# Finite Volume Methods for Hydrodynamics

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# Finite Volume Methods for hydrodynamics

## Overview

- **Linear hyperbolic systems and nonlinear scalar equations:** Riemann problem, Burgers equation, shocks and rarefactions. Explicit time integration, CFL condition, TVD concept and TVDLF method.
- **Finite Volume discretization:** integral versus differential form, from 1D to multi-D.
- **Euler equations:** gas dynamics in 1D, solution of the Riemann problem. TVDLF simulations.
- **Roe solver for Euler equations:** characteristic based shock-capturing schemes. Comparison with TVDLF.

# Linear Hyperbolic Systems

- constant coefficient linear system

$$\vec{q}_t + A\vec{q}_x = 0$$

  $\Rightarrow$ with $\vec{q}(x,t) \in \Re^m$ and matrix $A \in \Re^{m\times m}$

- **hyperbolic** when $A$ is diagonalizable with real eigenvalues

  $\Rightarrow$ **strictly hyperbolic** when distinct

  $\Rightarrow$ $m$ right eigenvectors + $m$ real eigenvalues

$$A\vec{r}_p = \lambda_p\vec{r}_p \text{ with } p : 1, \ldots, m$$

- write as

$$[A]\,[\vec{r}_1 \mid \vec{r}_2 \mid \ldots \mid \vec{r}_m] = [\vec{r}_1 \mid \vec{r}_2 \mid \ldots \mid \vec{r}_m] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ldots & \\ & & & \lambda_m \end{bmatrix}$$

  $\Rightarrow$ or shorthand $A\,R = R\,\Lambda$ with diagonal matrix $\Lambda$

  $\Rightarrow$ **matrix $R$ with right eigenvectors as columns**

- The solution to system $\vec{q}_t + A\vec{q}_x = 0$ is equivalent to:

  $\Rightarrow$ pre-multiply with $R^{-1}$ or:

$$(R^{-1}\vec{q})_t + R^{-1}(R\,\Lambda\,R^{-1})\vec{q}_x = 0$$

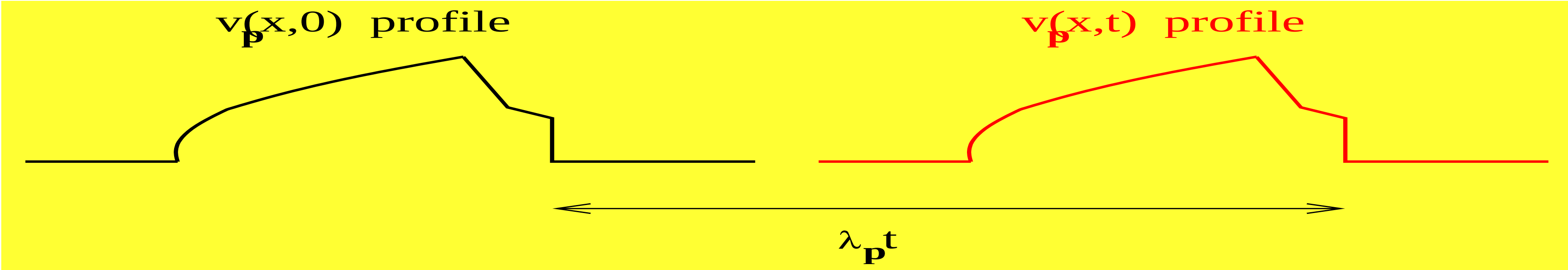
  $\Rightarrow$ redefine $\vec{v} \equiv R^{-1}\vec{q}$ to get

$$\vec{v}_t + \Lambda\vec{v}_x = 0$$

  $\Rightarrow$ $m$ **independent constant coefficient linear advection equations!**

- Each advection equation has trivial analytic solution:

$$v_p(x,t) = v_p(x - \lambda_p t, 0)$$




$\Rightarrow$ **analytic solution to the full linear hyperbolic system is**

$\Rightarrow$ $\vec{q}(x,t) = \sum_{p=1}^{m} v_p(x - \lambda_p t, 0)\vec{r}_p$

$\Rightarrow$ **depends on initial data at $m$ discrete points**

$\Rightarrow$ superposition of $m$ waves, advected independently without distortion

- nomenclature: $\vec{v}$ are 'characteristic variables'

$\Rightarrow$ curves $x = x_o + \lambda_p t$ are "p-characteristics"

# The Riemann problem for a linear hyperbolic system

- Riemann Problem for linear hyperbolic system:

  $\Rightarrow$ initial data $\vec{q}(x,0) = \begin{cases} \vec{q}_l & x < 0 \\ \vec{q}_r & x > 0 \end{cases}$

  $\Rightarrow$ **decompose initial data in terms of right eigenvectors as**

$$\vec{q}_l = \sum_{p=1}^{m} \alpha_p \vec{r}_p$$

$$\vec{q}_r = \sum_{p=1}^{m} \beta_p \vec{r}_p$$

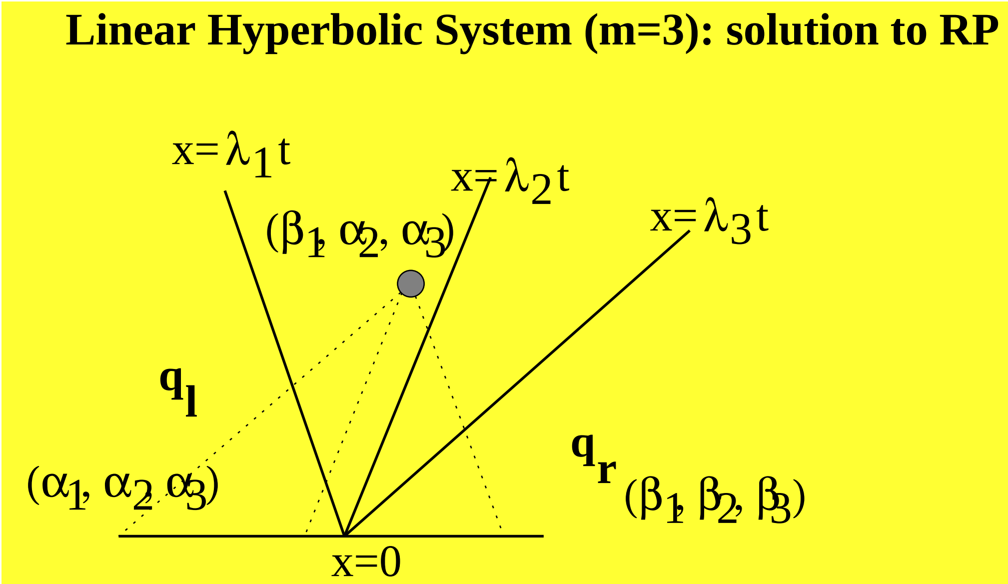
- then at $t=0$, characteristic variables are $v_p(x,0) = \begin{cases} \alpha_p & x < 0 \\ \beta_p & x > 0 \end{cases}$

  $\Rightarrow$ hence $v_p(x,t) = \begin{cases} \alpha_p & \text{if } x - \lambda_p t < 0 \\ \beta_p & \text{if } x - \lambda_p t > 0 \end{cases}$

- the solution to the Riemann problem for linear hyperbolic system is then:

$$\vec{q}(x,t) = \sum_{\text{all p where } x-\lambda_p t<0} \alpha_p \vec{r}_p + \sum_{\text{all p where } x-\lambda_p t>0} \beta_p \vec{r}_p$$

$\Rightarrow$ graphically illustrated for $m = 3$ with $\lambda_1 < \lambda_2 < \lambda_3$




- **constant states separated by discontinuities**

$\Rightarrow$ traveling at characteristic speeds

$\Rightarrow$ Note: jumps are eigenvectors of matrix $A$

## Scalar nonlinear conservation law

- nonlinear scalar conservation law for $u(x,t)$ written as

$$u_t + (f(u))_x = 0$$

- **inviscid Burgers equation** for $f(u) = u^2/2$

  $\Rightarrow$ quasi-linear form (assuming differentiability):

$$u_t + u\,u_x = 0$$

  $\Rightarrow$ characteristic speed from Jacobian, i.e. derivative, $f_u = u \equiv f'(u)$

  $\Rightarrow$ similar to **linear advection equation** $u_t + v\,u_x = 0$ (fixed $v$), which has trivial solution: $u(x,0)$ advected with speed $v$

- **Nonlinearity** in inviscid Burgers $u_t + u\, u_x = 0$

  $\Rightarrow$ advection with local speed $u$

- Consider $t = 0$ triangular pulse (width $2x_0$, height $h_0$) given by

$$u(x,0) = \begin{cases} u_0 & x \leq -x_0 \\ u_0 + h_0 \frac{x_0 + x}{x_0} & -x_0 < x \leq 0 \\ u_0 + h_0 \frac{x_0 - x}{x_0} & 0 < x \leq x_0 \\ u_0 & x > x_0 \end{cases}$$

  $\Rightarrow$ **wave steepening and shock formation** expected!

- tip of triangle experiences fastest rightward advection

  $\Rightarrow$ conserving total area underneath triangle, front edge steepens.

- discontinuity forms at time $t_s = x_0 / h_0$

  $\Rightarrow$ tip of triangle catches up rightmost point of front edge

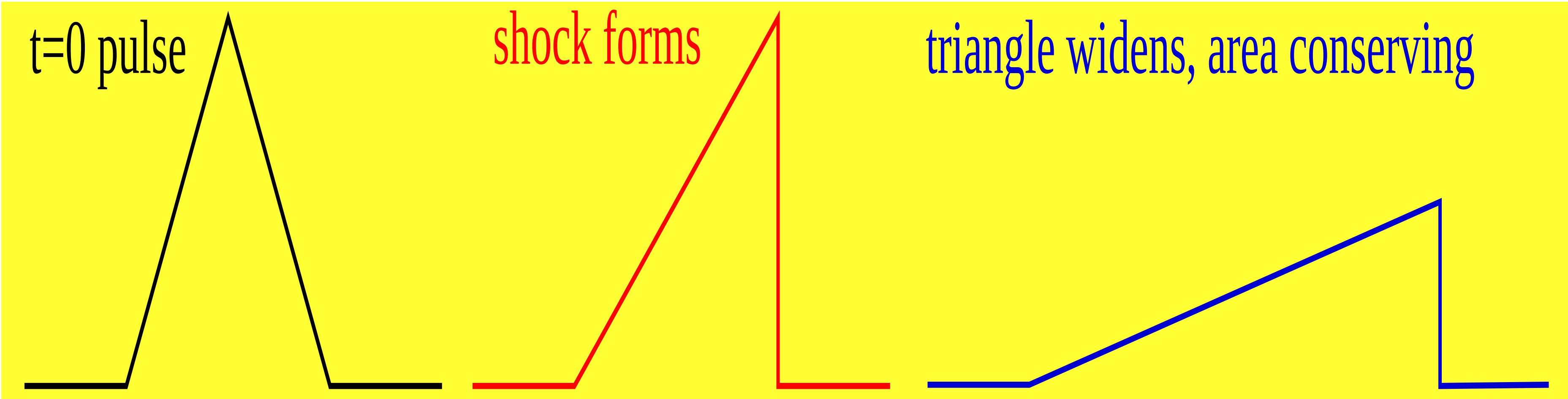
  $\Rightarrow$ **discrete equivalent of conservation law across discontinuity**

  $\Rightarrow$ **Rankine-Hugoniot relation** for left $u_l$ and right $u_r$ values

$$f(u_l) - f(u_r) = s\,(u_l - u_r)$$

- for inviscid Burgers case, find shock speed $s = (u_l + u_r)/2$
- fully analytic solution to triangular pulse problem

$\Rightarrow$ after shock forms, base of triangle widens due to the speed difference between left edge traveling with $u_0$, and shocked right edge traveling at speed $s(t)$. In accord with conservation, the height of the triangle must therefore decrease in time.




$\Rightarrow$ **Assignment: try different numerical schemes to simulate this evolution**

# The Riemann problem for Burgers

- specific initial condition separating 2 constant states
- Riemann problem for scalar conservation law

$$u = u_l \text{ for } x \leq 0$$
$$u = u_r \text{ for } x > 0$$

- Rankine-Hugoniot states

$$f(u_l) - f(u_r) = s\,(u_l - u_r)$$

$\Rightarrow$ symmetric in arguments $u_l$ and $u_r$

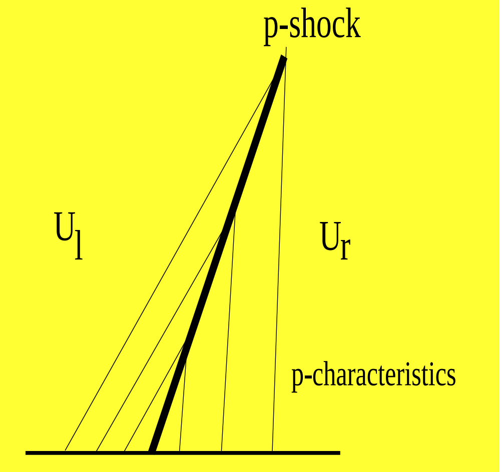
- For inviscid Burgers: only shock expected for $u_l > u_r$

$\Rightarrow$ extra condition for admissable shock: **Lax entropy condition**

$$f'(u_l) > s > f'(u_r)$$

$\Rightarrow$ **shock speed between characteristic speeds of 2 states**

$\Rightarrow$ characteristics 'go into the shock'



- **Rarefaction** waves for Burgers equation

  $\Rightarrow$ when $u_l < u_r$ expect right state 'runs away' from left

  $\Rightarrow$ try **'centered simple wave'** $u(\xi) = u(x/t)$

  $\Rightarrow$ conservation law translates into

$$f'(u)\,\frac{du}{d\xi} = \xi\,\frac{du}{d\xi}$$

  $\Rightarrow$ rarefaction wave: for Burgers:

$$u(x,t) = u(x/t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t < x < u_r t \\ u_r & x > u_r t \end{cases}$$

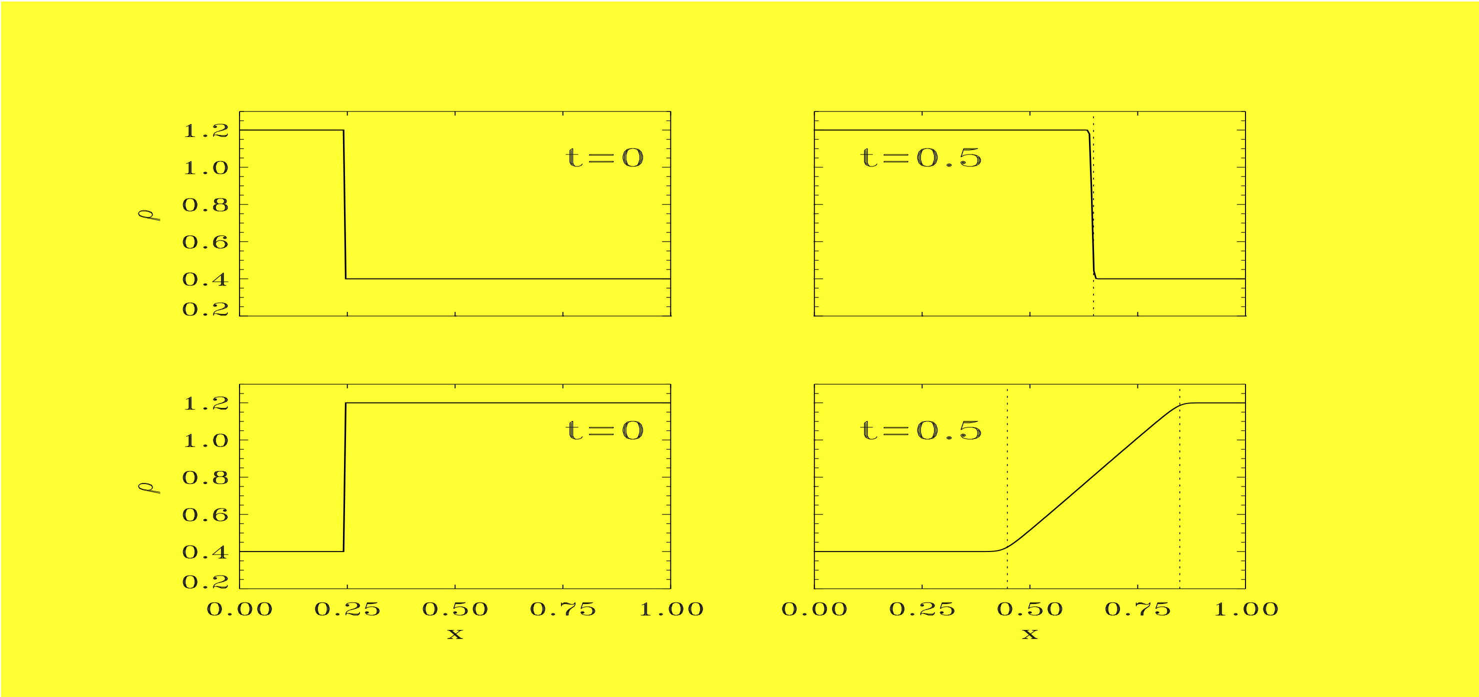
  $\Rightarrow$ $u$ decreases (density: medium gets rarefied) when signal passes

- **Riemann problem for inviscid Burgers has two cases:**

  $\Rightarrow$ $\rho_l > \rho_r$: <u>shock</u> traveling at speed $s = \frac{\rho_l + \rho_r}{2}$

  $\Rightarrow$ $\rho_l < \rho_r$: <u>rarefaction wave</u> $\rho(x,t) = \rho(x/t) = \begin{cases} \rho_l & x < \rho_l t \\ x/t & \rho_l t < x < \rho_r t \\ \rho_r & x > \rho_r t \end{cases}$

  $\Rightarrow$ Numerically with TVDLF scheme (see later):



# Numerical methods for nonlinear conservation law

- introduce spatial $x_i = i\Delta x$ and temporal $t^n = n\Delta t$ steps

  $\Rightarrow$ try **explicit** scheme on Burgers, directly discretize $u_t + u\,u_x = 0$ to

$$u_i^{n+1} - u_i^n + \frac{\Delta t}{\Delta x} u_i^n \left( u_i^n - u_{i-1}^n \right) = 0$$

  $\Rightarrow$ initial data $[1, 1, 1, 0, 0, 0]$ remains solution to this scheme

  $\Rightarrow$ **WRONG !!!!** should be traveling shock at speed $s = 0.5$

- reason: above scheme non-conservative (but ok for continuous data!)

  $\Rightarrow$ **conservative scheme** is of form

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x} \left[ \mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2} \right]$$

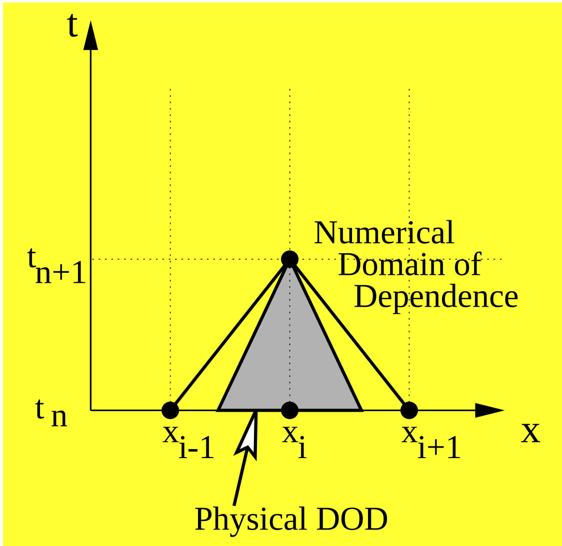
  $\Rightarrow$ numerical fluxes $\mathbf{F}_{i+1/2}$: time-average fluxes over cell edges $x_i + \frac{1}{2}\Delta x \equiv x_{i+\frac{1}{2}}$

# Explicit time integration

- calculate fluxes (and sources) from known time level $t^n$
- Explicit: $\Delta t$ restricted by Courant, Friedrichs, Lewy condition
  *domain of dependence of discretization must include*
  *PDE domain of dependence*

  $\Rightarrow \Delta t \leq$ crossing time of cells by fastest wave

  $\Rightarrow \Delta t \leq \Delta x / c^{\max}$ with maximal physical speed $c^{\max}$



# Lax-Friedrichs and Max-Cormack methods

- Better explicit discretization: **first order** Lax-Friedrichs

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{\Delta t}{2\Delta x} \left( f_{i+1}^n - f_{i-1}^n \right)$$

$\Rightarrow$ scheme is **conditionally stable**: restriction by CFL condition

$$\left| \frac{\Delta t}{\Delta x} f'(u_i) \right| \leq 1$$

$\Rightarrow$ conservative scheme, identify numerical flux as

$$F_{i+1/2}^{\mathrm{LF}} = \frac{1}{2} \left\{ f_{i+1} + f_i - \frac{\Delta x}{\Delta t} \left[ u_{i+1} - u_i \right] \right\}$$

- first order accuracy: local truncation error $\propto \Delta t$

- another, multilevel method: **two-step MacCormack method:**

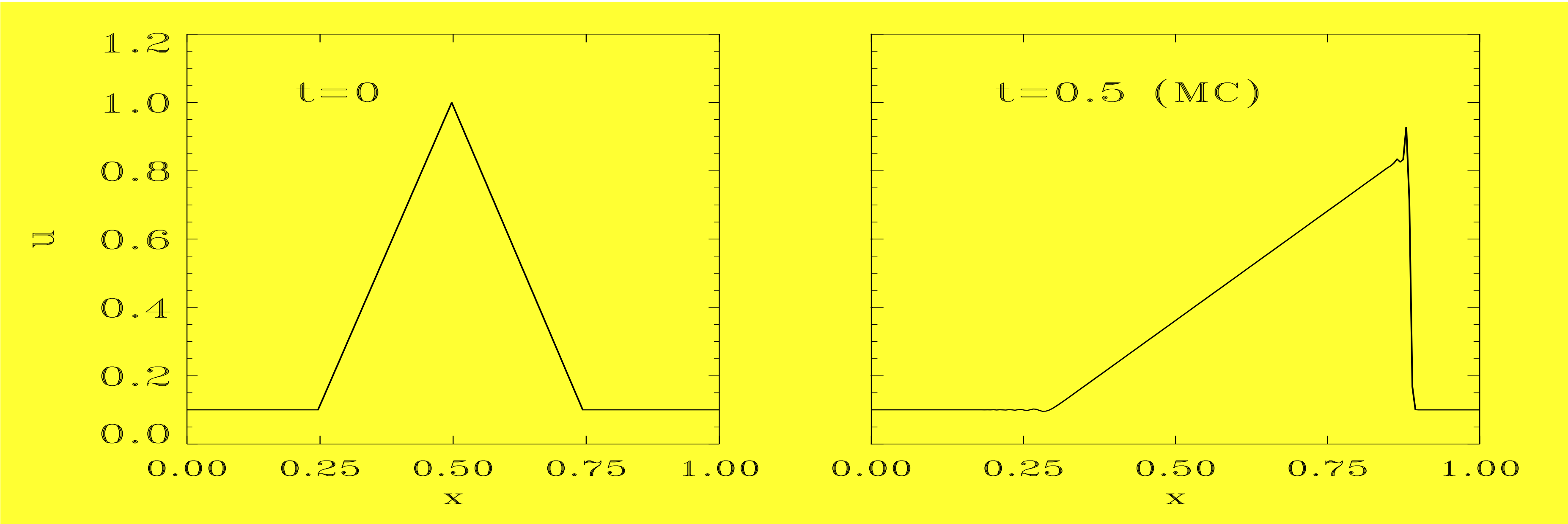
$$\text{'predictor':} \qquad u_i^* = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1}^n - f_i^n\right)$$

$$\text{'corrector':} \qquad u_i^{n+1} = \tfrac{1}{2}(u_i^n + u_i^*) - \tfrac{1}{2}\frac{\Delta t}{\Delta x}\left(f_i^* - f_{i-1}^*\right)$$

$\Rightarrow$ step size in predictor now $\Delta t$

$\Rightarrow$ second order accurate: local truncation error $\propto \Delta t^2$

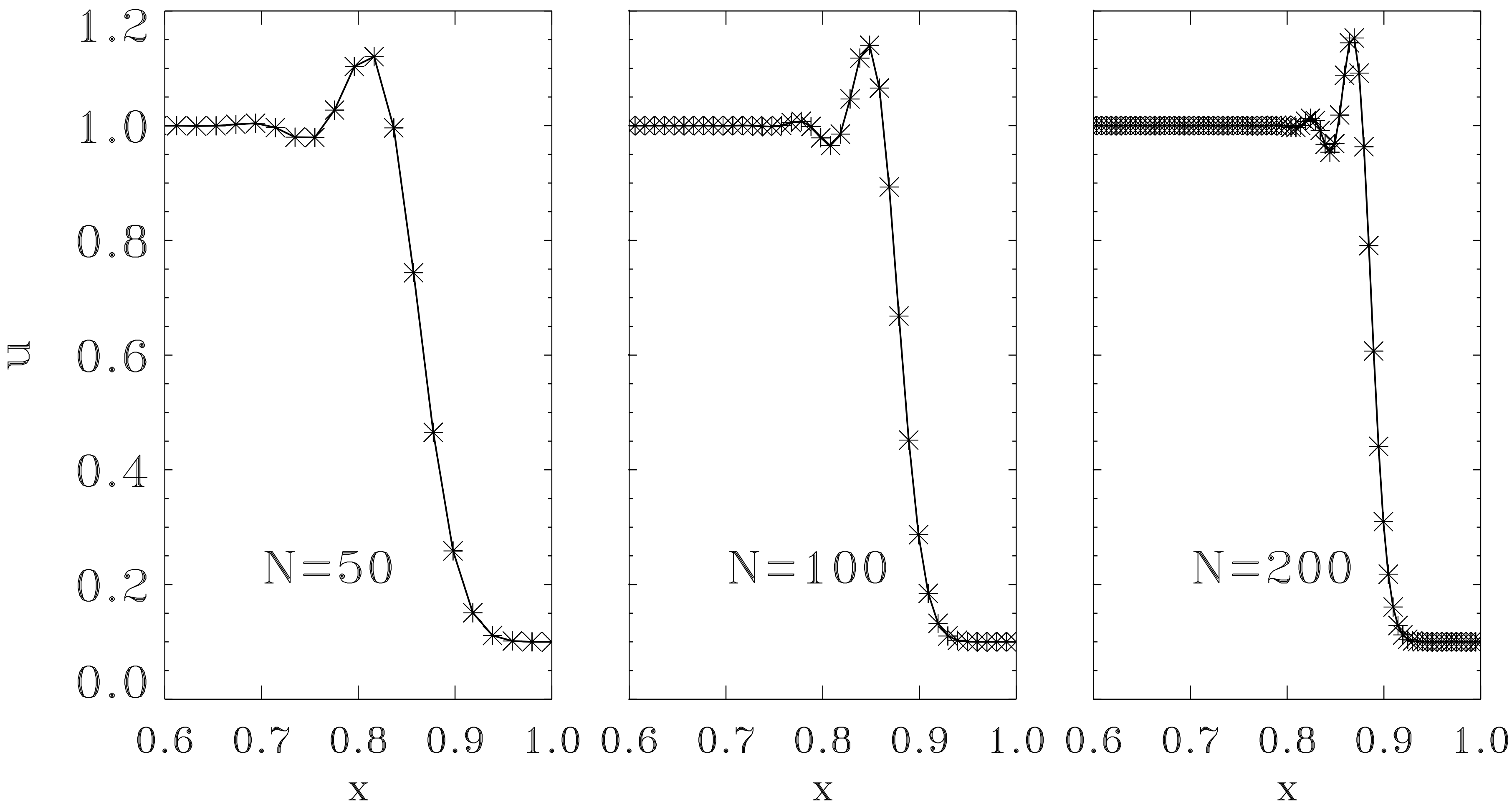
- Numerically simulate triangular pulse for Burgers




$\Rightarrow$ in agreement with theory, except for the 'wiggles'

- MacCormack method: **dispersive**

  $\Rightarrow$ manifests **Gibbs phenomenon**: for **linear advection of discontinuity**




  $\Rightarrow$ **non-monotonicity preserving**: monotone $u(x, 0)$ develops extrema

# Total Variation Diminishing concept

- **total variation** of function $u(x)$ on domain $[0,1]$ defined as

$$TV(u) \equiv \int_0^1 |\frac{du}{dx}| \, \mathrm{d}x$$

$\Rightarrow$ total variation of numerical approximation of $u$

$$TV(u^n) = \sum_{i=0}^{N} |u_{i+1}^n - u_i^n|$$

- scheme is **total variation diminishing** (TVD) in time if

$$TV(u^{n+1}) \leq TV(u^n) \qquad \forall n$$

$\Rightarrow$ solution scalar conservation law has TVD property $\forall t_2 > t_1$

$$TV(u(x,t_2)) \leq TV(u(x,t_1))$$

- **a TVD scheme is clearly monotonicity preserving!**

  $\Rightarrow$ a new local extremum would raise TV

- Harten: any scheme written in general form

$$u_i^{n+1} = u_i^n + A_{i+1/2} \underbrace{\left(u_{i+1}^n - u_i^n\right)}_{\Delta u_{i+1/2}^n} - B_{i-1/2} \underbrace{\left(u_i^n - u_{i-1}^n\right)}_{\Delta u_{i-1/2}^n}$$

  $\Rightarrow$ is TVD when coefficients $A_{i+1/2}$ and $B_{i-1/2}$ obey

$$A_{i+1/2} \geq 0$$
$$B_{i-1/2} \geq 0$$
$$0 \leq A_{i+1/2} + B_{i+1/2} \leq 1$$

- **first order Lax-Friedrichs scheme is TVD**, since rewrites as

$$u_i^{n+1} = u_i^n + \frac{1}{2}\left(1 - \frac{\Delta t}{\Delta x}\frac{f_{i+1}^n - f_i^n}{\Delta u_{i+\frac{1}{2}}}\right)\Delta u_{i+\frac{1}{2}} - \frac{1}{2}\left(1 + \frac{\Delta t}{\Delta x}\frac{f_i^n - f_{i-1}^n}{\Delta u_{i-\frac{1}{2}}}\right)\Delta u_{i-\frac{1}{2}}$$

$\Rightarrow$ TVD requirements translate to CFL condition

$$\left|\frac{\Delta t}{\Delta x}\frac{f_{i+1}^n - f_i^n}{u_{i+1}^n - u_i^n}\right| \leq 1$$

$\Rightarrow$ **generalize Lax-Friedrichs scheme to second order, keep TVD property**

## TVDLF scheme

- Recall: numerical flux for first-order Lax-Friedrichs is

$$F^{\mathrm{LF}}_{i+1/2} = \frac{1}{2}\left\{ f_{i+1} + f_i - \frac{\Delta x}{\Delta t}\left[u_{i+1} - u_i\right] \right\}$$

$\Rightarrow$ can improve scheme by changing to **Local Lax-Friedrichs** flux

$$F^{\mathrm{LLF}}_{i+1/2} = \frac{1}{2}\left\{ f_{i+1} + f_i - \left|\alpha_{i+\frac{1}{2}}\right| \left[u_{i+1} - u_i\right] \right\}$$

$\Rightarrow$ $\alpha_{i+1/2} = \frac{f^n_{i+1} - f^n_i}{u^n_{i+1} - u^n_i}$, proxy for local characteristic speed $f'(u)$

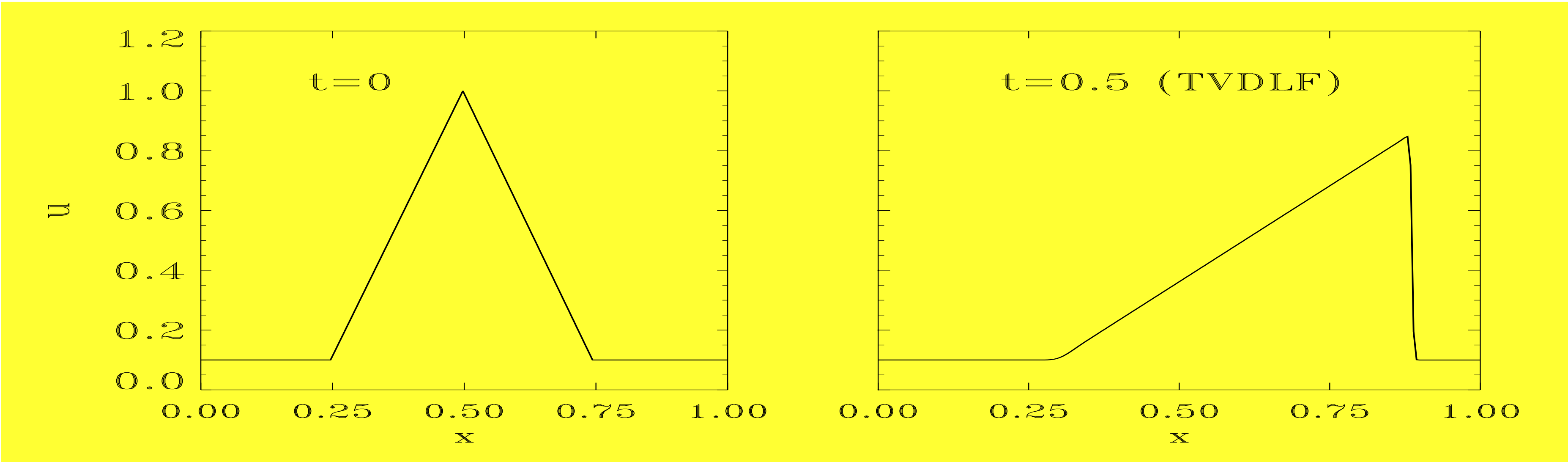
$\Rightarrow$ still in accord with TVD, if CFL condition satisfied

- turn into **second order accurate TVDLF scheme** by

  $\Rightarrow$ use predictor-corrector approach (raise temporal accuracy)

  $\Rightarrow$ use some form of linear interpolation in space (but keep TVD!)

- **Numerically simulate triangular pulse for Burgers**




  $\Rightarrow$ **Riemann problem for $u_l > u_r$**

  $\Rightarrow$ **Riemann problem for $u_l < u_r$**

# TVDLF scheme

- Robust, **general scheme to ANY hyperbolic system, ensures TVD (for scalar)**

  $\Rightarrow$ Predictor-corrector approach for temporal advance

$$\mathbf{U}^{n+1/2} = \mathbf{U}^n + \frac{\Delta t}{2}\left[-\nabla \cdot \mathbf{F}(\mathbf{U}^n) + \mathbf{S}(\mathbf{U}^n)\right]$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t\left[-\nabla \cdot \mathbf{F}(\mathbf{U}^{n+1/2}) + \mathbf{S}(\mathbf{U}^{n+1/2})\right]$$

  $\Rightarrow$ CFL condition links $\Delta t$ with $\Delta x$ (explicit scheme)

- **slope limited linear reconstruction** and flux expression

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left\{\mathbf{F}(\mathbf{U}^L_{i+\frac{1}{2}}) + \mathbf{F}(\mathbf{U}^R_{i+\frac{1}{2}}) - \mid c^{\max}\left(\frac{\mathbf{U}^L_{i+\frac{1}{2}} + \mathbf{U}^R_{i+\frac{1}{2}}}{2}\right) \mid \left[\mathbf{U}^R_{i+\frac{1}{2}} - \mathbf{U}^L_{i+\frac{1}{2}}\right]\right\}$$

  $\Rightarrow$ scalar $c^{\max}$ denotes maximal physical propagation speed

  $\Rightarrow$ for 1D HD $c^{\max}$ is $|v_x|$ plus sound speed, for MHD $c^{\max} = |v_x| + c_f$

# Integral form of conservation law

- consider cell $[x_1, x_2]$ and quantity $u(x,t)$ within cell

  $\Rightarrow$ flux over cell edge $f(u)$ changes total mass from $t_1$ to $t_2$ by

$$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(x_1,t)\,dt - \int_{t_1}^{t_2} f(x_2,t)\,dt$$

  $\Rightarrow$ integral form of scalar conservation law

$$u_t + \left(f(u)\right)_x = 0$$

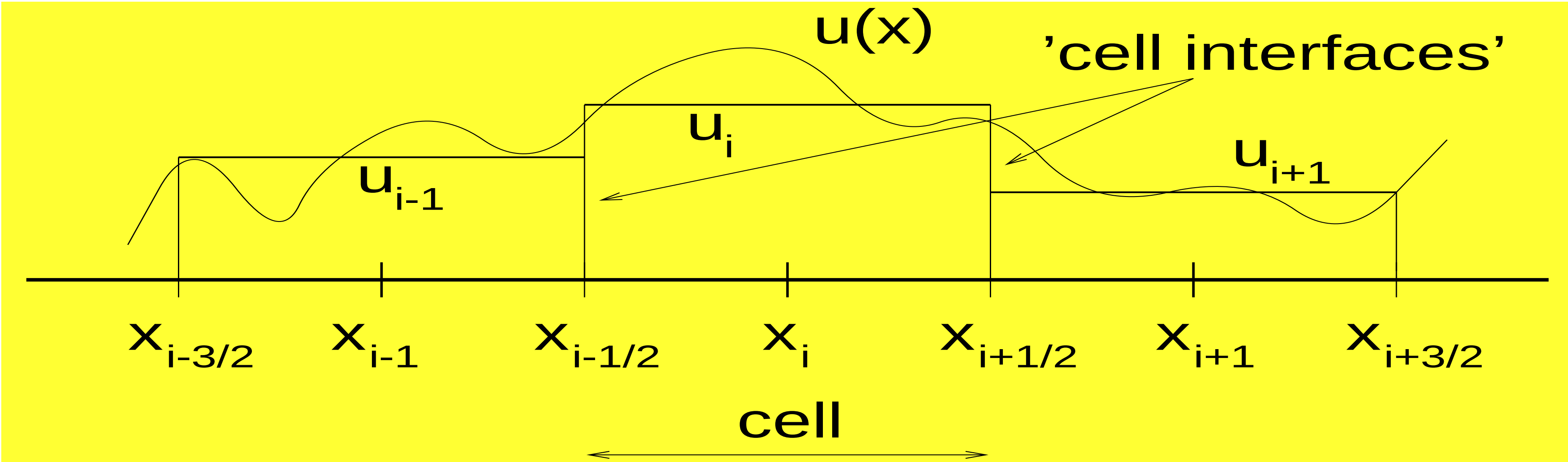
- **integral form more general: allows for discontinuous solutions**

  $\Rightarrow$ differential form assumes differentiable functions

- **Finite Volume** method in 1D for system of conservation laws

  $\Rightarrow$ interpret $\mathbf{U}_i$ as average value of $\mathbf{U}(x,t)$ in $[x_{i-1/2}, x_{i+1/2}]$:

$$\mathbf{U}_i(t) \equiv \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x,t)\,\mathrm{d}x\,,$$



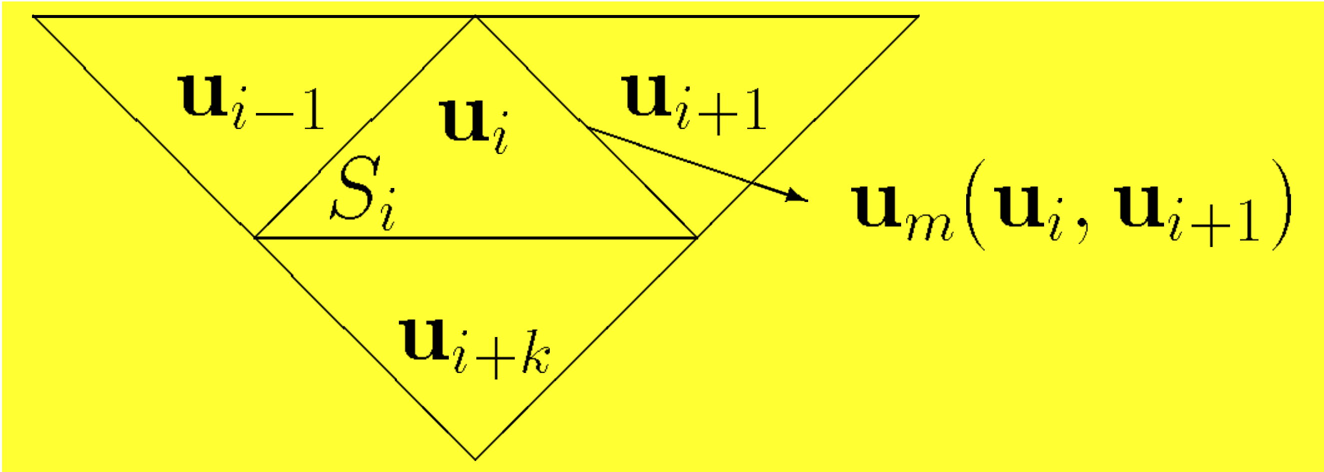

  $\Rightarrow$ update volume averages by

$$\frac{d\mathbf{U}_i}{dt} + \frac{1}{\Delta x_i}\left(\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}\right) = 0\,.$$

  $\Rightarrow$ discretized equation is integral law, weak solutions obey conservation

# Finite Volume on unstructured grids

- Finite volume approach: natural on general 2D and 3D unstructured grids




- multidimensional set of conservation laws

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = 0$$

$\Rightarrow$ in 3D, the $\nabla \cdot \mathbf{F}$ with three Cartesian coordinate axes

$$\nabla \cdot \mathbf{F} = \frac{\partial \mathbf{F}_x}{\partial x} + \frac{\partial \mathbf{F}_y}{\partial y} + \frac{\partial \mathbf{F}_z}{\partial z}$$

- discretize space in control volumes $V_i$

  $\Rightarrow$ bounding surfaces $\partial V_i$, unit normal $\mathbf{n} = (n_x,\, n_y,\, n_z)$

$$\frac{d\int_{V_i} \mathbf{U}(\mathbf{x},t)\,d\mathbf{x}}{dt} = -\int_{\partial V_i} \mathbf{F}\cdot\mathbf{n}\,dS$$
$$= -\int_{\partial V_i} \left(\mathbf{F}_x n_x + \mathbf{F}_y n_y + \mathbf{F}_z n_z\right)\,dS$$

  $\Rightarrow$ introduce ($8\times 8$ for MHD) matrix $T(\mathbf{n})$ which rotates vector quantities to local orthogonal coordinate system $\mathbf{n}$, $\mathbf{t}$, $\mathbf{s}\equiv\mathbf{n}\times\mathbf{t}$, where latter are tangential unit vectors within $\partial V_i$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \sin\theta\,\cos\varphi & \sin\theta\,\sin\varphi & \cos\theta & 0 & 0 & 0 & 0\\
0 & \cos\theta\,\cos\varphi & \cos\theta\,\sin\varphi & -\sin\theta & 0 & 0 & 0 & 0\\
0 & -\sin\varphi & \cos\varphi & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \sin\theta\,\cos\varphi & \sin\theta\,\sin\varphi & \cos\theta\\
0 & 0 & 0 & 0 & 0 & \cos\theta\,\cos\varphi & \cos\theta\,\sin\varphi & -\sin\theta\\
0 & 0 & 0 & 0 & 0 & -\sin\varphi & \cos\varphi & 0
\end{pmatrix}.$$

- HD and MHD equations must be unchanged under rotation

$$\mathbf{F}_x n_x + \mathbf{F}_y n_y + \mathbf{F}_z n_z = T^{-1}(\mathbf{n})\mathbf{F}_x\left(T(\mathbf{n})\mathbf{U}\right)$$

$\Rightarrow$ obtain essentially 1D problem in direction normal to volume boundary

$$\frac{d\int_{V_i}\mathbf{U}(\mathbf{x},t)\,d\mathbf{x}}{dt} = -\int_{\partial V_i} T^{-1}(\mathbf{n})\mathbf{F}_x\left(T(\mathbf{n})\mathbf{U}\right)\,dS$$

- control volumes with multiple, flat surface segments

$\Rightarrow$ integral over boundary into discrete sum over its sides

$\Rightarrow$ **only information of grid: volumes $V_i$, and geometry of cells: number of bounding surface segments, their surface area and their normal directions.**

## The Euler equations

- **conservation laws for 1D dynamics of compressible gas**

$$\begin{cases} \rho_t + (\rho\, v)_x & = \; 0 \\ m_t + (m\, v + p)_x & = \; 0 \\ e_t + (e\, v + p\, v)_x & = \; 0 \end{cases}$$

- vector of conserved quantities $U = \begin{pmatrix} \rho \\ m \\ e \end{pmatrix}$

  $\Rightarrow$ total energy density related to pressure by

$$e = \underbrace{\frac{\rho v^2}{2}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{thermal energy}}$$

  $\Rightarrow$ ratio of specific heats $\gamma$

- internal energy considerations
  - $\Rightarrow$ specific ($\equiv$ per unit mass) internal energy $e_i^s$
  - $\Rightarrow$ $\rho e_i^s = p/(\gamma - 1)$
- for ideal gas: temperature defined as $p = \mathcal{R}\rho T$ with gas constant $\mathcal{R}$
  - $\Rightarrow$ $e_i^s(T) = \frac{\mathcal{R}T}{\gamma-1} = \frac{(c_p-c_v)T}{\frac{c_p}{c_v}-1} = c_v T$
  - $\Rightarrow$ $c_v$ specific heat at constant volume
- generally $\gamma = \frac{\alpha+2}{\alpha}$, where $\alpha$ is the total number of degrees of freedom over which internal energy can be distributed
  - $\Rightarrow$ for molecules: translational, rotational, vibrational
  - $\Rightarrow$ monoatomic gas: only 3 translational DOF $\rightarrow \gamma = 5/3$

- deduce equation for ‘entropy’ $s = p\rho^{-\gamma}$

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0$$

$\Rightarrow$ since $v(x,t)$: Not in conservation form!

$\Rightarrow$ like advection equation

$\Rightarrow$ **$s$ constant along characteristics $\frac{dx}{dt} = v$: Riemann Invariant**

- equivalent to the ‘characteristic’ equation

$\Rightarrow$ along $\frac{dx}{dt} = v$, find

$$dp - c_s^2\, d\rho = 0$$

$\Rightarrow$ with $dp = p_t\, dt + p_x\, dx$ and $c_s^2 = \gamma p/\rho$

- write system as $U_t + (F(U))_x = 0$ with flux vector

$$F = \begin{pmatrix} m \\ \frac{m^2}{\rho}\frac{3-\gamma}{2} + (\gamma-1)e \\ \frac{em}{\rho}\gamma - \frac{\gamma-1}{2}\frac{m^3}{\rho^2} \end{pmatrix}$$

$\Rightarrow$ **Flux Jacobian** becomes

$$\frac{\partial F}{\partial U} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{m^2}{\rho^2}\frac{\gamma-3}{2} & \frac{m}{\rho}(3-\gamma) & \gamma-1 \\ -\gamma\frac{em}{\rho^2} + (\gamma-1)\frac{m^3}{\rho^3} & \frac{e\gamma}{\rho} + (1-\gamma)\frac{3}{2}\frac{m^2}{\rho^2} & \frac{m\gamma}{\rho} \end{pmatrix}$$

$\Rightarrow$ **3 eigenvalues/right eigenvectors**

- eigenvalue $\lambda_1 = \frac{m}{\rho} - \sqrt{\frac{\gamma p}{\rho}} = v - c_s$

  $\Rightarrow$ eigenvector $\vec{r}_1 = \begin{pmatrix} 1 \\ v - c_s \\ \frac{v^2}{2} - v c_s + \frac{c_s^2}{\gamma - 1} \end{pmatrix}$

- eigenvalue $\lambda_2 = \frac{m}{\rho} = v$

  $\Rightarrow$ eigenvector $\vec{r}_2 = \begin{pmatrix} 1 \\ v \\ \frac{v^2}{2} \end{pmatrix}$

- eigenvalue $\lambda_3 = \frac{m}{\rho} + \sqrt{\frac{\gamma p}{\rho}} = v + c_s$

  $\Rightarrow$ eigenvector $\vec{r}_3 = \begin{pmatrix} 1 \\ v + c_s \\ \frac{v^2}{2} + v c_s + \frac{c_s^2}{\gamma - 1} \end{pmatrix}$

- **Rankine-Hugoniot relations for Euler system**

$$F(U_l) - F(U_r) = s\,(U_l - U_r)$$

$$\Rightarrow \left\{ \begin{array}{rcl} m_l - m_r & = & s(\rho_l - \rho_r) \\ \left[\frac{m_l^2}{\rho_l}\frac{3-\gamma}{2} + (\gamma-1)e_l\right] - \left[\frac{m_r^2}{\rho_r}\frac{3-\gamma}{2} + (\gamma-1)e_r\right] & = & s(m_l - m_r) \\ \left[\frac{e_l m_l}{\rho_l}\gamma - \frac{\gamma-1}{2}\frac{m_l^3}{\rho_l^2}\right] - \left[\frac{e_r m_r}{\rho_r}\gamma - \frac{\gamma-1}{2}\frac{m_r^3}{\rho_r^2}\right] & = & s(e_l - e_r) \end{array} \right.$$

$\Rightarrow$ for given right state: 3 equations for 4 unknowns $s$, $U_l$

- verify that **Contact Discontinuity obeys RH**

$\Rightarrow$ $s = v$, $v_l = v_r = v$, $p_l = p_r = p$ while $\rho_l \neq \rho_r$

$\Rightarrow$ ‘2-wave’ for eigenvalue $\lambda_2 = v$ has (generalized Riemann) invariants $v$ and $p$

- **general solution to Riemann Problem:**
  - $\Rightarrow$ given two states $U_l$ and $U_r$
  - $\Rightarrow$ find intermediate state $U_{mr}$ connected to $U_r$ by a '3-wave'
  - $\Rightarrow$ which is such that its velocity and pressure

$$U_{mr} = \underbrace{\begin{pmatrix} \rho_{mr} \\ m_{mr} \\ e_{mr} \end{pmatrix}}_{\text{conservative}} \equiv \underbrace{\begin{pmatrix} \rho_{mr} \\ v_* \\ p_* \end{pmatrix}}_{\text{primitive}}$$

  match the velocity and pressure of intermediate state

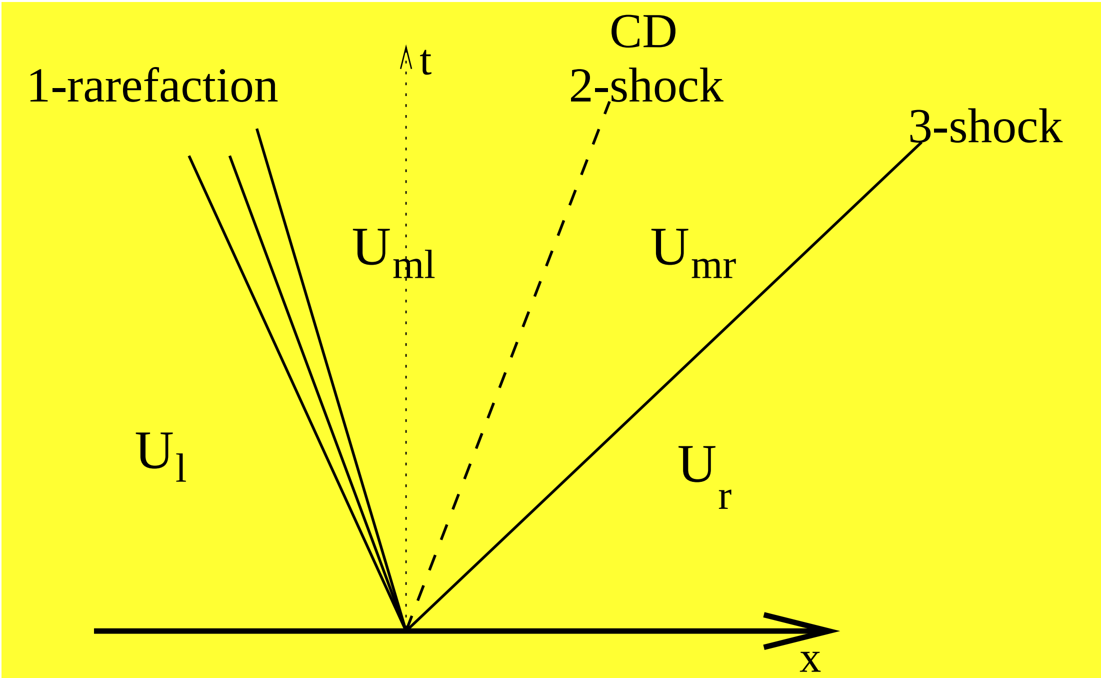
$$U_{ml} = \begin{pmatrix} \rho_{ml} \\ v_* \\ p_* \end{pmatrix}$$

  connected to $U_l$ by a '1-wave'

  - $\Rightarrow$ the states $U_{ml}$ and $U_{mr}$ can be connected by a '2-shock' (contact discontinuity)

- note: counts ok:

  $\Rightarrow$ 6 equations for 6 unknowns $(s_1, \rho_{ml}, v_*, p_*)$ and $(s_3, \rho_{mr}, v_*, p_*)$




  $\Rightarrow$ again **only entropy-satisfying shocks allowed**

- ingredients to solve RP: L – $R_1$ or $S_1$ – $M_l$ – CD – $M_r$ – $R_3$ or $S_3$ – R

- Euler system in terms of **primitive** variables

$$\begin{pmatrix} \rho \\ v \\ p \end{pmatrix}_t + \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \gamma p & v \end{pmatrix} \begin{pmatrix} \rho \\ v \\ p \end{pmatrix}_x = 0$$

$\Rightarrow$ possible to deduce

$$v_t + (v \pm c)\, v_x \pm \frac{1}{\sqrt{\gamma p \rho}} (p_t + (v \pm c) p_x) = 0$$

- since $\frac{2c}{\gamma-1} = \frac{2}{\gamma-1}\sqrt{\frac{\gamma p}{\rho}}$ and under constant $s = p\rho^{-\gamma}$
  $\Rightarrow$ can be rewritten to

$$\left(v \pm \frac{2c}{\gamma - 1}\right)_t + (v \pm c)\left(v \pm \frac{2c}{\gamma - 1}\right)_x = 0$$

- found (generalized) Riemann Invariants
  $\Rightarrow$ **for '1-wave' $v - c$: invariants are $s$ and $v + \frac{2c}{\gamma-1}$**
  $\Rightarrow$ **for '2-wave' $v$: invariants are $v$ and $p$**
  $\Rightarrow$ **for '3-wave' $v + c$: invariants are $s$ and $v - \frac{2c}{\gamma-1}$**

- can be written as **'characteristic' equations**

  $\Rightarrow$ $dp - \rho c\, dv = 0$ along $\frac{dx}{dt} = v - c$

  $\Rightarrow$ $dp - c^2\, d\rho = 0$ along $\frac{dx}{dt} = v$

  $\Rightarrow$ $dp + \rho c\, dv = 0$ along $\frac{dx}{dt} = v + c$

  $\Rightarrow$ could be used to solve IVP in $(x, t)$ space

- back to Rankine-Hugoniot relations for Euler system

$$F(U_l) - F(U_r) = s\,(U_l - U_r)$$

$\Rightarrow$ **consider again stationary shock** $s = 0 \rightarrow m_l = m_r$

$\Rightarrow$ two remaining equations result in

$$\frac{v_l^2}{2} + \frac{c_l^2}{\gamma - 1} = \frac{v_r^2}{2} + \frac{c_r^2}{\gamma - 1} = \frac{\gamma + 1}{2(\gamma - 1)} c_*^2$$

$\Rightarrow$ last equality for sonic point where $v_* = c_*$

$\Rightarrow$ again leads to $c_*^2 = v_l v_r$ **Prandtl Meyer relation**

$\Rightarrow$ stationary shock separates super- from subsonic state (w.r.t. $c_*$)!

- further analysis of stationary shock introduces $M_l = \frac{v_l}{c_l}$

$$\frac{v_l}{v_r} = \frac{(\gamma+1)M_l^2}{(\gamma-1)M_l^2+2}$$

$\Rightarrow$ and since $m_l = m_r$ we get for the density ratio

$$\frac{\rho_l}{\rho_r} = \frac{(\gamma-1)M_l^2+2}{(\gamma+1)M_l^2}$$

$\Rightarrow$ pressure ratio can be shown to obey

$$\frac{p_l}{p_r} = \frac{\gamma+1}{1-\gamma+2\gamma M_l^2}$$

$\Rightarrow$ **for stationary shock: all jumps depend on $\gamma$ and $M_l$ only**

- moving shock: Galilean transformation
  - $\Rightarrow$ leaves all thermodynamic quantities unchanged
  - $\Rightarrow$ change to parameters $\alpha = \frac{\gamma+1}{\gamma-1}$ and $P = \frac{p_l}{p_r}$
  - $\Rightarrow$ stationary shock obeys

$$\frac{v_l}{v_r} = \frac{\alpha + P}{\alpha P + 1} = \frac{\rho_r}{\rho_l}$$

  - $\Rightarrow$ three parameters for a moving shock: $\alpha$, $P$, shock speed $s$ give

$$\frac{v_l - s}{v_r - s} = \frac{\alpha + P}{\alpha P + 1} = \frac{\rho_r}{\rho_l}$$

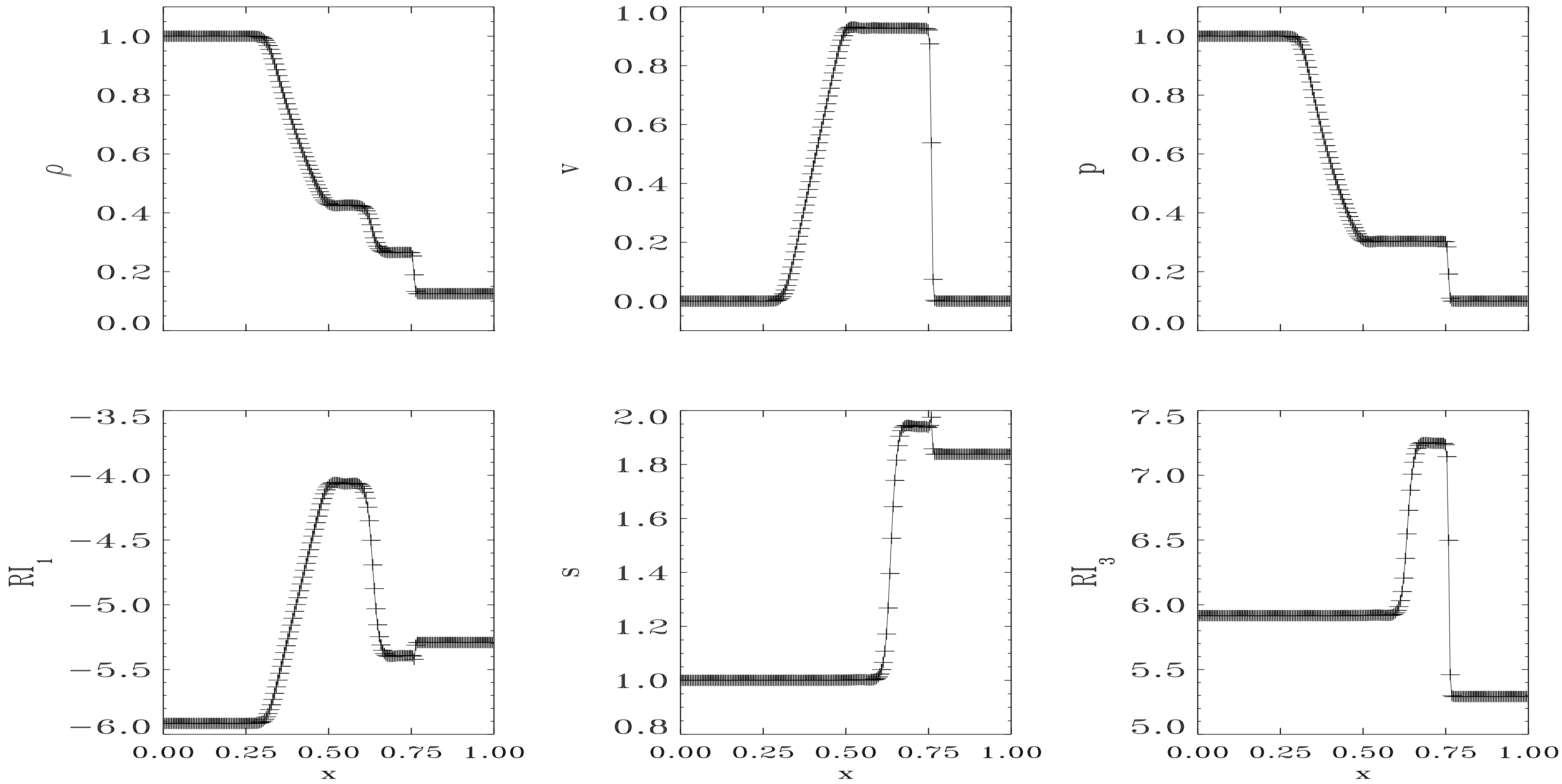
  - $\Rightarrow$ while also

$$(s - v_l)^2 = c_l^2 \left[1 + \frac{\gamma + 1}{2\gamma}\left(\frac{p_r}{p_l} - 1\right)\right]$$

## Numerical tests

- Perform series of Riemann Problem calculations for 1D Euler
  - $\Rightarrow$ always use **2nd order accurate, conservative, TVDLF discretization**
  - $\Rightarrow$ TVDLF is 'Total Variation Diminishing Lax-Friedrichs' scheme
  - $\Rightarrow$ monotonicity preserving, but diffusive especially at CD
  - $\Rightarrow$ 200 grid points on $[0, 1]$, $\gamma = 1.4$
  - $\Rightarrow$ BCs: $\partial x = 0$
- Start with **classical 'Sod' problem**
  - $\Rightarrow$ $U_l = (\rho_l,\ v_l,\ p_l) = (1, 0, 1)$ and $U_r = (0.125, 0, 0.1)$
  - $\Rightarrow$ **'shock tube problem'**: **diaphragm separates 2 gases at rest**

- Sod problem at $t = 0.15$

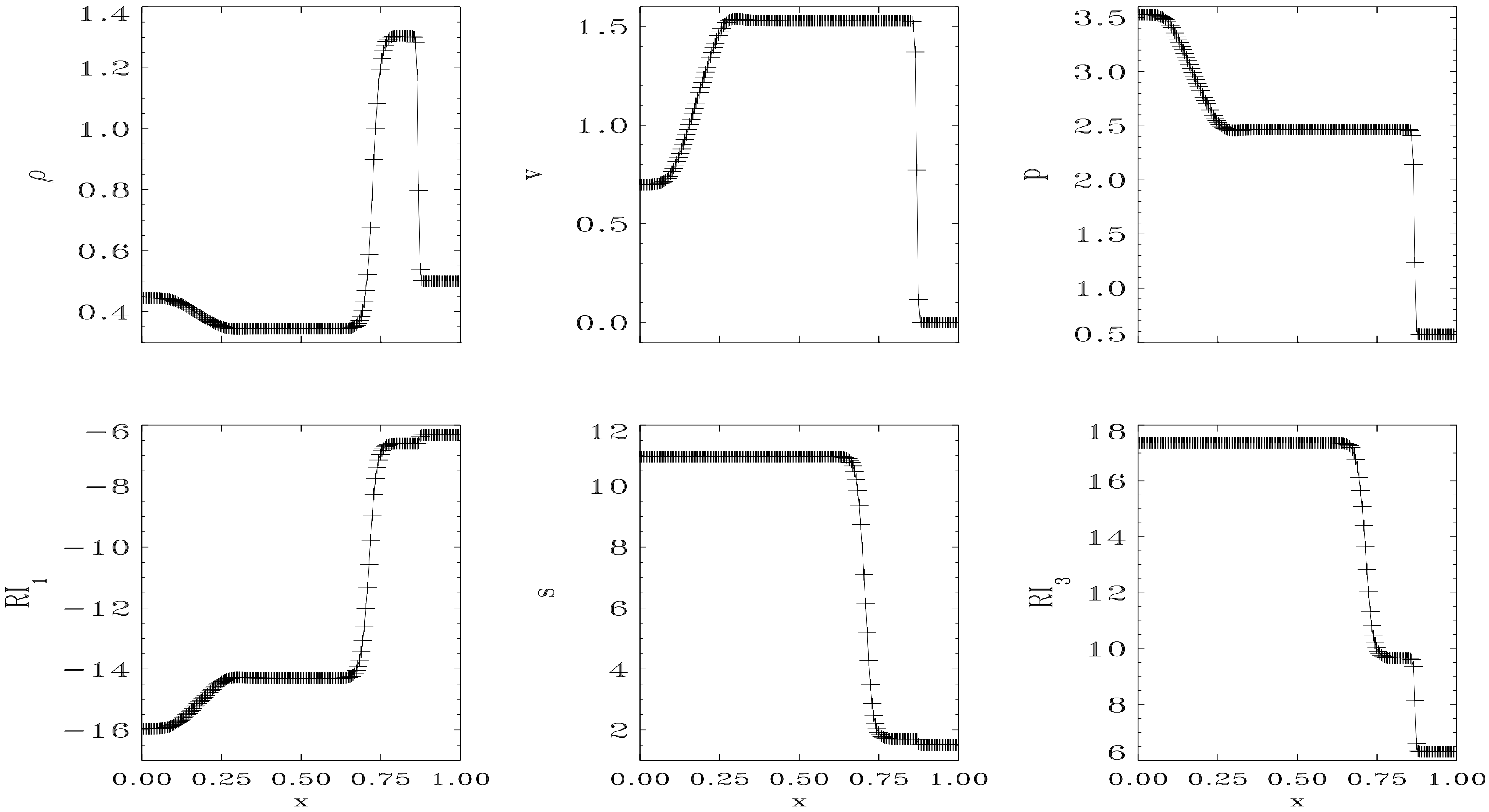



$\Rightarrow$ note $R_1$ where Riemann Invariants $s$ and $v + 2c/(\gamma - 1)$ are constant

$\Rightarrow$ CD spread over many cells

- **test case from Lax**: initial rightwardly moving left state, till $t = 0.15$

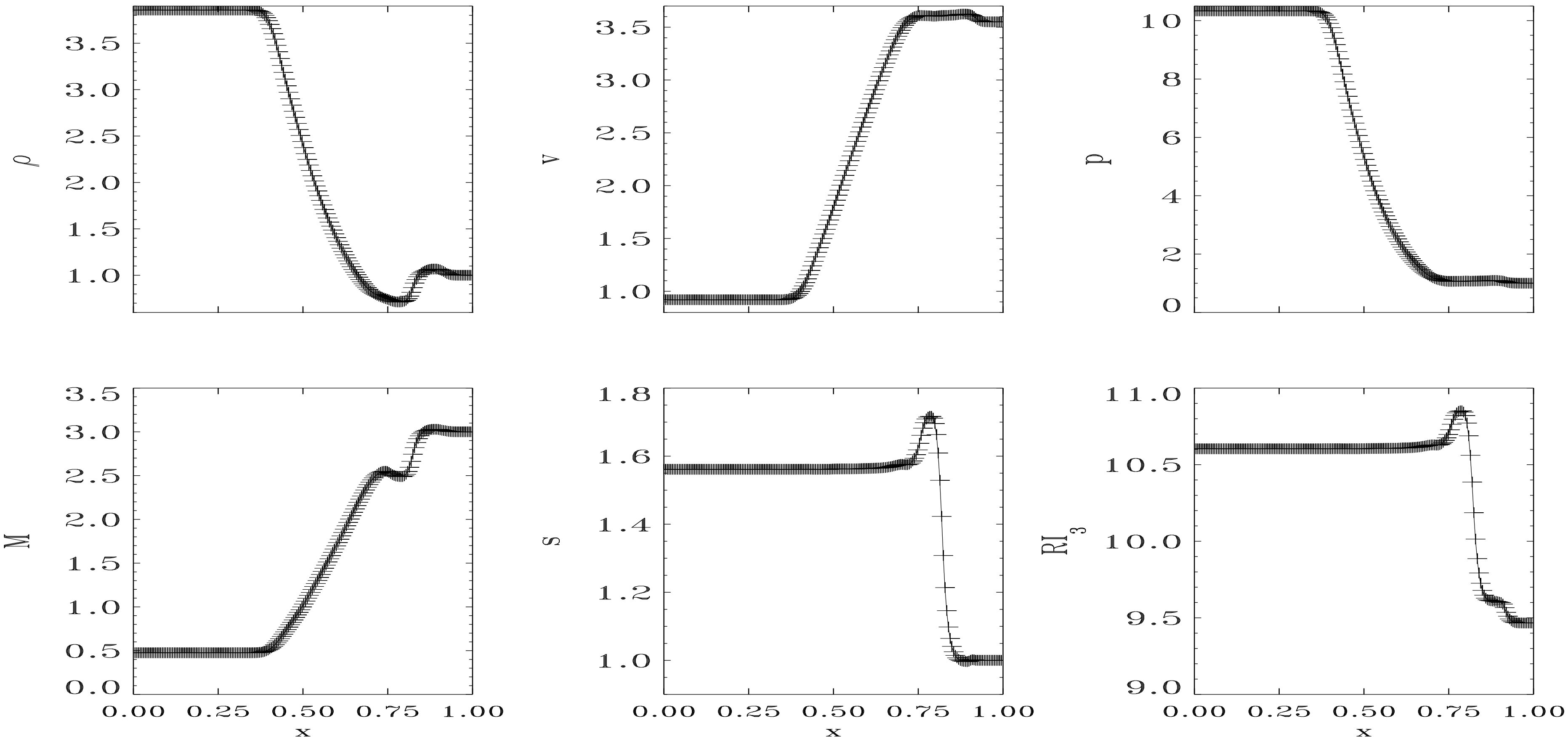
  $\Rightarrow U_l = (\rho_l,\, v_l,\, p_l) = (0.445, 0.698, 3.528)$ and $U_r = (0.5, 0, 0.571)$



- Sod and Lax test case: remain subsonic $M = v/c_s < 1$

  $\Rightarrow$ **Arora & Roe Mach 3 test case** considers

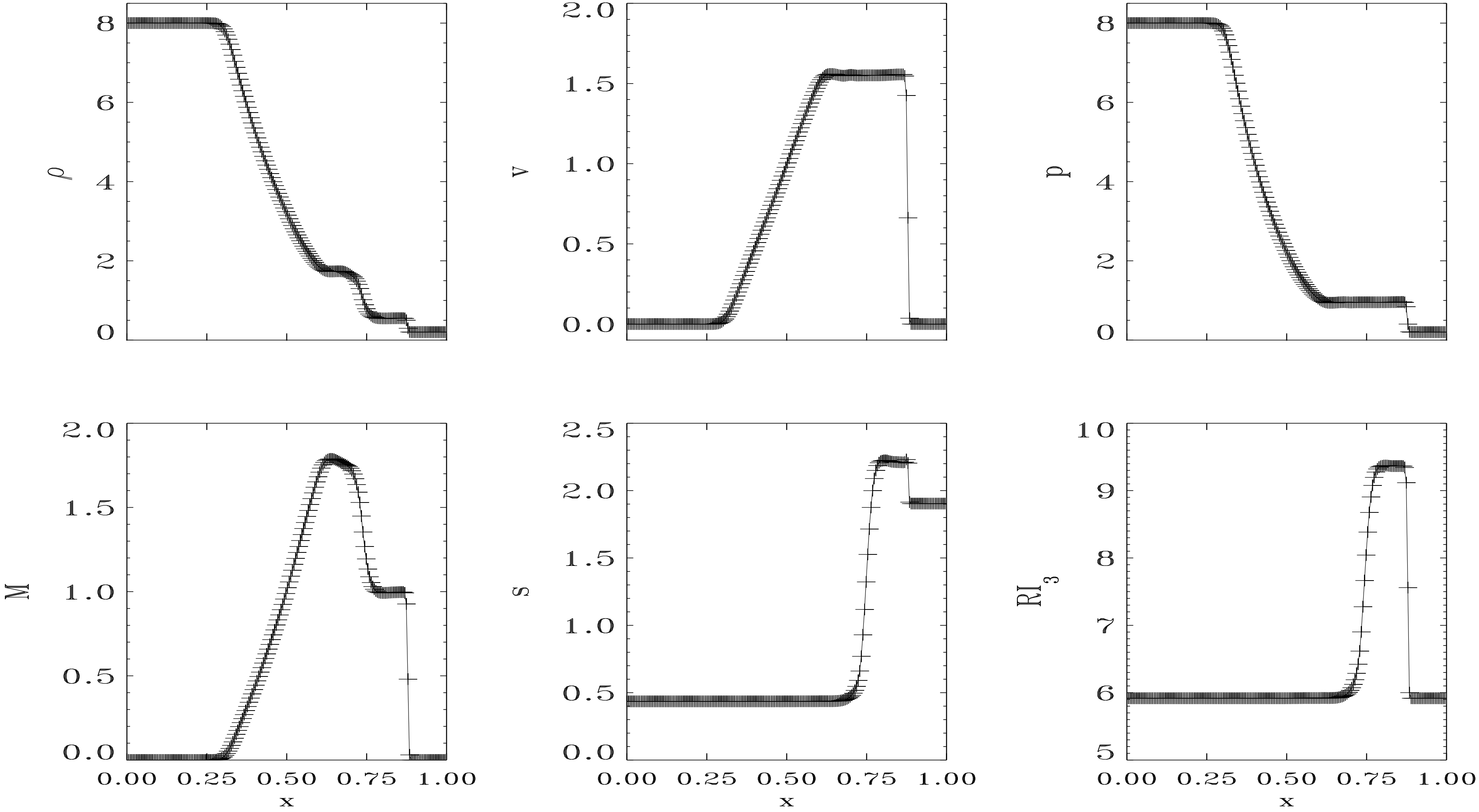
  $\Rightarrow$ $U_l = (\rho_l,\, v_l,\, p_l) = (3.857, 0.92, 10.333)$ and $U_r = (1, 3.55, 1)$




  $\Rightarrow$ solution at $t = 0.09$

- **supersonic shock tube problem** at time $t = 0.1562$

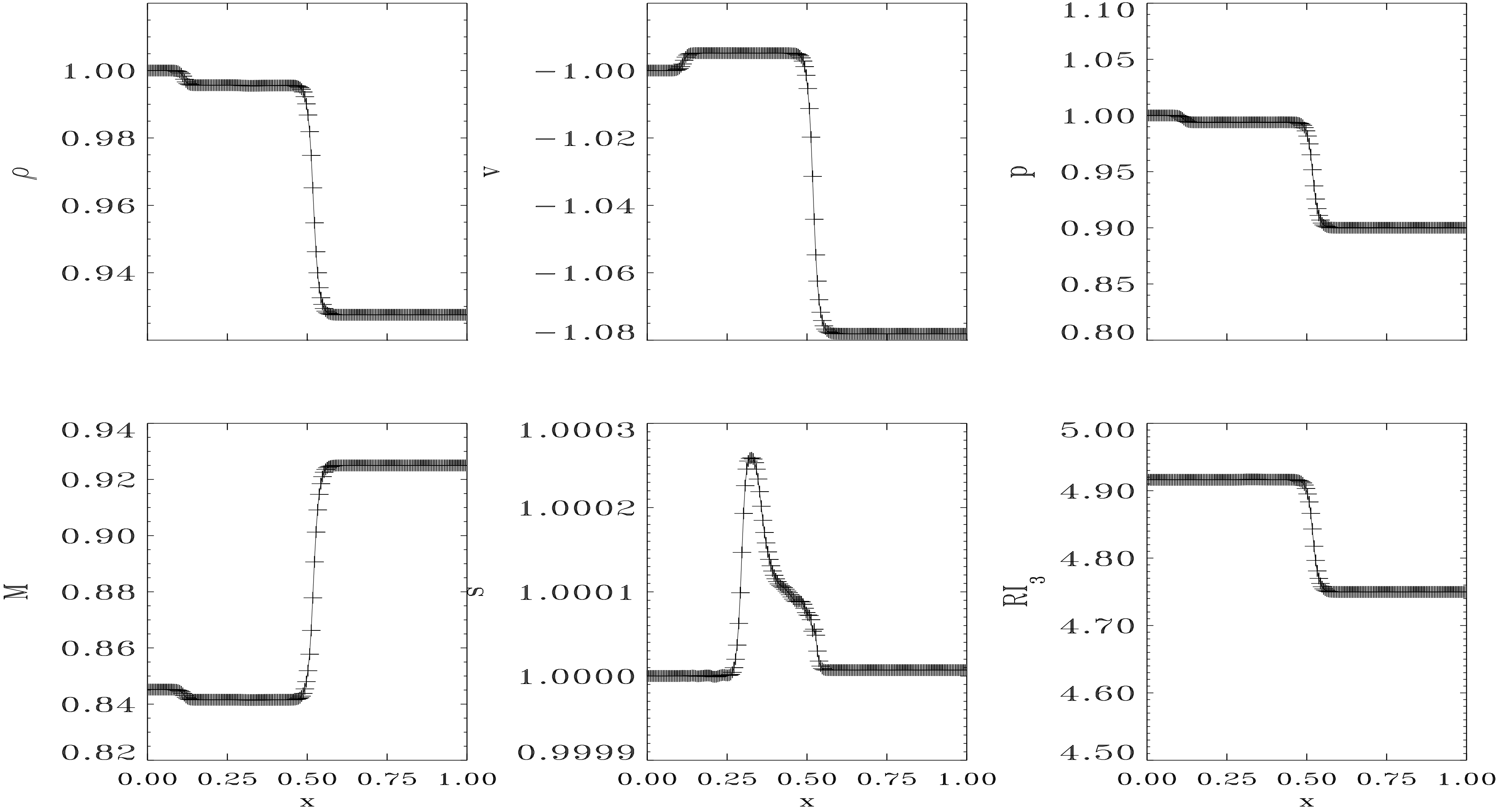
  $\Rightarrow U_l = (\rho_l,\, v_l,\, p_l) = (8, 0, 8)$ and $U_r = (0.2, 0, 0.2)$




$\Rightarrow$ better behaviour at contact than in Mach 3 case

- case of a **slowly moving very weak shock**, show $t = 0.175$

  $\Rightarrow U_l = (\rho_l, v_l, p_l) = (1, -1, 1)$ and $U_r = (0.9275, -1.0781, 0.9)$

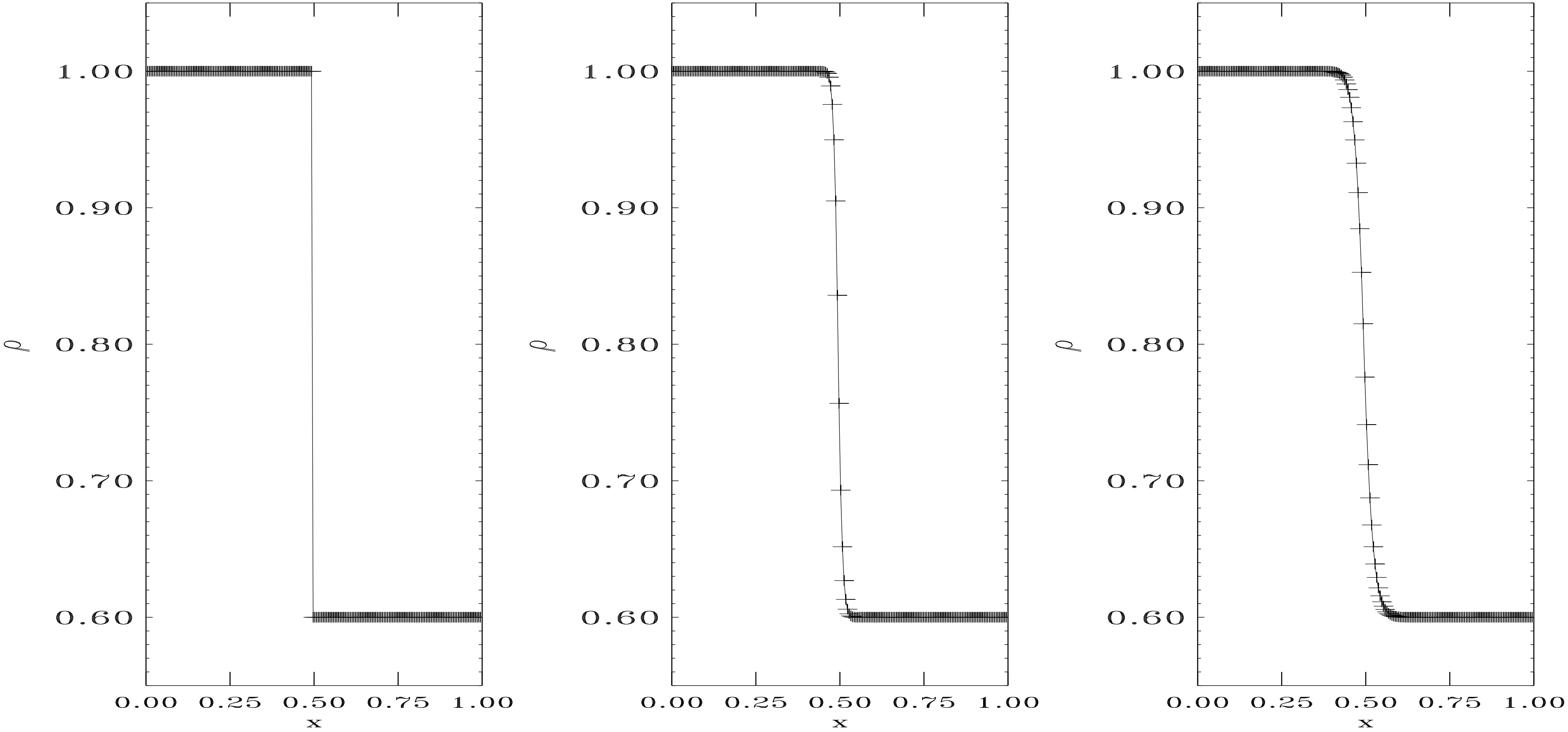



$\Rightarrow$ leftward rarefaction and rightward shock: (too) many cells in shock!

- stationary contact discontinuity

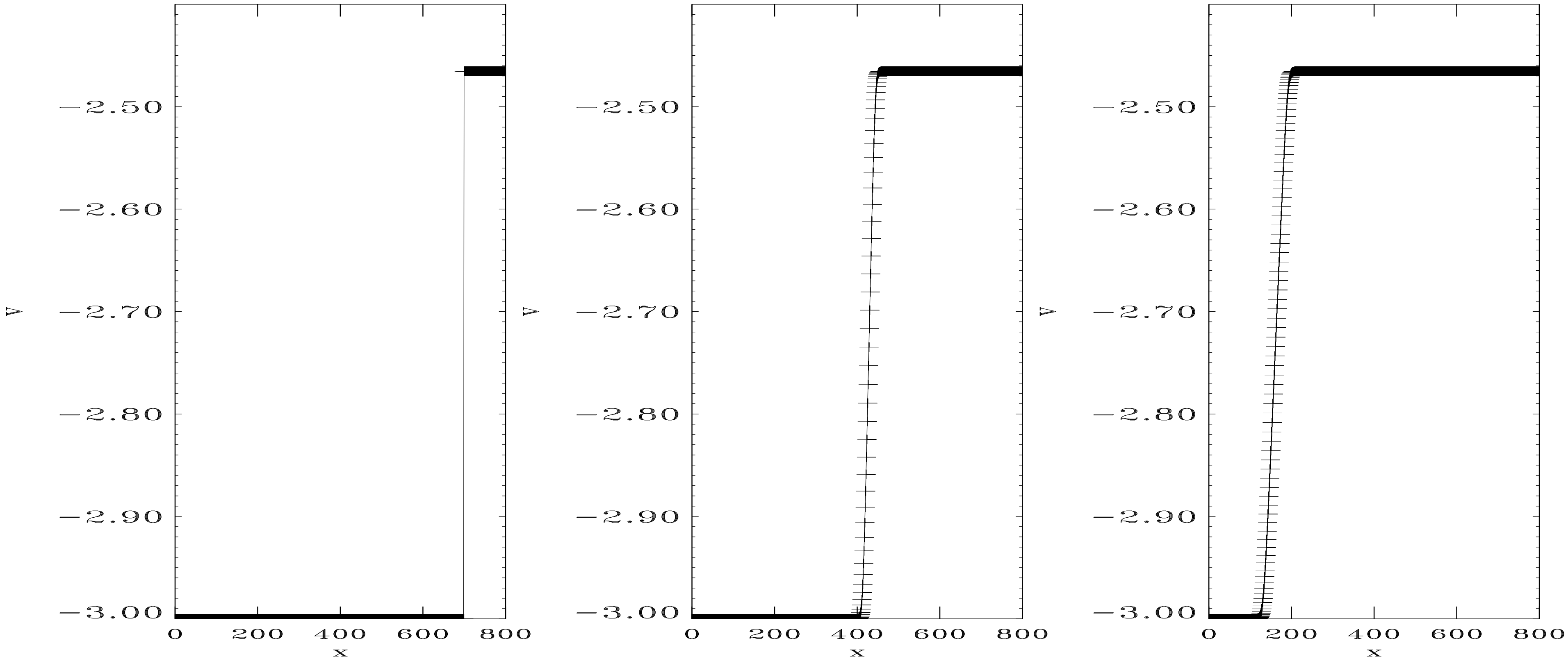
  $\Rightarrow U_l = (\rho_l,\, v_l,\, p_l) = (1, 0, 0.5)$ and $U_r = (0.6, 0, 0.5)$

  $\Rightarrow t = 0$ and $t = 0.1$ and $t = 1$ solution




$\Rightarrow$ diffusion obvious: increasingly (too) many cells in CD!

- **recognizing a rarefaction wave**
  - $\Rightarrow$ 800 cells from $[0, 800]$ with $\gamma = 5/3$
  - $\Rightarrow$ $U_l = (\rho_l,\, v_l,\, p_l) = (1, -3, 10)$ and $U_r = (0.87469, -2.46537, 8)$
  - $\Rightarrow$ $t = 0$ and $t = 40$ and $t = 80$ solution, plot $v$



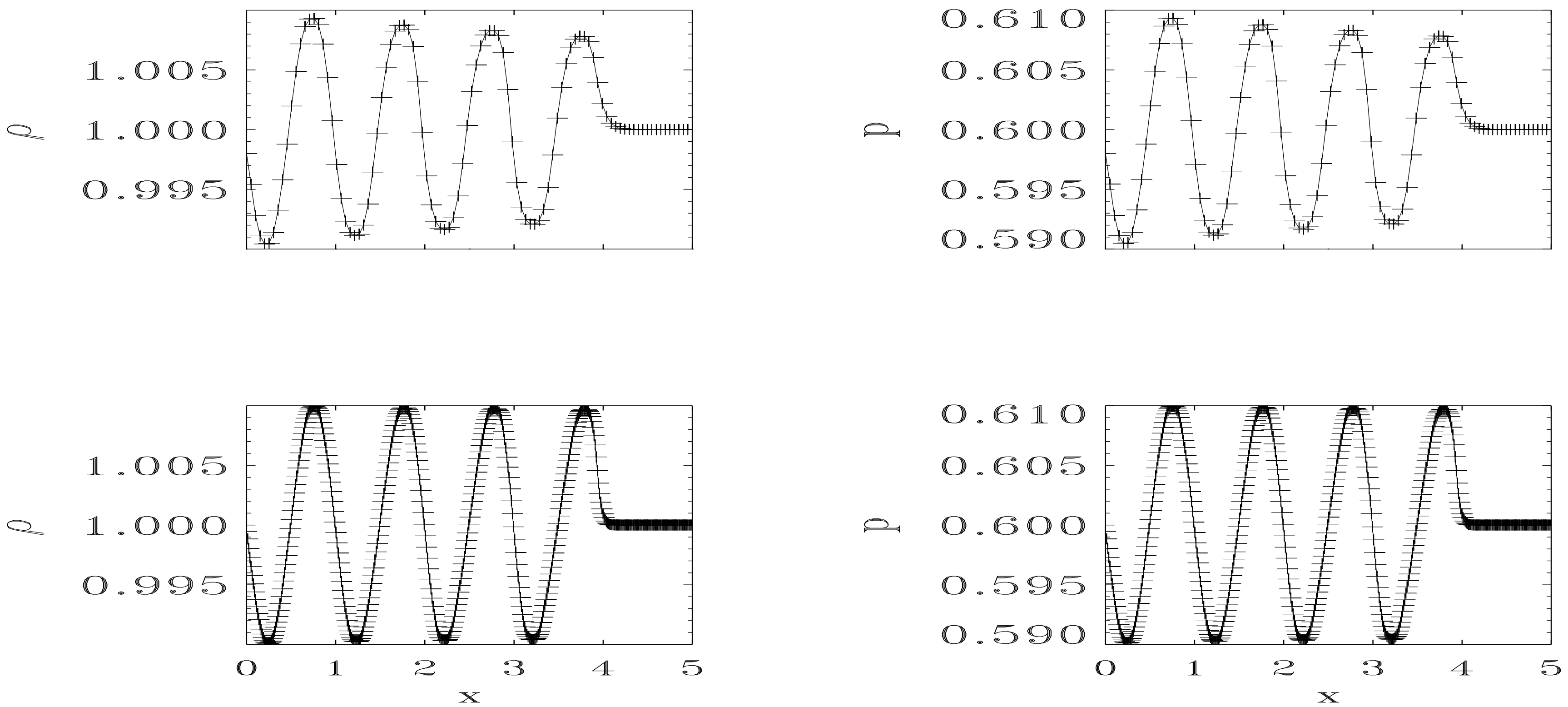

  - $\Rightarrow$ two states with same entropy: rarefaction emerges

- Linear sound waves: time dependent driver $v = A \sin(2\pi t/P)$ at $x = 0$

  $\Rightarrow$ density $\rho(t=0) = 1$, $v(t=0) = 0$, $p(t=0) = 0.6$ with $\gamma = 5/3$

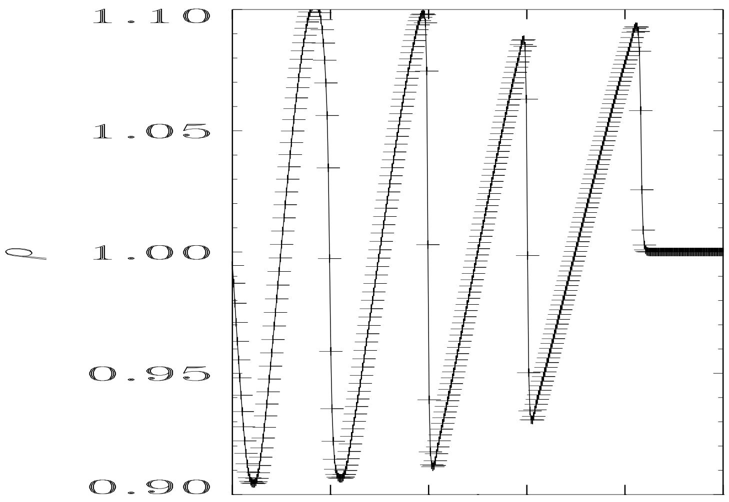
  $\Rightarrow$ $A = 0.02$ with $P = 1$ **generates sound waves** (amplitude $0.01$)

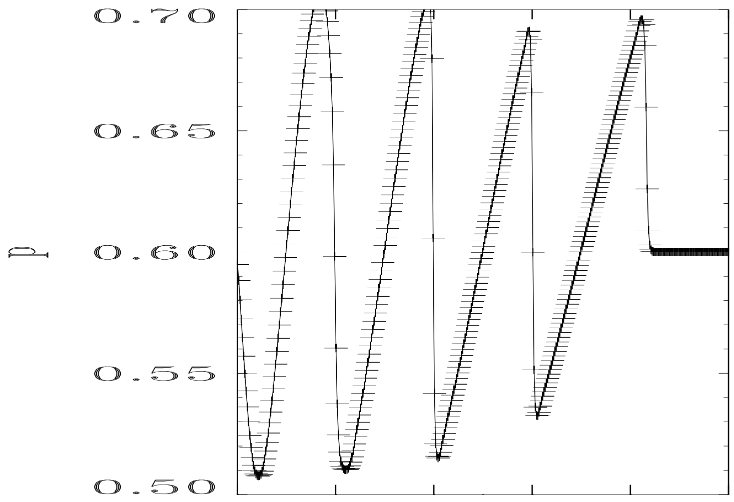
  $\Rightarrow$ compare TVDLF for 100 versus 400 cells at $t = 4$




$\Rightarrow$ to follow linear dynamics: need high resolution to battle diffusion!

- sound wave steepening and shock formation: take amplitude $A = 0.2$




$\Rightarrow$ **nonlinear shock formation well captured**

- Caution to use these methods for pure linear wave processes

$\Rightarrow$ high resolution prerequisite

$\Rightarrow$ seperate true physical diffusion from numerical effects

$\Rightarrow$ note that 10 % variations already imply nonlinear effects!

# Approximate Riemann solver based methods

- modern high resolution, shock-capturing schemes for Euler
  - $\Rightarrow$ capitalize on known solution of the Riemann problem
  - $\Rightarrow$ originally developed by Godunov
- always use conservative scheme of form

$$\frac{dU_i}{dt} + \frac{1}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0$$

  - $\Rightarrow$ cell values $U_i$ change through fluxes across cell edges
  - $\Rightarrow$ edge-centered numerical flux $F_{i+1/2}(U_{i-p}, U_{i-p+1}, \ldots, U_{i+q})$

## The Godunov scheme

- values $U_i^n$ for time $t = t_n$
  - $\Rightarrow$ consider piecewise constant values in cells
  - $\Rightarrow$ serve as initial condition to solve $U_t + (F(U))_x = 0$ for $t > t_n$
  - $\Rightarrow$ restrict timestep to $\Delta t_{n+1} < \frac{\Delta x}{2\max|\lambda|}$
  - $\Rightarrow$ with $\lambda$ eigenvalue of flux Jacobian $F_U$
  - $\Rightarrow$ then exact solution given by solving RP at cell interfaces
  - $\Rightarrow$ restriction on timestep ensures no wave interaction within $\Delta t_{n+1}$
- Godunov scheme
  - $\Rightarrow$ denote exact RP solution for state $U_i^n$ and $U_{i+1}^n$ as $\hat{U}\left(\frac{x-x_{i+1/2}}{t}, U_i^n, U_{i+1}^n\right)$
  - $\Rightarrow$ numerical flux

$$F_{i+1/2}(U_i, U_{i+1}) = F(\hat{U}(0, U_i^n, U_{i+1}^n))$$

  - $\Rightarrow$ need an exact Riemann solver

## The Roe solver

- due to piecewise constant representation

  $\Rightarrow$ Godunov scheme 1st order accurate

- exact solution to RP is complicated

  $\Rightarrow$ scheme is not exact due to piecewise constant representation

  $\Rightarrow$ might as well solve RP in approximate fashion

- schemes exploiting **approximate Riemann solver**

  $\Rightarrow$ use linearization of the nonlinear problem

  $\Rightarrow$ recall: exact solution for linear hyperbolic system known

## Roe-type approximate Riemann solver

- general procedure to solve system $\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = 0$

  $\Rightarrow$ local Riemann problem from left and right interface values $\mathbf{U}_l$ and $\mathbf{U}_r$.

  $\Rightarrow$ instead of exact nonlinear solution, solve a linear Riemann problem

$$\mathbf{U}_t + (\mathbf{G}(\mathbf{U}))_x = 0$$

  $\Rightarrow$ $\mathbf{G}(\mathbf{U}) = \mathbf{F}(\mathbf{U}_r) + A(\mathbf{U} - \mathbf{U}_r)$ includes constant matrix $A = A(\mathbf{U}_l, \mathbf{U}_r)$

  $\Rightarrow$ matrix must satisfy conditions

  1. $\mathbf{F}(\mathbf{U}_l) - \mathbf{F}(\mathbf{U}_r) = A(\mathbf{U}_l, \mathbf{U}_r)(\mathbf{U}_l - \mathbf{U}_r)$,
  2. $A(\mathbf{U}_l, \mathbf{U}_r) \to \mathbf{F}_{\mathbf{U}}(\mathbf{U}_r)$ as $\mathbf{U}_l \to \mathbf{U}_r$,
  3. $A(\mathbf{U}_l, \mathbf{U}_r)$ has only real eigenvalues,
  4. $A(\mathbf{U}_l, \mathbf{U}_r)$ has a complete system of eigenvectors.

  $\Rightarrow$ exact solution obtained when initial states obey Rankine-Hugoniot relations

  $\Rightarrow$ consistency and solvability of the linear Riemann problem.

- if Roe matrix $A$ found, Roe scheme uses the numerical flux

$$\mathbf{F}_{i+1/2}\left(\mathbf{U}_i,\mathbf{U}_{i+1}\right)=\mathbf{F}(\mathbf{U}_i)+A_{i+1/2}\left(\hat{\mathbf{U}}-\mathbf{U}_i\right)$$

$\Rightarrow$ $\hat{\mathbf{U}}=\hat{\mathbf{U}}(0,\mathbf{U}_i,\mathbf{U}_{i+1})$ is exact solution of linear Riemann problem

- Latter solution easy: when $A_{i+1/2}\vec{r^p}=\lambda_p\vec{r^p}$ write

$$\mathbf{U}_{i+1}-\mathbf{U}_i=\sum\alpha_p\vec{r^p}$$

$\Rightarrow$ can show that the solution along the $(x,t)$ ray $(x-x_{i+\frac{1}{2}})/t=0$

$$\hat{\mathbf{U}}=\frac{\mathbf{U}_i+\mathbf{U}_{i+1}}{2}+\frac{1}{2}\left[\sum_{\lambda_p<0}-\sum_{\lambda_p>0}\right]\alpha_p\vec{r^p}$$

- fill in for Roe flux, use first Roe condition to get

$$\boxed{\mathbf{F}_{i+1/2}=\frac{1}{2}\left(\mathbf{F}(\mathbf{U}_i)+\mathbf{F}(\mathbf{U}_{i+1})\right)-\frac{1}{2}\sum\mid\lambda_p\mid\alpha_p\vec{r^p}}$$

$\Rightarrow$ **'upwinding': characteristic speeds and wave directionality taken into account**

- **Needed: Roe matrix $A$, its eigenvalues $\lambda_p$, its right eigenvectors $\vec{r}^p$, and the wave strengths $\alpha_p$**

  $\Rightarrow$ when left eigenvectors $\vec{l}^p$ given, wave strengths

$$\alpha_p = \vec{l}^p \cdot (\mathbf{U}_{i+1} - \mathbf{U}_i)$$

- In practice: use for $A$ the Flux Jacobian $\mathbf{F}_{\mathbf{U}}$, evaluated in average state, e.g. arithmetic average of $\mathbf{U}_l$ and $\mathbf{U}_r$

  $\Rightarrow$ then not all Roe conditions fullfilled though

  $\Rightarrow$ all ingredients known: eigenvalues (characteristic speeds), eigenvectors from $R$, left eigenvectors from $R^{-1}$

- Note: wave strengths can also be computed from primitive jumps

$$\alpha_p = \vec{\mathbf{l}}^p \cdot (\mathbf{V}_{i+1} - \mathbf{V}_i)$$

  $\Rightarrow$ left eigenvectors $\vec{\mathbf{l}}^p$ are found from rows of

$$\mathbf{R}^{-1} = R^{-1}\mathbf{U}_{\mathbf{V}}$$

  $\Rightarrow$ where transformation matrix relates primitive with conservative formulation

$$d\mathbf{U} = \mathbf{U}_{\mathbf{V}} d\mathbf{V}$$

- solver completely determined once matrix $A_{i+1/2}$ that satisfies the Roe conditions is constructed

$\Rightarrow$ recall that $U = \begin{pmatrix} \rho \\ m \\ e \end{pmatrix}$ and $F = \begin{pmatrix} m \\ \frac{m^2}{\rho}\frac{3-\gamma}{2} + (\gamma-1)e \\ \frac{em}{\rho}\gamma - \frac{\gamma-1}{2}\frac{m^3}{\rho^2} \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(e+p) \end{pmatrix}$

$\Rightarrow$ introduce vector $Z = \begin{pmatrix} \sqrt{\rho} \\ \sqrt{\rho} v \\ (e+p)/\sqrt{\rho} \end{pmatrix}$

$\Rightarrow$ find that

$$U = \begin{pmatrix} z_1 z_1 \\ z_1 z_2 \\ \frac{1}{\gamma} z_1 z_3 + \frac{\gamma-1}{2\gamma} z_2 z_2 \end{pmatrix}$$

$\Rightarrow$ and similarly

$$F = \begin{pmatrix} z_1 z_2 \\ \frac{\gamma+1}{2\gamma} z_2 z_2 + \frac{\gamma-1}{\gamma} z_1 z_3 \\ z_2 z_3 \end{pmatrix}$$

$\Rightarrow$ both $U$ and $F$ are quadratic functions of elements of $Z$

- define difference $\delta a = a_{i+1} - a_i$
  - $\Rightarrow$ verify existence of matrices $B$ and $C$ such that
  - $\Rightarrow$ $\delta U = B\delta Z$ and $\delta F = C\delta Z$
  - $\Rightarrow$ matrices $B$ and $C$ have elements linear in $\bar{z} = \frac{1}{2}\left(z_i + z_{i+1}\right)$
- matrix $A_{i+1/2} \equiv C\, B^{-1}$ then satisfies
  - $\Rightarrow$ $A_{i+1/2}\delta U = \delta F$
  - $\Rightarrow$ **first Roe condition**

- define so-called **Roe-averages** as

$$\bar{v} \equiv \frac{\sqrt{\rho_i}v_i + \sqrt{\rho_{i+1}}v_{i+1}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}$$

$$\bar{h} \equiv \frac{\sqrt{\rho_i}h_i + \sqrt{\rho_{i+1}}h_{i+1}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}$$

$\Rightarrow$ latter uses the total specific enthalpy $h = \frac{e+p}{\rho}$

$\Rightarrow$ can write matrix $A_{i+1/2}$ as

$$A_{i+1/2} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\bar{v}^2 & (3-\gamma)\bar{v} & \gamma - 1 \\ \frac{\gamma-1}{2}\bar{v}^3 - \bar{v}\bar{h} & \bar{h} - (\gamma-1)\bar{v}^2 & \gamma\bar{v} \end{pmatrix}$$

$\Rightarrow$ by inspection, this is equal to $F_U(\bar{U})$

$\Rightarrow$ **flux Jacobian evaluated at Roe averaged state** $\bar{U}$

$\Rightarrow$ thus satisfies all other Roe conditions

- can be shown that matrix satisfying all Roe conditions is unique

- all ingredients for Roe flux now known
  - $\Rightarrow$ eigenvalues $\lambda_1 = \bar{v} - \bar{c}$, $\lambda_2 = \bar{v}$, $\lambda_3 = \bar{v} + \bar{c}$
  - $\Rightarrow$ uses sound speed from $\bar{c}^2 = (\gamma - 1)\left(\bar{h} - \frac{\bar{v}^2}{2}\right)$
  - $\Rightarrow$ recall eigenvectors $\vec{r}_1 = \begin{pmatrix} 1 \\ \bar{v} - \bar{c} \\ \bar{h} - \bar{v}\bar{c} \end{pmatrix}$ $\vec{r}_2 = \begin{pmatrix} 1 \\ \bar{v} \\ \frac{\bar{v}^2}{2} \end{pmatrix}$ $\vec{r}_3 = \begin{pmatrix} 1 \\ \bar{v} + \bar{c} \\ \bar{h} + \bar{v}\bar{c} \end{pmatrix}$
  - $\Rightarrow$ coefficients $\alpha_p = \vec{r}^p \cdot (U_{i+1} - U_i)$
  - $\Rightarrow$ with orthogonal left eigenvectors $\vec{l}^p$ from $\vec{l}^p \cdot \vec{r}_q = \delta^p_q$ given by

$$\vec{l}^1 = \left(\frac{\bar{v}}{4\bar{c}}(2 + (\gamma - 1)\frac{\bar{v}}{\bar{c}}),\ -\frac{1}{2\bar{c}}(1 + (\gamma - 1)\frac{\bar{v}}{\bar{c}}),\ \frac{\gamma - 1}{2}\frac{1}{\bar{c}^2}\right)$$

$$\vec{l}^2 = \left(1 - \frac{\gamma - 1}{2}\frac{\bar{v}^2}{\bar{c}^2},\ (\gamma - 1)\frac{\bar{v}}{\bar{c}^2},\ -(\gamma - 1)\frac{1}{\bar{c}^2}\right)$$

$$\vec{l}^3 = \left(-\frac{\bar{v}}{4\bar{c}}(2 - (\gamma - 1)\frac{\bar{v}}{\bar{c}}),\ \frac{1}{2\bar{c}}(1 - (\gamma - 1)\frac{\bar{v}}{\bar{c}}),\ \frac{\gamma - 1}{2}\frac{1}{\bar{c}^2}\right)$$

- coefficients $\alpha_p$ expressed in primitive variable differences

$$\alpha_1 = \frac{1}{2\bar{c}^2}\left(\Delta p - \bar{c}\bar{\rho}\Delta v\right)$$

$$\alpha_2 = \Delta\rho - \frac{1}{\bar{c}^2}\Delta p$$

$$\alpha_3 = \frac{1}{2\bar{c}^2}\left(\Delta p + \bar{c}\bar{\rho}\Delta v\right)$$

$\Rightarrow$ with $\bar{\rho} = \sqrt{\rho_i \rho_{i+1}}$

$\Rightarrow$ correspondence with characteristic equations

## Numerical tests

- Perform series of Riemann Problem calculations for 1D Euler

  $\Rightarrow$ **compare 2nd order conservative TVDLF with Roe-based TVD**

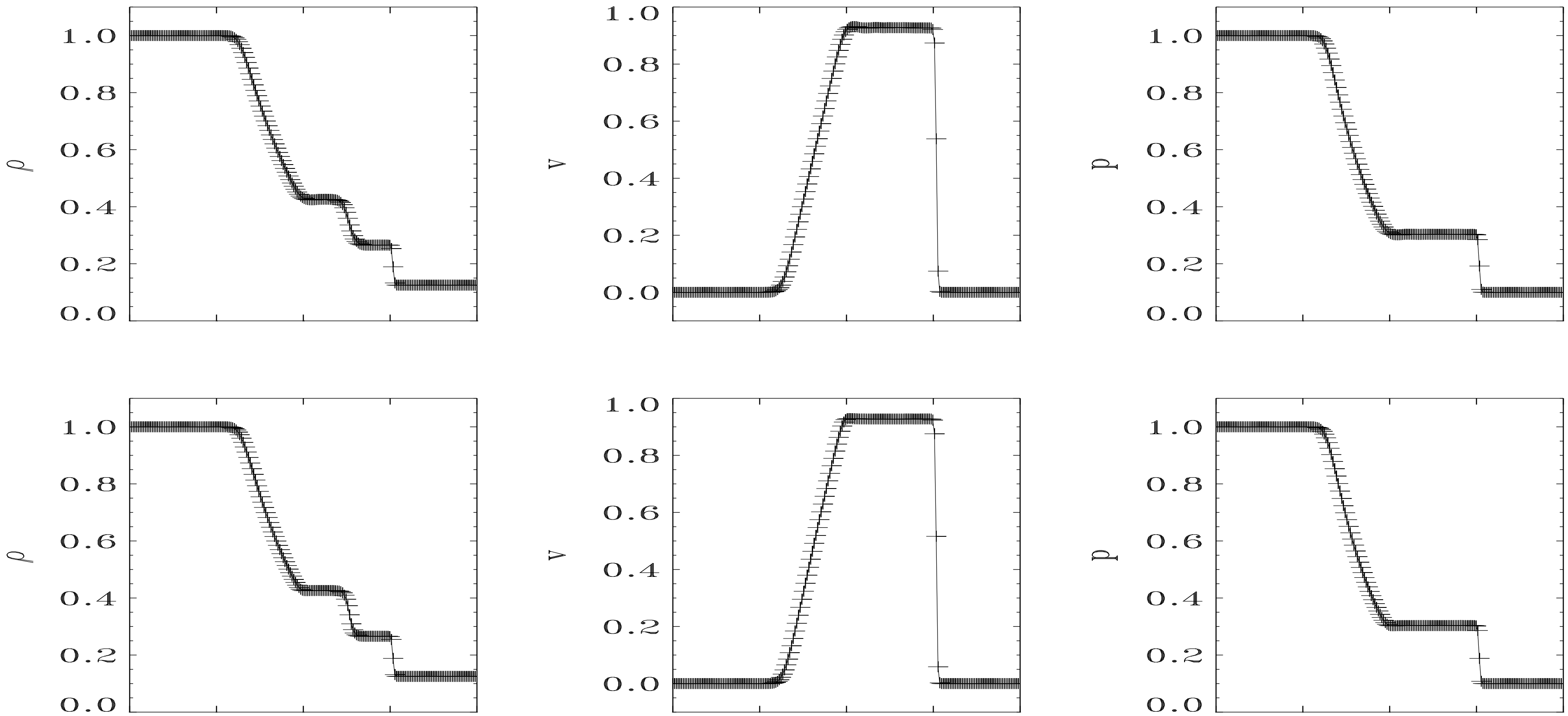
  $\Rightarrow$ 200 grid points on $[0,1]$, $\gamma = 1.4$ and BCs: $\partial x = 0$

- Start with classical 'Sod' problem

  $\Rightarrow$ $U_l = (\rho_l,\, v_l,\, p_l) = (1, 0, 1)$ and $U_r = (0.125, 0, 0.1)$

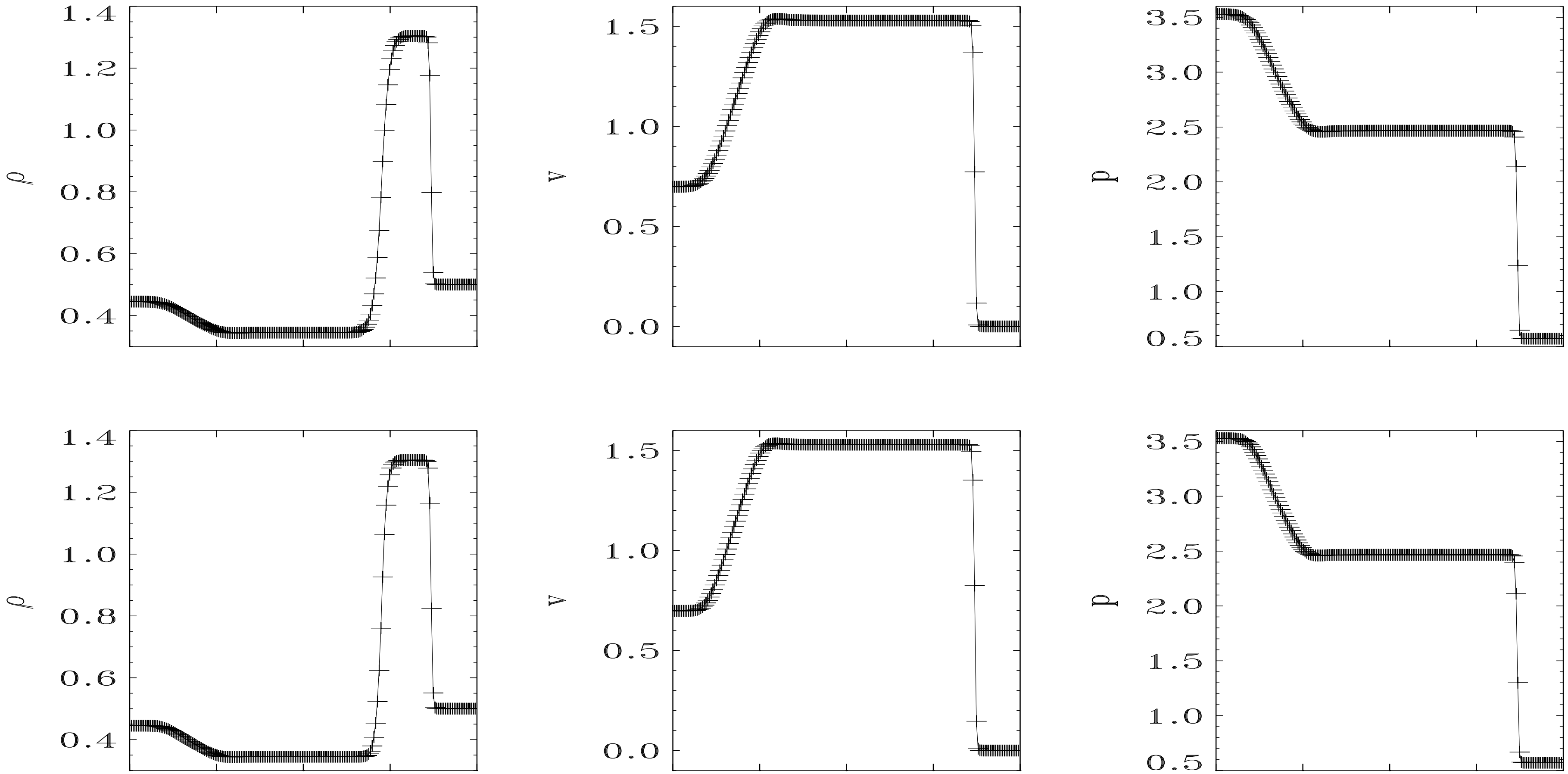
  $\Rightarrow$ **'shock tube problem'**: diaphragm separates 2 gases at rest

- Sod shock tube $t = 0.15$: compare <u>TVDLF</u> (top) with <u>Roe</u> (bottom)




$\Rightarrow$ Roe: slight improvement for CD

- Lax test case, TVDLF (top) versus TVD (bottom)

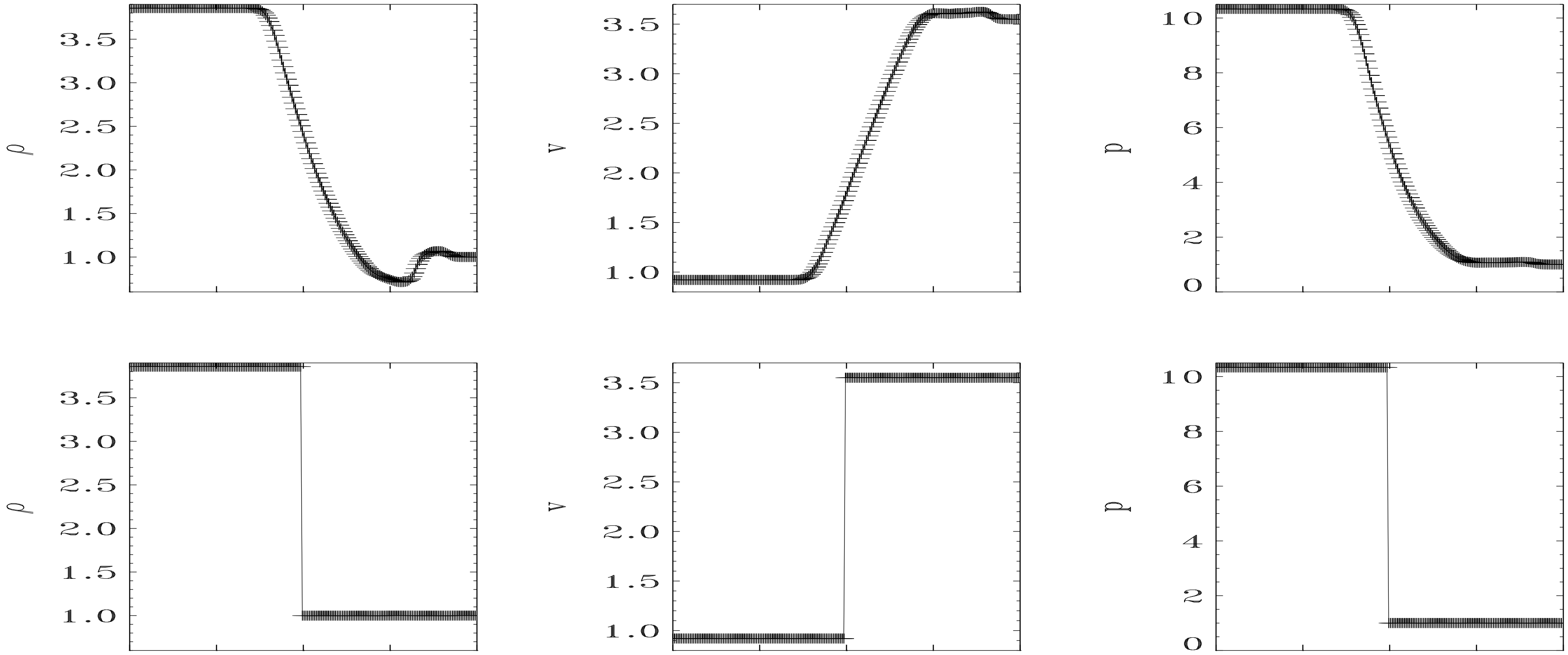



$\Rightarrow$ slight improvement for CD

- Arora & Roe Mach 3 test case considers

  $\Rightarrow U_l = (\rho_l,\, v_l,\, p_l) = (3.857, 0.92, 10.333)$ and $U_r = (1, 3.55, 1)$

- Mach 3 test at $t = 0.09$, TVDLF (top) versus Roe (bottom)




$\Rightarrow$ Roe gives completely wrong solution!!!!

- Erroneous Mach 3 result due to presence of $M = 1$ point

  $\Rightarrow$ $R_1$-rarefaction (expansion fan) goes transonic

  $\Rightarrow$ eigenvalue $\lambda_1 = \bar{v} - \bar{c} = 0$ at sonic point

  $\Rightarrow$ what if simultaneously $\alpha_2 = \alpha_3 = 0$?

  $\Rightarrow$ Roe flux reduces to central discretization

$$F_{i+1/2} = \frac{1}{2}\left(F(U_i) + F(U_{i+1})\right)$$

  $\Rightarrow$ discontinuities are insufficiently smeared out

  $\Rightarrow$ entropy-violating solution may occur (must increase through shock)

- **‘sonic entropy fix’ for Roe scheme**

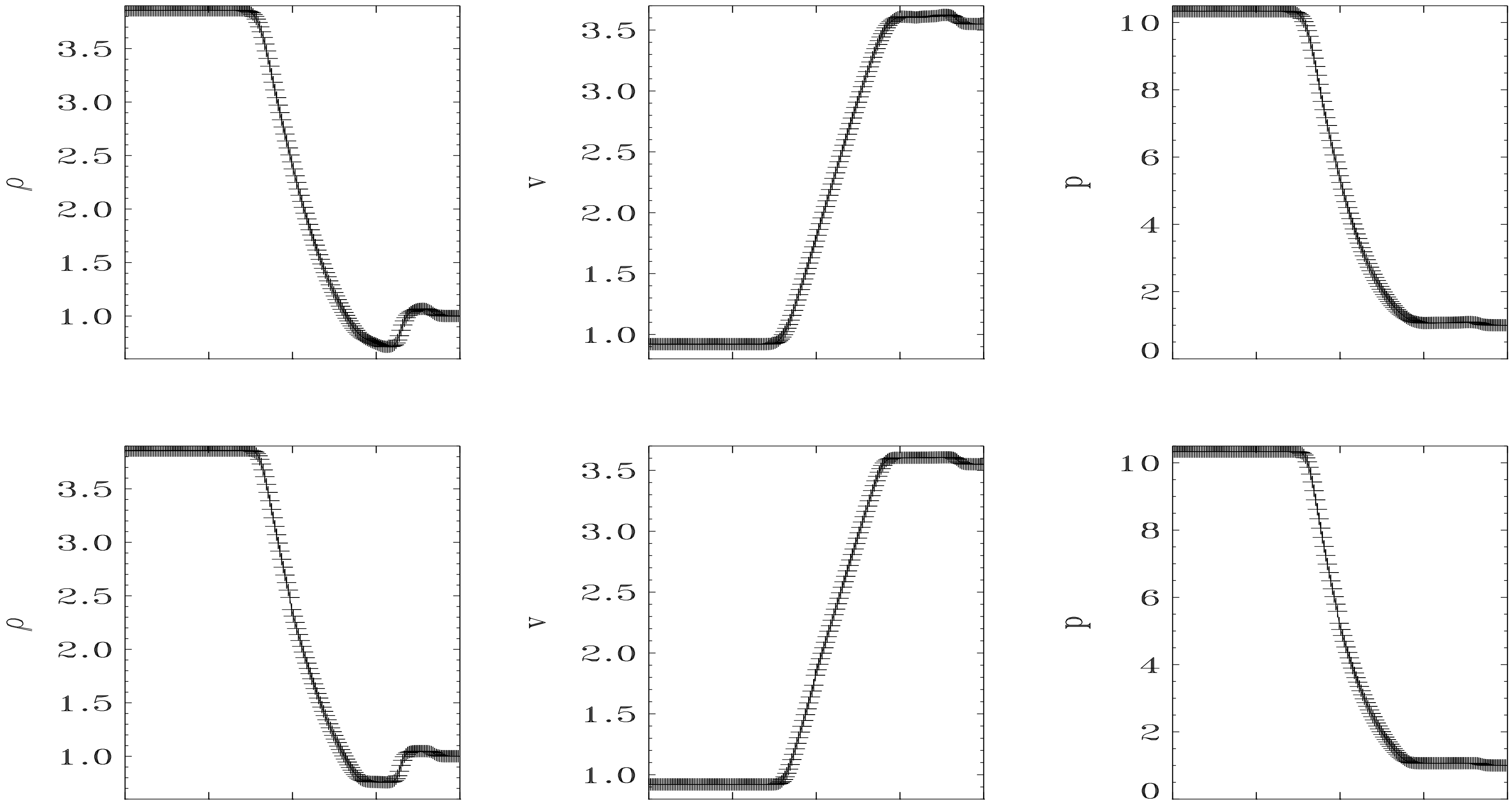
  $\Rightarrow$ replace eigenvalues $\lambda_1$ and $\lambda_3$ in the vicinity of zero

  $\Rightarrow$ if $\mid \lambda_1 \mid < \epsilon$ or $\mid \lambda_3 \mid < \epsilon$

$$\lambda_p \rightarrow \frac{1}{2}\left(\frac{\lambda_p^2}{\epsilon} + \epsilon\right)$$

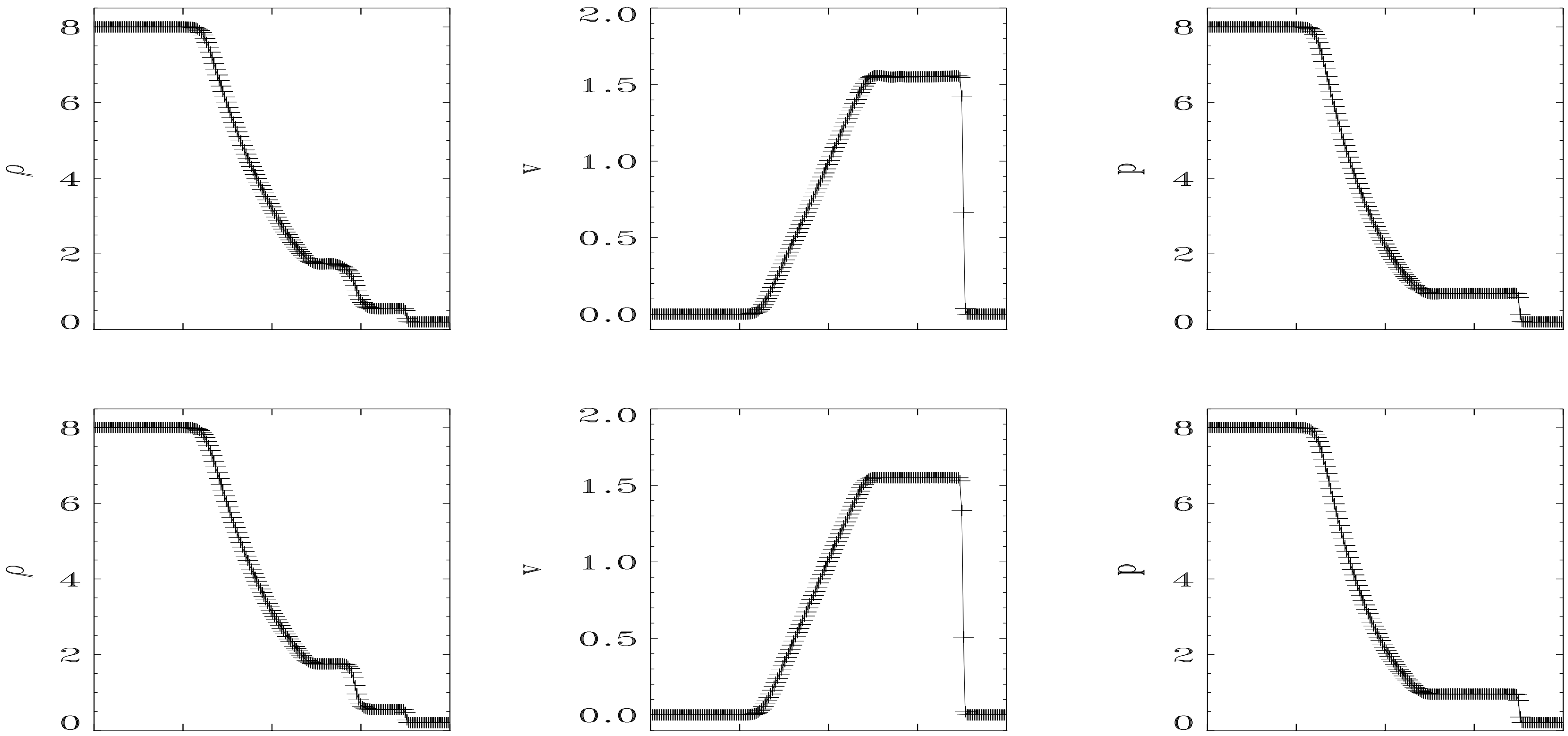
  $\Rightarrow$ with $\epsilon$ a small value, at play where $\mid v \mid \simeq c$

- Repeat Mach 3 with entropy fix:




$\Rightarrow$ **Roe**: (bottom) improvement over TVDLF (top)

- supersonic shock tube test:

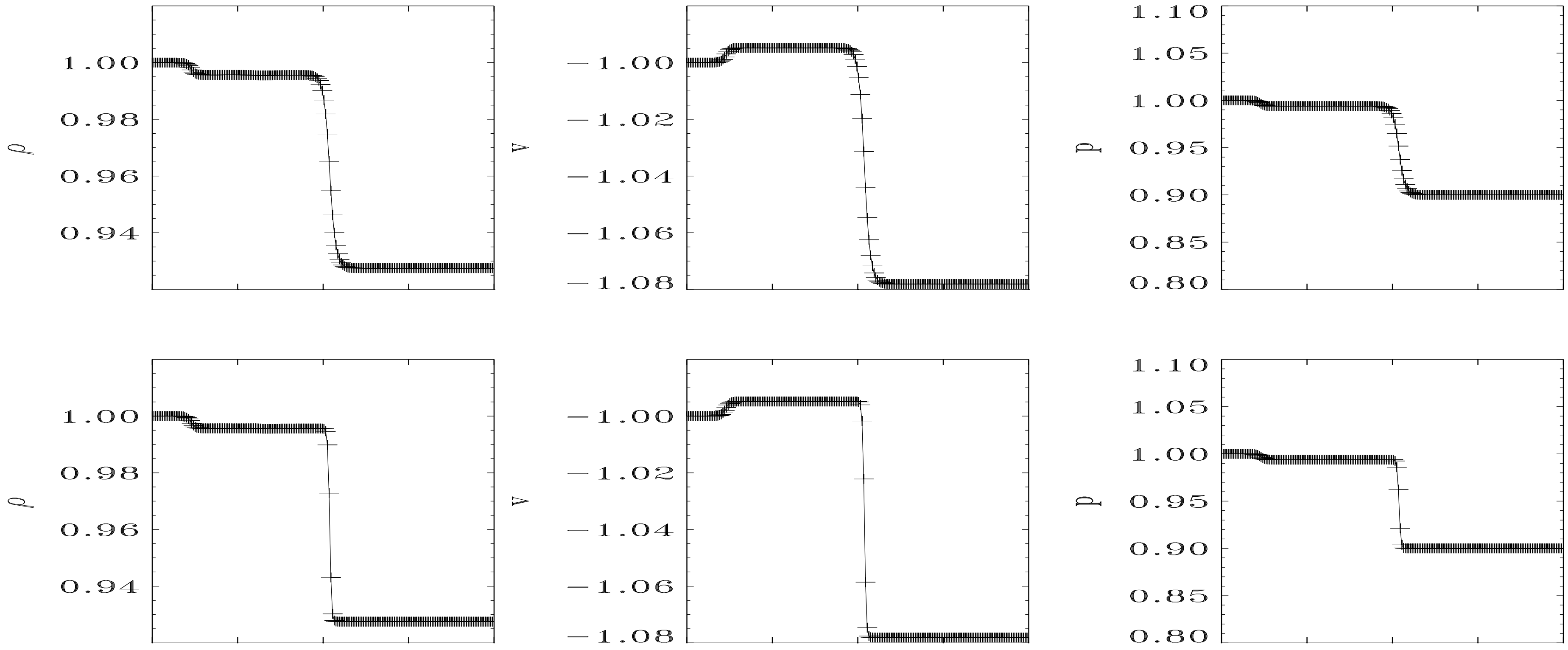



$\Rightarrow$ improvement over TVDLF, better at CD

- case of a slowly moving very weak shock, show $t = 0.175$

  $\Rightarrow$ $U_l = (\rho_l,\, v_l,\, p_l) = (1, -1, 1)$ and $U_r = (0.9275, -1.0781, 0.9)$

  $\Rightarrow$ leftward rarefaction and rightward shock

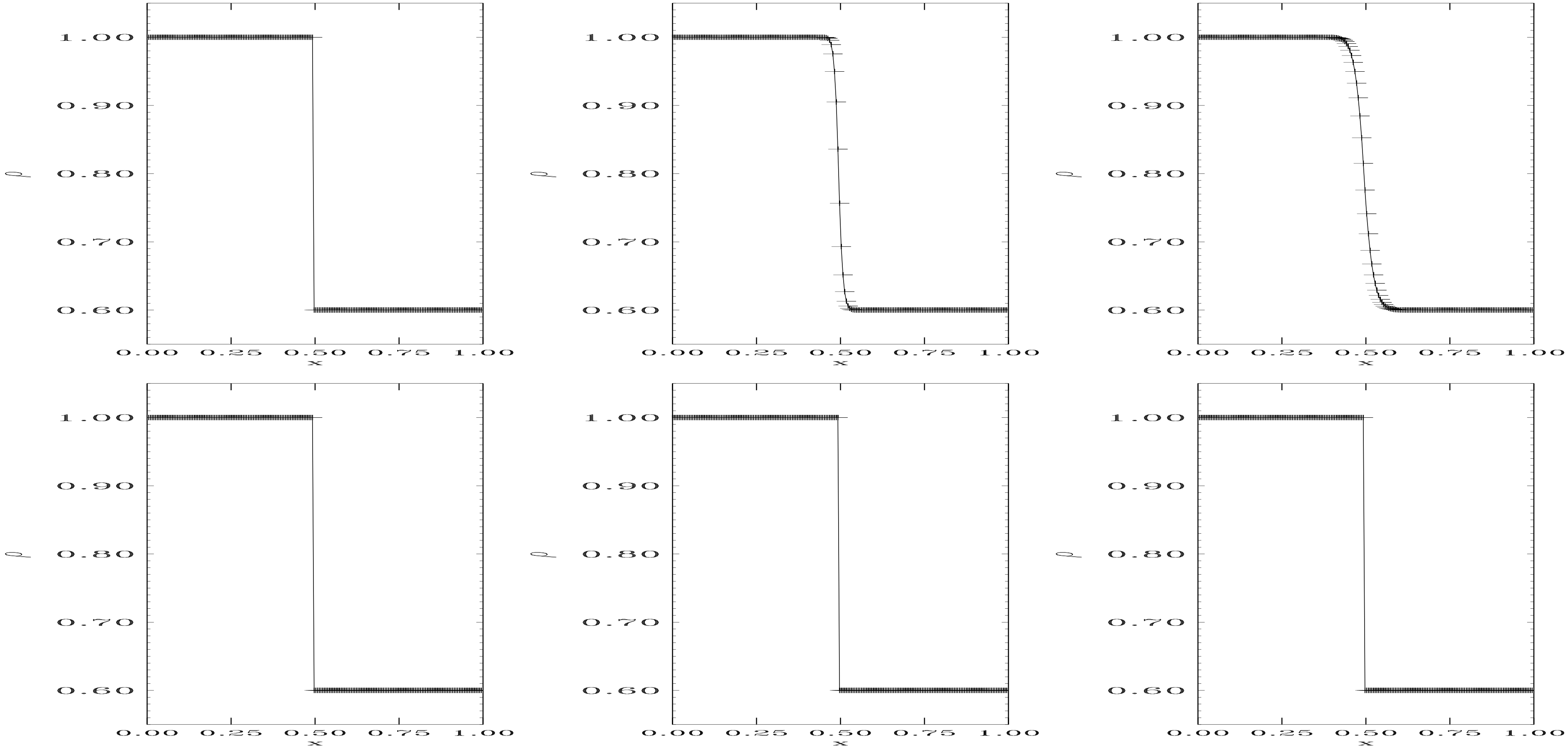



$\Rightarrow$ **Roe (bottom) much better** than TVDLF (top: many cells in shock)!

- stationary contact discontinuity :

  $\Rightarrow$ $U_l = (\rho_l,\ v_l,\ p_l) = (1, 0, 0.5)$ and $U_r = (0.6, 0, 0.5)$

  $\Rightarrow$ $t = 0$ and $t = 0.1$ and $t = 1$ solution




$\Rightarrow$ **Roe (bottom): CD recognized as steady solution, no diffusion!!!**

- summarizing
  - $\Rightarrow$ non-trivial calculation of Roe flux (more involved than TVDLF)
  - $\Rightarrow$ need for entropy fix for transonic expansion fans
  - $\Rightarrow$ better representation of CD, especially stationary CD
  - $\Rightarrow$ ok with slowly traveling weak shocks
- note: conclusions for 2nd order variants of TVDLF and Roe-based TVD
  - $\Rightarrow$ both heavily used in multi-D HD simulations

## Sedov blast wave

- astrophysical application: 'supernova' explosion or 'blast wave'

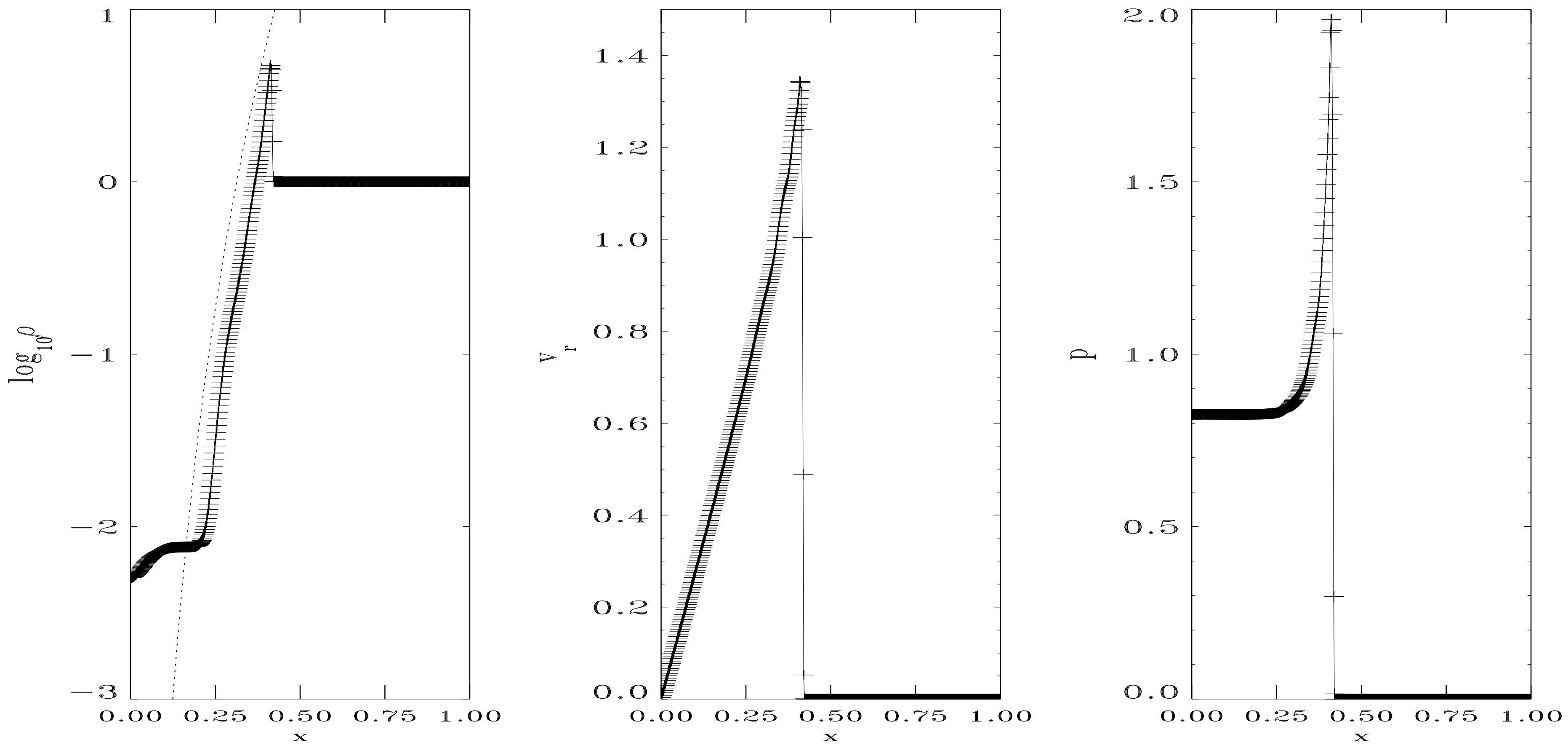
  $\Rightarrow$ modeled as 1D Euler Riemann Problem in spherical symmetry

- initial conditions $v_r = 0$, $\rho = 1$, and $\gamma = 1.4$

  $\Rightarrow$ extreme $p$ jump $p_{exp} = 763.944$ for $r \in [0, 0.05]$ and $p_{ext} = 10^{-5}$

  $\Rightarrow$ take 512 grid points on $r = [0, 1]$ run till time $t = 0.1$ with TVD

- good agreement with analytical result




$\Rightarrow$ $p$ goes to constant value at center

$\Rightarrow$ $v_r$ goes to linear profile in $r/R(t)$ with $R(t)$ shock position

$\Rightarrow$ density goes to profile $(r/R(t))^{3/(\gamma-1)}$ (overplotted)