\}$$

Example-1

Beta$(4,\ 12)$

p is closer to 0 than it is to 1

Beta$(2, 5)$

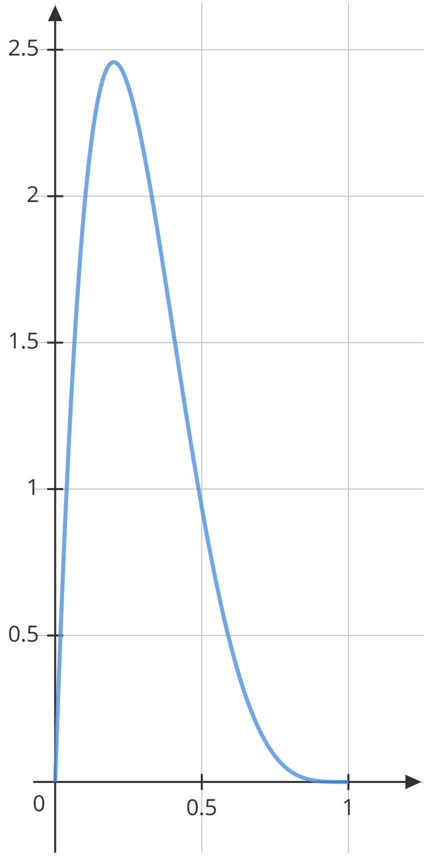


$$D = \{1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\}$$

Example-1

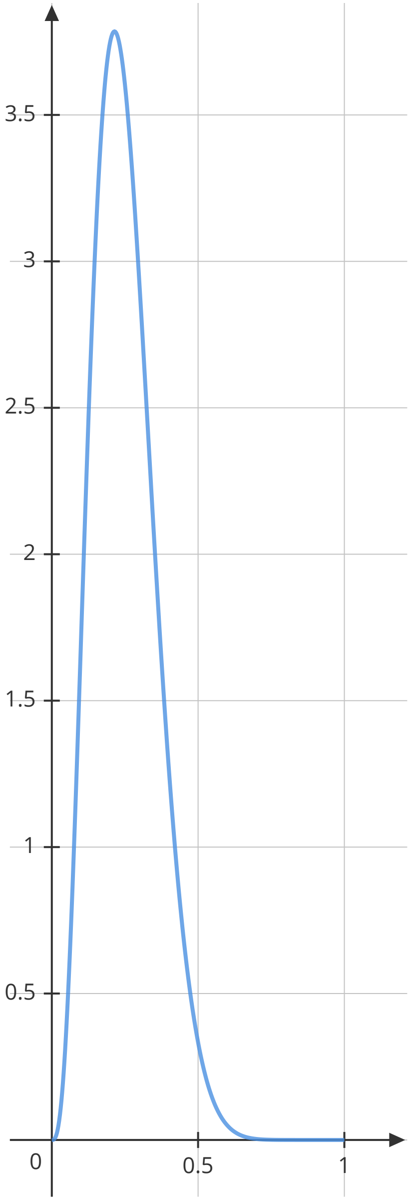
p is closer to 0 than it is to 1

Beta$(2, 5)$



$$D = \{1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\}$$

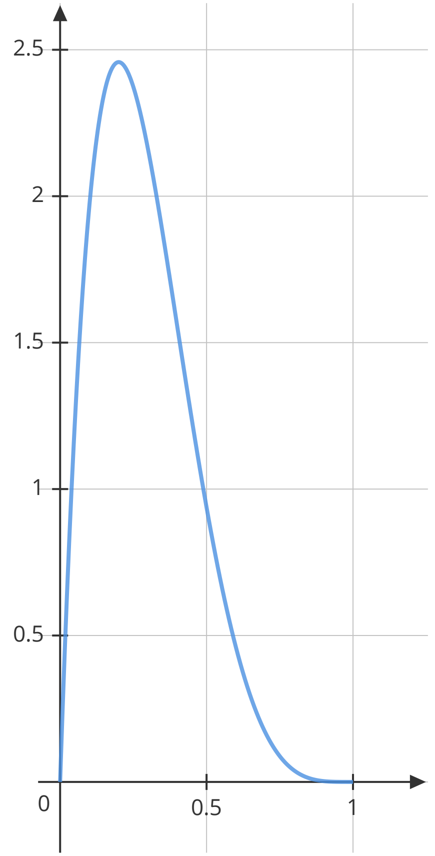
Beta$(4,\ 12)$



Example-1

p is closer to 0 than it is to 1

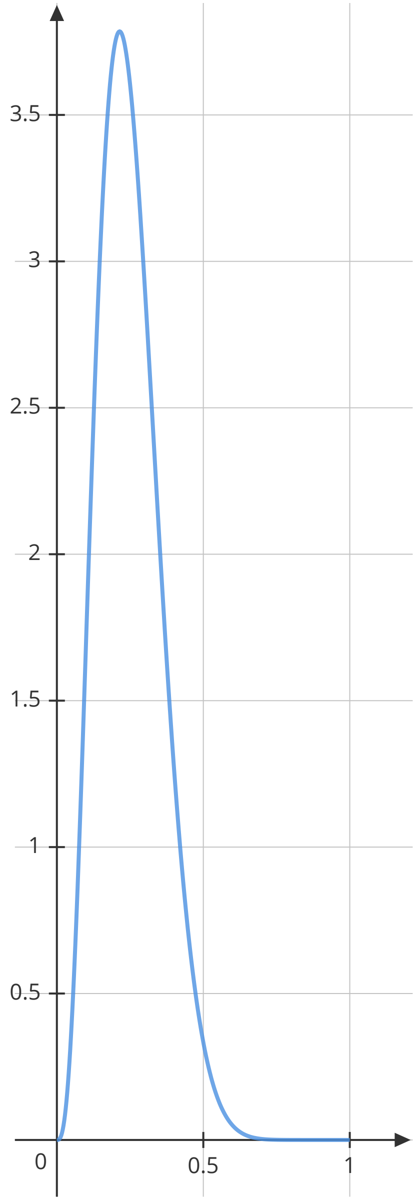
Beta$(2, 5)$



$$D_{\texttt{pseudo}} = \{1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

$$D = \{1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\}$$

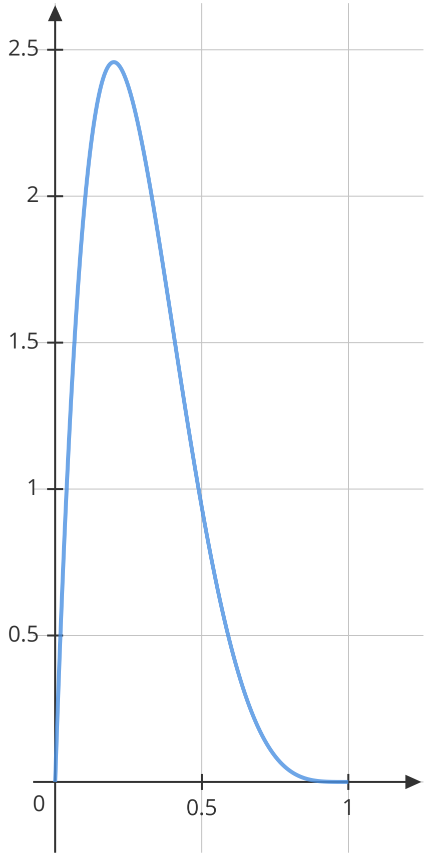
Beta$(4,\ 12)$



Example-2

p is closer to 0 than it is to 1

Beta$(2, 5)$

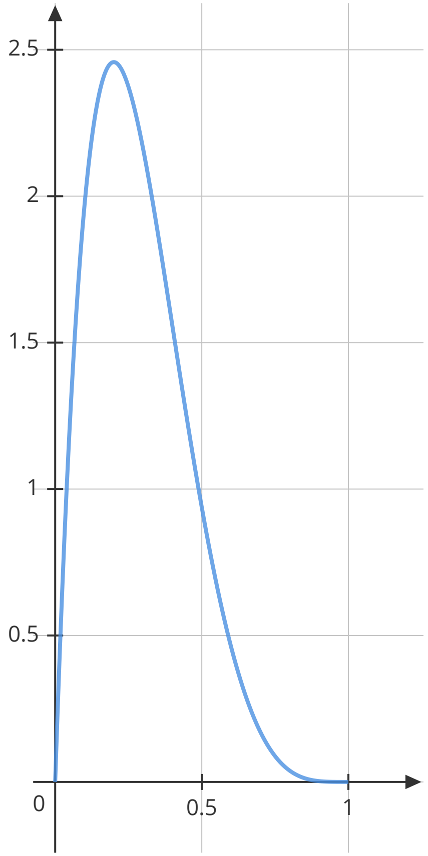


$$D = \{0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0\}$$

Example-2

p is closer to 0 than it is to 1

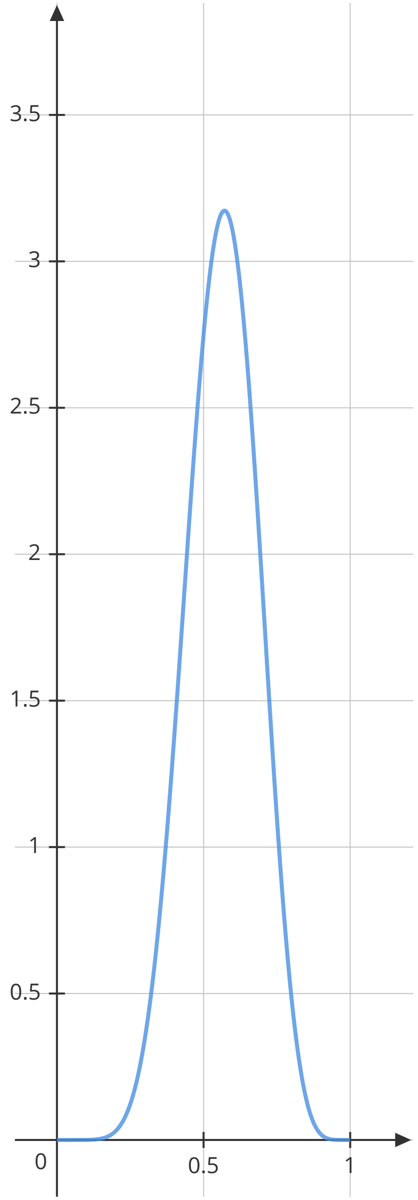
Beta$(2, 5)$



$$D_{\texttt{pseudo}} = \{1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

$$D = \{0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0\}$$

Beta$(9,\ 7)$

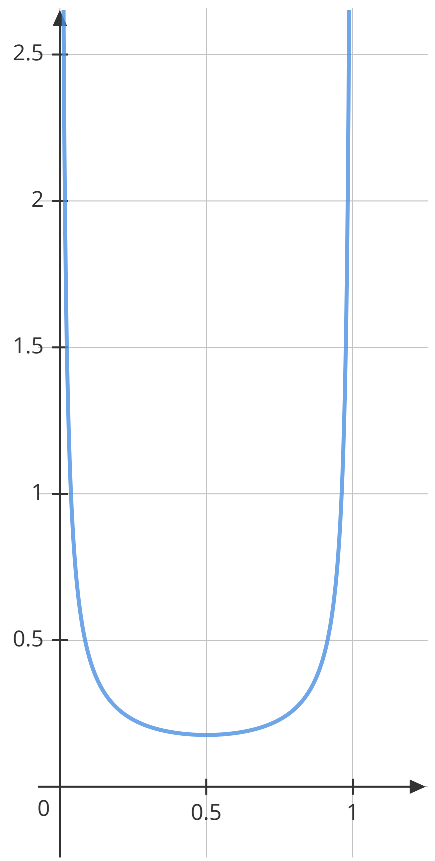


Example-3

p is extremely close to either 0 or 1

Example-3

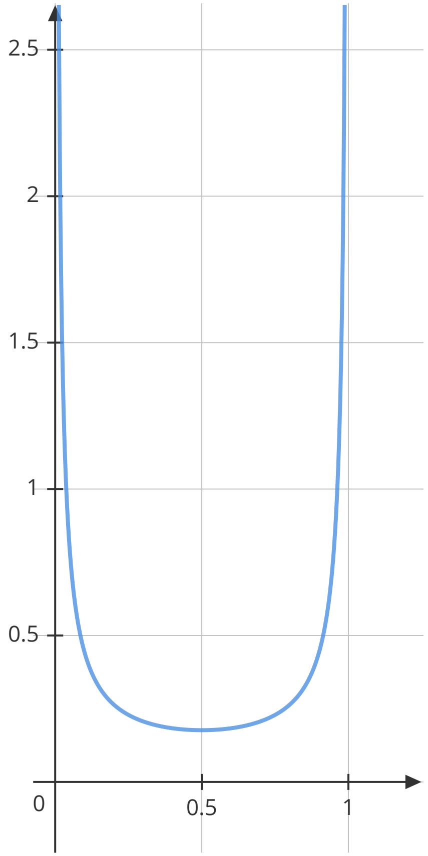
p is extremely close to either 0 or 1



Example-3

p is extremely close to either 0 or 1

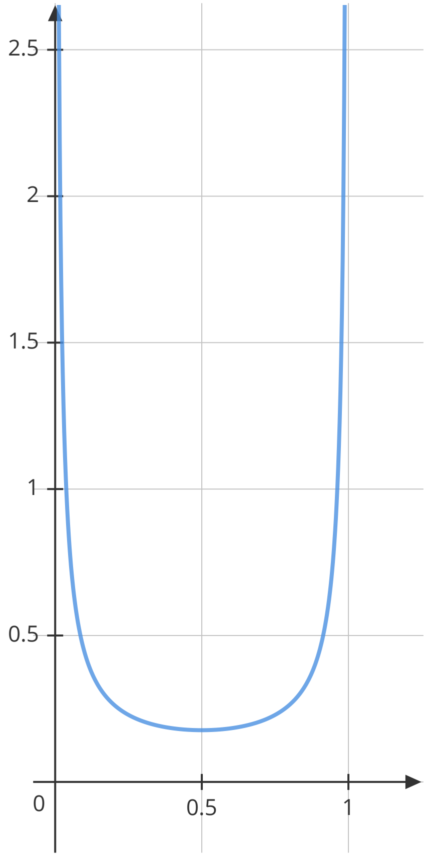
Beta$(0.1, 0.1)$



Example-3

p is extremely close to either 0 or 1

Beta$(0.1, 0.1)$



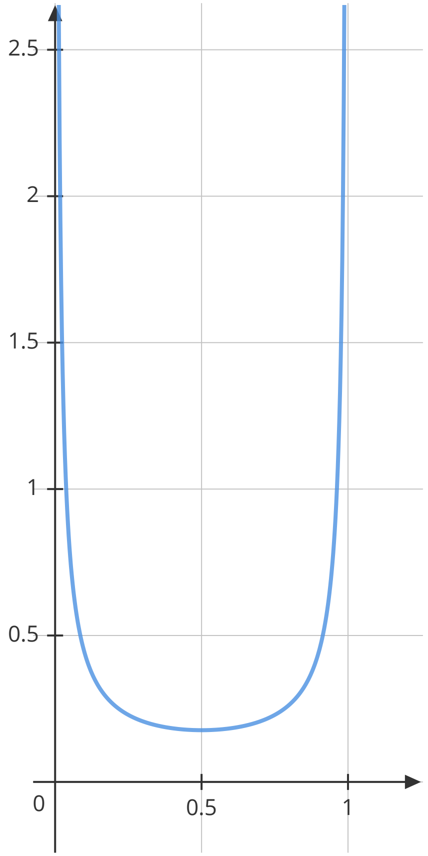
$$D = \{1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0\}$$

Example-3

Beta$(8.1,\ 1.1)$

p is extremely close to either 0 or 1

Beta$(0.1, 0.1)$

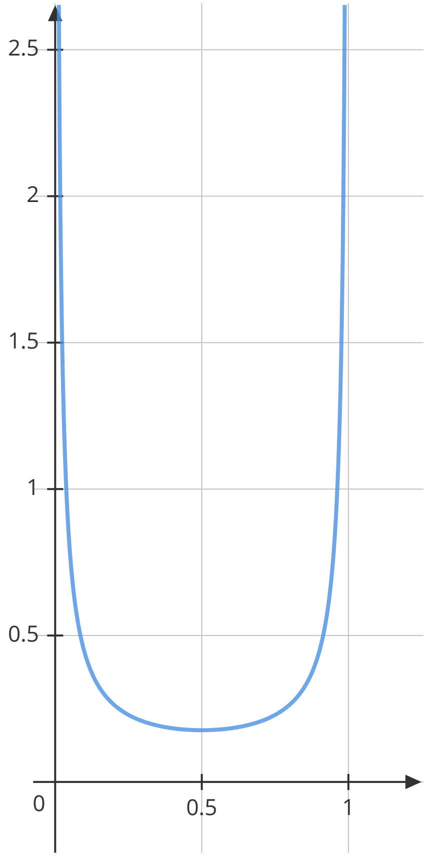


$$D = \{1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0\}$$

Example-3

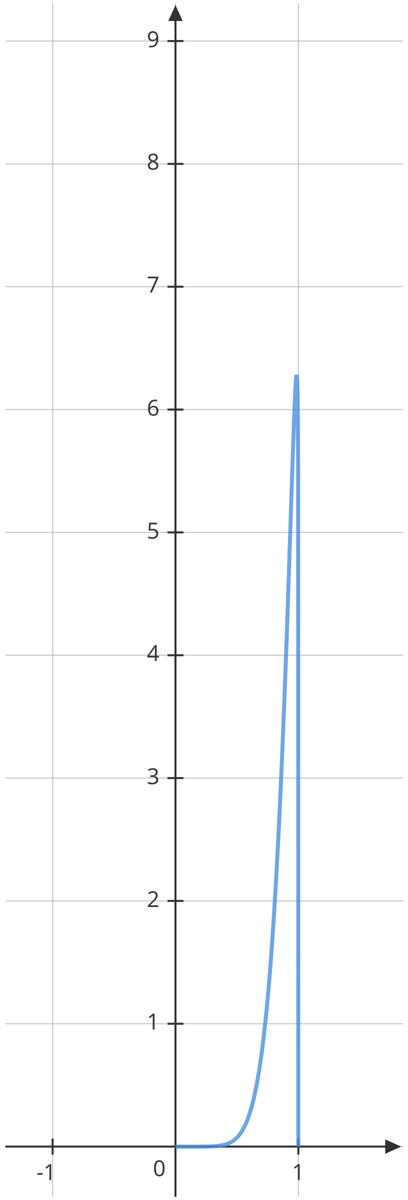
p is extremely close to either 0 or 1

Beta$(0.1, 0.1)$



$$D = \{1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0\}$$

Beta$(8.1,\ 1.1)$



Point Estimates

$$P(\theta \mid D)$$

Point Estimates

Posterior: Beta$(\alpha + n_h, \beta + n_t)$

$P(\theta \mid D)$

Point Estimates

Posterior: $\text{Beta}(\alpha + n_h, \beta + n_t)$

$$P(\theta \mid D) \rightarrow \text{Mean}$$

$$\text{Mean} = \frac{\alpha + n_h}{\alpha + \beta + n}$$

Point Estimates

$$\begin{tikzcd} & \text{Mean} \\ P(\theta \mid D) \arrow[ur] \arrow[dr] & \\ & \text{Mode (MAP)} \end{tikzcd}$$

Posterior: Beta$(\alpha + n_h, \beta + n_t)$

$$\text{Mean} = \frac{\alpha + n_h}{\alpha + \beta + n}$$

$$\text{Mode} = \frac{\alpha + n_h - 1}{\alpha + \beta + n - 2}$$

$$\alpha + n_h > 1$$
$$\beta + n_t > 1$$

Point Estimates

Posterior: Beta$(\alpha + n_h, \beta + n_t)$

Mean

$P(\theta \mid D)$

Mode (MAP)

$$\text{Mean} = \frac{\alpha + n_h}{\alpha + \beta + n}$$

$$\text{Mode} = \frac{\alpha + n_h - 1}{\alpha + \beta + n - 2}$$

$\alpha + n_h > 1$

$\beta + n_t > 1$

$$\widehat{\theta} = \arg\max_{\theta} \; P(\theta \mid D)$$

Maximum A Posteriori estimate