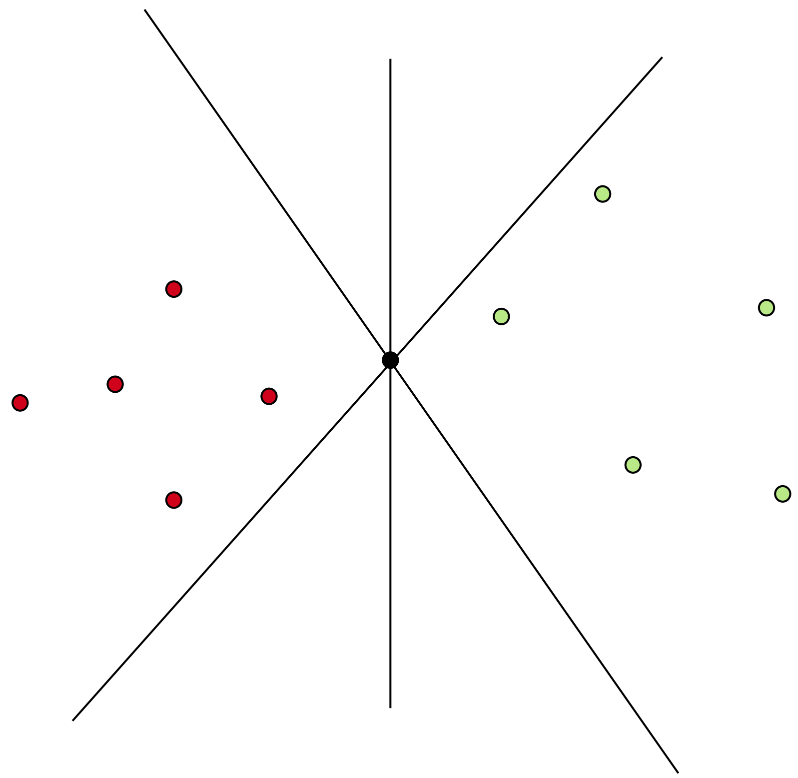
Support Vector Machines

Machine Learning Techniques

Outline

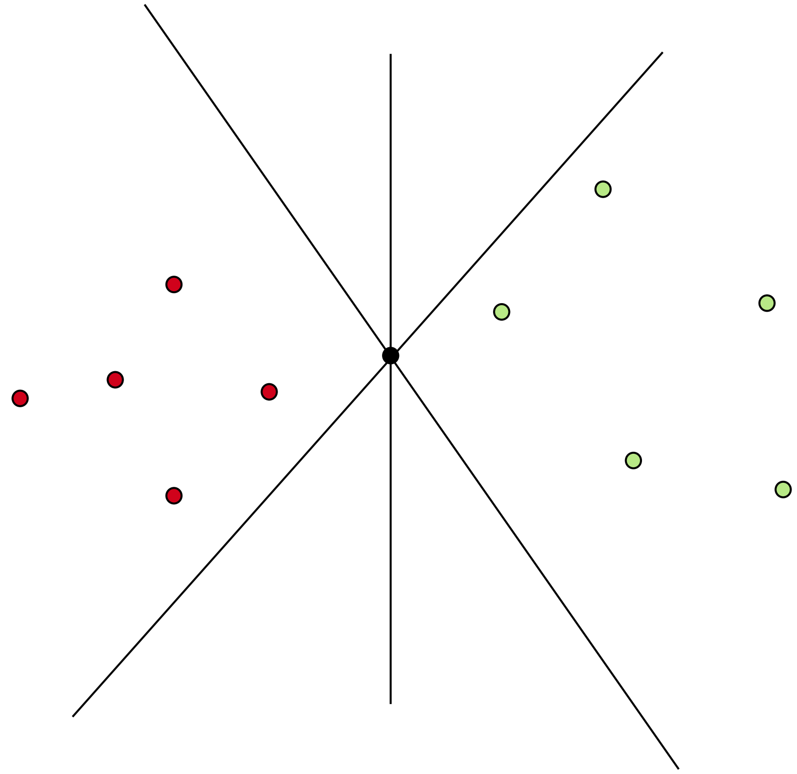
- Margin
- Max-margin classifier
- Duality
- Weight vector
- Support vectors
- Hard-margin Linear-SVM
- Soft-margin SVM

From Perceptrons to SVM



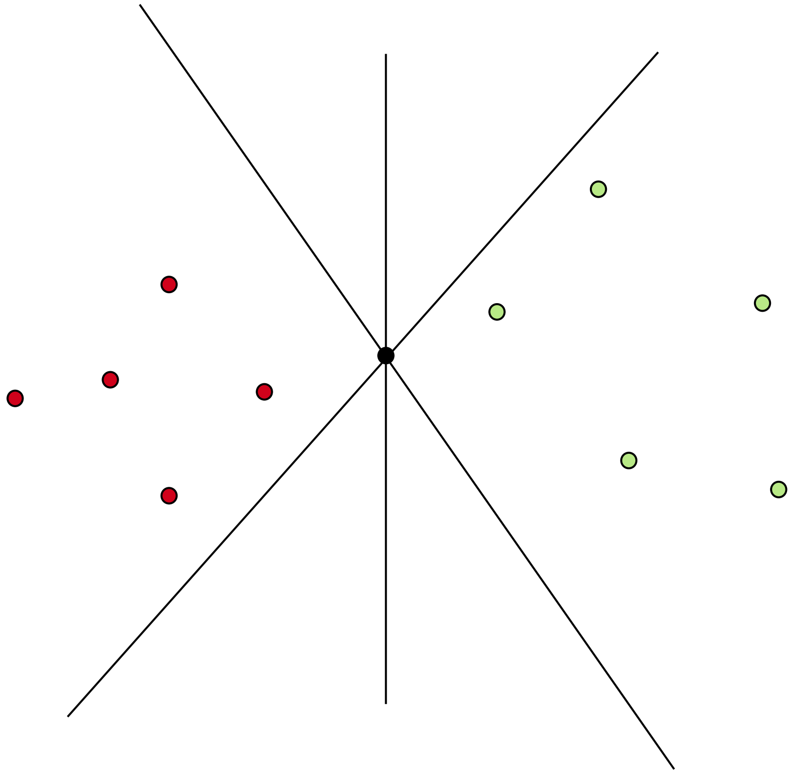
For linearly separable data with γ margin:

From Perceptrons to SVM



For linearly separable data with γ margin:
(1) Infinite number of valid linear classifiers.

From Perceptrons to SVM



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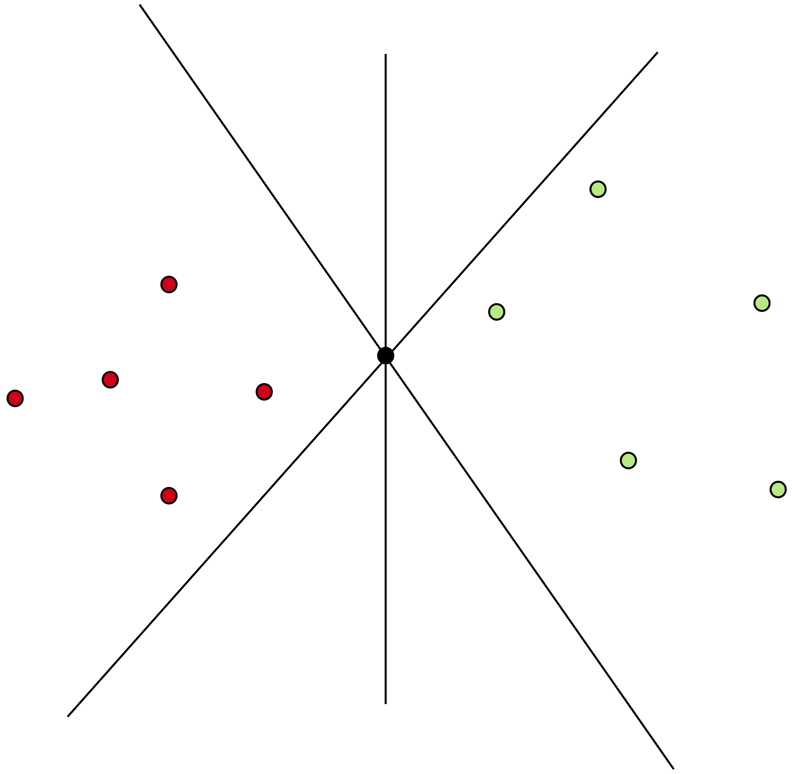

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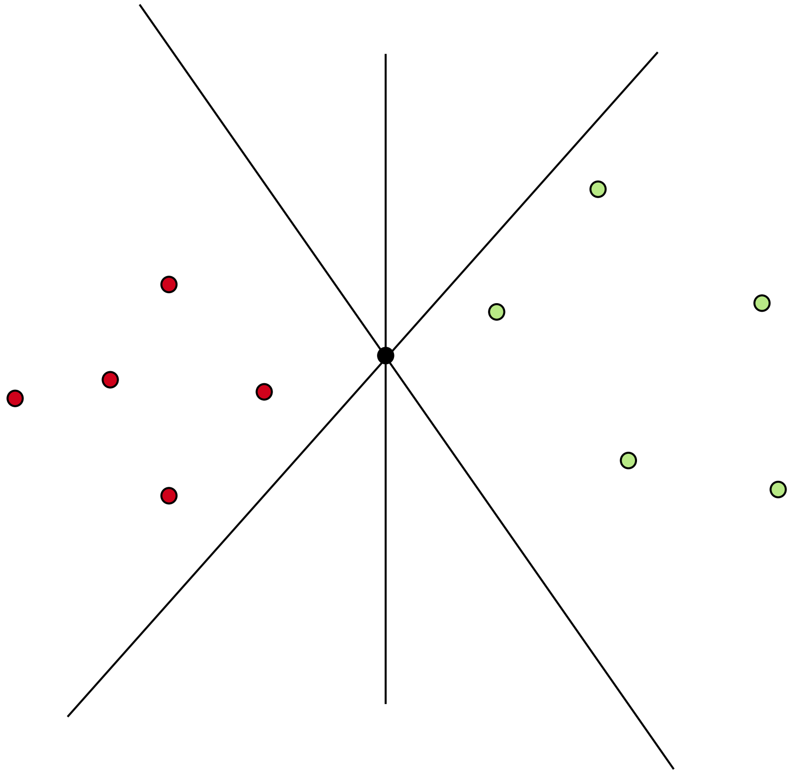

For linearly separable data with γ margin:

(1) Infinite number of valid linear classifiers.

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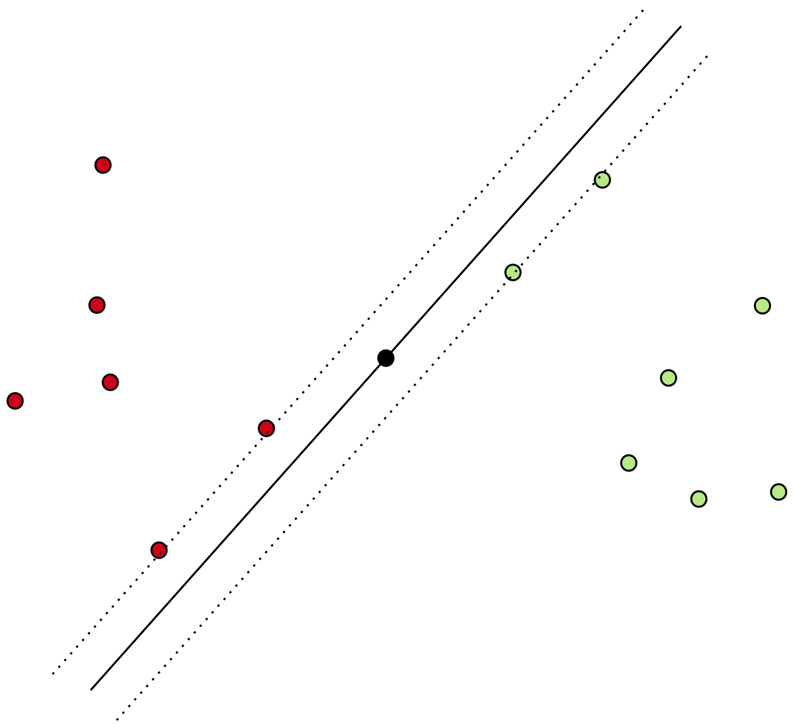
(3) Is it the "best"?

(4) What is a good notion of "best"?

Margin



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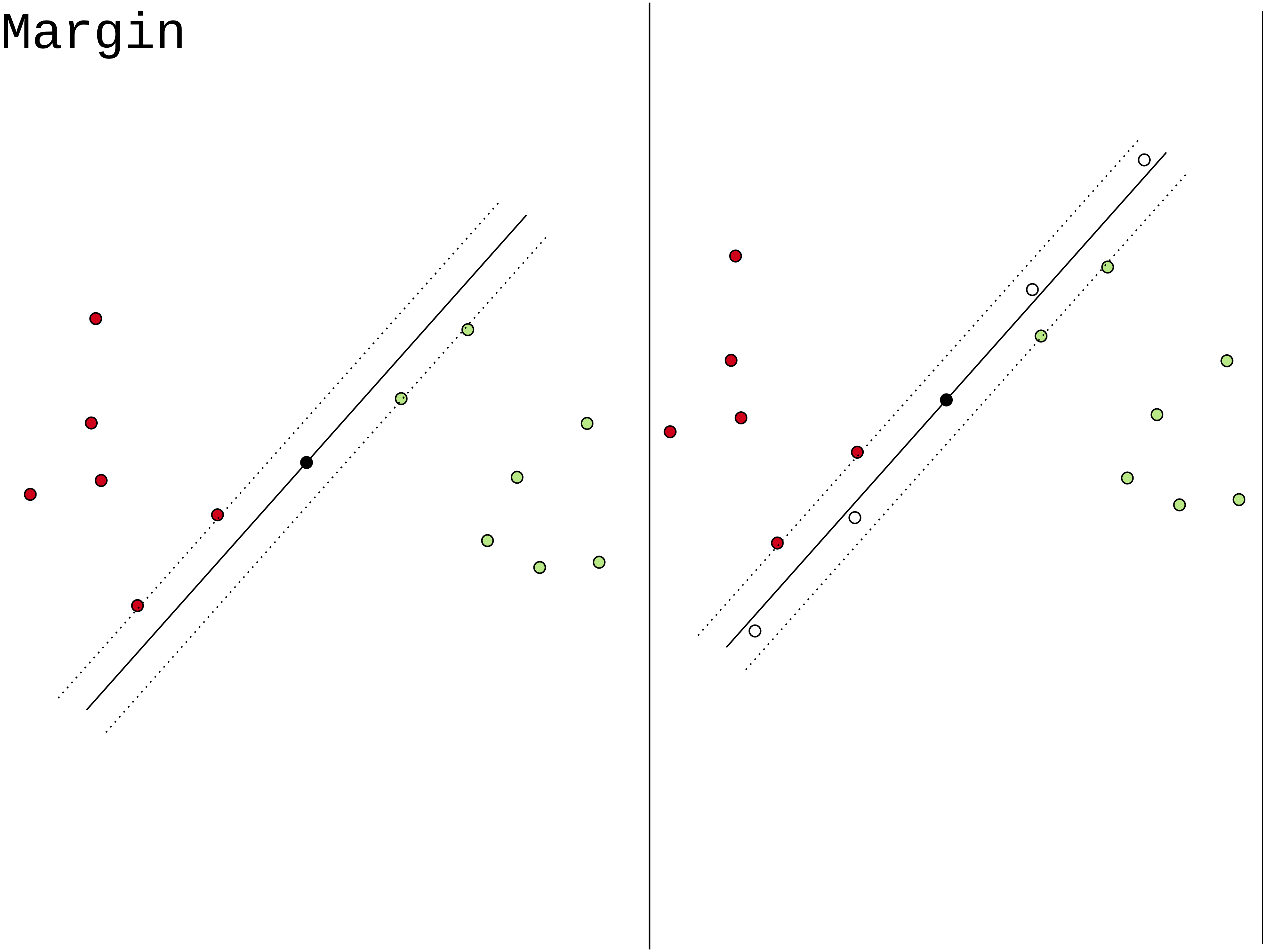

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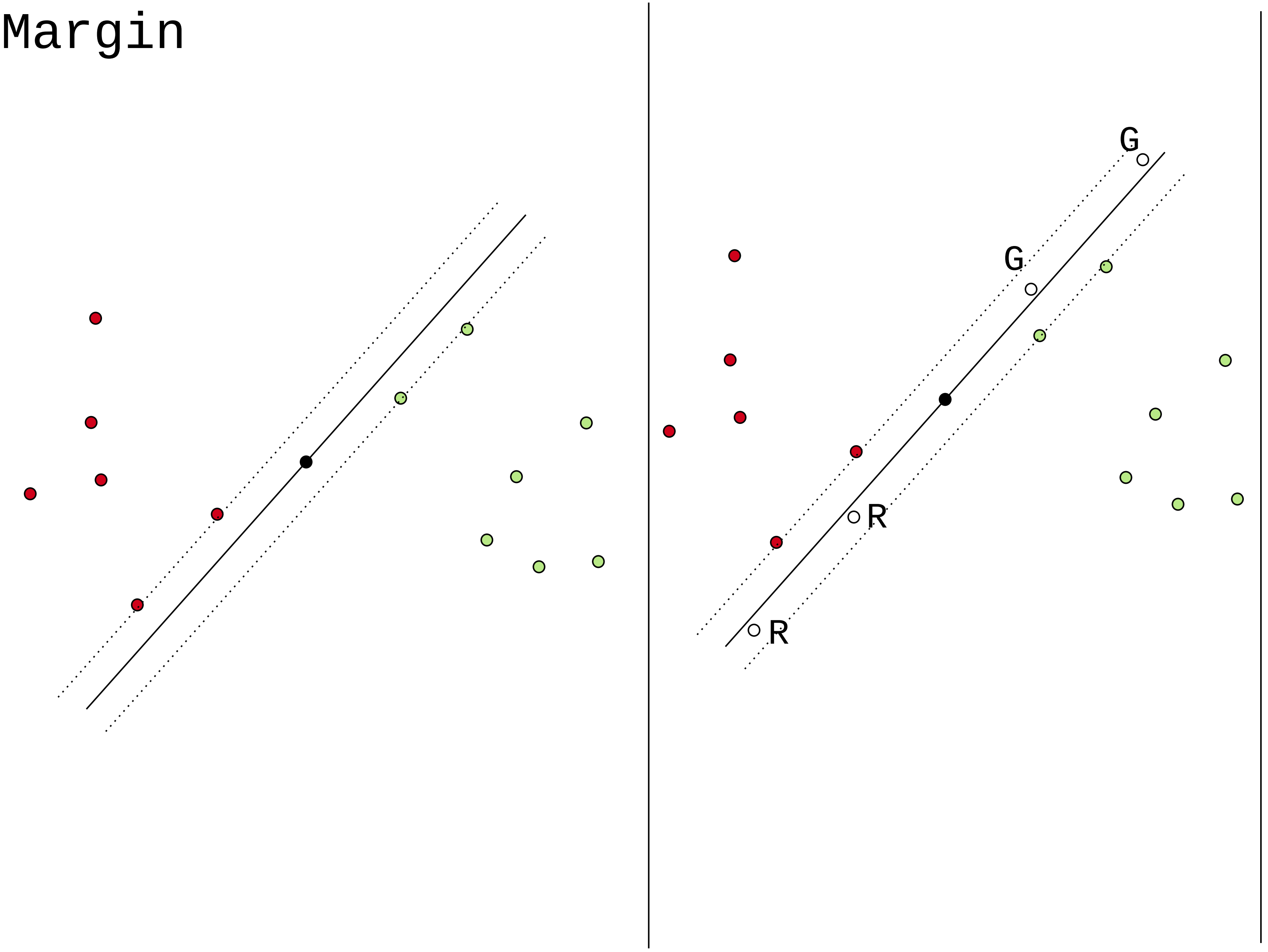

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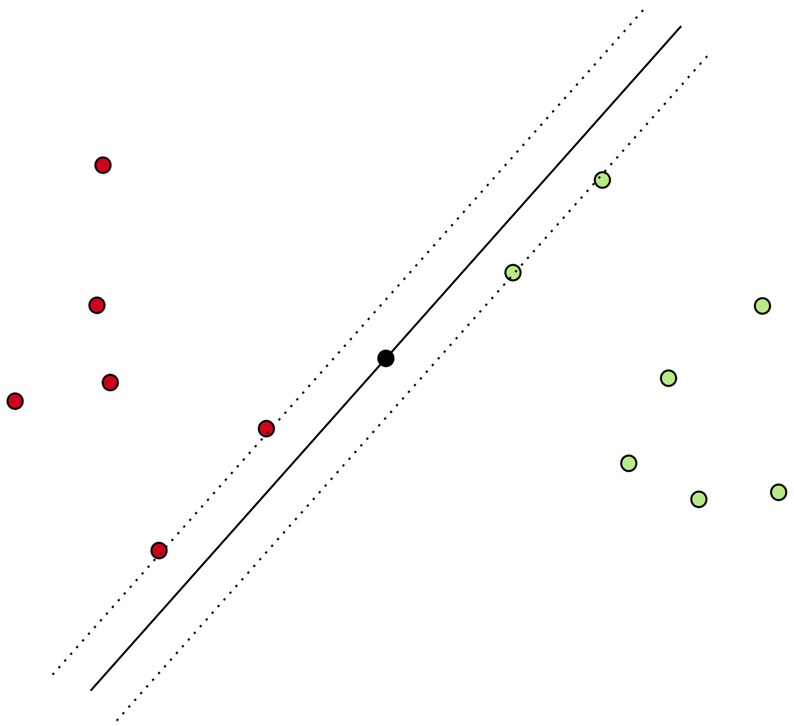
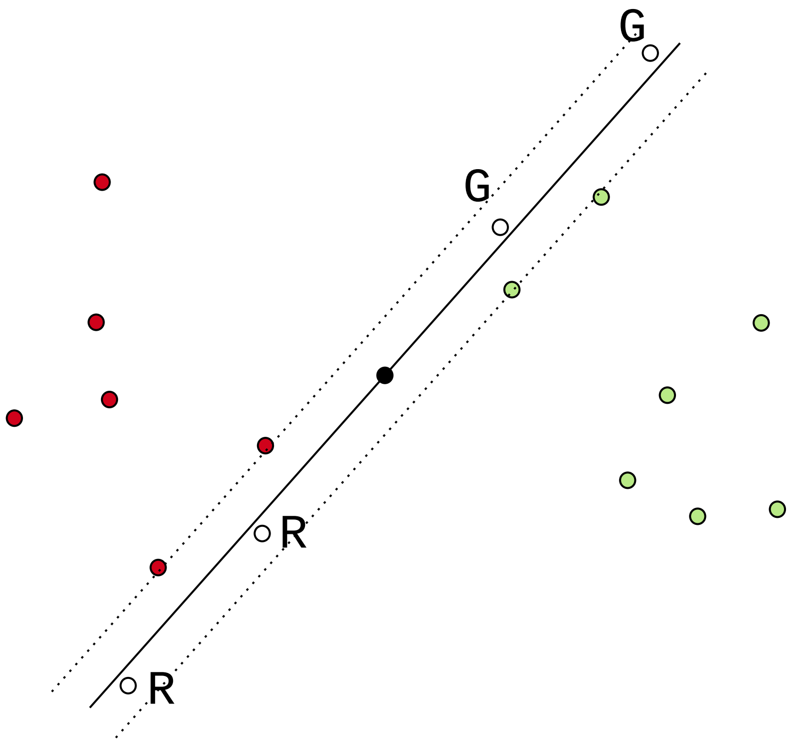
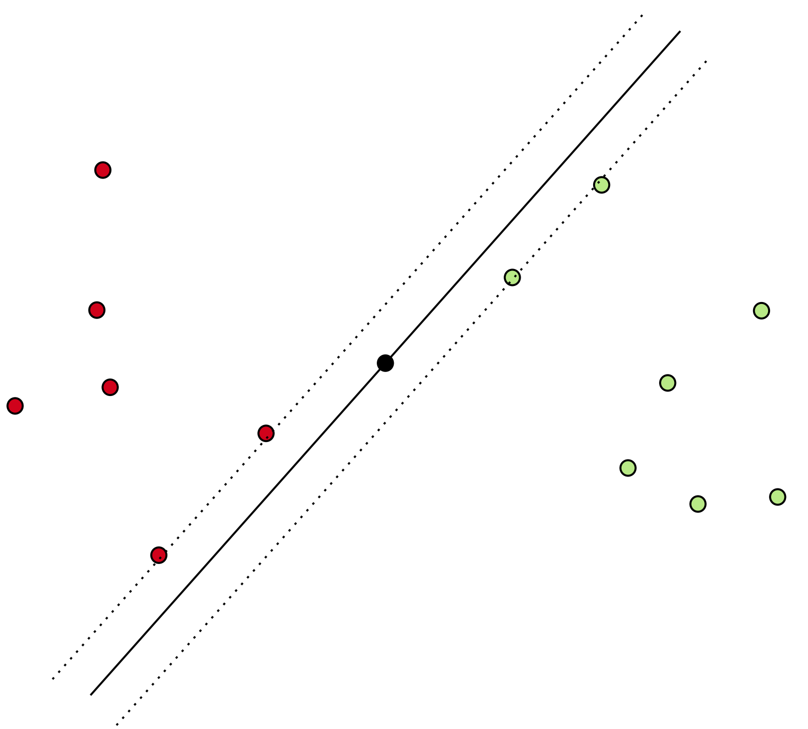
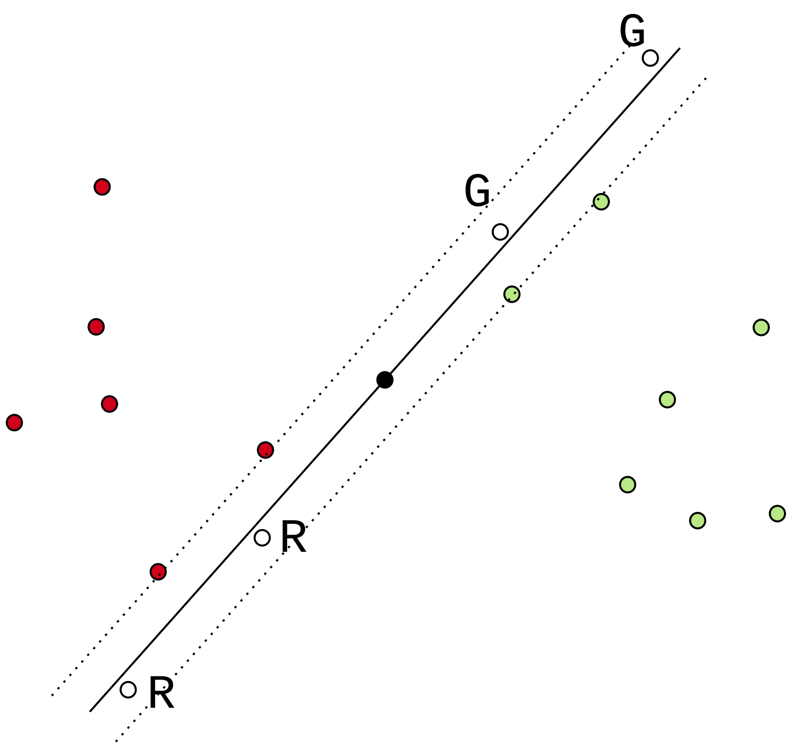




Small margin
Doesn't generalize well

Margin





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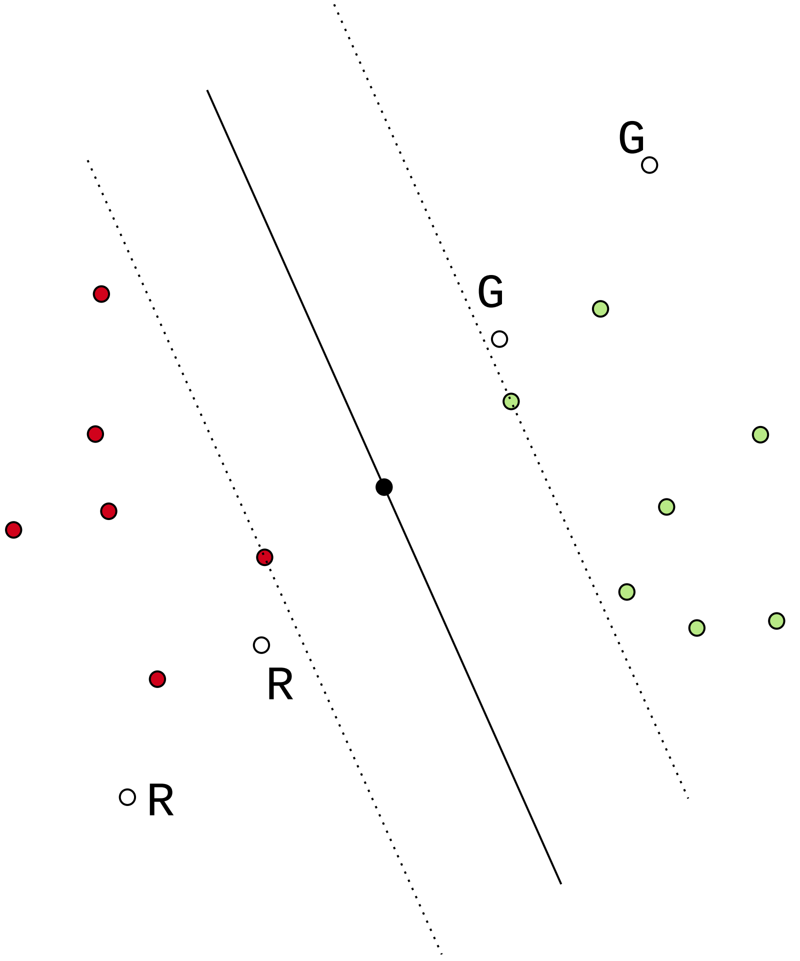
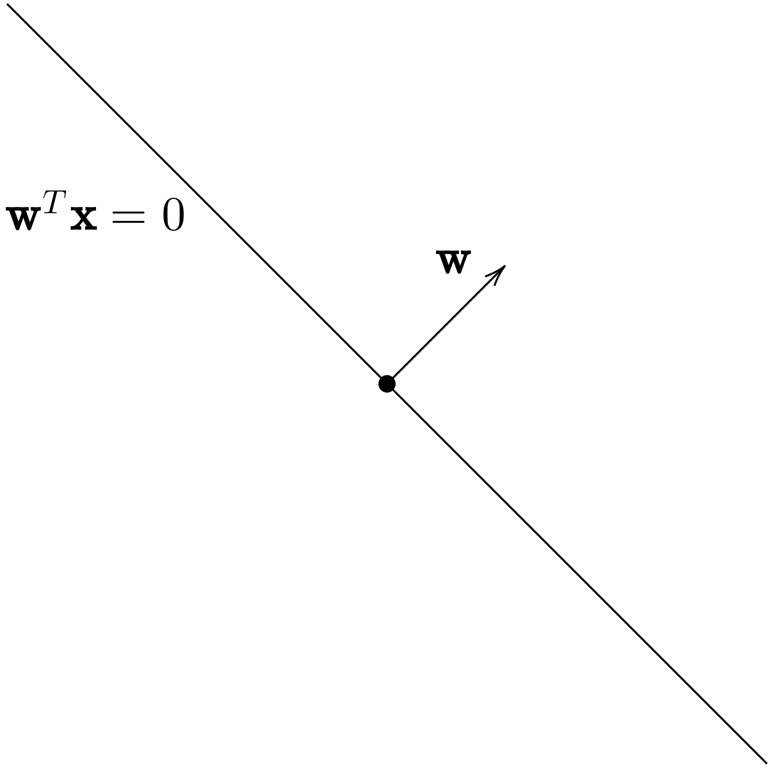


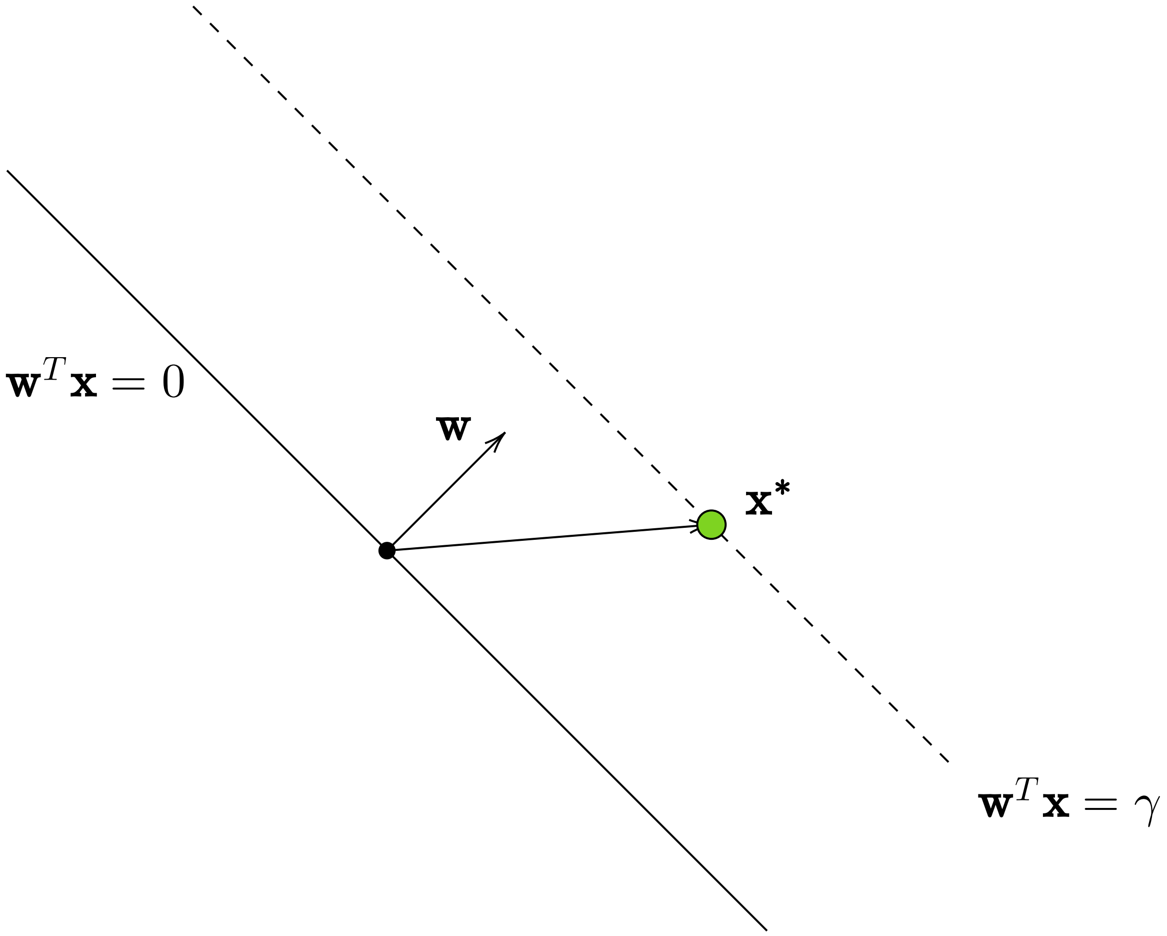
Large margin
Better generalization

Computing the Margin



For any linear classifier represented by $\mathbf{w}$:

Computing the Margin



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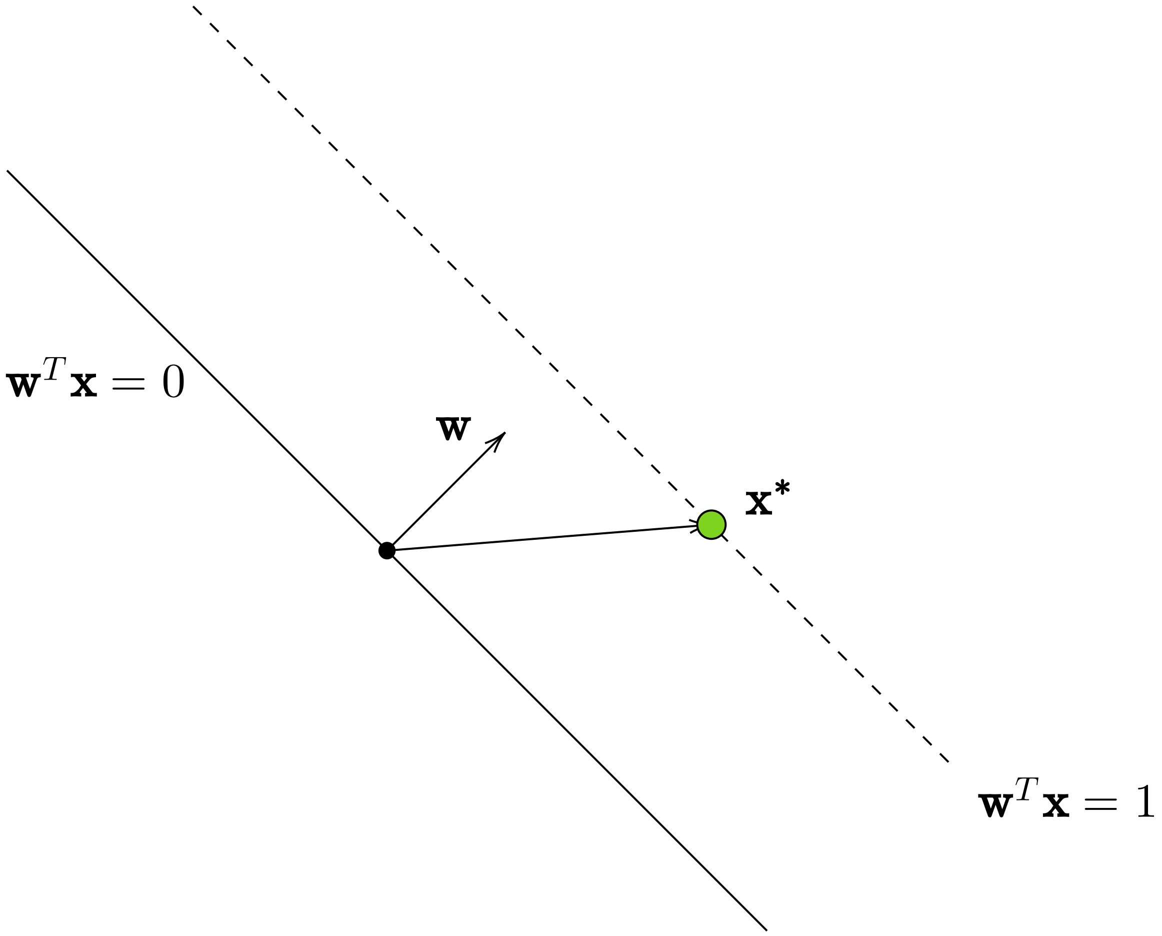

For any linear classifier represented by $\mathbf{w}$:

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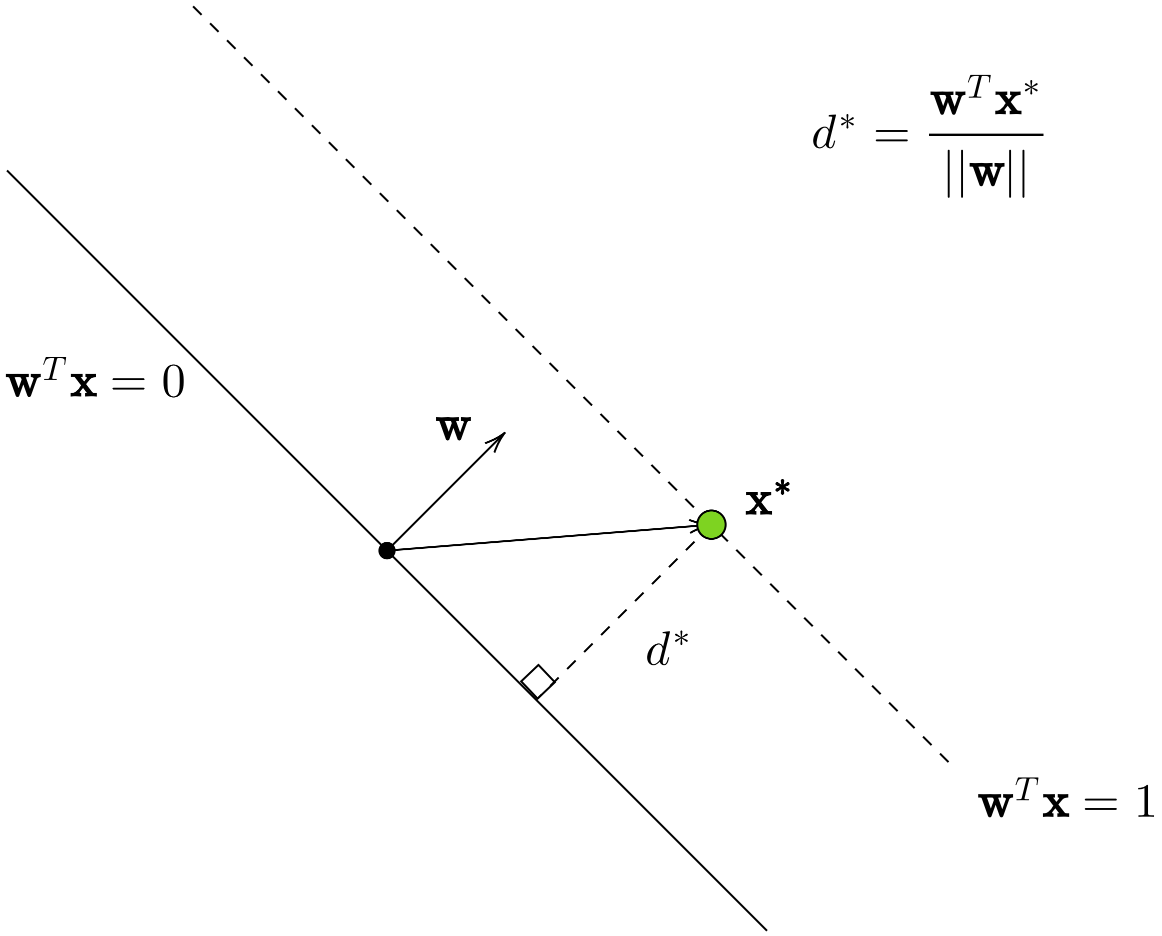

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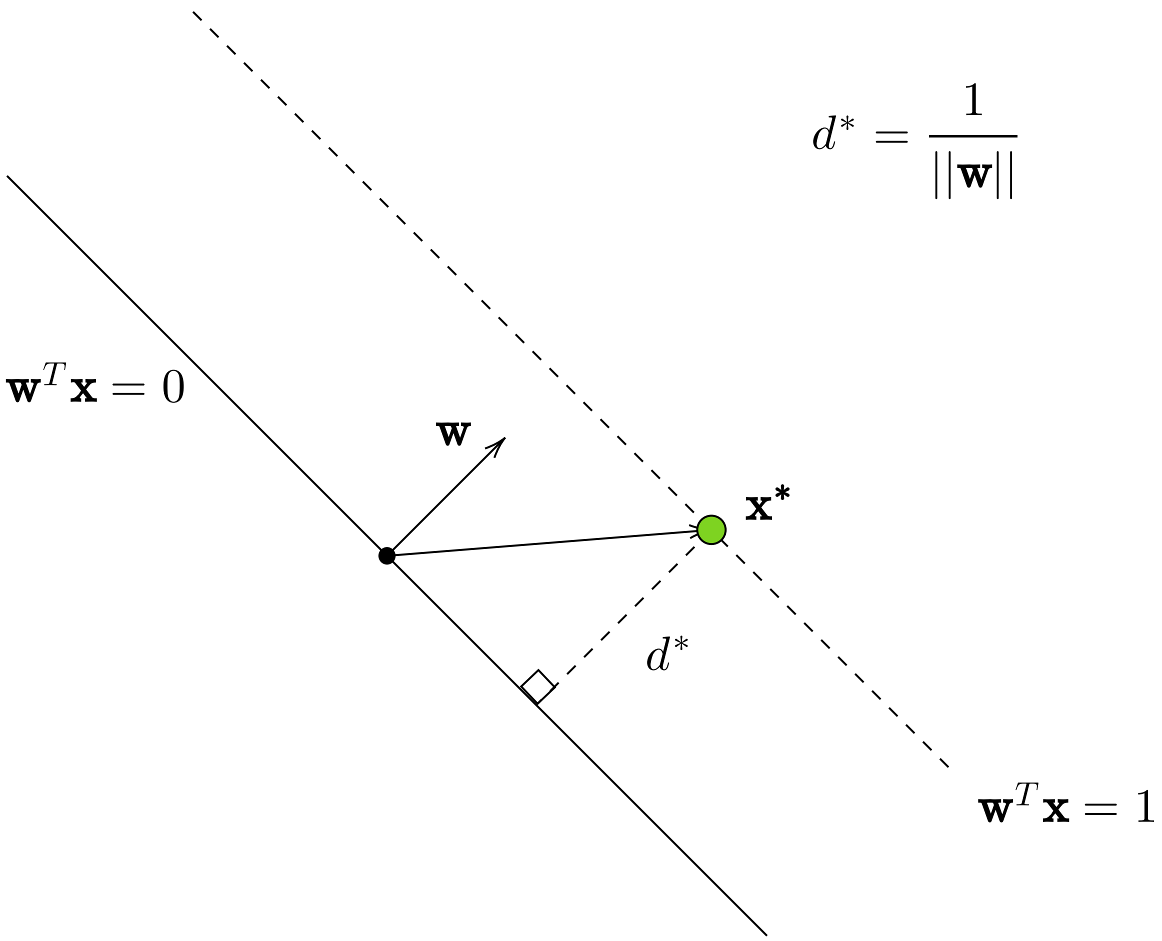

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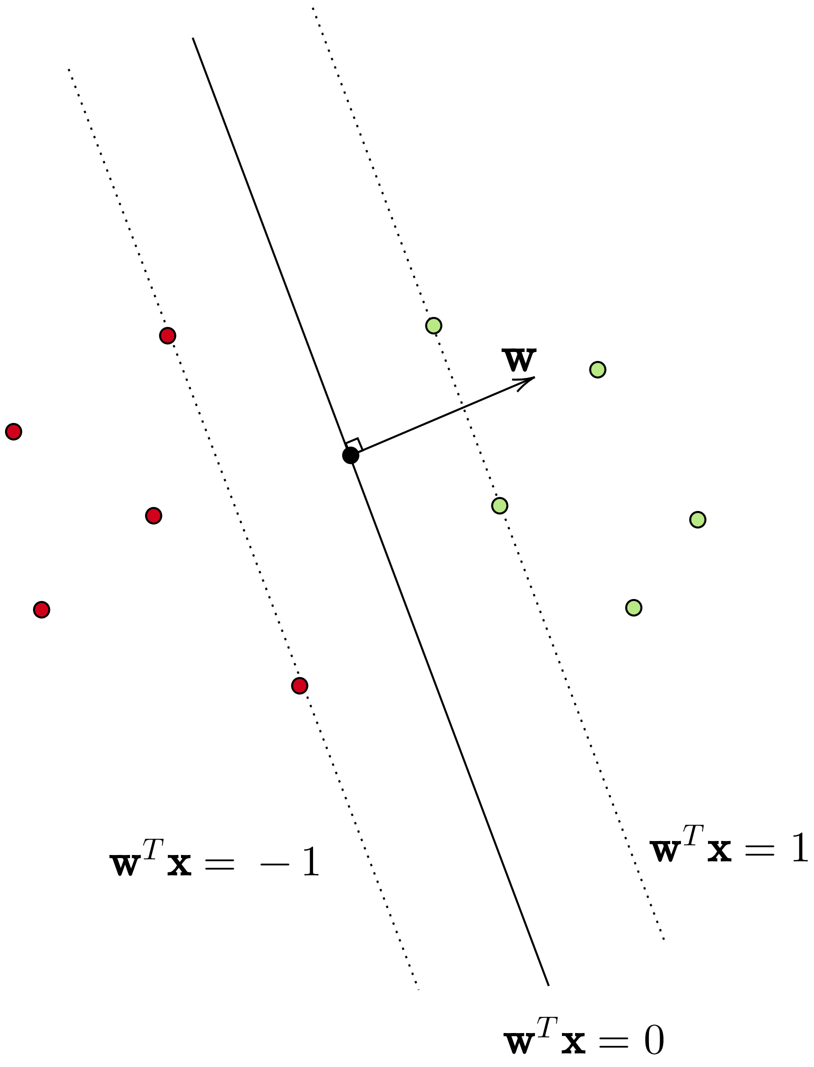
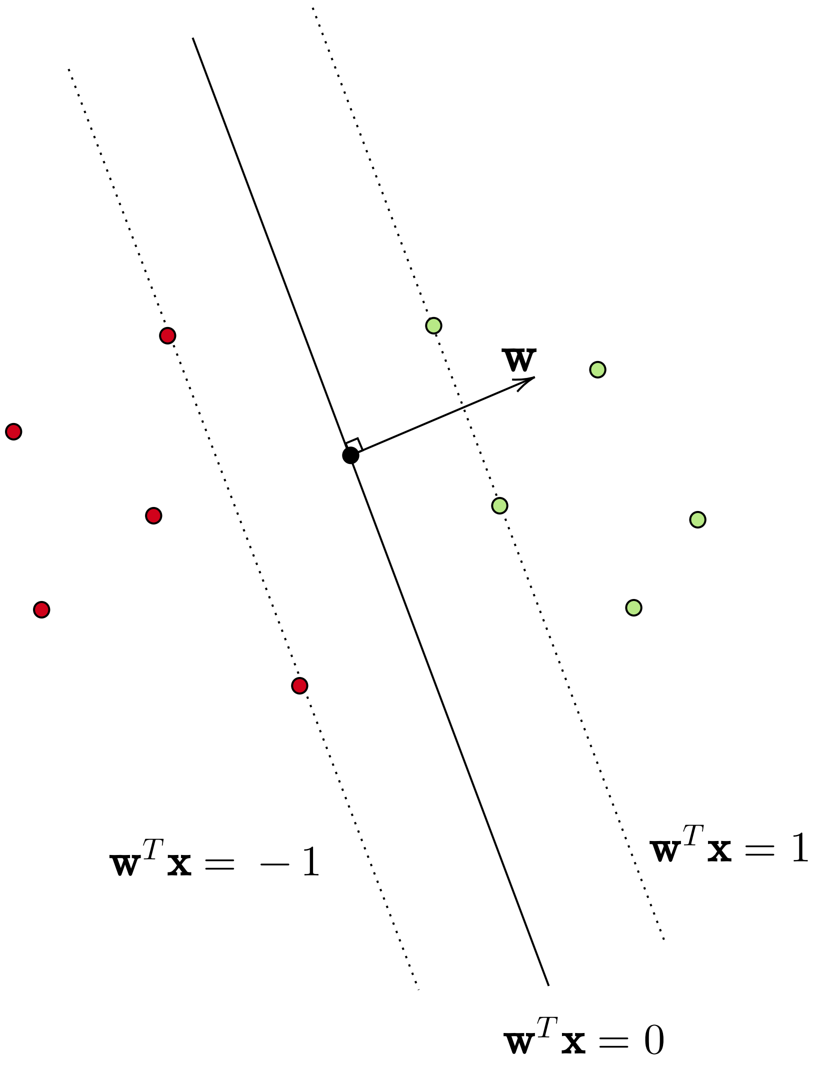
(4) This is the (geometric) margin for this linear classifier.

Beyond the "margin"



$$\left(\mathbf{w}^T\mathbf{x}_i\right)y_i \geqslant 1, \quad 1 \leqslant i \leqslant n$$

Max-Margin Classifier



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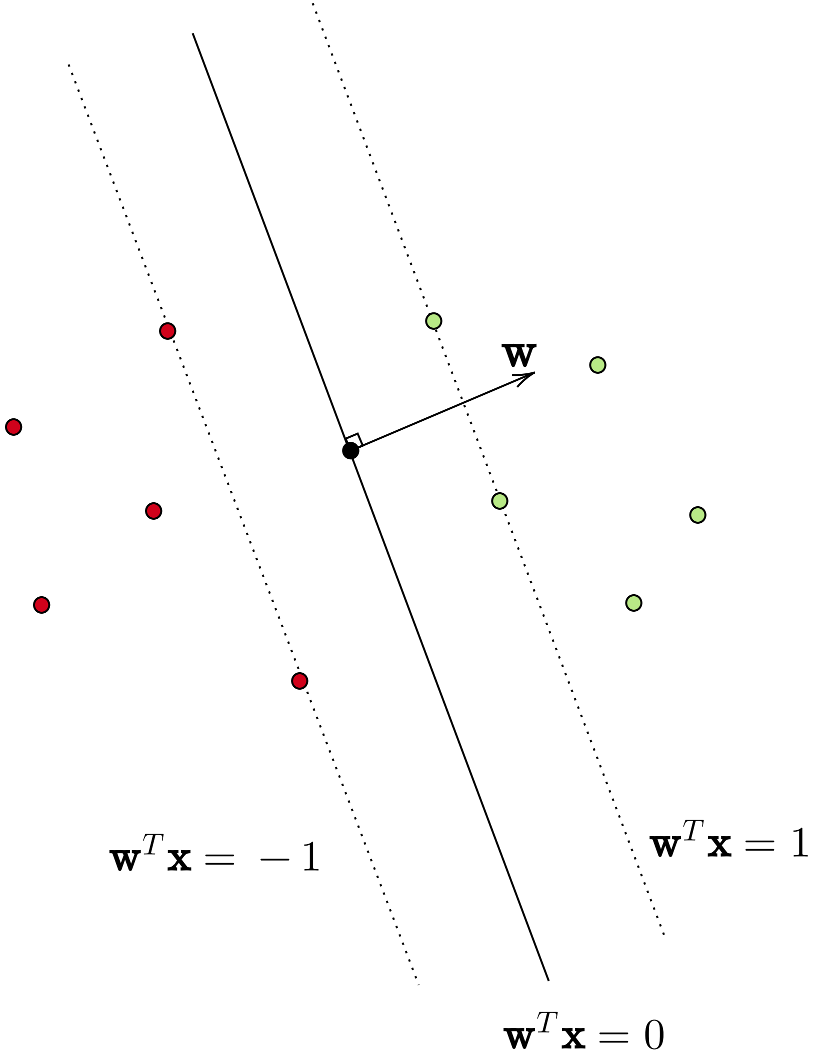

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2}$$

sub. to

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Primal and Dual

$$\min_{\mathbf{w}} \quad f(\mathbf{w})$$

sub. to

$$g(\mathbf{w}) \leqslant 0$$

Primal and Dual

$$\max_{\alpha \geqslant 0} \quad f(\mathbf{w}) + \alpha g(\mathbf{w}) = \begin{cases} f(\mathbf{w}), & g(\mathbf{w}) \leqslant 0 \\ \infty, & g(\mathbf{w}) > 0 \end{cases}$$

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$$d \times n$$

$$\mathbf{X} = \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix}$$

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$$n \times n$$

$$\mathbf{Y} = \begin{bmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{bmatrix}$$

$$\mathbf{X}\mathbf{Y}\boldsymbol{\alpha} = \begin{bmatrix} | & & | \\ y_1\mathbf{x}_1 & \cdots & y_n\mathbf{x}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i (y_i \mathbf{x}_i) = \mathbf{w}$$

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Advantages of the Dual

$$\boxed{\max_{\boldsymbol{\alpha} \geqslant 0} \ \boldsymbol{\alpha}^T \mathbf{1} - \frac{\boldsymbol{\alpha}^T \left(\mathbf{Y}^T \mathbf{X}^T \mathbf{X} \mathbf{Y} \right) \boldsymbol{\alpha}}{2}}$$

(1) $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^d$, if $n \ll d$, we are solving for fewer variables in the dual

(2) Simpler constraints, just bounds.

(3) The appearance of $\mathbf{X}^T \mathbf{X} \implies$ kernels.

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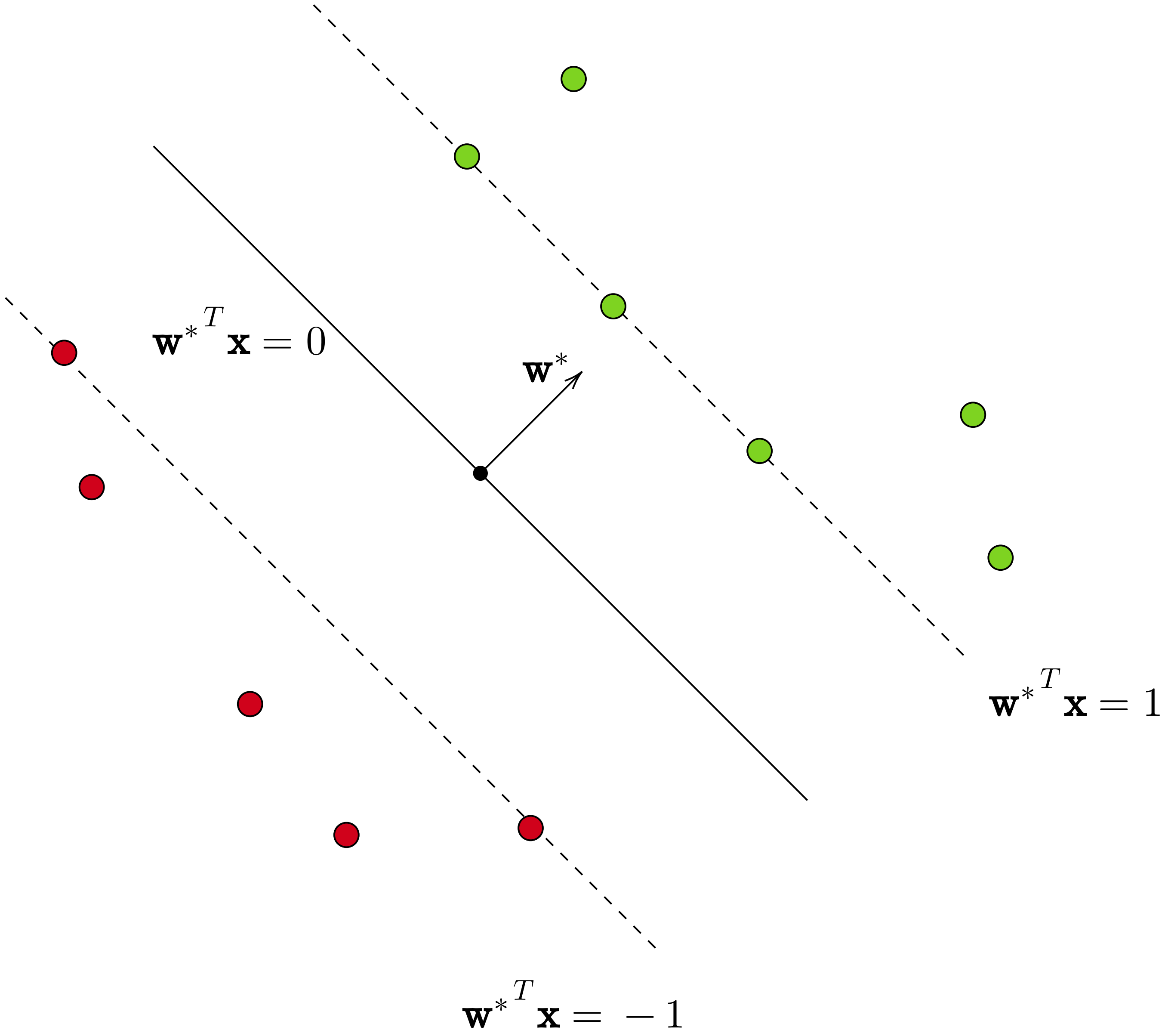

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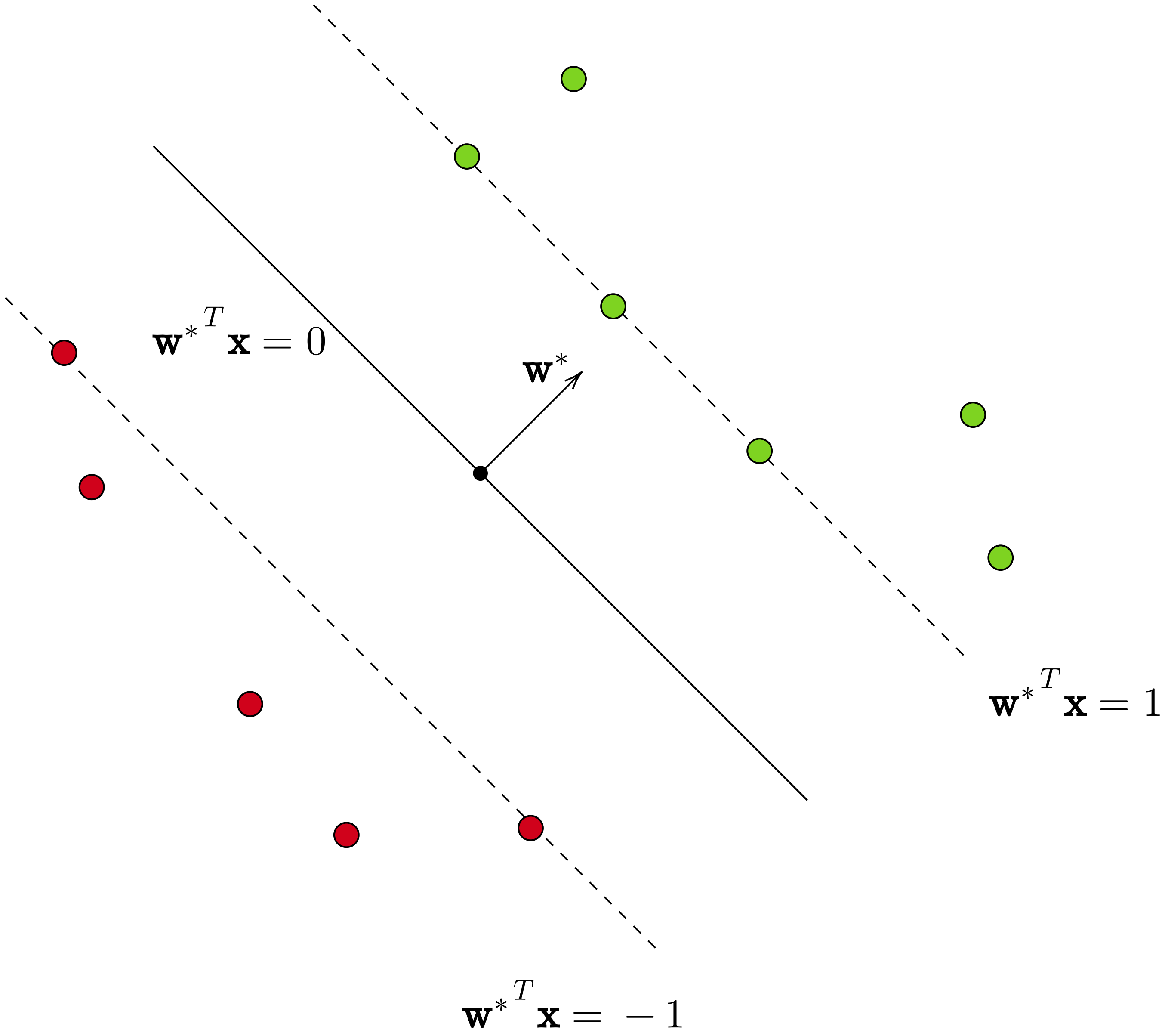

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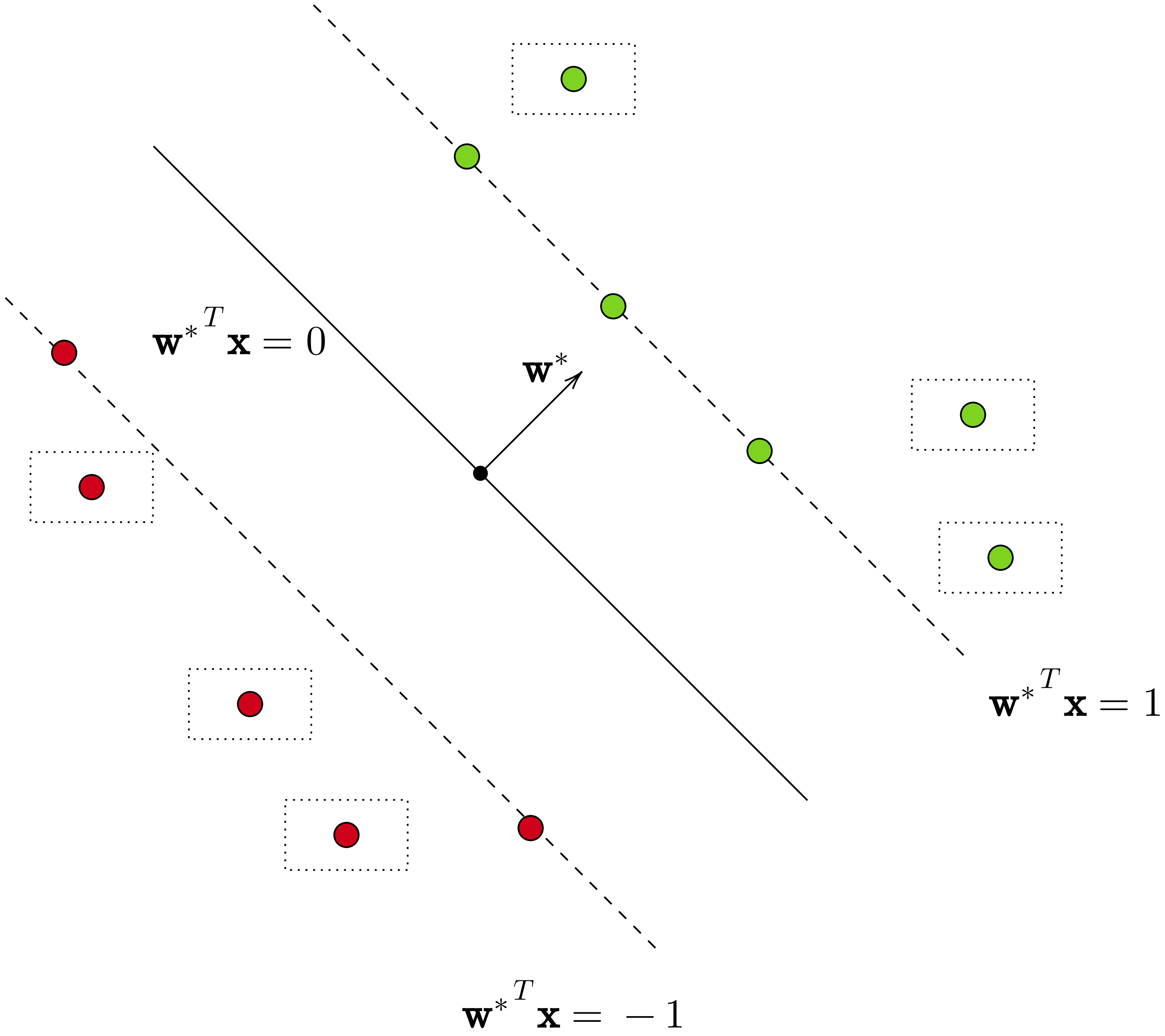
$$\alpha_i^* \left[1 - \left({\mathbf{w}^*}^T \mathbf{x}_i \right) y_i \right] = 0$$

$$y_i \left({\mathbf{w}^*}^T \mathbf{x}_i \right) \neq 1$$

$$\alpha_i^* = 0$$



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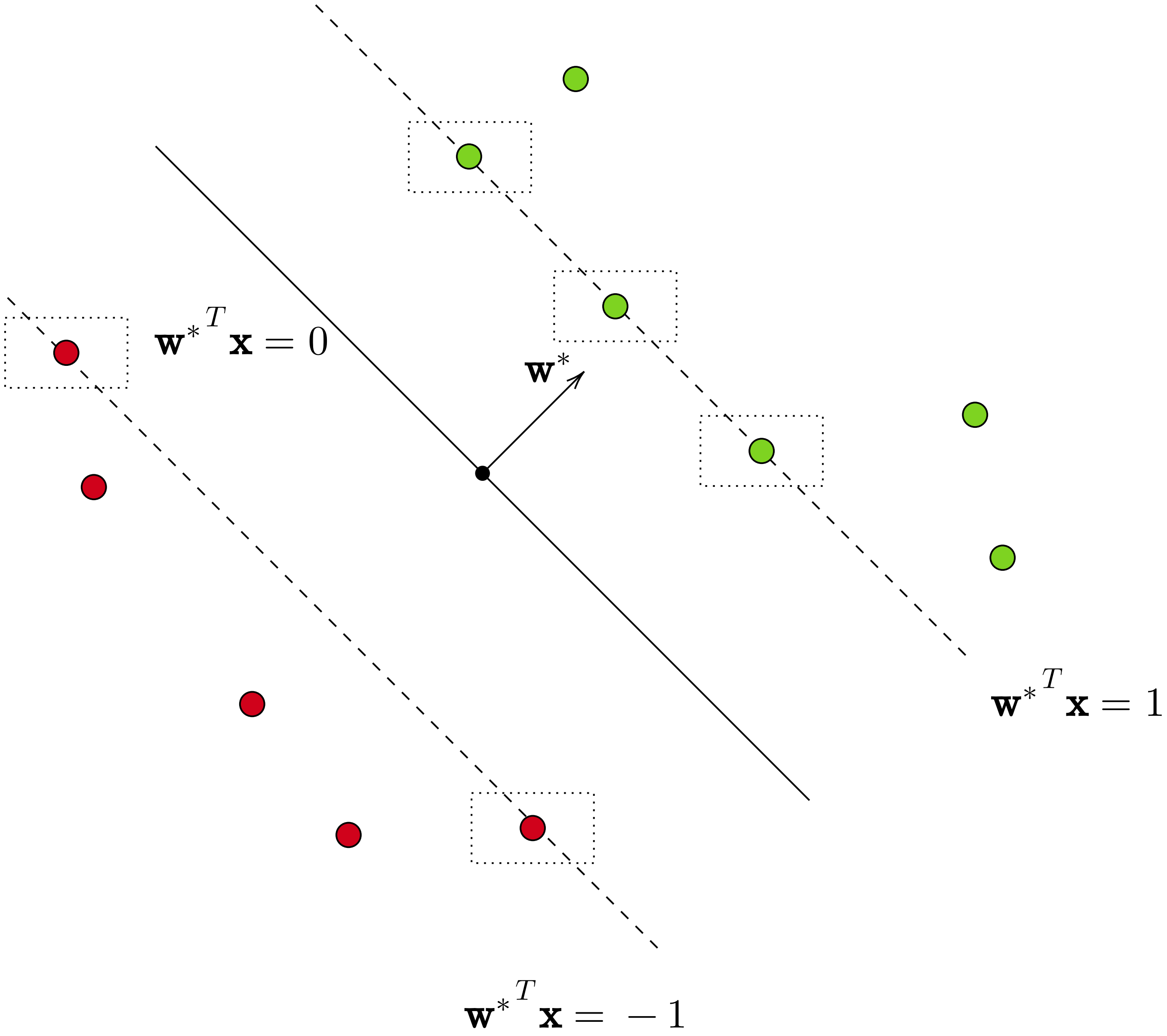
$$\begin{tikzcd}
& \alpha_i^*\Big[1 - \big(\mathbf{w}^{*T}\mathbf{x}_i\big) y_i\Big] = 0 \arrow[dl, "y_i\left(\mathbf{w}^{*T}\mathbf{x}_i\right) \neq 1"'] \arrow[dr, "y_i\left(\mathbf{w}^{*T}\mathbf{x}_i\right) = 1"] & \\
\alpha_i^* = 0 & & \alpha_i^* \geqslant 0
\end{tikzcd}$$



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$$\alpha_i^* > 0$$

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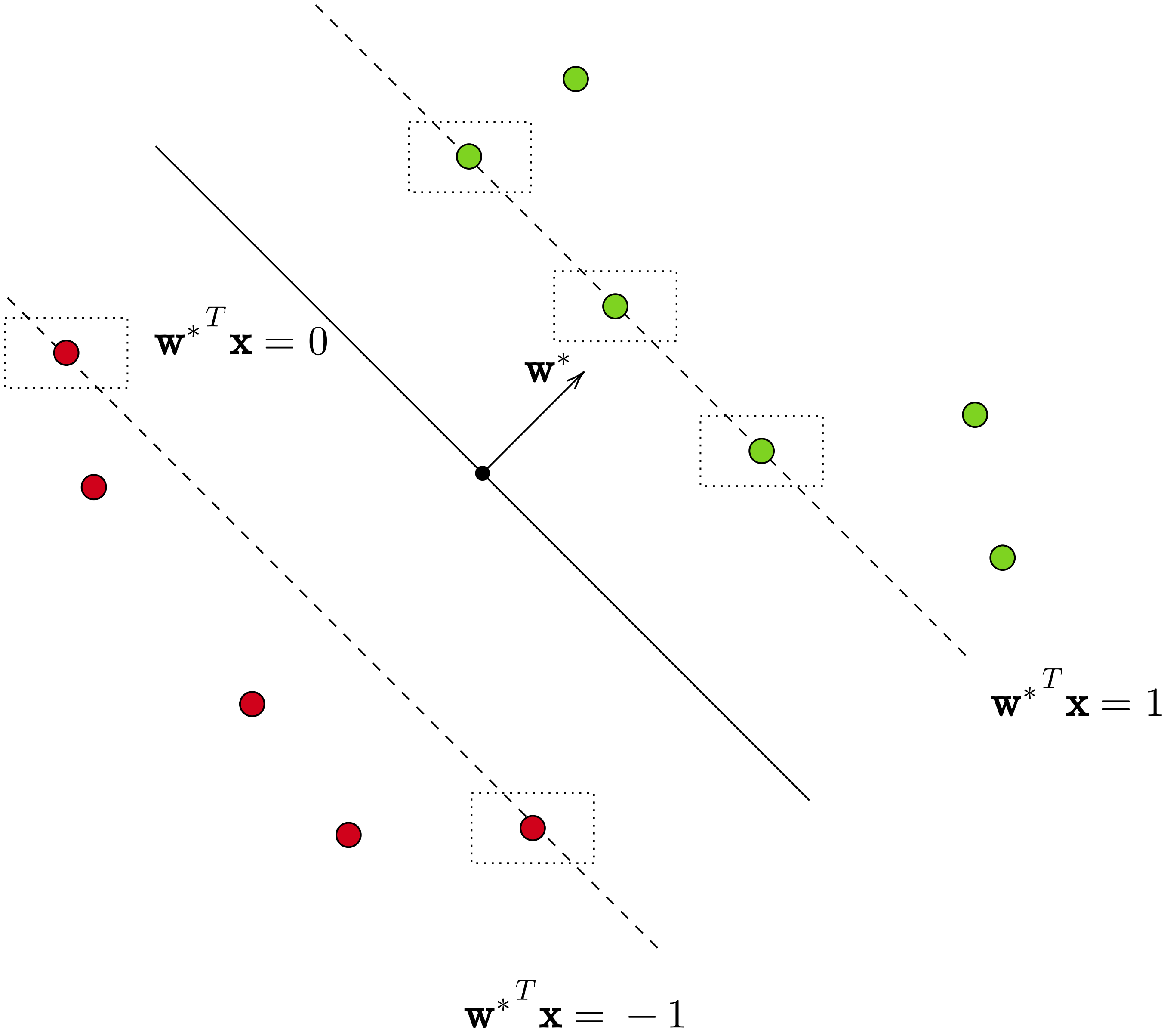

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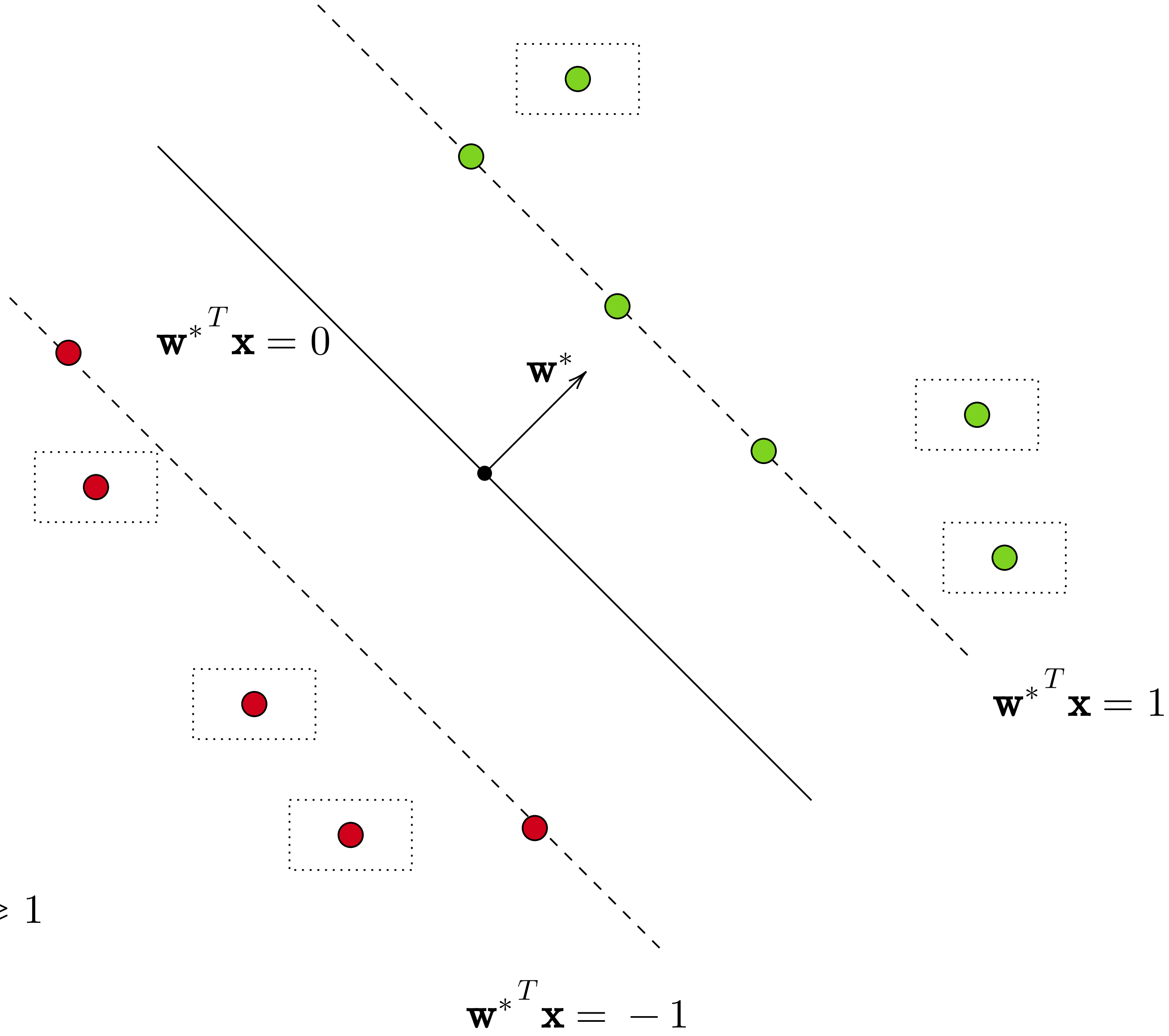
$$\alpha_i^* > 0 \implies y_i \left({\mathbf{w}^*}^T \mathbf{x}_i \right) = 1$$

$$\alpha_i^* = 0 \implies y_i \left({\mathbf{w}^*}^T \mathbf{x}_i \right) \geqslant 1$$



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Complementary Slackness

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Every support vector lies on one of the two supporting hyperplanes $(\mathbf{w}^*)^T \mathbf{x} = \pm 1$

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Every point that is **not** on one of the two supporting hyperplanes has $\alpha_i^* = 0$.

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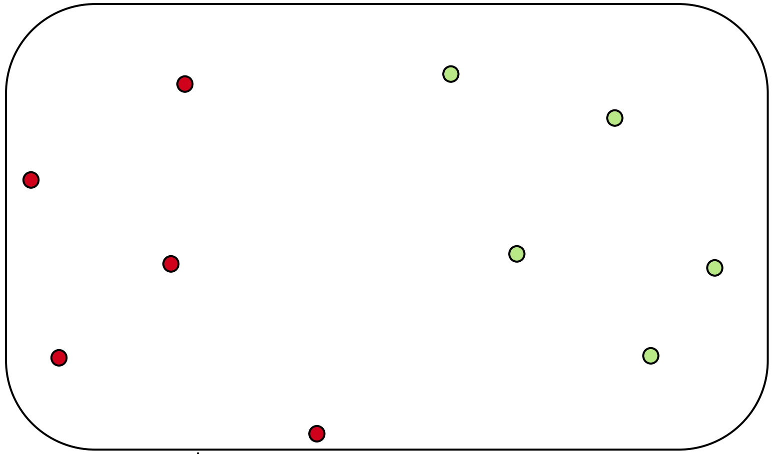
Every support vector lies on one of the two supporting hyperplanes $\left(\mathbf{w}^*\right)^T \mathbf{x} = \pm 1$

Is every point on one of the two supporting hyperplanes a support vector?

Every point that is **not** on one of the two supporting hyperplanes has $\alpha_i^* = 0$.

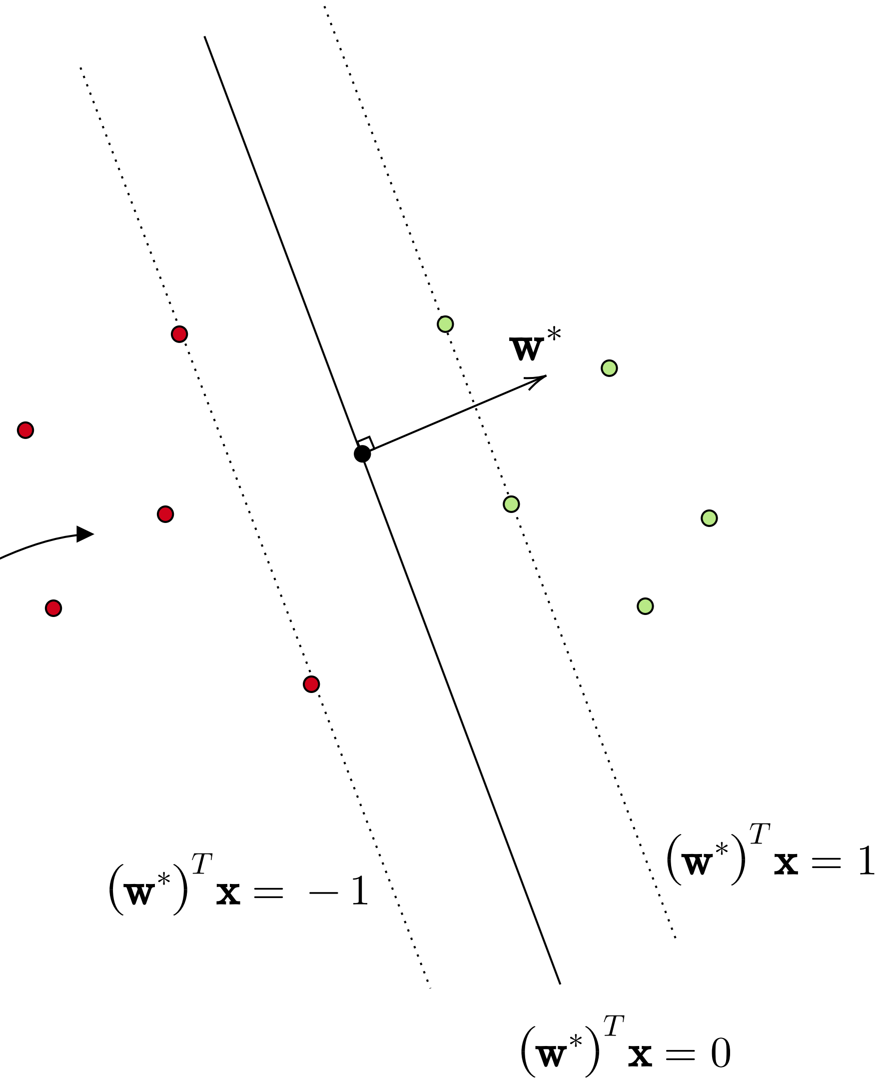
Hard-Margin, Linear-SVM



$$\widehat{y} = \begin{cases} 1, & (\mathbf{w}^*)^T \mathbf{x} \geqslant 0 \\ -1, & (\mathbf{w}^*)^T \mathbf{x} < 0 \end{cases}$$

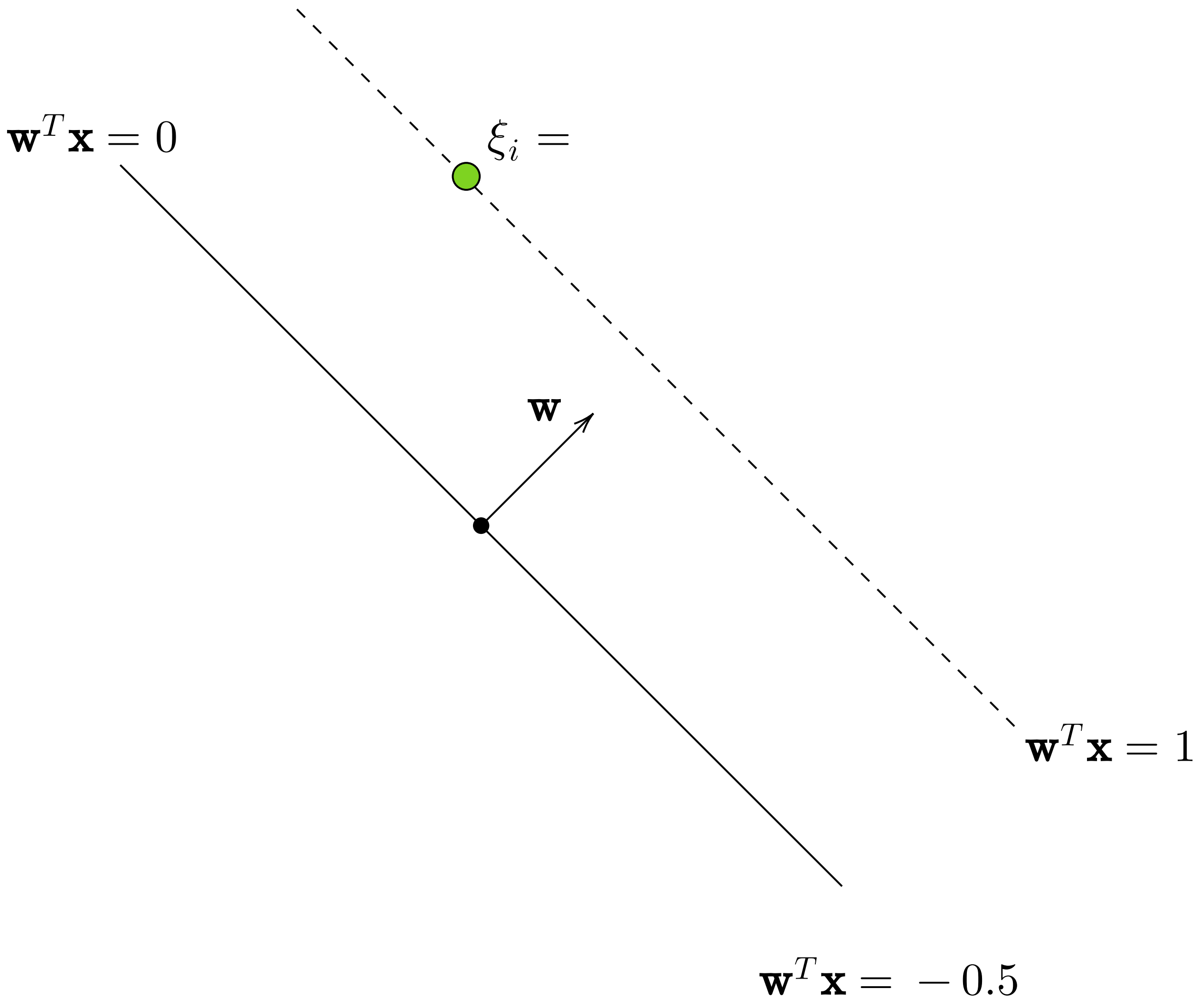
SVM Solver

$$\boldsymbol{\alpha}^* = \begin{bmatrix} \alpha_1^* \\ \vdots \\ \alpha_n^* \end{bmatrix}$$



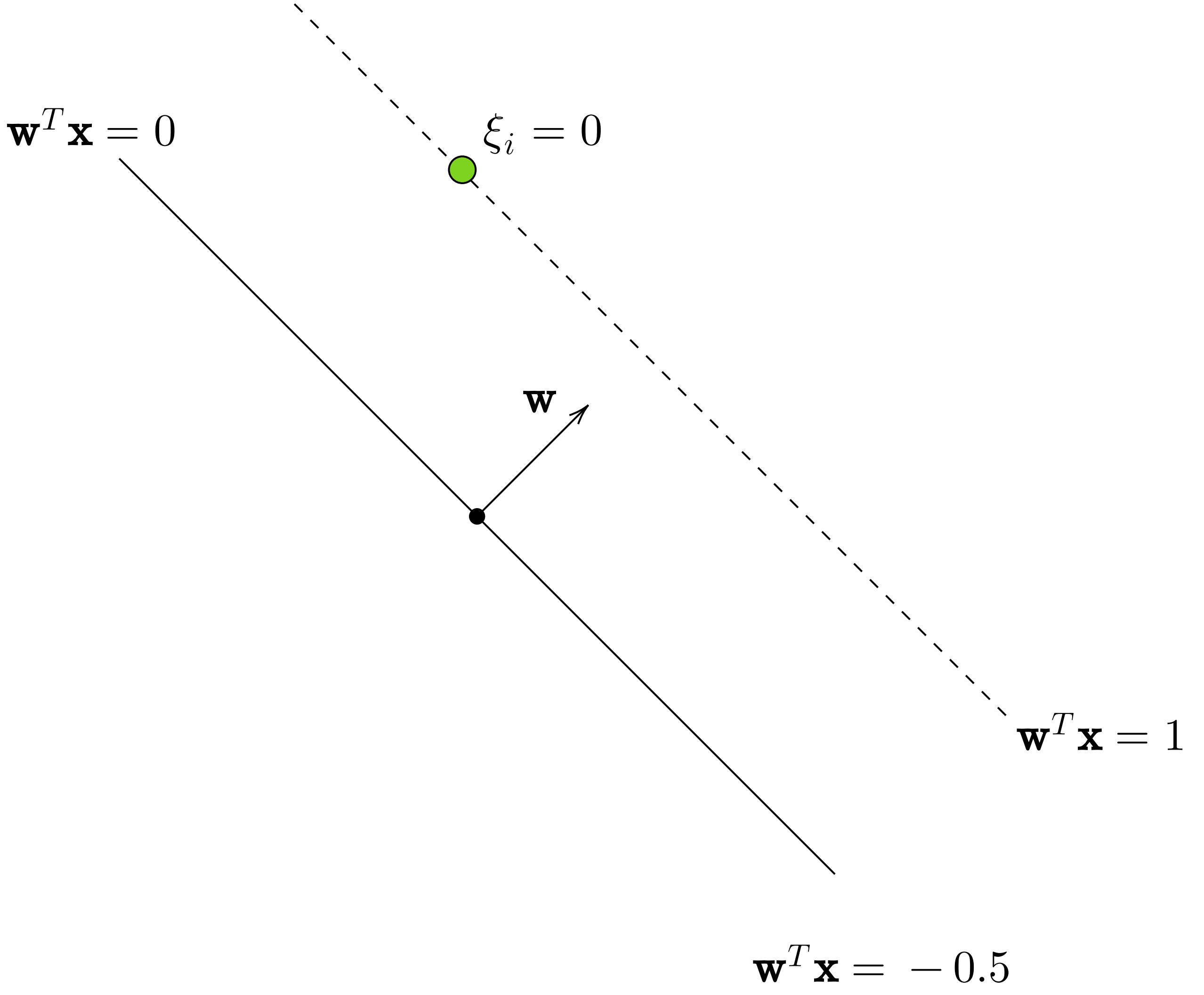
$$\mathbf{w}^* = \sum_{i=1}^{n} \alpha_i^* \mathbf{x}_i y_i$$

Soft-Margin



$$\left(\mathbf{w}^T\mathbf{x}_i\right)y_i + \xi_i \geqslant 1$$
$$\xi_i \geqslant 0$$

Soft-Margin



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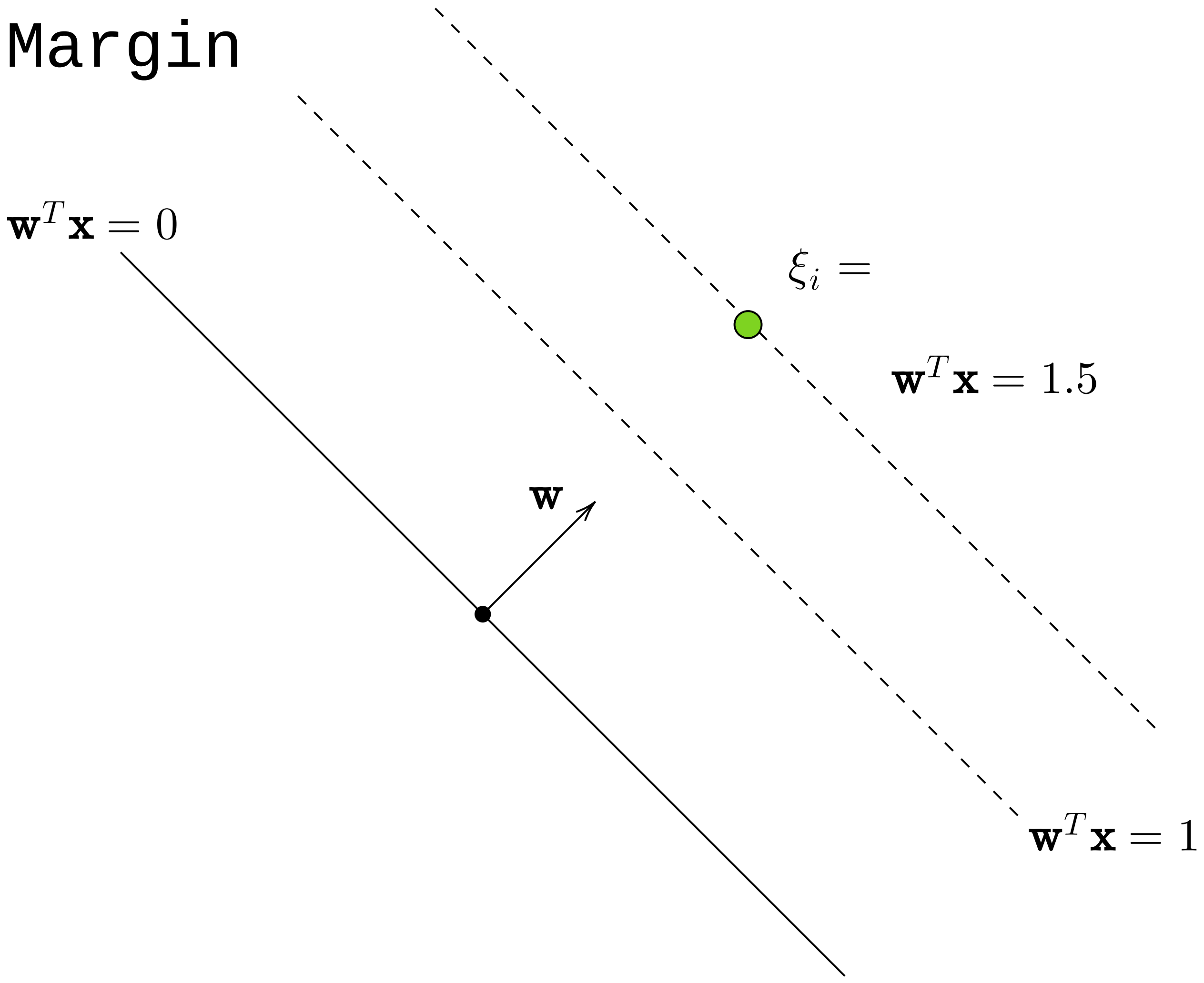

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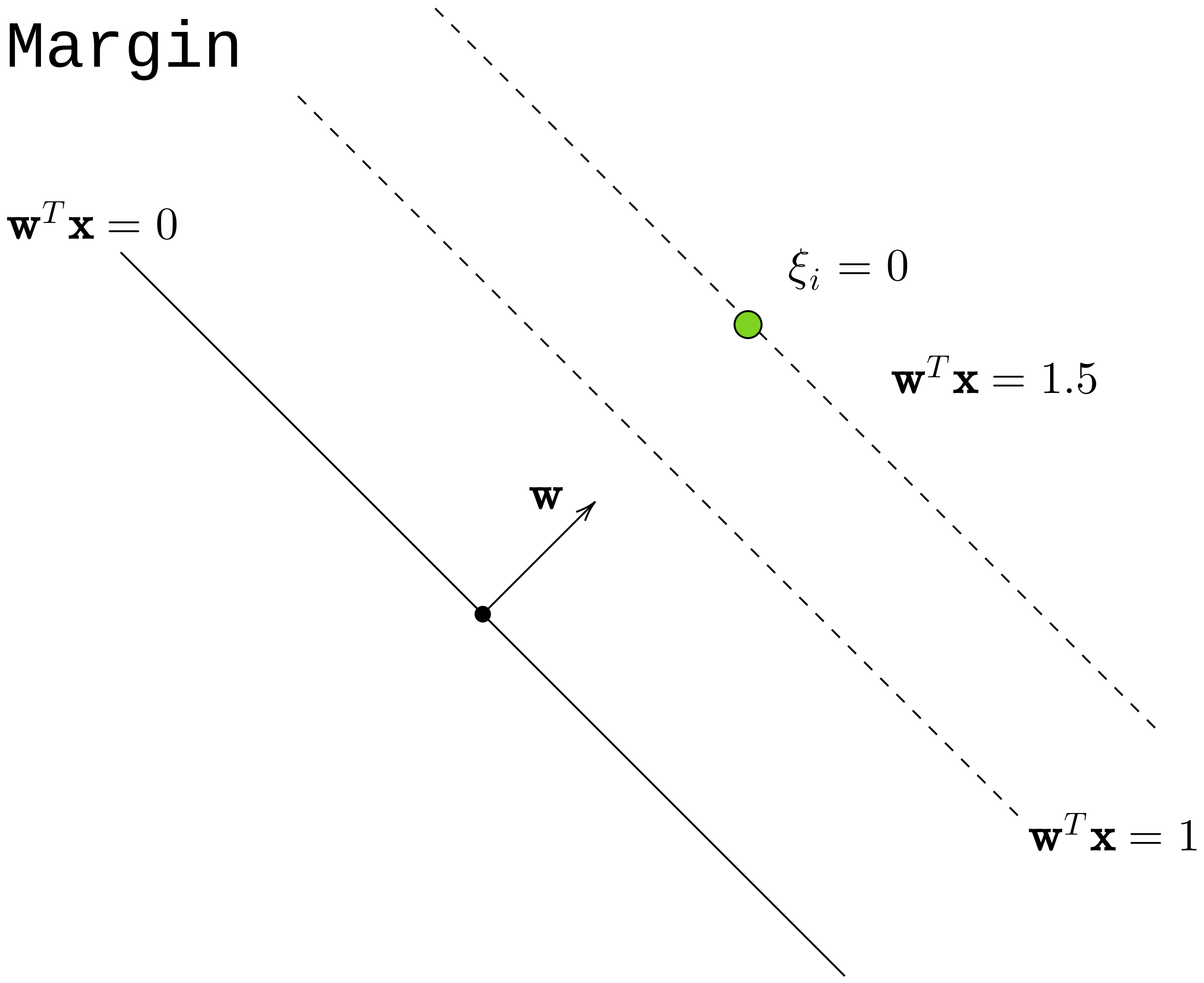

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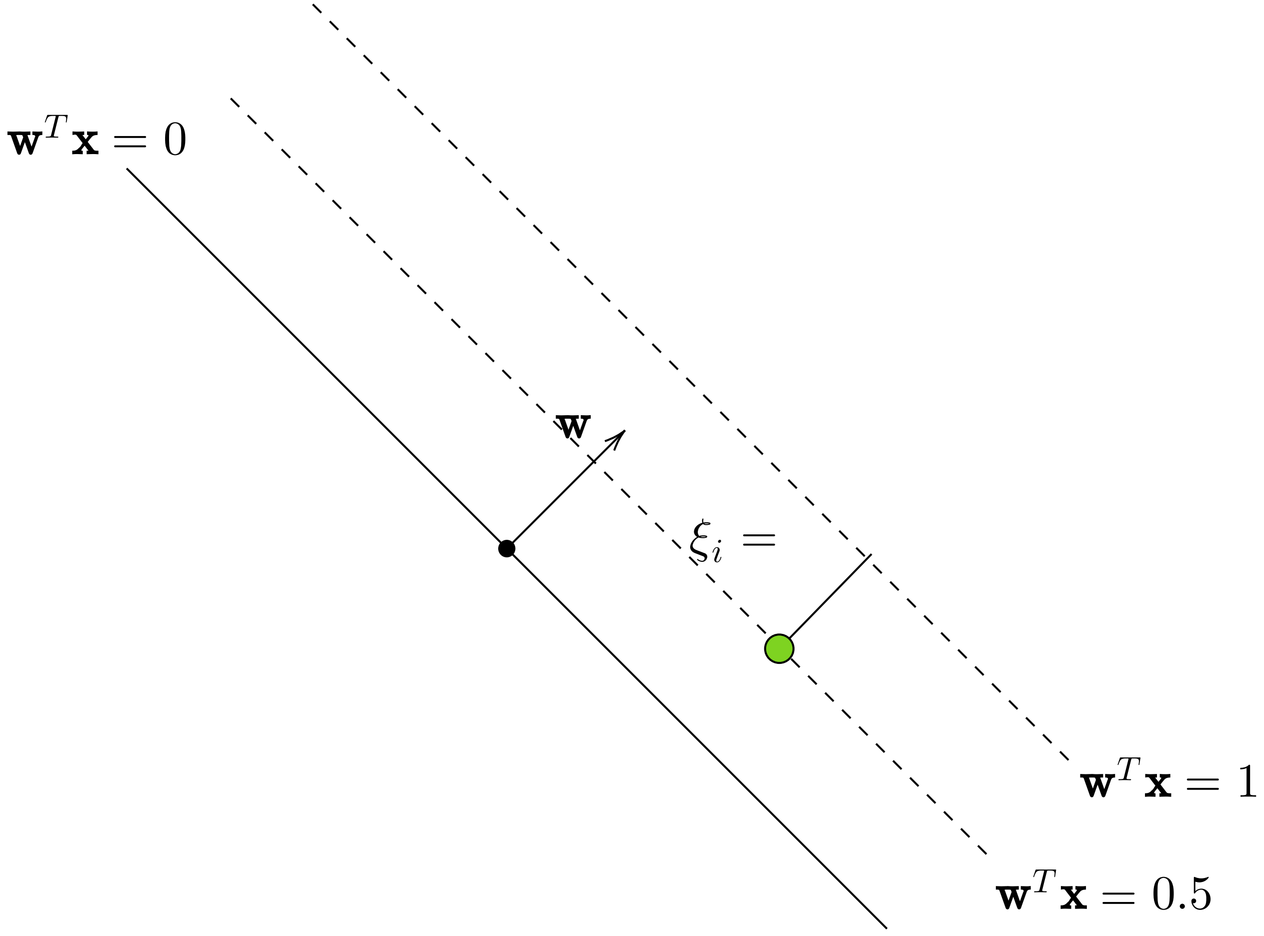

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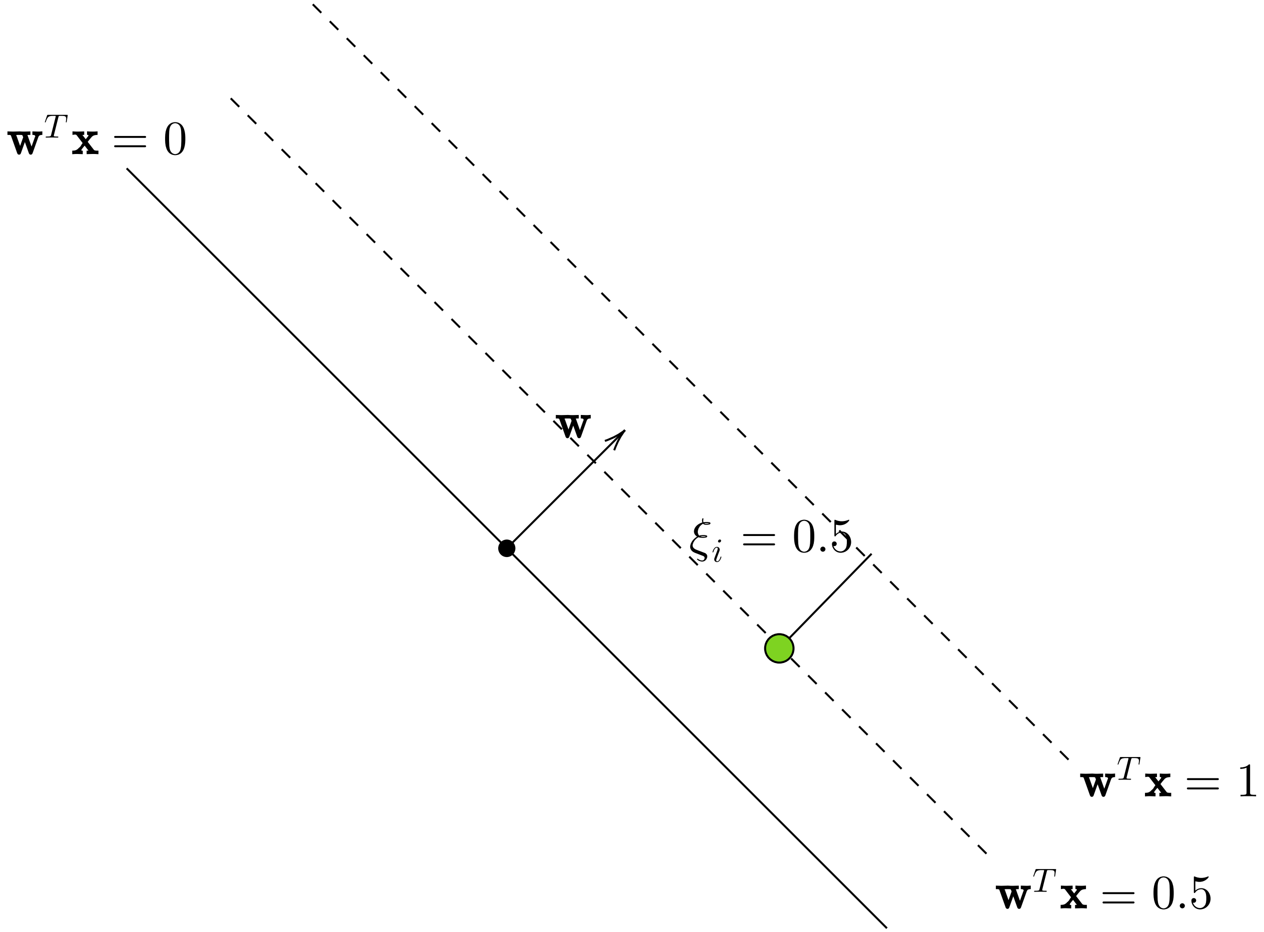

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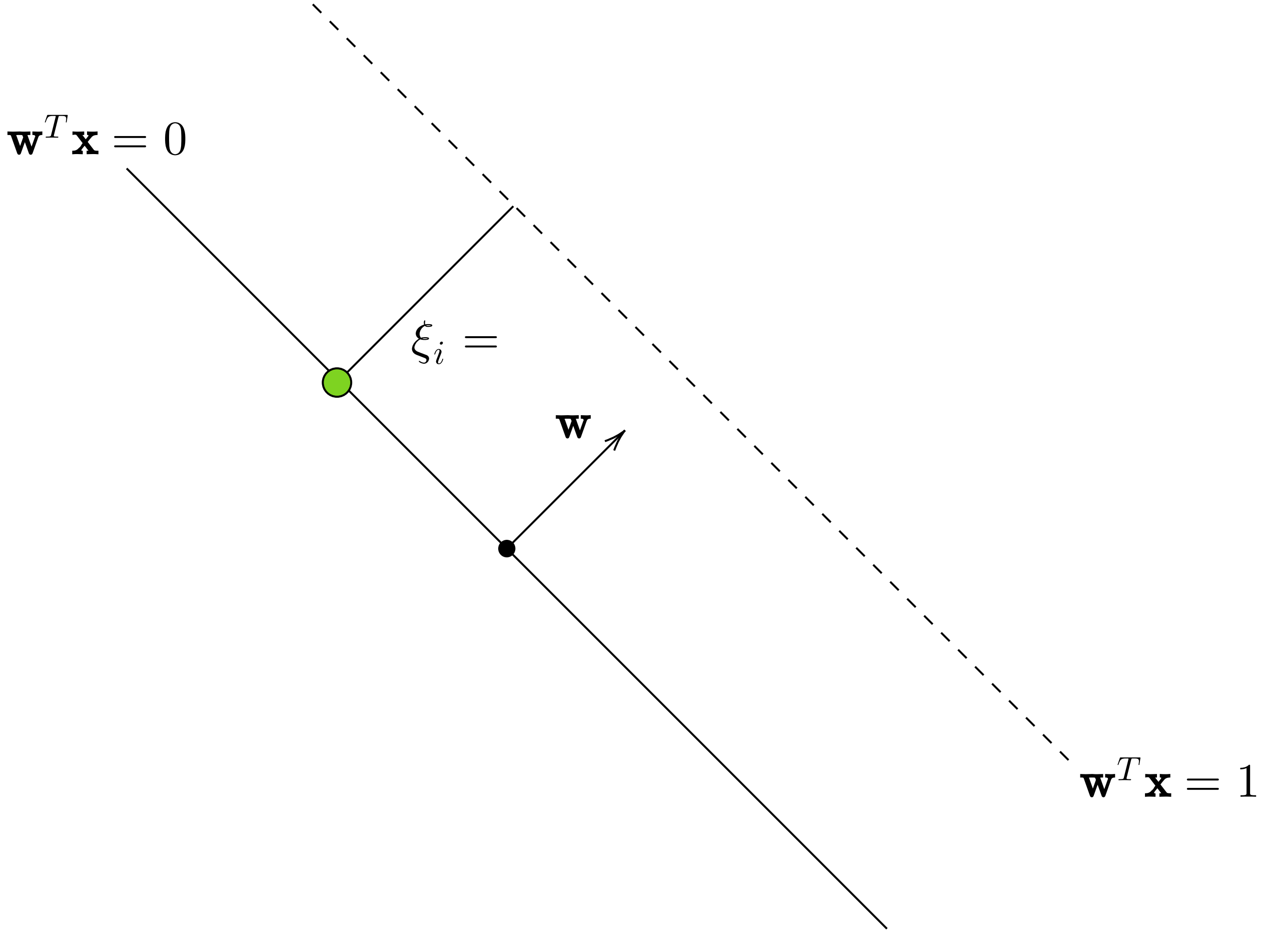

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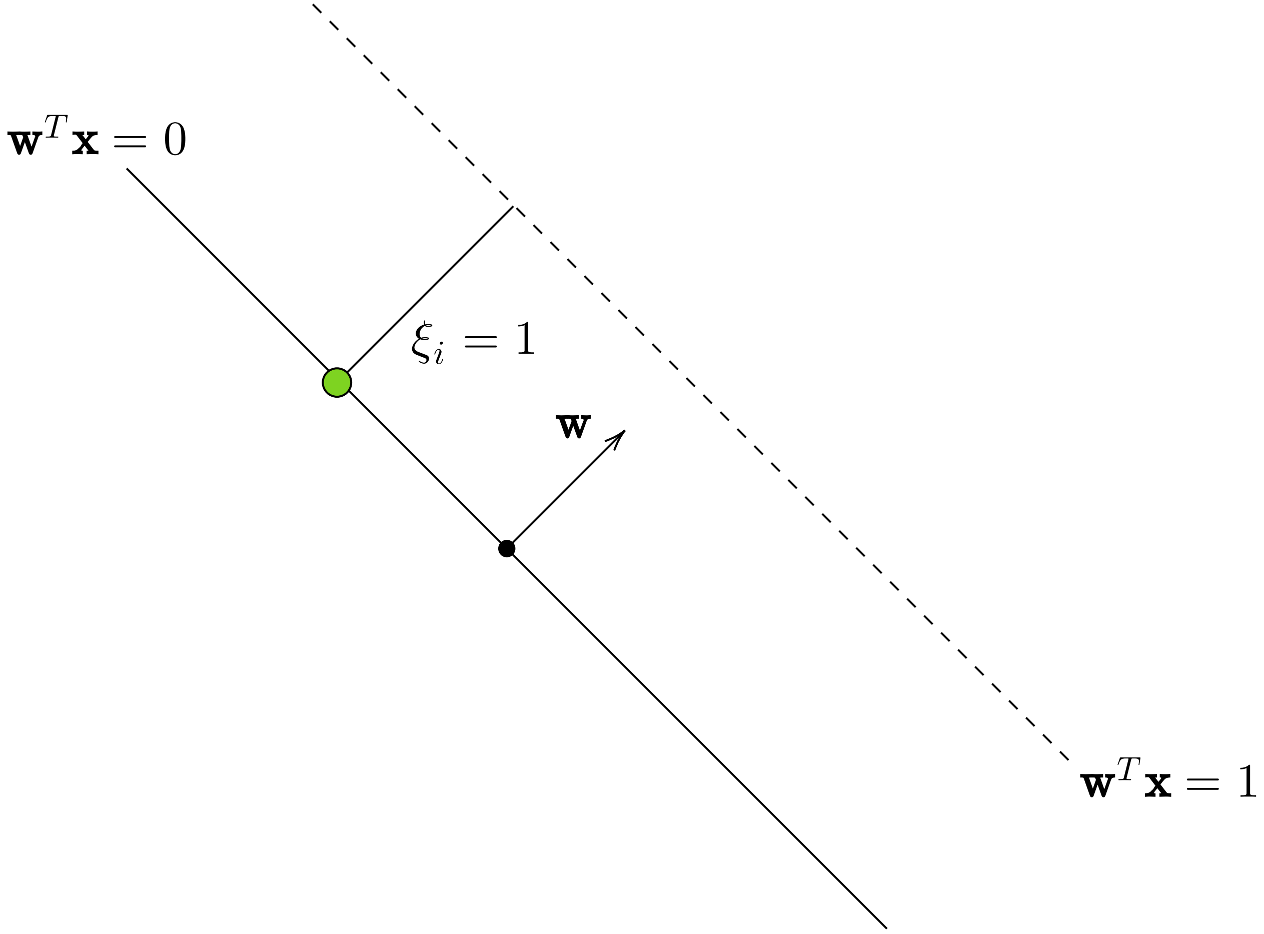

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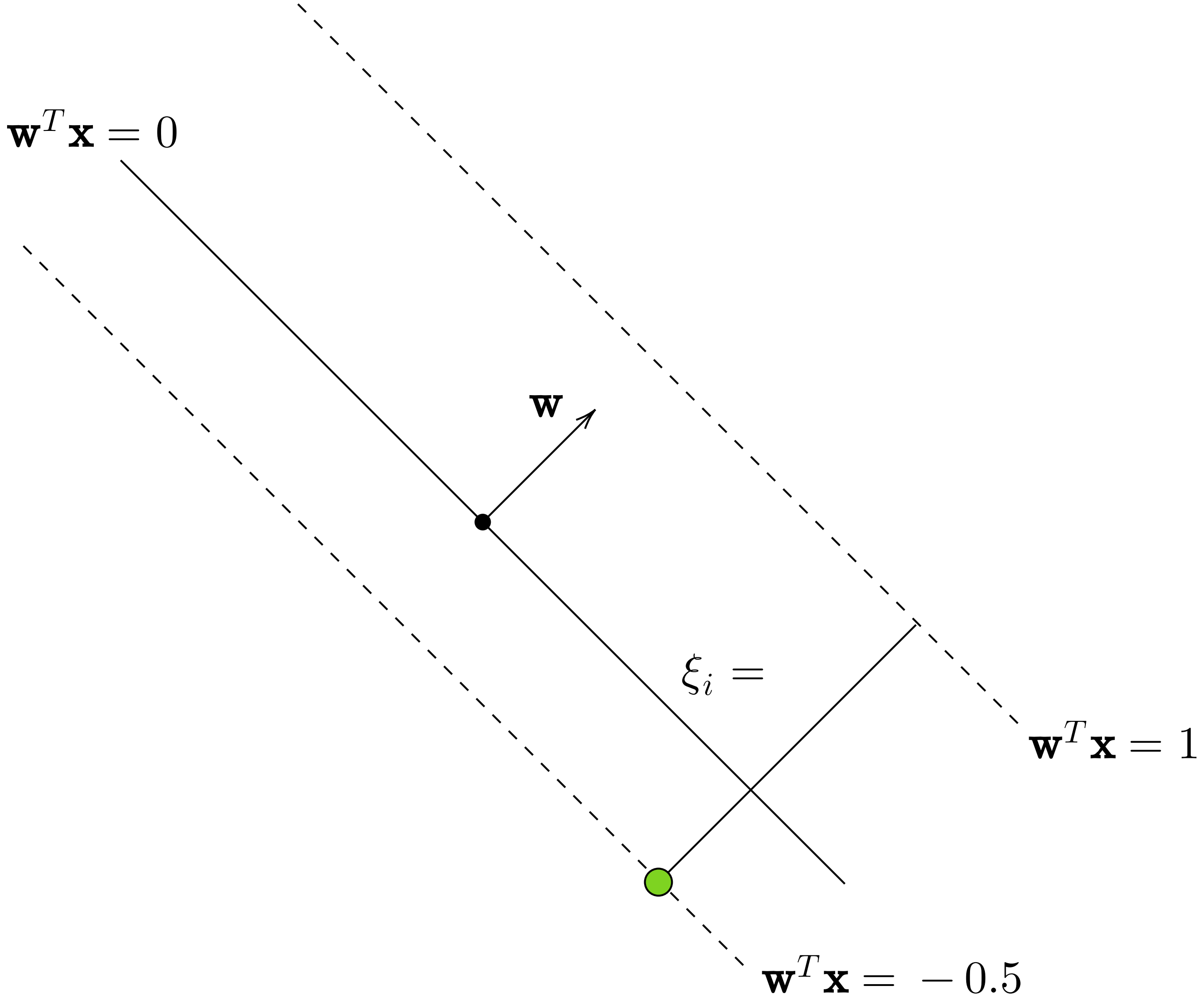

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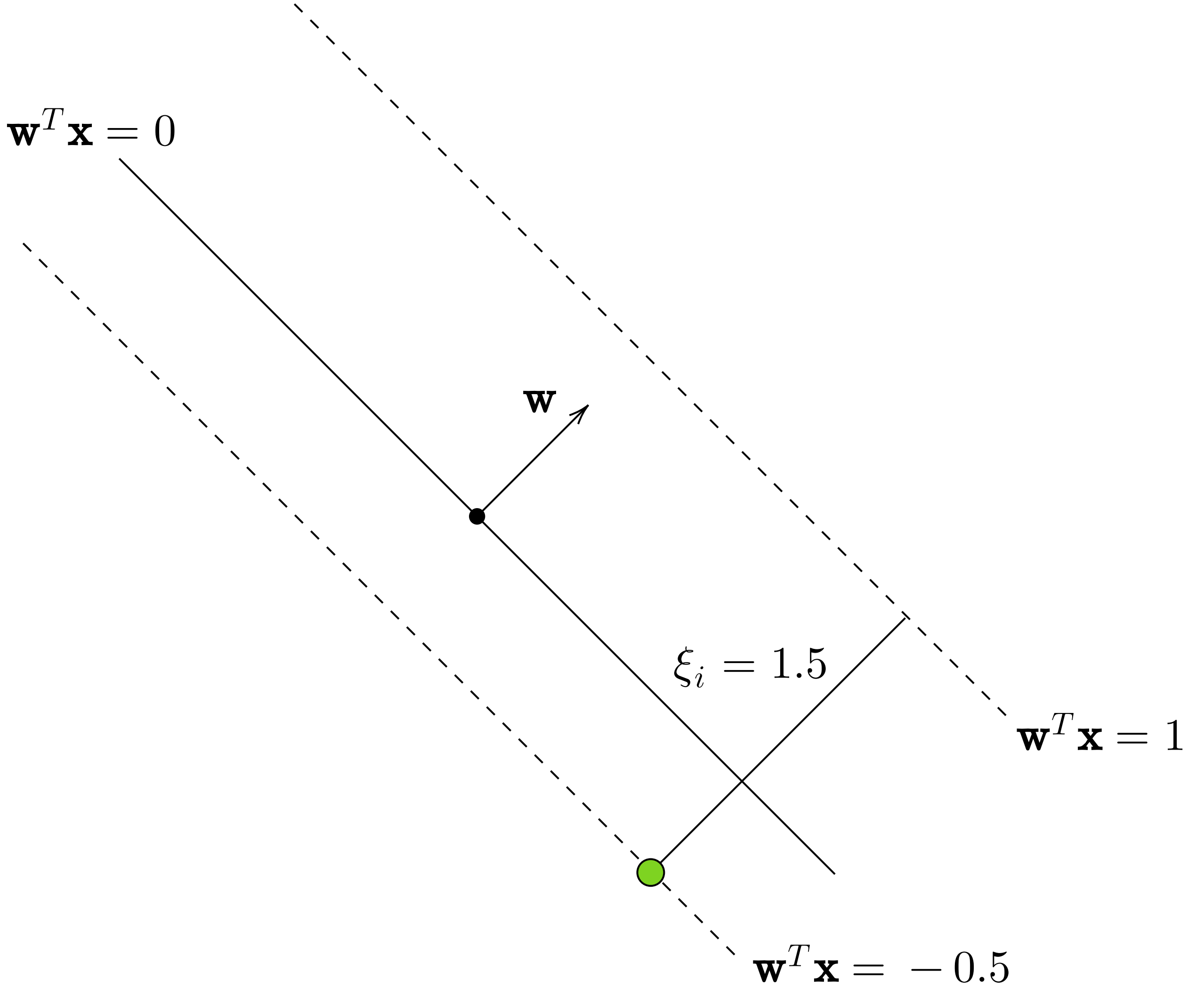

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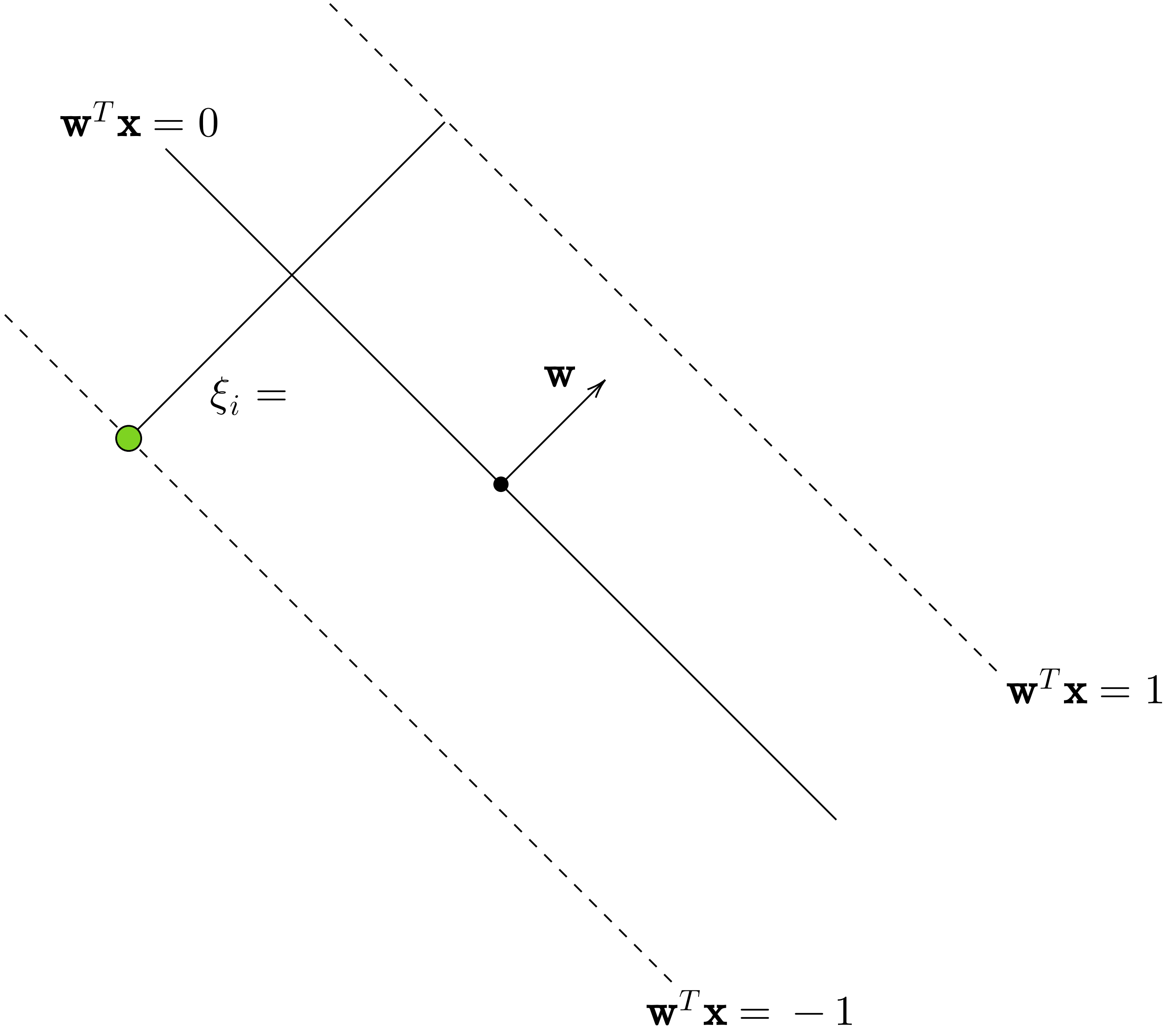

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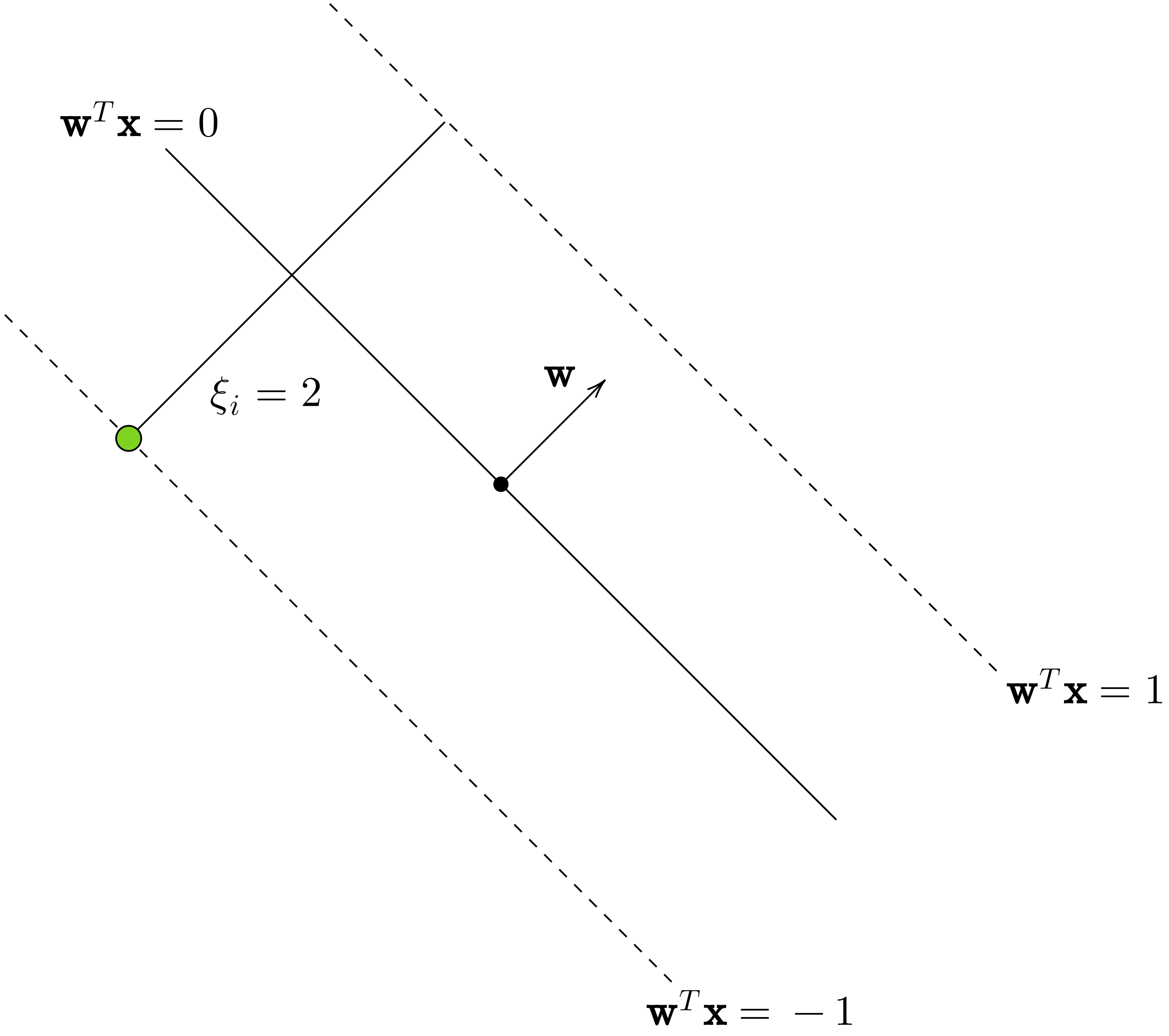

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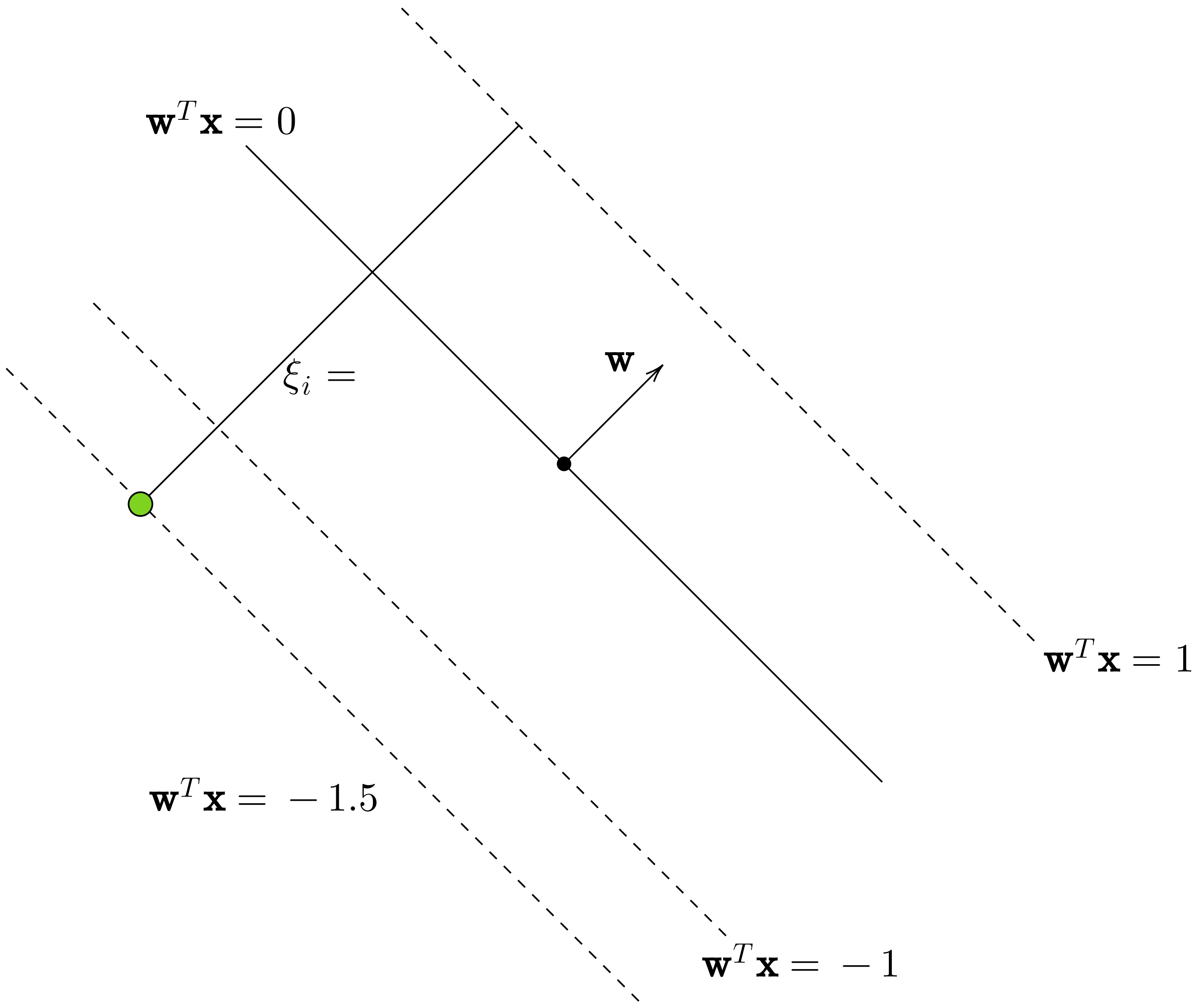

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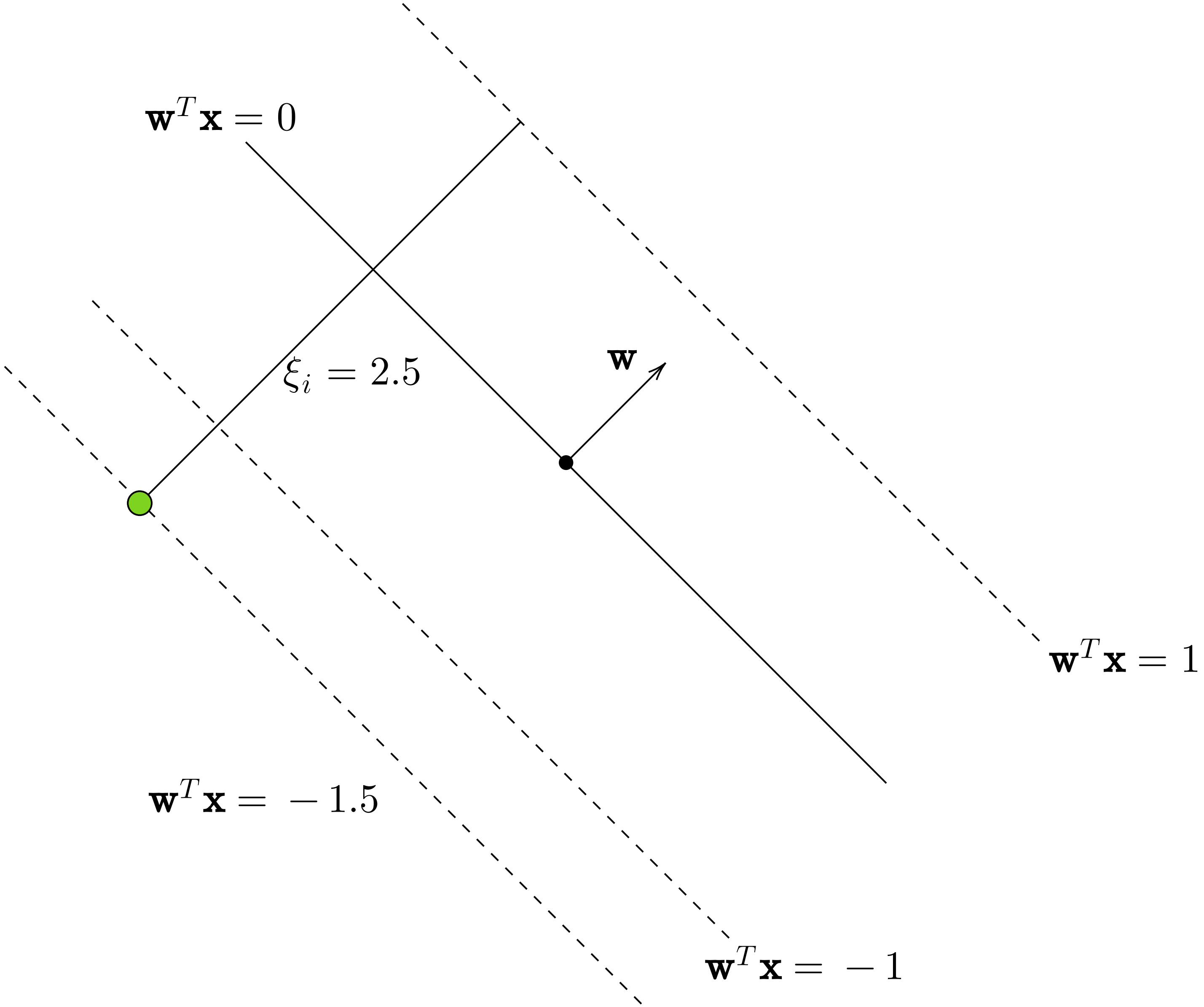

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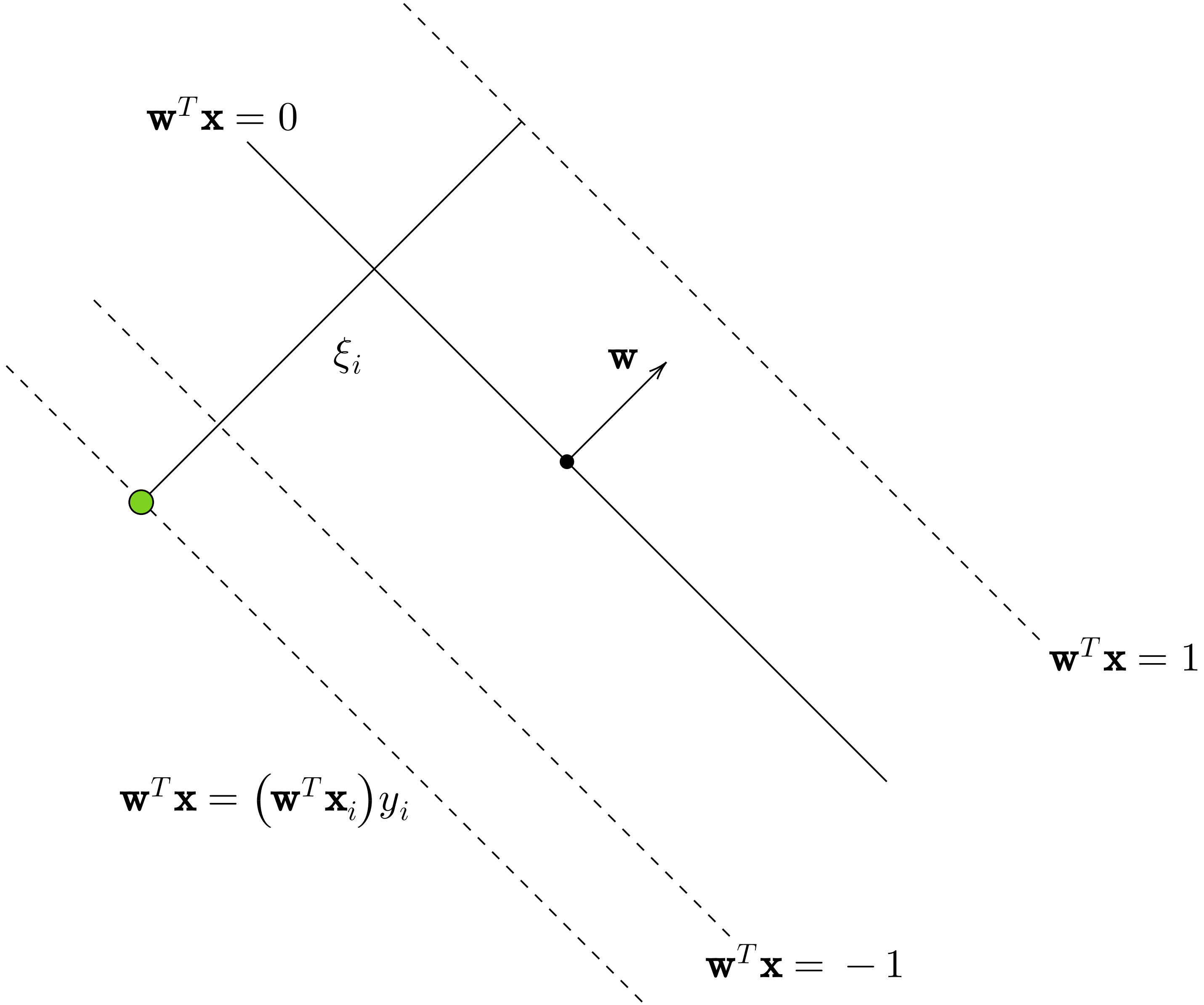

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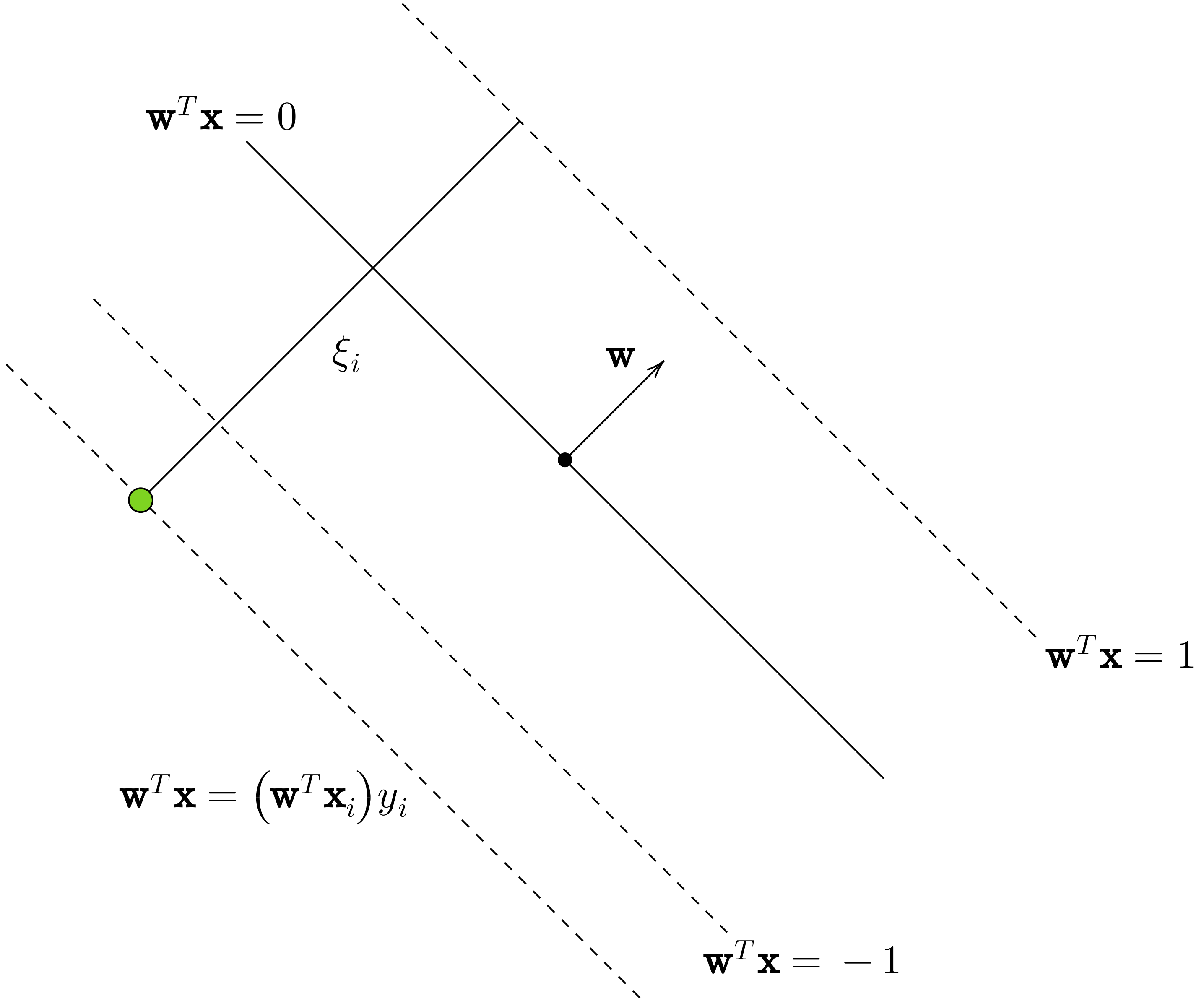


$$\big(\mathbf{w}^T\mathbf{x}_i\big)y_i + \xi_i \geqslant 1$$

$$\xi_i \geqslant 0$$

$$\xi_i = \max\Big(1 - \big(\mathbf{w}^T\mathbf{x}_i\big)y_i, 0\Big)$$

Soft-Margin, Linear-SVM

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \xi_i$$

sub. to

$$\begin{aligned} \big(\mathbf{w}^T \mathbf{x}_i\big) y_i + \xi_i \geqslant 1, &\quad 1 \leqslant i \leqslant n \\ \xi_i \geqslant 0, &\quad 1 \leqslant i \leqslant n \end{aligned}$$

Soft-Margin, Linear-SVM: Hinge-loss formulation

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \xi_i$$

sub. to

$$\left(\mathbf{w}^T \mathbf{x}_i\right) y_i + \xi_i \geqslant 1, \quad 1 \leqslant i \leqslant n$$

$$\xi_i \geqslant 0, \quad 1 \leqslant i \leqslant n$$

$$\equiv$$

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \max\Big(0,\ 1 - \left(\mathbf{w}^T \mathbf{x}_i\right) y_i\Big)$$

Soft-Margin, Linear-SVM: Hinge-loss formulation

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \xi_i$$

sub. to

$$\left(\mathbf{w}^T \mathbf{x}_i\right) y_i + \xi_i \geqslant 1, \quad 1 \leqslant i \leqslant n$$

$$\xi_i \geqslant 0, \quad 1 \leqslant i \leqslant n$$

$$\equiv$$

Model | Data

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \max\Big(0,\ 1 - \left(\mathbf{w}^T \mathbf{x}_i\right) y_i\Big)$$

Regularization | Hinge Loss

Soft-Margin, SVM: Hinge-loss formulation

$$\min_{\mathbf{w}} \quad \underbrace{\frac{||\mathbf{w}||^2}{2}}_{(1)} + C \cdot \underbrace{\sum_{i=1}^{n} \max\Big(0,\ 1 - \big(\mathbf{w}^T \mathbf{x}_i\big) y_i\Big)}_{(2)}$$

Terms in the objective:

(1) $\frac{||\mathbf{w}||^2}{2}$ controls the width of the margin

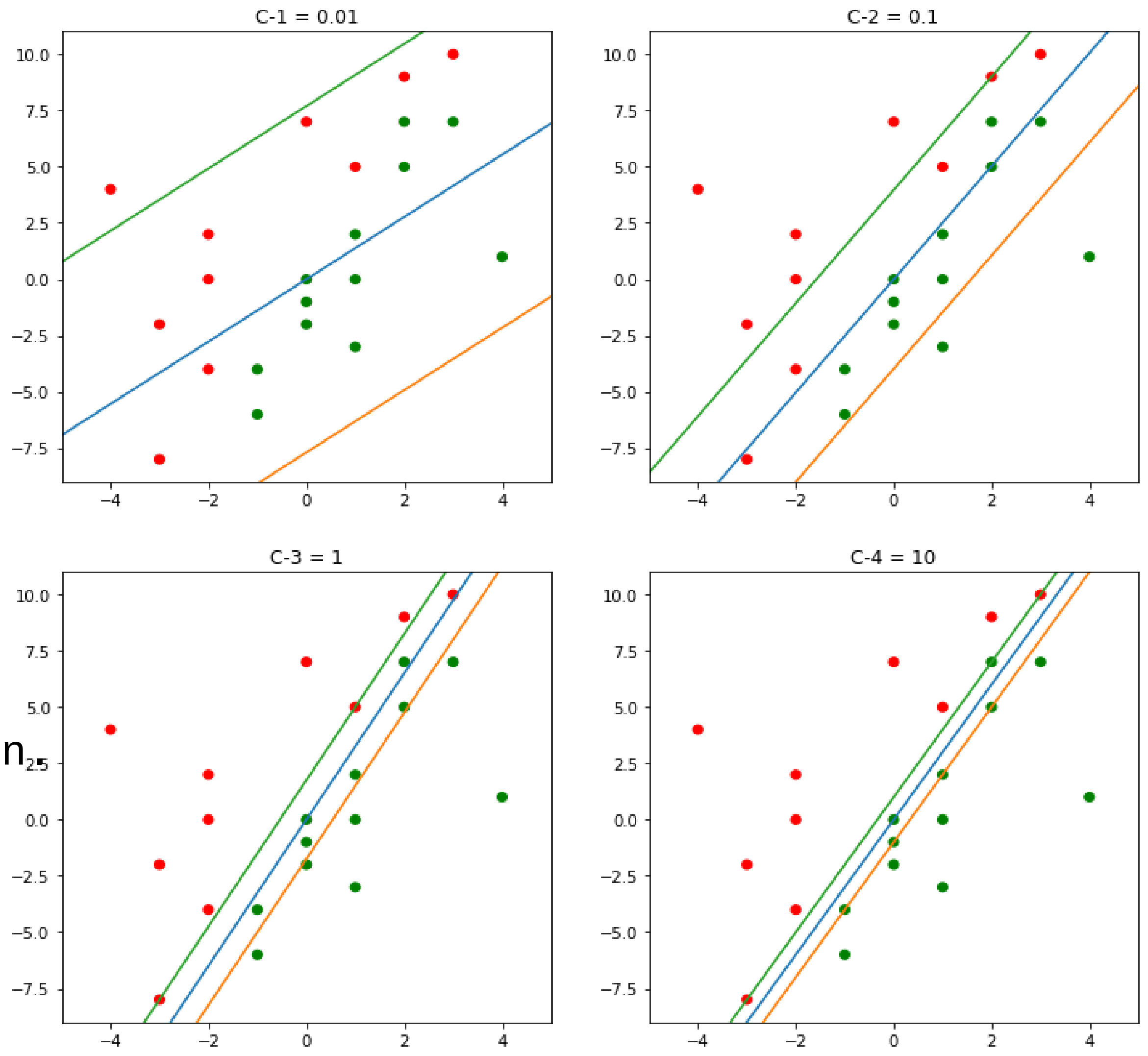
Smaller the value of $||\mathbf{w}||$, wider the margin

(2) $\sum_{i=1}^{n} \max\Big(0,\ 1 - \big(\mathbf{w}^T \mathbf{x}_i\big) y_i\Big)$ is the hinge-loss. Wider the margin, larger the loss.

Soft-Margin, SVM: Hinge-loss formulation

$$\min_{\mathbf{w}} \quad \frac{||\mathbf{w}||^2}{2} + C \cdot \sum_{i=1}^{n} \max\Big(0,\ 1 - \big(\mathbf{w}^T \mathbf{x}_i\big) y_i\Big)$$

(1) (2)



- (1) and (2) work in opposite directions
- If $||\mathbf{w}||$ decreases, the margin becomes wider, which increases the hinge-loss.
- C controls the tradeoff between (1) and (2):
  - If C is small, we are fine with a wide margin.
  - If C is large, we prefer a narrow margin.
  - If $C \to \infty$, we do not tolerate bribery at all.

Miscellaneous

Terminology used

(1) Hard-margin, Linear-SVM
(2) Hard-margin, Kernel-SVM
(3) Soft-margin, Linear-SVM
(4) Soft-margin, Kernel-SVM

Additional points

- Discriminative model
- Weight vector is a sparse linear combination of data-points
- The dual is a quadratic programming problem