Estimation, MLE

Machine Learning Techniques

Karthik Thiagarajan

Story so far

Unsupervised Learning

- Representation learning
  - PCA
  - Kernel PCA
- Clustering
  - Lloyd's algorithm (K-means)
- Estimation

Story so far

Comprehension via Compression

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- PCA
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  - Where does the data come from?
  - What is the process that generates the data?
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  - Where does the data come from?
  - What is the process that generates the data?
  - What are the parameters that govern this process?

- How is it different from clustering and representation learning?
  - Present → Future
    - Start with what is given and try to explain patterns in it
  - Present → Past
    - Explore the process that could have given rise to the data

Generative Process

$$D = \{x_1, \cdots, x_n\}$$

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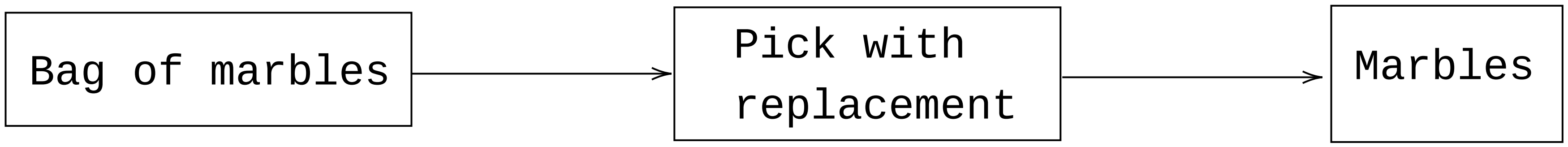

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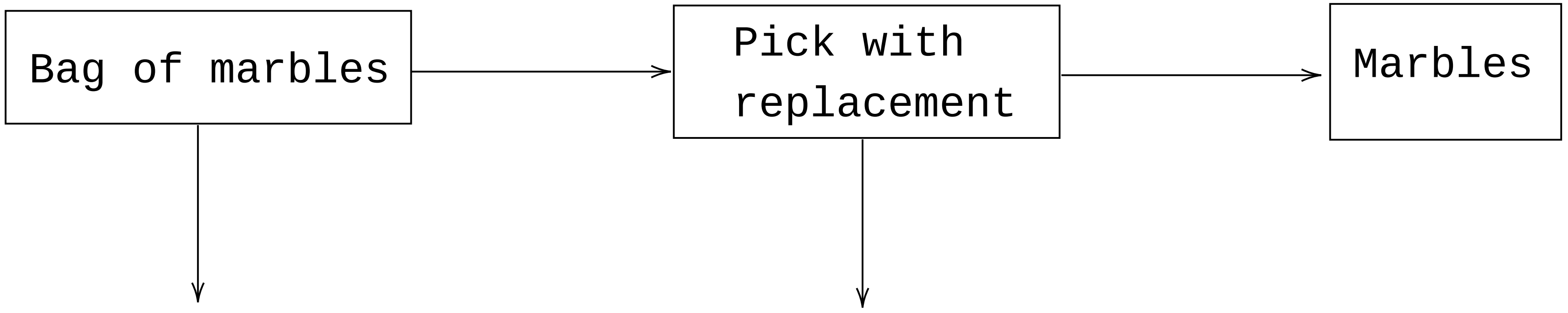

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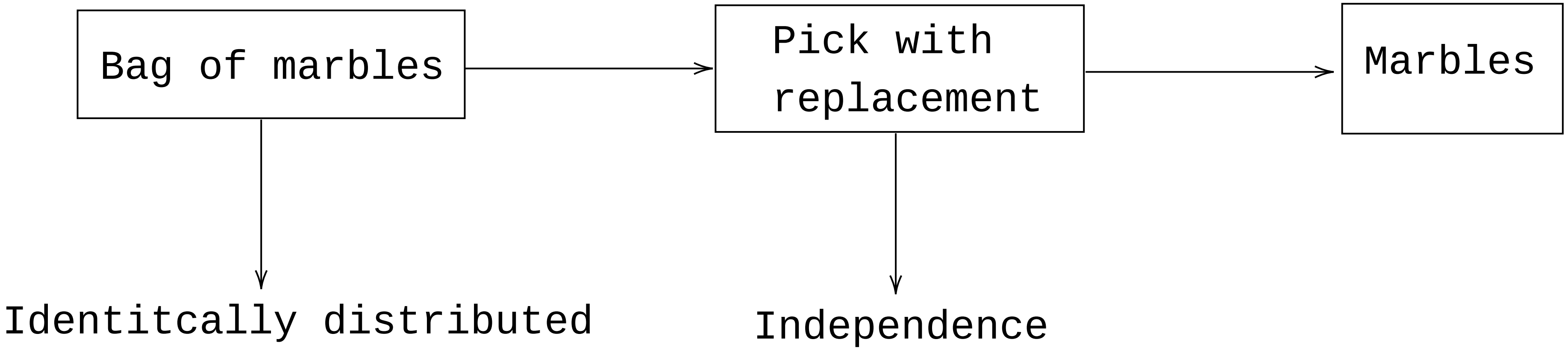

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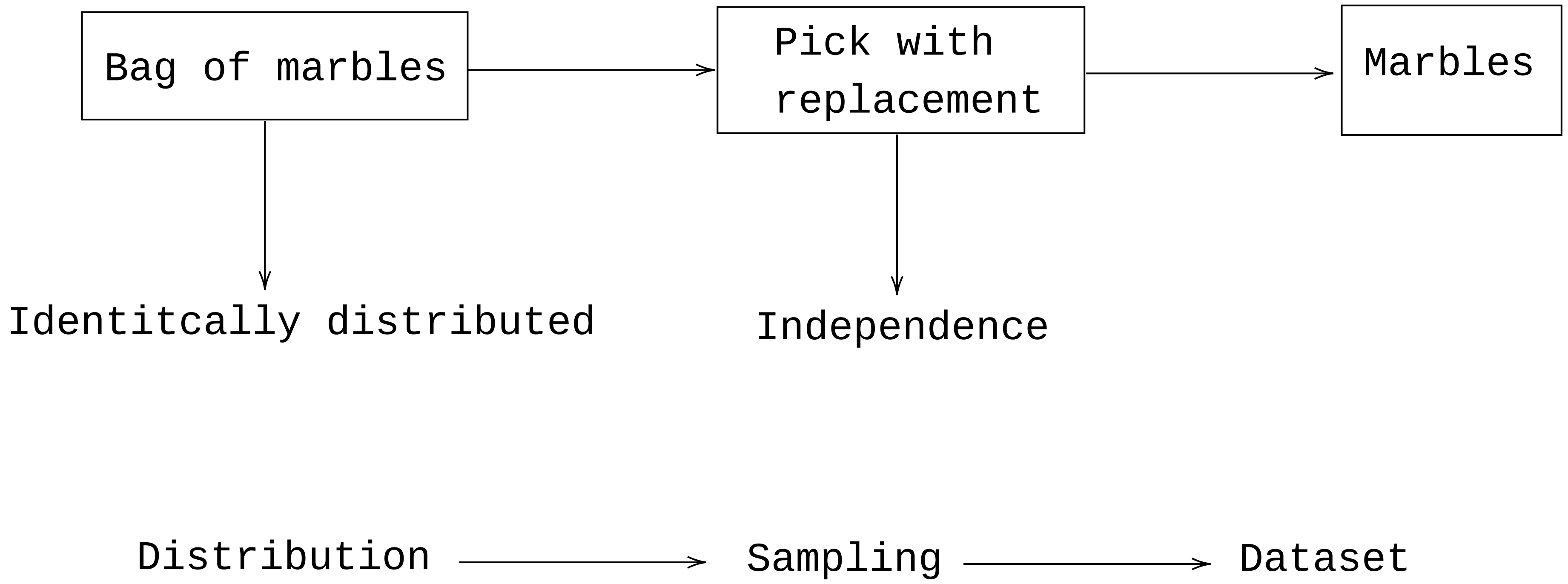
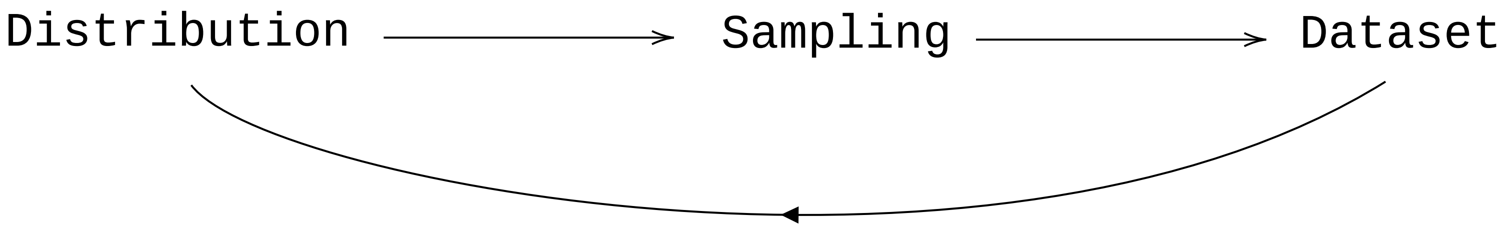


Maximum Likelihood



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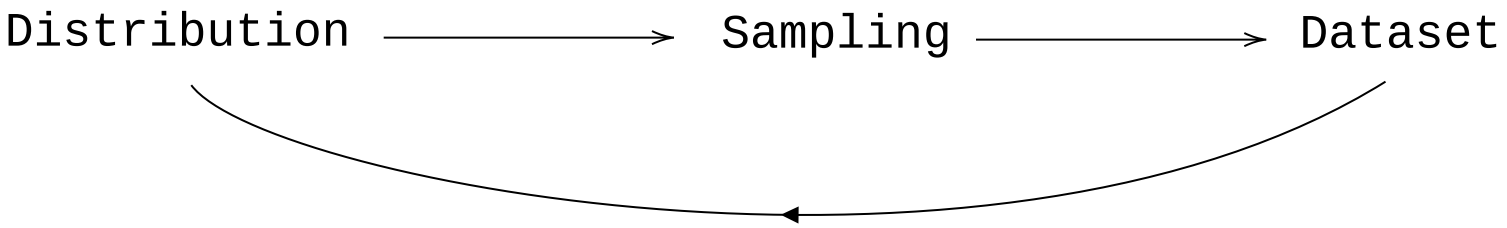

(1) Choose a distribution
(2) Estimate the parameters

Maximum Likelihood

Distribution → Sampling → Dataset

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$$L(\theta;\{x_1,\,\cdots,x_n\}) = P(\{x_1,\,\cdots,x_n\};\theta)$$

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\text{Distribution} \arrow[r] & \text{Sampling} \arrow[r] & \text{Dataset} \arrow[ll, bend left]
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$$\max_{\theta} \quad L(\theta; \{x_1, \cdots, x_n\}$$

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$$\widehat{\theta} = \arg\max_{\theta} \quad L(\theta; \{x_1, \cdots, x_n\}$$

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Likelihood and Log-Likelihood

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Likelihood and Log-Likelihood

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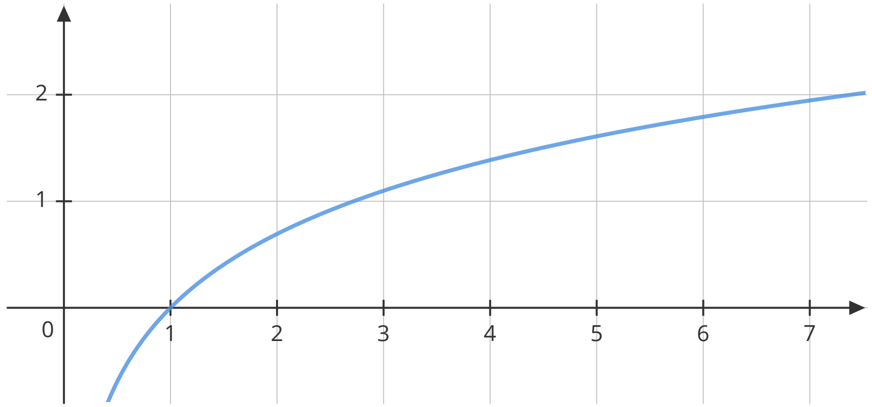
$$f(x) \leqslant f(x^*) \iff \log(f(x)) \leqslant \log\big(f(x^*)\big)$$



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