TD 1: mercredi 13/09/2017
Equation de transport à coefficients constants

For a given $a \in \mathbb{R}$, we consider the following linear transport equation in one dimension :

$$\begin{cases} \partial_t \bar{u} + a\ \partial_x \bar{u} = 0, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ \bar{u}(x,0) = u_0(x), & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u_0 \in L^\infty(\mathbb{R})$. Without loss of generality, we assume that $a > 0$. We refer to the chapter 2, subsection 2.2.1, for the continuous framework of this equation. Here we focus on finding u a discrete approximation of $\bar{u}$ thanks to discrete schemes. As in chapter 3, we introduce a discretization of the domain using a regular mesh : $(x_j, t_n) = (j\Delta x, n\Delta t)$, $\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, where Δx, respectively Δt, denotes the space step, respectively the time step. We also denote u_j^n the approximation of $\bar{u}(x_j, t_n)$.

Def: We will say a scheme is L^∞ stable if we can prove the estimate

$$\sup_j \left|u_j^{n+1}\right| \leq \sup_j \left|u_j^n\right|.$$

Def: We will say a scheme is L^2 stable if we can prove the estimate

$$\Delta x \sum_j \left|u_j^{n+1}\right|^2 \leq \Delta x \sum_j \left|u_j^n\right|^2.$$

1 Lax-Wendroff scheme

We first focus on the *Lax-Wendroff* scheme :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\ \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{a^2 \Delta t}{2}\ \frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{\Delta x^2} = 0\ . \tag{2}$$

Q1. Truncation error

The exact solution $\bar{u}$ of (1) is generally not a solution of the scheme (2). The truncation error estimates the difference. Let us assume that the solution of (1) is such that $\bar{u} \in C^3(\mathbb{R} \times \mathbb{R}^+)$.

1. Prove that, for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $\partial_{tt}\bar{u} = a^2\ \partial_{xx}\bar{u}$.
2. Compute the Taylor expansions ("développements limités avec reste de Taylor-Lagrange") at a convenient order of $\bar{u}(x_j, t_{n+1})$, $\bar{u}(x_{j+1}, t_n)$, and $\bar{u}(x_{j-1}, t_n)$ at the point (x_j, t_n).
3. Assuming that enough partial derivatives of $\bar{u}$ are bounded in L^∞ norm by some constant $\mathrm{C} \in \mathbb{R}_*^+$, prove that the absolute value of the truncation error of the Lax-Wendroff scheme is second order both in time and space.

Q2. L^∞ stability

1. Show that, for any non-negative values α, β, γ such that $\alpha + \beta + \gamma = 1$, then
$$\forall x, y, z \in \mathbb{R}, \min(x,y,z) \leq \alpha x + \beta y + \gamma z \leq \max(x,y,z).$$
2. Using (2), find α, β, γ such that $u_j^{n+1} = \alpha u_j^n + \beta u_{j+1}^n + \gamma u_{j-1}^n$.
3. Provide a necessary and sufficient condition on Δt, Δx and a ensuring the non-negativity of the coefficients α, β, γ found at the previous question. Show the L^∞ stability domain of the scheme is degenerated.

2 Schemes overview

- *Centered explicit scheme*

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}\ =\ 0. \tag{3}$$

- *Centered implicit scheme*

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x}\ =\ 0. \tag{4}$$

- *Upwind scheme*

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \dfrac{u_j^n-u_{j-1}^n}{\Delta x}\ =\ 0\ ,\ \text{if } a>0, \\ \dfrac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \dfrac{u_{j+1}^n-u_j^n}{\Delta x}\ =\ 0\ ,\ \text{if } a<0. \end{cases} \tag{5}$$

- *Lax-Friedrichs*

$$\frac{2u_j^{n+1}-u_{j+1}^n-u_{j-1}^n}{2\Delta t}+a\ \frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}\ =\ 0. \tag{6}$$

- *Beam-Warming* if $a>0$,

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \frac{3u_j^n-4u_{j-1}^n+u_{j-2}^n}{2\Delta x}-\frac{a^2\Delta t}{2}\ \frac{u_j^n-2u_{j-1}^n+u_{j-2}^n}{\Delta x^2}\ =\ 0. \tag{7}$$

Q3. We assume that u_0 is a periodic function. Unlike the other schemes, the *centered implicit* scheme does not allow, for a given space index j and a given time index n, to express explicitly u_j^{n+1} in function of the $(u_k^n)_k$. A linear system has to be solved. Construct the matrix of the linear system, prove it is invertible (let A be its matrix: show that $AU=0\Rightarrow U=0$ by computing U^tAU).
Show the L^2 stability unconditionally.

Q4. A finite volume scheme for equation (1) can be written

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\ \frac{f_{j+\frac12}^n-f_{j-\frac12}^n}{\Delta x}\ =\ 0, \tag{8}$$

where $f_{j\pm\frac12}^n$ denotes a numerical flux. We still denote $\omega=\dfrac{a\Delta t}{\Delta x}$.
Check that the *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming* scheme can be seen as a finite volume scheme with

Lax-Wendroff	$f_{j+\frac12}^n=u_j^n+\frac12(1-\omega)(u_{j+1}^n-u_j^n)$
upwind	$f_{j+\frac12}^n=u_j^n$
Lax-Friedrichs	$f_{j+\frac12}^n=\dfrac{u_{j+1}^n+u_j^n}{2}-\dfrac{u_{j+1}^n-u_j^n}{2w}$
Beam-Warming	$f_{j+\frac12}^n=u_j^n+\frac12(1-\omega)(u_j^n-u_{j-1}^n)$

We sum up in the table below some properties of each scheme :

scheme	**stability**	**truncation error**
Lax-Wendroff	L^2 stable under CFL $\lvert a\rvert\Delta t\le\Delta x$ [L^∞ stable if $\lvert a\rvert\Delta t=\Delta x$]	$\mathcal{O}\left((\Delta t)^2+(\Delta x)^2\right)$
centered explicit	unstable	$\mathcal{O}\left(\Delta t+(\Delta x)^2\right)$
centered implicit	unconditionally L^2 stable	$\mathcal{O}\left(\Delta t+(\Delta x)^2\right)$
upwind	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t\le\Delta x$	$\mathcal{O}\left(\Delta t+(\Delta x)^2\right)$
Lax-Friedrichs	L^2 and L^∞ stable under CFL $\lvert a\rvert\Delta t\le\Delta x$	$\mathcal{O}\left(\Delta t+\dfrac{(\Delta x)^2}{\Delta t}\right)$
Beam-Warming	L^2 stable under CFL $\lvert a\rvert\Delta t\le 2\Delta x$	$\mathcal{O}\left((\Delta t)^2+(\Delta x)^2\right)$

Q5. Do you see one advantage to use the *Beam-Warming* scheme ?

Q6. For the following schemes: *Lax-Wendroff*, *upwind*, *Lax-Friedrichs* and *Beam-Warming*, show that if $a\Delta t = \Delta x$, the numerical solution u_j^n is equal to the analytical solution at the discretization point (x_j, t_n).

Q7. By using the same tools as the ones used for the Lax-Wendroff scheme in section one, for each scheme of the table above, check its stability properties and its truncation error.

Q8. Assuming $a > 0$, we introduce the third order scheme,

$$O3 = (1-\delta)LW + \delta BW \quad , \quad \delta = \frac{1+\omega}{3} \tag{9}$$

where LW denotes the Lax-Wendroff scheme and BW denotes the Beam-Warming scheme. Check that this scheme is of order 3 in space and in time.