MEASURE THEORY

Volume 3

D.H.Fremlin

By the same author:

Topological Riesz Spaces and Measure Theory, Cambridge University Press, 1974.
Consequences of Martin's Axiom, Cambridge University Press, 1982.

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Measure Theory, vol. 1, Torres Fremlin, 2000.
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MEASURE THEORY

Volume 3

Measure Algebras

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Dedicated by the Author

to the Publisher

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General introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (§§114-115). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

As I write this Introduction (December 2001), the plan of the work is as follows:

Volume 1: The Irreducible Minimum
Volume 2: Broad Foundations
Volume 3: Measure Algebras
Volume 4: Topological Measure Spaces
Volume 5: Set-theoretic Measure Theory.

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 will assume a wide-ranging competence over large parts of analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

The work as a whole is not yet complete; and when it is finished, it will undoubtedly be too long to be printed as a single volume in any reasonable format. I am therefore publishing it one part at a time. However, drafts of most of the rest are available on the Internet; see `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm` for detailed instructions. For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a $>$ those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 132E is the fifth paragraph in the second section of the third chapter of Volume 1, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(121A)' as an instruction, or even a recommendation, to lift Volume 1 off the shelf and hunt for §121. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or the notation has suddenly become opaque, local cross-references may help you to fill in the gaps.

Each volume will have an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I recommend you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

Introduction to Volume 3

One of the first things one learns, as a student of measure theory, is that sets of measure zero are frequently 'negligible' in the straightforward sense that they can safely be ignored. This is not quite a universal principle, and one of my purposes in writing this treatise is to call attention to the exceptional cases in which 'negligible' sets are important. But very large parts of the theory, including some of the topics already treated in Volume 2, can be expressed in an appropriately abstract language in which negligible sets have been factored out. This is what the present volume is about. A 'measure algebra' is a quotient of an algebra of measurable sets by an ideal of negligible sets; that is, the elements of the measure algebra are equivalence classes of measurable sets. At the cost of an extra layer of abstraction, we obtain a language which can give concise and elegant expression to a substantial proportion of the ideas of measure theory, and which offers insights almost everywhere in the subject.

It is here that I embark wholeheartedly on 'pure' measure theory. I think it is fair to say that the applications of measure theory to other branches of mathematics are more often through measure *spaces* rather than measure *algebras*. Certainly there will be in this volume many theorems of wide importance outside measure theory; but typically their usefulness will be in forms translated back into the language of the first two volumes. But it is also fair to say that the language of measure algebras is the only reasonable way to discuss large parts of a subject which, as pure mathematics, can bear comparison with any.

In the structure of this volume I can distinguish seven 'working' and two 'accessory' chapters. The 'accessory' chapters are 31 and 35. In these I develop the theories of Boolean algebras and Riesz spaces (= vector lattices) which are needed later. As in Volume 2 you have a certain amount of choice in the order in which you take the material. Everything except Chapter 35 depends on Chapter 31, and everything except Chapters 31 and 35 depends on Chapter 32. Chapters 33, 34 and 36 can be taken in any order, but Chapter 36 relies on Chapter 35. (I do not mean that Chapter 33 is never referred to in Chapter 34, nor even that no results from Chapter 33 are relied on in the later chapters. What I mean is that their most important ideas are accessible without learning the material of Chapter 33 properly.) Chapter 37 depends on Chapters 35 and 36. Chapter 38 would be difficult to make sense of without some notion of what has been done in Chapter 33. Chapter 39 uses fragments of Chapters 35 and 36.

The first half of the volume follows almost the only line permitted by the structure of the subject. If we are going to study measure algebras at all, we must know the relevant facts about Boolean algebras (Chapter 31) and how to translate what we know about measure spaces into the new language (Chapter 32). Then we must get a proper grip on the two most important theorems: Maharam's theorem on the classification of measure algebras (Chapter 33) and the von Neumann-Maharam lifting theorem (Chapter 34). Since I am now writing for readers who are committed – I hope, happily committed – to learning as much as they can about the subject, I take the space to push these ideas as far as they can easily go, giving a full classification of closed subalgebras of probability algebras, for instance (§333), and investigating special types of lifting (§§345-346). I mention here three sections interpolated into Chapter 34 (§§342-344) which attack a subtle and important question: when can we expect homomorphisms between measure algebras to be realizable in terms of transformations between measure spaces, as discussed briefly in §235 and elsewhere.

Chapters 36 and 37 are devoted to re-working the ideas of Chapter 24 on 'function spaces' in the more abstract context now available, and relating them to the general Riesz spaces of Chapter 35. I am concerned here not to develop new structures, nor even to prove striking new theorems, but rather to offer new ways of looking at the old ones. Only in the Ergodic Theorem (§372) do I come to a really important new result. Chapter 38 looks at two questions, both obvious ones to ask if you have been trained in twentieth-century pure mathematics: what does the automorphism group of a measure algebra look like, and inside such an automorphism group, what do the conjugacy classes look like? (The second question is a fancy way of asking how to decide, given two automorphisms of one of the structures considered in this volume, whether they are really different, or just copies of each other obtained by looking at the structure a different way up.) Finally, in Chapter 39, I discuss what is known about the question of which Boolean algebras can appear as measure algebras.

Concerning the prerequisites for this volume, we certainly do not need everything in Volume 2. The important chapters there are 21, 23, 24, 25 and 27. If you are approaching this volume without having read the earlier parts of this treatise, you will need the Radon-Nikodým theorem and product measures (of arbitrary families of probability spaces), for Maharam's theorem; a simple version of the martingale theorem, for the lifting theorem; and an acquaintance with L^p spaces (particularly, with L^0 spaces) for Chapter 36. But I would recommend the results-only versions of Volumes 1 and 2 in case some reference is totally obscure. Outside measure theory, I call on quite a lot of terms from general topology, but none of the ideas needed are difficult (Baire's and Tychonoff's theorems are the deepest); they are sketched in §§3A3 and 3A4. We do need some functional analysis for Chapters 36 and 39, but very little more than was already used in Volume 2, except that I now call on versions of the Hahn-Banach theorem (§3A5).

In this volume I assume that readers have substantial experience in both real and abstract analysis, and I make few concessions which would not be appropriate when addressing active researchers, except that perhaps I am a little gentler when calling on ideas from set theory and general topology than I should be with my own colleagues, and I continue to include all the easiest exercises I can think of. I do maintain my practice of giving proofs in very full detail, not so much because I am trying to make them easier, but because one of my purposes here is to provide a complete account of the ideas of the subject. I hope that the result will be accessible to most doctoral students who are studying topics in, or depending on, measure theory.

Chapter 31

Boolean algebras

The theory of measure algebras naturally depends on certain parts of the general theory of Boolean algebras. In this chapter I collect those results which will be useful later. Since many students encounter the formal notion of Boolean algebra for the first time in this context, I start at the beginning; and indeed I include in the Appendix (§3A2) a brief account of the necessary part of the theory of rings, as not everyone will have had time for this bit of abstract algebra in an undergraduate course. But unless you find the algebraic theory of Boolean algebras so interesting that you wish to study it for its own sake – in which case you should perhaps turn to Sikorski 64 or Koppelberg 89 – I do not think it would be very sensible to read the whole of this chapter before proceeding to the main work of the volume in Chapter 32. Probably §311 is necessary to get an idea of what a Boolean algebra looks like, and a glance at the statements of the theorems in §312 would be useful, but the later sections can wait until you have need of them, on the understanding that apparently innocent formal manipulations may depend on concepts which take some time to master. I hope that the cross-references will be sufficiently well-targeted to make it possible to read this material in parallel with its applications.

311 Boolean algebras

In this section I try to give a sufficient notion of the character of abstract Boolean algebras to make the calculations which will appear on almost every page of this volume seem both elementary and natural. The principal result is of course M.H.Stone's theorem: every Boolean algebra can be expressed as an algebra of sets (311E). So the section divides naturally into the first part, proving Stone's theorem, and the second, consisting of elementary consequences of the theorem and a little practice in using the insights it offers.

311A Definitions (a) A **Boolean ring** is a ring $(\mathfrak{A},+,.)$ in which $a^2=a$ for every $a\in\mathfrak{A}$.

(b) A **Boolean algebra** is a Boolean ring $\mathfrak{A}$ with a multiplicative identity $1=1_{\mathfrak{A}}$; I allow $1=0$ in this context.

Remark For notes on those parts of the elementary theory of rings which we shall need, see §3A2.

I hope that the rather arbitrary use of the word 'algebra' here will give no difficulties; it gives me the freedom to insist that the ring $\{0\}$ should be accepted as a Boolean algebra.

311B Examples (a) For any set X, $(\mathcal{P}X,\triangle,\cap)$ is a Boolean algebra; its zero is $\emptyset$ and its multiplicative identity is X. **P** We have to check the following, which are all easily established, using Venn diagrams or otherwise:

$A\triangle B\subseteq X$ for all A, $B\subseteq X$,

$(A\triangle B)\triangle C=A\triangle(B\triangle C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is a semigroup;

$A\triangle\emptyset=\emptyset\triangle A=A$ for every $A\subseteq X$,

so that $\emptyset$ is the identity in $(\mathcal{P}X,\triangle)$;

$A\triangle A=\emptyset$ for every $A\subseteq X$,

so that every element of $\mathcal{P}X$ is its own inverse in $(\mathcal{P}X,\triangle)$, and $(\mathcal{P}X,\triangle)$ is a group;

$A\triangle B=B\triangle A$ for all A, $B\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is an abelian group;

$A\cap B\subseteq X$ for all A, $B\subseteq X$,

$(A\cap B)\cap C=A\cap(B\cap C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\cap)$ is a semigroup;

$A\cap(B\triangle C)=(A\cap B)\triangle(A\cap C)$, $(A\triangle B)\cap C=(A\cap C)\triangle(B\cap C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\triangle,\cap)$ is a ring;

$A \cap A = A$ for every $A \subseteq X$,

so that $(\mathcal{P}X, \triangle, \cap)$ is a Boolean ring;

$A \cap X = X \cap A = A$ for every $A \subseteq X$,

so that $(\mathcal{P}X, \triangle, \cap)$ is a Boolean algebra and X is its identity. **Q**

(b) Recall that an 'algebra of subsets of X' (136E) is a family $\Sigma \subseteq \mathcal{P}X$ such that $\emptyset \in \Sigma$, $X \setminus E \in \Sigma$ for every $E \in \Sigma$, and $E \cup F \in \Sigma$ for all E, $F \in \Sigma$. In this case $(\Sigma, \triangle, \cap)$ is a Boolean algebra with zero $\emptyset$ and identity X. **P** If E, $F \in \Sigma$, then

$$E \cap F = X \setminus ((X \setminus E) \cup (X \setminus F)) \in \Sigma,$$

$$E \triangle F = (E \cap (X \setminus F)) \cup (F \cap (X \setminus E)) \in \Sigma.$$

Because $\emptyset$ and $X = X \setminus \emptyset$ both belong to Σ, we can work through the identities in (a) above to see that Σ, like $\mathcal{P}X$, is a Boolean algebra. **Q**

(c) Consider the ring $\mathbb{Z}_2 = \{0, 1\}$, with its ring operations $+_2$, $\cdot$ given by setting

$$0 +_2 0 = 1 +_2 1 = 0, \quad 0 +_2 1 = 1 +_2 0 = 1,$$

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

I leave it to you to check, if you have not seen it before, that this is a ring. Because $0 \cdot 0 = 0$ and $1 \cdot 1 = 1$, it is a Boolean algebra.

311C Proposition Let $\mathfrak{A}$ be a Boolean ring.

(a) $a + a = 0$, that is, $a = -a$, for every $a \in \mathfrak{A}$.

(b) $ab = ba$ for all a, $b \in \mathfrak{A}$.

proof (a) If $a \in \mathfrak{A}$, then

$$a + a = (a + a)(a + a) = a^2 + a^2 + a^2 + a^2 = a + a + a + a,$$

so we must have $0 = a + a$.

(b) Now for any a, $b \in \mathfrak{A}$,

$$a + b = (a + b)(a + b) = a^2 + ab + ba + b^2 = a + ab + ba + b,$$

so

$$0 = ab + ba = ab + ab$$

and $ab = ba$.

311D Lemma Let $\mathfrak{A}$ be a Boolean ring, I an ideal of $\mathfrak{A}$ (3A2E), and $a \in \mathfrak{A} \setminus I$. Then there is a ring homomorphism $\phi : \mathfrak{A} \to \mathbb{Z}_2$ such that $\phi a = 1$ and $\phi d = 0$ for every $d \in I$.

proof (a) Let $\mathcal{I}$ be the family of those ideals J of $\mathfrak{A}$ which include I and do not contain a. Then $\mathcal{I}$ has a maximal element K say. **P** Apply Zorn's lemma. Since $I \in \mathcal{I}$, $\mathcal{I} \neq \emptyset$. If $\mathcal{J}$ is a non-empty totally ordered subset of $\mathcal{I}$, then set $J^* = \bigcup \mathcal{J}$. If b, $c \in J^*$ and $d \in \mathfrak{A}$, then there are J_1, $J_2 \in \mathcal{J}$ such that $b \in J_1$ and $c \in J_2$; now $J = J_1 \cup J_2$ is equal to one of J_1, J_2, so belongs to $\mathcal{J}$, and 0, $b + c$, bd all belong to J, so all belong to J^*. Thus $J^* \triangleleft \mathfrak{A}$; of course $I \subseteq J^*$ and $a \notin J^*$, so $J^* \in \mathcal{I}$ and is an upper bound for $\mathcal{J}$ in $\mathcal{I}$. As $\mathcal{J}$ is arbitrary, the hypotheses of Zorn's lemma are satisfied and $\mathcal{I}$ has a maximal element. **Q**

(b) For $b \in \mathfrak{A}$ set $K_b = \{d : d \in \mathfrak{A}, bd \in K\}$. The following are easy to check:

(i) $K \subseteq K_b$ for every $b \in \mathfrak{A}$, because K is an ideal.

(ii) $K_b \triangleleft \mathfrak{A}$ for every $b \in \mathfrak{A}$. **P** $0 \in K \subseteq K_b$. If d, $d' \in K_b$ and $c \in \mathfrak{A}$ then

$$b(d + d') = bd + bd', \quad b(dc) = (bd)c$$

belong to K, so $d + d'$, $dc \in K_b$. **Q**

(iii) If $b \in \mathfrak{A}$ and $a \notin K_b$, then $K_b \in \mathcal{I}$ so $K_b = K$.

(iv) Now $a^2 = a \notin K$, so $a \notin K_a$ and $K_a = K$.

(v) If $b \in \mathfrak{A} \setminus K$ then $b \notin K_a$, that is, $ba = ab \notin K$, and $a \notin K_b$; consequently $K_b = K$.

(vi) If b, $c \in \mathfrak{A} \setminus K$ then $c \notin K_b$ so $bc \notin K$.

(vii) If b, $c \in \mathfrak{A} \setminus K$ then

$$bc(b+c) = b^2c + bc^2 = bc + bc = 0 \in K,$$

so $b + c \in K_{bc}$. By (vi) and (v), $K_{bc} = K$ so $b + c \in K$.

(c) Now define $\phi : \mathfrak{A} \to \mathbb{Z}_2$ by setting $\phi d = 0$ if $d \in K$, $\phi d = 1$ if $d \in \mathfrak{A} \setminus K$. Then ϕ is a ring homomorphism. **P**

(i) If b, $c \in K$ then $b + c$, $bc \in K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(ii) If $b \in K$, $c \in \mathfrak{A} \setminus K$ then

$$c = (b + b) + c = b + (b + c) \notin K$$

so $b + c \notin K$, while $bc \in K$, so

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(iii) Similarly,

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c$$

if $b \in \mathfrak{A} \setminus K$ and $c \in K$.

(iv) If b, $c \in \mathfrak{A} \setminus K$, then by (b-vi) and (b-vii) we have $b + c \in K$, $bc \notin K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 1 = \phi b\, \phi c.$$

Thus ϕ is a ring homomorphism. **Q**

(d) Finally, if $d \in I$ then $d \in K$ so $\phi d = 0$; and $\phi a = 1$ because $a \notin K$.

311E M.H.Stone's Theorem: first form Let $\mathfrak{A}$ be any Boolean ring, and let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Then we have an injective ring homomorphism $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$, setting $\widehat{a} = \{z : z \in Z, z(a) = 1\}$. If $\mathfrak{A}$ is a Boolean algebra, then $\widehat{1_{\mathfrak{A}}} = Z$.

proof (a) If a, $b \in \mathfrak{A}$, then

$$\widehat{a+b} = \{z : z(a+b) = 1\} = \{z : z(a) +_2 z(b) = 1\} = \{z : \{z(a), z(b)\} = \{0,1\}\} = \widehat{a} \triangle \widehat{b},$$

$$\widehat{ab} = \{z : z(ab) = 1\} = \{z : z(a)z(b) = 1\} = \{z : z(a) = z(b) = 1\} = \widehat{a} \cap \widehat{b}.$$

Thus $a \mapsto \widehat{a}$ is a ring homomorphism.

(b) If $a \in \mathfrak{A}$ and $a \neq 0$, then by 311D, with $I = \{0\}$, there is a $z \in Z$ such that $z(a) = 1$, that is, $z \in \widehat{a}$; so that $\widehat{a} \neq \emptyset$. This shows that the kernel of $a \mapsto \widehat{a}$ is $\{0\}$, so that the homomorphism is injective (3A2Db).

(c) If $\mathfrak{A}$ is a Boolean algebra, and $z \in Z$, then there is some $a \in \mathfrak{A}$ such that $z(a) = 1$, so that $z(1_{\mathfrak{A}})z(a) = z(1_{\mathfrak{A}}a) \neq 0$ and $z(1_{\mathfrak{A}}) \neq 0$; thus $\widehat{1_{\mathfrak{A}}} = Z$.

311F Remarks (a) For any Boolean ring $\mathfrak{A}$, I will say that the **Stone space** of $\mathfrak{A}$ is the set Z of non-zero ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, and the canonical map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is the **Stone representation**.

(b) Because the map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is an injective ring homomorphism, $\mathfrak{A}$ is isomorphic, as Boolean ring, to its image $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$, which is a subring of $\mathcal{P}Z$. Thus the Boolean rings $\mathcal{P}X$ of 311Ba are leading examples in a very strong sense.

(c) I have taken the set Z of the Stone representation to be actually the set of homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Of course we could equally well take any set which is in a natural one-to-one correspondence with Z; a popular choice is the set of maximal ideals of $\mathfrak{A}$, since a subset of $\mathfrak{A}$ is a maximal ideal iff it is the kernel of a member of Z, which is then uniquely defined.

311G The operations $\cup$, $\setminus$, $\triangle$ on a Boolean ring Let $\mathfrak{A}$ be a Boolean ring.

(a) Using the Stone representation, we can see that the elementary operations $\cup$, $\cap$, $\setminus$, $\triangle$ of set theory all correspond to operations on $\mathfrak{A}$. If we set

$$a \cup b = a + b + ab, \quad a \cap b = ab, \quad a \setminus b = a + ab, \quad a \triangle b = a + b$$

for a, $b \in \mathfrak{A}$, then we see that

$$\widehat{a \cup b} = \widehat{a} \triangle \widehat{b} \triangle (\widehat{a} \cap \widehat{b}) = \widehat{a} \cup \widehat{b},$$

$$\widehat{a \cap b} = \widehat{a} \cap \widehat{b},$$

$$\widehat{a \setminus b} = \widehat{a} \setminus \widehat{b},$$

$$\widehat{a \triangle b} = \widehat{a} \triangle \widehat{b}.$$

Consequently all the familiar rules for manipulation of $\cap$, $\cup$, etc. will apply also to $\cap$, $\cup$, and we shall have, for instance,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

for any members a, b, c of any Boolean ring $\mathfrak{A}$.

(b) Still importing terminology from elementary set theory, I will say that a set $A \subseteq \mathfrak{A}$ is **disjoint** if $a \cap b = 0$, that is, $ab = 0$, for all distinct a, $b \in A$; and that an indexed family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **disjoint** if $a_i \cap a_j = 0$ for all distinct i, $j \in I$. (Just as I allow $\emptyset$ to be a member of a disjoint family of sets, I allow $0 \in A$ or $a_i = 0$ in the present context.)

(c) A **partition of unity** in $\mathfrak{A}$ will be *either* a disjoint set $C \subseteq \mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c = 0$ for every $c \in C$ *or* a disjoint family $\langle c_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c_i = 0$ for every $i \in I$. (In the first case I allow $0 \in C$, and in the second I allow $c_i = 0$.)

(d) If C and D are two partitions of unity, I say that C **refines** D if for every $c \in C$ there is a $d \in D$ such that $cd = d$. Note that if C refines D and D refines E then C refines E. **P** If $c \in C$, there is a $d \in D$ such that $cd = c$; now there is an $e \in E$ such that $de = d$; in this case,

$$ce = (cd)e = c(de) = cd = c;$$

as c is arbitrary, C refines E. **Q**

311H The order structure of a Boolean ring Again treating a Boolean ring $\mathfrak{A}$ as an algebra of sets, we have a natural ordering on it, setting $a \subseteq b$ if $ab = a$, so that $a \subseteq b$ iff $\widehat{a} \subseteq \widehat{b}$. This translation makes it obvious that $\subseteq$ is a partial ordering on $\mathfrak{A}$, with least element 0, and with greatest element 1 iff $\mathfrak{A}$ is a Boolean algebra. Moreover, $\mathfrak{A}$ is a lattice (definition: 2A1Ad), with $a \cup b = \sup\{a, b\}$, $a \cap b = \inf\{a, b\}$ for all a, $b \in \mathfrak{A}$. Generally, for $a_0, \dots, a_n \in \mathfrak{A}$,

$$\sup_{i \le n} a_i = a_0 \cup \dots \cup a_n, \quad \inf_{i \le n} a_i = a_0 \cap \dots \cap a_n;$$

suprema and infima of finite subsets $\mathfrak{A}$ correspond to unions and intersections of the corresponding families in the Stone space. (But suprema and infima of *infinite* subsets of $\mathfrak{A}$ are a very different matter; see §313 below.)

It may be obvious, but it is nevertheless vital to recognise that when $\mathfrak{A}$ is a ring of sets then $\subseteq$ agrees with $\subseteq$.

311I The topology of a Stone space: Theorem Let Z be the Stone space of a Boolean ring $\mathfrak{A}$, and let $\mathfrak{T}$ be

$$\{G : G \subseteq Z \text{ and for every } z \in G \text{ there is an } a \in \mathfrak{A} \text{ such that } z \in \widehat{a} \subseteq G\}.$$

Then $\mathfrak{T}$ is a topology on Z, under which Z is a locally compact zero-dimensional Hausdorff space, and $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ is precisely the set of compact open subsets of Z. $\mathfrak{A}$ is a Boolean algebra iff Z is compact.

proof (a) Because $\mathcal{E}$ is closed under $\cap$, and $\bigcup\mathcal{E} = Z$ (recall that Z is the set of surjective homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that every $z \in Z$ is somewhere non-zero and belongs to some $\widehat{a}$), $\mathcal{E}$ is a topology base, and $\mathfrak{T}$ is a topology.

(b) $\mathfrak{T}$ is Hausdorff. **P** Take any distinct z, $w \in Z$. Then there is an $a \in \mathfrak{A}$ such that $z(a) \ne w(a)$; let us take it that $z(a) = 1$, $w(a) = 0$. There is also a $b \in \mathfrak{A}$ such that $w(b) = 1$, so that $w(b + ab) = w(b) +_2 w(a)w(b) = 1$ and $w \in (b + ab)\widehat{\ }$; also

$$a(b + ab) = ab + a^2b = ab + ab = 0,$$

so

$$\widehat{a} \cap (b + ab)\widehat{\ } = (a(b + ab))\widehat{\ } = \widehat{0} = \emptyset,$$

and $\widehat{a}$, $(b + ab)\widehat{\ }$ are disjoint members of $\mathfrak{T}$ containing z, w respectively. **Q**

(c) If $a \in \mathfrak{A}$ then $\widehat{a}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on Z containing $\widehat{a}$. For each $b \in \mathfrak{A}$, $z_0(b) = \lim_{z\to\mathcal{F}} z(b)$ must be defined in $\mathbb{Z}_2$, since one of the sets $\{z : z(b) = 0\}$, $\{z : z(b) = 1\}$ must belong to $\mathcal{F}$. If b, $c \in \mathfrak{A}$, then the set

$$F = \{z : z(b) = z_0(b),\ z(c) = z_0(c),\ z(b + c) = z_0(b + c),\ z(bc) = z_0(bc)\}$$

belongs to $\mathcal{F}$, so is not empty; take any $z_1 \in F$; then

$$z_0(b + c) = z_1(b + c) = z_1(b) +_2 z_1(c) = z_0(b) +_2 z_0(c),$$

$$z_0(bc) = z_1(bc) = z_1(b)z_1(c) = z_0(b)z_0(c).$$

As b, c are arbitrary, $z_0 : \mathfrak{A} \to \mathbb{Z}_2$ is a ring homomorphism. Also $z_0(a) = 1$, because $\widehat{a} \in \mathcal{F}$, so $z_0 \in \widehat{a}$. Now let G be any open subset of Z containing z_0; then there is a $b \in \mathfrak{A}$ such that $z_0 \subseteq \widehat{b} \subseteq G$; since $\lim_{z\to\mathcal{F}} z(b) = z_0(b) = 1$, we must have $\widehat{b} = \{z : z(b) = 1\} \in \mathcal{F}$ and $G \in \mathcal{F}$. Thus $\mathcal{F}$ converges to z_0. As $\mathcal{F}$ is arbitrary, $\widehat{a}$ is compact (2A3R). **Q**

(d) This shows that $\widehat{a}$ is a compact open set for every $a \in \mathfrak{A}$. Moreover, since every point of Z belongs to some $\widehat{a}$, every point of Z has a compact neighbourhood, and Z is locally compact. Every $\widehat{a}$ is closed (because it is compact, or otherwise), so $\mathcal{E}$ is a base for $\mathfrak{T}$ consisting of open-and-closed sets, and $\mathfrak{T}$ is zero-dimensional.

(e) Now suppose that $E \subseteq Z$ is an open compact set. If $E = \emptyset$ then $E = \widehat{0}$. Otherwise, set

$$\mathcal{G} = \{\widehat{a} : a \in \mathfrak{A},\ \widehat{a} \subseteq E\}.$$

Then $\mathcal{G}$ is a family of open subsets of Z and $\bigcup\mathcal{G} = E$, because E is open. But E is also compact, so there is a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $E = \bigcup\mathcal{G}_0$. Express $\mathcal{G}_0$ as $\{\widehat{a}_0, \dots , \widehat{a}_n\}$. Then

$$E = \widehat{a}_0 \cup \ldots \cup \widehat{a}_n = (a_0 \cup \ldots \cup a_n)\widehat{\ }.$$

This shows that every compact open subset of Z is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$.

(f) Finally, if $\mathfrak{A}$ is a Boolean algebra then $Z = \widehat{1}$ is compact, by (c); while if Z is compact then (e) tells us that $Z = \widehat{a}$ for some $a \in \mathfrak{A}$, and of course this a must be a multiplicative identity for $\mathfrak{A}$, so that $\mathfrak{A}$ is a Boolean algebra.

311J We have a kind of converse of Stone's theorem.

Proposition Let X be a locally compact zero-dimensional Hausdorff space. Then the set $\mathfrak{A}$ of open-and-compact subsets of X is a subring of $\mathcal{P}Z$. If Z is the Stone space of $\mathfrak{A}$, there is a unique homeomorphism $\theta : Z \to X$ such that $\widehat{a} = \theta^{-1}[a]$ for every $a \in \mathfrak{A}$.

proof (a) Because X is Hausdorff, all its compact sets are closed, so every member of $\mathfrak{A}$ is closed. Consequently $a \cup b$, $a \setminus b$, $a \cap b$ and $a\triangle b$ belong to $\mathfrak{A}$ for all a, $b \in \mathfrak{A}$, and $\mathfrak{A}$ is a subring of $\mathcal{P}X$.

It will be helpful to know that $\mathfrak{A}$ is a base for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then (because X is locally compact) there is a compact set $K \subseteq X$ such that $x \in \operatorname{int} K$; now (because X is zero-dimensional) there is an open-and-closed set $a \subseteq X$ such that $x \in a \subseteq G \cap \operatorname{int} K$; because a is a closed subset of a compact subset of X, it is compact, and belongs to $\mathfrak{A}$, while $x \in a \subseteq G$. **Q**

(b) Let $R \subseteq Z \times X$ be the relation

$$\{(z,x)\text{: for every } a \in \mathfrak{A},\ x \in a \iff z(a) = 1\}.$$

Then R is the graph of a bijective function $\theta : Z \to X$.

P (i) If $z \in Z$ and x, $x' \in X$ are distinct, then, because X is Hausdorff, there is an open set $G \subseteq X$ containing x and not containing x'; because $\mathfrak{A}$ is a base for the topology of X, there is an $a \in \mathfrak{A}$ such that $x \in a \subseteq G$, so that $x' \notin a$. Now either $z(a) = 1$ and $(z, x') \notin R$, or $z(a) = 0$ and $(z, x) \notin R$. Thus R is the graph of a function θ with domain included in Z and taking values in X.

(ii) If $z \in Z$, there is an $a_0 \in \mathfrak{A}$ such that $z(a_0) = 1$. Consider $\mathcal{A} = \{a : z(a) = 1\}$. This is a family of closed subsets of X containing the compact set a_0, and $a \cap b \in \mathcal{A}$ for all a, $b \in \mathcal{A}$. So $\bigcap \mathcal{A}$ is not empty (3A3Db); take $x \in \bigcap \mathcal{A}$. Then $x \in a$ whenever $z(a) = 1$. On the other hand, if $z(a) = 0$, then

$$z(a_0 \setminus a) = z(a_0 \triangle (a \cap a_0)) = z(a_0) +_2 z(a_0)z(a) = 1,$$

so $x \in a_0 \setminus a$ and $x \notin a$. Thus $(z, x) \in R$ and $\theta(z) = x$ is defined. As z is arbitrary, the domain of θ is the whole of Z.

(iii) If $x \in X$, define $z : \mathfrak{A} \to \mathbb{Z}_2$ by setting $z(a) = 1$ if $x \in a$, 0 otherwise. It is elementary to check that z is a ring homomorphism form $\mathfrak{A}$ to $\mathbb{Z}_2$. To see that it takes the value 1, note that because $\mathfrak{A}$ is a base for the topology of X there is an $a \in \mathfrak{A}$ such that $x \in a$, so that $z(a) = 1$. So $z \in Z$, and of course $(z, x) \in R$. As x is arbitrary, θ is surjective.

(iv) If z, $z' \in Z$ and $\theta(z) = \theta(z')$, then, for any $a \in \mathfrak{A}$,

$$z(a) = 1 \iff \theta(z) \in a \iff \theta(z') \in a \iff z'(a) = 1,$$

so $z = z'$. Thus θ is injective. **Q**

(c) For any $a \in \mathfrak{A}$,

$$\theta^{-1}[a] = \{z : \theta(z) \in a\} = \{z : z(a) = 1\} = \widehat{a}.$$

It follows that θ is a homeomorphism. **P** (i) If $G \subseteq X$ is open, then (because $\mathfrak{A}$ is a base for the topology of X) $G = \bigcup\{a : a \in \mathfrak{A},\ a \subseteq G\}$ and

$$\theta^{-1}[G] = \bigcup\{\theta^{-1}[a] : a \in \mathfrak{A},\ a \subseteq G\} = \bigcup\{\widehat{a} : a \in \mathfrak{A},\ a \subseteq G\}$$

is an open subset of Z. As G is arbitrary, θ is continuous. (ii) On the other hand, if $G \subseteq X$ and $\theta^{-1}[G]$ is open, then $\theta^{-1}[G]$ is of the form $\bigcup_{a \in \mathcal{A}} \widehat{a}$ for some $\mathcal{A} \subseteq \mathfrak{A}$, so that $G = \bigcup \mathcal{A}$ is an open set in X. Accordingly θ is a homeomorphism. **Q**

(d) Finally, I must check the uniqueness of θ. But of course if $\tilde{\theta} : Z \to X$ is any function such that $\tilde{\theta}^{-1}[a] = \widehat{a}$ for every $a \in \mathfrak{A}$, then the graph of $\tilde{\theta}$ must be R, so $\tilde{\theta} = \theta$.

311K Remark Thus we have a correspondence between Boolean rings and zero-dimensional locally compact Hausdorff spaces which is (up to isomorphism, on the one hand, and homeomorphism, on the other) one-to-one. Every property of Boolean rings which we study will necessarily correspond to some property of zero-dimensional locally compact Hausdorff spaces.

311L Complemented distributive lattices I have introduced Boolean algebras through the theory of rings; this seems to be the quickest route to them from an ordinary undergraduate course in abstract algebra. However there are alternative approaches, taking the order structure rather than the algebraic operations as fundamental, and for the sake of an application in Chapter 35 I give the details of one of these.

Proposition Let $\mathfrak{A}$ be a lattice such that

(i) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, $c \in \mathfrak{A}$;

(ii) there is a bijection $a \mapsto a' : \mathfrak{A} \to \mathfrak{A}$ which is order-reversing, that is, $a \leq b$ iff $b' \leq a'$, and such that $a'' = a$ for every a;

(iii) $\mathfrak{A}$ has a least element 0 and $a \wedge a' = 0$ for every $a \in \mathfrak{A}$.

Then $\mathfrak{A}$ has a Boolean algebra structure for which $a \subseteq b$ iff $a \leq b$.

proof (a) Write 1 for $0'$; if $a \in \mathfrak{A}$, then $a' \geq 0$ so $a = a'' \leq 0' = 1$, and 1 is the greatest element of $\mathfrak{A}$.

If a, $b \in \mathfrak{A}$ then, because $'$ is an order-reversing bijection, $a' \vee b' = (a \wedge b)'$. **P** For $c \in \mathfrak{A}$,

$$\begin{aligned} a' \vee b' \leq c &\iff a' \leq c \,\&\, b' \leq c \iff c' \leq a \,\&\, c' \leq b \\ &\iff c' \leq a \wedge b \iff (a \wedge b)' \leq c. \textbf{ Q} \end{aligned}$$

Similarly, $a' \wedge b' = (a \vee b)'$. If a, b, $c \in \mathfrak{A}$ then

$$(a \wedge b) \vee c = ((a' \vee b') \wedge c')' = ((a' \wedge c') \vee (b' \wedge c'))' = (a \vee c) \wedge (b \vee c).$$

(b) Define addition and multiplication on $\mathfrak{A}$ by setting

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b$$

for a, $b \in \mathfrak{A}$.

(c)(i) If a, $b \in \mathfrak{A}$ then

$$\begin{aligned} (a + b)' &= (a' \vee b) \wedge (a \vee b') = (a' \wedge a) \vee (a' \wedge b') \vee (b \wedge a) \vee (b \wedge b') \\ &= 0 \vee (a' \wedge b') \vee (b \wedge a) = (a' \wedge b') \vee (a \wedge b). \end{aligned}$$

So if a, b, $c \in \mathfrak{A}$ then

$$\begin{aligned} (a + b) + c &= ((a + b) \wedge c') \vee ((a + b)' \wedge c) \\ &= (((a \wedge b') \vee (a' \wedge b)) \wedge c') \vee (((a' \wedge b') \vee (a \wedge b)) \wedge c) \\ &= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c); \end{aligned}$$

as this last formula is symmetric in a, b and c, it is also equal to $a + (b + c)$. Thus addition is associative.

(ii) For any $a \in \mathfrak{A}$,

$$a + 0 = 0 + a = (a' \wedge 0) \vee (a \wedge 0') = 0 \vee (a \wedge 1) = a,$$

so 0 is the additive identity of $\mathfrak{A}$. Also

$$a + a = (a \wedge a') \vee (a' \wedge a) = 0 \vee 0 = 0$$

so each element of $\mathfrak{A}$ is its own additive inverse, and $(\mathfrak{A}, +)$ is a group. It is abelian because $\vee$, $\wedge$ are commutative.

(d) Because $\wedge$ is associative and commutative, $(\mathfrak{A}, \cdot)$ is a commutative semigroup; also 1 is its identity, because $a \wedge 1 = a$ for every $a \in \mathfrak{A}$. As for the distributive law in $\mathfrak{A}$,

$$\begin{aligned} ab + ac &= (a \wedge b \wedge (a \wedge c)') \vee ((a \wedge b)' \wedge a \wedge c) \\ &= (a \wedge b \wedge (a' \vee c')) \vee ((a' \vee b') \wedge a \wedge c) \\ &= (a \wedge b \wedge a') \vee (a \wedge b \wedge c') \vee (a' \wedge a \wedge c) \vee (b' \wedge a \wedge c) \\ &= (a \wedge b \wedge c') \vee (b' \wedge a \wedge c) \\ &= a \wedge ((b \wedge c') \vee (b' \wedge c)) = a(b + c) \end{aligned}$$

for all a, b, $c \in \mathfrak{A}$. Thus $(\mathfrak{A}, +, \cdot)$ is a ring; because $a \wedge a = a$ for every a, it is a Boolean ring.

(e) For a, $b \in \mathfrak{A}$,

$$a \subseteq b \iff ab = a \iff a \wedge b = a \iff a \leq b,$$

so the order relations of $\mathfrak{A}$ coincide.

Remark It is the case that the Boolean algebra structure of $\mathfrak{A}$ is uniquely determined by its order structure, but I delay the proof to the next section (312L).

311X Basic exercises (a) Let $A_0, \dots, A_n$ be sets. Show that

$$A_0 \triangle \dots \triangle A_n = \{x : \#(\{i : i \leq n,\, x \in A_i\}) \text{ is odd}\}.$$

(b) Let X be a set, and $\Sigma \subseteq \mathcal{P}X$. Show that the following are equiveridical: (i) Σ is an algebra of subsets of X; (ii) Σ is a subring of $\mathcal{P}X$ (that is, contains $\emptyset$ and is closed under $\triangle$ and $\cap$) and contains X; (iii) $\emptyset \in \Sigma$, $X \setminus E \in \Sigma$ for every $E \in \Sigma$, and $E \cap F \in \Sigma$ for all E, $F \in \Sigma$.

(c) Let $\mathfrak{A}$ be any Boolean ring. Let $a \mapsto a'$ be any bijection between $\mathfrak{A}$ and a set B disjoint from $\mathfrak{A}$. Set $\mathfrak{B} = \mathfrak{A} \cup B$, and extend the addition and multiplication of $\mathfrak{A}$ to form binary operations on $\mathfrak{B}$ by using the formulae

$$a + b' = a' + b = (a + b)', \quad a' + b' = a + b,$$

$$a'b = b + ab, \quad ab' = a + ab, \, a'b' = (a + b + ab)'.$$

Show that $\mathfrak{B}$ is a Boolean algebra and that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$.

>(d) Let $\mathfrak{A}$ be a Boolean ring, and K a finite subset of $\mathfrak{A}$. Show that the subring of $\mathfrak{A}$ generated by K has at most $2^{2^{\#(K)}}$ members, being the set of sums of products of members of K.

>(e) Show that any finite Boolean ring is isomorphic to $\mathcal{P}X$ for some finite set X (and, in particular, is a Boolean algebra).

(f) Let $\mathfrak{A}$ be any Boolean ring. Show that

$$a \cup (b \cap c) = (a \cap b) \cup (a \cap c), \quad a \cup (b \cap c) = (a \cap b) \cup (a \cap c)$$

for all a, b, $c \in \mathfrak{A}$ directly from the definitions in 311G, without using Stone's theorem.

>(g) Let $\mathfrak{A}$ be any Boolean ring. Show that if we regard the Stone space Z of $\mathfrak{A}$ as a subset of $\{0,1\}^{\mathfrak{A}}$, then the topology of Z (311I) is just the subspace topology induced by the ordinary product topology of $\{0,1\}^{\mathfrak{A}}$.

(h) Let I be any set, and set $X = \{0,1\}^I$ with its usual topology (3A3K). Show that for a subset E of X the following are equiveridical: (i) E is open-and-compact; (ii) E is determined by coordinates in a finite subset of I (definition: 254M); (iii) E belongs to the algebra of subsets of X generated by $\{E_i : i \in I\}$, where $E_i = \{x : x(i) = 1\}$ for each i.

(i) Let $(\mathfrak{A}, \le)$ be a lattice such that (α) $\mathfrak{A}$ has a least element 0 and a greatest element 1 (β) for every a, b, $c \in \mathfrak{A}$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (γ) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\le$ agrees with $\subseteq$.

311Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ the Boolean algebra constructed by the method of 311Xc. Show that the Stone space of $\mathfrak{B}$ can be identified with the one-point compactification (3A3O) of the Stone space of $\mathfrak{A}$.

(b) Let $(\mathfrak{A}, \vee, \wedge, 0, 1)$ be such that (i) $(\mathfrak{A}, \vee)$ is a commutative semigroup with identity 0 (ii) $(\mathfrak{A}, \wedge)$ is a commutative semigroup with identity 1 (iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, $c \in \mathfrak{A}$ (iv) $a \vee a = a \wedge a = a$ for every $a \in \mathfrak{A}$ (v) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$, $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\vee = \cup$, $\wedge = \cap$.

311 Notes and comments My aim in this section has been to get as quickly as possible to Stone's theorem, since this is surely the best route to a picture of general Boolean algebras; they are isomorphic to algebras of sets. This means that all their elementary algebraic properties – indeed, all their first-order properties – can be effectively studied in the context of elementary set theory. In 311G-311H I describe a few of the ways in which the Stone representation suggests algebraic properties of Boolean algebras.

You should not, however, come too readily to the conclusion that Boolean algebras will never be able to surprise you. In this book, in particular, we shall need to work a good deal with suprema and infima of infinite sets in Boolean algebras, for the ordering of 311H; and even though this corresponds to the ordering $\subseteq$ of ordinary sets, we find that $(\sup A)^{\widehat{}}$ is sufficiently different from $\bigcup_{a \in A} \widehat{a}$ to need new kinds of intuition.

(The point is that $\bigcup_{a\in A}\widehat{a}$ is an open set in the Stone space, but need not be compact if A is infinite, so may well be smaller than $(\sup A)\widehat{\ }$, even when $\sup A$ is defined in $\mathfrak{A}$.) There is also the fact that Stone's theorem depends crucially on a fairly strong form of the axiom of choice (employed through Zorn's Lemma in the argument of 311D). Of course I shall be using the axiom of choice without scruple throughout this volume. But it should be clear that such results as 312B-312C in the next section cannot possibly need the axiom of choice for their proofs, and that to use Stone's theorem in such a context is slightly misleading.

Nevertheless, it is so useful to be able to regard a Boolean algebra as an algebra of sets – especially when dealing with only finitely many elements of the algebra at a time – that henceforth I will almost always use the symbols $\triangle$, $\cap$ for the addition and multiplication of a Boolean ring, and will use $\cup$, $\setminus$, $\subseteq$ without further comment, just as if I were considering $\cup$, $\setminus$ and $\subseteq$ in the Stone space. (In 311Gb I have given a definition of 'disjointness' in a Boolean algebra based on the same idea.) Even without the axiom of choice this approach can be justified, once we have observed that finitely-generated Boolean algebras are finite (311Xd), since relatively elementary methods show that any finite Boolean algebra is isomorphic to $\mathcal{P}X$ for some finite set X.

I have taken a Boolean algebra to be a particular kind of commutative ring with identity. Of course there are other approaches. If we wish to think of the order relation as primary, then 311L and 311Xi are reasonably natural. Other descriptions can be based on a list of the properties of the binary operations $\cup$, $\cap$ and the complementation operation $a \mapsto a' = 1 \setminus a$, as in 311Yb. I give extra space to 311L only because this is well adapted to an application in 352Q below.

312 Homomorphisms

I continue the theory of Boolean algebras with a section on subalgebras, ideals and homomorphisms. From now on, I will relegate Boolean rings which are not algebras to the exercises; I think there is no need to set out descriptions of the trifling modifications necessary to deal with the extra generality. The first part of the section (312A-312K) concerns the translation of the basic concepts of ring theory into the language which I propose to use for Boolean algebras. 312L shows that the order relation on a Boolean algebra defines the algebraic structure, and in 312M-312N I give a fundamental result on the extension of homomorphisms. I end the section with results relating the previous ideas to the Stone representation of a Boolean algebra (312O-312S).

312A Subalgebras Let $\mathfrak{A}$ be a Boolean algebra. I will use the phrase **subalgebra of** $\mathfrak{A}$ to mean a subring of $\mathfrak{A}$ containing its multiplicative identity $1 = 1_{\mathfrak{A}}$.

312B Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Then the following are equiveridical, that is, if one is true so are the others:

(i) $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$;

(ii) $0 \in \mathfrak{B}$, $a \cup b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$;

(iii) $\mathfrak{B} \neq \emptyset$, $a \cap b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$.

proof (a)(i)$\Rightarrow$(iii) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and a, $b \in \mathfrak{B}$, then of course we shall have

$$0,\ 1 \in \mathfrak{B}, \text{ so } \mathfrak{B} \neq \emptyset,$$

$$a \cap b \in \mathfrak{B}, \quad 1 \setminus a = 1 \triangle a \in \mathfrak{B}.$$

(b)(iii)$\Rightarrow$(ii) If (iii) is true, then there is some $b_0 \in \mathfrak{B}$; now $1 \setminus b_0 \in \mathfrak{B}$, so

$$0 = b_0 \cap (1 \setminus b_0) \in \mathfrak{B}.$$

If a, $b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

So (ii) is true.

(c)(ii)⇒(i) If (ii) is true, then for any a, $b \in \mathfrak{B}$,

$$a \cap b = 1 \setminus ((1 \setminus a) \cup (1 \setminus b)) \in \mathfrak{B},$$

$$a \triangle b = (a \cap (1 \setminus b)) \cup (b \cap (1 \setminus a)) \in \mathfrak{B},$$

so (because also $0 \in \mathfrak{B}$) $\mathfrak{B}$ is a subring of $\mathfrak{A}$, and

$$1 = 1 \setminus 0 \in \mathfrak{B},$$

so $\mathfrak{B}$ is a subalgebra.

Remark Thus an algebra of subsets of a set X, as defined in 136E or 311Bb, is just a subalgebra of the Boolean algebra $\mathcal{P}X$.

312C Ideals in Boolean algebras: Proposition If $\mathfrak{A}$ is a Boolean algebra, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all a, $b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

proof (a) Suppose that I is an ideal. Then of course $0 \in I$. If a, $b \in I$ then $a \cap b \in I$ so $a \cup b = (a \triangle b) \triangle (a \cap b) \in I$. If $b \in I$ and $a \subseteq b$ then $a = a \cap b \in I$.

(b) Now suppose that I satisfies the conditions proposed. If a, $b \in I$ then

$$a \triangle b \subseteq a \cup b \in I$$

so $a \triangle b \in I$, while of course $-a = a \in I$, and also $0 \in I$, by hypothesis; thus I is a subgroup of $(\mathfrak{A}, \triangle)$. Finally, if $a \in I$ and $b \in \mathfrak{A}$ then

$$a \cap b \subseteq a \in I,$$

so $b \cap a = a \cap b \in I$; thus I is an ideal.

Remark Thus what I have called an 'ideal of subsets of X' in 232Xc is just an ideal in the Boolean algebra $\mathcal{P}X$.

312D Principal ideals Of course, while an ideal I in a Boolean algebra $\mathfrak{A}$ is necessarily a subring, it is not as a rule a subalgebra, except in the special case $I = \mathfrak{A}$. But if we say that a **principal ideal** of $\mathfrak{A}$ is the ideal $\mathfrak{A}_a$ generated by a single element a of $\mathfrak{A}$, we have a special phenomenon.

312E Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any element of $\mathfrak{A}$. Then the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a is just $\{b : b \in \mathfrak{A}, b \subseteq a\}$, and (with the inherited operations $\cap \restriction \mathfrak{A}_a \times \mathfrak{A}_a$, $\triangle \restriction \mathfrak{A}_a \times \mathfrak{A}_a$) is a Boolean algebra in its own right, with multiplicative identity a.

proof $b \subseteq a$ iff $b \cap a = a$, so that

$$\mathfrak{A}_a = \{b : b \subseteq a\} = \{b \cap a : b \in \mathfrak{A}\}$$

is an ideal of $\mathfrak{A}$, and of course it is the smallest ideal of $\mathfrak{A}$ containing a. Being an ideal, it is a subring; the idempotent relation $b \cap b = b$ is inherited from $\mathfrak{A}$, so it is a Boolean ring; and a is plainly its multiplicative identity.

312F Boolean homomorphisms Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras. I will use the phrase **Boolean homomorphism** to mean a function $\pi : \mathfrak{A} \to \mathfrak{B}$ which is a ring homomorphism (that is, $\pi(a \triangle b) = \pi a \triangle \pi b$, $\pi(a \cap b) = \pi a \cap \pi b$ for all a, $b \in \mathfrak{A}$) which is uniferent, that is, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

312G Proposition Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be Boolean algebras.

(a) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{B}$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ are Boolean homomorphisms, then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a Boolean homomorphism.

(c) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijective Boolean homomorphism, then $\pi^{-1} : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism.

proof These are all immediate consequences of the corresponding results for ring homomorphisms (3A2D).

312H Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function. Then the following are equiveridical:

(i) π is a Boolean homomorphism;

(ii) $\pi(a \cap b) = \pi a \cap \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iii) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iv) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi a \cap \pi b = 0_{\mathfrak{B}}$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0_{\mathfrak{A}}$, and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

proof (i)⇒(iv) If π is a Boolean homomorphism then of course $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$; also, given that $a \cap b = 0$ in $\mathfrak{A}$,

$$\pi a \cap \pi b = \pi(a \cap b) = \pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}},$$

$$\pi(a \cup b) = \pi(a \triangle b) = \pi a \triangle \pi b = \pi a \cup \pi b.$$

(iv)⇒(iii) Assume (iv), and take a, $b \in \mathfrak{A}$. Then

$$\pi a = \pi(a \cap b) \cup \pi(a \setminus b), \quad \pi b = \pi(a \cap b) \cup \pi(b \setminus a),$$

so

$$\pi(a \cup b) = \pi a \cup \pi(b \setminus a) = \pi(a \cap b) \cup \pi(a \setminus b) \cup \pi(b \setminus a) = \pi a \cup \pi b.$$

Taking $b = 1 \setminus a$, we must have

$$1_{\mathfrak{B}} = \pi(1_{\mathfrak{A}}) = \pi a \cup \pi(1_{\mathfrak{A}} \setminus a), \quad 0_{\mathfrak{B}} = \pi a \cap \pi(1_{\mathfrak{A}} \setminus a),$$

so $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$. Thus (iii) is true.

(iii)⇒(ii) If (iii) is true and a, $b \in \mathfrak{A}$, then

$$\begin{aligned}\pi(a \cup b) &= \pi(1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cap (1_{\mathfrak{A}} \setminus b))) \\ &= 1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cap (1_{\mathfrak{B}} \setminus \pi b))) = \pi a \cup \pi b.\end{aligned}$$

So (ii) is true.

(ii)⇒(i) If (ii) is true, then

$$\begin{aligned}\pi(a \triangle b) &= \pi((1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cap (1_{\mathfrak{A}} \setminus b)) \cap (1_{\mathfrak{A}} \setminus (a \cap b))) \\ &= (1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cap (1_{\mathfrak{B}} \setminus \pi b)) \cap (1_{\mathfrak{B}} \setminus (\pi a \cap \pi b))) = \pi a \triangle \pi b\end{aligned}$$

for all a, $b \in \mathfrak{A}$, so π is a ring homomorphism; and now

$$\pi(1_{\mathfrak{A}}) = \pi(1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus \pi(0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus 0_{\mathfrak{B}} = 1_{\mathfrak{B}},$$

so that π is a Boolean homomorphism.

312I Proposition If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi a \subseteq \pi b$ whenever $a \subseteq b$ in $\mathfrak{A}$.

proof

$$a \subseteq b \implies a \cap b = a \implies \pi a \cap \pi b = \pi a \implies \pi a \subseteq \pi b.$$

312J Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any member of $\mathfrak{A}$. Then the map $b \mapsto a \cap b$ is a surjective Boolean homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

proof This is an elementary verification.

312K Quotient algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Then the quotient ring $\mathfrak{A}/I$ (3A2F) is a Boolean algebra, and the canonical map $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{A}/I$ is a Boolean homomorphism, so that

$$(a \triangle b)^\bullet = a^\bullet \triangle b^\bullet, \quad (a \cup b)^\bullet = a^\bullet \cup b^\bullet, \quad (a \cap b)^\bullet = a^\bullet \cap b^\bullet, \quad (a \setminus b)^\bullet = a^\bullet \setminus b^\bullet$$

for all a, $b \in \mathfrak{A}$.

(b) The order relation on $\mathfrak{A}/I$ is defined by the formula

$$a^\bullet \subseteq b^\bullet \iff a \setminus b \in I.$$

For any $a \in \mathfrak{A}$,

$$\{u : u \subseteq a^\bullet\} = \{b^\bullet : b \subseteq a\}.$$

proof (a) Of course the map $a \mapsto a^\bullet = \{a \triangle b : b \in I\}$ is a ring homomorphism (3A2F). Because

$$(a^\bullet)^2 = (a^2)^\bullet = a^\bullet$$

for every $a \in \mathfrak{A}$, $\mathfrak{A}/I$ is a Boolean ring; because $1^\bullet$ is a multiplicative identity, it is a Boolean algebra, and $a \mapsto a^\bullet$ is a Boolean homomorphism. The formulae given are now elementary.

(b) We have

$$a^\bullet \subseteq b^\bullet \iff a^\bullet \setminus b^\bullet = 0 \iff a \setminus b \in I.$$

Now

$$\{u : u \subseteq a^\bullet\} = \{u \cap a^\bullet : u \in \mathfrak{A}/I\} = \{(b \cap a)^\bullet : b \in \mathfrak{A}\} = \{b^\bullet : b \subseteq a\}.$$

312L The above results are both repetitive and nearly trivial. Now I come to something with a little more meat to it.

Proposition If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijection such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$, then π is a Boolean algebra isomorphism.

proof (a) Because π is surjective, there must be c_0, $c_1 \in \mathfrak{A}$ such that $\pi c_0 = 0_{\mathfrak{B}}$, $\pi c_1 = 1_{\mathfrak{B}}$; now $\pi(0_{\mathfrak{A}}) \subseteq \pi c_0$, $\pi c_1 \subseteq \pi(1_{\mathfrak{A}})$, so we must have $\pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}}$, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

(b) If $a \in \mathfrak{A}$, then $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **P** There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cup \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \setminus a) \subseteq \pi c \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}};$$

as π is injective, $c \cap a = c \setminus a = 0_{\mathfrak{A}}$ and $c = 0_{\mathfrak{A}}$, $\pi c = 0_{\mathfrak{B}}$, $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **Q**

(c) If $a \in \mathfrak{A}$, then $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **P** It may be clear to you that this is just a dual form of (b). If not, I repeat the argument in the form now appropriate. There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cap \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cup a) \supseteq \pi c \cup \pi a = 1_{\mathfrak{B}}, \quad \pi(c \cup (1_{\mathfrak{A}} \setminus a)) \supseteq \pi c \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}};$$

as π is injective, $c \cup a = c \cup (1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{A}}$ and $c = 1_{\mathfrak{A}}$, $\pi c = 1_{\mathfrak{B}}$, $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **Q**

(d) Putting (b) and (c) together, we have $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for every $a \in \mathfrak{A}$. Now $\pi(a \cup b) = \pi a \cup \pi b$ for every a, $b \in \mathfrak{A}$. **P** Surely $\pi a \cup \pi b \subseteq \pi(a \cup b)$. Let $c \in \mathfrak{A}$ be such that $\pi c = \pi(a \cup b) \setminus (\pi a \cup \pi b)$. Then

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \cap b) \subseteq \pi c \cap \pi b = 0_{\mathfrak{B}},$$

so $c \cap a = c \cap b = 0$ and $c \subseteq 1_{\mathfrak{A}} \setminus (a \cup b)$; accordingly

$$\pi c \subseteq \pi(1_{\mathfrak{A}} \setminus (a \cup b)) = 1_{\mathfrak{B}} \setminus \pi(a \cup b);$$

as also $\pi c \subseteq \pi(a \cup b)$, $\pi c = 0_{\mathfrak{B}}$ and $\pi(a \cup b) = \pi a \cup \pi b$. **Q**

(e) So the conditions of 312H(iii) are satisfied and π is a Boolean homomorphism; being bijective, it is an isomorphism.

312M I turn next to a fundamental lemma on the construction of homomorphisms. We need to start with a proper description of a certain type of subalgebra.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$; let c be any member of $\mathfrak{A}$. Then

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\}$$

is a subalgebra of $\mathfrak{A}$; it is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

proof We have to check the following:

$$a = (a \cap c) \cup (a \setminus c) \in \mathfrak{A}_1$$

for every $a \in \mathfrak{A}_0$, so $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$; in particular, $0 \in \mathfrak{A}_1$.

$$1 \setminus ((a \cap c) \cup (b \setminus c)) = ((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c) \in \mathfrak{A}_1$$

for all a, $b \in \mathfrak{A}_0$, so $1 \setminus d \in \mathfrak{A}_1$ for every $d \in \mathfrak{A}_1$.

$$(a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c) = ((a \cup a') \cap c) \cup ((b \cup b') \setminus c) \in \mathfrak{A}_1$$

for all a, b, a', $b' \in \mathfrak{A}_0$, so $d \cup d' \in \mathfrak{A}_1$ for all d, $d' \in \mathfrak{A}_1$. Thus $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}$ (using 312B).

$$c = (1 \cap c) \cup (0 \setminus c) \in \mathfrak{A}_1,$$

so $\mathfrak{A}_1$ includes $\mathfrak{A}_0 \cup \{c\}$; and finally it is clear that any subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0 \cup \{c\}$, being closed under $\cap$, $\cup$ and complementation, must include $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

312N Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, and $c \in \mathfrak{A}$. If $v \in \mathfrak{B}$ is such that $\pi a \subseteq v \subseteq \pi b$ whenever a, $b \in \mathfrak{A}_0$ and $a \subseteq c \subseteq b$, then there is a unique Boolean homomorphism π_1 from the subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π and $\pi_1 c = v$.

proof (a) The basic fact we need to know is that if a, a', b, $b' \in \mathfrak{A}_0$ and

$$(a \cap c) \cup (b \setminus c) = d = (a' \cap c) \cup (b' \setminus c),$$

then

$$(\pi a \cap v) \cup (\pi b \setminus v) = (\pi a' \cap v) \cup (\pi b' \setminus v).$$

P We have

$$a \cap c = d \cap c = a' \cap c.$$

Accordingly $(a \triangle a') \cap c = 0$ and $c \subseteq 1 \setminus (a \triangle a')$. Consequently (since $a \triangle a'$ surely belongs to $\mathfrak{A}_0$)

$$v \subseteq \pi(1 \setminus (a \triangle a')) = 1 \setminus (\pi a \triangle \pi a'),$$

and

$$\pi a \cap v = \pi a' \cap v.$$

Similarly,

$$b \setminus c = d \setminus c = b' \setminus c,$$

so

$$(b \triangle b') \setminus c = 0, \quad b \triangle b' \subseteq c, \quad \pi(b \triangle b') \subseteq v$$

and

$$\pi b \setminus v = \pi b' \setminus v.$$

Putting these together, we have the result. **Q**

(b) Consequently, we have a function π_1 defined by writing

$$\pi_1((a \cap c) \cup (b \setminus c)) = (\pi a \cap v) \cup (\pi b \setminus c)$$

for all a, $b \in \mathfrak{A}_0$; and 312M tells us that the domain of π_1 is just $\mathfrak{A}_1$. Now π_1 is a Boolean homomorphism. **P** This amounts to running through the proof of 312M again.

(i) If a, $b \in \mathfrak{A}_0$, then

$$\begin{aligned}\pi_1(1 \setminus ((a \cap c) \cup (b \setminus c))) &= \pi_1(((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c))\\ &= (\pi(1 \setminus a) \cap v) \cup (\pi(1 \setminus b) \setminus v)\\ &= ((1 \setminus \pi a) \cap v) \cup ((1 \setminus \pi b) \setminus v)\\ &= 1 \setminus ((\pi a \cap v) \cup (\pi b \setminus v)) = 1 \setminus \pi_1((a \cap c) \cup (b \setminus c)).\end{aligned}$$

So $\pi_1(1\setminus d)=1\setminus\pi_1 d$ for every $d\in\mathfrak{A}_1$.

(ii) If a, b, a', $b'\in\mathfrak{A}_0$, then

$$\begin{aligned}\pi_1((a\cap c)\cup(b\setminus c)\cup(a'\cap c)\cup(b'\setminus c))&=\pi_1(((a\cup a')\cap c)\cup((b\cup b')\setminus c))\\&=(\pi(a\cup a')\cap v)\cup(\pi(b\cup b')\setminus v)\\&=((\pi a\cup\pi a')\cap v)\cup((\pi b\cup\pi b')\setminus v)\\&=(\pi a\cap v)\cup(\pi b\setminus v)\cup(\pi a'\cap v)\cup(\pi b'\setminus v)\\&=\pi_1((a\cap c)\cup(b\setminus c))\cup\pi_1((a'\cap v)\cup(b'\setminus v)).\end{aligned}$$

So $\pi_1(d\cup d')=\pi_1 d\cup\pi_1 d'$ for all d, $d'\in\mathfrak{A}_1$.

By 312H(iii), π_1 is a Boolean homomorphism. **Q**

(c) If $a\in\mathfrak{A}_0$, then

$$\pi_1 a=\pi_1((a\cap c)\cup(a\setminus c))=(\pi a\cap v)\cup(\pi a\setminus v)=\pi a,$$

so π_1 extends π. As for the action of π_1 on c,

$$\pi_1 c=\pi_1((1\cap c)\cup(0\setminus c))=(\pi 1\cap v)\cup(\pi 0\setminus v)=(1\cap v)\cup(0\setminus v)=v,$$

as required.

(d) Finally, the formula of (b) is the only possible definition for any Boolean homomorphism from $\mathfrak{A}_1$ to $\mathfrak{B}$ which will extend π and take c to v. So π_1 is unique.

312O Homomorphisms and Stone spaces Because the Stone space Z of a Boolean algebra $\mathfrak{A}$ (311E) can be constructed explicitly from the algebraic structure of $\mathfrak{A}$, it must in principle be possible to describe any feature of the Boolean structure of $\mathfrak{A}$ in terms of Z. In the next few paragraphs I work through the most important identifications.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\widehat{a}\subseteq Z$ for the open-and-closed set corresponding to $a\in\mathfrak{A}$. Then there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G\subseteq Z$, given by the formulae

$$G=\bigcup_{a\in I}\widehat{a},\quad I=\{a:\widehat{a}\subseteq G\}.$$

proof (a) For any ideal $I\lhd\mathfrak{A}$, set $H(I)=\bigcup_{a\in I}\widehat{a}$; then $H(I)$ is a union of open subsets of Z, so is open. For any open set $G\subseteq Z$, set $J(G)=\{a:a\in\mathfrak{A},\,\widehat{a}\subseteq G\}$; then $J(G)$ satisfies the conditions of 312C, so is an ideal of $\mathfrak{A}$.

(b) If $I\lhd\mathfrak{A}$, then $J(H(I))=I$. **P** (i) If $a\in I$, then $\widehat{a}\subseteq H(I)$ so $a\in J(H(I))$. (ii) If $a\in J(H(I))$, then $\widehat{a}\subseteq H(I)=\bigcup_{b\in I}\widehat{b}$. Because $\widehat{a}$ is compact and all the $\widehat{b}$ are open, there must be finitely many $b_0,\dots,b_n\in I$ such that $\widehat{a}\subseteq\widehat{b}_0\cup\ldots\cup\widehat{b}_n$. But now $a\subseteq b_0\cup\ldots\cup b_n\in I$, so $a\in I$. **Q**

(c) If $G\subseteq Z$ is open, then $H(J(G))=G$. **P** (i) If $z\in G$, then (because $\{\widehat{a}:a\in\mathfrak{A}\}$ is a base for the topology of Z) there is an $a\in\mathfrak{A}$ such that $z\in\widehat{a}\subseteq G$; now $a\in J(G)$ and $z\in H(J(G))$. (ii) If $z\in H(J(G))$, there is an $a\in J(G)$ such that $z\in\widehat{a}$; now $\widehat{a}\subseteq G$, so $z\in G$. **Q**

This shows that the maps $G\mapsto J(G)$, $I\mapsto H(I)$ are two halves of a one-to-one correspondence, as required.

312P Theorem Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z, W; write $\widehat{a}\subseteq Z$, $\widehat{b}\subseteq W$ for the open-and-closed sets corresponding to $a\in\mathfrak{A}$, $b\in\mathfrak{B}$. Then we have a one-to-one correspondence between Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{B}$ and continuous functions $\phi:W\to Z$, given by the formula

$$\pi a=b\iff\phi^{-1}[\widehat{a}]=\widehat{b}.$$

proof (a) Recall that I have constructed Z, W as the sets of Boolean homomorphisms from $\mathfrak{A}$, $\mathfrak{B}$ to $\mathbb{Z}_2$ (311E). So if $\pi:\mathfrak{A}\to\mathfrak{B}$ is any Boolean homomorphism, and $w\in W$, $\psi_\pi(w)=w\pi$ is a Boolean homomorphism from $\mathfrak{A}$ to $\mathbb{Z}_2$ (312Gb), and belongs to Z. Now $\psi_\pi^{-1}[\widehat{a}]=\widehat{\pi a}$ for every $a\in\mathfrak{A}$. **P**

$$\psi_\pi^{-1}[\widehat{a}] = \{w : \psi_\pi(w) \in \widehat{a}\} = \{w : w\pi \in \widehat{a}\} = \{w : w\pi(a) = 1\} = \{w : w \in \widehat{\pi a}\}. \textbf{ Q}$$

Consequently ψ_π is continuous. **P** Let G be any open subset of Z. Then $G = \bigcup\{\widehat{a} : \widehat{a} \subseteq G\}$, so

$$\psi_\pi^{-1}[G] = \bigcup\{\psi_\pi^{-1}[\widehat{a}] : \widehat{a} \subseteq G\} = \bigcup\{\widehat{\pi a} : \widehat{a} \subseteq G\}$$

is open. As G is arbitrary, ψ_π is continuous. **Q**

(b) If $\phi : W \to Z$ is continuous, then for any $a \in \mathfrak{A}$ the set $\phi^{-1}[\widehat{a}]$ must be an open-and-closed set in W; consequently there is a unique member of $\mathfrak{B}$, call it $\theta_\phi a$, such that $\phi^{-1}[\widehat{a}] = \widehat{\theta_\phi a}$. Observe that, for any $w \in W$ and $a \in \mathfrak{A}$,

$$w(\theta_\phi a) = 1 \iff w \in \widehat{\theta_\phi a} \iff \phi(w) \in \widehat{a} \iff (\phi(w))(a) = 1,$$

so $\phi(w) = w\theta_\phi$.

Now θ_ϕ is a Boolean homomorphism. **P** (i) If a, $b \in \mathfrak{A}$ then

$$\theta_\phi(a \cup b)\widehat{\ } = \phi^{-1}[(a \cup b)\widehat{\ }] = \phi^{-1}[\widehat{a} \cup \widehat{b}] = \phi^{-1}[\widehat{a}] \cup \phi^{-1}[\widehat{b}] = \widehat{\theta_\phi a} \cup \widehat{\theta_\phi b} = (\theta_\phi a \cup \theta_\phi b)\widehat{\ },$$

so $\theta_\phi(a \cup b) = \theta_\phi a \cup \theta_\phi b$. (ii) If $a \in \mathfrak{A}$, then

$$\theta_\phi(1 \setminus a)\widehat{\ } = \phi^{-1}[(1 \setminus a)\widehat{\ }] = \phi^{-1}[Z \setminus \widehat{a}] = W \setminus \phi^{-1}[\widehat{a}] = W \setminus \widehat{\theta_\phi a} = (1 \setminus \theta_\phi a)\widehat{\ },$$

so $\theta_\phi(1 \setminus a) = 1 \setminus \theta_\phi a$. (iii) By 312H, θ_ϕ is a Boolean homomorphism. **Q**

(c) For any Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$, $\pi = \theta_{\psi_\pi}$. **P** For $a \in \mathfrak{A}$,

$$(\theta_{\psi_\pi} a)\widehat{\ } = \psi_\pi^{-1}[\widehat{a}] = \widehat{\pi a},$$

so $\theta_{\psi_\pi} a = a$. **Q**

(d) For any continuous function $\phi : W \to Z$, $\phi = \psi_{\theta_\phi}$. **P** For any $w \in W$,

$$\psi_{\theta_\phi}(w) = w\theta_\phi = \phi(w). \textbf{ Q}$$

(e) Thus $\pi \mapsto \psi_\pi$, $\phi \mapsto \theta_\phi$ are the two halves of a one-to-one correspondence, as required.

312Q Theorem Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean algebras, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be Boolean homomorphisms, with corresponding continuous functions $\phi : W \to Z$ and $\psi : V \to W$. Then the Boolean homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : V \to Z$.

proof For any $a \in \mathfrak{A}$,

$$\widehat{\theta\pi a} = (\theta(\pi a))\widehat{\ } = \psi^{-1}[\widehat{\pi a}] = \psi^{-1}[\phi^{-1}[\widehat{a}]] = (\phi\psi)^{-1}[\widehat{a}].$$

312R Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi : W \to Z$. Then

(a) π is injective iff ϕ is surjective;

(b) π is surjective iff ϕ is injective.

proof (a) If $a \in \mathfrak{A}$, then

$$\begin{aligned}\widehat{a} \cap \phi[W] = \emptyset &\iff \phi(w) \notin \widehat{a} \text{ for every } w \in W\\ &\iff (\phi(w))(a) = 0 \text{ for every } w \in W\\ &\iff w(\pi a) = 0 \text{ for every } w \in W\\ &\iff \pi a = 0.\end{aligned}$$

Now W is compact, so $\phi[W]$ is also compact, therefore closed, and

$$\begin{aligned}\phi \text{ is not surjective} &\iff Z \setminus \phi[W] \neq \emptyset \\ &\iff \text{ there is a non-zero } a \in \mathfrak{A} \text{ such that } \widehat{a} \subseteq Z \setminus \phi[W] \\ &\iff \text{ there is a non-zero } a \in \mathfrak{A} \text{ such that } \pi a = 0 \\ &\iff \pi \text{ is not injective}\end{aligned}$$

(3A2Db).

(b)(i) If π is surjective and w, w' are distinct members of W, then there is a $b \in \mathfrak{B}$ such that $w \in \widehat{b}$ and $w' \notin \widehat{b}$. Now $b = \pi a$ for some $a \in \mathfrak{A}$, so $\phi(w) \in \widehat{a}$ and $\phi(w') \notin \widehat{a}$, and $\phi(w) \neq \phi(w')$. As w and w' are arbitrary, ϕ is injective.

(ii) If ϕ is injective and $b \in \mathfrak{B}$, then $K = \phi[\widehat{b}]$, $L = \phi[W \setminus \widehat{b}]$ are disjoint compact subsets of Z. Consider $I = \{a : a \in \mathfrak{A},\ L \cap \widehat{a} = \emptyset\}$. Then $\bigcup_{a \in I} \widehat{a} = Z \setminus L \supseteq K$. Because K is compact and every $\widehat{a}$ is open, there is a finite family $a_0, \dots, a_n \in I$ such that $K \subseteq \widehat{a}_0 \cup \ldots \cup \widehat{a}_n$. Set $a = a_0 \cup \ldots \cup a_n$. Then $\widehat{a} = \widehat{a}_0 \cup \ldots \cup \widehat{a}_n$ includes K and is disjoint from L. So $\widehat{\pi a} = \phi^{-1}[\widehat{a}]$ includes $\widehat{b}$ and is disjoint from $W \setminus \widehat{b}$; that is, $\widehat{\pi a} = \widehat{b}$ and $\pi a = b$. As b is arbitrary, π is surjective.

312S Principal ideals If $\mathfrak{A}$ is a Boolean algebra and $a \in \mathfrak{A}$, we have a natural surjective Boolean homomorphism $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$, the principal ideal generated by a (312J). Writing Z for the Stone space of $\mathfrak{A}$ and Z_a for the Stone space of $\mathfrak{A}_a$, this homomorphism must correspond to an injective continuous function $\phi : Z_a \to Z$ (312Rb). Because Z_a is compact and Z is Hausdorff, ϕ must be a homeomorphism between Z_a and its image $\phi[Z_a] \subseteq Z$ (3A3Dd). To identify $\phi[Z_a]$, note that it is compact, therefore closed, and that

$$\begin{aligned}Z \setminus \phi[Z_a] &= \bigcup\{\widehat{b} : b \in \mathfrak{A},\ \widehat{b} \cap \phi[Z_a] = \emptyset\} \\ &= \bigcup\{\widehat{b} : \phi^{-1}[\widehat{b}] = \emptyset\} = \bigcup\{\widehat{b} : b \cap a = 0\} = Z \setminus \widehat{a},\end{aligned}$$

so that $\phi[Z_a] = \widehat{a}$. It is therefore natural to identify Z_a with the open-and-closed set $\widehat{a} \subseteq Z$.

312X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subring of $\mathfrak{A}$ iff $0 \in \mathfrak{B}$ and $a \cup b$, $a \setminus b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ iff $1 \in \mathfrak{B}$ and $a \setminus b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $I \subseteq A \subseteq \mathfrak{A}$ are such that $1 \in A$, $a \cap b \in I$ for all a, $b \in I$ and $a \setminus b \in A$ whenever a, $b \in A$ and $b \subseteq a$. Show that A includes the subalgebra of $\mathfrak{A}$ generated by I. (*Hint*: 136Xf.)

(d) Show that if $\mathfrak{A}$ is a Boolean ring, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all a, $b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a function such that (i) $\phi(a) \subseteq \phi(b)$ whenever $a \subseteq b$ (ii) $\phi(a) \cap \phi(b) = 0_{\mathfrak{B}}$ whenever $a \cap b = 0_{\mathfrak{A}}$ (iii) $\phi(a) \cup \phi(b) \cup \phi(c) = 1_{\mathfrak{B}}$ whenever $a \cup b \cup c = 1_{\mathfrak{A}}$. Show that ϕ is a Boolean homomorphism.

(f) Let $\mathfrak{A}$ be a Boolean ring, and a any member of $\mathfrak{A}$. Show that the map $b \mapsto a \cap b$ is a ring homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

(g) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Boolean rings, and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the Boolean algebras constructed from them by the method of 311Xc. Show that any ring homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ has a unique extension to a Boolean homomorphism from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a ring homomorphism, and $c \in \mathfrak{A}$. Show that if $v \in \mathfrak{B}$ is such that $\pi a \setminus v = \pi b \cap v = 0$ whenever a, $b \in \mathfrak{A}_0$ and $a \setminus c = b \cap c = 0$, then there is a unique ring homomorphism π_1 from the subring $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π_0 and $\pi_1 c = v$.

(i) Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. Show that there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae $G = \bigcup_{a\in I} \widehat{a}$, $I = \{a : \widehat{a} \subseteq G\}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\mathfrak{A}$ is the subalgebra of itself generated by $\mathfrak{A}_0 \cup \{c\}$, where $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$ and $c \in \mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$ and Z_0 the Stone space of $\mathfrak{A}_0$. Let $\psi : Z \to Z_0$ be the continuous surjection corresponding to the embedding of $\mathfrak{A}_0$ in $\mathfrak{A}$. Show that $\psi\restriction\widehat{c}$ and $\psi\restriction Z \setminus \widehat{c}$ are injective.

Now let $\mathfrak{B}$ be another Boolean algebra, with Stone space W, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, with corresponding function $\phi : W \to Z_0$. Show that there is a continuous function $\phi_1 : W \to Z$ such that $\psi\phi_1 = \phi$ iff there is an open-and-closed set $V \subseteq W$ such that $\phi[V] \subseteq \psi[\widehat{c}]$ and $\phi[W \setminus V] \subseteq \psi[Z \setminus \widehat{c}]$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z, and I an ideal of $\mathfrak{A}$, corresponding to an open set $G \subseteq Z$. Show that the Stone space of the quotient algebra $\mathfrak{A}/I$ may be identified with $Z \setminus G$.

312Y Further exercises (a) Find a function $\phi : \mathcal{P}\{0, 1, 2\} \to \mathbb{Z}_2$ such that $\phi(1 \setminus a) = 1 \setminus \phi a$ for every $a \in \mathcal{P}\{0, 1, 2\}$ and $\phi(a) \subseteq \phi(b)$ whenever $a \subseteq b$, but ϕ is not a Boolean homomorphism.

(b) Let $\mathfrak{A}$ be the Boolean ring of finite subsets of $\mathbb{N}$. Show that there is a bijection $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$ but π is not a ring homomorphism.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings, with Stone spaces Z, W. Show that we have a one-to-one correspondence between ring homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : H \to Z$, where $H \subseteq W$ is an open set, such that $\phi^{-1}[K]$ is compact for every compact set $K \subseteq Z$, given by the formula $\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b}$.

(d) Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean rings, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be ring homomorphisms, with corresponding continuous functions $\phi : H \to Z$ and $\psi : G \to W$. Show that the ring homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : \psi^{-1}[H] \to Z$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism, with associated continuous function $\phi : H \to Z$. Show that π is injective iff $\phi[H]$ is dense in Z, and that π is surjective iff ϕ is injective and $H = W$.

(f) Let $\mathfrak{A}$ be a Boolean ring and $a \in \mathfrak{A}$. Show that the Stone space of the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a can be identified with the compact open set $\widehat{a}$ in the Stone space of $\mathfrak{A}$. Show that the identity map is a ring homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}$, and corresponds to the identity function on $\widehat{a}$.

312 Notes and comments The definitions of 'subalgebra' and 'Boolean homomorphism' (312A, 312F), like that of 'Boolean algebra', are a trifle arbitrary, but will be a convenient way of mandating appropriate treatment of multiplicative identities. I run through the work of 312A-312J essentially for completeness; once you are familiar with Boolean algebras, they should all seem obvious. 312L has a little bit more to it. It shows that the order structure of a Boolean algebra defines the ring structure, in a fairly strong sense.

I call 312N a 'lemma', but actually it is the most important result in this section; it is the basic tool we have for extending a homomorphism from a subalgebra to a slightly larger one, and with Zorn's Lemma (another 'lemma' which deserves a capital L) will provide us with general methods of constructing homomorphisms.

In 312O-312S I describe the basic relationships between the Boolean homomorphisms and continuous functions on Stone spaces. 312P-312Q show that, in the language of category theory, the Stone representation provides a 'contravariant functor' from the category of Boolean algebras with Boolean homomorphisms to the category of topological spaces with continuous functions. Using 311I-311J, we know exactly which topological spaces appear, the zero-dimensional compact Hausdorff spaces; and we know also that the functor is faithful, that is, that we can recover Boolean algebras and homomorphisms from the corresponding topological spaces and continuous functions. There is an agreeable duality in 312R. All of this can be done for Boolean rings, but there are some extra complications (312Yc-312Yf).

To my mind, the very essence of the theory of Boolean algebras is the fact that they are abstract rings, but at the same time can be thought of 'locally' as algebras of sets. Consequently we can bring two quite separate kinds of intuition to bear. 312N gives an example of a ring-theoretic problem, concerning the extension of homomorphisms, which has a resolution in terms of the order relation, a concept most naturally described in terms of algebras-of-sets. It is very much a matter of taste and habit, but I myself find that a Boolean homomorphism is easiest to think of in terms of its action on finite subalgebras, which are directly representable as $\mathcal{P}X$ for some finite X (311Xe); the corresponding continuous map between Stone spaces is less helpful. I offer 312Xj, the Stone-space version of 312N, for you to test your own intuitions on.

313 Order-continuous homomorphisms

Because a Boolean algebra has a natural partial order (311H), we have corresponding notions of upper bounds, lower bounds, suprema and infima. These are particularly important in the Boolean algebras arising in measure theory, and the infinitary operations 'sup' and 'inf' require rather more care than the basic binary operations '$\cup$', '$\cap$', because intuitions from elementary set theory are sometimes misleading. I therefore take a section to work through the most important properties of these operations, together with the homomorphisms which preserve them.

313A Relative complementation: Proposition Let $\mathfrak{A}$ be a Boolean algebra, e a member of $\mathfrak{A}$, and A a non-empty subset of $\mathfrak{A}$.

(a) If $\sup A$ is defined in $\mathfrak{A}$, then $\inf\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \sup A$.

(b) If $\inf A$ is defined in $\mathfrak{A}$, then $\sup\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \inf A$.

proof (a) Writing a_0 for $\sup A$, we have $e \setminus a_0 \subseteq e \setminus a$ for every $a \in A$, so $e \setminus a_0$ is a lower bound for $C = \{e \setminus a : a \in A\}$. Now suppose that c is any lower bound for C. Then (because A is not empty) $c \subseteq e$, and

$$a = (a \setminus e) \cup (e \setminus (e \setminus a)) \subseteq (a_0 \setminus e) \cup (e \setminus c)$$

for every $a \in A$. Consequently $a_0 \subseteq (a_0 \setminus e) \cup (e \setminus c)$ is disjoint from c and

$$c = c \cap e \subseteq e \setminus a_0.$$

Accordingly $e \setminus a_0$ is the greatest lower bound of C, as claimed.

(b) This time set $a_0 = \inf A$, $C = \{e \setminus a : a \in A\}$. As before, $e \setminus a_0$ is surely an upper bound for C. If c is any upper bound for C, then

$$e \setminus c \subseteq e \setminus (e \setminus a) = e \cap a \subseteq a$$

for every $a \in A$, so $e \setminus c \subseteq a_0$ and $e \setminus a_0 \subseteq c$. As c is arbitrary, $e \setminus a_0$ is indeed the least upper bound of C.

Remark In the arguments above I repeatedly encourage you to treat $\cap$, $\cup$, $\setminus$, $\subseteq$ as if they were the corresponding operations and relation of basic set theory. This is perfectly safe so long as we take care that every manipulation so justified has only finitely many elements of the Boolean algebra in hand at once.

313B General distributive laws: Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\sup A$ is defined in $\mathfrak{A}$, then $\sup\{e \cap a : a \in A\}$ is defined and equal to $e \cap \sup A$.

(b) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\inf A$ is defined in $\mathfrak{A}$, then $\inf\{e \cup a : a \in A\}$ is defined and equal to $e \cup \inf A$.

(c) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\sup A$, $\sup B$ are defined in $\mathfrak{A}$. Then $\sup\{a \cap b : a \in A, b \in B\}$ is defined and is equal to $\sup A \cap \sup B$.

(d) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\inf A$, $\inf B$ are defined in $\mathfrak{A}$. Then $\inf\{a \cup b : a \in A, b \in B\}$ is defined and is equal to $\inf A \cup \inf B$.

proof (a) Set

$$B = \{e \setminus a : a \in A\}, \quad C = \{e \setminus b : b \in B\} = \{e \cap a : a \in A\}.$$

Using 313A, we have

$$\inf B = e \setminus \sup A, \quad \sup C = e \setminus \inf B = e \cap \sup A,$$

as required.

(b) Set $a_0 = \inf A$, $B = \{e \cup a : a \in A\}$. Then $e \cup a_0 \subseteq e \cup a$ for every $a \in A$, so $e \cup a_0$ is a lower bound for B. If c is any lower bound for B, then $c \setminus e \subseteq a$ for every $a \in A$, so $c \setminus e \subseteq a_0$ and $c \subseteq e \cup a_0$; thus $e \cup a_0$ is the greatest lower bound for B, as claimed.

(c) By (a), we have

$$a \cap \sup B = \sup_{b \in B} a \cap b$$

for every $a \in A$, so

$$\sup_{a \in A, b \in B} a \cap b = \sup_{a \in A}(a \cap \sup B) = \sup A \cap \sup B,$$

using (a) again.

(d) Similarly, using (b) twice,

$$\inf_{a \in A, b \in B} a \cup b = \inf_{a \in A}(a \cup \inf B) = \inf A \cup \inf B.$$

313C As always, it is worth developing a representation of the concepts of sup and inf in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; for $a \in \mathfrak{A}$ write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

(a) If $A \subseteq \mathfrak{A}$ and $a_0 \in \mathfrak{A}$ then $a_0 = \sup A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \overline{\bigcup_{a \in A} \widehat{a}}$.

(b) If $A \subseteq \mathfrak{A}$ is non-empty and $a_0 \in \mathfrak{A}$ then $a_0 = \inf A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \operatorname{int} \bigcap_{a \in A} \widehat{a}$.

(c) If $A \subseteq \mathfrak{A}$ is non-empty then $\inf A = 0$ in $\mathfrak{A}$ iff $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z.

proof (a) For any $b \in \mathfrak{A}$,

$$\begin{aligned} b \text{ is an upper bound for } A &\iff \widehat{a} \subseteq \widehat{b} \text{ for every } a \in A \\ &\iff \bigcup_{a \in A} \widehat{a} \subseteq \widehat{b} \iff \overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{b} \end{aligned}$$

because $\widehat{b}$ is certainly closed in Z. It follows at once that if $\widehat{a}_0$ is actually equal to $\overline{\bigcup_{a \in A} \widehat{a}}$ then a_0 must be the least upper bound of A in $\mathfrak{A}$. On the other hand, if $a_0 = \sup A$, then $\overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{a}_0$. **?** If $\widehat{a}_0 \neq \overline{\bigcup_{a \in A} \widehat{a}}$, then $\widehat{a}_0 \setminus \overline{\bigcup_{a \in A} \widehat{a}}$ is a non-empty open set in Z, so includes $\widehat{b}$ for some non-zero $b \in \mathfrak{A}$; now $\widehat{a} \subseteq \widehat{a}_0 \setminus \widehat{b}$, so $a \subseteq a_0 \setminus b$ for every $a \in A$, and $a_0 \setminus b$ is an upper bound for A strictly less than a_0. **X** Thus $\widehat{a}_0$ must be exactly $\overline{\bigcup_{a \in A} \widehat{a}}$.

(b) Take complements: setting $a_1 = 1 \setminus a_0$, we have

$$a_0 = \inf A \iff a_1 = \sup_{a \in A} 1 \setminus a$$

(by 313A)

$$\iff \widehat{a}_1 = \overline{\bigcup_{a \in A} Z \setminus \widehat{a}}$$

$$\iff \widehat{a}_0 = Z \setminus \overline{\bigcup_{a \in A} Z \setminus \widehat{a}} = \operatorname{int} \bigcap_{a \in A} \widehat{a}.$$

(c) Since $\bigcap_{a \in A} \widehat{a}$ is surely a closed set, it is nowhere dense iff it has empty interior, that is, iff $0 = \inf A$.

313D I started the section with the results above because they are easily stated and of great importance. But I must now turn to some new definitions, and I think it may help to clarify the ideas involved if I give them in their own natural context, even though this is far more general than we have any immediate need for here.

Definitions Let P be a partially ordered set and C a subset of P.

(a) C is **order-closed** if $\sup A \in C$ whenever A is a non-empty upwards-directed subset of C such that $\sup A$ is defined in P, and $\inf A \in C$ whenever A is a non-empty downwards-directed subset of C such that $\inf A$ is defined in P.

(b) C is **sequentially order-closed** if $\sup_{n\in\mathbb{N}} p_n \in C$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in C such that $\sup_{n\in\mathbb{N}} p_n$ is defined in P, and $\inf_{n\in\mathbb{N}} p_n \in C$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in C such that $\inf_{n\in\mathbb{N}} p_n$ is defined in P.

Remark I hope it is obvious that an order-closed set is sequentially order-closed.

313E Order-closed subalgebras and ideals Of course, in the very special cases of a subalgebra or ideal of a Boolean algebra, the concepts 'order-closed' and 'sequentially order-closed' have expressions simpler than those in 313D. I spell them out.

(a) Let $\mathfrak{B}$ be a subalgebra of a Boolean algebra $\mathfrak{A}$.

(i) The following are equiveridical:

(α) $\mathfrak{B}$ is order-closed in $\mathfrak{A}$;

(β) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\sup B$ is defined in $\mathfrak{A}$;

(β') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\inf B$ is defined in $\mathfrak{A}$;

(γ) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and upwards-directed and $\sup B$ is defined in $\mathfrak{A}$;

(γ') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and downwards-directed and $\inf B$ is defined in $\mathfrak{A}$.

P Of course (β) $\Rightarrow$ (γ). If (γ) is true and $B \subseteq \mathfrak{B}$ is any set with a supremum in $\mathfrak{A}$, then $B' = \{0\} \cup \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in B\}$ is a non-empty upwards-directed set with the same upper bounds as B, so $\sup B = \sup B' \in \mathfrak{B}$. Thus ($\gamma$) $\Rightarrow$ (β) and (β), (γ) are equiveridical. Next, if (β) is true and $B \subseteq \mathfrak{B}$ is a set with an infimum in $\mathfrak{A}$, then $B' = \{1 \setminus b : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$ and $\sup B' = 1 \setminus \inf B$ is defined, so $\sup B'$ and $\inf B$ belong to $\mathfrak{B}$. Thus (β) $\Rightarrow$ (β'). In the same way, (γ') $\iff$ (β') $\Rightarrow$ (β) and (β), (β'), (γ), (γ') are all equiveridical. But since we also have (α) $\iff$ (γ)&(γ'), (α) is equiveridical with the others. **Q**

Replacing the sets B above by sequences, the same arguments provide conditions for $\mathfrak{B}$ to be sequentially order-closed, as follows.

(ii) The following are equiveridical:

(α) $\mathfrak{B}$ is sequentially order-closed in $\mathfrak{A}$;

(β) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(β') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$.

(b) Now suppose that I is an ideal of $\mathfrak{A}$. Then if $A \subseteq I$ is non-empty all lower bounds of A necessarily belong to I; so that

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ is non-empty, upwards-directed and has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I with a supremum in $\mathfrak{A}$.

Moreover, because I is closed under $\cup$,

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in I with a supremum in $\mathfrak{A}$.

(c) If $\mathfrak{A} = \mathcal{P}X$ is a power set, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is just a σ-algebra of sets, while a sequentially order-closed ideal of $\mathfrak{A}$ is a what I have called a σ-ideal of sets (112Db). If $\mathfrak{A}$ is itself a σ-algebra of sets, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is a 'σ-subalgebra' in the sense of 233A.

Accordingly I will normally use the phrases **σ-subalgebra**, **σ-ideal** for sequentially order-closed subalgebras and ideals of Boolean algebras.

313F Order-closures and generated sets (a) It is an immediate consequence of the definitions that

(i) if $\mathcal{S}$ is any non-empty family of subalgebras of a Boolean algebra $\mathfrak{A}$, then $\bigcap\mathcal{S}$ is a subalgebra of $\mathfrak{A}$;

(ii) if $\mathcal{F}$ is any non-empty family of order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is an order-closed subset of P;

(iii) if $\mathcal{F}$ is any non-empty family of sequentially order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is a sequentially order-closed subset of P.

(b) Consequently, given any Boolean algebra $\mathfrak{A}$ and a subset B of $\mathfrak{A}$, we have a smallest subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including B, being the intersection of all the subalgebras of $\mathfrak{A}$ which include B; a smallest σ-subalgebra $\mathfrak{B}_\sigma$ of $\mathfrak{A}$ including B, being the intersection of all the σ-subalgebras of $\mathfrak{A}$ which include B; and a smallest order-closed subalgebra $\mathfrak{B}_\tau$ of $\mathfrak{A}$ including B, being the intersection of all the order-closed subalgebras of $\mathfrak{A}$ which include B. We call $\mathfrak{B}$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}_\tau$ the subalgebra, σ-subalgebra and order-closed subalgebra **generated** by B. (I shall return to this in 331E.)

(c) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$, then the smallest order-closed subset $\overline{\mathfrak{B}}$ of $\mathfrak{A}$ which includes $\mathfrak{B}$ is again a subalgebra of $\mathfrak{A}$ (so is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$). **P** (i) The set $\{b : 1 \setminus b \in \overline{\mathfrak{B}}\}$ is order-closed (use 313A) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus $1 \setminus b \in \overline{\mathfrak{B}}$ for every $b \in \overline{\mathfrak{B}}$. (ii) If $c \in \mathfrak{B}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed (use 313Bb) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever $b \in \overline{\mathfrak{B}}$ and $c \in \mathfrak{B}$. (iii) If $c \in \overline{\mathfrak{B}}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed and includes $\mathfrak{B}$ (by (ii)), so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever b, $c \in \overline{\mathfrak{B}}$. (iv) By 312B, $\overline{\mathfrak{B}}$ is a subalgebra of $\mathfrak{A}$. **Q**

313G This is a convenient moment at which to spell out an abstract version of the Monotone Class Theorem (136B).

Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that $1 \in I \subseteq A \subseteq \mathfrak{A}$ and that

$$a \cap b \in I \text{ for all } a,\, b \in I,$$

$$b \setminus a \in A \text{ whenever } a,\, b \in A \text{ and } a \subseteq b.$$

Then A includes the subalgebra of $\mathfrak{A}$ generated by I.

(b) If moreover $\sup_{n\in\mathbb{N}} a_n \in A$ for every non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A with a supremum in $\mathfrak{A}$, then A includes the σ-subalgebra of $\mathfrak{A}$ generated by I.

(c) And if $\sup C \in A$ whenever $C \subseteq A$ is an upwards-directed set with a supremum in $\mathfrak{A}$, then A includes the order-closed subalgebra of $\mathfrak{A}$ generated by I.

proof (a)(i) Let $\mathfrak{P}$ be the family of all sets J such that $I \subseteq J \subseteq A$ and $a \cap b \in J$ for all a, $b \in J$. Then $I \in \mathfrak{P}$ and if $\mathfrak{Q} \subseteq \mathfrak{P}$ is upwards-directed and not empty, $\bigcup\mathfrak{Q} \in \mathfrak{P}$. By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\mathfrak{B}$.

(ii) Now

$$\mathfrak{B} = \{c : c \in \mathfrak{A},\ c \cap b \in A \text{ for every } b \in \mathfrak{B}\}.$$

P If $c \in \mathfrak{B}$, then of course $c \cap b \in \mathfrak{B} \subseteq A$ for every $b \in \mathfrak{B}$, because $\mathfrak{B} \in \mathfrak{P}$. If $c \in \mathfrak{A} \setminus \mathfrak{B}$, consider

$$J = \mathfrak{B} \cup \{c \cap b : b \in \mathfrak{B}\}.$$

Then $c = c \cap 1 \in J$ so J properly includes $\mathfrak{B}$ and cannot belong to $\mathfrak{P}$. On the other hand, if b_1, $b_2 \in \mathfrak{B}$,

$$b_1 \cap b_2 \in \mathfrak{B} \subseteq J, \quad (c \cap b_1) \cap b_2 = b_1 \cap (c \cap b_2) = (c \cap b_1) \cap (c \cap b_2) = c \cap (b_1 \cap b_2) \in J,$$

so $c_1 \cap c_2 \in J$ for all c_1, $c_2 \in J$; and of course $I \subseteq \mathfrak{B} \subseteq J$. So J cannot be a subset of A, and there must be a $b \in \mathfrak{B}$ such that $c \cap b \notin A$. **Q**

(iii) Consequently $c \setminus b \in \mathfrak{B}$ whenever b, $c \in \mathfrak{B}$ and $b \subseteq c$. **P** If $a \in \mathfrak{B}$, then $b \cap a$, $c \cap a \in \mathfrak{B} \subseteq A$ and $b \cap a \subseteq c \cap a$, so

$$(c \setminus b) \cap a = (c \cap a) \setminus (b \cap a) \in A$$

by the hypothesis on A. By (ii), $c \setminus b \in \mathfrak{B}$. **Q**

(iv) It follows that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. **P** If $b \in \mathfrak{B}$, then

$$b \subseteq 1 \in I \subseteq \mathfrak{B},$$

so $1 \setminus b \in \mathfrak{B}$. If a, $b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

$0 = 1 \setminus 1 \in \mathfrak{B}$, so that the conditions of 312B(ii) are satisfied. **Q**

Now the subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in A, as required.

(b) Now suppose that $\sup_{n\in\mathbb{N}} a_n$ belongs to A whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$. Then $\mathfrak{B}$, as defined in part (a) of the proof, is a σ-subalgebra of $\mathfrak{A}$. **P** Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}$ with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\langle b_n \cap b\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A with a supremum $c \cap b$ in $\mathfrak{A}$ (313Ba). So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$, by the criterion in (a-ii) above. As $\langle b_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{B}$ is a σ-subalgebra, by 313Ea. **Q**

Accordingly the σ-subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

(c) Finally, if $\sup C \in A$ whenever C is a non-empty upwards-directed subset of A with a least upper bound in $\mathfrak{A}$, $\mathfrak{B}$ is order-closed. **P** Let $C \subseteq \mathfrak{B}$ be a non-empty upwards-directed set with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\{c \cap b : c \in C\}$ is a non-empty upwards-directed set in A with supremum $c \cap b$ in $\mathfrak{A}$. So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$. As C is arbitrary, $\mathfrak{B}$ is order-closed in $\mathfrak{A}$. **Q**

Accordingly the order-closed subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

313H Definitions It is worth distinguishing various types of supremum- and infimum-preserving function. Once again, I do this in almost the widest possible context. Let P and Q be two partially ordered sets, and $\phi : P \to Q$ an **order-preserving** function, that is, a function such that $\phi(p) \leq \phi(q)$ in Q whenever $p \leq q$ in P.

(a) I say that ϕ is **order-continuous** if (i) $\phi(\sup A) = \sup_{p\in A} \phi(p)$ whenever A is a non-empty upwards-directed subset of P and $\sup A$ is defined in P (ii) $\phi(\inf A) = \inf_{p\in A} \phi(p)$ whenever A is a non-empty downwards-directed subset of P and $\inf A$ is defined in P.

(b) I say that ϕ is **sequentially order-continuous** or **σ-order-continuous** if (i) $\phi(p) = \sup_{n\in\mathbb{N}} \phi(p_n)$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in P and $p = \sup_{n\in\mathbb{N}} p_n$ in P (ii) $\phi(p) = \inf_{n\in\mathbb{N}} \phi(p_n)$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in P and $p = \inf_{n\in\mathbb{N}} p_n$ in P.

Remark You may feel that one of the equivalent formulations in Proposition 313Lb gives a clearer idea of what is really being demanded of ϕ in the ordinary cases we shall be looking at.

313I Proposition Let P, Q and R be partially ordered sets, and $\phi : P \to Q$, $\psi : Q \to R$ order-preserving functions.

(a) $\psi\phi : P \to R$ is order-preserving.

(b) If ϕ and ψ are order-continuous, so is $\psi\phi$.

(c) If ϕ and ψ are sequentially order-continuous, so is $\psi\phi$.

(d) ϕ is order-continuous iff $\phi^{-1}[B]$ is order-closed for every order-closed $B \subseteq Q$.

proof (a)-(c) I think the only point that needs remarking is that if $A \subseteq P$ is upwards-directed, then $\phi[A] \subseteq Q$ is upwards-directed, because ϕ is order-preserving. So if $\sup A$ is defined in P and ϕ, ψ are order-continuous, we shall have

$$\psi(\phi(\sup A)) = \psi(\sup \phi[A]) = \sup \psi[\phi[A]].$$

(d)(i) Suppose that ϕ is order-continuous and that $B \subseteq Q$ is order-closed. Let $A \subseteq \phi^{-1}[B]$ be a non-empty upwards-directed set with supremum $p \in P$. Then $\phi[A] \subseteq B$ is non-empty and upwards-directed, because ϕ is order-preserving, and $\phi(p) = \sup \phi[A]$ because ϕ is order-continuous. Because B is order-closed,

$\phi(p) \in B$ and $p \in \phi^{-1}[B]$. Similarly, if $A \subseteq \phi^{-1}[B]$ is non-empty and downwards-directed, and $\inf A$ is defined in P, then $\phi(\inf A) = \inf \phi[A] \in B$ and $\inf A \in \phi^{-1}[B]$. Thus $\phi^{-1}[B]$ is order-closed; as B is arbitrary, ϕ satisfies the condition.

(ii) Now suppose that $\phi^{-1}[B]$ is order-closed in P whenever $B \subseteq Q$ is order-closed in Q. Let $A \subseteq P$ be a non-empty upwards-directed subset of P with a supremum $p \in P$. Then $\phi(p)$ is an upper bound of $\phi[A]$. Let q be any upper bound of $\phi[A]$ in Q. Consider $B = \{r : r \leq q\}$; then $B \subseteq Q$ is upwards-directed and order-closed, so $\phi^{-1}[B]$ is order-closed. Also $A \subseteq \phi^{-1}[B]$ is non-empty and upwards-directed and has supremum p, so $p \in \phi^{-1}[B]$ and $\phi(p) \in B$, that is, $\phi(p) \leq q$. As q is arbitrary, $\phi(p) = \sup \phi[A]$. Similarly, $\phi(\inf A) = \inf \phi[A]$ whenever $A \subseteq P$ is non-empty, downwards-directed and has an infimum in P; so ϕ is order-continuous.

313J It is useful to introduce here the following notion.

Definition Let $\mathfrak{A}$ be a Boolean algebra. A set $D \subseteq \mathfrak{A}$ is **order-dense** if for every non-zero $a \in \mathfrak{A}$ there is a non-zero $d \in D$ such that $d \subseteq a$.

Remark Many authors use the simple word 'dense' where I have insisted on the phrase 'order-dense'. In the work of this treatise it will be important to distinguish clearly between this concept of 'dense' set and the topological concept (2A3U); typically, in those contexts in which both appear, an order-dense set can be in some sense much smaller than a topologically dense set.

313K Lemma If $\mathfrak{A}$ is a Boolean algebra and $D \subseteq \mathfrak{A}$ is order-dense, then for any $a \in \mathfrak{A}$ there is a disjoint $C \subseteq D$ such that $\sup C = a$; in particular, $a = \sup\{d : d \in D, d \subseteq a\}$ and there is a partition of unity $C \subseteq D$.

proof Set $D_a = \{d : d \in D, d \subseteq a\}$. Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint sets $C \subseteq D_a$, we have a maximal $C \in \mathcal{C}$. Now if $b \in \mathfrak{A}$ and $b \not\supseteq a$, there is a $d \in D$ such that $0 \neq d \subseteq a \setminus b$. Because C is maximal, there must be a $c \in C$ such that $c \cap d \neq 0$, so that $c \not\subseteq b$. Turning this round, any upper bound of C must include a, so that $a = \sup C$. It follows at once that $a = \sup D_a$.

Taking $a = 1$ we obtain a partition of unity included in D.

313L Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) π is order-preserving.

(b) The following are equiveridical:

(i) π is order-continuous;

(ii) whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$, then $\inf \pi[A] = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is non-empty and upwards-directed and $\sup A = 1$ in $\mathfrak{A}$, then $\sup \pi[A] = 1$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(v) whenever $A \subseteq \mathfrak{A}$ and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(vi) whenever $C \subseteq \mathfrak{A}$ is a partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(c) The following are equiveridical:

(i) π is sequentially order-continuous;

(ii) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} a_n = 0$ in $\mathfrak{A}$, then $\inf_{n\in\mathbb{N}} \pi a_n = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is countable and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ is countable and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(v) whenever $C \subseteq \mathfrak{A}$ is a countable partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

proof (a) This is 312I.

(b)(i)⇒(ii) is trivial, as $\pi 0 = 0$.

(ii)⇒(iv) Assume (ii), and let A be any subset of $\mathfrak{A}$ such that $c = \sup A$ is defined in $\mathfrak{A}$. If $A = \emptyset$, then $c = 0$ and $\sup \pi[A] = 0 = \pi c$. Otherwise, set

$$A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}, \quad C = \{c \setminus a : a \in A'\}.$$

Then A' is upwards-directed and has the same upper bounds as A, so $c = \sup A'$ and $0 = \inf C$, by 313Aa. Also C is downwards-directed, so $\inf \pi[C] = 0$ in $\mathfrak{B}$. But now

$$\pi[C] = \{\pi c \setminus \pi a : a \in A'\} = \{\pi c \setminus b : b \in \pi[A']\},$$

$$\pi[A'] = \{\pi a_0 \cup \ldots \cup \pi a_n : a_0, \ldots , a_n \in A\} = \{b_0 \cup \ldots \cup b_n : b_0, \ldots , b_n \in \pi[A]\},$$

because π is a Boolean homomorphism. Again using 313Aa and the fact that $b \subseteq \pi c$ for every $b \in \pi[A']$, we get

$$\pi c = \sup \pi[A'] = \sup \pi[A].$$

As A is arbitrary, (iv) is satisfied.

(iv)⇒(v) If $A \subseteq \mathfrak{A}$ and $c = \inf A$ is defined in $\mathfrak{A}$, then $1 \setminus c = \sup_{a\in A} 1 \setminus a$, so

$$\pi c = 1 \setminus \pi(1 \setminus c) = 1 \setminus \sup_{a\in A} \pi(1 \setminus a) = \inf_{a\in A} 1 \setminus \pi(1 \setminus a) = \inf_{a\in A} \pi a.$$

(v)⇒(ii) is trivial, because $\pi 0 = 0$.

(iv)⇒(iii) is similarly trivial.

(iii)⇒(vi) Assume (iii), and let C be a partition of unity in $\mathfrak{A}$. Then $C' = \{c_0 \cup \ldots \cup c_n : c_0, \ldots , c_n \in C\}$ is upwards-directed and has supremum 1, so $\sup \pi[C'] = 1$. But (because π is a Boolean homomorphism) $\pi[C]$ and $\pi[C']$ have the same upper bounds, so $\sup \pi[C] = 1$, as required.

(vi)⇒(ii) Assume (vi), and let $A \subseteq \mathfrak{A}$ be a set with infimum 0. Set

$$D = \{d : d \in \mathfrak{A}, \exists\, a \in A, d \cap a = 0\}.$$

Then D is order-dense in $\mathfrak{A}$. **P** If $e \in \mathfrak{A} \setminus \{0\}$, then there is an $a \in A$ such that $e \not\subseteq a$, so that $e \setminus a$ is a non-zero member of D included in e. **Q** Consequently there is a partition of unity $C \subseteq D$, by 313K. But now if b is any lower bound for $\pi[A]$ in $\mathfrak{B}$, we must have $b \cap \pi d = 0$ for every $d \in D$, so $\pi c \subseteq 1 \setminus b$ for every $c \in C$, and $1 \setminus b = 1$, $b = 0$. Thus $\inf \pi[A] = 0$. As A is arbitrary, (ii) is satisfied.

(v)&(iv)⇒(i) is trivial.

(c) We can use nearly identical arguments, remembering only to interpolate the word ‘countable’ from time to time. I spell out the new version of (ii)⇒(iii), even though it requires no more than an adaptation of the language. Assume (ii), and let A be a countable subset of $\mathfrak{A}$ with a supremum $c \in \mathfrak{A}$. If $A = \emptyset$, then $c = 0$ so $\pi c = 0 = \sup \pi[A]$. Otherwise, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A; set $a'_n = a_0 \cup \ldots \cup a_n$ and $c_n = c \setminus a'_n$ for each n. Then $\langle a'_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, with supremum c, and $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing, with infimum 0; so $\inf_{n\in\mathbb{N}} \pi c_n = 0$ and

$$\sup_{n\in\mathbb{N}} \pi a_n = \sup_{n\in\mathbb{N}} \pi a'_n = \pi c.$$

For (v)⇒(ii), however, a different idea is involved. Assume (v), and suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. Set $c_0 = 1 \setminus a_0$, $c_n = a_{n-1} \setminus a_n$ for $n \ge 1$; then $C = \{c_n : n \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$ (because if $c \cap c_n = 0$ for every n, then $c \subseteq a_n$ for every n), so $\pi[C]$ is a partition of unity in $\mathfrak{B}$. Now if $b \le \pi a_n$ for every n, $b \cap \pi c_n$ for every n, so $b = 0$; thus $\inf_{n\in\mathbb{N}} \pi a_n = 0$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is satisfied.

313M The following result is perfectly elementary, but it will save a moment later on to have it spelt out.

Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism.

(a) If $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$, then $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$.

(b) If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $C \subseteq \mathfrak{A}$, then the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{B}$ generated by $\pi[C]$ includes $\pi[\mathfrak{C}]$.

(c) Now suppose that π is surjective and that $C \subseteq \mathfrak{A}$ is such that the order-closed subalgebra of $\mathfrak{A}$ generated by C is $\mathfrak{A}$ itself. Then the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$ is $\mathfrak{B}$.

proof (a) Setting $\mathfrak{C} = \pi^{-1}[\mathfrak{D}]$: if a, $a' \in \mathfrak{C}$ then $\pi(a \cap b) = \pi a \cap \pi b$, $\pi(a \triangle a') = \pi a \triangle \pi a' \in \mathfrak{D}$, so $a \cap a'$, $a \triangle a' \in \mathfrak{C}$; $\pi 1 = 1 \in \mathfrak{D}$ so $1 \in \mathfrak{C}$; thus $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$. By 313Id, $\mathfrak{C}$ is order-closed.

(b) By (a), $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$. It includes C so includes $\mathfrak{C}$, and $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$.

(c) In the language of (b), we have $\mathfrak{C} = \mathfrak{A}$, so $\mathfrak{D}$ must be $\mathfrak{B}$.

313N Definition The phrase **regular embedding** is sometimes used to mean an injective order-continuous Boolean homomorphism; a subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is said to be **regularly embedded** in $\mathfrak{A}$ if the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous, that is, if whenever $b \in \mathfrak{B}$ is the supremum (in $\mathfrak{B}$) of $B \subseteq \mathfrak{B}$, then b is also the supremum in $\mathfrak{A}$ of B; and similarly for infima. One important case is when $\mathfrak{B}$ is order-dense (313O); another is in 314G-314H below.

313O Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$. In particular, if $B \subseteq \mathfrak{B}$ and $c \in \mathfrak{B}$ then $c = \sup B$ in $\mathfrak{B}$ iff $c = \sup B$ in $\mathfrak{A}$.

proof I have to show that the identity homomorphism $\iota : \mathfrak{B} \to \mathfrak{A}$ is order-continuous. **?** Suppose, if possible, otherwise. By 313L(b-ii), there is a non-empty set $B \subseteq \mathfrak{B}$ such that $\inf B = 0$ in $\mathfrak{B}$ but $B = \iota[B]$ has a non-zero lower bound $a \in \mathfrak{A}$. In this case, however (because $\mathfrak{B}$ is order-dense) there is a non-zero $d \in \mathfrak{B}$ with $d \subseteq a$, in which case d is a non-zero lower bound for B in $\mathfrak{B}$. **X**

313P The most important use of these ideas to us concerns quotient algebras (313Q); I approach by means of a superficially more general result.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism with kernel I.

(a)(i) If π is order-continuous then I is order-closed.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is order-closed then π is order-continuous.

(b)(i) If π is sequentially order-continuous then I is a σ-ideal.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is a σ-ideal then π is sequentially order-continuous.

proof (a)(i) If $A \subseteq I$ is upwards-directed and has a supremum $c \in \mathfrak{A}$, then $\pi c = \sup \pi[A] = 0$, so $c \in I$. As remarked in 313Eb, this shows that I is order-closed.

(ii) We are supposing that the identity map from $\pi[\mathfrak{A}]$ to $\mathfrak{B}$ is order-continuous, so it will be enough to show that π is order-continuous when regarded as a map from $\mathfrak{A}$ to $\pi[\mathfrak{A}]$. Suppose that $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and that $\inf A = 0$. **?** Suppose, if possible, that 0 is not the greatest lower bound of $\pi[A]$ in $\pi[\mathfrak{A}]$. Then there is a $c \in \mathfrak{A}$ such that $0 \neq \pi c \subseteq \pi a$ for every $a \in A$. Now

$$\pi(c \setminus a) = \pi c \setminus \pi a = 0$$

for every $a \in A$, so $c \setminus a \in I$ for every $a \in A$. The set $C = \{c \setminus a : a \in A\}$ is upwards-directed and has supremum c; because I is order-closed, $c = \sup C \in I$, and $\pi c = 0$, contradicting the specification of c. **X** Thus $\inf \pi[A] = 0$ in either $\pi[\mathfrak{A}]$ or $\mathfrak{B}$. As A is arbitrary, π is order-continuous, by the criterion (ii) of 313Lb.

(b) Argue in the same way, replacing each set A by a sequence.

313Q Corollary Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$; write π for the canonical map from $\mathfrak{A}$ to $\mathfrak{A}/I$.

(a) π is order-continuous iff I is order-closed.

(b) π is sequentially order-continuous iff I is a σ-ideal.

proof $\pi[\mathfrak{A}] = \mathfrak{A}/I$ is surely regularly embedded in $\mathfrak{A}/I$.

313R For order-continuous homomorphisms, at least, there is an elegant characterization in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Let Z and W be their Stone spaces, and $\phi : W \to Z$ the corresponding continuous function (312P). Then the following are equiveridical:

(i) π is order-continuous;

(ii) $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense set $M \subseteq Z$;

(iii) $\operatorname{int} \phi[H] \neq \emptyset$ for every non-empty open set $H \subseteq W$.

proof (a)(i)⇒(iii) Suppose that π is order-continuous. **?** Suppose, if possible, that $H \subseteq W$ is a non-empty open set and $\operatorname{int}\phi[H] = \emptyset$. Let $b \in \mathfrak{B} \setminus \{0\}$ be such that $\widehat{b} \subseteq H$. Then $\phi[\widehat{b}]$ has empty interior; but also it is a closed set, so its complement is dense. Set $A = \{a : a \in \mathfrak{A},\ \widehat{a} \cap \phi[\widehat{b}] = \emptyset\}$. Then $\bigcup_{a\in\mathfrak{A}} \widehat{a} = Z \setminus \phi[\widehat{b}]$ is a dense open set, so $\sup A = 1$ in $\mathfrak{A}$ (313Ca). Because π is order-continuous, $\sup \pi[A] = 1$ in $\mathfrak{B}$ (313L(b-iii)), and there is an $a \in A$ such that $\pi a \cap b \neq 0$. But this means that $\widehat{b} \cap \phi^{-1}[\widehat{a}] \neq \emptyset$ and $\phi[\widehat{b}] \cap \widehat{a} \neq \emptyset$, contrary to the definition of A. **X**

Thus there is no such set H, and (iii) is true.

(b)(iii)⇒(ii) Now assume (iii). If $M \subseteq Z$ is nowhere dense, set $N = \phi^{-1}[\overline{M}]$, so that $N \subseteq W$ is a closed set. If $H = \operatorname{int} N$, then $\operatorname{int}\phi[H] \subseteq \operatorname{int}\overline{M} = \emptyset$, so (iii) tells us that H is empty; thus N and $\phi^{-1}[M]$ are nowhere dense, as required by (ii).

(c)(ii)⇒(i) Assume (ii), and let $A \subseteq \mathfrak{A}$ be a non-empty set such that $\inf A = 0$ in $\mathfrak{A}$. Then $M = \bigcap_{a\in A} \widehat{a}$ has empty interior in Z (313Cb), so (being closed) is nowhere dense, and $\phi^{-1}[M]$ is also nowhere dense. If $b \in \mathfrak{B} \setminus \{0\}$, then

$$\widehat{b} \not\subseteq \phi^{-1}[M] = \bigcap_{a\in A} \phi^{-1}[\widehat{a}] = \bigcap_{a\in A} \widehat{\pi a},$$

so b is not a lower bound for $\pi[A]$. This shows that $\inf \pi[A] = 0$ in $\mathfrak{B}$. As A is arbitrary, π is order-continuous (313L(b-ii)).

313X Basic exercises (a) Use 313C to give alternative proofs of 313A and 313B.

(b) Let P be a partially ordered set. Show that there is a topology on P for which the closed sets are just the order-closed sets.

(c) Let P be a partially ordered set, $Q \subseteq P$ an order-closed set, and R a subset of Q which is order-closed in Q when Q is given the partial ordering induced by that of P. Show that R is order-closed in P.

>(d) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $1 \in I \subseteq \mathfrak{A}$ and that $a \cap b \in I$ for all a, $b \in I$. (i) Let $\mathfrak{B}$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$ and $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. (ii) Let $\mathfrak{B}_\sigma$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$, $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$ and $\sup_{n\in\mathbb{N}} b_n \in A$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$. Show that $\mathfrak{B}_\sigma$ is a σ-subalgebra of $\mathfrak{A}$. (iii) Let $\mathfrak{B}_\tau$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$, $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$ and $\sup B \in A$ whenever B is a non-empty upwards-directed subset of A with a supremum in $\mathfrak{A}$. Show that $\mathfrak{B}_\tau$ is an order-closed subalgebra of $\mathfrak{A}$. (iv) Hence give a proof of 313G not relying on Zorn's Lemma or any other use of the axiom of choice.

(e) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Let $\mathfrak{B}_\sigma$ be the smallest sequentially order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. Show that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$.

>(f) Let X be a set, and $\mathcal{A}$ a subset of $\mathcal{P}X$. Show that $\mathcal{A}$ is an order-closed subalgebra of $\mathcal{P}X$ iff it is of the form $\{f^{-1}[F] : F \subseteq Y\}$ for some set Y, function $f : X \to Y$.

(g) Let P and Q be partially ordered sets, and $\phi : P \to Q$ an order-preserving function. Show that ϕ is sequentially order-continuous iff $\phi^{-1}[C]$ is sequentially order-closed in $\mathfrak{A}$ for every sequentially order-closed $C \subseteq \mathfrak{B}$.

(h) For partially ordered sets P and Q, let us call a function $\phi : P \to Q$ **monotonic** if it is *either* order-preserving *or* order-reversing. State and prove definitions and results corresponding to 313H, 313I and 313Xg for general monotonic functions.

>(i) Let $\mathfrak{A}$ be a Boolean algebra. Show that the operations $(a,b) \mapsto a \cup b$ and $(a,b) \mapsto a \cap b$ are order-continuous operations from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$, if we give $\mathfrak{A} \times \mathfrak{A}$ the product partial order, saying that $(a,b) \leq (a',b')$ iff $a \subseteq a'$ and $b \subseteq b'$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. Show that if a subalgebra of $\mathfrak{A}$ is order-dense then it is dense in the topology of 313Xb.

>(k) Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A}$ any disjoint set. Show that there is a partition of unity in $\mathfrak{A}$ including A.

>(l) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two order-continuous Boolean homomorphisms. Show that $\{a : \pi_1 a = \pi_2 a\}$ is an order-closed subalgebra of $\mathfrak{A}$.

(m) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two Boolean homomorphisms. Suppose that π_1 and π_2 agree on some order-dense subset of $\mathfrak{A}$, and that one of them is order-continuous. Show that they are equal. (*Hint*: if π_1 is order-continuous, $\pi_2 a \supseteq \pi_1 a$ for every a.)

(n) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Show that π is order-continuous iff $\pi\restriction\mathfrak{A}_0 : \mathfrak{A}_0 \to \mathfrak{B}$ is order-continuous.

>(o) Let $\mathfrak{A}$ be a Boolean algebra. For $A \subseteq \mathfrak{A}$ set $A^\perp = \{b : a \cap b = 0 \ \forall \ a \in A\}$. (i) Show that $A^\perp$ is an order-closed ideal of $\mathfrak{A}$. (ii) Show that a set $A \subseteq \mathfrak{A}$ is an order-closed ideal of $\mathfrak{A}$ iff $A = A^{\perp\perp}$. (iii) Show that if $I \subseteq \mathfrak{A}$ is an order-closed ideal then $\{a^\bullet : a \in I^\perp\}$ is an order-dense ideal in the quotient algebra $\mathfrak{A}/I$.

(p) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W; let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a Boolean homomorphism, and $\phi : W \to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is order-continuous; (ii) $\operatorname{int}\phi^{-1}[F] = \phi^{-1}[\operatorname{int}F]$ for every closed $F \subseteq Z$ (iii) $\overline{\phi^{-1}[G]} = \phi^{-1}[\overline{G}]$ for every open $G \subseteq Z$.

313Y Further exercises (a) Prove 313A-313C for general Boolean rings.

(b) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi : W \to Z$. Show that π is sequentially order-continuous iff $\phi^{-1}[M]$ is nowhere dense for every nowhere dense zero set $M \subseteq Z$.

(c) Let P be any partially ordered set, and let $\mathfrak{T}$ be the topology of 313Xb. (i) Show that a sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P is $\mathfrak{T}$-convergent to $p \in P$ iff every subsequence of $\langle p_n\rangle_{n\in\mathbb{N}}$ has a monotonic sub-subsequence with supremum or infimum equal to p. (ii) Show that a subset A of P is sequentially order-closed, in the sense of 313Db, iff the $\mathfrak{T}$-limit of any $\mathfrak{T}$-convergent sequence in A belongs to A. (iii) Suppose that A is an upwards-directed subset of P with supremum $p_0 \in P$. For $a \in A$ set $F_a = \{p : a \le p \in A\}$, and let $\mathcal{F}$ be the filter on P generated by $\{F_a : a \in A\}$. Show that $\mathcal{F} \to p_0$ for $\mathfrak{T}$. (iv) Show that if Q is another partially ordered set, endowed with a topology $\mathfrak{S}$ in the same way, then a monotonic function $\phi : P \to Q$ is order-continuous iff it is continuous for the topologies $\mathfrak{T}$ and $\mathfrak{S}$, and is sequentially order-continuous iff it is sequentially continuous for these topologies.

(d) Let U be a Banach lattice (242G, 354Ab). Show that its norm is order-continuous in the sense of 242Yc (or 354Dc) iff its restriction to $\{u : u \ge 0\}$ is order-continuous in the sense of 313Ha.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism and $\phi : W \to Z$ the corresponding continuous function. Show that $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$ iff ϕ is **irreducible**, that is, $\phi[F] \ne \phi[W]$ for any proper closed subset F of W.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism and $\phi : W \to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is injective and order-continuous; (ii) for $M \subseteq Z$, M is nowhere dense iff $\phi^{-1}[M]$ is nowhere dense.

313 Notes and comments I give 'elementary' proofs of 313A-313B because I believe that they help to exhibit the relevant aspects of the structure of Boolean algebras; but various abbreviations are possible, notably if we allow ourselves to use the Stone representation (313Xa). 313A and 313Ba-b can be expressed by saying that the Boolean operations $\cup$, $\cap$ and $\setminus$ are (separately) order-continuous. Of course, $\setminus$ is order-reversing, rather than order-preserving, in the second variable; but the natural symmetry in the concept of partial order means that the ideas behind 313H-313I can be applied equally well to order-reversing functions (313Xh). In fact, $\cup$ and $\cap$ can be regarded as order-continuous functions on the product space (313Bc-d, 313Xi). Clearly 313Bc-d can be extended into forms valid for any finite sequence $A_0, \dots, A_n$ of subsets of $\mathfrak{A}$ in place of A, B. But if we seek to go to infinitely many subsets of $\mathfrak{A}$ we find ourselves saying something new; see 316G-316J below.

Proposition 313C, and its companions 313R, 313Xp and 313Yb, are worth studying not only as a useful technique, but also in order to understand the difference between $\sup A$, where A is a set in a Boolean algebra, and $\bigcup \mathcal{A}$, where $\mathcal{A}$ is a family of sets. Somehow $\sup A$ can be larger, and $\inf A$ smaller, than one's first intuition might suggest, corresponding to the fact that not every subset of the Stone space corresponds to an element of the Boolean algebra.

I should like to use the words 'order-closed' and 'sequentially order-closed' to mean closed, or sequentially closed, for some more or less canonical topology. The difficulty is that while a great many topologies can be defined from a partial order (one is described in 313Xb and 313Yc, and another in 367Yc), none of them has such pre-eminence that it can be called 'the' order-topology. Accordingly there is a degree of arbitrariness in the language I use here. Nevertheless (sequentially) order-closed subalgebras and ideals are of such importance that they seem to deserve a concise denotation. The same remarks apply to (sequential) order-continuity. Concerning the term 'order-dense' in 313J, this has little to do with density in any topological sense, but the word 'dense', at least, is established in this context.

With all these definitions, there is a good deal of scope for possible interrelations. The most important to us is 313Q, which will be used repeatedly (typically, with $\mathfrak{A}$ an algebra of sets), but I think it is worth having the expanded version in 313P available.

I take the opportunity to present an abstract form of an important lemma on σ-algebras generated by families closed under $\cap$ (136B, 313Gb). This time round I use the Zorn's Lemma argument in the text and suggest the alternative, 'elementary' method in the exercises (313Xd). The two methods are opposing extremes in the sense that the Zorn's Lemma argument looks for maximal subalgebras included in A (which are not unique, and have to be picked out using the axiom of choice) and the other approach seeks minimal subalgebras including I (which are uniquely defined, and can be described without the axiom of choice).

Note that the concept of 'order-closed' algebra of sets is not particularly useful; there are too few order-closed subalgebras of $\mathcal{P}X$ and they are of too simple a form (313Xf). It is in abstract Boolean algebras that the idea becomes important. In the most important partially ordered sets of measure theory, the sequentially order-closed sets are the same as the order-closed sets (see, for instance, 316Fb below), and most of the important order-closed subalgebras dealt with in this chapter can be thought of as σ-subalgebras which are order-closed because they happen to lie in the right kind of algebra.

314 Order-completeness

The results of §313 are valid in all Boolean algebras, but of course are of most value when many suprema and infima exist. I now set out the most useful definitions which guarantee the existence of suprema and infima (314A) and work through their elementary relationships with the concepts introduced so far (314C-314J). I then embark on the principal theorems concerning order-complete Boolean algebras: the extension theorem for homomorphisms to a Dedekind complete algebra (314K), the Loomis-Sikorski representation of a Dedekind σ-complete algebra as a quotient of a σ-algebra of sets (314M), the characterization of Dedekind complete algebras in terms of their Stone spaces (314S), and the idea of 'Dedekind completion' of a Boolean algebra (314T-314U). On the way I describe 'regular open algebras' (314O-314Q).

314A Definitions Let P be a partially ordered set.

(a) P is **Dedekind complete**, or **order-complete**, or **conditionally complete** if every non-empty subset of P with an upper bound has a least upper bound.

(b) P is **Dedekind σ-complete**, or **σ-order-complete**, if (i) every countable non-empty subset of P with an upper bound has a least upper bound (ii) every countable non-empty subset of P with a lower bound has a greatest lower bound.

314B Remarks (a) I give these definitions in the widest possible generality because they are in fact of great interest for general partially ordered sets, even though for the moment we shall be concerned only with Boolean algebras. Indeed I have already presented the same idea in the context of Riesz spaces (241F).

(b) You will observe that the definition in (a) of 314A is asymmetric, unlike that in (b). This is because the inverted form of the definition is equivalent to that given; that is, P is Dedekind complete (on the definition 314Aa) iff every non-empty subset of P with a lower bound has a greatest lower bound. **P** (i) Suppose that P is Dedekind complete, and that $B \subseteq P$ is non-empty and bounded below. Let A be the set of lower bounds for B. Then A has at least one upper bound (since any member of B is an upper bound for A) and is not empty; so $a_0 = \sup A$ is defined. Now if $b \in B$, b is an upper bound for A, so $a_0 \le b$; thus $a_0 \in A$ and must be the greatest member of A, that is, the greatest lower bound of B. (ii) Similarly, if every non-empty subset of P with a lower bound has a greatest lower bound, P is Dedekind complete. **Q**

(c) In the special case of Boolean algebras, we do not need both halves of the definition 314Ab; in fact we have, for any Boolean algebra $\mathfrak{A}$,

$$\begin{aligned} &A \text{ is Dedekind } \sigma\text{-complete} \\ &\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a least upper bound} \\ &\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a greatest lower bound.} \end{aligned}$$

P Because $\mathfrak{A}$ has a least element 0 and a greatest element 1, every subset of $\mathfrak{A}$ has upper and lower bounds; so the two one-sided conditions together are equivalent to saying that $\mathfrak{A}$ is Dedekind σ-complete. I therefore have to show that they are equiveridical. Now if $A \subseteq \mathfrak{A}$ is a non-empty countable set, so is $B = \{1 \setminus a : a \in A\}$, and

$$\inf A = 1 \setminus \sup B, \quad \sup A = 1 \setminus \inf B$$

whenever the right-hand-sides are defined (313A). So if the existence of a supremum (resp. infimum) of B is guaranteed, so is the existence of an infimum (resp. supremum) of A. **Q**

The real point here is of course that $(\mathfrak{A}, \subseteq)$ is isomorphic to $(\mathfrak{A}, \supseteq)$.

(d) Most specialists in Boolean algebra speak of 'complete', or 'σ-complete', Boolean algebras. I prefer the longer phrases 'Dedekind complete' and 'Dedekind σ-complete' because we shall be studying metrics on Boolean algebras and shall need the notion of metric completeness as well as that of order-completeness.

(e) I have had to make some rather arbitrary choices in the definition here. The principal examples of partially ordered set to which we shall apply these definitions are Boolean algebras and Riesz spaces, which are all lattices. Consequently it is not possible to distinguish in these contexts between the property of Dedekind completeness, as defined above, and the weaker property, which we might call 'monotone order-completeness',

> (i) whenever $A \subseteq P$ is non-empty, upwards-directed and bounded above then A has a least upper bound in P (ii) whenever $A \subseteq P$ is non-empty, downwards-directed and bounded below then A has a greatest lower bound in P.

(See 314Xa below. 'Monotone order-completeness' is the property involved in 314Ya, for instance.) Nevertheless I am prepared to say, on the basis of my own experience of working with other partially ordered sets, that 'Dedekind completeness', as I have defined it, is at least of sufficient importance to deserve a name. Note that it does not imply that P is a lattice, since it allows two elements of P to have no common upper bound.

(f) The phrase **complete lattice** is sometimes used to mean a Dedekind complete lattice with greatest and least elements; equivalently, a Dedekind complete partially ordered set with greatest and least elements. Thus a Dedekind complete Boolean algebra is a complete lattice in this sense, but $\mathbb{R}$ is not.

(g) The most important Dedekind complete Boolean algebras (at least from the point of view of measure theory) are the 'measure algebras' of the next chapter. I shall not pause here to give other examples, but will proceed directly with the general theory.

314C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a σ-ideal of $\mathfrak{A}$. Then the quotient Boolean algebra $\mathfrak{A}/I$ is Dedekind σ-complete.

proof I use the description in 314Bc. Let $B \subseteq \mathfrak{A}/I$ be a non-empty countable set. For each $u \in B$, choose an $a_u \in \mathfrak{A}$ such that $u = a_u^\bullet = a_u + I$. Then $c = \sup_{u \in B} a_u$ is defined in $\mathfrak{A}$; consider $v = c^\bullet$ in $\mathfrak{A}/I$. Because the map $a \mapsto a^\bullet$ is sequentially order-continuous (313Qb), $v = \sup B$. As B is arbitrary, $\mathfrak{A}/I$ is Dedekind σ-complete.

314D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Then $\Sigma \cap \mathcal{I}$ is a σ-ideal of the Boolean algebra Σ, and $\Sigma/\Sigma \cap \mathcal{I}$ is Dedekind σ-complete.

proof Of course Σ is Dedekind σ-complete, because if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence in Σ then $\bigcup_{n \in \mathbb{N}} E_n$ is the least upper bound of $\{E_n : n \in \mathbb{N}\}$ in Σ. It is also easy to see that $\Sigma \cap \mathcal{I}$ is a σ-ideal of Σ, since $F \cap \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{I}$ whenever $F \in \Sigma$ and $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\Sigma \cap \mathcal{I}$. So 314C gives the result.

314E Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind complete, then all its order-closed subalgebras and principal ideals are Dedekind complete.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then all its σ-subalgebras and principal ideals are Dedekind σ-complete.

proof All we need to note is that if $\mathfrak{C}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, and $B \subseteq \mathfrak{C}$ is such that $b = \sup B$ is defined in $\mathfrak{A}$, then $b \in \mathfrak{C}$ (see 313E(a-i-β)), so b is still the supremum of B in $\mathfrak{C}$; while the same is true if $\mathfrak{C}$ is a σ-subalgebra and $B \subseteq \mathfrak{C}$ is countable, using 313E(a-ii-β).

314F I spell out some further connexions between the concepts 'order-closed set', 'order-continuous function' and 'Dedekind complete Boolean algebra' which are elementary without being quite transparent.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a)(i) If $\mathfrak{A}$ is Dedekind complete and π is order-continuous, then $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind complete and π is injective and $\pi[\mathfrak{A}]$ is order-closed then π is order-continuous.

(b)(i) If $\mathfrak{A}$ is Dedekind σ-complete and π is sequentially order-continuous, then $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind σ-complete and π is injective and $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$ then π is sequentially order-continuous.

proof (a)(i) If $B \subseteq \pi[\mathfrak{A}]$, then $a_0 = \sup(\pi^{-1}[B])$ is defined in $\mathfrak{A}$; now

$$\pi a_0 = \sup(\pi[\pi^{-1}[B]]) = \sup B$$

in $\mathfrak{B}$ (313L(b-iv)), and of course $\pi a_0 \in \pi[\mathfrak{A}]$. By 313E(a-i-$\beta$) again, this is enough to show that $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) Suppose that $A \subseteq \mathfrak{A}$ and $\inf A = 0$ in $\mathfrak{A}$. Then $\pi[A]$ has an infimum b_0 in $\mathfrak{B}$, which belongs to $\pi[\mathfrak{A}]$ because $\pi[\mathfrak{A}]$ is an order-closed subalgebra of $\mathfrak{B}$ (313E(a-i-β')). Now if $a_0 \in \mathfrak{A}$ is such that $\pi a_0 = b_0$, we have

$$\pi(a \cap a_0) = \pi a \cap \pi a_0 = \pi a$$

for every $a \in A$, so (because π is injective) $a \cap a_0 = a_0$ and $a_0 \subseteq a$ for every $a \in A$. But this means that $a_0 = 0$ and $b_0 = \pi 0 = 0$. As A is arbitrary, π is order-continuous (313L(b-ii)).

(b) Use the same arguments, but with sequences in place of the sets B, A above.

314G Corollary (a) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$ (definition: 313N).

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof (a) Apply 314F(a-ii) to the identity map from $\mathfrak{B}$ to $\mathfrak{A}$.

(b) Let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$. By 313Mb, $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$. On the other hand, the identity homomorphism $\iota : \mathfrak{C} \to \mathfrak{A}$ is order-continuous, by (a), so $\pi\iota : \mathfrak{C} \to \mathfrak{B}$ is order-continuous, and $\pi[\mathfrak{C}] = \pi\iota[\mathfrak{C}]$ is order-closed in $\mathfrak{B}$, by 314F(a-i). But since $\pi[C]$ is surely included in $\pi[\mathfrak{C}]$, $\mathfrak{D}$ is also included in $\pi[\mathfrak{C}]$. Accordingly $\pi[\mathfrak{C}] = \mathfrak{D}$, as claimed.

314H Corollary Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$.

(a) If $\mathfrak{A}$ is Dedekind complete, then $\mathfrak{B}$ is order-closed iff it is Dedekind complete in itself and is regularly embedded in $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then $\mathfrak{B}$ is a σ-subalgebra iff it is Dedekind σ-complete in itself and the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous.

proof Put 314E and 314F together.

314I Corollary (a) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ which is Dedekind complete in itself, then $\mathfrak{B} = \mathfrak{A}$.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{B}$ is an injective Boolean homomorphism and $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$, then π is an isomorphism.

proof (a) Being order-dense, $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$ (313O), so this is a special case of 314Ha.

(b) Because $\pi[\mathfrak{A}]$ is order-dense, it is regularly embedded in $\mathfrak{B}$; also, the kernel of π is $\{0\}$, which is surely order-closed in $\mathfrak{A}$, so 313P(a-ii) tells us that π is order-continuous. By 314F(a-i), $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$; being order-dense, it must be the whole of $\mathfrak{B}$ (313K). Thus π is surjective; being injective, it is an isomorphism.

314J When we come to applications of the extension procedure in 312N, the following will sometimes be needed.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$. Take any $c \in \mathfrak{A}$, and set

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\},$$

the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ (312M).

(a) Suppose that $\mathfrak{A}$ is Dedekind complete. If $\mathfrak{A}_0$ is order-closed in $\mathfrak{A}$, so is $\mathfrak{A}_1$.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. If $\mathfrak{A}_0$ is a σ-subalgebra of $\mathfrak{A}$, so is $\mathfrak{A}_1$.

proof (a) Let D be any subset of $\mathfrak{A}_1$. Set

$$E = \{e : e \in \mathfrak{A}, \text{ there is some } d \in D \text{ such that } e \subseteq d\},$$

$$A = \{a : a \in \mathfrak{A}_0,\, a \cap c \in E\}, \quad B = \{b : b \in \mathfrak{A}_0,\, b \setminus c \in E\}.$$

Because $\mathfrak{A}$ is Dedekind complete, $a^* = \sup A$ and $b^* = \sup B$ are defined in $\mathfrak{A}$; because $\mathfrak{A}_0$ is order-closed, both belong to $\mathfrak{A}_0$, so $d^* = (a^* \cap c) \cup (b^* \setminus c)$ belongs to $\mathfrak{A}_1$.

Now if $d \in D$, it is expressible as $(a \cap c) \cup (b \setminus c)$ for some a, $b \in \mathfrak{A}_0$; since $a \in A$ and $b \in B$, we have $a \subseteq a^*$ and $b \subseteq b^*$, so $d \subseteq d^*$. Thus d^* is an upper bound for D. On the other hand, if d' is any other upper bound for D in $\mathfrak{A}$, it is also an upper bound for E, so we must have

$$a^* \cap c = \sup_{a \in A} a \cap c \subseteq d', \quad b^* \setminus c = \sup_{b \in B} b \setminus c \subseteq d',$$

and $d^* \subseteq d'$. Thus $d^* = \sup D$. This shows that the supremum of any subset of $\mathfrak{A}_1$ belongs to $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is order-closed.

(b) The argument is the same, except that we replace D by a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$, and A, B by sequences $\langle a_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_0$ such that $d_n = (a_n \cap c) \cup (b_n \setminus c)$ for every n.

314K Extension of homomorphisms The following is one of the most striking properties of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra. Let $\mathfrak{A}_0$ be a Boolean subalgebra of $\mathfrak{A}$ and $\pi_0 : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism. Then there is a Boolean homomorphism $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ extending π_0.

proof (a) Let P be the set of all Boolean homomorphisms π such that $\operatorname{dom}\pi$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0$ and π extends π_0. Identify each member of P with its graph, which is a subset of $\mathfrak{A} \times \mathfrak{B}$, and order P by inclusion, so that $\pi \subseteq \theta$ means just that θ extends π. Then any non-empty totally ordered subset Q of P has an upper bound in P. **P** Let π^* be the simple union of these graphs. (i) If (a, b) and (a, b') both belong to π^*, then there are π, $\pi' \in Q$ such that $\pi a = b$, $\pi' a = b'$; now either $\pi \subseteq \pi'$ or $\pi' \subseteq \pi$; in either case, $\theta = \pi \cup \pi' \in Q$, so that

$$b = \pi a = \theta a = \pi' a = b'.$$

This shows that π^* is a function. (ii) Because $Q \neq \emptyset$,

$$\operatorname{dom}\pi_0 \subseteq \operatorname{dom}\pi \subseteq \operatorname{dom}\pi^*$$

for some $\pi \in Q$; thus π^* extends π_0 (and, in particular, $0 \in \operatorname{dom}\pi^*$). (iii) Now suppose that a, $a' \in \operatorname{dom}(\pi^*)$. Then there are π, $\pi' \in Q$ such that $a \in \operatorname{dom}\pi$, $a' \in \operatorname{dom}\pi'$; once again, $\theta = \pi \cup \pi' \in Q$, so that a, $a' \in \operatorname{dom}\theta$, and

$$a \cap a' \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*, \quad 1 \setminus a \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*,$$

$$\pi^*(a \cap a') = \theta(a \cap a') = \theta a \cap \theta a' = \pi^* a \cap \pi^* a',$$

$$\pi^*(1 \setminus a) = \theta(1 \setminus a) = 1 \setminus \theta a = 1 \setminus \pi^* a.$$

(iv) This shows that $\operatorname{dom}\pi^*$ is a subalgebra of $\mathfrak{A}$ and that π^* is a Boolean homomorphism, that is, that $\pi^* \in P$; and of course π^* is an upper bound for Q in P. **Q**

(b) By Zorn's Lemma, P has a maximal element π_1 say.

? Suppose, if possible, that $\mathfrak{A}_1 = \operatorname{dom}\pi_1$ is not the whole of $\mathfrak{A}$; take $c \in \mathfrak{A} \setminus \mathfrak{A}_1$. Set $A = \{a : a \in \mathfrak{A}_1, a \subseteq c\}$. Because $\mathfrak{B}$ is Dedekind complete, $d = \sup \pi_1[A]$ is defined in $\mathfrak{B}$. If $a' \in \mathfrak{A}$ and $c \subseteq a'$, then of course $a \subseteq a'$ and $\pi_1 a \subseteq \pi_1 a'$ whenever $a \in A$, so that $\pi_1 a'$ is an upper bound for $\pi_1[A]$, and $d \subseteq \pi_1 a'$.

But this means that there is an extension of π_1 to a Boolean homomorphism π on the Boolean subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1 \cup \{c\}$ (312N). And this π must be a member of P properly extending π_1, which is supposed to be maximal. **X**

Thus $\operatorname{dom}\pi_1 = \mathfrak{A}$ and π_1 is an extension of π_0 to $\mathfrak{A}$, as required.

314L The Loomis-Sikorski representation of a Dedekind σ-complete Boolean algebra The construction in 314D is not only the commonest way in which new Dedekind σ-complete Boolean algebras appear, but is adequate to describe them all. I start with an elementary general fact.

Lemma Let X be any topological space, and write $\mathcal{M}$ for the family of meager subsets of X. Then $\mathcal{M}$ is a σ-ideal of subsets of X.

proof The point is that if $A \subseteq X$ is nowhere dense, so is every subset of A; this is obvious, since if $B \subseteq A$ then $\overline{B} \subseteq \overline{A}$ so $\operatorname{int}\overline{B} \subseteq \operatorname{int}\overline{A} = \emptyset$. So if $B \subseteq A \in \mathcal{M}$, let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets with union A; then $\langle B \cap A_n\rangle_{n\in\mathbb{N}}$ is a sequence of nowhere dense sets with union B, so $B \in \mathcal{M}$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with union A, then for each n we may choose a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of nowhere dense sets with union A_n; then the countable family $\langle A_{nm}\rangle_{n,m\in\mathbb{N}}$ may be re-indexed as a sequence of nowhere dense sets with union A, so $A \in \mathcal{M}$. Finally, $\emptyset$ is nowhere dense, so belongs to $\mathcal{M}$.

314M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the σ-ideal of meager subsets of Z. Then $\Sigma = \{E \triangle A : E \in \mathcal{E}, A \in \mathcal{M}\}$ is a σ-algebra of subsets of Z, $\mathcal{M}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic, as Boolean algebra, to $\Sigma/\mathcal{M}$.

proof (a) I start by showing that Σ is a σ-algebra. **P** Of course $\emptyset = \emptyset\triangle\emptyset \in \Sigma$. If $F \in \Sigma$, express it as $E\triangle A$ where $E \in \mathcal{E}$, $A \in \mathcal{M}$; then $Z \setminus F = (Z \setminus E)\triangle A \in \Sigma$.

If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, express each F_n as $E_n\triangle A_n$, where $E_n \in \mathcal{E}$ and $A_n \in \mathcal{M}$. Now each E_n is expressible as $\widehat{a}_n$, where $a_n \in \mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind σ-complete, $a = \sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$. Set $E = \widehat{a} \in \mathcal{E}$. By 313Ca, $E = \overline{\bigcup_{n\in\mathbb{N}} E_n}$, so the closed set $E \setminus \bigcup_{n\in\mathbb{N}} E_n$ has empty interior and is nowhere dense. Accordingly, setting $A = E\triangle\bigcup_{n\in\mathbb{N}} F_n$, we have

$$A \subseteq (E \setminus \textstyle\bigcup_{n\in\mathbb{N}} E_n) \cup \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M},$$

so that $\bigcup_{n\in\mathbb{N}} F_n = E\triangle A \in \Sigma$. Thus Σ is closed under countable unions and is a σ-algebra. **Q**

Evidently $\mathcal{M} \subseteq \Sigma$, because $\emptyset \in \mathcal{E}$.

(b) For each $F \in \Sigma$, there is exactly one $E \in \mathcal{E}$ such that $F\triangle E \in \mathcal{M}$. **P** There is surely some $E \in \mathcal{E}$ such that F is expressible as $E\triangle A$ where $A \in \mathcal{M}$, so that $F\triangle E = A \in \mathcal{M}$. If E' is any other member of $\mathcal{E}$, then $E'\triangle E$ is a non-empty open set in X, while $E'\triangle E \subseteq A \cup (F\triangle E')$; by Baire's theorem for compact Hausdorff spaces (3A3G), $A \cup (F\triangle E') \notin \mathcal{M}$ and $F\triangle E' \notin \mathcal{M}$. Thus E is unique. **Q**

(c) Consequently the maps $E \mapsto E^\bullet : \mathcal{E} \to \Sigma/\mathcal{M}$ and $a \mapsto \widehat{a}^\bullet : \mathfrak{A} \to \Sigma/\mathcal{M}$ are bijections. But since they are also Boolean homomorphisms, they are isomorphisms, and $\mathfrak{A} \cong \Sigma/\mathcal{M}$, as claimed.

314N Corollary A Boolean algebra $\mathfrak{A}$ is Dedekind σ-complete iff it is isomorphic to a quotient $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of sets and $\mathcal{I}$ is a σ-ideal of Σ.

proof Put 314D and 314M together.

314O Regular open algebras For Boolean algebras which are Dedekind complete in the full sense, there is another general method of representing them, which leads to further very interesting ideas.

Definition Let X be a topological space. A **regular open set** in X is an open set $G \subseteq X$ such that $G = \operatorname{int}\overline{G}$.

Note that if $F \subseteq X$ is any closed set, then $G = \operatorname{int} F$ is a regular open set, because $G \subseteq \overline{G} \subseteq F$ so

$$G \subseteq \operatorname{int}\overline{G} \subseteq \operatorname{int} F = G$$

and $G = \operatorname{int}\overline{G}$.

314P Theorem Let X be any topological space, and write $\mathfrak{G}$ for the set of regular open sets in X. Then $\mathfrak{G}$ is a Dedekind complete Boolean algebra, with $1_{\mathfrak{G}} = X$ and $0_{\mathfrak{G}} = \emptyset$, and with Boolean operations given by

$$G \cap_{\mathfrak{G}} H = G \cap H, \quad G \triangle_{\mathfrak{G}} H = \operatorname{int}\overline{G\triangle H}, \quad G \cup_{\mathfrak{G}} H = \operatorname{int}\overline{G \cup H}, \quad G \setminus_{\mathfrak{G}} H = G \setminus \overline{H},$$

with Boolean ordering given by

$$G \subseteq_{\mathfrak{G}} H \iff G \subseteq H,$$

and with suprema and infima given by

$$\sup \mathcal{H} = \operatorname{int}\overline{\textstyle\bigcup \mathcal{H}}, \quad \inf \mathcal{H} = \operatorname{int} \textstyle\bigcap \mathcal{H} = \operatorname{int}\overline{\bigcap \mathcal{H}}$$

for all non-empty $\mathcal{H} \subseteq \mathfrak{G}$.

Remark I use the expressions

$$\cap_{\mathfrak{G}} \quad \cup_{\mathfrak{G}} \quad \triangle_{\mathfrak{G}} \quad \setminus_{\mathfrak{G}} \quad \subseteq_{\mathfrak{G}}$$

in case the distinction between

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

and

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

is insufficiently marked.

proof I base the proof on the study of an auxiliary algebra of sets which involves some of the ideas already used in 314M.

(a) Let $\mathcal{I}$ be the family of nowhere dense subsets of X. Then $\mathcal{I}$ is an ideal of subsets of X. **P** Of course $\emptyset \in \mathcal{I}$. If $A \subseteq B \in \mathcal{I}$ then $\operatorname{int}\overline{A} \subseteq \operatorname{int}\overline{B} = \emptyset$. If A, $B \in \mathcal{I}$ and G is a non-empty open set, then $G \setminus \overline{A}$ is a non-empty open set and $(G \setminus \overline{A}) \setminus \overline{B}$ is non-empty; accordingly G cannot be a subset of $\overline{A} \cup \overline{B} = \overline{A \cup B}$. This shows that $\operatorname{int}\overline{A \cup B} = \emptyset$, so that $A \cup B \in \mathcal{I}$. **Q**

(b) For any set $A \subseteq X$, write ∂A for the boundary of A, that is, $\overline{A} \setminus \operatorname{int} A$. Set

$$\Sigma = \{E : E \subseteq X,\ \partial E \in \mathcal{I}\}.$$

The Σ is an algebra of subsets of X. **P** (i) $\partial\emptyset = \emptyset \in \mathcal{I}$ so $\emptyset \in \Sigma$. (ii) If A, $B \subseteq X$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$, while $\operatorname{int}(A \cup B) \supseteq \operatorname{int} A \cup \operatorname{int} B$; so $\partial(A \cup B) \subseteq \partial A \cup \partial B$. So if E, $F \in \Sigma$, $\partial(E \cup F) \subseteq \partial E \cup \partial F \in \mathcal{I}$ and $E \cup F \in \Sigma$. (iii) If $A \subseteq X$, then

$$\partial(X \setminus A) = \overline{X \setminus A} \setminus \operatorname{int}(X \setminus A) = (X \setminus \operatorname{int} A) \setminus (X \setminus \overline{A}) = \overline{A} \setminus \operatorname{int} A = \partial A.$$

So if $E \in \Sigma$, $\partial(X \setminus E) = \partial E \in \mathcal{I}$ and $X \setminus E \in \Sigma$. **Q**

If $A \in \mathcal{I}$, then of course $\partial A = \overline{A} \in \mathcal{I}$, so $A \in \Sigma$; accordingly $\mathcal{I}$ is an ideal in the Boolean algebra Σ, and we can form the quotient $\Sigma/\mathcal{I}$.

It will be helpful to note that every open set belongs to Σ, since if G is open then $\partial G = \overline{G} \setminus G$ cannot include any non-empty open set (since any open set meeting $\overline{G}$ must meet G).

(c) For each $E \in \Sigma$, set $V_E = \operatorname{int}\overline{E}$; then V_E is the unique member of $\mathfrak{G}$ such that $E \triangle V_E \in \mathcal{I}$. **P** (i) Being the interior of a closed set, $V_E \in \mathfrak{G}$. Since $\operatorname{int} E \subseteq V_E \subseteq \overline{E}$, $E \triangle V_E \subseteq \partial E \in \mathcal{I}$. (ii) If $G \in \mathfrak{G}$ is such that $E \triangle G \in \mathcal{I}$, then

$$G \setminus \overline{V_E} \subseteq G \setminus V_E \subseteq (G \triangle E) \cup (V_E \triangle E) \in \mathcal{I},$$

so $G \setminus \overline{V_E}$, being open, must be actually empty, and $G \subseteq \overline{V_E}$; but this means that $G \subseteq \operatorname{int}\overline{V_E} = V_E$. Similarly, $V_E \subseteq G$ and $V_E = G$. This shows that V_E is unique. **Q**

(d) It follows that the map $G \mapsto G^\bullet : \mathfrak{G} \to \Sigma/\mathcal{I}$ is a bijection, and we have a Boolean algebra structure on $\mathfrak{G}$ defined by the Boolean algebra structure of $\Sigma/\mathcal{I}$. What this means is that for each of the binary Boolean operations $\cap_{\mathfrak{G}}$, $\triangle_{\mathfrak{G}}$, $\cup_{\mathfrak{G}}$, $\setminus_{\mathfrak{G}}$ and for G, $H \in \mathfrak{G}$ we must have $G *_{\mathfrak{G}} H = \operatorname{int}\overline{G * H}$, writing $*_{\mathfrak{G}}$ for the operation on the algebra $\mathfrak{G}$ and $*$ for the corresponding operation on Σ or $\mathcal{P}X$.

(e) Before working through the identifications, it will be helpful to observe that if $\mathcal{H}$ is any non-empty subset of $\mathfrak{G}$, then $\operatorname{int}\bigcap\mathcal{H} = \operatorname{int}\overline{\bigcap\mathcal{H}}$. **P** Set $G = \operatorname{int}\overline{\bigcap\mathcal{H}}$. For every $H \in \mathcal{H}$, $G \subseteq \overline{H}$ so $G \subseteq \operatorname{int}\overline{H} = H$; thus

$$G \subseteq \operatorname{int}\bigcap\mathcal{H} \subseteq \operatorname{int}\overline{\bigcap\mathcal{H}} = G,$$

so $G = \operatorname{int}\overline{\bigcap\mathcal{H}}$. **Q** Consequently $\operatorname{int}\bigcap\mathcal{H}$, being the interior of a closed set, belongs to $\mathfrak{G}$.

(f)(i) If G, $H \in \mathfrak{G}$ then their intersection in the algebra $\mathfrak{G}$ is

$$G \cap_{\mathfrak{G}} H = \operatorname{int}\overline{G \cap H} = \operatorname{int}(G \cap H) = G \cap H,$$

using (d) for the first equality and (e) for the second.

(ii) Of course $X \in \mathfrak{G}$ and $X^\bullet = 1_{\Sigma/\mathcal{I}}$, so $X = 1_{\mathfrak{G}}$.

(iii) If $G \in \mathfrak{G}$ then its complement $1_{\mathfrak{G}} \setminus_{\mathfrak{G}} G$ in $\mathfrak{G}$ is

$$\operatorname{int}\overline{X \setminus G} = \operatorname{int}(X \setminus G) = X \setminus \overline{G}.$$

(iv) If G, $H \in \mathfrak{G}$, then the relative complement in $\mathfrak{G}$ is

$$G \setminus_{\mathfrak{G}} H = G \cap_{\mathfrak{G}} (1_{\mathfrak{G}} \setminus_{\mathfrak{G}} H) = G \cap (X \setminus \overline{H}) = G \setminus \overline{H} = \operatorname{int}(G \setminus H).$$

(v) If G, $H \in \mathfrak{G}$, then $G \cup_{\mathfrak{G}} H = \operatorname{int}\overline{G \cup H}$ and $G \triangle_{\mathfrak{G}} H = \operatorname{int}\overline{G \triangle H}$, by the remarks in (d).

(g) We must note that for G, $H \in \mathfrak{G}$,

$$G \subseteq_{\mathfrak{G}} H \iff G \cap_{\mathfrak{G}} H = G \iff G \cap H = G \iff G \subseteq H;$$

that is, the ordering of the Boolean algebra $\mathfrak{G}$ is just the partial ordering induced on $\mathfrak{G}$ by the Boolean ordering $\subseteq$ of $\mathcal{P}X$ or Σ.

(h) If $\mathcal{H}$ is any non-empty subset of $\mathfrak{G}$, consider $G_0 = \operatorname{int}\bigcap\mathcal{H}$ and $G_1 = \operatorname{int}\overline{\bigcup\mathcal{H}}$.

$G_0 = \inf\mathcal{H}$ in $\mathfrak{G}$. **P** By (e), $G_0 \in \mathfrak{G}$. Of course $G_0 \subseteq H$ for every $H \in \mathcal{H}$, so G_0 is a lower bound for $\mathcal{H}$. If G is any lower bound for $\mathcal{H}$ in $\mathfrak{G}$, then $G \subseteq H$ for every $H \in \mathcal{H}$, so $G \subseteq \bigcap\mathcal{H}$; but also G is open, so $G \subseteq \operatorname{int}\bigcap\mathcal{H} = G_0$. Thus G_0 is the greatest lower bound for $\mathcal{H}$. **Q**

$G_1 = \sup\mathcal{H}$ in $\mathfrak{G}$. **P** Being the interior of a closed set, $G_1 \in \mathfrak{G}$, and of course

$$H = \operatorname{int}\overline{H} \subseteq \operatorname{int}\overline{\bigcup\mathcal{H}} = G_1$$

for every $H \in \mathcal{H}$, so G_1 is an upper bound for $\mathcal{H}$ in $\mathfrak{G}$. If G is any upper bound for $\mathcal{H}$ in $\mathfrak{G}$, then

$$G = \operatorname{int}\overline{G} \supseteq \operatorname{int}\overline{\bigcup\mathcal{H}} = G_1;$$

thus G_1 is the least upper bound for $\mathcal{H}$ in $\mathfrak{G}$. **Q**

This shows that every non-empty $\mathcal{H} \subseteq \mathfrak{G}$ has a supremum and an infimum in $\mathfrak{G}$; consequently $\mathfrak{G}$ is Dedekind complete, and the proof is finished.

314Q Remarks (a) $\mathfrak{G}$ is called the **regular open algebra** of the topological space X.

(b) Note that the map $E \mapsto V_E : \Sigma \to \mathfrak{G}$ of part (c) of the proof above is a Boolean homomorphism, if $\mathfrak{G}$ is given its Boolean algebra structure. Its kernel is of course $\mathcal{I}$; the induced map $E^\bullet \mapsto V_E : \Sigma/\mathcal{I} \to \mathfrak{G}$ is just the inverse of the isomorphism $G \mapsto G^\bullet : \mathfrak{G} \to \Sigma/\mathcal{I}$.

***314R** I interpolate a lemma corresponding to 313R.

Lemma Let X and Y be topological spaces, and $f : X \to Y$ a continuous function such that $f^{-1}[M]$ is nowhere dense in X for every nowhere dense $M \subseteq Y$. Then we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathrm{RO}(Y)$ of Y to the regular open algebra $\mathrm{RO}(X)$ of X defined by setting $\pi H = \operatorname{int}\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}(Y)$.

proof (a) By the remark in 314O, the formula for πH always defines a member of $\mathrm{RO}(X)$; and of course π is order-preserving.

Observe that if $H \in \mathrm{RO}(Y)$, then $f^{-1}[H]$ is open, so $f^{-1}[H] \subseteq \pi H$. It will be convenient to note straight away that if $V \subseteq Y$ is a dense open set then $f^{-1}[V]$ is dense in X. **P** $M = Y \setminus V$ is nowhere dense, so $f^{-1}[M]$ is nowhere dense and its complement $f^{-1}[V]$ is dense. **Q**

(b) If H_1, $H_2 \in \mathrm{RO}(Y)$ then $\pi(H_1 \cap H_2) = \pi H_1 \cap \pi H_2$. **P** Because π is order-preserving, $\pi(H_1 \cap H_2) \subseteq \pi H_1 \cap \pi H_2$. **?** Suppose, if possible, that they are not equal. Then (because $\pi(H_1 \cap H_2)$ is a regular open set) $G = \pi H_1 \cap \pi H_2 \setminus \overline{\pi(H_1 \cap H_2)}$ is non-empty. Set $M = \overline{f[G]}$. Then $f^{-1}[M] \supseteq G$ is not nowhere dense, so $H = \operatorname{int} M$ must be non-empty. Now $G \subseteq \pi H_1 \subseteq \overline{f^{-1}[H_1]}$, so

$$f[G] \subseteq f[\overline{f^{-1}[H_1]}] \subseteq \overline{f[f^{-1}[H_1]]} \subseteq \overline{H}_1,$$

so $M \subseteq \overline{H}_1$ and $H \subseteq \operatorname{int}\overline{H}_1 = H_1$. Similarly, $H \subseteq H_2$, and $f^{-1}[H] \subseteq f^{-1}[H_1 \cap H_2] \subseteq \pi(H_1 \cap H_2)$. But also $H \cap f[G]$ is not empty, so

$$\emptyset \neq G \cap f^{-1}[H] \subseteq G \cap \pi(H_1 \cap H_2),$$

which is impossible. **XQ**

(c) If $H \in \mathrm{RO}(Y)$ and $H' = Y \setminus \overline{H}$ is its complement in $\mathrm{RO}(Y)$ then $\pi H' = X \setminus \overline{\pi H}$ is the complement of πH in $\mathrm{RO}(X)$. **P** By (b), πH and $\pi H'$ are disjoint. Now $H \cup H'$ is a dense open subset of Y, so

$$\pi H \cup \pi H' \supseteq f^{-1}[H] \cup f^{-1}[H'] = f^{-1}[H \cup H']$$

is dense in X, and the regular open set $\pi H'$ must include the complement of πH in $\mathrm{RO}(X)$. **Q**

Putting (b) and (c) together, we see that the conditions of 312H(ii) are satisfied, so that π is a Boolean homomorphism.

(d) To see that it is order-continuous, let $\mathcal{H} \subseteq \mathrm{RO}(Y)$ be a non-empty set with supremum Y. Then $H_0 = \bigcup\mathcal{H}$ is a dense open subset of Y (see the formula in 314P). So

$$\bigcup_{H\in\mathcal{H}} \pi H \supseteq \bigcup_{H\in\mathcal{H}} f^{-1}[H] = f^{-1}[H_0]$$

is dense in X, and $\sup_{H\in\mathcal{H}} \pi H = X$ in $\mathrm{RO}(X)$. By 313L(b-iii), π is order-continuous.

314S It is now easy to characterize the Stone spaces of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathfrak{G}$ for the regular open algebra of Z. Then the following are equiveridical:

(i) $\mathfrak{A}$ is Dedekind complete;

(ii) Z is extremally disconnected (definition: 3A3Ae);

(iii) $\mathcal{E} = \mathfrak{G}$.

proof (i)$\Rightarrow$(ii) If $\mathfrak{A}$ is Dedekind complete, let G be any open set in Z. Set $A = \{a : a \in \mathfrak{A}, \widehat{a} \subseteq G\}$, $a_0 = \sup A$. Then $G = \bigcup\{\widehat{a} : a \in A\}$, because $\mathcal{E}$ is a base for the topology of Z, so $\widehat{a}_0 = \overline{G}$, by 313Ca. Consequently $\overline{G}$ is open. As G is arbitrary, Z is extremally disconnected.

(ii)$\Rightarrow$(iii) If $E \in \mathcal{E}$, then of course $E = \overline{E} = \operatorname{int}\overline{E}$, so E is a regular open set. Thus $\mathcal{E} \subseteq \mathfrak{G}$. On the other hand, suppose that $G \subseteq Z$ is a regular open set. Because Z is extremally disconnected, $\overline{G}$ is open; so $G = \operatorname{int}\overline{G} = \overline{G}$ is open-and-closed, and belongs to $\mathcal{E}$. Thus $\mathcal{E} = \mathfrak{G}$.

(iii)$\Rightarrow$(i) Since $\mathfrak{G}$ is Dedekind complete (314P), $\mathcal{E}$ and $\mathfrak{A}$ are also Dedekind complete Boolean algebras.

Remark Note that if the conditions above are satisfied, either 312L or the formulae in 314P show that the Boolean structures of $\mathcal{E}$ and $\mathfrak{G}$ are identical.

314T I come now to a construction of great importance, both as a foundation for further constructions and as a source of insight into the nature of Dedekind completeness.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z; for $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of $\mathfrak{A}$. Let $\widehat{\mathfrak{A}}$ be the regular open algebra of Z (314P).

(a) The map $a \mapsto \widehat{a}$ is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) If $\mathfrak{B}$ is any Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a unique order-continuous Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$ such that $\pi_1\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$.

proof (a)(i) Setting $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$, every member of $\mathcal{E}$ is open-and-closed, so is surely equal to the interior of its closure, and is a regular open set; thus $\widehat{a} \in \widehat{\mathfrak{A}}$ for every $a \in \mathfrak{A}$. The formulae in 314P tell us that if a, $b \in \mathfrak{A}$, then $\widehat{a} \cap \widehat{b}$, taken in $\widehat{\mathfrak{A}}$, is just the set-theoretic intersection $\widehat{a} \cap \widehat{b} = (a \cap b)\widehat{\ }$; while $1 \setminus \widehat{a}$, taken in $\widehat{\mathfrak{A}}$, is

$$Z \setminus \overline{\widehat{a}} = Z \setminus \widehat{a} = (1 \setminus a)\widehat{\ }.$$

And of course $\widehat{0} = \emptyset$ is the zero of $\widehat{\mathfrak{A}}$. Thus the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ preserves $\cap$ and complementation, so is a Boolean homomorphism (312H). Of course it is injective.

(ii) If $A \subseteq \mathfrak{A}$ is non-empty and $\inf A = 0$, then $\bigcap_{a\in A} \widehat{a}$ is nowhere dense in Z (313Cc), so

$$\inf\{\widehat{a} : a \in A\} = \operatorname{int}(\bigcap_{a\in A} \widehat{a}) = \emptyset$$

(314P again). As A is arbitrary, the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is order-continuous.

(iii) If $G \in \widehat{\mathfrak{A}}$ is not empty, then there is a non-empty member of $\mathcal{E}$ included in it, by the definition of the topology of Z (311I). So $\mathcal{E}$ is an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) Now suppose that $\mathfrak{B}$ is a Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism. Write $\iota a = \widehat{a}$ for $a \in \mathfrak{A}$, so that $\iota : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is an isomorphism between $\mathfrak{A}$ and the order-dense subalgebra $\mathcal{E}$ of $\widehat{\mathfrak{A}}$. Accordingly $\pi\iota^{-1} : \mathcal{E} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, being the composition of the order-continuous Boolean homomorphisms π and ι^{-1}. By 314K, it has an extension to a Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$, and $\pi_1\iota = \pi$, that is, $\pi_1\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$. Now π_1 is order-continuous. **P** Suppose that $\mathcal{H} \subseteq \widehat{\mathfrak{A}}$ has supremum 1 in $\widehat{\mathfrak{A}}$. Set

$$\mathcal{H}' = \{E : E \in \mathcal{E},\ E \subseteq H \text{ for some } H \in \mathcal{H}\}.$$

Because $\mathcal{E}$ is order-dense in $\widehat{\mathfrak{A}}$,

$$H = \sup_{E\in\mathcal{E},E\subseteq H} E = \sup_{E\in\mathcal{H}',E\subseteq H} E$$

for every $H \in \mathcal{H}$ (313K), and $\sup \mathcal{H}' = 1$ in $\widehat{\mathfrak{A}}$. It follows at once that $\sup \mathcal{H}' = 1$ in $\mathcal{E}$, so $\sup \pi_1[\mathcal{H}'] = \sup(\pi\iota^{-1})[\mathcal{H}'] = 1$. Since any upper bound for $\pi_1[\mathcal{H}]$ must also be an upper bound for $\pi_1[\mathcal{H}']$, $\sup \pi_1[\mathcal{H}] = 1$ in $\mathfrak{B}$. As $\mathcal{H}$ is arbitrary, π_1 is order-continuous (313L(b-iii)). **Q**

If $\pi_1' : \widehat{\mathfrak{A}} \to \mathfrak{B}$ is any other Boolean homomorphism such that $\pi_1'\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$, then π_1 and π_1' agree on $\mathcal{E}$, and the argument just above shows that π_1' is also order-continuous. But if $G \in \widehat{\mathfrak{A}}$, G is the supremum (in $\widehat{\mathfrak{A}}$) of $\mathcal{F} = \{E : E \in \mathcal{E},\ E \subseteq G\}$, so

$$\pi_1'G = \sup_{E\in\mathcal{F}} \pi_1'E = \sup_{E\in\mathcal{F}} \pi_1 E = \pi_1 G.$$

As G is arbitrary, $\pi_1' = \pi_1$. Thus π_1 is unique.

314U The Dedekind completion of a Boolean algebra For any Boolean algebra $\mathfrak{A}$, I will say that the Boolean algebra $\widehat{\mathfrak{A}}$ constructed in 314T is the **Dedekind completion** of $\mathfrak{A}$.

When using this concept I shall frequently suppress the distinction between $a \in \mathfrak{A}$ and $\widehat{a} \in \widehat{\mathfrak{A}}$, and treat $\mathfrak{A}$ as itself an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

314V Upper envelopes (a) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. For $a \in \mathfrak{A}$, write

$$\mathrm{upr}(a, \mathfrak{C}) = \inf\{c : c \in \mathfrak{C},\ a \subseteq c\}$$

if the infimum is defined in $\mathfrak{C}$. (The most important cases are when $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}$ is order-closed in $\mathfrak{A}$, so that $\mathfrak{C}$ is Dedekind complete (314E) and $\mathrm{upr}(a, \mathfrak{C})$ is defined for every $a \in \mathfrak{A}$; but others also arise.)

(b) If $A \subseteq \mathfrak{A}$ is such that $\mathrm{upr}(a, \mathfrak{C})$ is defined for every $a \in A$, $a_0 = \sup A$ is defined in $\mathfrak{A}$ and $c_0 = \sup_{a\in A} \mathrm{upr}(a, \mathfrak{C})$ is defined in $\mathfrak{C}$, then $c_0 = \mathrm{upr}(a_0, \mathfrak{C})$. **P** If $c \in \mathfrak{C}$ then

$$\begin{aligned} c_0 \subseteq c &\iff \mathrm{upr}(a, \mathfrak{C}) \subseteq c \text{ for every } a \in A \\ &\iff a \subseteq c \text{ for every } a \in A \iff a_0 \subseteq c. \text{ } \mathbf{Q} \end{aligned}$$

In particular, $\mathrm{upr}(a \cup a', \mathfrak{C}) = \mathrm{upr}(a, \mathfrak{C}) \cup \mathrm{upr}(a', \mathfrak{C})$ whenever the right-hand side is defined.

(c) If $a \in \mathfrak{A}$ is such that $\mathrm{upr}(a, \mathfrak{C})$ is defined, then $\mathrm{upr}(a \cap c, \mathfrak{C}) = c \cap \mathrm{upr}(a, \mathfrak{C})$ for every $c \in \mathfrak{C}$. **P** For $c' \in \mathfrak{C}$,

$$\begin{aligned} a \cap c \subseteq c' &\iff a \subseteq c' \cup (1 \setminus c) \\ &\iff \mathrm{upr}(a, \mathfrak{C}) \subseteq c' \cup (1 \setminus c) \iff c \cap \mathrm{upr}(a, \mathfrak{C}) \subseteq c'. \text{ } \mathbf{Q} \end{aligned}$$

314X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind complete (β) every non-empty upwards-directed subset of $\mathfrak{A}$ with an upper bound has a least upper bound (γ) every non-empty downwards-directed subset of $\mathfrak{A}$ with a lower bound has a greatest lower bound. (ii) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind σ-complete (β) every non-decreasing sequence in $\mathfrak{A}$ with an upper bound has a least upper bound (γ) every non-increasing sequence in $\mathfrak{A}$ with a lower bound has a greatest lower bound.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that any principal ideal of $\mathfrak{A}$ is order-closed. Show that $\mathfrak{A}$ is Dedekind complete iff every order-closed ideal is principal.

>(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the σ-subalgebra of $\mathfrak{A}$ generated by C, show that $\pi[\mathfrak{C}]$ is the σ-subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $\{a \cap b : b \in \mathfrak{B}\}$ is an order-closed subalgebra of $\mathfrak{A}_a$.

(e) Find a proof of 314Tb which does not appeal to 314K.

>(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Show that $\mathfrak{A}$ is isomorphic to the Dedekind completion of $\mathfrak{B}$.

(g) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$. Show that an element a of $\mathfrak{A}$ belongs to $\mathfrak{C}$ iff $\operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1 \setminus \operatorname{upr}(a, \mathfrak{C})$ iff $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \cap \operatorname{upr}(a, \mathfrak{C}) = 0$.

>(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$, and $a_0 \in \mathfrak{A}$, $c_0 \in \mathfrak{C}$. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and $\pi a_0 = c_0$ (ii) $1 \setminus \operatorname{upr}(1 \setminus a_0, \mathfrak{C}) \subseteq c_0 \subseteq \operatorname{upr}(a_0, \mathfrak{C})$.

314Y Further exercises (a) Let P be a Dedekind complete partially ordered set. Show that a set $Q \subseteq P$ is order-closed iff $\sup R$, $\inf R$ belong to Q whenever $R \subseteq Q$ is a totally ordered subset of Q with upper and lower bounds in P. (*Hint*: show by induction on κ that if $A \subseteq Q$ is upwards-directed and bounded above and $\#(A) \le \kappa$ then $\sup A \in Q$.)

(b) Let P be a lattice. Show that P is Dedekind complete iff every non-empty totally ordered subset of P with an upper bound in P has a least upper bound in P. (*Hint*: if $A \subseteq P$ is non-empty and bounded below in P, let B be the set of lower bounds of A and use Zorn's Lemma to find a maximal element of B.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with an order-closed subalgebra $\mathfrak{A}_0$ and an element c such that the subalgebra generated by $\mathfrak{A}_0 \cup \{c\}$ is not order-closed.

(d) Let X be any topological space. Let $\mathcal{M}$ be the σ-ideal of meager subsets of X, and set

$$\widehat{\mathcal{B}} = \{G \triangle A : G \subseteq X \text{ is open, } A \in \mathcal{M}\}.$$

(i) Show that $\widehat{\mathcal{B}}$ is a σ-algebra of subsets of X, and that $\widehat{\mathcal{B}}/\mathcal{M}$ is Dedekind complete. (Members of $\widehat{\mathcal{B}}$ are said to be the subsets of X **with the Baire property**; $\widehat{\mathcal{B}}$ is the **Baire property algebra** of X.) (ii) Show that if $A \subseteq X$ and $\bigcup\{G : G \subseteq X \text{ is open, } A \cap G \in \widehat{\mathcal{B}}\}$ is dense, then $A \in \widehat{\mathcal{B}}$. (iii) Show that there is a largest open set $V \in \mathcal{M}$. (iv) Let $\mathfrak{G}$ be the regular open algebra of X. Show that the map $G \mapsto G^\bullet$ is an order-continuous Boolean homomorphism from $\mathfrak{G}$ onto $\widehat{\mathcal{B}}/\mathcal{M}$, so induces a Boolean isomorphism between the principal ideal of $\mathfrak{G}$ generated by $X \setminus \overline{V}$ and $\widehat{\mathcal{B}}/\mathcal{M}$. ($\widehat{\mathcal{B}}/\mathcal{M}$ is the **category algebra** of X; it is a Dedekind complete Boolean algebra. X is called a **Baire space** if $V = \emptyset$; in this case $\mathfrak{G} \cong \widehat{\mathcal{B}}/\mathcal{M}$.)

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n \rangle_{n\in\mathbb{N}}$ any sequence in $\mathfrak{A}$. For $n \in \mathbb{N}$ set $E_n = \{x : x \in \{0,1\}^{\mathbb{N}}, x(n) = 1\}$, and let $\mathcal{B}$ be the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by $\{E_n : n \in \mathbb{N}\}$. ($\mathcal{B}$ is the 'Borel σ-algebra' of $\{0,1\}^{\mathbb{N}}$; see 4A2Wd.) Show that there is a unique sequentially order-continuous Boolean homomorphism $\theta : \mathcal{B} \to \mathfrak{A}$ such that $\theta(E_n) = a_n$ for every $n \in \mathbb{N}$. (*Hint*: define a suitable function ϕ from the Stone space Z of $\mathfrak{A}$ to $\{0,1\}^{\mathbb{N}}$, and consider $\{E : E \subseteq \{0,1\}^{\mathbb{N}}, \phi^{-1}[E]$ has the Baire property in $Z\}$.) Show that $\theta[\mathcal{B}]$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{a_n : n \in \mathbb{N}\}$.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Show that $\mathfrak{A}$ is Dedekind σ-complete iff $\overline{G}$ is open whenever G is a cozero set in Z. (Such spaces are called **basically disconnected** or **quasi-Stonian**.)

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind complete Boolean algebras and $D \subseteq \mathfrak{A} \setminus \{0\}$ an order-dense set. Suppose that $\phi : D \to \mathfrak{B}$ is such that (i) $\phi[D]$ is order-dense in $\mathfrak{B}$ (ii) for all d, $d' \in D$, $d \cap d' = 0$ iff $\phi d \cap \phi d' = 0$. Show that ϕ has a unique extension to a Boolean isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(h) Let $\mathfrak{A}$ be any Boolean algebra. Let $\mathcal{J}$ be the family of order-closed ideals in $\mathfrak{A}$. Show that (i) $\mathcal{J}$ is a Dedekind complete Boolean algebra with operations defined by the formulae $I \cap J = I \cap J$, $1 \setminus J = \{a : a \cap b = 0 \text{ for every } b \in J\}$ (ii) the map $a \mapsto \mathfrak{A}_a$, the principal ideal generated by a, is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\mathcal{J}$ (iii) $\mathcal{J}$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

314 Notes and comments At the risk of being tiresomely long-winded, I have taken the trouble to spell out a large proportion of the results in this section and the last in their 'sequential' as well as their 'unrestricted' forms. The point is that while (in my view) the underlying ideas are most clearly and dramatically expressed in terms of order-closed sets, order-continuous functions and Dedekind complete algebras, a large proportion of the applications in measure theory deal with sequentially order-closed sets, sequentially order-continuous functions and Dedekind σ-complete algebras. As a matter of simple technique, therefore, it is necessary to master both, and for the sake of later reference I generally give the statements of both versions in full. Perhaps the points to look at most keenly are just those where there is a difference in the ideas involved, as in 314Bb, or in which there is only one version given, as in 314M and 314T.

If you have seen the Hahn-Banach theorem (3A5A), it may have been recalled to your mind by Theorem 314K; in both cases we use an order relation and a bit of algebra to make a single step towards an extension of a function, and Zorn's lemma to turn this into the extension we seek. A good part of this section has turned out to be on the borderland between the theory of Boolean algebra and general topology; naturally enough, since (as always with the general theory of Boolean algebra) one of our first concerns is to establish connexions between algebras and their Stone spaces.

I think 314T is the first substantial 'universal mapping theorem' in this volume; it is by no means the last. The idea of the construction $\widehat{\mathfrak{A}}$ is not just that we obtain a Dedekind complete Boolean algebra in which $\mathfrak{A}$ is embedded as an order-dense subalgebra, but that we simultaneously obtain a theorem on the canonical extension to $\widehat{\mathfrak{A}}$ of order-continuous Boolean homomorphisms defined on $\mathfrak{A}$. This characterization is enough to define the pair $(\widehat{\mathfrak{A}}, a \mapsto \widehat{a})$ up to isomorphism, so the exact method of construction of $\widehat{\mathfrak{A}}$ becomes of secondary importance. The one used in 314T is very natural (at least, if we believe in Stone spaces), but there are others (see 314Yh), with different virtues.

314K and 314T both describe circumstances in which we can find extensions of Boolean homomorphisms. Clearly such results are fundamental in the theory of Boolean algebras, but I shall not attempt any systematic presentation here. 314Ye can also be regarded as belonging to this family of ideas.

315 Products and free products

I describe here two algebraic constructions of fundamental importance. They are very different in character, indeed may be regarded as opposites, despite the common use of the word 'product'. The first part of the section (315A-315G) deals with the easier construction, the 'simple product'; the second part (315H-315P) with the 'free product'.

315A Products of Boolean algebras (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras. Set $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$, with the natural ring structure

$$a \triangle b = \langle a(i) \triangle b(i)\rangle_{i\in I},$$

$$a \cap b = \langle a(i) \cap b(i)\rangle_{i\in I}$$

for a, $b \in \mathfrak{A}$. Then $\mathfrak{A}$ is a ring (3A2H); it is a Boolean ring because

$$a \cap a = \langle a(i) \cap a(i)\rangle_{i\in I} = a$$

for every $a \in \mathfrak{A}$; and it is a Boolean algebra because if we set $1_{\mathfrak{A}} = \langle 1_{\mathfrak{A}_i}\rangle_{i\in I}$, then $1_{\mathfrak{A}} \cap a = a$ for every $a \in \mathfrak{A}$. I will call $\mathfrak{A}$ the **simple product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$.

I should perhaps remark that when $I = \emptyset$ then $\mathfrak{A}$ becomes $\{\emptyset\}$, to be interpreted as the singleton Boolean algebra.

(b) The Boolean operations on $\mathfrak{A}$ are now defined by the formulae

$$a \cup b = \langle a(i) \cup b(i)\rangle_{i\in I}, \quad a \setminus b = \langle a(i) \setminus b(i)\rangle_{i\in I}$$

for all a, $b \in \mathfrak{A}$.

315B Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their simple product.

(a) The maps $a\mapsto\pi_i(a)=a(i):\mathfrak{A}\to\mathfrak{A}_i$ are all Boolean homomorphisms.

(b) If $\mathfrak{B}$ is any other Boolean algebra, then a map $\phi:\mathfrak{B}\to\mathfrak{A}$ is a Boolean homomorphism iff $\pi_i\phi:\mathfrak{B}\to\mathfrak{A}_i$ is a Boolean homomorphism for every $i\in I$.

proof Verification of these facts amounts just to applying the definitions with attention.

315C Products of partially ordered sets (a) It is perhaps worth spelling out the following elementary definition. If $\langle P_i\rangle_{i\in I}$ is any family of partially ordered sets, its **product** is the set $P=\prod_{i\in I}P_i$ ordered by saying that $p\le q$ iff $p(i)\le q(i)$ for every $i\in I$; it is easy to check that P is now a partially ordered set.

(b) The point is that if $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then the ordering of $\mathfrak{A}$ is just the product partial order:

$$a\subseteq b\iff a\cap b=a\iff a(i)\cap b(i)=a(i)\ \forall\ i\in I\iff a(i)\subseteq b(i)\ \forall\ i\in I.$$

Now we have the following elementary, but extremely useful, general facts about products of partially ordered sets.

315D Proposition Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P.

(a) For any non-empty set $A\subseteq P$ and $q\in P$,

(i) $\sup A=q$ in P iff $\sup_{p\in A}p(i)=q(i)$ in P_i for every $i\in I$,

(ii) $\inf A=q$ in P iff $\inf_{p\in A}p(i)=q(i)$ in P_i for every $i\in I$.

(b) The coordinate maps $p\mapsto\pi_i(p)=p(i):P\to P_i$ are all order-preserving and order-continuous.

(c) For any partially ordered set Q and function $\phi:Q\to P$, ϕ is order-preserving iff $\pi_i\phi$ is order-preserving for every $i\in I$.

(d) For any partially ordered set Q and order-preserving function $\phi:Q\to P$,

(i) ϕ is order-continuous iff $\pi_i\phi$ is order-continuous for every i,

(ii) ϕ is sequentially order-continuous iff $\pi_i\phi$ is sequentially order-continuous for every i.

(e)(i) P is Dedekind complete iff every P_i is Dedekind complete.

(ii) P is Dedekind σ-complete iff every P_i is Dedekind σ-complete.

proof All these are elementary verifications. Of course parts (b), (d) and (e) rely on (a).

315E Factor algebras as principal ideals Because Boolean algebras have least elements, we have a second type of canonical map associated with their products. If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Boolean algebras with simple product $\mathfrak{A}$, define $\theta_i:\mathfrak{A}_i\to\mathfrak{A}$ by setting $(\theta_i a)(i)=a$, $(\theta_i a)(j)=0_{\mathfrak{A}_j}$ if $i\in I$, $a\in\mathfrak{A}_i$ and $j\in I\setminus\{i\}$. Each θ_i is a ring homomorphism, and is a Boolean isomorphism between $\mathfrak{A}_i$ and the principal ideal of $\mathfrak{A}$ generated by $\theta_i(1_{\mathfrak{A}_i})$. The family $\langle\theta_i(1_{\mathfrak{A}_i})\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$.

Associated with these embeddings is the following important result.

315F Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Suppose

either (i) that I is finite

or (ii) that I is countable and $\mathfrak{A}$ is Dedekind σ-complete

or (iii) that $\mathfrak{A}$ is Dedekind complete.

Then the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_{e_i}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal of $\mathfrak{A}$ generated by e_i for each i.

proof The given map is a Boolean homomorphism because each of the maps $a\mapsto a\cap e_i:\mathfrak{A}\to\mathfrak{A}_{e_i}$ is (312J). It is injective because $\sup_{i\in I}e_i=1$, so if $a\in\mathfrak{A}\setminus\{0\}$ there is an i such that $a\cap e_i\ne 0$. It is surjective because $\langle e_i\rangle_{i\in I}$ is disjoint and if $c\in\prod_{i\in I}\mathfrak{A}_{e_i}$ then $a=\sup_{i\in I}c(i)$ is defined in $\mathfrak{A}$ and

$$a\cap e_j=\sup_{i\in I}c(i)\cap e_j=c(j)$$

for every $j\in I$ (using 313Ba). The three alternative versions of the hypotheses of this proposition are designed to ensure that the supremum is always well-defined in $\mathfrak{A}$.

315G Algebras of sets and their quotients The Boolean algebras of measure theory are mostly presented as algebras of sets or quotients of algebras of sets, so it is perhaps worth spelling out the ways in which the product construction applies to such algebras.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The simple product $\prod_{i\in I}\Sigma_i$ may be identified with the algebra

$$\Sigma=\{E : E\subseteq X,\,\{x:(x,i)\in E\}\in\Sigma_i \text{ for every } i\in I\}$$

of subsets of $X=\{(x,i):i\in I,\,x\in X_i\}$, with the canonical homomorphisms $\pi_i:\Sigma\to\Sigma_i$ being given by

$$\pi_iE=\{x:(x,i)\in E\}$$

for each $E\in\Sigma$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where

$$\mathcal{J}=\{E:E\in\Sigma,\,\{x:(x,i)\in E\}\in\mathcal{J}_i \text{ for every } i\in I\},$$

and the canonical homomorphisms $\tilde\pi_i:\Sigma/\mathcal{J}\to\Sigma_i/\mathcal{J}_i$ are given by the formula $\tilde\pi_i(E^\bullet)=(\pi_iE)^\bullet$ for every $E\in\Sigma$.

proof (a) It is easy to check that Σ is a subalgebra of $\mathcal{P}X$, and that the map $E\mapsto\langle\pi_iE\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ is a Boolean isomorphism.

(b) Again, it is easy to check that $\mathcal{J}$ is an ideal of Σ, that the proposed formula for $\tilde\pi_i$ does indeed define a map from $\Sigma/\mathcal{J}$ to $\Sigma_i/\mathcal{J}_i$, and that $E^\bullet\mapsto\langle\tilde\pi_iE^\bullet\rangle_{i\in I}$ is an isomorphism between $\Sigma/\mathcal{J}$ and $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$.

315H Free products I come now to the second construction of this section.

(a) Definition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. For each $i\in I$, let Z_i be the Stone space of $\mathfrak{A}_i$. Set $Z=\prod_{i\in I}Z_i$, with the product topology. Then the **free product** of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is the algebra $\mathfrak{A}$ of open-and-closed sets in Z; I will denote it by $\bigotimes_{i\in I}\mathfrak{A}_i$.

(b) For $i\in I$ and $a\in\mathfrak{A}_i$, the set $\widehat{a}\subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z\mapsto z(i):Z\to Z_i$ is continuous, $\varepsilon_i(a)=\{z:z(i)\in\widehat{a}\}$ is open-and-closed, so belongs to $\mathfrak{A}$. In this context I will call $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the **canonical map**.

(c) The topological space Z may be identified with the Stone space of the Boolean algebra $\mathfrak{A}$. **P** By Tychonoff's theorem (3A3J), Z is compact. If $z\in Z$ and G is an open subset of Z containing z, then there are J, $\langle G_j\rangle_{i\in J}$ such that J is a finite subset of I, G_j is an open subset of Z_j for each $j\in J$, and

$$z\in\{w:w\in Z,\,w(j)\in G_j \text{ for every } j\in J\}\subseteq G.$$

Because each Z_j is zero-dimensional, we can find an open-and-closed set $E_j\subseteq Z_j$ such that $z(j)\in E_j\subseteq G_j$. Now

$$E=Z\cap\bigcap_{j\in J}\{w:w(j)\in E_j\}$$

is a finite intersection of open-and-closed subsets of Z, so is open-and-closed; and $z\in E\subseteq G$. As z and G are arbitrary, Z is zero-dimensional. Finally, Z, being the product of Hausdorff spaces, is Hausdorff. So the result follows from 311J. **Q**

315I Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $\mathfrak{A}$.

(a) The canonical map $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is a Boolean homomorphism for every $i\in I$.

(b) For any Boolean algebra $\mathfrak{B}$ and any family $\langle\phi_i\rangle_{i\in I}$ such that ϕ_i is a Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{B}$ for every i, there is a unique Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi_i=\phi\varepsilon_i$ for each i.

proof These are both consequences of 312P-312Q. As in 315H, write Z_i for the Stone space of $\mathfrak{A}$, and Z for $\prod_{i\in I}Z_i$, identified with the Stone space of $\mathfrak{A}$, as observed in 315Hc. The maps $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ are defined as the homomorphisms corresponding to the continuous maps $z\mapsto\tilde\varepsilon_i(z)=z(i):Z\to Z_i$, so (a) is surely true.

Now suppose that we are given a Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ for each $i\in I$. Let W be the Stone space of $\mathfrak{B}$, and let $\tilde\phi_i:W\to Z_i$ be the continuous function corresponding to ϕ_i. By 3A3Ib, the map $w\mapsto\tilde\phi(w)=\langle\tilde\phi_i(w)\rangle_{i\in I}:W\to Z$ is continuous, and corresponds to a Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$;

because $\tilde{\phi}_i = \tilde{\varepsilon}_i\tilde{\phi}$, $\phi\varepsilon_i = \phi_i$ for each i. Moreover, ϕ is the only Boolean homomorphism with this property, because if $\psi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism such that $\psi\varepsilon_i = \phi_i$ for every i, then ψ corresponds to a continuous map $\tilde{\psi} : W \to Z$, and we must have $\tilde{\varepsilon}_i\tilde{\psi} = \tilde{\phi}_i$ for each i, so that $\tilde{\psi} = \tilde{\phi}$ and $\psi = \phi$. This proves (b).

315J Of course 315I is the defining property of the free product (see 315Xg below). I list a few further basic facts.

Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their free product; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps.

(a) $\mathfrak{A}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$.

(b) Write C for the set of those members of $\mathfrak{A}$ expressible in the form $\inf_{j\in J}\varepsilon_j(a_j)$, where $J \subseteq I$ is finite and $a_j \in \mathfrak{A}_j$ for every j. Then every member of $\mathfrak{A}$ is expressible as the supremum of a disjoint finite subset of C. In particular, C is order-dense in $\mathfrak{A}$.

(c) Every ε_i is order-continuous.

(d) $\mathfrak{A} = \{0_{\mathfrak{A}}\}$ iff there is some $i \in I$ such that $\mathfrak{A}_i = \{0_{\mathfrak{A}_i}\}$.

(e) Now suppose that $\mathfrak{A}_i \neq \{0_{\mathfrak{A}_i}\}$ for every $i \in I$.

(i) ε_i is injective for every $i \in I$.

(ii) If $J \subseteq I$ is finite and a_j is a non-zero member of $\mathfrak{A}_j$ for each $j \in J$, then $\inf_{j\in J}\varepsilon_j(a_j) \neq 0$.

(iii) If i, j are distinct members of I and $a \in \mathfrak{A}_i$, $b \in \mathfrak{A}_j$, then $\varepsilon_i(a) = \varepsilon_j(b)$ iff either $a = 0_{\mathfrak{A}_i}$ and $b = 0_{\mathfrak{A}_j}$ or $a = 1_{\mathfrak{A}_i}$ and $b = 1_{\mathfrak{A}_j}$.

proof As usual, write Z_i for the Stone space of $\mathfrak{A}_i$, and $Z = \prod_{i\in I} Z_i$, identified with the Stone space of $\mathfrak{A}$ (315Hc).

(a) Write $\mathfrak{A}'$ for the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$. Then $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}'$ is a Boolean homomorphism for each i, so by 315Ib there is a Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}'$ such that $\phi\varepsilon_i = \varepsilon_i$ for each i. Now, regarding ϕ as a Boolean homomorphism from $\mathfrak{A}$ to itself, the uniqueness assertion of 315Ib (with $\mathfrak{B} = \mathfrak{A}$) shows that ϕ must be the identity, so that $\mathfrak{A}' = \mathfrak{A}$.

(b) Write $\mathcal{D}$ for the set of finite partitions of unity in $\mathfrak{A}$ consisting of members of C, and A for the set of members of $\mathfrak{A}$ expressible in the form $\sup D'$ where D' is a subset of a member of $\mathcal{D}$. Then A is a subalgebra of $\mathfrak{A}$. **P** (i) $1_{\mathfrak{A}} \in C$ (set $J = \emptyset$ in the definition of members of C) so $\{1_{\mathfrak{A}}\} \in \mathcal{D}$ and $0_{\mathfrak{A}}$, $1_{\mathfrak{A}} \in A$. (ii) Note that if c, $d \in C$ then $c \cap d \in C$. (iii) If a, $b \in A$, express them as $\sup D'$, $\sup E'$ where $D' \subseteq D \in \mathcal{D}$, $E' \subseteq E \in \mathcal{D}$. Then

$$F = \{d \cap e : d \in D,\, e \in E\} \in \mathcal{D},$$

so

$$1_{\mathfrak{A}} \setminus a = \sup D \setminus D' \in A,$$

$$a \cup b = \sup\{f : f \in F,\, f \subseteq a \cup b\} \in A. \text{ } \mathbf{Q}$$

Also, $\varepsilon_i[\mathfrak{A}_i] \subseteq A$ for each $i \in I$. **P** If $a \in \mathfrak{A}_i$, then $\{\varepsilon_i(a), \varepsilon_i(1_{\mathfrak{A}_i} \setminus a)\} \in \mathcal{D}$, so $\varepsilon_i(a) \in A$. **Q**

So (a) tells us that $A = \mathfrak{A}$, and every member of $\mathfrak{A}$ is a finite disjoint union of members of C.

(c) If $i \in I$ and $A \subseteq \mathfrak{A}_i$ and $\inf A = 0$ in $\mathfrak{A}_i$, take any non-zero $c \in \mathfrak{A}$. By (b), we can find a finite $J \subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ such that $c' = \inf_{j\in J}\varepsilon_j(a_j) \subseteq c$ and $c' \neq 0$. Regarding c' as a subset of Z, we have a point $z \in c'$. Adding i to J and setting $a_i = 1_{\mathfrak{A}_i}$ if necessary, we may suppose that $i \in J$. Now $c' \neq 0_{\mathfrak{A}}$ so $a_i \neq 0_{\mathfrak{A}_i}$ and there is an $a \in A$ such that $a_i \not\subseteq a$, so there is a $t \in \widehat{a}_i \setminus \widehat{a}$. In this case, setting $w(i) = t$, $w(j) = z(j)$ for $j \neq i$, we have $w \in c' \setminus \varepsilon_i(a)$, and c', c are not included in $\varepsilon_i(a)$. As c is arbitrary, this shows that $\inf \varepsilon_i[A] = 0$. As A is arbitrary, ε_i is order-continuous.

(d) The point is that $\mathfrak{A} = \{0_{\mathfrak{A}}\}$ iff $Z = \emptyset$, which is so iff some Z_i is empty.

(e)(i) Because no Z_i is empty, all the coordinate maps from Z to Z_i are surjective, so the corresponding homomorphisms ε_i are injective (312Ra).

(ii) Because J is finite,

$$\inf_{j\in J}\varepsilon_j(a_j)=\{z:z\in Z,\,z(j)\in\widehat{a}_j\text{ for every }j\in J\}$$

is not empty.

(iii) If $\varepsilon_i(a)=\varepsilon_j(b)=0_{\mathfrak{A}}$ then (using (i)) $a=0_{\mathfrak{A}_i}$ and $b=0_{\mathfrak{A}_j}$; if $\varepsilon_i(a)=\varepsilon_j(b)=1_{\mathfrak{A}}$ then $a=1_{\mathfrak{A}_i}$ and $b=1_{\mathfrak{A}_j}$. **?** If $\varepsilon_i(a)=\varepsilon_j(b)\in\mathfrak{A}\setminus\{0_{\mathfrak{A}},1_{\mathfrak{A}}\}$, then there are $t\in\widehat{a}$ and $u\in Z_j\setminus\widehat{b}$. Now there is a $z\in Z$ such that $z(i)=t$ and $z(j)=u$, so that $z\in\varepsilon_i(a)\setminus\varepsilon_j(b)$. **X**

315K Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, and $\langle J_k\rangle_{k\in K}$ any partition of I. Then the free product $\mathfrak{A}$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is isomorphic to the free product $\mathfrak{B}$ of $\langle\mathfrak{B}_k\rangle_{k\in K}$, where each $\mathfrak{B}_k$ is the free product of $\langle\mathfrak{A}_i\rangle_{i\in J_k}$.

proof Write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$, $\varepsilon'_i:\mathfrak{A}\to\mathfrak{B}_k$ and $\delta_k:\mathfrak{B}_k\to\mathfrak{B}$ for the canonical maps when $k\in K$, $i\in J_k$. Then the homomorphisms $\delta_k\varepsilon'_i:\mathfrak{A}_i\to\mathfrak{B}$ correspond to a homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\delta_k\varepsilon'_i$ whenever $i\in J_k$. Next, for each k, the homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$, for $i\in J_k$, correspond to a homomorphism $\psi_k:\mathfrak{B}_k\to\mathfrak{A}$ such that $\psi_k\varepsilon'_i=\varepsilon_i$ for $i\in J_k$; and the family $\langle\psi_k\rangle_{k\in K}$ corresponds to a homomorphism $\psi:\mathfrak{B}\to\mathfrak{A}$ such that $\psi\delta_k=\psi_k$ for $k\in K$. Consequently

$$\psi\phi\varepsilon_i=\psi\delta_k\varepsilon'_i=\psi_k\epsilon'_i=\epsilon_i$$

whenever $k\in K$, $i\in J_k$. Once again using the uniqueness assertion in 315Ib, $\psi\phi$ is the identity homomorphism on $\mathfrak{A}$. On the other hand, if we look at $\phi\psi:\mathfrak{B}\to\mathfrak{B}$, then we see that

$$\phi\psi\delta_k\varepsilon'_i=\phi\psi_k\epsilon'_i=\phi\epsilon_i=\delta_k\varepsilon'_i$$

whenever $k\in K$, $i\in J_k$. Now, for given k, $\{b:b\in\mathfrak{B}_k,\,\phi\psi\delta_kb=\delta_kb\}$ is a subalgebra of $\mathfrak{B}_k$ including $\bigcup_{i\in J_k}\varepsilon'_i[\mathfrak{A}_i]$, and must be the whole of $\mathfrak{B}_k$, by 315Ja. So $\{b:b\in\mathfrak{B},\,\phi\psi b=b\}$ is a subalgebra of $\mathfrak{B}$ including $\bigcup_{k\in K}\delta_k[\mathfrak{B}_k]$, and is the whole of $\mathfrak{B}$. Thus $\phi\psi$ is the identity on $\mathfrak{B}$ and ϕ, ψ are the two halves of an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

315L Algebras of sets and their quotients Once again I devote a paragraph to spelling out the application of the construction to the algebras most important to us.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The free product $\bigotimes_{i\in I}\Sigma_i$ may be identified with the algebra Σ of subsets of $X=\prod_{i\in I}X_i$ generated by the set $\{\varepsilon_i(E):i\in I,\,E\in\Sigma_i\}$, where $\varepsilon_i(E)=\{x:x\in X,\,x(i)\in E\}$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\bigotimes_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where $\mathcal{J}$ is the ideal of Σ generated by $\{\varepsilon_i(E):i\in I,\,E\in\mathcal{J}_i\}$; the corresponding canonical maps $\tilde\varepsilon_i:\Sigma_i/\mathcal{J}_i\to\Sigma/\mathcal{J}$ being defined by the formula $\tilde\varepsilon_i(E^\bullet)=(\varepsilon_i(E))^\bullet$ for $i\in I$, $E\in\Sigma_i$.

proof I start by proving (b) in detail; the argument for (a) is then easy to extract. Write $\mathfrak{A}_i=\Sigma_i/\mathcal{J}_i$, $\mathfrak{A}=\Sigma/\mathcal{J}$.

(i) Fix $i\in I$ for the moment. By the definition of Σ, $\varepsilon_i(E)\in\Sigma$ for $E\in\Sigma_i$, and it is easy to check that $\varepsilon_i:\Sigma_i\to\Sigma$ is a Boolean homomorphism. Again, because $\varepsilon_i(E)\in\mathcal{J}$ whenever $E\in\mathcal{J}_i$, the kernel of the homomorphism $E\mapsto(\varepsilon_i(E))^\bullet:\Sigma_i\to\mathfrak{A}$ includes $\mathcal{J}_i$, so the formula for $\tilde\varepsilon_i$ defines a homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}$.

Now let $\mathfrak{C}=\bigotimes_{i\in I}\mathfrak{A}_i$ be the free product, and write $\varepsilon'_i:\mathfrak{A}_i\to\mathfrak{C}$ for the canonical homomorphisms. By 315I, there is a Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{A}$ such that $\phi\varepsilon'_i=\tilde\varepsilon_i$ for each i. The set

$$\{E:E\in\Sigma,\,E^\bullet\in\phi[\mathfrak{C}]\}$$

is a subalgebra of Σ including $\varepsilon_i[\Sigma_i]$ for every i, so is Σ itself, and ϕ is surjective.

(ii) We need a simple description of the ideal $\mathcal{J}$, as follows: a set $E\in\Sigma$ belongs to $\mathcal{J}$ iff there are a finite $K\subseteq I$ and a family $\langle F_k\rangle_{k\in K}$ such that $F_k\in\mathcal{J}_k$ for each k and $E\subseteq\bigcup_{k\in K}\varepsilon_k(F_k)$. For evidently such sets have to belong to $\mathcal{J}$, since the $\varepsilon_k(F_k)$ will be in $\mathcal{J}$, while the family of all these sets is an ideal containing $\varepsilon_i(F)$ whenever $i\in I$, $F\in\mathcal{J}_i$.

(iii) Now we can see that $\phi:\mathfrak{C}\to\mathfrak{A}$ is injective. **P** Take any non-zero $c\in\mathfrak{C}$. By 315Jb, we can find a finite $J\subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ in $\prod_{j\in J}\mathfrak{A}_j$ such that $0\neq\inf_{j\in J}\varepsilon'_ja_j\subseteq c$. Express each a_j as $E_j^\bullet$, where $E_j\in\Sigma_j$, and consider $E=X\cap\bigcap_{j\in J}\varepsilon_j(E_j)\in\Sigma$. Then

$$E^\bullet = \inf_{j\in J}\tilde\varepsilon_j a_j = \phi(\inf_{j\in J}\varepsilon'_j a_j) \subseteq \phi(c).$$

Also, because $\varepsilon'_j a_j \ne 0$, $E_j \notin \mathcal{J}_j$ for each j. But it follows that $E \notin \mathcal{J}$, because if $K \subseteq I$ is finite and $F_k \in \mathcal{J}_k$ for each $k \in K$, set $E_i = X_i$ for $i \in I\setminus J$, $F_i = \emptyset$ for $i \in I \setminus K$; then there is an $x \in X$ such that $x(i) \in E_i \setminus F_i$ for each $i \in I$, so that $x \in E \setminus \bigcup_{k\in K} F_k$. By the criterion of (ii), $E \notin \mathcal{J}$. So

$$0 \ne E^\bullet \subseteq \phi(c).$$

As c is arbitrary, the kernel of ϕ is $\{0\}$, and ϕ is injective. **Q**

So $\phi : \mathfrak{C} \to \mathfrak{A}$ is the required isomorphism.

(iv) This proves (b). Reading through the arguments above, it is easy to see the simplifications which compose a proof of (a), reading Σ_i for $\mathfrak{A}_i$ and $\{\emptyset\}$ for $\mathcal{J}_i$.

315M Notation Free products are sufficiently surprising that I think it worth taking a moment to look at a pair of examples relevant to the kinds of application I wish to make of the concept in the next chapter. First let me introduce a somewhat more direct notation which seems appropriate for the free product of finitely many factors. If $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras, I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product, and for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ I write $a \otimes b$ for $\varepsilon_1(a) \cap \varepsilon_2(b)$, where $\varepsilon_1 : \mathfrak{A} \to \mathfrak{A}\otimes\mathfrak{B}$, $\varepsilon_2 : \mathfrak{B} \to \mathfrak{A}\otimes\mathfrak{B}$ are the canonical maps. Observe that $(a_1 \otimes b_1) \cap (a_2 \otimes b_2) = (a_1 \cap a_2) \otimes (b_1 \cap b_2)$, and that the maps $a \mapsto a \otimes b_0$, $b \mapsto a_0 \otimes b$ are always ring homomorphisms. Now 315J(e-ii) tells us that $a \otimes b = 0$ only when one of a, b is 0. In the context of 315L, we can identify $E \otimes F$ with $E \times F$ for $E \in \Sigma_1$, $F \in \Sigma_2$, and $E^\bullet \otimes F^\bullet$ with $(E\times F)^\bullet$.

315N Lemma Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras.

(a) Any element of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\in I} a_i \otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$.

(b) If $c \in \mathfrak{A}\otimes\mathfrak{B}$ is non-zero there are non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ such that $a \otimes b \subseteq c$.

proof (a) Let C be the set of elements of $\mathfrak{A}\otimes\mathfrak{B}$ representable in this form. Then C is a subalgebra of $\mathfrak{A}\otimes\mathfrak{B}$. **P** (i) If $\langle a_i\rangle_{i\in I}$, $\langle a'_j\rangle_{j\in J}$ are finite partitions of unity in $\mathfrak{A}$, and b_i, b'_j members of $\mathfrak{B}$ for $i \in I$ and $j \in J$, the $\langle a_i \cap a'_j\rangle_{i\in I, j\in J}$ is a partition of unity in $\mathfrak{A}$, and

$$\begin{aligned}(\sup_{i\in I} a_i \otimes b_i) \cap (\sup_{j\in J} a'_j \otimes b'_j) &= \sup_{i\in I, j\in J} (a_i \otimes b_i) \cap (a'_j \otimes b'_j)\\ &= \sup_{i\in I, j\in J} (a_i \cap a'_j) \otimes (b_i \cap b'_j) \in C.\end{aligned}$$

So $c \cap c' \in C$ for all c, $c' \in C$. (ii) If $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i \in \mathfrak{B}$ for each i, then

$$1 \setminus \sup_{i\in I} a_i \otimes b_i = (\sup_{i\in I} a_i \otimes 1) \setminus (\sup_{i\in I} a_i \otimes b_i) = \sup_{i\in I} a_i \otimes (1 \setminus b_i) \in C.$$

Thus $1 \setminus c \in C$ for every $c \in C$. **Q**

Since $a \otimes 1 = (a \otimes 1) \cup ((1 \setminus a) \otimes 0)$ and $1 \otimes b$ belong to C for every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, C must be the whole of $\mathfrak{A}\otimes\mathfrak{B}$, by 315Ja.

(b) Now this follows at once, just as in 315Jb.

315O Example $\mathfrak{A} = \mathcal{P}\mathbb{N}\otimes\mathcal{P}\mathbb{N}$ is not Dedekind σ-complete. **P** Consider $A = \{\{n\}\otimes\{n\} : n \in \mathbb{N}\} \subseteq \mathfrak{A}$. **?** If A has a least upper bound c in $\mathfrak{A}$, then c is expressible as a supremum $\sup_{j\le k} a_j \otimes b_j$, by 315Jb. Because k is finite, there must be distinct m, n such that $\{j : m \in a_j\} = \{j : n \in a_j\}$. Now $\{n\}\times\{n\} \subseteq c$, so there is a $j \le k$ such that

$$(a_j \cap \{n\}) \otimes (b_j \cap \{n\}) = (\{n\}\otimes\{n\}) \cap (a_j \otimes b_j) \ne 0,$$

so that neither $a_j \cap \{n\}$ nor $b_j \cap \{n\}$ is empty, that is, $n \in a_j \cap b_j$. But this means that $m \in a_j$, so that

$$(a_j \otimes b_j) \cap (\{m\}\otimes\{n\}) = (a_j \cap \{m\}) \otimes (b_j \cap \{n\}) \ne 0,$$

and $c \cap (\{m\}\otimes\{n\}) \ne 0$, even though $a \cap (\{m\}\otimes\{n\}) = 0$ for every $a \in A$. **X** Thus we have found a countable subset of $\mathfrak{A}$ with no supremum in $\mathfrak{A}$, and $\mathfrak{A}$ is not Dedekind σ-complete. **Q**

315P Example Now let $\mathfrak{A}$ be any non-trivial atomless Boolean algebra, and $\mathfrak{B}$ the free product $\mathfrak{A}\otimes\mathfrak{A}$. Then the identity homomorphism from $\mathfrak{A}$ to itself induces a homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ given by setting

$\phi(a \otimes b) = a \cap b$ for every $a, b \in \mathfrak{A}$. The point I wish to make is that ϕ is not order-continuous. **P** Let C be the set $\{a \otimes b : a, b \in \mathfrak{A}, a \cap b = 0\}$. Then $\phi(c) = 0_{\mathfrak{A}}$ for every $c \in C$. If $d \in \mathfrak{B}$ is non-zero, then by 315Nb there are non-zero $a, b \in \mathfrak{A}$ such that $a \otimes b \subseteq d$; now, because $\mathfrak{A}$ is atomless, there is a non-zero $a' \subseteq a$ such that $a \setminus a' \neq 0$. At least one of $b \setminus a'$, $b \setminus (a \setminus a')$ is non-zero; suppose the former. Then $a' \otimes (b \setminus a')$ is a non-zero member of C included in d. As d is arbitrary, this shows that $\sup C = 1_{\mathfrak{B}}$. So

$$\sup_{c \in C} \phi(c) = 0_{\mathfrak{A}} \neq 1_{\mathfrak{A}} = \phi(\sup C),$$

and ϕ is not order-continuous. **Q**

Thus the free product (unlike the product, see 315Dd) does not respect order-continuity.

315X Basic exercises (a) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$, and $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ the coordinate homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\pi'_i : \mathfrak{A}' \to \mathfrak{A}_i$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle \phi_i \rangle_{i \in I}$ of homomorphisms from $\mathfrak{B}$ to the $\mathfrak{A}_i$ there is a unique homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}'$ such that $\phi_i = \pi'_i \phi$ for every i. Show that there is a unique isomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$ such that $\pi'_i \psi = \pi_i$ for every $i \in I$.

(b) Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets, with product partially ordered set P. Show that P is a lattice iff every P_i is a lattice, and that in this case it is the product lattice in the sense that $p \vee q = \langle p(i) \vee q(i) \rangle_{i \in I}$, $p \wedge q = \langle p(i) \wedge q(i) \rangle_{i \in I}$ for all $p, q \in P$.

(c) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}$. For each $i \in I$ let Z_i be the Stone space of $\mathfrak{A}_i$, and let Z be the Stone space of $\mathfrak{A}$. Show that the coordinate maps from $\mathfrak{A}$ onto $\mathfrak{A}_i$ induce homeomorphisms between the Z_i and open-and-closed subsets Z_i^* of Z. Show that $\langle Z_i^* \rangle_{i \in I}$ is disjoint. Show that $\bigcup_{i \in I} Z_i^*$ is dense in Z, and is equal to Z iff $\{i : \mathfrak{A}_i \neq \{0\}\}$ is finite.

(d) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. Suppose that for each $i \in I$ we are given an ideal I_i of $\mathfrak{A}_i$. Show that $I = \prod_{i \in I} I_i$ is an ideal of $\mathfrak{A}$, and that $\mathfrak{A}/I$ may be identified, as Boolean algebra, with $\prod_{i \in I} \mathfrak{A}_i / I_i$.

(e) Let $\langle X_i \rangle_{i \in I}$ be any family of topological spaces. Let X be their disjoint union $\{(x, i) : i \in I, x \in X_i\}$, with the disjoint union topology; that is, a set $G \subseteq X$ is open in X iff $\{x : (x, i) \in G\}$ is open in X_i for every $i \in I$. (i) Show that the algebra of open-and-closed subsets of X can be identified, as Boolean algebra, with the simple product of the algebras of open-and-closed sets of the X_i. (ii) Show that the regular open algebra of X can be identified, as Boolean algebra, with the simple product of the regular open algebras of the X_i.

(f) Show that the topological product of any family of zero-dimensional spaces is zero-dimensional.

(g) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be any family of Boolean algebras, with free product $\mathfrak{A}$, and $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ the canonical homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\varepsilon'_i : \mathfrak{A}_i \to \mathfrak{A}'$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle \phi_i \rangle_{i \in I}$ of homomorphisms from the $\mathfrak{A}_i$ to $\mathfrak{B}$ there is a unique homomorphism $\phi : \mathfrak{A}' \to \mathfrak{B}$ such that $\phi_i = \phi \varepsilon'_i$ for every i. Show that there is a unique isomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$ such that $\varepsilon'_i = \psi \varepsilon_i$ for every $i \in I$.

(h) Let I be any set, and let $\mathfrak{A}$ be the algebra of open-and-closed sets of $\{0,1\}^I$; for each $i \in I$ set $a_i = \{x : x \in \{0,1\}^I, x(i) = 1\} \in \mathfrak{A}$. Show that for any Boolean algebra $\mathfrak{B}$, any family $\langle b_i \rangle_{i \in I}$ in $\mathfrak{B}$ there is a unique Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi(a_i) = b_i$ for every $i \in I$.

(i) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$, $\langle \mathfrak{B}_j \rangle_{j \in J}$ be two families of Boolean algebras. Show that there is a natural injective homomorphism $\phi : \prod_{i \in I} \mathfrak{A}_i \otimes \prod_{j \in J} \mathfrak{B}_j \to \prod_{i \in I, j \in J} \mathfrak{A}_i \otimes \mathfrak{B}_j$ defined by saying that

$$\phi(a \otimes b) = \langle a(i) \otimes b(j) \rangle_{i \in I, j \in J}$$

for $a \in \prod_{i \in I} \mathfrak{A}_i$, $b \in \prod_{j \in J} \mathfrak{B}_j$. Show that ϕ is surjective if I and J are finite.

(j) Let $\langle J(i) \rangle_{i \in I}$ be a family of sets, with product $Q = \prod_{i \in I} J(i)$. Let $\langle \mathfrak{A}_{ij} \rangle_{i \in I, j \in J(i)}$ be a family of Boolean algebras. Describe a natural injective homomorphism $\phi : \bigotimes_{i \in I} \prod_{j \in J(i)} \mathfrak{A}_{ij} \to \prod_{q \in Q} \bigotimes_{i \in I} \mathfrak{A}_{i,q(i)}$.

(k) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with partitions of unity $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$. Show that $\langle a_i\otimes b_j\rangle_{i\in I,j\in J}$ is a partition of unity in $\mathfrak{A}\otimes\mathfrak{B}$.

(l) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $a\in\mathfrak{A}$, $b\in\mathfrak{B}$. Write $\mathfrak{A}_a$, $\mathfrak{B}_b$ for the corresponding principal ideals. Show that there is a canonical isomorphism between $\mathfrak{A}_a\otimes\mathfrak{B}_b$ and the principal ideal of $\mathfrak{A}\otimes\mathfrak{B}$ generated by $a\otimes b$.

(m) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with free product $\bigotimes_{i\in I}\mathfrak{A}_i$, and $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the canonical maps. Show that $\varepsilon_i[\mathfrak{A}_i]$ is an order-closed subalgebra of $\mathfrak{A}$ for every i.

(n) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j\in J}a_j\ne 0$ whenever $J\subseteq I$ is finite and $a_j\in\mathfrak{A}_j\setminus\{0\}$ for every $j\in J$. Show that in this case the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{A}_i$ is isomorphic to the free product $\bigotimes_{i\in I}\mathfrak{A}_i$.

(o) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that for each $i\in I$ we are given a Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ with kernel $K_i\lhd\mathfrak{A}_i$. Show that the ϕ_i induce a Boolean homomorphism $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ with kernel generated by the images of the K_i. Show that if every ϕ_i is surjective, so is ϕ.

(p) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of non-trivial Boolean algebras. Show that if $J\subseteq I$ and $\mathfrak{B}_j$ is a subalgebra of $\mathfrak{A}_j$ for each $j\in J$, then $\bigotimes_{j\in J}\mathfrak{B}_j$ is canonically embedded as a subalgebra of $\bigotimes_{i\in I}\mathfrak{A}_i$.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, neither $\{0\}$. Show that any element of $\mathfrak{A}\otimes\mathfrak{B}$ is uniquely expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, with no a_i equal to 0, and $b_i\ne b_j$ in $\mathfrak{B}$ for $i\ne j$.

315Y Further exercises (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that we are given Boolean homomorphisms $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ for each i; let $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ be the induced homomorphism. (i) Show that if every ϕ_i is order-continuous, so is ϕ. (ii) Show that if every ϕ_i is sequentially order-continuous, so is ϕ.

(b) Let $\langle Z_i\rangle_{i\in I}$ be any family of topological spaces with product Z. For $i\in I$, $z\in Z$ set $\tilde\varepsilon_i(z)=z(i)$. Show that if $M\subseteq Z_i$ is nowhere dense in Z_i then $\tilde\varepsilon_i^{-1}[M]$ is nowhere dense in Z. Use this to prove 315Jc.

(c) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and suppose that we are given subalgebras $\mathfrak{B}_i$ of $\mathfrak{A}_i$ for each i; set $\mathfrak{A}=\bigotimes_{i\in I}\mathfrak{A}_i$ and $\mathfrak{B}=\bigotimes_{i\in I}\mathfrak{B}_i$, and let $\phi:\mathfrak{B}\to\mathfrak{A}$ be the homomorphism induced by the embeddings $\mathfrak{B}_i\subseteq\mathfrak{A}_i$. (i) Show that if every $\mathfrak{B}_i$ is order-closed in $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is order-closed in $\mathfrak{A}$. (ii) Show that if every $\mathfrak{B}_i$ is a σ-subalgebra of $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is a σ-subalgebra in $\mathfrak{A}$.

(d) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. Let $\mathfrak{G}_i$, $\mathfrak{G}$ be the corresponding regular open algebras. Show that $\mathfrak{G}$ can be identified with the Dedekind completion of $\bigotimes_{i\in I}\mathfrak{G}_i$.

(e) Use the ideas of 315Xh and 315L to give an alternative construction of 'free product', for which 315I and 315J(e-ii) are true, which does not depend on the concept of Stone space nor on any other use of the axiom of choice. (*Hint*: show that for any Boolean algebra $\mathfrak{A}$ there is a canonical surjection from the algebra $\mathcal{E}_{\mathfrak{A}}$ onto $\mathfrak{A}$, where $\mathcal{E}_J$ is the algebra of subsets of $\{0,1\}^J$ generated by sets of the form $\{x:x(j)=1\}$; show that for such algebras $\mathcal{E}_J$, at least, the method of 315H-315I can be used; now apply the method of 315L to describe $\bigotimes_{i\in I}\mathfrak{A}_i$ as a quotient of $\mathcal{E}_J$ where $J=\{(a,i):i\in I,\,a\in\mathfrak{A}_i\}$. Finally check that if no $\mathfrak{A}_i$ is trivial, then nor is the free product.)

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that $\mathfrak{A}\otimes\mathfrak{B}$ is Dedekind complete iff *either* $\mathfrak{A}=\{0\}$ *or* $\mathfrak{B}=\{0\}$ *or* $\mathfrak{A}$ is finite and $\mathfrak{B}$ is Dedekind complete *or* $\mathfrak{B}$ is finite and $\mathfrak{A}$ is Dedekind complete.

(g) Let $\langle P_i\rangle_{i\in I}$ be any family of partially ordered spaces. (i) Give a construction of a partially ordered space P, together with a family of order-preserving maps $\varepsilon_i:P_i\to P$, such that whenever Q is a partially ordered set and $\phi_i:P_i\to Q$ is order-preserving for every $i\in I$, there is a unique order-preserving map $\phi:P\to Q$ such that $\phi_i=\phi\varepsilon_i$ for every i. (ii) Show that ϕ will be order-continuous iff every ϕ_i is. (iii) Show that P will be Dedekind complete iff every P_i is, but (except in trivial cases) is not a lattice.

315 Notes and comments In this section I find myself asking for slightly more sophisticated algebra than seems necessary elsewhere. The point is that simple products and free products are best regarded as defined by the properties described in 315B and 315I. That is, it is sometimes right to think of *a* simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras as being a structure $(\mathfrak{A},\langle\pi_i\rangle_{i\in I})$ where $\mathfrak{A}$ is a Boolean algebra, $\pi_i:\mathfrak{A}\to\mathfrak{A}_i$ is a homomorphism for every $i\in I$, and every family of homomorphisms from a Boolean algebra $\mathfrak{B}$ to the $\mathfrak{A}_i$ can be uniquely represented by a single homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. Similarly, reversing the direction of the homomorphisms, we can speak of *a* free product (it would be natural to say 'coproduct') $(\mathfrak{A},\langle\varepsilon_i\rangle_{i\in I})$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$. On such definitions, it is elementary that any two simple products, or free products, are isomorphic in the obvious sense (315Xa, 315Xg), and very general arguments from abstract algebra, not restricted to Boolean algebras (see Bourbaki 68, IV.3.2), show that they exist. (But in order to prove such basic facts as that the π_i are surjective, or that the ε_i are, except when the construction collapses altogether, injective, we do of course have to look at the special properties of Boolean algebras.) Now in the case of simple products, the Cartesian product construction is so direct and so familiar that there seems no need to trouble our imaginations with any other. But in the case of free products, things are more complicated. I have given primacy to the construction in terms of Stone spaces because I believe that this is the fastest route to effective mental pictures. But in some ways this approach seems to be inappropriate. If you take what in my view is a tenable position, and say that a Boolean algebra is best regarded as the limit of its finite subalgebras, then you might prefer a construction of a free product as a limit of free products of finitely many finite subalgebras. Or you might feel that it is wrong to rely on the axiom of choice to prove a result which certainly does not need it (see 315Ye).

Because I believe that the universal mapping theorem 315I is the right basis for the study of free products, I am naturally led to use it as the starting point for proofs of theorems about free products, as in 315K. But 315J(e-ii) seems to lie deeper. (Note, for instance, that in 315L we *do* need the axiom of choice, in part (c) of the proof, since without it the product $\prod_{i\in I}X_i$ could be empty.)

Both 'simple product' and 'free product' are essentially algebraic constructions involving the category of Boolean algebras and Boolean homomorphisms, and any relationships with such concepts as order-continuity must be regarded as accidental. 315Cb and 315D show that simple products behave very straightforwardly when the homomorphisms involved are order-continuous. 315P, 315Xm and 315Ya-315Yc show that free products are much more complex and subtle.

For finite products, we have a kind of distributivity; $(\mathfrak{A}\times\mathfrak{B})\otimes\mathfrak{C}$ can be identified with $(\mathfrak{A}\otimes\mathfrak{C})\times(\mathfrak{B}\otimes\mathfrak{C})$ (315Xi, 315Xj). There are contexts in which this makes it seem more natural to write $\mathfrak{A}\oplus\mathfrak{B}$ in place of $\mathfrak{A}\times\mathfrak{B}$, and indeed I have already spoken of a 'direct sum' of measure spaces (214K) in terms which correspond closely to the simple product of algebras of sets described in 315Ga. Generally, the simple product corresponds to disjoint unions of Stone spaces (315Xc) and the free product to products of Stone spaces. But the simple product is indeed the product Boolean algebra, in the ordinary category sense; the universal mapping theorem 315B is exactly of the type we expect from products of topological spaces (3A3Ib) or partially ordered sets (315Dc), etc. It is the 'free product' which is special to Boolean algebras. The nearest analogy that I know of elsewhere is with the concept of 'tensor product' of linear spaces (cf. §253).

316 Further topics

I introduce two special properties of Boolean algebras which will be of great importance in the rest of this volume: the countable chain condition (316A-316F) and weak (σ,∞)-distributivity (316G-316K). I end the section with brief notes on atoms in Boolean algebras (316L-316M).

316A Definitions (a) A Boolean algebra $\mathfrak{A}$ is **ccc**, or satisfies the **countable chain condition**, if every disjoint subset of $\mathfrak{A}$ is countable.

(b) A topological space X is **ccc**, or satisfies the **countable chain condition**, or has **Souslin's property**, if every disjoint collection of open sets in X is countable.

316B Theorem A Boolean algebra $\mathfrak{A}$ is ccc iff its Stone space Z is ccc.

proof (a) If $\mathfrak{A}$ is ccc and $\mathcal{G}$ is a disjoint family of open sets in Z, then for each $G \in \mathcal{G}' = \mathcal{G} \setminus \{\emptyset\}$ we can find a non-zero $a_G \in \mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}_G$ is included in G. Now $\{a_G : G \in \mathcal{G}'\}$ is a disjoint family in $\mathfrak{A}$, so is countable; since $a_G \neq a_H$ for distinct G, $H \in \mathcal{G}'$, $\mathcal{G}'$ and $\mathcal{G}$ must be countable. As $\mathcal{G}$ is arbitrary, Z is ccc.

(b) If Z is ccc and $A \subseteq \mathfrak{A}$ is disjoint, then $\{\widehat{a} : a \in A\}$ is a disjoint family of open subsets of Z, so must be countable, and A is countable. As A is arbitrary, $\mathfrak{A}$ is ccc.

316C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Then the quotient algebra $\mathfrak{B} = \mathfrak{A}/\mathcal{I}$ is ccc iff every disjoint family in $\mathfrak{A} \setminus \mathcal{I}$ is countable.

proof (a) Suppose that $\mathfrak{B}$ is ccc and that A is a disjoint family in $\mathfrak{A} \setminus \mathcal{I}$. Then $\{a^\bullet : a \in A\}$ is a disjoint family in $\mathfrak{B}$, therefore countable, and $a^\bullet \neq b^\bullet$ when a, b are distinct members of A; so A is countable.

(b) Now suppose that $\mathfrak{B}$ is not ccc. Then there is an uncountable disjoint set $B \subseteq \mathfrak{B}$. Of course $B \setminus \{0\}$ is still uncountable, so may be enumerated as $\langle b_\xi \rangle_{\xi<\kappa}$, where κ is an uncountable cardinal (2A1K), so that $\omega_1 \leq \kappa$. For each $\xi < \omega_1$, choose $a_\xi \in \mathfrak{A}$ such that $a_\xi^\bullet = b_\xi$. Of course $a_\xi \notin \mathcal{I}$. If $\eta < \xi < \omega_1$, then $b_\eta \cap b_\xi = 0$, so $a_\xi \cap a_\eta \in \mathcal{I}$. Because $\xi < \omega_1$, it is countable; because $\mathcal{I}$ is a σ-ideal, and $\mathfrak{A}$ is Dedekind σ-complete,

$$d_\xi = \sup_{\eta<\xi} a_\xi \cap a_\eta \in \mathcal{I},$$

so that

$$c_\xi = a_\xi \setminus d_\xi \in \mathfrak{A} \setminus \mathcal{I}.$$

But now of course

$$c_\xi \cap c_\eta \subseteq c_\xi \cap a_\eta \subseteq c_\xi \cap d_\xi = 0$$

whenever $\eta < \xi < \omega_1$, so $\{c_\xi : \xi < \omega_1\}$ is an uncountable disjoint family in $\mathfrak{A} \setminus \mathcal{I}$.

Remark An ideal $\mathcal{I}$ satisfying the conditions of this proposition is said to be ω_1**-saturated** in $\mathfrak{A}$.

316D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Then the quotient algebra $\Sigma/\mathcal{I}$ is ccc iff every disjoint family in $\Sigma \setminus \mathcal{I}$ is countable.

316E Proposition Let $\mathfrak{A}$ be a ccc Boolean algebra. Then for any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ such that B has the same upper and lower bounds as A.

proof (a) Set

$$D = \bigcup_{a\in A}\{d : d \subseteq a\}.$$

Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint subsets of D, we have a maximal $C_0 \in \mathcal{C}$. For each $c \in C_0$ choose a $b_c \in A$ such that $c \subseteq b_c$, and set $B_0 = \{b_c : c \in C_0\}$. Because $\mathfrak{A}$ is ccc, C_0 is countable, so B_0 is also countable. **?** If there is an upper bound e for B_0 which is not an upper bound for A, take $a \in A$ such that $c' = a \setminus e \neq 0$; then $c' \in D$ and $c' \cap c = c' \cap b_c = 0$ for every $c \in C_0$, so $C_0 \cup \{c'\} \in \mathcal{C}$; but C_0 was supposed to be maximal in $\mathcal{C}$. **X** Thus every upper bound for B_0 is also an upper bound for A.

(b) Similarly, there is a countable set $B_1' \subseteq A' = \{1 \setminus a : a \in A\}$ such that every upper bound for B_1' is an upper bound for A'. Set $B_1 = \{1 \setminus b : b \in B_1'\}$; then B_1 is a countable subset of A and every lower bound for B_1 is a lower bound for A. Try $B = B_0 \cup B_1$. Then B is a countable subset of A and every upper (resp. lower) bound for B is an upper (resp. lower) bound for A; so that B must have exactly the same upper and lower bounds as A has.

316F Corollary Let $\mathfrak{A}$ be a ccc Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind σ-complete it is Dedekind complete.

(b) If $A \subseteq \mathfrak{A}$ is sequentially order-closed it is order-closed.

(c) If Q is any partially ordered set and $\phi : \mathfrak{A} \to Q$ is a sequentially order-continuous order-preserving function, it is order-continuous.

(d) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, it is order-continuous.

proof (a) If A is any subset of $\mathfrak{A}$, let $B \subseteq A$ be a countable set with the same upper bounds as A; then $\sup B$ is defined in $\mathfrak{A}$ and must be $\sup A$.

(b) Suppose that $B \subseteq A$ is non-empty and upwards-directed and has a supremum a in $\mathfrak{A}$. Then there is a non-empty countable set $C \subseteq B$ with the same upper bounds as B. Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence running over C. Because B is upwards-directed, we can choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively such that

$$b_0 = c_0, \quad b_{n+1} \in B,\, b_{n+1} \supseteq b_n \cup c_{n+1} \text{ for every } n \in \mathbb{N}.$$

Now any upper bound for $\{b_n : n \in \mathbb{N}\}$ must also be an upper bound for $\{c_n : n \in \mathbb{N}\} = C$, so is an upper bound for the whole set B. But this means that $a = \sup_{n\in\mathbb{N}} b_n$. As $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A, and A is sequentially order-closed, $a \in A$.

In the same way, if $B \subseteq A$ is downwards-directed and has an infimum in $\mathfrak{A}$, this is also the infimum of some non-increasing sequence in B, so must belong to A. Thus A is order-closed.

(c)(i) Suppose that $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum $a_0 \in \mathfrak{A}$. As in (b), there is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with supremum a_0. Because ϕ is sequentially order-continuous, $\phi a_0 = \sup_{n\in\mathbb{N}} \phi c_n$ in Q. But this means that ϕa_0 must be the least upper bound of $\phi[A]$.

(ii) Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum a_0, there is a non-increasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in A with infimum a_0, so that

$$\inf \phi[A] = \inf_{n\in\mathbb{N}} \phi c_n = \phi a_0.$$

Putting this together with (i), we see that ϕ is order-continuous, as claimed.

(d) This is a special case of (c).

316G Definition Let $\mathfrak{A}$ be a Boolean algebra. We say that $\mathfrak{A}$ is **weakly** (σ,∞)**-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$ all with infimum 0 in $\mathfrak{A}$, there is a set $A \subseteq \mathfrak{A}$, also with infimum 0, such that for every $a \in A$, $n \in \mathbb{N}$ there is an $a' \in A_n$ such that $a' \subseteq a$.

316H Remarks (a) Note that for any Boolean algebra $\mathfrak{A}$, and any sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of $\mathfrak{A}$, the set

$$B = \{b : b \in \mathfrak{A}, \forall\, n \in \mathbb{N}\ \exists\ a' \in A_n \text{ such that } a' \subseteq b\}$$

is downwards-directed. **P** If b_1, $b_2 \in B$ and $n \in \mathbb{N}$, there are b_1', $b_2' \in A_n$ such that $b_1 \supseteq b_1'$ and $b_2 \supseteq b_2'$; now there is a $b' \in A_n$ such that $b' \subseteq b_1' \cap b_2'$, and $b_1 \cap b_2 \supseteq b'$. As n is arbitrary, $b_1 \cap b_2 \in B$; as b_1 and b_2 are arbitrary, B is downwards-directed. **Q** Also, B is not empty (as $1 \in B$). If $\mathfrak{A}$ is weakly (σ,∞)-distributive and $\inf A_n = 0$ for every n, then there is an $A \subseteq B$ such that $\inf A = 0$, so in this case $\inf B = 0$.

Thus a Boolean algebra $\mathfrak{A}$ is weakly (σ,∞)-distributive iff

> whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$ all with infimum 0 in $\mathfrak{A}$, there is a downwards-directed set $A \subseteq \mathfrak{A}$, also with infimum 0, such that for every $a \in A$, $n \in \mathbb{N}$ there is an $a' \in A_n$ such that $a' \subseteq a$.

(b) I have expressed the definition above in terms which are adapted to general Boolean algebras. More often than not, however, the algebras we are concerned with will be Dedekind σ-complete, and in this case the condition

> for every $a \in A$, $n \in \mathbb{N}$ there is an $a' \in A_n$ such that $a' \subseteq a$

can be replaced by

> for every $a \in A$ there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \in A_n$ for every n and $\sup_{n\in\mathbb{N}} a_n \subseteq a$;

so that the whole conclusion

> there is a set $A \subseteq \mathfrak{A}$, also with infimum 0, such that for every $a \in A$, $n \in \mathbb{N}$ there is an $a' \in A_n$ such that $a_n \subseteq a$

can be replaced by

$\inf\{\sup_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\} = 0.$

Note that the set

$$B' = \{\sup_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}$$

is also downwards-directed.

316I As usual, a characterization of the property in terms of the Stone spaces is extremely valuable.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then $\mathfrak{A}$ is weakly (σ,∞)-distributive iff every meager set in Z is nowhere dense.

proof (a) Suppose that $\mathfrak{A}$ is weakly (σ,∞)-distributive and that M is a meager subset of Z. Then M can be expressed as $\bigcup_{n\in\mathbb{N}} M_n$ where each M_n is nowhere dense. Set $B_n = \{b : b \in \mathfrak{A}, \widehat{b} \cap M_n = \emptyset\}$. Then $\bigcup_{b\in B_n} \widehat{b} = Z \setminus \overline{M}_n$ is dense in Z, so $\sup B_n = 1$ in $\mathfrak{A}$ (313Ca). Set $A_n = \{1 \setminus b : b \in B_n\}$; then $\inf A_n = 0$ (313Aa). Also B_n is upwards-directed (indeed, closed under $\cup$), so A_n is downwards-directed. Let $A \subseteq \mathfrak{A}$ be a set with infimum 0 such that every member of A includes members of every A_n, and set $B = \{1 \setminus a : a \in A\}$. Then $\sup B = 1$ and every member of B is included in members of every B_n, so that $\widehat{b} \cap M_n = \emptyset$ for every $b \in B$, $n \in \mathbb{N}$. Now $\bigcup_{b\in B} \widehat{b}$ is a dense open subset of Z (313Ca again) disjoint from M, so M is nowhere dense, as claimed.

(b) Suppose that every meager set in Z is nowhere dense, and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed sets in $\mathfrak{A}$, all with infimum 0. Then $M_n = \bigcap_{a\in A_n} \widehat{a}$ is nowhere dense for each n (313Cc), so $M = \bigcup_{n\in\mathbb{N}} M_n$ is meager, therefore nowhere dense. Set $B = \{b : b \in \mathfrak{A}, \widehat{b} \cap M = \emptyset\}$, $A = \{1 \setminus b : b \in B\}$; then $\sup B = 1$ and $\inf A = 0$. If $c \in A$, $n \in \mathbb{N}$ then $\widehat{c} \supseteq M \supseteq M_n$, so

$$(Z \setminus \widehat{c}) \setminus \textstyle\bigcup_{a\in A_n} (Z \setminus \widehat{a}) = M_n \setminus \widehat{c} = \emptyset;$$

because $Z \setminus \widehat{c}$ is compact and $Z \setminus \widehat{a}$ is open for every $a \in A_n$, there must be finitely many $a_0, \dots, a_k \in A_n$ such that $\widehat{c} \supseteq \bigcap_{i\le k} \widehat{a}_i$, that is, $c \supseteq \inf_{i\le k} a_i$. Because A_n is downwards-directed, there must be an $a' \in A_n$ such that $a' \subseteq a_i$ for every $i \le k$, so that $c \supseteq a'$. But this is exactly what is required of A in the definition 316G. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

316J It will be convenient later to have spelt out an elementary inversion of the definition in 316G.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete weakly (σ,∞)-distributive Boolean algebra. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of $\mathfrak{A}$ such that $c_n = \sup A_n$ is defined in $\mathfrak{A}$ for each $n \in \mathbb{N}$, then $\inf_{n\in\mathbb{N}} c_n = \sup\{\inf_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}$.

proof Set $c = \inf_{n\in\mathbb{N}} c_n$, $A = \{\inf_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}$. If $a_n \in A_n$ for every $n \in \mathbb{N}$, then $\inf_{n\in\mathbb{N}} a_n \subseteq c$, just because $a_n \subseteq c_n$ for every n; thus c is an upper bound for A. On the other hand, if $0 \ne d \subseteq c$, $d = \sup\{a \cap d : a \in A_n\}$, so $\inf_{a\in A_n} d \setminus a = 0$, for every n (313Ba, 313Aa). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there must therefore be a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \in A_n$ for every $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} d \setminus a_n \ne d$, that is, $d \cap \sup_{n\in\mathbb{N}} a_n \ne 0$. This shows that $c \setminus d$ cannot be and upper bound for A; as d is arbitrary, $c = \sup A$, as claimed.

316K The regular open algebra of $\mathbb{R}$ For examples of weakly (σ,∞)-distributive algebras, I think we can wait for Chapter 32 (see also 392I). But the standard example of an algebra which is *not* weakly (σ,∞)-distributive is of such importance that (even though it has nothing to do with measure theory, narrowly defined) I think it right to describe it here.

Proposition The algebra $\mathfrak{G}$ of regular open subsets of $\mathbb{R}$ (314O) is not weakly (σ,∞)-distributive.

proof Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, set

$$A_n = \{G : G \in \mathfrak{G},\ q_i \in G \text{ for every } i \le n\}.$$

Then A_n is downwards-directed, and

$$\inf A_n = \operatorname{int} \textstyle\bigcap A_n = \operatorname{int}\{q_i : i \le n\} = \emptyset.$$

But if $A \subseteq \mathfrak{G}$ is such that

for every $n \in \mathbb{N}$, $G \in A$ there is an $H \in A_n$ such that $H \subseteq G$,

then we must have $\mathbb{Q} \subseteq G$ for every $G \in A$, so that

$$\mathbb{R} = \operatorname{int} \overline{\mathbb{Q}} \subseteq \operatorname{int} \overline{G} = G$$

for every $G \in A$, and $A \subseteq \{\mathbb{R}\}$; which means that $\inf A \neq \emptyset$ in $\mathfrak{G}$, and 316G cannot be satisfied.

316L Atoms in Boolean algebras (a) If $\mathfrak{A}$ is a Boolean algebra, an **atom** in $\mathfrak{A}$ is a non-zero $a \in \mathfrak{A}$ such that the only elements included in a are 0 and a.

(b) A Boolean algebra is **atomless** if it has no atoms.

(c) A Boolean algebra is **purely atomic** if every non-zero element includes an atom.

316M Proposition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z.

(a) There is a one-to-one correspondence between atoms a of $\mathfrak{A}$ and isolated points $z \in Z$, given by the formula $\widehat{a} = \{z\}$.

(b) $\mathfrak{A}$ is atomless iff Z has no isolated points.

(c) $\mathfrak{A}$ is purely atomic iff the isolated points of Z form a dense subset of Z.

proof (a)(i) If z is an isolated point in Z, then $\{z\}$ is an open-and-closed subset of Z, so is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$; now if $b \subseteq a$, $\widehat{b}$ must be either $\emptyset$ or $\{z\}$, so b must be either a or 0, and a is an atom.

(ii) If $a \in \mathfrak{A}$ and $\widehat{a}$ has two points z and w, then (because Z is Hausdorff, 311I) there is an open set G containing z but not w. Now there is a $c \in \mathfrak{A}$ such that $z \in \widehat{c} \subseteq G$, so that $a \cap c$ must be different from both 0 and a, and a is not an atom.

(b) This follows immediately from (a).

(c) From (a) we see that $\mathfrak{A}$ is purely atomic iff $\widehat{a}$ contains an isolated point for every non-zero $a \in \mathfrak{A}$; since every non-empty open set in Z includes a non-empty set of the form $\widehat{a}$, this happens iff every non-empty open set in Z contains an isolated point, that is, iff the set of isolated points is dense.

316X Basic exercises >(a) Show that any subalgebra of a ccc Boolean algebra is ccc.

(b) Show that any principal ideal of a ccc Boolean algebra is ccc.

(c) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. Show that $\mathfrak{A}$ is ccc iff every $\mathfrak{A}_i$ is ccc and $\{i : \mathfrak{A}_i \neq \{0\}\}$ is countable.

>(d) Let X be a separable topological space. Show that X is ccc.

>(e) Show that the regular open algebra of a topological space X is ccc iff X is ccc, so that, in particular, the regular open algebra of $\mathbb{R}$ is ccc.

(f) Show that if $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is ccc iff $\mathfrak{B}$ is.

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that it is ccc iff there is no family $\langle a_\xi \rangle_{\xi < \omega_1}$ in $\mathfrak{A}$ such that $a_\xi \subset a_\eta$ whenever $\xi < \eta < \omega_1$.

(h) Let $\mathfrak{A}$ be any Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is ccc iff there is no uncountable disjoint family in $\mathfrak{A} \setminus \mathcal{I}$.

(i) Let $\mathfrak{A}$ be a ccc Boolean algebra. Show that if $\mathcal{I}$ is a σ-ideal of $\mathfrak{A}$, then it is order-closed, and $\mathfrak{A}/\mathcal{I}$ is ccc.

(j) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is ccc; (ii) every σ-ideal of $\mathfrak{A}$ is order-closed; (iii) every σ-subalgebra of $\mathfrak{A}$ is order-closed; (iv) every sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to another Boolean algebra is order-continuous. (*Hint*: 313Q.)

(k) Show that any principal ideal of a weakly (σ,∞)-distributive Boolean algebra is a weakly (σ,∞)-distributive Boolean algebra.

(l) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff every $\mathfrak{A}_i$ is.

>(m) Show that if $\mathfrak{A}$ is a weakly (σ,∞)-distributive Boolean algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is regularly embedded in $\mathfrak{A}$, then $\mathfrak{B}$ is weakly (σ,∞)-distributive.

(n) Show that if $\mathfrak{A}$ is a weakly (σ,∞)-distributive Boolean algebra and $\mathcal{I}$ is an order-closed ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathcal{I}$ is weakly (σ,∞)-distributive.

>(o) (i) Show that if $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $\mathfrak{B}$ is. (ii) Let X be a zero-dimensional compact Hausdorff space. Show that the regular open algebra of X is weakly (σ,∞)-distributive iff the algebra of open-and-closed subsets of X is.

(p) Let $\mathfrak{A}$ be a Boolean algebra. Show that it is weakly (σ,∞)-distributive iff whenever $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a partition C of unity such that for every $c\in C$, $n\in\mathbb{N}$ there is a finite set $I\subseteq C_n$ such that $c\subseteq\sup I$.

>(q) Show that the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$, with its usual topology, is not weakly (σ,∞)-distributive.

(r) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Show that $\mathfrak{A}$ and $\mathfrak{B}$ have the same atoms, so that $\mathfrak{A}$ is atomless, or purely atomic, iff $\mathfrak{B}$ is.

(s) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a regularly embedded subalgebra of $\mathfrak{A}$. Show that (i) every atom of $\mathfrak{A}$ is included in an atom of $\mathfrak{B}$ (ii) if $\mathfrak{A}$ is purely atomic, so is $\mathfrak{B}$ (iii) if $\mathfrak{B}$ is atomless, so is $\mathfrak{A}$.

>(t) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra. Show that it is isomorphic to $\mathcal{P}A$, where A is the set of atoms of $\mathfrak{A}$.

(u) Let $\mathfrak{A}$ be a Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that (i) if $\mathfrak{A}$ is atomless, so is $\mathfrak{A}/\mathcal{I}$ (ii) if $\mathfrak{A}$ is purely atomic, so is $\mathfrak{A}/\mathcal{I}$.

(v) Let $\mathfrak{A}$ be a Boolean algebra. Show that (i) if $\mathfrak{A}$ is atomless, so is every principal ideal of $\mathfrak{A}$ (ii) if $\mathfrak{A}$ is purely atomic, so is every principal ideal of $\mathfrak{A}$.

(w) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}$. Show that (i) $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is (ii) $\mathfrak{A}$ is atomless iff every $\mathfrak{A}_i$ is.

>(x) Show that any purely atomic Boolean algebra is weakly (σ,∞)-distributive.

(y) Let $\mathfrak{A}$ be a Boolean algebra. Show that there is a one-to-one correspondence between atoms a of $\mathfrak{A}$ and order-continuous Boolean homomorphisms $\phi:\mathfrak{A}\to\mathbb{Z}_2$, defined by saying that ϕ corresponds to a iff $\phi(a)=1$.

316Y Further exercises (a) Let I be any set. Show that $\{0,1\}^I$, with its usual topology, is ccc. (*Hint*: show that if $E\subseteq\{0,1\}^I$ is a non-empty open-and-closed set, then $\mu E>0$, where μ is the usual measure on $\{0,1\}^I$.)

(b) Show that the Stone space of the regular open algebra of $\mathbb{R}$ is separable. More generally, show that if a topological space X is separable so is the Stone space of its regular open algebra.

(c) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Show that $\mathfrak{A}$ is ccc iff every nowhere dense subset of Z is included in a nowhere dense zero set.

(d) Let X be a zero-dimensional topological space. Show that X is ccc iff the regular open algebra of X is ccc iff the algebra of open-and-closed subsets of X is ccc.

(e) Set $X = \{0,1\}^{\omega_1}$, and for $\xi < \omega_1$ set $E_\xi = \{x : x \in X,\, x(\xi) = 1\}$. Let Σ be the algebra of subsets of X generated by $\{E_\xi : \xi < \omega_1\} \cup \{\{x\} : x \in X\}$, and $\mathcal{I}$ the σ-ideal of Σ generated by $\{E_\xi \cap E_\eta : \xi < \eta < \omega_1\} \cup \{\{x\} : x \in X\}$. Show that $\Sigma/\mathcal{I}$ is not ccc, but that there is no uncountable disjoint family in $\Sigma \setminus \mathcal{I}$.

(f) Let X be a regular topological space and $\mathfrak{G}$ its regular open algebra. Show that $\mathfrak{G}$ is weakly (σ,∞)-distributive iff every meager set in X is nowhere dense.

(g) Let $\mathfrak{A}$ be a Boolean algebra. $\mathfrak{A}$ is **weakly σ-distributive** if whenever $\langle a_{mn}\rangle_{m,n\in\mathbb{N}}$ is a double sequence in $\mathfrak{A}$ such that $\langle a_{mn}\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 for every $m \in \mathbb{N}$, then

$$\inf\{a : \forall\, m \in \mathbb{N}\, \exists\, n \in \mathbb{N},\ a_{mn} \subseteq a\} = 0.$$

(Dedekind complete weakly σ-distributive algebras are also called **ω^ω-bounding**.) $\mathfrak{A}$ has the **Egorov property** if whenever $\langle a_{mn}\rangle_{m,n\in\mathbb{N}}$ is a double sequence in $\mathfrak{A}$ such that $\langle a_{mn}\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 for every $m \in \mathbb{N}$, then there is a non-increasing sequence $\langle a_m\rangle_{m\in\mathbb{N}}$ such that $\inf_{m\in\mathbb{N}} a_m = 0$ and for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $a_m \supseteq a_{mn}$. (i) Show that if $\mathfrak{A}$ has the Egorov property it is weakly σ-distributive. (ii) Show that if $\mathfrak{A}$ is weakly (σ,∞)-distributive it is weakly σ-distributive. (iii) Show that if $\mathfrak{A}$ is ccc then it is weakly (σ,∞)-distributive iff it has the Egorov property iff it is weakly σ-distributive. (iv) Show that $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ does not have the Egorov property, even though it is weakly (σ,∞)-distributive. (*Hint*: try $a_{mn} = \{f : f(m) \ge n\}$.)

(h) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. (i) Show that $\mathfrak{A}$ is weakly σ-distributive iff the union of any sequence of nowhere dense zero sets in Z is nowhere dense. (ii) Show that $\mathfrak{A}$ has the Egorov property iff the union of any sequence of nowhere dense zero sets in Z is included in a nowhere dense zero set.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete weakly (σ,∞)-distributive Boolean algebra, Z its Stone space, $\mathcal{E}$ the algebra of open-and-closed subsets of Z, $\mathcal{M}$ the σ-ideal of meager subsets of Z, and Σ the Baire property algebra $\{E \triangle M : E \in \mathcal{E},\, M \in \mathcal{M}\}$, as in 314M. (i) Suppose that $f : Z \to \mathbb{R}$ is a Σ-measurable function. Show that there is a dense open set $G \subseteq Z$ such that $f{\restriction}G$ is continuous. (ii) Now suppose that $\mathfrak{A}$ is Dedekind complete. Show that if $f : Z \to \mathbb{R}$ is a function such that $f{\restriction}G$ is continuous for some dense open set $G \subseteq Z$, then f is Σ-measurable; and that if f is also bounded, there is a continuous function $g : Z \to \mathbb{R}$ such that $\{z : f(z) \ne g(z)\}$ is meager. (*Hint*: the graph of g will be the closure of the graph of $f{\restriction}G$; because Z is extremally disconnected, this is the graph of a function.)

(j) (i) Let X be a non-empty separable Hausdorff space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (ii) Let (X,ρ) be a non-empty metric space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (iii) Let I be any infinite set. Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is not weakly (σ,∞)-distributive. Show that the regular open algebra of $\{0,1\}^I$ is not weakly (σ,∞)-distributive.

(k) For any set X, write

$$\mathcal{C}_X = \{I : I \subseteq X \text{ is finite}\} \cup \{X \setminus I : I \subseteq X \text{ is finite}\}.$$

(i) Show that $\mathcal{C}_X$ is an algebra of subsets of X (the **finite-cofinite algebra**). (ii) Show that a Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the finite-cofinite algebra of some set. (iii) Show that a Dedekind σ-complete Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the countable-cocountable algebra of some set (211R).

(l) Let $\langle \mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, none of them $\{0\}$, with free product $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is purely atomic and $\{i : \mathfrak{A}_i \ne \{0,1\}\}$ is finite. (ii) Show that $\mathfrak{A}$ is atomless iff either some $\mathfrak{A}_i$ is atomless or $\{i : \mathfrak{A}_i \ne \{0,1\}\}$ is infinite.

(m) Show that a Boolean algebra is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ iff it is countable, atomless and not $\{0\}$.

(n) Show that a Boolean algebra is isomorphic to the regular open algebra of $\mathbb{R}$ iff it is atomless, Dedekind complete, has a countable order-dense subalgebra and is not $\{0\}$.

(o) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$. Show that there is an injective Boolean homomorphism $\pi : \mathfrak{G} \to \mathcal{P}\mathbb{N}$. (*Hint*: the Stone space of $\mathfrak{G}$ is separable.) Show that there is a Boolean homomorphism $\phi : \mathcal{P}\mathbb{N} \to \mathfrak{G}$ such that $\phi\pi$ is the identity on $\mathcal{P}\mathbb{N}$. (*Hint*: 314K.)

(p) Write $[\mathbb{N}]^{<\omega}$ for the ideal of $\mathcal{P}\mathbb{N}$ consisting of the finite subsets of $\mathbb{N}$. Show that $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is atomless, weakly (σ,∞)-distributive and not ccc.

(q) Let X be a Hausdorff space and $\mathfrak{G}$ its regular open algebra. (i) Show that the atoms of $\mathfrak{G}$ are precisely the sets $\{x\}$ where x is an isolated point in X. (ii) Show that $\mathfrak{G}$ is atomless iff X has no isolated points. (iii) Show that $\mathfrak{G}$ is purely atomic iff the set of isolated points in X is dense in X.

316 Notes and comments The phrase 'countable chain condition' is perhaps unfortunate, since the disjoint sets to which the definition 316A refers could more naturally be called 'antichains'; but there is in fact a connexion between countable chains and countable antichains (316Xg). While some authors speak of the 'countable antichain condition' or 'cac', the term 'ccc' has become solidly established. In the Boolean algebra context, it could equally well be called the 'countable sup property' (316E).

The countable chain condition can be thought of as a restriction on the 'width' of a Boolean algebra; it means that the algebra cannot spread too far laterally (see 316Xc), though it may be indefinitely complex in other ways. Generally it means that in a wide variety of contexts we need look only at countable families and monotonic sequences, rather than arbitrary families and directed sets (316E, 316F, 316Yg). Many of the ideas of 316B-316F have already appeared in 215B; see 322G below.

I remarked in the notes to §313 that the distributive laws described in 313B have important generalizations, of which 'weak (σ,∞)-distributivity' and its cousin 'weak σ-distributivity' (316Yg) are two. They are characteristic of the measure algebras which are the chief subject of this volume. The 'Egorov property' (316Yg) is an alternative formulation applicable to ccc spaces.

Of course every property of Boolean algebras has a reflection in a topological property of their Stone spaces; happily, the concepts of this section correspond to reasonably natural topological expressions (316B, 316I, 316M, 316Yh).

With four new properties (ccc, weakly (σ,∞)-distributive, atomless, purely atomic) to incorporate into the constructions of the last few sections, a very large number of questions can be asked; most are elementary. Any subalgebra of a ccc algebra is ccc (316Xa). All four properties are inherited by order-dense subalgebras and principal ideals (316Xb, 316Xf, 316Xk, 316Xo, 316Xs, 316Xv); with the exception of the countable chain condition (316Xc), they are inherited by simple products (316Xl, 316Xw); with the exception of atomlessness, they are inherited by regularly embedded subalgebras (316Xm, 316Xs), and, in particular, by order-closed subalgebras of Dedekind complete algebras. As for quotient algebras (equivalently, homomorphic images), all four properties are inherited by order-continuous images (316Xi, 316Xn, 316Xu). The countable chain condition is so important that it is worth noting that a sequentially order-continuous image of a ccc algebra is ccc (316Xi), and that there is a useful necessary and sufficient condition for a sequentially order-continuous image of a σ-complete algebra to be ccc (316C, 316D, 316Xh; but see also 316Ye). To see that sequentially order-continuous images do not inherit weak (σ,∞)-distributivity, recall that the regular open algebra of $\mathbb{R}$ is isomorphic to the quotient of the Baire-property algebra $\widehat{\mathcal{B}}$ of $\mathbb{R}$ by the meager ideal $\mathcal{M}$ (314Yd); but that $\widehat{\mathcal{B}}$ is purely atomic (since it contains all singletons), therefore weakly (σ,∞)-distributive (316Xx). Similarly, $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is a non-ccc image of a ccc algebra (316Yp).

The definitions here provide a language in which a remarkably interesting question can be asked: is the free product of ccc Boolean algebras always ccc? equivalently, is the product of ccc topological spaces always ccc? What is special about this question is that it cannot be answered within the ordinary rules of mathematics (even including the axiom of choice); it is undecidable, in the same way that the continuum hypothesis is. I will deal with a variety of undecidable questions in Volume 5; this particular one is treated in Jech 78 and Fremlin 84. Note that the free product of two weakly (σ,∞)-distributive algebras need not be weakly (σ,∞)-distributive (325Yd).

I have taken the opportunity to mention three of the most important of all Boolean algebras: the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316Ym), the regular open algebra of $\mathbb{R}$ (316K, 316Yn) and the quotient

$\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ (316Yp). A fourth algebra which belongs in this company is the Lebesgue measure algebra, which is atomless, ccc and weakly (σ,∞)-distributive (so that every countable subset of its Stone space Z is nowhere dense, and Z is a non-separable ccc space); but for this I wait for the next chapter.

Chapter 32

Measure algebras

I now come to the real work of this volume, the study of the Boolean algebras of equivalence classes of measurable sets. In this chapter I work through the 'elementary' theory, defining this to consist of the parts which do not depend on Maharam's theorem or the lifting theorem or non-trivial set theory.

§321 gives the definition of 'measure algebra', and relates this idea to its origin as the quotient of a σ-algebra of measurable sets by a σ-ideal of negligible sets, both in its elementary properties (following those of measure spaces treated in §112) and in an appropriate version of the Stone representation. §322 deals with the classification of measure algebras according to the scheme already worked out in §211 for measure spaces. §323 discusses the canonical topology and uniformity of a measure algebra. §324 contains results concerning Boolean homomorphisms between measure algebras, with the relationships between topological continuity, order-continuity and preservation of measure. §325 is devoted to the measure algebras of product measures, and their abstract characterization. Finally, §§326-327 address the properties of additive functionals on Boolean algebras, generalizing the ideas of Chapter 23.

321 Measure algebras

I begin by defining 'measure algebra' and relating this concept to the work of Chapter 31 and to the elementary properties of measure spaces.

321A Definition A **measure algebra** is a pair $(\mathfrak{A},\bar{\mu})$, where $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\bar{\mu}:\mathfrak{A}\to[0,\infty]$ is a function such that

$\bar{\mu}0=0$;

whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, $\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)=\sum_{n=0}^{\infty}\bar{\mu}a_n$;

$\bar{\mu}a>0$ whenever $a\in\mathfrak{A}$ and $a\neq 0$.

321B Elementary properties of measure algebras Corresponding to the most elementary properties of measure spaces (112C in Volume 1), we have the following basic properties of measure algebras. Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra.

(a) If a, $b\in\mathfrak{A}$ and $a\cap b=0$ then $\bar{\mu}(a\cup b)=\bar{\mu}a+\bar{\mu}b$. **P** Set $a_0=a$, $a_1=b$, $a_n=0$ for $n\geq 2$; then

$$\bar{\mu}(a\cup b)=\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)=\sum_{n=0}^{\infty}\bar{\mu}a_n=\bar{\mu}a+\bar{\mu}b. \textbf{ Q}$$

(b) If a, $b\in\mathfrak{A}$ and $a\subseteq b$ then $\bar{\mu}a\leq\bar{\mu}b$. **P**

$$\bar{\mu}a\leq\bar{\mu}a+\bar{\mu}(b\setminus a)=\bar{\mu}b. \textbf{ Q}$$

(c) For any a, $b\in\mathfrak{A}$, $\bar{\mu}(a\cup b)\leq\bar{\mu}a+\bar{\mu}b$. **P**

$$\bar{\mu}(a\cup b)=\bar{\mu}a+\bar{\mu}(b\setminus a)\leq\bar{\mu}a+\bar{\mu}b. \textbf{ Q}$$

(d) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)\leq\sum_{n=0}^{\infty}\bar{\mu}a_n$. **P** For each n, set $b_n=a_n\setminus\sup_{i<n}a_i$. Inducing on n, we see that $\sup_{i\leq n}a_i=\sup_{i\leq n}b_i$ for each n, so $\sup_{n\in\mathbb{N}}a_n=\sup_{n\in\mathbb{N}}b_n$ and

$$\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)=\bar{\mu}(\sup_{n\in\mathbb{N}}b_n)=\sum_{n=0}^{\infty}\bar{\mu}b_n\leq\sum_{n=0}^{\infty}\bar{\mu}a_n$$

because $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint. **Q**

(e) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)=\lim_{n\to\infty}\bar{\mu}a_n$. **P** Set $b_0=a_0$, $b_n=a_n\setminus a_{n-1}$ for $n\geq 1$. Then

$$\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \bar\mu(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^{\infty}\bar\mu b_n$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n}\bar\mu b_i = \lim_{n\to\infty}\bar\mu(\sup_{i\le n} b_i) = \lim_{n\to\infty}\bar\mu a_n. \ \mathbf{Q}$$

(f) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}}\bar\mu a_n < \infty$, then $\bar\mu(\inf_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty}\bar\mu a_n$. **P** (Cf. 112Cf.) Set $a = \inf_{n\in\mathbb{N}} a_n$. Take $k\in\mathbb{N}$ such that $\bar\mu a_k < \infty$. Set $b_n = a_k\setminus a_n$ for $n\in\mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\sup_{n\in\mathbb{N}} b_n = a_k\setminus a$ (313Ab). Because $\bar\mu a_k$ is finite,

$$\bar\mu a = \bar\mu a_k - \bar\mu(a_k\setminus a) = \bar\mu a_k - \lim_{n\to\infty}\bar\mu b_n$$

(by (e) above)

$$= \lim_{n\to\infty}\bar\mu(a_k\setminus b_n) = \lim_{n\to\infty}\bar\mu a_n. \ \mathbf{Q}$$

321C Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $A\subseteq\mathfrak{A}$ a non-empty upwards-directed set. If $\sup_{a\in A}\bar\mu a < \infty$, then $\sup A$ is defined in $\mathfrak{A}$ and $\bar\mu(\sup A) = \sup_{a\in A}\bar\mu a$.

proof (Compare 215A.) Set $\gamma = \sup_{a\in A}\bar\mu a$, and for each $n\in\mathbb{N}$ choose $a_n\in A$ such that $\bar\mu a_n \ge \gamma - 2^{-n}$. Next, choose $\langle b_n\rangle_{n\in\mathbb{N}}$ in A such that $b_{n+1}\supseteq b_n\cup a_n$ for each n, and set $b = \sup_{n\in\mathbb{N}} b_n$. Then

$$\bar\mu b = \lim_{n\to\infty}\bar\mu b_n \le \gamma, \quad \bar\mu a_n \le \bar\mu b \text{ for every } n\in\mathbb{N},$$

so $\bar\mu b = \gamma$.

If $a\in A$, then for every $n\in\mathbb{N}$ there is an $a_n'\in A$ such that $a\cup a_n\subseteq a_n'$, so that

$$\bar\mu(a\setminus b) \le \bar\mu(a\setminus a_n) \le \bar\mu(a_n'\setminus a_n) = \bar\mu a_n' - \bar\mu a_n \le \gamma - \bar\mu a_n \le 2^{-n}.$$

This means that $\bar\mu(a\setminus b) = 0$, so $a\setminus b = 0$ and $a\subseteq b$. Accordingly b is an upper bound of A, and is therefore $\sup A$; since we already know that $\bar\mu b = \gamma$, the proof is complete.

321D Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty upwards-directed set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A) = \sup_{a\in A}\bar\mu a$.

proof If $\sup_{a\in A}\bar\mu a = \infty$, this is trivial; otherwise it follows from 321C.

321E Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a disjoint set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A) = \sum_{a\in A}\bar\mu a$.

proof If $A=\emptyset$ then $\sup A = 0$ and the result is trivial. Otherwise, set $B = \{a_0\cup\ldots\cup a_n : a_0,\ldots,a_n\in A$ are distinct$\}$. Then B is upwards-directed, and $\sup_{b\in B}\bar\mu b = \sum_{a\in A}\bar\mu a$ because A is disjoint. Also B has the same upper bounds as A, so $\sup B = \sup A$ and

$$\bar\mu(\sup A) = \bar\mu(\sup B) = \sup_{b\in B}\bar\mu b = \textstyle\sum_{a\in A}\bar\mu a.$$

321F Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty downwards-directed set. If $\inf_{a\in A}\bar\mu a < \infty$, then $\inf A$ is defined in $\mathfrak{A}$ and $\bar\mu(\inf A) = \inf_{a\in A}\bar\mu a$.

proof Take $a_0\in A$ with $\bar\mu a_0 < \infty$, and set $B = \{a_0\setminus a : a\in A\}$. Then B is upwards-directed, and $\sup_{b\in B}\bar\mu b \le \bar\mu a_0 < \infty$, so $\sup B$ is defined. Accordingly $\inf A = a_0\setminus\sup B$ is defined (313Aa), and

$$\bar\mu(\inf A) = \bar\mu a_0 - \bar\mu(\sup B) = \bar\mu a_0 - \sup_{b\in B}\bar\mu b$$

$$= \inf_{b\in B}\bar\mu(a_0\setminus b) = \inf_{a\in A}\bar\mu(a_0\cap a) = \inf_{a\in A}\bar\mu a.$$

321G Subalgebras If $(\mathfrak{A},\bar{\mu})$ is a measure algebra, and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ is a measure algebra. **P** As remarked in 314Eb, $\mathfrak{B}$ is Dedekind σ-complete. If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$, then the supremum $b=\sup_{n\in\mathbb{N}} b_n$ is the same whether taken in $\mathfrak{B}$ or $\mathfrak{A}$, so that we have $\bar{\mu}b=\sum_{n=0}^{\infty}\bar{\mu}b_n$. **Q**

321H The measure algebra of a measure space I introduce the abstract notion of 'measure algebra' because I believe that this is the right language in which to formulate the questions addressed in this volume. However it is very directly linked with the idea of 'measure space', as the next two results show.

Proposition Let (X,Σ,μ) be a measure space, and $\mathcal{N}$ the ideal of μ-negligible sets. Let $\mathfrak{A}$ be the Boolean algebra quotient $\Sigma/\Sigma\cap\mathcal{N}$. Then we have a functional $\bar{\mu}:\mathfrak{A}\to[0,\infty]$ defined by setting

$$\bar{\mu}E^\bullet=\mu E \text{ for every } E\in\Sigma,$$

and $(\mathfrak{A},\bar{\mu})$ is a measure algebra. The canonical map $E\mapsto E^\bullet:\Sigma\to\mathfrak{A}$ is sequentially order-continuous.

proof (a) By 314C, $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra. By 313Qb, $E\mapsto E^\bullet$ is sequentially order-continuous, because $\mathcal{N}\cap\Sigma$ is a σ-ideal of Σ.

(b) If E, $F\in\Sigma$ and $E^\bullet=F^\bullet$ in $\mathfrak{A}$, then $E\triangle F\in\mathcal{N}$, so

$$\mu E\le\mu F+\mu(E\setminus F)=\mu F\le\mu E+\mu(F\setminus E)=\mu E$$

and $\mu E=\mu F$. Accordingly the given formula does indeed define a function $\bar{\mu}:\mathfrak{A}\to[0,\infty]$.

(c) Now

$$\bar{\mu}0=\bar{\mu}\emptyset^\bullet=\mu\emptyset=0.$$

If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, choose for each $n\in\mathbb{N}$ an $E_n\in\Sigma$ such that $E_n^\bullet=a_n$. Set $F_n=E_n\setminus\bigcup_{i<n}E_i$; then

$$F_n^\bullet=E_n^\bullet\setminus\sup_{i<n}E_i^\bullet=a_n\setminus\sup_{i<n}a_i=a_n$$

for each n, so $\bar{\mu}a_n=\mu F_n$ for each n. Now set $E=\bigcup_{n\in\mathbb{N}}E_n=\bigcup_{n\in\mathbb{N}}F_n$; then $E^\bullet=\sup_{n\in\mathbb{N}}F_n^\bullet=\sup_{n\in\mathbb{N}}a_n$. So

$$\bar{\mu}(\sup_{n\in\mathbb{N}}a_n)=\mu E=\sum_{n=0}^{\infty}\mu F_n=\sum_{n=0}^{\infty}\bar{\mu}a_n.$$

Finally, if $a\ne0$, then there is an $E\in\Sigma$ such that $E^\bullet=a$, and $E\notin\mathcal{N}$, so $\bar{\mu}a=\mu E>0$. Thus $(\mathfrak{A},\bar{\mu})$ is a measure algebra.

321I Definition For any measure space (X,Σ,μ) I will call $(\mathfrak{A},\bar{\mu})$, as constructed above, the **measure algebra of** (X,Σ,μ).

321J The Stone representation of a measure algebra Just as with Dedekind σ-complete Boolean algebras (314N), every measure algebra is obtainable from the construction above.

Theorem Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra. Then it is isomorphic, as measure algebra, to the measure algebra of some measure space.

proof (a) We know from 314M that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a quotient algebra $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of the Stone space Z of $\mathfrak{A}$, and $\mathcal{M}$ is the ideal of meager subsets of Z. Let $\pi:\Sigma/\mathcal{M}\to\mathfrak{A}$ be the canonical isomorphism, and set $\theta E=\pi E^\bullet$ for each $E\in\Sigma$; then $\theta:\Sigma\to\mathfrak{A}$ is a sequentially order-continuous surjective Boolean homomorphism with kernel $\mathcal{M}$.

(b) For $E\in\Sigma$, set

$$\nu E=\bar{\mu}(\theta E).$$

Then (Z,Σ,ν) is a measure space. **P** (i) We know already that Σ is a σ-algebra of subsets of Z. (ii)

$$\nu\emptyset=\bar{\mu}(\theta\emptyset)=\bar{\mu}0=0.$$

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, then (because θ is a Boolean homomorphism) $\langle\theta E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and (because θ is sequentially order-continuous) $\theta(\bigcup_{n\in\mathbb{N}}E_n)=\sup_{n\in\mathbb{N}}\theta E_n$; so

$$\nu(\bigcup_{n\in\mathbb{N}} E_n) = \bar{\mu}(\sup_{n\in\mathbb{N}} \theta E_n) = \sum_{n=0}^{\infty} \bar{\mu}(\theta E_n) = \sum_{n=0}^{\infty} \nu E_n. \textbf{ Q}$$

(c) For $E \in \Sigma$,

$$\nu E = 0 \iff \bar{\mu}(\theta E) = 0 \iff \theta E = 0 \iff E \in \mathcal{M}.$$

So the measure algebra of (Z, Σ, ν) is just $\Sigma/\mathcal{M}$, with

$$\bar{\nu} E^\bullet = \nu E = \bar{\mu}(\theta E) = \bar{\mu}(\pi E^\bullet)$$

for every $E \in \Sigma$. Thus the Boolean algebra isomorphism π is also an isomorphism between the measure algebras $(\Sigma/\mathcal{M}, \bar{\nu})$ and $(\mathfrak{A}, \bar{\mu})$, and $(\mathfrak{A}, \bar{\mu})$ is represented in the required form.

321K Definition I will call the measure space (Z, Σ, ν) constructed in the proof of 321J the **Stone space** of the measure algebra $(\mathfrak{A}, \bar{\mu})$.

For later reference, I repeat the description of this space as developed in 311E, 311I, 314M and 321J. Z is a compact Hausdorff space, being the Stone space of $\mathfrak{A}$. $\mathfrak{A}$ can be identified with the algebra of open-and-closed sets in Z. The ideal of ν-negligible sets coincides with the ideal of meager subsets of Z; in particular, ν is complete. The measurable sets are precisely those expressible in the form $E = \widehat{a} \triangle M$ where $a \in \mathfrak{A}$, $\widehat{a} \subseteq Z$ is the corresponding open-and-closed set, and M is meager; in this case $\nu E = \bar{\mu} a$ and $a = \theta E$ is the member of $\mathfrak{A}$ corresponding to E.

For the most important classes of measure algebras, more can be said; see 322M *et seq.* below.

321X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $a \in \mathfrak{A}$. Show that $(\mathfrak{A}_a, \bar{\mu} \restriction \mathfrak{A}_a)$ is a measure algebra, writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(b) Let $(X, \Sigma, \bar{\mu})$ be a measure space, and $\mathfrak{A}$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{E^\bullet : E \in \mathrm{T}\}$ is a σ-subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$ then $\{E : E \in \Sigma,\ E^\bullet \in \mathfrak{B}\}$ is a σ-subalgebra of Σ.

321Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $I \lhd \mathfrak{A}$ a σ-ideal. For $u \in \mathfrak{A}/I$ set $\bar{\mu} u = \inf\{\bar{\mu} a : a \in \mathfrak{A},\ a^\bullet = u\}$. Show that $(\mathfrak{A}, \bar{\mu})$ is a measure algebra.

321 Notes and comments The idea behind taking the quotient $\Sigma/\mathcal{N}$, where Σ is the algebra of measurable sets and $\mathcal{N}$ is the ideal of negligible sets, is just that if negligible sets can be ignored – as is the case for a very large proportion of the results of measure theory – then two measurable sets can be counted as virtually the same if they differ by a negligible set, that is, if they represent the same member of the measure algebra. The definition in 321A is designed to be an exact characterization of these quotient algebras, taking into account the measures with which they are endowed. In the course of the present chapter I will work through many of the basic ideas dealt with in Volumes 1 and 2 to show how they can be translated into theorems about measure algebras, as I have done in 321B-321F. It is worth checking these correspondences carefully, because some of the ideas mutate significantly in translation. In measure algebras, it becomes sensible to take seriously the suprema and infima of uncountable sets (see 321C-321F).

I should perhaps remark that while the Stone representation (321J-321K) is significant, it is not the most important method of representing measure algebras, which is surely Maharam's theorem, to be dealt with in the next chapter. Nevertheless, the Stone representation is a canonical one, and will appear at each point that we meet a new construction involving measure algebras, just as the ordinary Stone representation of Boolean algebras can be expected to throw light on any aspect of Boolean algebra.

322 Taxonomy of measure algebras

Before going farther with the general theory of measure algebras, I run through those parts of the classification of measure spaces in §211 which have expressions in terms of measure algebras. The most important concepts at this stage are those of 'semi-finite', 'localizable' and 'σ-finite' measure algebra (322Ac-322Ae); these correspond exactly to the same terms applied to measure spaces (322B). I briefly investigate the Boolean-algebra properties of semi-finite and σ-finite measure algebras (322F, 322G), with mentions of completions and c.l.d. versions (322D), subspace measures (322I-322J), direct sums of measure spaces (322K, 322L) and subalgebras of measure algebras (322M). It turns out that localizability of a measure algebra is connected in striking ways to the properties of the canonical measure on its Stone space (322N). I end the section with a description of the 'localization' of a semi-finite measure algebra (322O-322P) and with some further properties of Stone spaces (322Q).

322A Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that $(\mathfrak{A},\bar\mu)$ is a **probability algebra** if $\bar\mu 1 = 1$.

(b) $(\mathfrak{A},\bar\mu)$ is **totally finite** if $\bar\mu 1 < \infty$.

(c) $(\mathfrak{A},\bar\mu)$ is **σ-finite** if there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar\mu a_n < \infty$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} a_n = 1$. Note that in this case $\langle a_n\rangle_{n\in\mathbb{N}}$ can be taken *either* to be non-decreasing (consider $a'_n = \sup_{i<n} a_i$) *or* to be disjoint (consider $a''_n = a_n \setminus a'_n$).

(d) $(\mathfrak{A},\bar\mu)$ is **semi-finite** if whenever $a \in \mathfrak{A}$ and $\bar\mu a = \infty$ there is a non-zero $b \subseteq a$ such that $\bar\mu b < \infty$.

(e) $(\mathfrak{A},\bar\mu)$ is **localizable** or **Maharam** if it is semi-finite and the Boolean algebra $\mathfrak{A}$ is Dedekind complete.

322B The first step is to relate these concepts to the corresponding ones for measure spaces.

Theorem Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then

(a) (X,Σ,μ) is a probability space iff $(\mathfrak{A},\bar\mu)$ is a probability algebra;
(b) (X,Σ,μ) is totally finite iff $(\mathfrak{A},\bar\mu)$ is;
(c) (X,Σ,μ) is σ-finite iff $(\mathfrak{A},\bar\mu)$ is;
(d) (X,Σ,μ) is semi-finite iff $(\mathfrak{A},\bar\mu)$ is;
(e) (X,Σ,μ) is localizable iff $(\mathfrak{A},\bar\mu)$ is;
(f) if $E \in \Sigma$, then E is an atom for μ iff $E^\bullet$ is an atom in $\mathfrak{A}$;
(g) (X,Σ,μ) is atomless iff $\mathfrak{A}$ is;
(h) (X,Σ,μ) is purely atomic iff $\mathfrak{A}$ is.

proof (a), (b) are trivial, since $\bar\mu 1 = \mu X$.

(c)(i) If μ is σ-finite, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of finite measure covering X; then $\bar\mu E_n^\bullet < \infty$ for every n, and

$$\sup_{n\in\mathbb{N}} E_n^\bullet = (\bigcup_{n\in\mathbb{N}} E_n)^\bullet = 1,$$

so $(\mathfrak{A},\bar\mu)$ is σ-finite.

(ii) If $(\mathfrak{A},\bar\mu)$ is σ-finite, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar\mu a_n < \infty$ for every n and $\sup_{n\in\mathbb{N}} a_n = 1$. For each n, choose $E_n \in \Sigma$ such that $E_n^\bullet = a_n$. Set $E = \bigcup_{n\in\mathbb{N}} E_n$; then $E^\bullet = \sup_{n\in\mathbb{N}} a_n = 1$, so E is conegligible. Now $(X\setminus E, E_0, E_1, \dots)$ is a sequence of sets of finite measure covering X, so μ is σ-finite.

(d)(i) Suppose that μ is semi-finite and that $a \in \mathfrak{A}$, $\bar\mu a = \infty$. Then there is an $E \in \Sigma$ such that $E^\bullet = a$, so that $\mu E = \bar\mu a = \infty$. As μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Set $b = F^\bullet$; then $b \subseteq a$ and $0 < \bar\mu b < \infty$.

(ii) Suppose that $(\mathfrak{A},\bar\mu)$ is semi-finite and that $E \in \Sigma$, $\mu E = \infty$. Then $\bar\mu E^\bullet = \infty$, so there is a $b \subseteq E^\bullet$ such that $0 < \bar\mu b < \infty$. Let $F \in \Sigma$ be such that $F^\bullet = b$. Then $F \cap E \in \Sigma$, $F \cap E \subseteq E$ and $(F\cap E)^\bullet = E^\bullet \cap b = b$, so that $\mu(F\cap E) = \bar\mu b \in \,]0,\infty[$.

(e)(i) Note first that if $\mathcal{E} \subseteq \Sigma$ and $F \in \Sigma$, then

$$\begin{aligned} &E \setminus F \text{ is negligible for every } E \in \mathcal{E} \\ &\iff E^\bullet \subseteq F^\bullet = 0 \text{ for every } E \in \mathcal{E} \\ &\iff F^\bullet \text{ is an upper bound for } \{E^\bullet : E \in \mathcal{E}\}. \end{aligned}$$

So if $\mathcal{E} \subseteq \Sigma$ and $H \in \Sigma$, then H is an essential supremum of $\mathcal{E}$ in Σ, in the sense of 211G, iff $H^\bullet$ is the supremum of $A = \{E^\bullet : E \in \mathcal{E}\}$ in $\mathfrak{A}$. **P** Writing $\mathcal{F}$ for

$$\{F : F \in \Sigma,\ E \setminus F \text{ is negligible for every } E \in \mathcal{E}\},$$

we see that $B = \{F^\bullet : F \in \mathcal{F}\}$ is just the set of upper bounds of A, and that H is an essential supremum of $\mathcal{E}$ iff $H \in \mathcal{F}$ and $H^\bullet$ is a lower bound for B; that is, iff $H^\bullet = \sup A$. **Q**

(ii) Thus $\mathfrak{A}$ is Dedekind complete iff every family in Σ has an essential supremum in Σ. Since we already know that $(\mathfrak{A}, \bar\mu)$ is semi-finite iff μ is, we see that $(\mathfrak{A}, \bar\mu)$ is localizable iff μ is.

(f) This is immediate from the definitions in 211I and 316L, if we remember always that $\{b : b \subseteq E^\bullet\} = \{F^\bullet : F \in \Sigma,\ F \subseteq E\}$ (312Kb).

(g), (h) follow at once from (f).

322C I copy out the relevant parts of Theorem 211L in the new context.

Theorem (a) A probability algebra is totally finite.
(b) A totally finite measure algebra is σ-finite.
(c) A σ-finite measure algebra is localizable.
(d) A localizable measure algebra is semi-finite.

proof All except (c) are trivial; and (c) may be deduced from 211Lc-211Ld, 322Bc, 322Be and 321J, or from 316Fa and 322G below.

322D Of course not all the definitions in §211 are directly relevant to measure algebras. The concepts of 'complete', 'locally determined' and 'strictly localizable' measure space do not correspond in any direct way to properties of the measure algebras. Indeed, completeness is just irrelevant, as the next proposition shows.

Proposition Let (X, Σ, μ) be a measure space, with completion $(X, \hat\Sigma, \hat\mu)$ and c.l.d. version $(X, \tilde\Sigma, \tilde\mu)$ (213E). Write $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$ for the measure algebras of μ, $\hat\mu$ and $\tilde\mu$ respectively.
(a) The embedding $\Sigma \subseteq \hat\Sigma$ corresponds to an isomorphism between $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{A}_1, \bar\mu_1)$.
(b)(i) The embedding $\Sigma \subseteq \tilde\Sigma$ defines an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_2$. Setting $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$, $\pi\restriction\mathfrak{A}^f$ is a measure-preserving bijection between $\mathfrak{A}^f$ and $\mathfrak{A}_2^f = \{c : c \in \mathfrak{A}_2,\ \bar\mu_2 c < \infty\}$.
(ii) π is injective iff μ is semi-finite, and in this case $\bar\mu_2(\pi a) = \bar\mu a$ for every $a \in \mathfrak{A}$.
(iii) If μ is localizable, π is a bijection.

proof For $E \in \Sigma$, I write E° for its image in $\mathfrak{A}$; for $F \in \hat\Sigma$, I write F^* for its image in $\mathfrak{A}_1$; and for $G \in \tilde\Sigma$, I write $F^\bullet$ for its image in $\mathfrak{A}_2$.

(a) This is nearly trivial. The map $E \mapsto E^* : \Sigma \to \mathfrak{A}_1$ is a Boolean homomorphism, being the composition of the Boolean homomorphisms $E \mapsto E : \Sigma \to \hat\Sigma$ and $F \mapsto F^* : \hat\Sigma \to \mathfrak{A}_1$. Its kernel is $\{E : E \in \Sigma,\ \hat\mu E = 0\} = \{E : E \in \Sigma,\ \mu E = 0\}$, so it induces an injective Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}_1$ given by the formula $\phi(E^\circ) = E^*$ for every $E \in \Sigma$ (312F, 3A2G). To see that ϕ is surjective, take any $b \in \mathfrak{A}_1$. There is an $F \in \hat\Sigma$ such that $F^* = b$, and there is an $E \in \Sigma$ such that $E \subseteq F$ and $\hat\mu(F \setminus E) = 0$, so that

$$\pi(E^\circ) = E^* = F^* = b.$$

Thus π is a Boolean algebra isomorphism. It is a measure algebra isomorphism because for any $E \in \Sigma$

$$\bar\mu_1\phi(E^\circ) = \bar\mu_1 E^* = \hat\mu E = \mu E = \bar\mu E^\circ.$$

(b)(i) The map $E \mapsto E^\bullet : \Sigma \to \mathfrak{A}_2$ is a Boolean homomorphism with kernel $\{E : E \in \Sigma,\ \tilde{\mu}E = 0\} \supseteq \{E : E \in \Sigma,\ \mu E = 0\}$, so induces a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_2$, defined by saying that $\pi E^\circ = E^\bullet$ for every $E \in \Sigma$.

If $a \in \mathfrak{A}^f$, it is expressible as E° where $\mu E < \infty$. Then $\tilde{\mu}E = \mu E$ (213Fa), so $\pi a = E^\bullet$ belongs to $\mathfrak{A}_2^f$, and $\bar{\mu}_2(\pi a) = \bar{\mu}a$. If a, a' are distinct members of $\mathfrak{A}^f$, then

$$\bar{\mu}_2(\pi a \triangle \pi a') = \bar{\mu}_2\pi(a \triangle a') = \bar{\mu}(a \triangle a') > 0,$$

so $\pi a \neq \pi a'$; thus $\pi{\restriction}\mathfrak{A}^f$ is an injective map from $\mathfrak{A}^f$ to $\mathfrak{A}_2^f$. If $c \in \mathfrak{A}_2^f$, then $c = G^\bullet$ where $\tilde{\mu}G < \infty$; by 213Fc, there is an $E \in \Sigma$ such that $E \subseteq G$, $\mu E = \tilde{\mu}G$ and $\tilde{\mu}(G \setminus E) = 0$, so that $E^\circ \in \mathfrak{A}^f$ and

$$\pi E^\circ = E^\bullet = G^\bullet = c.$$

As c is arbitrary, $\phi[\mathfrak{A}^f] = \mathfrak{A}_2^f$.

Finally, π is order-continuous. **P** Let $A \subseteq \mathfrak{A}$ be a non-empty downwards-directed set with infimum 0, and $b \in \mathfrak{A}_2$ a lower bound for $\pi[A]$. **?** If $b \neq 0$, then (because $(\mathfrak{A}_2, \bar{\mu}_2)$ is semi-finite) there is a $b_0 \in \mathfrak{A}_2^f$ such that $0 \neq b_0 \subseteq b$. Let $a_0 \in \mathfrak{A}$ be such that $\pi a_0 = b_0$. Then $a_0 \neq 0$, so there is an $a \in A$ such that $a \not\supseteq a_0$, that is, $a \cap a_0 \neq a_0$. But now, because $\pi{\restriction}\mathfrak{A}^f$ is injective,

$$b_0 = \pi a_0 \neq \pi(a \cap a_0) = \pi a \cap \pi a_0 = \pi a \cap b_0,$$

and $b_0 \not\subseteq \pi a$, which is impossible. **X** Thus $b = 0$, and 0 is the only lower bound of $\pi[A]$. As A is arbitrary, π is order-continuous (313L(b-ii)). **Q**

(ii) (α) If μ is semi-finite, then $\tilde{\mu}E = \mu E$ for every $E \in \Sigma$ (213Hc), so

$$\bar{\mu}_2(\pi E^\circ) = \bar{\mu}_2 E^\bullet = \tilde{\mu}E = \mu E = \bar{\mu}E^\circ$$

for every $E \in \Sigma$. In particular,

$$\pi a = 0 \Longrightarrow 0 = \bar{\mu}_2(\pi a) = \bar{\mu}a \Longrightarrow a = 0,$$

so π is injective. (β) If μ is not semi-finite, there is an $E \in \Sigma$ such that $\mu E = \infty$ but $\mu H = 0$ whenever $H \in \Sigma$, $H \subseteq E$ and $\mu H < \infty$; so that $\tilde{\mu}E = 0$ and

$$E^\circ \neq 0, \quad \pi E^\circ = E^\bullet = 0.$$

So in this case π is not injective.

(iii) Now suppose that μ is localizable. Then for every $G \in \tilde{\Sigma}$ there is an $E \in \Sigma$ such that $\tilde{\mu}(E \triangle G) = 0$, by 213Hb; accordingly $\pi E^\circ = E^\bullet = G^\bullet$. As G is arbitrary, π is surjective; and we know from (ii) that π is injective, so it is a bijection, as claimed.

322E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff it has a partition of unity consisting of elements of finite measure.

(b) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $a = \sup\{b : b \subseteq a,\ \bar{\mu}b < \infty\}$ and $\bar{\mu}a = \sup\{\bar{\mu}b : b \subseteq a,\ \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$.

proof Set $\mathfrak{A}^f = \{b : b \in \mathfrak{A},\ \bar{\mu}b < \infty\}$.

(a)(i) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then $\mathfrak{A}^f$ is order-dense in $\mathfrak{A}$, so there is a partition of unity consisting of members of $\mathfrak{A}^f$ (313K).

(ii) If there is a partition of unity $C \subseteq \mathfrak{A}^f$, and $\bar{\mu}a = \infty$, then there is a $c \in C$ such that $a \cap c \neq 0$, and now $a \cap c \subseteq a$ and $0 < \bar{\mu}(a \cap c) < \infty$; as a is arbitrary, $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

(b) Of course $\mathfrak{A}^f$ is upwards-directed, by 321Bc, and we are supposing that its supremum is 1. If $a \in \mathfrak{A}$, then

$$B = \{b : b \in \mathfrak{A}^f,\ b \subseteq a\} = \{a \cap b : b \in \mathfrak{A}^f\}$$

is upwards-directed and has supremum a (313Ba), so $\bar{\mu}a = \sup_{b \in B} \bar{\mu}b$, by 321D.

Remark Compare 213A.

322F Proposition If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}$ is a weakly (σ, ∞)-distributive Boolean algebra.

proof Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set

$$A = \{\sup_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}.$$

If $c \in \mathfrak{A}\setminus\{0\}$, let $b \subseteq c$ be such that $0 < \bar\mu b < \infty$. For each $n \in \mathbb{N}$, $\inf_{a\in A_n} \bar\mu(b\cap a_n) = 0$, by 321F; so we may choose $a_n \in A_n$ such that $\bar\mu(b\cap a_n) \le 2^{-n-2}\bar\mu b$. Set $a = \sup_{n\in\mathbb{N}} a_n \in A$. Then

$$\bar\mu(b\cap a) \le \sum_{n=0}^{\infty} \bar\mu(b\cap a_n) < \bar\mu b,$$

so $b \not\subseteq a$ and $c \not\subseteq a$. As c is arbitrary, $\inf A = 0$; as $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

322G Corresponding to 215B, we have the following description of σ-finite algebras.

Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then the following are equiveridical:
(i) $(\mathfrak{A},\bar\mu)$ is σ-finite;
(ii) $\mathfrak{A}$ is ccc;
(iii) *either* $\mathfrak{A} = \{0\}$ *or* there is a functional $\bar\nu : \mathfrak{A} \to [0,1]$ such that $(\mathfrak{A},\bar\nu)$ is a probability algebra.

proof (i) $\iff$ (ii) By 321J, it is enough to consider the case in which $(\mathfrak{A},\bar\mu)$ is the measure algebra of a measure space (X,Σ,μ), and μ is semi-finite, by 322Bd. We know that $\mathfrak{A}$ is ccc iff there is no uncountable disjoint set in $\Sigma\setminus\mathcal{N}$, where $\mathcal{N}$ is the ideal of negligible sets (316D). But 215B(iii) shows that this is equivalent to μ being σ-finite, which is equivalent to $(\mathfrak{A},\bar\mu)$ being σ-finite, by 322Bc.

(i)$\Rightarrow$(iii) If $(\mathfrak{A},\bar\mu)$ is σ-finite, and $\mathfrak{A} \ne \{0\}$, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$ such that $\bar\mu a_n < \infty$ for every n and $\sup_{n\in\mathbb{N}} a_n = 1$. Then $\bar\mu a_n > 0$ for some n, so there are $\gamma_n > 0$ such that $\sum_{n=0}^{\infty}\gamma_n\bar\mu a_n = 1$. (Set $\gamma'_n = 2^{-n}/(1+\bar\mu a_n)$, $\gamma_n = \gamma'_n/(\sum_{i=0}^{\infty}\gamma'_i\bar\mu a_i)$.) Set $\nu a = \sum_{n=0}^{\infty}\gamma_n\bar\mu(a\cap a_n)$ for every $a \in \mathfrak{A}$; it is easy to check that $(\mathfrak{A},\bar\nu)$ is a probability algebra.

(iii)$\Rightarrow$(i) is a consequence of (ii)$\Rightarrow$(i).

322H Principal ideals If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $a \in \mathfrak{A}$, then it is easy to see (using 314Eb) that $(\mathfrak{A}_a,\bar\mu{\restriction}\mathfrak{A}_a)$ is a measure algebra, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

322I Subspace measures General subspace measures give rise to complications in the measure algebra (see 322Xg, 322Yd). But subspaces with measurable envelopes (132D, 213K) are manageable.

Proposition Let (X,Σ,μ) be a measure space, and $A \subseteq X$ a set with a measurable envelope E. Let μ_A be the subspace measure on A, and Σ_A its domain; let $(\mathfrak{A},\bar\mu)$ be the measure algebra of (X,Σ,μ) and $(\mathfrak{A}_A,\bar\mu_A)$ the measure algebra of (A,Σ_A,μ_A). Set $a = E^\bullet$ and let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then we have an isomorphism between $(\mathfrak{A}_a,\bar\mu{\restriction}\mathfrak{A}_a)$ and $(\mathfrak{A}_A,\bar\mu_A)$ given by the formula

$$F^\bullet \mapsto (F\cap A)^\circ$$

whenever $F \in \Sigma$ and $F \subseteq E$, writing $F^\bullet$ for the equivalence class of F in $\mathfrak{A}$ and $(F\cap A)^\circ$ for the equivalence class of $F\cap A$ in $\mathfrak{A}_A$.

proof Set $\Sigma_E = \{E\cap F : F \in \Sigma\}$. For F, $G \in \Sigma_E$,

$$F^\bullet = G^\bullet \iff \mu(F\triangle G) = 0 \iff \mu_A(A\cap(F\triangle G)) = 0 \iff (F\cap A)^\circ = (G\cap A)^\circ,$$

because E is a measurable envelope of A. Accordingly the given formula defines an injective function from the image $\{F^\bullet : F \in \Sigma_E\}$ of Σ_E in $\mathfrak{A}$ to $\mathfrak{A}_A$; but this image is just the principal ideal $\mathfrak{A}_a$. It is easy to check that the map is a Boolean homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}_A$, and it is a Boolean isomorphism because $\Sigma_A = \{F\cap A : F \in \Sigma_E\}$. Finally, it is measure-preserving because

$$\bar\mu F^\bullet = \mu F = \mu^*(F\cap A) = \mu_A(F\cap A) = \bar\mu_A(F\cap A)^\circ$$

for every $F \in \Sigma_E$, again using the fact that E is a measurable envelope of A.

322J Corollary Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar\mu)$.
(a) If $E \in \Sigma$, then the measure algebra of the subspace measure μ_E can be identified with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$.
(b) If $A \subseteq X$ is a set of full outer measure (in particular, if $\mu^*A = \mu X < \infty$), then the measure algebra of the subspace measure μ_A can be identified with $\mathfrak{A}$.

322K Simple products (a) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be any indexed family of measure algebras. Let $\mathfrak{A}$ be the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$ (315A), and for $a\in\mathfrak{A}$ set $\bar{\mu}a=\sum_{i\in I}\bar{\mu}_ia(i)$. Then it is easy to check (using 315D(e-ii)) that $(\mathfrak{A},\bar{\mu})$ is a measure algebra; I will call it the **simple product** of the family $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$. Each of the $\mathfrak{A}_i$ corresponds to a principal ideal $\mathfrak{A}_{e_i}$ say in $\mathfrak{A}$, where $e_i\in\mathfrak{A}$ corresponds to $1_{\mathfrak{A}_i}\in\mathfrak{A}_i$ (315E), and the Boolean isomorphism between $\mathfrak{A}_i$ and $\mathfrak{A}_{e_i}$ is a measure algebra isomorphism between $(\mathfrak{A}_i,\bar{\mu}_i)$ and $(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$.

(b) If $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of measure spaces, with direct sum (X,Σ,μ) (214K), then the measure algebra $(\mathfrak{A},\bar{\mu})$ of (X,Σ,μ) can be identified with the simple product of the measure algebras $(\mathfrak{A}_i,\bar{\mu}_i)$ of the (X_i,Σ_i,μ_i). **P** If, as in 214K, we set $X=\{(x,i):i\in I,\,x\in X_i\}$, and for $E\subseteq X$, $i\in I$ we set $E_i=\{x:(x,i)\in E\}$, then the Boolean isomorphism $E\mapsto\langle E_i\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ induces a Boolean isomorphism from $\mathfrak{A}$ to $\prod_{i\in I}\mathfrak{A}_i$, which is also a measure algebra isomorphism, because

$$\bar{\mu}E^\bullet=\mu E=\sum_{i\in I}\mu_iE_i=\sum_{i\in I}\bar{\mu}_iE_i^\bullet$$

for every $E\in\Sigma$. **Q**

(c) A product of measure algebras is semi-finite, or localizable, or atomless, or purely atomic, iff every factor is. (Compare 214Jb.)

(d) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra.

(i) If $\langle e_i\rangle_{i\in I}$ is any partition of unity in $\mathfrak{A}$, then $(\mathfrak{A},\bar{\mu})$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals. **P** By 315F(iii), the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_i$. Because $\langle e_i\rangle_{i\in I}$ is disjoint and $a=\sup_{i\in I}a\cap e_i$, $\bar{\mu}a=\sum_{i\in I}\bar{\mu}(a\cap e_i)$ (321E), for every $a\in\mathfrak{A}$. So $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a measure algebra isomorphism between $(\mathfrak{A},\bar{\mu})$ and $\prod_{i\in I}(\mathfrak{A}_i,\bar{\mu}\restriction\mathfrak{A}_{e_i})$. **Q**

(ii) In particular, since $\mathfrak{A}$ has a partition of unity consisting of elements of finite measure (322Ea), $(\mathfrak{A},\bar{\mu})$ is isomorphic to a simple product of totally finite measure algebras. Each of these is isomorphic to the measure algebra of a totally finite measure space, so $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra of a direct sum of totally finite measure spaces, which is strictly localizable.

Thus every localizable measure algebra is isomorphic to the measure algebra of a strictly localizable measure space. (See also 322N below.)

***322L Strictly localizable spaces** The following fact is occasionally useful.

Proposition Let (X,Σ,μ) be a strictly localizable measure space with $\mu X>0$, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, there is a disjoint family $\langle X_i\rangle_{i\in I}$ in Σ, with union X, such that $X_i^\bullet=a_i$ for every $i\in I$ and

$$\Sigma=\{E:E\subseteq X,\,E\cap X_i\in\Sigma\;\forall\,i\in I,$$

$$\mu E=\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma;$$

that is, the isomorphism between $\mathfrak{A}$ and the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ of its principal ideals (315F) corresponds to an isomorphism between (X,Σ,μ) and the direct sum of the subspace measures on X_i.

proof (a) Suppose to begin with that $\mu X<\infty$. In this case $J=\{i:a_i\ne0\}$ must be countable (322G). For each $i\in J$, choose $E_i\in\Sigma$ such that $E_i^\bullet=a_i$, and set $F_i=E_i\setminus\bigcup_{j\in J,j\ne i}E_j$; then $F_i^\bullet=a_i$ for each $i\in J$, and $\langle F_i\rangle_{i\in J}$ is disjoint. Because $\mu X>0$, J is non-empty; fix some $j_0\in J$ and set

$$\begin{aligned}X_i&=F_{j_0}\cup(X\setminus\bigcup_{j\in J}F_j)\text{ if }i=j_0,\\&=F_i\text{ for }i\in J\setminus\{j_0\},\\&=\emptyset\text{ for }i\in I\setminus J.\end{aligned}$$

Then $\langle X_i\rangle_{i\in I}$ is a disjoint family in Σ, $\bigcup_{i\in I}X_i=X$ and $X_i^\bullet=a_i$ for every i. Moreover, because only countably many of the X_i are non-empty, we certainly have

$$\Sigma=\{E:E\subseteq X,\,E\cap X_i\in\Sigma\;\forall\,i\in I,$$

$$\mu E = \sum_{i\in I}\mu(E\cap X_i) \text{ for every } E\in\Sigma.$$

(b) For the general case, start by taking a decomposition $\langle Y_j\rangle_{j\in J}$ of X. We can suppose that no Y_j is negligible, because there is certainly some j_0 such that $\mu Y_{j_0} > 0$, and we can if necessary replace Y_{j_0} by $Y_{j_0}\cup\bigcup\{Y_j : \mu Y_j = 0\}$. For each j, we can identify the measure algebra of the subspace measure on Y_j with the principal ideal $\mathfrak{A}_{b_j}$ generated by $b_j = Y_j^\bullet$ (322I). Now $\langle a_i\cap b_j\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_{b_j}$, so by (a) just above we can find a disjoint family $\langle X_{ji}\rangle_{i\in I}$ in Σ such that $\bigcup_{i\in I}X_{ji} = Y_j$, $X_{ji}^\bullet = a_i\cap b_j$ for every i and

$$\Sigma\cap\mathcal{P}Y_j = \{E : E\subseteq Y_j,\ E\cap X_{ji}\in\Sigma\ \forall\ i\in I\},$$

$$\mu E = \sum_{i\in I}\mu(E\cap X_{ji}) \text{ for every } E\in\Sigma\cap\mathcal{P}Y_j.$$

Set $X_i = \bigcup_{j\in I}X_{ji}$ for every $i\in I$. Then $\langle X_i\rangle_{i\in I}$ is disjoint and covers X. Because $X_i\cap Y_j = X_{ji}$ is measurable for every j, $X_i\in\Sigma$. Because $X_i^\bullet\supseteq a_i\cap b_j$ for every j, and $\langle b_j\rangle_{j\in J}$ is a partition of unity in $\mathfrak{A}$ (322Kb), $X_i^\bullet\supseteq a_i$ for each i; because $\langle X_i^\bullet\rangle_{i\in I}$ is disjoint and $\sup_{i\in I}a_i = 1$, $X_i^\bullet = a_i$ for every i. If $E\subseteq X$ is such that $E\cap X_i\in\Sigma$ for every i, then $E\cap X_{ji}\in\Sigma$ for all $i\in I$ and $j\in J$, so $E\cap Y_j\in\Sigma$ for every $j\in J$ and $E\in\Sigma$. If $E\in\Sigma$, then

$$\begin{aligned}\mu E &= \sum_{j\in J}\mu(E\cap Y_j) = \sum_{j\in J, i\in I}\mu(E\cap X_{ji}) = \sum_{i\in I}\sup_{K\subseteq J\text{ is finite}}\sum_{j\in K}\mu(E\cap X_{ji})\\ &\le \sum_{i\in I}\mu(E\cap X_i) = \sup_{K\subseteq I\text{ is finite}}\sum_{i\in K}\mu(E\cap X_i)\le\mu E;\end{aligned}$$

so $\mu E = \sum_{i\in I}\mu(E\cap X_i)$. Thus $\langle X_i\rangle_{i\in I}$ is a suitable family.

322M Subalgebras: Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Set $\bar\nu = \bar\mu\restriction\mathfrak{B}$.

(a) $(\mathfrak{B},\bar\nu)$ is a measure algebra.
(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, or a probability algebra, so is $(\mathfrak{B},\bar\nu)$.
(c) If $(\mathfrak{A},\bar\mu)$ is σ-finite and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is σ-finite.
(d) If $(\mathfrak{A},\bar\mu)$ is localizable and $\mathfrak{B}$ is order-closed and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is localizable.
(e) If $(\mathfrak{B},\bar\nu)$ is a probability algebra, or totally finite, or σ-finite, so is $(\mathfrak{A},\bar\mu)$.

proof (a) By 314Eb, $\mathfrak{B}$ is Dedekind σ-complete, and the identity map $\pi:\mathfrak{B}\to\mathfrak{A}$ is sequentially order-continuous; so that $\bar\nu = \bar\mu\pi$ will be countably additive and $(\mathfrak{B},\bar\nu)$ will be a measure algebra.

(b) This is trivial.

(c) Use 322G. Every disjoint subset of $\mathfrak{B}$ is disjoint in $\mathfrak{A}$, therefore countable, because $\mathfrak{A}$ is ccc; so $\mathfrak{B}$ is also ccc and $(\mathfrak{B},\bar\nu)$ (being semi-finite) is σ-finite.

(d) By 314Ea, $\mathfrak{B}$ is Dedekind complete; we are supposing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so it is localizable.

(e) This is elementary.

322N The Stone space of a localizable measure algebra I said above that the concepts of 'strictly localizable' and 'locally determined' measure space have no equivalents in the theory of measure algebras. But when we look at the canonical measure on the Stone space of a measure algebra, we can of course hope that properties of the measure algebra will be reflected in the properties of this measure, as happens in the next theorem.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, Z the Stone space of $\mathfrak{A}$, and ν the standard measure on Z constructed by the method of 321J-321K. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is localizable;
(ii) ν is localizable;
(iii) ν is locally determined;
(iv) ν is strictly localizable.

proof Write Σ for the domain of ν, that is,

$$\{E\triangle A : E \subseteq Z \text{ is open-and-closed, } A \subseteq Z \text{ is meager}\},$$

and $\mathcal{M}$ for the ideal of meager subsets of Z, that is, the ideal of ν-negligible sets (314M, 321K). Then $a \mapsto \widehat{a}^\bullet : \mathfrak{A} \to \Sigma/\mathcal{M}$ is an isomorphism between $(\mathfrak{A}, \bar{\mu})$ and the measure algebra of (Z, Σ, ν) (314M). Note that because any subset of a meager set is meager, ν is surely complete.

(a)(i) $\iff$ (ii) is a consequence of 322Be.

(b)(ii)$\Rightarrow$(iii) Suppose that ν is localizable. Of course it is semi-finite. Let $V \subseteq Z$ be a set such that $V \cap E \in \Sigma$ whenever $E \in \Sigma$ and $\nu E < \infty$. Because ν is localizable, there is a $W \in \Sigma$ which is an essential supremum in Σ of $\{V \cap E : E \in \Sigma, \nu E < \infty\}$, that is, $W^\bullet = \sup\{(V \cap E)^\bullet : \nu E < \infty\}$ in $\Sigma/\mathcal{M}$. I claim that $W\triangle V$ is nowhere dense. **P** Let $G \subseteq Z$ be a non-empty open set. Then there is a non-zero $a \in \mathfrak{A}$ such that $\widehat{a} \subseteq G$. Because $(\mathfrak{A}, \bar{\mu})$ is semi-finite, we may suppose that $\bar{\mu}a < \infty$. Now

$$(W \cap \widehat{a})^\bullet = W^\bullet \cap \widehat{a}^\bullet = \sup_{\nu E<\infty}(V \cap E)^\bullet \cap \widehat{a}^\bullet = \sup_{\nu E<\infty}(V \cap E \cap \widehat{a})^\bullet = (V \cap \widehat{a})^\bullet,$$

so $(W\triangle V) \cap \widehat{a}$ is negligible, therefore meager. But we know that $\mathfrak{A}$ is weakly (σ, ∞)-distributive (322F), so that meager sets in Z are nowhere dense (316I), and there is a non-empty open set $H \subseteq \widehat{a} \setminus (W\triangle V)$. Now $H \subseteq G \setminus \overline{W\triangle V}$. As G is arbitrary, $\operatorname{int}\overline{W\triangle V} = \emptyset$ and $W\triangle V$ is nowhere dense. **Q**

But this means that $W\triangle V \in \mathcal{M} \subseteq \Sigma$ and $V = W\triangle(W\triangle V) \in \Sigma$. As V is arbitrary, ν is locally determined.

(c)(iii)$\Rightarrow$(iv) Assume that ν is locally determined. Because $(\mathfrak{A}, \bar{\mu})$ is semi-finite, there is a partition of unity $C \subseteq \mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{C} = \{\widehat{c} : c \in C\}$. This is a disjoint family of sets of finite measure for ν. Now suppose that $F \in \Sigma$ and $\nu F > 0$. Then there is an open-and-closed set $E \subseteq Z$ such that $F\triangle E$ is meager, and E is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$. Since

$$\bar{\mu}a = \nu\widehat{a} = \nu F > 0,$$

there is some $c \in C$ such that $a \cap c \neq 0$, and now

$$\nu(F \cap \widehat{c}) = \bar{\mu}(a \cap c) > 0.$$

This means that ν satisfies the conditions of 213O and must be strictly localizable.

(d)(iv)$\Rightarrow$(ii) This is just 211Ld.

322O Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then there is a unique extension of $\bar{\mu}$ to a functional $\tilde{\mu}$ on $\widehat{\mathfrak{A}}$ such that $(\widehat{\mathfrak{A}}, \tilde{\mu})$ is a localizable measure algebra. The embedding $\mathfrak{A} \subseteq \widehat{\mathfrak{A}}$ identifies the ideals $\{a : a \in \mathfrak{A}, \bar{\mu}a < \infty\}$ and $\{a : a \in \widehat{\mathfrak{A}}, \tilde{\mu}a < \infty\}$.

proof (I write the argument out as if $\mathfrak{A}$ were actually a subalgebra of $\widehat{\mathfrak{A}}$.) For $c \in \widehat{\mathfrak{A}}$, set

$$\tilde{\mu}c = \sup\{\bar{\mu}a : a \in \mathfrak{A}, a \subseteq c\}.$$

Evidently $\tilde{\mu}$ is a function from $\widehat{\mathfrak{A}}$ to $[0, \infty]$ extending $\bar{\mu}$, so $\tilde{\mu}0 = 0$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $\tilde{\mu}c > 0$ whenever $c \neq 0$, because any such c includes a non-zero member of $\mathfrak{A}$. If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{A}}$ with supremum c, then $\tilde{\mu}c = \sum_{n=0}^\infty \tilde{\mu}c_n$. **P** Let A be the set of all members of $\mathfrak{A}$ expressible as $a = \sup_{n\in\mathbb{N}} a_n$ where $a_n \in \mathfrak{A}$, $a_n \subseteq c_n$ for every $n \in \mathbb{N}$. Now

$$\begin{aligned}\sup_{a\in A} \bar{\mu}a &= \sup\{\sum_{n=0}^\infty \bar{\mu}a_n : a_n \in \mathfrak{A},\ a_n \subseteq c_n \text{ for every } n \in \mathbb{N}\} \\ &= \sum_{n=0}^\infty \sup\{\bar{\mu}a_n : a_n \subseteq c_n\} = \sum_{n=0}^\infty \tilde{\mu}c_n.\end{aligned}$$

Also, because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $c_n = \sup\{a : a \in \mathfrak{A}, a \subseteq c_n\}$ for each n, and $\sup A$, taken in $\widehat{\mathfrak{A}}$, must be c. But this means that if $a' \in \mathfrak{A}$, $a' \subseteq c$ then $a' = \sup_{a\in A} a' \cap a$ in $\widehat{\mathfrak{A}}$ and therefore also in $\mathfrak{A}$; so that

$$\bar{\mu}a' = \sup_{a\in A} \bar{\mu}(a' \cap a) \leq \sup_{a\in A} \bar{\mu}a.$$

Accordingly

$$\tilde{\mu}c = \sup_{a\in A}\bar{\mu}a = \sum_{n=0}^{\infty}\tilde{\mu}c_n. \quad \mathbf{Q}$$

This shows that $(\widehat{\mathfrak{A}},\tilde{\mu})$ is a measure algebra. It is semi-finite because $(\mathfrak{A},\bar{\mu})$ is and every non-zero element of $\widehat{\mathfrak{A}}$ includes a non-zero element of $\mathfrak{A}$, which in turn includes a non-zero element of finite measure. Since $\widehat{\mathfrak{A}}$ is Dedekind complete, $(\widehat{\mathfrak{A}},\bar{\mu})$ is localizable.

If $\bar{\mu}a$ is finite, then surely $\tilde{\mu}a = \bar{\mu}a$ is finite. If $\tilde{\mu}c$ is finite, then $\{A : a \in \mathfrak{A},\, a \subseteq c\}$ is upwards-directed and $\sup_{a\in A}\bar{\mu}A = \tilde{\mu}c$ is finite, so $b = \sup A$ is defined in $\mathfrak{A}$ and $\bar{\mu}b = \tilde{\mu}c$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $b = c$ (313K, 313O) and $c \in \mathfrak{A}$, with $\bar{\mu}c = \tilde{\mu}c$.

322P Definition Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. I will call $(\widehat{\mathfrak{A}},\tilde{\mu})$, as constructed above, the **localization** of $(\mathfrak{A},\bar{\mu})$. Of course it is unique just in so far as the Dedekind completion of $\mathfrak{A}$ is.

322Q Further properties of Stone spaces: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and (Z,Σ,ν) its Stone space.

(a) Meager sets in Z are nowhere dense; every $E \in \Sigma$ is uniquely expressible as $G\triangle M$ where $G \subseteq Z$ is open-and-closed and M is nowhere dense, and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) The c.l.d. version $\tilde{\nu}$ of ν is strictly localizable, and has the same negligible sets as ν.

(c) If $(\mathfrak{A},\bar{\mu})$ is totally finite then $\nu E = \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}$ for every $E \in \Sigma$.

proof (a) I have already remarked (in the proof of 322N) that $\mathfrak{A}$ is weakly (σ,∞)-distributive, so that meager sets in Z are nowhere dense. But we know that every member of Σ is expressible as $G\triangle M$ where G is open-and-closed and M is meager, therefore nowhere dense. Moreover, the expression is unique, because if $G\triangle M = G'\triangle M'$ then $G\triangle G' \subseteq M \cup M'$ is open and nowhere dense, therefore empty, so $G = G'$ and $M = M'$.

Now let $a \in \mathfrak{A}$ be such that $\widehat{a} = G$, and consider $B = \{b : b \in \mathfrak{A},\, \widehat{b} \subseteq E\}$. Then $\sup B = a$ in $\mathfrak{A}$. $\mathbf{P}$ If $b \in B$, then $\widehat{b}\setminus\widehat{a} \subseteq M$ is nowhere dense, therefore empty; so a is an upper bound for B. **?** If a is not the supremum of B, then there is a non-zero $c \subseteq a$ such that $b \subseteq a\setminus c$ for every $b \in B$. But now $\widehat{c}$ cannot be empty, so $\widehat{c}\setminus\overline{M}$ is non-empty, and there is a non-zero $d \in \mathfrak{A}$ such that $\widehat{d} \subseteq \widehat{c}\setminus\overline{M}$. In this case $d \in B$ and $d \not\subseteq a\setminus c$. $\mathbf{X}$ Thus $a = \sup B$. $\mathbf{Q}$

It follows that

$$\begin{aligned}\nu E = \nu G = \bar{\mu}a &= \sup_{b\in B}\bar{\mu}b\\ &= \sup_{b\in B}\nu\widehat{b} \le \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\} \le \nu E\end{aligned}$$

and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) This is the same as part (c) of the proof of 322N. We have a disjoint family $\mathcal{C}$ of sets of finite measure for ν such that whenever $E \in \Sigma$, $\nu E > 0$ there is a $C \in \mathcal{C}$ such that $\mu(C\cap E) > 0$. Now if $\tilde{\nu}F$ is defined and not 0, there is an $E \in \Sigma$ such that $E \subseteq F$ and $\nu E > 0$ (213Fc), so that there is a $C \in \mathcal{C}$ such that $\nu(E\cap C) > 0$; since $\nu C < \infty$, we have

$$\tilde{\nu}(F\cap C) \ge \tilde{\nu}(E\cap C) = \nu(E\cap C) > 0.$$

And of course $\tilde{\nu}C < \infty$ for every $C \in \mathcal{C}$. This means that $\mathcal{C}$ witnesses that $\tilde{\nu}$ satisfies the conditions of 213O, so that $\tilde{\nu}$ is strictly localizable.

Any ν-negligible set is surely $\tilde{\nu}$-negligible. If M is $\tilde{\nu}$-negligible then it is nowhere dense. $\mathbf{P}$ If $G \subseteq Z$ is open and not empty then there is a non-empty open-and-closed set $H_1 \subseteq G$, and now $H_1 \in \Sigma$, so there is a non-empty open-and-closed set $H \subseteq H_1$ such that νH is finite (because ν is semi-finite). In this case $H\cap M$ is ν-negligible, therefore nowhere dense, and $H \not\subseteq \overline{M}$. But this means that $G \not\subseteq \overline{M}$; as G is arbitrary, M is nowhere dense. $\mathbf{Q}$ Accordingly $M \in \mathcal{M}$ and is ν-negligible.

Thus ν and $\tilde{\nu}$ have the same negligible sets.

(c) Because $\nu Z < \infty$,

$$\begin{aligned}\nu E = \nu Z - \nu(Z \setminus E) &= \nu Z - \sup\{\nu H : H \subseteq Z \setminus E \text{ is open-and-closed}\}\\ &= \inf\{\nu(Z \setminus H) : H \subseteq Z \setminus E \text{ is open-and-closed}\}\\ &= \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}.\end{aligned}$$

322X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Let I_∞ be the set of those $a \in \mathfrak{A}$ which are 'purely infinite', that is, $\bar{\mu}a = \infty$ and $\bar{\mu}b = \infty$ for every non-zero $b \subseteq a$. Show that I_∞ is a σ-ideal of $\mathfrak{A}$. Show that there is a function $\bar{\mu}_{sf} : \mathfrak{A}/I_\infty \to [0, \infty]$ defined by setting $\bar{\mu}_{sf}a^\bullet = \sup\{\bar{\mu}b : b \subseteq a, \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$. Show that $(\mathfrak{A}/I_\infty, \bar{\mu}_{sf})$ is a semi-finite measure algebra.

(b) Let (X, Σ, μ) be a measure space and let μ_{sf} be the 'semi-finite version' of μ, as defined in 213Xc. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of (X, Σ, μ). Show that the measure algebra of (X, Σ, μ_{sf}) is isomorphic to the measure algebra $(\mathfrak{A}/I_\infty, \bar{\mu}_{sf})$ of (a) above.

(c) Let (X, Σ, μ) be a measure space and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version. Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be the corresponding measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{A}_2$ the canonical homomorphism, as in 322Db. Show that the kernel of π is the ideal I_∞, as described in 322Xa, so that $\mathfrak{A}/I_\infty$ is isomorphic, as Boolean algebra, to $\pi[\mathfrak{A}] \subseteq \mathfrak{A}_2$. Show that this isomorphism identifies $\bar{\mu}_{sf}$, as described in 322Xa, with $\bar{\mu}_2 \restriction \pi[\mathfrak{A}]$.

(d) Give a direct proof of 322G, not relying on 215B and 321J.

>(e) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra, A a non-empty subset of $\mathfrak{A}$, and $c \in \mathfrak{A}$ such that $\bar{\mu}c < \infty$. Show that (i) $c_0 = \sup\{a \cap c : a \in A\}$ is defined in $\mathfrak{A}$ (ii) there is a countable set $B \subseteq A$ such that $c_0 = \sup\{a \cap c : a \in B\}$.

(f) Let (X, Σ, μ) be a measure space and ν an indefinite-integral measure over μ (§234). Show that the measure algebra of ν can be identified, as Boolean algebra, with a principal ideal of the measure algebra of μ.

(g) Let (X, Σ, μ) be a measure space and A any subset of X; let μ_A be the subspace measure on A and Σ_A its domain. Write $(\mathfrak{A}, \bar{\mu})$ for the measure algebra of (X, Σ, μ) and $(\mathfrak{A}_A, \bar{\mu}_A)$ for the measure algebra of (A, Σ_A, μ_A). Show that the formula $F^\bullet \mapsto (F \cap A)^\bullet$ defines a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_A$ which has kernel $I = \{F^\bullet : F \in \Sigma, F \cap A = \emptyset\}$. Show that for any $a \in \mathfrak{A}$, $\bar{\mu}_A(\pi a) = \min\{\bar{\mu}b : b \in \mathfrak{A}, a \setminus b \in I\}$.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. Suppose that $(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B})$ is semi-finite. Show that $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra and (Z, Σ, ν) its Stone space. Show that the c.l.d. version of ν is strictly localizable.

322Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Set $\mathcal{N} = \{N : \exists F \in \mathcal{I}, N \subseteq F\}$. Show that $\mathcal{N}$ is a σ-ideal of subsets of X. Set $\hat{\Sigma} = \{E \triangle N : E \in \Sigma, N \in \mathcal{N}\}$. Show that $\hat{\Sigma}$ is a σ-algebra of subsets of X and that $\hat{\Sigma}/\mathcal{N}$ is isomorphic to $\Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and (Z, Σ, ν) its Stone space. Let $\tilde{\nu}$ be the c.l.d. version of ν, and $\tilde{\Sigma}$ its domain. Show that $\tilde{\Sigma}$ is precisely the Baire property algebra $\{G \triangle A : G \subseteq Z \text{ is open}, A \subseteq Z \text{ is meager}\}$, so that $\tilde{\Sigma}/\mathcal{M}$ can be identified with the regular open algebra of Z (314Yd) and the measure algebra of $\tilde{\nu}$ can be identified with the localization of $\mathfrak{A}$.

(c) Give an example of a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ with a σ-subalgebra $\mathfrak{B}$ such that $(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B})$ is semi-finite and atomless, but $\mathfrak{A}$ is not atomless.

(d) Let (X, Σ, μ) be a measure space and $A \subseteq X$ a subset; let μ_A be the subspace measure on A, $\mathfrak{A}$ and $\mathfrak{A}_A$ the measure algebras of μ and μ_A, and $\pi : \mathfrak{A} \to \mathfrak{A}_A$ the canonical homomorphism, as described in 322Xg. (i) Show that if μ_A is semi-finite, then π is order-continuous. (ii) Show that if μ is semi-finite but μ_A is not, then π is not order-continuous.

(e) Show that if $(\mathfrak{A},\bar\mu)$ is a measure algebra, with Stone space (Z,Σ,ν), then ν has locally determined negligible sets in the sense of 213I.

(f) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and (Z,Σ,ν) its Stone space. (i) Show that a function $f:Z\to\mathbb{R}$ is Σ-measurable iff there is a conegligible set $G\subseteq X$ such that $f\restriction G$ is continuous. (*Hint*: 316Yi.) (ii) Show that $f:Z\to[0,1]$ is Σ-measurable iff there is a continuous function $g:Z\to[0,1]$ such that $f=g$ ν-a.e.

322 Notes and comments I have taken this leisurely tour through the concepts of Chapter 21 partly to recall them (or persuade you to look them up) and partly to give you practice in the elementary manipulations of measure algebras. The really vital result here is the correspondence between 'localizability' in measure spaces and measure algebras. Part of the object of this volume (particularly in Chapter 36) is to try to make sense of the properties of localizable measure spaces, as discussed in Chapter 24 and elsewhere, in terms of their measure algebras. I hope that 322Be has already persuaded you that the concept really belongs to measure algebras, and that the formulation in terms of 'essential suprema' is a dispensable expedient.

I have given proofs of 322C and 322G depending on the realization of an arbitrary measure algebra as the measure algebra of a measure space, and the corresponding theorems for measure spaces, because this seems the natural approach from where we presently stand; but I am sympathetic to the view that such proofs must be inappropriate, and that it is in some sense better style to look for arguments which speak only of measure algebras (322Xd).

For any measure algebra $(\mathfrak{A},\bar\mu)$, the set $\mathfrak{A}^f$ of elements of finite measure is an ideal of $\mathfrak{A}$; consequently it is order-dense iff it includes a partition of unity (322E). In 322F we have something deeper: any semi-finite measure algebra must be weakly (σ,∞)-distributive when regarded as a Boolean algebra, and this has significant consequences in its Stone space, which are used in the proofs of 322N and 322Q. Of course a result of this kind must depend on the semi-finiteness of the measure algebra, since any Dedekind σ-complete Boolean algebra becomes a measure algebra if we give every non-zero element the measure ∞. It is natural to look for algebraic conditions on a Boolean algebra sufficient to make it 'measurable', in the sense that it should carry a semi-finite measure; this is an unresolved problem to which I will return in Chapter 39.

Subspace measures, simple products, direct sums, principal ideals and order-closed subalgebras give no real surprises; I spell out the details in 322I-322M and 322Xg-322Xh. It is worth noting that completing a measure space has no effect on its measure algebra (322D, 322Ya). We see also that from the point of view of measure algebras there is no distinction to be made between 'localizable' and 'strictly localizable', since every localizable measure algebra is representable as the measure algebra of a strictly localizable measure space (322Kd). (But strict localizability does have implications for some processes starting in the measure algebra; see 322L.) It is nevertheless remarkable that the canonical measure on the Stone space of a semi-finite measure algebra is localizable iff it is strictly localizable (322N). This canonical measure has many other interesting properties, which I skim over in 322Q, 322Xi, 322Yb and 322Yf. In Chapter 21 I discussed a number of methods of improving measure spaces, notably 'completions' (212C) and 'c.l.d. versions' (213E). Neither of these is applicable in any general way to measure algebras. But in fact we have a more effective construction, at least for semi-finite measure algebras, that of 'localization' (322O-322P); I say that it is more effective just because localizability is more important than completeness or local determinedness, being of vital importance in the behaviour of function spaces (241Gb, 243Gb, 245Ec, 363M, 364O, 365J, 367N, 369A, 369C). Note that the localization of a semi-finite measure algebra does in fact correspond to the c.l.d. version of a certain measure (322Q, 322Yb). But of course $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ do *not* have the same Stone spaces, even when $\widehat{\mathfrak{A}}$ can be effectively represented as the measure algebra of a measure on the Stone space of $\mathfrak{A}$. What is happening in 322Yb is that we are using all the open sets of Z to represent members of $\widehat{\mathfrak{A}}$, not just the open-and-closed sets, which correspond to members of $\mathfrak{A}$.

323 The topology of a measure algebra

I take a short section to discuss one of the fundamental tools for studying totally finite measure algebras, the natural metric that each carries. The same ideas, suitably adapted, can be applied to an arbitrary measure algebra, where we have a topology corresponding closely to the topology of convergence in measure on the function space L^0. Most of the section consists of an analysis of the relations between this topology and the order structure of the measure algebra.

323A The pseudometrics ρ_a (a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$. For $a\in\mathfrak{A}^f$ and $b,\,c\in\mathfrak{A}$, write $\rho_a(b,c)=\bar\mu(a\cap(b\triangle c))$. Then ρ_a is a pseudometric on $\mathfrak{A}$. **P** (i) Because $\bar\mu a<\infty$, ρ_a takes values in $[0,\infty[$. (ii) If $b,\,c,\,d\in\mathfrak{A}$ then $b\triangle d\subseteq(b\triangle c)\cup(c\triangle d)$, so

$$\begin{aligned}\rho_a(b,d)=\bar\mu(a\cap(b\triangle d))&\le\bar\mu((a\cap(b\triangle c))\cup(a\cap(c\triangle d)))\\&\le\bar\mu(a\cap(b\triangle c))+\bar\mu(a\cap(c\triangle d))=\rho_a(b,c)+\rho_a(c,d).\end{aligned}$$

(iii) If $b,\,c\in\mathfrak{A}$ then

$$\rho_a(b,c)=\bar\mu(a\cap(b\triangle c))=\bar\mu(a\cap(c\triangle b))=\rho_a(c,b).\ \mathbf{Q}$$

(b) Now the **measure-algebra topology** of the measure algebra $(\mathfrak{A},\bar\mu)$ is that generated by the family $\mathrm{P}=\{\rho_a:a\in\mathfrak{A}^f\}$ of pseudometrics on $\mathfrak{A}$. Similarly the **measure-algebra uniformity** on $\mathfrak{A}$ is that generated by P.

(For a general discussion of topologies defined by pseudometrics, see 2A3F *et seq.* For the associated uniformities see §3A4.)

(c) Note that P is upwards-directed, since $\rho_{a\cup a'}\ge\max(\rho_a,\rho_{a'})$ for all $a,\,a'\in\mathfrak{A}^f$.

(d) When $(\mathfrak{A},\bar\mu)$ is totally finite, it is more natural to work from the **measure metric** $\rho=\rho_1$, with $\rho(a,b)=\bar\mu(a\triangle b)$, since this by itself defines the measure-algebra topology and uniformity.

***(e)** Even when $(\mathfrak{A},\bar\mu)$ is not totally finite, we still have a metric ρ on $\mathfrak{A}^f$ defined by setting $\rho(a,b)=\bar\mu(a\triangle b)$ for all $a,\,b\in\mathfrak{A}^f$, which is sometimes useful (323Xg). Note however that the topology on $\mathfrak{A}^f$ defined from ρ is not in general the topology induced by the measure-algebra topology of $\mathfrak{A}$.

323B Proposition Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Then the operations $\cup$, $\cap$, $\setminus$ and $\triangle$ are all uniformly continuous.

proof The point is that for any $b,\,c,\,b',\,c'\in\mathfrak{A}$ we have

$$(b*c)\triangle(b'*c')\subseteq(b\triangle b')\cup(c\triangle c')$$

for any of the operations $*=\cup,\ \cap$ etc.; so that if $a\in\mathfrak{A}^f$ then

$$\rho_a(b*c,b'*c')\le\rho_a(b,b')+\rho_a(c,c').$$

Consequently the operation $*$ must be uniformly continuous.

323C Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra. Then $\bar\mu:\mathfrak{A}\to[0,\infty[$ is uniformly continuous.

(b) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then $\bar\mu:\mathfrak{A}\to[0,\infty]$ is lower semi-continuous.

(c) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. If $a\in\mathfrak{A}$ and $\bar\mu a<\infty$, then $b\mapsto\bar\mu(b\cap a):\mathfrak{A}\to\mathbb{R}$ is uniformly continuous.

proof (a) For any $a,\,b\in\mathfrak{A}$,

$$|\bar\mu a-\bar\mu b|\le\bar\mu(a\triangle b)=\rho_1(a,b).$$

(b) Suppose that $b\in\mathfrak{A}$ and $\bar\mu b>\alpha\in\mathbb{R}$. Then there is an $a\subseteq b$ such that $\alpha<\bar\mu a<\infty$ (322Eb). If $c\in\mathfrak{A}$ is such that $\rho_a(b,c)<\bar\mu a-\alpha$, then

$$\bar\mu c \ge \bar\mu(a \cap c) = \bar\mu a - \bar\mu(a \cap (b \setminus c)) > \alpha.$$

Thus $\{b : \bar\mu b > \alpha\}$ is open; as α is arbitrary, $\bar\mu$ is lower semi-continuous.

(c) $|\bar\mu(a \triangle b) - \bar\mu(a \triangle c)| \le \rho_a(b, c)$ for all b, $c \in \mathfrak{A}$.

323D The following facts are basic to any understanding of the relationship between the order structure and topology of a measure algebra.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) Let $B \subseteq \mathfrak{A}$ be a non-empty upwards-directed set. For $b \in B$ set $F_b = \{c : b \subseteq c \in B\}$.

(i) $\{F_b : b \in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$.

(ii) If $\sup B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\uparrow)$; in particular, it belongs to the topological closure of B.

(b) Let $B \subseteq \mathfrak{A}$ be a non-empty downwards-directed set. For $b \in B$ set $F_b = \{c : b \supseteq c \in B\}$.

(i) $\{F_b : b \in B\}$ generates a Cauchy filter $\mathcal{F}(B\downarrow)$ on $\mathfrak{A}$.

(ii) If $\inf B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\downarrow)$; in particular, it belongs to the topological closure of B.

(c)(i) Closed subsets of $\mathfrak{A}$ are order-closed in the sense of 313D.

(ii) An order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.

(d) Now suppose that $(\mathfrak{A}, \bar\mu)$ is semi-finite.

(i) The sets $\{b : b \subseteq c\}$, $\{b : b \supseteq c\}$ are closed for every $c \in \mathfrak{A}$.

(ii) If $B \subseteq \mathfrak{A}$ is non-empty and upwards-directed and e is a cluster point of $\mathcal{F}(B\uparrow)$, then $e = \sup B$.

(iii) If $B \subseteq \mathfrak{A}$ is non-empty and downwards-directed and e is a cluster point of $\mathcal{F}(B\downarrow)$, then $e = \inf B$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) (α) If b, $c \in B$ then there is a $d \in B$ such that $b \cup c \subseteq d$, so that $F_d \subseteq F_b \cap F_c$; consequently

$$\mathcal{F}(B\uparrow) = \{F : F \subseteq \mathfrak{A},\ \exists\, b \in B,\ F_b \subseteq F\}$$

is a filter on $\mathfrak{A}$. (β) Let $a \in \mathfrak{A}^f$, $\epsilon > 0$. Then there is a $b \in B$ such that $\bar\mu(a \cap c) \le \bar\mu(a \cap b) + \frac{1}{2}\epsilon$ for every $c \in B$, and $F_b \in \mathcal{F}(B\uparrow)$. If now c, $c' \in F_b$, $c \triangle c' \subseteq (c \setminus b) \cup (c' \setminus b)$, so

$$\rho_a(c, c') \le \bar\mu(a \cap c \setminus b) + \bar\mu(a \cap c' \setminus b) = \bar\mu(a \cap c) + \bar\mu(a \cap c') - 2\bar\mu(a \cap b) \le \epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow)$ is Cauchy.

(ii) Suppose that $e = \sup B$ is defined in $\mathfrak{A}$. Let $a \in \mathfrak{A}^f$, $\epsilon > 0$. By 313Ba, $a \cap e = \sup_{b\in B} a \cap b$; but $\{a \cap b : b \in B\}$ is upwards-directed, so $\bar\mu(a \cap e) = \sup_{b\in B} \bar\mu(a \cap b)$, by 321D. Let $b \in B$ be such that $\bar\mu(a \cap b_0) \ge \bar\mu(a \cap e) - \epsilon$. Then for any $c \in F_b$, $e \triangle c \subseteq e \setminus b$, so

$$\rho_a(e, c) = \bar\mu(a \cap (e \triangle c)) \le \bar\mu(a \cap (e \setminus b)) = \bar\mu(a \cap e) - \bar\mu(a \cap b) \le \epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow) \to e$.

Because $B \in \mathcal{F}(B\uparrow)$, e surely belongs to the topological closure of B.

(b) Either repeat the arguments above, with appropriate inversions, using 321F in place of 321D, or apply (a) to the set $\{1 \setminus b : b \in B\}$.

(c)(i) This follows at once from (a) and (b) and the definition in 313D.

(ii) If $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-dense subalgebra and $a \in \mathfrak{A}$, then $B = \{b : b \in \mathfrak{B},\ b \subseteq a\}$ is upwards-directed and has supremum a (313K); by (a-ii), $a \in \overline{B} \subseteq \overline{\mathfrak{B}}$. As a is arbitrary, $\mathfrak{B}$ is topologically dense.

(d)(i) Set $F = \{b : b \subseteq c\}$. If $d \in \mathfrak{A} \setminus F$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite) there is an $a \in \mathfrak{A}^f$ such that $\delta = \bar\mu(a \cap d \setminus c) > 0$; now if $b \in F$,

$$\rho_a(d, b) \ge \bar\mu(a \cap d \setminus b) \ge \delta,$$

so that d cannot belong to the closure of F. As d is arbitrary, F is closed. Similarly, $\{b : b \supseteq c\}$ is closed.

(ii) (α) If $b \in B$, then $e \in \overline{F_b}$, because $F_b \in \mathcal{F}(B\uparrow)$; but $\{c : b \subseteq c \in \mathfrak{A}\}$ is a closed set including F_b, so contains e, and $b \subseteq e$. As b is arbitrary, e is an upper bound for B. (β) If d is an upper bound of B, then $\{c : c \subseteq d\}$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains e. As d is arbitrary, this shows that e is the supremum of B, as claimed.

(iii) Use the same arguments as in (ii), but inverted.

323E Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If $\langle b_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum b, then $\langle b_n \rangle_{n\in\mathbb{N}} \to b$ for the measure-algebra topology.

(b) If $\langle b_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum b, then $\langle b_n \rangle_{n\in\mathbb{N}} \to b$ for the measure-algebra topology.

proof I call this a 'corollary' because it is the special case of 323Da-323Db in which B is the set of terms of a monotonic sequence; but it is probably easier to work directly from the definition in 323A, and use 321Be or 321Bf to see that $\lim_{n\to\infty} \rho_a(b_n, b) = 0$ whenever $\bar{\mu}a < \infty$.

323F The following is a useful calculation.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $\langle c_n \rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that the sum $\sum_{n=0}^{\infty} \bar{\mu}(c_n \triangle c_{n+1})$ is finite. Set $d_0 = \sup_{n\in\mathbb{N}} \inf_{m\geq n} c_m$, $d_1 = \inf_{n\in\mathbb{N}} \sup_{m\geq n} c_m$. Then $d_0 = d_1$ and, writing d for their common value, $\lim_{n\to\infty} \bar{\mu}(c_n \triangle d) = 0$.

proof Write $\alpha_n = \bar{\mu}(c_n \triangle c_{n+1})$, $\beta_n = \sum_{k=n}^{\infty} \alpha_k$ for $n \in \mathbb{N}$; we are supposing that $\lim_{n\to\infty} \beta_n = 0$. Set $b_n = \sup_{m\geq n} c_m \triangle c_{m+1}$; then

$$\bar{\mu}b_n \leq \sum_{m=n}^{\infty} \bar{\mu}(c_m \triangle c_{m+1}) = \beta_n$$

for each n. If $m \geq n$, then

$$c_m \triangle c_n \subseteq \sup_{n\leq k<m} c_k \triangle c_{k+1} \subseteq b_n,$$

so

$$c_n \setminus b_n \subseteq c_m \subseteq c_n \cup b_n.$$

Consequently

$$c_n \setminus b_n \subseteq \inf_{k\geq m} c_k \subseteq \sup_{k\geq m} c_k \subseteq c_n \cup b_n$$

for every $m \geq n$, and

$$c_n \setminus b_n \subseteq d_0 \subseteq d_1 \subseteq c_n \cup b_n,$$

so that

$$c_n \triangle d_0 \subseteq b_n, \quad c_n \triangle d_1 \subseteq b_n, \quad d_1 \setminus d_0 \subseteq b_n.$$

As this is true for every n,

$$\lim_{n\to\infty} \bar{\mu}(c_n \triangle d_i) \leq \lim_{n\to\infty} \bar{\mu}b_n = 0$$

for both i, and

$$\bar{\mu}(d_1 \triangle d_0) \leq \inf_{n\in\mathbb{N}} \bar{\mu}b_n = 0,$$

so that $d_1 = d_0$.

323G The classification of measure algebras: Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, $\mathfrak{T}$ its measure-algebra topology and $\mathcal{U}$ its measure-algebra uniformity.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(b) $(\mathfrak{A}, \bar{\mu})$ is σ-finite iff $\mathfrak{T}$ is metrizable, and in this case $\mathcal{U}$ is also metrizable.

(c) $(\mathfrak{A}, \bar{\mu})$ is localizable iff $\mathfrak{T}$ is Hausdorff and $\mathfrak{A}$ is complete under $\mathcal{U}$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and that b, c are distinct members of $\mathfrak{A}$. Then there is an $a \subseteq b \triangle c$ such that $0 < \bar{\mu}a < \infty$, and now $\rho_a(b, c) > 0$. As b and c are arbitrary, $\mathfrak{T}$ is Hausdorff (2A3L).

(ii) Suppose that $\mathfrak{T}$ is Hausdorff and that $b \in \mathfrak{A}$ has $\bar{\mu}b = \infty$. Then $b \neq 0$ so there must be an $a \in \mathfrak{A}^f$ such that $\bar{\mu}(a \cap b) = \rho_a(0, b) > 0$; in which case $a \cap b \subseteq b$ and $0 < \bar{\mu}(a \cap b) < \infty$. As b is arbitrary, $\bar{\mu}$ is semi-finite.

(b)(i) Suppose that $\bar{\mu}$ is σ-finite. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1. Set

$$\rho(b, c) = \sum_{n=0}^{\infty} \frac{\rho_{a_n}(b, c)}{1 + 2^n \bar{\mu}a_n}$$

for b, $c \in \mathfrak{A}$. Then ρ is a metric on $\mathfrak{A}$, because if $\rho(b, c) = 0$ then $a_n \cap (b \triangle c) = 0$ for every n, so $b \triangle c = 0$ and $b = c$.

If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, take n such that $\bar{\mu}(a \setminus a_n) \leq \frac{1}{2}\epsilon$. If b, $c \in \mathfrak{A}$ and $\rho(b, c) \leq \epsilon/2(1 + 2^n\bar{\mu}a_n)$, then $\rho_{a_n}(b, c) \leq \frac{1}{2}\epsilon$ so

$$\begin{aligned}\rho_a(b, c) &= \rho_{a\setminus a_n}(b, c) + \rho_{a\cap a_n}(b, c) \leq \bar{\mu}(a \setminus a_n) + \rho_{a_n}(b, c)\\ &\leq \frac{1}{2}\epsilon + (1 + 2^n\bar{\mu}a_n)\rho(b, c) \leq \epsilon.\end{aligned}$$

In the other direction, given $\epsilon > 0$, take $n \in \mathbb{N}$ such that $2^{-n} \leq \frac{1}{2}\epsilon$; then $\rho(b, c) \leq \epsilon$ whenever $\rho_{a_n}(b, c) \leq \epsilon/2(n + 1)$.

This shows that $\mathcal{U}$ is the same as the metrizable uniformity defined by $\{\rho\}$; accordingly $\mathfrak{T}$ is also defined by ρ.

(ii) Now suppose that $\mathfrak{T}$ is metrizable, and let ρ be a metric defining $\mathfrak{T}$. For each $n \in \mathbb{N}$ there must be $a_{n0}, \ldots, a_{nk_n} \in \mathfrak{A}^f$ and $\delta_n > 0$ such that

$$\rho_{a_{ni}}(b, 1) \leq \delta_n \text{ for every } i \leq k_n \implies \rho(b, 1) \leq 2^{-n}.$$

Set $d = \sup_{n\in\mathbb{N}, i\leq k_n} a_{ni}$. Then $\rho_{a_{ni}}(d, 1) = 0$ for every n, i, so $\rho(d, 1) \leq 2^{-n}$ for every n and $d = 1$. Thus 1 is the supremum of countably many elements of finite measure and $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(c)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable. Then $\mathfrak{T}$ is Hausdorff, by (a). Let $\mathcal{F}$ be a Cauchy filter on $\mathfrak{A}$. For each $a \in \mathfrak{A}^f$, choose a sequence $\langle F_n(a) \rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\rho_a(b, c) \leq 2^{-n}$ whenever b, $c \in F_n(a)$ and $n \in \mathbb{N}$. Choose $c_{an} \in \bigcap_{k\leq n} F_k(a)$ for each n; then $\rho_a(c_{an}, c_{a,n+1}) \leq 2^{-n}$ for each n. Set $d_a = \sup_{n\in\mathbb{N}} \inf_{k\geq n} a \cap c_{ak}$. Then

$$\lim_{n\to\infty} \rho_a(d_a, c_{an}) = \lim_{n\to\infty} \bar{\mu}(d_a \triangle (a \cap c_{an})) = 0,$$

by 323F.

If a, $b \in \mathfrak{A}^f$ and $a \subseteq b$, then $d_a = a \cap d_b$. **P** For each $n \in \mathbb{N}$, $F_n(a)$ and $F_n(b)$ both belong to $\mathcal{F}$, so must have a point e in common; now

$$\begin{aligned}\rho_a(d_a, d_b) &\leq \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_a(e, c_{bn}) + \rho_a(c_{bn}, d_b)\\ &\leq \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_b(e, c_{bn}) + \rho_b(c_{bn}, d_b)\\ &\leq \rho_a(d_a, c_{an}) + 2^{-n} + 2^{-n} + \rho_b(c_{bn}, d_b)\\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently $\rho_a(d_a, d_b) = 0$, that is,

$$d_a = a \cap d_a = a \cap d_b. \quad \mathbf{Q}$$

Set $d = \sup\{d_b : b \in \mathfrak{A}^f\}$; this is defined because $\mathfrak{A}$ is Dedekind complete. Then $\mathcal{F} \to d$. **P** If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, then

$$a \cap d = \sup_{b\in\mathfrak{A}^f} a \cap d_b = \sup_{b\in\mathfrak{A}^f} a \cap b \cap d_{a\cup b} = \sup_{b\in\mathfrak{A}^f} a \cap b \cap d_a = a \cap d_a.$$

So if we choose $n \in \mathbb{N}$ such that $2^{-n} + \rho_a(c_{an}, d_a) \leq \epsilon$, then for any $e \in F_n(a)$ we shall have

$$\rho_a(e, d) \leq \rho_a(e, c_{an}) + \rho_a(c_{an}, d) \leq 2^{-n} + \rho_a(c_{an}, d_a) \leq \epsilon.$$

Thus

$$\{e : \rho_a(d,e) \le \epsilon\} \supseteq F_n(a) \in \mathcal{F}.$$

As a, ϵ are arbitrary, $\mathcal{F}$ converges to d. **Q** As $\mathcal{F}$ is arbitrary, $\mathfrak{A}$ is complete.

(ii) Now suppose that $\mathfrak{T}$ is Hausdorff and that $\mathfrak{A}$ is complete under $\mathcal{U}$. By (a), $(\mathfrak{A},\bar\mu)$ is semi-finite. Let B be any non-empty subset of $\mathfrak{A}$, and set $B' = \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in B\}$, so that B' is upwards-directed and has the same upper bounds as B. By 323Da, we have a Cauchy filter $\mathcal{F}(B'\uparrow)$; because $\mathfrak{A}$ is complete, this is convergent; and because $(\mathfrak{A},\bar\mu)$ is semi-finite, its limit must be $\sup B' = \sup B$, by 323Dd. As B is arbitrary, $\mathfrak{A}$ is Dedekind complete, so $(\mathfrak{A},\bar\mu)$ is localizable.

323H Closed subalgebras The ideas used in the proof of (c) above have many other applications, of which one of the most important is the following. You may find it helpful to read the next theorem first on the assumption that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then it is closed for the measure-algebra topology iff it is order-closed.

proof (a) If $\mathfrak{B}$ is closed, it must be order-closed, by 323Dc.

(b) Now suppose that $\mathfrak{B}$ is order-closed. I repeat the ideas of part (c-i) of the proof of 323G. Let e be any member of the closure of $\mathfrak{B}$ in $\mathfrak{A}$. For each $a \in \mathfrak{A}^f$, $n \in \mathbb{N}$ choose $c_{an} \in \mathfrak{B}$ such that $\rho_a(c_{an}, e) \le 2^{-n}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \bar\mu((a \cap c_{an}) \triangle (a \cap c_{a,n+1})) &= \sum_{n=0}^{\infty} \rho_a(c_{an}, c_{a,n+1}) \\ &\le \sum_{n=0}^{\infty} \rho_a(c_{an}, e) + \rho_a(e, c_{a,n+1}) < \infty.\end{aligned}$$

So if we set $e_a = \sup_{n\in\mathbb{N}} \inf_{k\ge n} c_{ak}$, then

$$\rho_a(e_a, c_{an}) = \rho_a(a \cap e_a, a \cap c_{an}) \to 0$$

as $n \to \infty$, by 323F, and $\rho_a(e, e_a) = 0$, that is, $a \cap e_a = a \cap e$. Also, because $\mathfrak{B}$ is order-closed, $\inf_{k\ge n} c_{ak} \in \mathfrak{B}$ for every n, and $e_a \in \mathfrak{B}$.

Because $\mathfrak{A}$ is Dedekind complete, we can set

$$e'_a = \inf\{e_b : b \in \mathfrak{A}^f,\ a \subseteq b\};$$

then $e'_a \in \mathfrak{B}$ and

$$e'_a \cap a = \inf_{b\supseteq a} e_b \cap a = \inf_{b\supseteq a} e_b \cap b \cap a = \inf_{b\supseteq a} e \cap b \cap a = e \cap a.$$

Now $e'_a \subseteq e'_b$ whenever $a \subseteq b$, so $B = \{e'_a : a \in \mathfrak{A}^f\}$ is upwards-directed, and

$$\sup B = \sup\{e'_a \cap a : a \in \mathfrak{A}^f\} = \sup\{e \cap a : a \in \mathfrak{A}^f\} = e$$

because $(\mathfrak{A},\bar\mu)$ is semi-finite. Accordingly $e \in \mathfrak{B}$. As e is arbitrary, $\mathfrak{B}$ is closed, as claimed.

323I Notation In the context of 323H, I will say simply that $\mathfrak{B}$ is a **closed subalgebra** of $\mathfrak{A}$.

323J Proposition If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then the topological closure $\overline{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{A}$ is precisely the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

proof Write $\mathfrak{B}_\tau$ for the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. By 313Fc, $\mathfrak{B}_\tau$ is a subalgebra of $\mathfrak{A}$, and is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$. Being an order-closed subalgebra of $\mathfrak{A}$, it is topologically closed, by 323H, and must include $\overline{\mathfrak{B}}$. On the other hand, $\overline{\mathfrak{B}}$, being topologically closed, is order-closed (323D(c-i)), so includes $\mathfrak{B}_\tau$. Thus $\overline{\mathfrak{B}} = \mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

323K I note some simple results for future reference.

Lemma If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then for any $a \in \mathfrak{A}$ the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{a\}$ is closed.

proof By 314Ja, $\mathfrak{C}$ is order-closed.

323L Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of measure algebras with simple product $(\mathfrak{A},\bar\mu)$ (322K). Then the measure-algebra topology on $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i$ defined by $\bar\mu$ is just the product of the measure-algebra topologies of the $\mathfrak{A}_i$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A. Write $\mathfrak{T}$ for the measure-algebra topology of $\mathfrak{A}$ and $\mathfrak{S}$ for the product topology. For $i\in I$, $d\in\mathfrak{A}_i^f$ define a pseudometric $\tilde\rho_{di}$ on $\mathfrak{A}$ by setting

$$\tilde\rho_{di}(b,c)=\rho_d(b(i),c(i))$$

whenever b, $c\in\mathfrak{A}$; then $\mathfrak{S}$ is defined by $\mathrm{P}=\{\tilde\rho_{di}:i\in I,\,a\in\mathfrak{A}_i^f\}$ (3A3Ig). Now each $\tilde\rho_{di}$ is one of the defining pseudometrics for $\mathfrak{T}$, since

$$\tilde\rho_{di}(b,c)=\bar\mu(\tilde d\cap(b\triangle c))$$

where $\tilde d(i)=d$, $\tilde d(j)=0$ for $j\ne i$. So $\mathfrak{S}\subseteq\mathfrak{T}$.

Now suppose that $a\in\mathfrak{A}^f$ and $\epsilon>0$. Then $\sum_{i\in I}\bar\mu_i a(i)=\bar\mu a$ is finite, so there is a finite set $J\subseteq I$ such that $\sum_{i\in I\setminus J}\bar\mu_i a(i)\le\frac{1}{2}\epsilon$. For each $j\in J$, $\tau_j=\tilde\rho_{a(j),j}$ belongs to P, and

$$\begin{aligned}\rho_a(b,c)&=\sum_{i\in I}\bar\mu_i(a(i)\cap(b(i)\triangle c(i)))\le\sum_{j\in J}\bar\mu_j(a(j)\cap(b(j)\triangle c(j)))+\frac{1}{2}\epsilon\\&=\sum_{j\in J}\tau_j(b(j),c(j))+\frac{1}{2}\epsilon\le\epsilon\end{aligned}$$

whenever b, c are such that $\tau_j(b(j),c(j))\le\epsilon/(1+2\#(J))$ for every $j\in J$. By 2A3H, the identity map from $(\mathfrak{A},\mathfrak{S})$ to $(\mathfrak{A},\mathfrak{T})$ is continuous, that is, $\mathfrak{T}\subseteq\mathfrak{S}$.

Putting these together, we see that $\mathfrak{S}=\mathfrak{T}$, as claimed.

323X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Show that the set $\{(a,b):a\subseteq b\}$ is a closed set in $\mathfrak{A}\times\mathfrak{A}$.

>(b) Let (X,Σ,μ) be a σ-finite measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{F^\bullet:F\in\mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then $\{F:F\in\Sigma,\,F^\bullet\in\mathfrak{B}\}$ is a σ-subalgebra of Σ.

(c) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra, and $C\subseteq\mathfrak{A}$ a set such that $\sup A$, $\inf A$ belong to C for all non-empty subsets A of C. Show that C is closed for the measure-algebra topology.

(d) (i) Show that if $(\mathfrak{A},\bar\mu)$ is any measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then its topological closure $\overline{\mathfrak{B}}$ is again a subalgebra. (ii) Use this fact instead of 313Fc to prove 323J.

(e) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $e\in\mathfrak{A}$; let $\mathfrak{A}_e$ be the principal ideal of $\mathfrak{A}$ generated by e, and $\bar\mu_e$ its measure (322H). Show that the topology on $\mathfrak{A}_e$ defined by $\bar\mu_e$ is just the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$.

>(f) Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. (i) Show that we have an injection $\chi:\mathfrak{A}\to L^0(\mu)$ (see §241) given by setting $\chi(E^\bullet)=(\chi E)^\bullet$ for every $E\in\Sigma$. (ii) Show that χ is a homeomorphism between $\mathfrak{A}$ and its image if $\mathfrak{A}$ is given its measure-algebra topology and $L^0(\mu)$ is given its topology of convergence in measure (245A).

(g) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{A}^f$ the ideal of elements of finite measure. For a, $b\in\mathfrak{A}^f$ set $\rho(a,b)=\bar\mu(a\triangle b)$. Show that $(\mathfrak{A}^f,\bar\mu)$ is a complete metric space and that the operations $\cup$, $\cap$, $\setminus$ and $\triangle$ are uniformly continuous on $\mathfrak{A}^f$, while $\bar\mu:\mathfrak{A}^f\to\mathbb{R}$ is also uniformly continuous. Show that the embedding $\mathfrak{A}^f\subseteq\mathfrak{A}$ is continuous for the measure-algebra topology on $\mathfrak{A}$. In the context of 323Xf, show that $\chi:\mathfrak{A}^f\to L^0(\mu)$ is an isometry between $\mathfrak{A}^f$ and a subset of $L^1(\mu)$.

323Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra. Show that a set $F\subseteq\mathfrak{A}$ is closed for the measure-algebra topology iff $e\in F$ whenever there are non-empty sets B, $C\subseteq\mathfrak{A}$ such that B is upwards-directed, C is downwards-directed, $\sup B=\inf C=e$ and $[b,c]\cap F\ne\emptyset$ for every $b\in B$, $c\in C$, writing $[b,c]=\{d:b\subseteq d\subseteq c\}$.

(b) Give an example to show that (a) is false for general localizable measure algebras.

(c) Give an example of a semi-finite measure algebra $(\mathfrak{A}, \bar{\mu})$ with an order-closed subalgebra which is not closed for the measure-algebra topology.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and write $\mathbb{B}$ for the family of closed subalgebras of $\mathfrak{A}$. For $\mathfrak{B}$, $\mathfrak{C} \in \mathbb{B}$ set $\rho(\mathfrak{B}, \mathfrak{C}) = \sup_{b\in\mathfrak{B}} \inf_{c\in\mathfrak{C}} \bar{\mu}(b \triangle c) + \sup_{c\in\mathfrak{C}} \inf_{b\in\mathfrak{B}} \bar{\mu}(b \triangle c)$. Show that $(\mathbb{B}, \rho)$ is a complete metric space.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Show that it is separable in its measure-algebra topology. (*Hint*: 245Yj.)

323Z Problem Find a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$, a σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ which converges, for the measure-algebra topology, to a member of $\mathfrak{A} \setminus \mathfrak{B}$.

323 Notes and comments The message of this section is that the topology of a measure algebra is essentially defined by its order and algebraic structure; see also 324F-324H below. Of course the results are really about semi-finite measure algebras, and indeed this whole volume, like the rest of measure theory, has little of interest to say about others; they are included only because they arise occasionally and it is not absolutely essential to exclude them. We therefore expect to be able to describe such things as closed subalgebras and continuous homomorphisms in terms of the ordering, as in 323H and 324G. For σ-finite algebras, indeed, there is an easy description of the topology in terms of the order (323Ya). I think the result of this section which I shall most often wish to quote is 323I: in most contexts, there is no need to distinguish between 'closed subalgebra' and 'order-closed subalgebra'. I conjecture, however, that a σ-subalgebra of a localizable measure algebra need not be topologically sequentially closed (323Z).

It is also the case that the topology of a measure algebra corresponds very closely indeed to the topology of convergence in measure. A description of this correspondence is in 323Xf. Indeed all the results of this section have analogues in the theory of topological Riesz spaces. I will enlarge on the idea here in §367. For the moment, however, if you look back to Chapter 24, you will see that 323B and 323G are closely paralleled by 245D and 245E, while 323Ya is related to 245L.

It is I think natural to ask whether there are any other topological Boolean algebras with the properties 323B-323D. In fact a question in this direction, the Control Measure Problem, is one of the most important questions outstanding in abstract measure theory. I will discuss it in §393; the particular form relevant to the present section is what I call 'CM_4' (393J).

324 Homomorphisms

In the course of Volume 2, I had occasion to remark that elementary measure theory was unusual among abstract topics in pure mathematics in not being dominated by any particular class of structure-preserving operators. We now come to what I think is one of the reasons for the gap: the most important operators of the theory are not between measure spaces at all, but between their measure algebras. In this section I run through the most elementary facts about Boolean homomorphisms between measure algebras. I start with results on the construction of such homomorphisms from functions between measure spaces (324A-324E), then investigate continuity and order-continuity of homomorphisms (324F-324H) before turning to measure-preserving homomorphisms (324I-324O).

324A Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ their measure algebras. Write $\hat{\Sigma}$ for the domain of the completion $\hat{\mu}$ of μ. Let $D \subseteq X$ be a set of full outer measure (definition: 132E), and let $\hat{\Sigma}_D$ be the subspace σ-algebra on D induced by $\hat{\Sigma}$. Let $\phi : D \to Y$ be a function such that $\phi^{-1}[F] \in \hat{\Sigma}_D$ for every $F \in \mathrm{T}$ and $\phi^{-1}[F]$ is μ-negligible whenever $\nu F = 0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by the formula

$$\pi F^\bullet = E^\bullet \text{ whenever } F \in \mathrm{T},\ E \in \Sigma \text{ and } (E \cap D) \triangle \phi^{-1}[F] \text{ is negligible.}$$

proof Let $F \in \mathrm{T}$. Then there is an $H \in \hat{\Sigma}$ such that $H \cap D = \phi^{-1}[F]$; now there is an $E \in \Sigma$ such that $E \triangle H$ is negligible, so that $(E \cap D) \triangle \phi^{-1}[F]$ is negligible. If E_1 is another member of Σ such that $(E_1 \cap D) \triangle \phi^{-1}[F]$ is negligible, then $(E \triangle E_1) \cap D$ is negligible, so is included in a negligible member G of Σ. Since $(E \triangle E_1) \setminus G$ belongs to Σ and is disjoint from D, it is negligible; accordingly $E \triangle E_1$ is negligible and $E^\bullet = E_1^\bullet$ in $\mathfrak{A}$.

What this means is that the formula offered defines a map $\pi : \mathfrak{B} \to \mathfrak{A}$. It is now easy to check that π is a Boolean homomorphism, because if

$$(E \cap D) \triangle \phi^{-1}[F], \quad (E' \cap D) \triangle \phi^{-1}[F']$$

are negligible, so are

$$((X \setminus E) \cap D) \triangle \phi^{-1}[Y \setminus F], \quad ((E \cup E') \cap D) \triangle \phi^{-1}[F \cup F'].$$

To see that π is sequentially order-continuous, let $\langle b_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{B}$. For each n we may choose an $F_n \in \mathrm{T}$ such that $F_n^\bullet = b_n$, and $E_n \in \Sigma$ such that $(E_n \cap D) \triangle \phi^{-1}[F_n]$ is negligible; now, setting $F = \bigcup_{n \in \mathbb{N}} F_n$, $E = \bigcup_{n \in \mathbb{N}} E_n$,

$$(E \cap D) \triangle \phi^{-1}[F] \subseteq \bigcup_{n \in \mathbb{N}} (E_n \cap D) \triangle \phi^{-1}[F_n]$$

is negligible, so

$$\pi(\sup_{n \in \mathbb{N}} b_n) = \pi(F^\bullet) = E^\bullet = \sup_{n \in \mathbb{N}} E_n^\bullet = \sup_{n \in \mathbb{N}} \pi b_n.$$

(Recall that the maps $E \mapsto E^\bullet$, $F \mapsto F^\bullet$ are sequentially order-continuous, by 321H.) So π is sequentially order-continuous (313L(c-iii)).

324B Corollary Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ their measure algebras. Let $\phi : X \to Y$ be a function such that $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$ and $\mu\phi^{-1}[F] = 0$ whenever $\nu F = 0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by the formula

$$\pi F^\bullet = (\phi^{-1}[F])^\bullet \text{ for every } F \in \mathrm{T}.$$

324C Remarks (a) In §235 and elsewhere in Volume 2 I spent a good deal of time on functions between measure spaces which satisfy the conditions of 324A. Indeed, I take the trouble to spell 324A out in such generality just in order to catch these applications. Some of the results of the present chapter (322D, 322Jb) can also be regarded as special cases of 324A.

(b) The question of which homomorphisms between the measure algebras of measure spaces (X, Σ, μ), (Y, T, ν) can be realized by functions between X and Y is important and deep; I will return to it in §§343-344.

(c) In the simplified context of 324B, I have actually defined a contravariant functor; the relevant facts are the following.

324D Proposition Let (X, Σ, μ), (Y, T, ν) and (Z, Λ, λ) be measure spaces, with measure algebras $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$, $(\mathfrak{C}, \bar{\lambda})$. Suppose that $\phi : X \to Y$ and $\psi : Y \to Z$ satisfy the conditions of 324B, that is,

$$\phi^{-1}[F] \in \Sigma \text{ if } F \in \mathrm{T}, \quad \mu\phi^{-1}[F] = 0 \text{ if } \nu F = 0,$$

$$\psi^{-1}[G] \in \mathrm{T} \text{ if } G \in \Lambda, \quad \mu\psi^{-1}[G] = 0 \text{ if } \lambda G = 0.$$

Let $\pi_\phi : \mathfrak{B} \to \mathfrak{A}$, $\pi_\psi : \mathfrak{C} \to \mathfrak{B}$ be the corresponding homomorphisms. Then $\psi\phi : X \to Z$ is another map of the same type, and $\pi_{\psi\phi} = \pi_\phi \pi_\psi : \mathfrak{C} \to \mathfrak{A}$.

proof The necessary checks are all elementary.

324E Stone spaces While in the context of general measure spaces the question of realizing homomorphisms is difficult, in the case of the Stone representation it is relatively straightforward.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W, as described in 321J-321K, and Σ, T their domains. If $\pi : \mathfrak{B} \to \mathfrak{A}$ is any order-continuous Boolean homomorphism, let $\phi : Z \to W$ be the corresponding continuous function, as described in 312P. Then $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$, $\mu\phi^{-1}[F] = 0$ whenever $\nu F = 0$, and (writing E^* for the member of $\mathfrak{A}$ corresponding to $E \in \Sigma$) $\pi F^* = (\phi^{-1}[F])^*$ for every $F \in \mathrm{T}$.

proof Recall that $E^* = a$ iff $E \triangle \widehat{a}$ is meager, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. In particular, $\mu E = 0$ iff E is meager. Now the point is that $\phi^{-1}[F]$ is nowhere dense in Z whenever F is a nowhere dense subset of W, by 313R. Consequently $\phi^{-1}[F]$ is meager whenever F is meager in W, since F is then just a countable union of nowhere dense sets. Thus we see already that $\mu\phi^{-1}[F] = 0$ whenever $\nu F = 0$. If F is any member of T, there is an open-and-closed set F_0 such that $F \triangle F_0$ is meager; now $\phi^{-1}[F_0]$ is open-and-closed, so $\phi^{-1}[F] = \phi^{-1}[F_0] \triangle \phi^{-1}[F \triangle F_0]$ belongs to Σ. Moreover, if $b \in \mathfrak{B}$ is such that $\widehat{b} = F_0$, and $a = \pi b$, then $\widehat{a} = \phi^{-1}[F_0]$, so

$$\pi F^* = \pi b = a = (\phi^{-1}[F_0])^* = (\phi^{-1}[F])^*,$$

as required.

324F I turn now to the behaviour of order-continuous homomorphisms between measure algebras.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $\mathfrak{B}, \bar{\nu})$ be measure algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) π is continuous iff it is continuous at 0 iff it is uniformly continuous.

(b) If $(\mathfrak{B}, \bar{\nu})$ is semi-finite and π is continuous, then it is order-continuous.

(c) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and π is order-continuous, then it is continuous.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a) Suppose that π is continuous at 0; I seek to show that it is uniformly continuous. Take $b \in \mathfrak{B}^f$ and $\epsilon > 0$. Then there are $a_0, \dots, a_n \in \mathfrak{A}^f$ and $\delta > 0$ such that

$$\bar{\nu}(b \cap \pi c) = \rho_b(\pi c, 0) \le \epsilon \text{ whenever } \max_{i \le n} \rho_{a_i}(c, 0) \le \delta;$$

setting $a = \sup_{i \le n} a_i$,

$$\bar{\nu}(b \cap \pi c) \le \epsilon \text{ whenever } \bar{\mu}(a \cap c) \le \delta.$$

Now suppose that $\rho_a(c, c') \le \delta$. Then $\bar{\mu}(a \cap (c \triangle c')) \le \delta$, so

$$\rho_b(\pi c, \pi c') = \bar{\nu}(b \cap (\pi c \triangle \pi c')) = \bar{\nu}(b \cap \pi(c \triangle c')) \le \epsilon.$$

As b, ϵ are arbitrary, π is uniformly continuous. The rest of the implications are elementary.

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then $0 \in \overline{A}$ (323D(b-ii)); because π is continuous, $0 \in \overline{\pi[A]}$. **?** If b is a non-zero lower bound for $\pi[A]$ in $\mathfrak{B}$, then (because $(\mathfrak{B}, \bar{\nu})$ is semi-finite) there is a $c \subseteq b$ with $0 < \bar{\nu}c < \infty$; now

$$\rho_c(\pi a, 0) = \bar{\nu}(c \cap \pi a) = \bar{\nu}c > 0$$

for every $a \in A$, so $0 \notin \overline{\pi[A]}$. **X**

Thus $\inf \pi[A] = 0$ in $\mathfrak{B}$; as A is arbitrary, π is order-continuous (313L(b-ii)).

(c) By (a), it will be enough to show that π is continuous at 0. Let $b \in \mathfrak{B}^f$, $\epsilon > 0$. **?** Suppose, if possible, that for every $a \in \mathfrak{A}^f$, $\delta > 0$ there is a $c \in \mathfrak{A}$ such that $\bar{\mu}(a \cap c) \le \delta$ but $\bar{\nu}(b \cap \pi c) \ge \epsilon$. For each $a \in \mathfrak{A}^f$, $n \in \mathbb{N}$ choose c_{an} such that $\bar{\mu}(a \cap c_{an}) \le 2^{-n}$ but $\bar{\nu}(b \cap \pi c_{an}) \ge \epsilon$. Set $c_a = \inf_{n \in \mathbb{N}} \sup_{m \ge n} c_{am}$; then

$$\bar{\mu}(a \cap c_a) \le \inf_{n \in \mathbb{N}} \sum_{m=n}^{\infty} \bar{\mu}(a \cap c_{an}) = 0,$$

so $c_a \cap a = 0$. On the other hand, because π is order-continuous, $\pi c_a = \inf_{n \in \mathbb{N}} \sup_{m \ge n} \pi c_{am}$, so that

$$\bar{\nu}(b \cap \pi c_a) = \lim_{n \to \infty} \bar{\nu}(b \cap \sup_{m \ge n} \pi c_{am}) \ge \epsilon.$$

This shows that

$$\rho_b(1 \setminus a, 0) = \bar{\nu}(b \cap \pi(1 \setminus a)) \ge \bar{\nu}(b \cap \pi c_a) \ge \epsilon.$$

But now observe that $A = \{1 \setminus a : a \in \mathfrak{A}^f\}$ is a downwards-directed subset of $\mathfrak{A}$ with infimum 0, because $(\mathfrak{A}, \bar{\mu})$ is semi-finite. So $\pi[A]$ is downwards-directed and has infimum 0, and 0 must be in the closure of $\pi[A]$, by 323D(b-ii); while we have just seen that $\rho_b(d, 0) \ge \epsilon$ for every $d \in \pi[A]$. **X**

Thus there must be $a \in \mathfrak{A}^f$, $\delta > 0$ such that

$$\rho_b(\pi c, 0) = \bar\nu(b \cap \pi c) \le \epsilon$$

whenever

$$\rho_a(c, 0) = \bar\mu(a \cap c) \le \delta.$$

As b, ϵ are arbitrary, π is continuous at 0 and therefore continuous.

324G Corollary If $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are semi-finite measure algebras, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is continuous iff it is order-continuous.

324H Corollary If $\mathfrak{A}$ is a Boolean algebra and $\bar\mu$, $\bar\nu$ are two measures both rendering $\mathfrak{A}$ a semi-finite measure algebra, then they endow $\mathfrak{A}$ with the same uniformity (and, of course, the same topology).

proof By 324G, the identity map from $\mathfrak{A}$ to itself is continuous whichever of the topologies we place on $\mathfrak{A}$; and by 324F it is therefore uniformly continuous.

324I Definition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras. A Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is **measure-preserving** if $\bar\nu(\pi a) = \bar\mu a$ for every $a \in \mathfrak{A}$.

324J Proposition Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ and $(\mathfrak{C}, \bar\lambda)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$, $\theta : \mathfrak{B} \to \mathfrak{C}$ measure-preserving Boolean homomorphisms. Then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism.

proof Elementary.

324K Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism.

(a) π is injective.

(b) $(\mathfrak{A}, \bar\mu)$ is totally finite iff $(\mathfrak{B}, \bar\nu)$ is, and in this case π is order-continuous, therefore continuous, and $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$.

(c) If $(\mathfrak{A}, \bar\mu)$ is semi-finite and $(\mathfrak{B}, \bar\nu)$ is σ-finite, then $(\mathfrak{A}, \bar\mu)$ is σ-finite.

(d) If $(\mathfrak{A}, \bar\mu)$ is σ-finite and π is sequentially order-continuous, then $(\mathfrak{B}, \bar\nu)$ is σ-finite.

(e) If $(\mathfrak{A}, \bar\mu)$ is semi-finite and π is order-continuous, then $(\mathfrak{B}, \bar\nu)$ is semi-finite.

(f) If $(\mathfrak{A}, \bar\mu)$ is atomless and semi-finite, and π is order-continuous, then $\mathfrak{B}$ is atomless.

(g) If $\mathfrak{B}$ is purely atomic and $(\mathfrak{A}, \bar\mu)$ is semi-finite, then $\mathfrak{A}$ is purely atomic.

proof (a) If $a \ne 0$ in $\mathfrak{A}$, then $\bar\nu\pi a = \bar\mu a > 0$ so $\pi a \ne 0$. By 3A2Db, π is injective.

(b) Because

$$\bar\nu 1_{\mathfrak{B}} = \bar\nu\pi 1_{\mathfrak{A}} = \bar\mu 1_{\mathfrak{A}},$$

$(\mathfrak{A}, \bar\mu)$ is totally finite iff $(\mathfrak{B}, \bar\nu)$ is. Now suppose that $A \subseteq \mathfrak{A}$ is downwards-directed and non-empty and that $\inf A = 0$. Then

$$\inf_{a\in A} \bar\nu\pi a = \inf_{a\in A} \bar\mu a = 0$$

by 321F. So $\bar\nu b = 0$ for any lower bound b of $\pi[A]$, and $\inf \pi[A] = 0$. As A is arbitrary, π is order-continuous.

By 324Fc, π is continuous. By 314Fa, $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$, that is, 'closed' in the sense of 323I.

(c) I appeal to 322G. If C is a disjoint family in $\mathfrak{A} \setminus \{0\}$, then $\langle \pi c\rangle_{c\in C}$ is a disjoint family in $\mathfrak{B} \setminus \{0\}$, so is countable, and C must be countable, because π is injective. Thus $\mathfrak{A}$ is ccc and (being semi-finite) $(\mathfrak{A}, \bar\mu)$ is σ-finite.

(d) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar\mu a_n < \infty$ for every n and $\sup_{n\in\mathbb{N}} a_n = 1$. Then $\bar\nu\pi a_n < \infty$ for every n and (because π is sequentially order-continuous) $\sup_{n\in\mathbb{N}} \pi a_n = 1$, so $(\mathfrak{B}, \bar\nu)$ is σ-finite.

(e) Setting $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$, $\sup \mathfrak{A}^f = 1$; because π is order-continuous, $\sup \pi[\mathfrak{A}^f] = 1$ in $\mathfrak{B}$. So if $\bar\nu b = \infty$, there is an $a \in \mathfrak{A}^f$ such that $\pi a \cap b \ne 0$, and now $0 < \bar\nu(b \cap \pi a) < \infty$.

(f) Take any non-zero $b \in \mathfrak{B}$. As in (e), there is an $a \in \mathfrak{A}$ such that $\bar{\mu}a < \infty$ and $a \cap b \neq 0$. If $a \cap b \neq b$, then surely b is not an atom. Otherwise, set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a,\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a, so $c_0 = \inf C$ is defined in $\mathfrak{A}$ (321F), and

$$\bar{\mu}c_0 = \inf_{c \in C} \bar{\mu}c \geq \bar{\nu}b > 0,$$

so $c_0 \neq 0$. Because $\mathfrak{A}$ is atomless, there is a $d \subseteq c_0$ such that neither d nor $c_0 \setminus d$ is zero, so that neither $c_0 \setminus d$ nor d can belong to C. But this means that $b \cap \pi d$ and $b \cap \pi(c_0 \setminus d)$ are both non-zero, so that again b is not an atom. As b is arbitrary, $\mathfrak{B}$ is atomless.

(g) Take any non-zero $a \in \mathfrak{A}$. Then there is an $a' \subseteq a$ such that $0 < \bar{\mu}a' < \infty$. Because $\mathfrak{B}$ is purely atomic, there is an atom b of $\mathfrak{B}$ with $b \subseteq \pi a'$. Set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a',\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a', so $c_0 = \inf C$ is defined in $\mathfrak{A}$, and

$$\bar{\mu}c_0 = \inf_{c \in C} \bar{\mu}c \geq \bar{\nu}b > 0,$$

so $c_0 \neq 0$. If $d \subseteq c_0$, then $b \cap \pi d$ must be either b or 0. If $b \cap \pi d = b$, then $d \in C$ and $d = c_0$. If $b \cap \pi d = 0$, then $c_0 \setminus d \in C$ and $d = 0$. Thus c_0 is an atom in $\mathfrak{A}$. As a is arbitrary, $\mathfrak{A}$ is purely atomic.

324L Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $(\mathfrak{B}, \bar{\nu})$ a measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof This is a special case of 314Gb.

324M Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$. Let $\phi : X \to Y$ be inverse-measure-preserving. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by setting $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

proof This is immediate from 324B.

324N Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W. If $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous measure-preserving Boolean homomorphism, and $\phi : Z \to W$ the corresponding continuous function, then ϕ is inverse-measure-preserving.

proof Use 324E. In the notation there, if $F \in \mathrm{T}$, then

$$\nu F = \bar{\nu}F^* = \bar{\mu}\pi F^* = \bar{\mu}\phi^{-1}[F]^* = \mu\phi^{-1}[F].$$

324O Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, $\mathfrak{A}_0$ a topologically dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi a = \bar{\mu}a$ for every $a \in \mathfrak{A}_0$. Then π has a unique extension to a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof Let ρ, σ be the standard metrics on $\mathfrak{A}$, $\mathfrak{B}$, as in 323Ad. Then for any a, $a' \in \mathfrak{A}_0$

$$\sigma(\pi a, \pi a') = \bar{\nu}(\pi a \triangle \pi a') = \bar{\nu}\pi(a \triangle a') = \bar{\mu}(a \triangle a') = \rho(a, a');$$

that is, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ is an isometry. Because $\mathfrak{A}_0$ is dense in the metric space $(\mathfrak{A}, \rho)$, while $\mathfrak{B}$ is complete under σ (323Gc), there is a unique continuous function $\hat{\pi} : \mathfrak{A} \to \mathfrak{B}$ extending π (3A4G). Now the operations

$$(a, a') \mapsto \hat{\pi}(a \cup a'), \quad (a, a') \mapsto \hat{\pi}a \cup \hat{\pi}a' : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B},$$

are continuous and agree on the dense subset $\mathfrak{A}_0 \times \mathfrak{A}_0$ of $\mathfrak{A} \times \mathfrak{A}$; because the topology of $\mathfrak{B}$ is Hausdorff, they agree on $\mathfrak{A} \times \mathfrak{A}$, that is, $\hat{\pi}(a \cup a') = \hat{\pi}a \cup \hat{\pi}a'$ for all a, $a' \in \mathfrak{A}$ (2A3Uc). Similarly, the operations

$$a \mapsto \hat{\pi}(1 \setminus a), \quad a \mapsto 1 \setminus \hat{\pi}a : \mathfrak{A} \to \mathfrak{B}$$

are continuous and agree on the dense subset $\mathfrak{A}_0$ of $\mathfrak{A}$, so they agree on $\mathfrak{A}$, that is, $\hat{\pi}(1 \setminus a) = 1 \setminus a$ for every $a \in \mathfrak{A}$. Thus $\hat{\pi}$ is a Boolean homomorphism. To see that it is measure-preserving, observe that

$$a \mapsto \bar{\mu}a = \rho(a, 0), \quad a \mapsto \bar{\nu}(\hat{\pi}a) = \sigma(\hat{\pi}a, 0) : \mathfrak{A} \to \mathbb{R}$$

are continuous and agree on $\mathfrak{A}_0$, so agree on $\mathfrak{A}$. Finally, $\hat{\pi}$ is the only measure-preserving Boolean homomorphism extending π, because any such map must be continuous (324Kb), and $\hat{\pi}$ is the only continuous extension of π.

***324P** The following fact will be useful in §386, by which time it will seem perfectly elementary; for the moment, it may be a useful exercise.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras such that $\bar{\mu}1 = \bar{\nu}1$. Suppose that $A \subseteq \mathfrak{A}$ and $\phi : A \to \mathfrak{B}$ are such that $\bar{\nu}(\inf_{i \le n} \phi a_i) = \bar{\mu}(\inf_{i \le n} a_i)$ for all $a_0, \dots, a_n \in A$. Let $\mathfrak{C}$ be the smallest closed subalgebra of $\mathfrak{A}$ including A. Then ϕ has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

proof (a) Let Ψ be the family of all functions ψ extending ϕ and having the same properties; that is, ψ is a function from a subset of $\mathfrak{A}$ to $\mathfrak{B}$, and $\bar{\nu}(\inf_{i \le n} \psi a_i) = \bar{\mu}(\inf_{i \le n} a_i)$ for all $a_0, \dots, a_n \in \operatorname{dom} \psi$. By Zorn's Lemma, Ψ has a maximal member θ. Write D for the domain of θ.

(b)(i) If c, $d \in D$ then $c \cap d \in D$. **P?** Otherwise, set $D' = D \cup \{c \cap d\}$ and extend θ to $\theta' : D' \to \mathfrak{B}$ by writing $\theta'(c \cap d) = \theta c \cap \theta d$. It is easy to check that $\theta' \in \Psi$, which is supposed to be impossible. **XQ**

Now

$$\bar{\nu}(\theta c \cap \theta d \cap \theta(c \cap d)) = \bar{\mu}(c \cap d) = \bar{\nu}(\theta c \cap \theta d) = \bar{\nu}\theta(c \cap d),$$

so $\theta(c \cap d) = \theta c \cap \theta d$.

(ii) If $d \in D$ then $1 \setminus d \in D$. **P?** Otherwise, set $D' = D \cup \{1 \setminus d\}$ and extend θ to D' by writing $\theta'(1 \setminus d) = 1 \setminus \theta d$. Once again, it is easy to check that $\theta' \in \Psi$, which is impossible. **XQ**

Consequently (since D is certainly not empty, even if C is), D is a subalgebra of $\mathfrak{A}$ (312B(iii)).

(iii) Since

$$\bar{\nu}\theta 1 = \bar{\mu}1 = \bar{\nu}1,$$

$\theta 1 = 1$. If $d \in D$ then

$$\bar{\nu}\theta(1 \setminus d) = \bar{\mu}(1 \setminus d) = \bar{\mu}1 - \bar{\mu}d = \bar{\nu}1 - \bar{\nu}\theta d = \bar{\nu}(1 \setminus \theta d),$$

while

$$\bar{\nu}(\theta d \cap \theta(1 \setminus d)) = \bar{\mu}(d \cap (1 \setminus d)) = 0,$$

so $\theta d \cap \theta(1 \setminus d)) = 0$, $\theta(1 \setminus d) \subseteq 1 \setminus \theta d$ and $\theta(1 \setminus d)$ must be equal to $1 \setminus \theta d$.

By 312H, $\theta : D \to \mathfrak{B}$ is a Boolean homomorphism.

(iv) Let $\mathfrak{D}$ be the topological closure of D in $\mathfrak{A}$. Then it is an order-closed subalgebra of $\mathfrak{A}$ (323J), so, with $\bar{\mu}\restriction\mathfrak{D}$, is a totally finite measure algebra in which D is a topologically dense subalgebra. By 324O, there is an extension of θ to a measure-preserving Boolean homomorphism from $\mathfrak{D}$ to $\mathfrak{B}$; of course this extension belongs to Ψ, so in fact $D = \mathfrak{D}$ is a closed subalgebra of $\mathfrak{A}$.

(c) Since $A \subseteq D$, $\mathfrak{C} \subseteq \mathfrak{D}$ and $\phi_1 = \theta\restriction\mathfrak{C}$ is a suitable extension of ϕ.

To see that ϕ_1 is unique, let $\phi_2 : \mathfrak{C} \to \mathfrak{B}$ be any other measure-preserving Boolean homomorphism extending ϕ. Set $C = \{a : \phi_1 a = \phi_2 a\}$; then C is a topologically closed subalgebra of $\mathfrak{A}$ including A, so is the whole of $\mathfrak{C}$, and $\phi_2 = \phi_1$.

324X Basic exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind σ-complete, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A}/I \to \mathfrak{B}$ given by setting $\phi(a^\bullet) = \phi a$ for every $a \in \mathfrak{A}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Show that *provided that* $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ *is semi-finite*, then the topology of $\mathfrak{B}$ induced by $\bar{\mu}\restriction\mathfrak{B}$ is just the subspace topology induced by the topology of $\mathfrak{A}$. (*Hint*: apply 324Fc to the embedding $\mathfrak{B}\subseteq\mathfrak{A}$.)

(c) Let (X,Σ,μ) be a measure space and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version. Let $\mathfrak{A}$, $\mathfrak{A}_2$ be the corresponding measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}_2$ the canonical homomorphism (see 322Db). Show that π is topologically continuous.

(d) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a bijective measure-preserving Boolean homomorphism. Show that $\pi^{-1}:\mathfrak{B}\to\mathfrak{A}$ is a measure-preserving homomorphism.

(e) Let $\bar{\mu}$ be counting measure on $\mathcal{P}\mathbb{N}$. Show that $(\mathcal{P}\mathbb{N},\bar{\mu})$ is a σ-finite measure algebra. Find a measure-preserving Boolean homomorphism from $\mathcal{P}\mathbb{N}$ to itself which is not sequentially order-continuous.

324Y Further exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind complete, and $\phi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have an order-continuous Boolean homomorphism $\pi:\mathfrak{A}/I\to\mathfrak{B}$ given by setting $\phi(a^\bullet)=\phi a$ for every $a\in\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathcal{Z}$ for the family of nowhere dense zero sets of Z; let $\mathcal{Z}_\sigma$ be the σ-ideal of subsets of Z generated by $\mathcal{Z}$. Show that $\Sigma=\{E\triangle U:E\in\mathcal{E},\,U\in\mathcal{Z}_\sigma\}$ is a σ-algebra of subsets of Z, and describe a canonical isomorphism between $\Sigma/\mathcal{Z}_\sigma$ and $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, with Stone spaces Z and W. Construct $\mathcal{Z}_\sigma\subseteq\Sigma\subseteq\mathcal{P}Z$ as in 324Yb, and let $\mathcal{W}_\sigma\subseteq\mathrm{T}\subseteq\mathcal{P}W$ be the corresponding structure defined from $\mathfrak{B}$. Let $\pi:\mathfrak{B}\to\mathfrak{A}$ be a sequentially order-continuous Boolean homomorphism, and $\phi:Z\to W$ the corresponding continuous map. Show that if $E^*\in\mathfrak{A}$ corresponds to $E\in\Sigma$, then $\pi F^*=\phi^{-1}[F]^*$ for every $F\in\mathrm{T}$. (*Hint*: 313Yb.)

(d) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a ccc Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ an injective Boolean homomorphism. Show that $\mathfrak{A}$ is ccc.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra, and $\pi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous Boolean homomorphism. Show that for every atom $b\in\mathfrak{B}$ there is an atom $a\in\mathfrak{A}$ such that $\pi a\supseteq b$. Hence show that if $\mathfrak{A}$ is atomless so is $\mathfrak{B}$, and that if $\mathfrak{B}$ is purely atomic and π is injective then $\mathfrak{A}$ is purely atomic.

(f) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be localizable measure algebras and $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$. Suppose that $\pi:\mathfrak{A}_0\to\mathfrak{B}$ is an order-continuous Boolean homomorphism such that $\bar{\nu}\pi a=\bar{\mu}a$ for every $a\in\mathfrak{A}_0$. Show that π has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. (i) Show that there is an injective order-preserving function $f:\mathfrak{A}\to\mathcal{P}\mathbb{N}$. (*Hint*: take a countable topologically dense subset D of $\mathfrak{A}$, and define $f:\mathfrak{A}\to\mathcal{P}(D\times\mathbb{N})$ by setting $f(a)=\{(d,q):\bar{\mu}(a\cap d)\ge q\}$.) (ii) Show that there is an order-preserving function $h:\mathcal{P}\mathbb{N}\to\mathfrak{A}$ such that $h(f(a))=a$ for every $a\in\mathfrak{A}$. (*Hint*: set $h(I)=\sup\{a:f(a)\subseteq I\}$.) Compare 316Yo.

(h) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, and $f:\mathfrak{A}\to\mathfrak{B}$ an isometry for the measure metrics. Show that $a\mapsto f(a)\triangle f(1)$ is a measure-preserving Boolean homomorphism.

324 Notes and comments If you examine the arguments of this section carefully, you will see that rather little depends on the measures named. Really this material deals with structures $(X,\Sigma,\mathcal{I})$ where X is a set, Σ is a σ-ideal of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ, corresponding to the family of measurable negligible sets. In this abstract form it is natural to think in terms of sequentially order-continuous homomorphisms, as in 324Yc. I have stated 324E in terms of order-continuous homomorphisms just for a slight gain in simplicity. But in fact, when there is a difference, it is likely that order-continuity, rather than sequential order-continuity, will be the more significant condition. Note that when the domain algebra is σ-finite, the two concepts coincide, because it is ccc (316Fd, 322G).

Of course I need to refer to measures when looking at such concepts as σ-finite measure algebra or measure-preserving homomorphism, but even here the real ideas involved are such notions as order-continuity and the countable chain condition, as you will see if you work through 324K. It is instructive to look at the translations of these facts into the context of inverse-measure-preserving functions; see 235Xe.

324H shows that we may speak of 'the' topology and uniformity of a Dedekind σ-complete Boolean algebra which carries any semi-finite measure; the topology of such an algebra is determined by its algebraic structure. Contrast this with the theory of normed spaces: two Banach spaces (e.g., ℓ^1 and ℓ^2) can be isomorphic as linear spaces, both being of algebraic dimension $\mathfrak{c}$, while they are not isomorphic as topological linear spaces. When we come to the theory of ordered linear topological spaces, however, we shall again find ourselves with operators whose algebraic properties guarantee continuity (355C, 367P).

325 Free products and product measures

In this section I aim to describe the measure algebras of product measures as defined in Chapter 25. This will involve the concept of 'free product' set out in §315. It turns out that we cannot determine the measure algebra of a product measure from the measure algebras of the factors (325B), unless the product measure is localizable; but that there is nevertheless a general construction of 'localizable measure algebra free product', applicable to any pair of semi-finite measure algebras (325D), which represents the measure algebra of the product measure in the most important cases (325Eb). In the second part of the section (325I-325M) I deal with measure algebra free products of probability algebras, corresponding to the products of probability spaces treated in §254.

325A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$. Let λ be the c.l.d. product measure on $X\times Y$, and Λ its domain; let $(\mathfrak{C},\bar\lambda)$ be the corresponding measure algebra.

(a)(i) The map $E\mapsto E\times Y:\Sigma\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

(ii) The map $F\mapsto X\times F:\mathrm{T}\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$.

(b) The map $(E,F)\mapsto E\times F:\Sigma\times\mathrm{T}\to\Lambda$ induces a Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$.

(c) $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$.

(d) For every $c\in\mathfrak{C}$,

$$\bar\lambda c=\sup\{\bar\lambda(c\cap\psi(a\otimes b)):a\in\mathfrak{A},\ b\in\mathfrak{B},\ \bar\mu a<\infty,\ \bar\nu b<\infty\}.$$

(e) If μ and ν are semi-finite, ψ is injective and $\bar\lambda\psi(a\otimes b)=\bar\mu a\cdot\bar\mu b$ for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$.

proof (a) By 251E, $E\times Y\in\Lambda$ for every $E\in\Sigma$, and $\lambda_0(E\times Y)=0$ whenever $\mu E=0$, where λ_0 is the primitive product measure described in 251A-251C; consequently $\lambda(E\times Y)=0$ whenever $\mu E=0$ (251F). Thus $E\mapsto(E\times Y)^\bullet:\Sigma\to\mathfrak{C}$ is a Boolean homomorphism with kernel including $\{E:\mu E=0\}$, so descends to a Boolean homomorphism $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$.

To see that ε_1 is order-continuous, let $A\subseteq\mathfrak{A}_1$ be a non-empty downwards-directed set with infimum 0. **?** If there is a non-zero lower bound c of $\varepsilon_1[A]$, express c as $W^\bullet$ where $W\in\Lambda$. We have $\lambda(W)>0$; by the definition of λ (251F), there are $G\in\Sigma$, $H\in\mathrm{T}$ such that $\mu G<\infty$, $\nu H<\infty$ and $\lambda(W\cap(G\times H))>0$. Of course $\inf_{a\in A}a\cap G^\bullet=0$ in $\mathfrak{A}$, so $\inf_{a\in A}\bar\mu(a\cap G^\bullet)=0$, by 321F; let $a\in A$ be such that $\bar\mu(a\cap G^\bullet)\cdot\nu H<\lambda(W\cap(G\times H))$. Express a as $E^\bullet$, where $E\in\Sigma$. Then $\lambda(W\setminus(E\times Y))=0$. But this means that

$$\lambda(W\cap(G\times H))\le\lambda((E\cap G)\times H)=\mu(E\cap G)\cdot\nu H=\bar\mu(a\cap G^\bullet)\cdot\nu H,$$

contradicting the choice of a. **X** Thus $\inf\varepsilon_1[A]=0$ in $\mathfrak{C}$; as A is arbitrary, ε_1 is order-continuous.

Similarly $\varepsilon_2:\mathfrak{B}\to\mathfrak{C}$, induced by $F\mapsto X\times F:\mathrm{T}\to\Lambda$, is order-continuous.

(b) Now there must be a corresponding Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$ such that $\psi(a\otimes b)=\varepsilon_1 a\cap\varepsilon_2 b$ for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$, that is,

$$\psi(E^\bullet \otimes F^\bullet) = (E \times Y)^\bullet \cap (X \times F)^\bullet = (E \times F)^\bullet$$

for every $E \in \Sigma$, $F \in \mathrm{T}$ (315I).

(c) Suppose that c, $e \in \mathfrak{C}$, $\bar\lambda e < \infty$ and $\epsilon > 0$. Express c, e as $U^\bullet$, $W^\bullet$ where U, $W \in \Lambda$. By 251Ie, there are $E_0, \dots, E_n \in \Sigma$, $F_0, \dots, F_n \in \mathrm{T}$, all of finite measure, such that $\lambda((U \cap W) \triangle \bigcup_{i \le n} E_i \times F_i) \le \epsilon$. Set

$$c_1 = (\bigcup_{i \le n} E_i \times F_i)^\bullet \in \psi[\mathfrak{A} \otimes \mathfrak{B}];$$

then

$$\bar\lambda(e \cap (c \mathbin{\triangle} c_1)) = \lambda(W \cap (U \triangle \bigcup_{i \le n} E_i \times F_i)) \le \epsilon.$$

As c, e and ϵ are arbitrary, $\psi[\mathfrak{A} \otimes \mathfrak{B}]$ is topologically dense in $\mathfrak{C}$.

(d) By the definition of λ, we have

$$\lambda W = \sup\{\lambda(W \cap (E \times F)) : E \in \Sigma,\ F \in \mathrm{T},\ \mu E < \infty,\ \nu F < \infty\}$$

for every $W \in \Lambda$; so all we have to do is express c as $W^\bullet$.

(e) Now suppose that μ and ν are semi-finite. Then $\lambda(E \times F) = \mu E \cdot \nu F$ for any $E \in \Sigma$, $F \in \mathrm{T}$ (251J), so $\bar\lambda\psi(a \otimes b) = \bar\mu a \cdot \bar\nu b$ for every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

To see that ψ is injective, take any non-zero $c \in \mathfrak{A} \otimes \mathfrak{B}$; then there must be non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ such that $a \otimes b \subseteq c$ (315Jb), so that

$$\bar\lambda\psi c \ge \bar\lambda\psi(a \otimes b) = \bar\mu a \cdot \bar\nu b > 0$$

and $\psi c \ne 0$.

325B Characterizing the measure algebra of a product space A very natural question to ask is, whether it is possible to define a 'measure algebra free product' of two abstract measure algebras in a way which will correspond to one of the constructions above. I give an example to show the difficulties involved.

Example There are complete locally determined localizable measure spaces (X, μ), (X', μ'), with isomorphic measure algebras, and a probability space (Y, ν) such that the measure algebras of the c.l.d. product measures on $X \times Y$, $X' \times Y$ are not isomorphic.

proof Let (X, Σ, μ) be the complete locally determined localizable not-strictly-localizable measure space described in 216E. Recall that, for $E \in \Sigma$, $\mu E = \#(\{\gamma : \gamma \in C,\ f_\gamma \in E\})$ if this is finite, ∞ otherwise (216Eb), where C is a set of cardinal greater than $\mathfrak{c}$. The map $E \mapsto \{\gamma : f_\gamma \in E\} : \Sigma \to \mathcal{P}C$ is surjective (216Ec), so descends to an isomorphism between $\mathfrak{A}$, the measure algebra of μ, and $\mathcal{P}C$. Let (X', Σ', μ') be C with counting measure, so that its measure algebra $(\mathfrak{A}', \bar\mu')$ is isomorphic to $(\mathfrak{A}, \bar\mu)$, while μ' is of course strictly localizable.

Let (Y, T, ν) be $\{0,1\}^C$ with its usual measure. Let λ, λ' be the c.l.d. product measures on $X \times Y$, $X' \times Y$ respectively, and $(\mathfrak{C}, \bar\lambda)$, $(\mathfrak{C}', \bar\lambda')$ the corresponding measure algebras. Then λ is not localizable (254U), so $(\mathfrak{C}, \bar\lambda)$ is not localizable (322Be). On the other hand, λ', being the c.l.d. product of strictly localizable measures, is strictly localizable (251N), therefore localizable, so $(\mathfrak{C}', \bar\lambda')$ is localizable, and is not isomorphic to $(\mathfrak{C}, \bar\lambda)$.

325C Thus there can be no universally applicable method of identifying the measure algebra of a product measure from the measure algebras of the factors. However, you have no doubt observed that the example above involves non-σ-finite spaces, and conjectured that this is not an accident. In contexts in which we know that all the algebras involved are localizable, there are positive results available, such as the following.

Theorem Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be semi-finite measure spaces, with measure algebras $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$. Let λ be the c.l.d. product measure on $X_1 \times X_2$, and $(\mathfrak{C}, \bar\lambda)$ the corresponding measure algebra. Let $(\mathfrak{B}, \bar\nu)$ be a localizable measure algebra, and $\phi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\phi_2 : \mathfrak{A}_2 \to \mathfrak{B}$ order-continuous Boolean homomorphisms such that $\bar\nu(\phi_1(a_1) \cap \phi_2(a_2)) = \bar\mu_1 a_1 \cdot \bar\mu_2 a_2$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$. Then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi(\psi(a_1 \otimes a_2)) = \phi_1(a_1) \cap \phi_2(a_2)$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, writing $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ for the canonical map described in 325A.

proof (a) Because ψ is injective, it is an isomorphism between $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ as actually a subalgebra of $\mathfrak{C}$. Now the Boolean homomorphisms ϕ_1, ϕ_2 correspond to a Boolean homomorphism $\theta : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{B}$. The point is that $\bar{\nu}\theta c = \bar{\lambda}c$ for every $c \in \mathfrak{A} \otimes \mathfrak{B}$. **P** By 315Jb, every member of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is expressible as $\sup_{i \le n} a_i \otimes a'_i$, where $a_i \in \mathfrak{A}_1$, $a'_i \in \mathfrak{A}_2$ and $\langle a_i \otimes a'_i \rangle_{i \le n}$ is disjoint. Now for each i we have

$$\bar{\nu}\theta(a_i \otimes a'_i) = \bar{\nu}(\phi_1(a_i) \cap \phi_2(a'_i)) = \bar{\mu}_1 a_i \cdot \bar{\mu}_2 a'_i = \bar{\lambda}(a_i \otimes a'_i),$$

by 325Ad. So

$$\bar{\nu}\theta(c) = \sum_{i=0}^n \bar{\nu}\theta(a_i \otimes a'_i) = \sum_{i=0}^n \bar{\lambda}(a_i \otimes a'_i) = \bar{\lambda}c. \text{ } \mathbf{Q}$$

(b) The following fact will underlie many of the arguments below. If $e \in \mathfrak{B}$, $\bar{\nu}e < \infty$ and $\epsilon > 0$, there are $e_1 \in \mathfrak{A}_1^f$, $e_2 \in \mathfrak{A}_2^f$ such that $\bar{\nu}(e \setminus \theta(e_1 \otimes e_2)) \le \epsilon$, writing $\mathfrak{A}_i^f = \{a : \bar{\mu}_i a < \infty\}$. **P** Because $(\mathfrak{A}_1, \bar{\mu}_1)$ is semi-finite, $\mathfrak{A}_1^f$ has supremum 1 in $\mathfrak{A}_1$; because ϕ_1 is order-continuous, $\sup\{\phi_1(a) : a \in \mathfrak{A}_1^f\} = 1$ in $\mathfrak{B}$, and $\inf\{e \setminus \phi_1(a) : a \in \mathfrak{A}_1^f\} = 0$ (313Aa). Because $\mathfrak{A}_1^f$ is upwards-directed, $\{e \setminus \phi_1(a) : a \in \mathfrak{A}_1^f\}$ is downwards-directed, so $\inf\{\bar{\nu}(e \setminus \phi(a) : a \in \mathfrak{A}_1^f\} = 0$ (321F). Let $e_1 \in \mathfrak{A}_1^f$ be such that $\bar{\nu}(e \setminus \phi_1(e_1)) \le \frac{1}{2}\epsilon$.

In the same way, there is an $e_2 \in \mathfrak{A}_2^f$ such that $\bar{\nu}(e \setminus \phi_2(e_2)) \le \frac{1}{2}\epsilon$. Consider $e' = e_1 \otimes e_2 \in \mathfrak{C}$. Then

$$\bar{\nu}(e \setminus \theta e') = \bar{\nu}(e \setminus (\phi_1(e_1) \cap \phi_2(e_2))) \le \bar{\nu}(e \setminus \phi_1(e_1)) + \bar{\nu}(e \setminus \phi_2(e_2)) \le \epsilon. \text{ } \mathbf{Q}$$

(c) The next step is to check that θ is uniformly continuous for the uniformities defined by $\bar{\nu}$, $\bar{\lambda}$. **P** Take any $e \in \mathfrak{B}^f$ and $\epsilon > 0$. By (b), there are e_1, e_2 such that $\bar{\lambda}(e_1 \otimes e_2) < \infty$ and $\bar{\nu}(e \setminus \theta(e_1 \otimes e_2)) \le \frac{1}{2}\epsilon$. Set $e' = e_1 \otimes e_2$. Now suppose that c, $c' \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$ and $\bar{\lambda}((c \triangle c') \cap e') \le \frac{1}{2}\epsilon$. Then

$$\bar{\nu}((\theta(c) \triangle \theta(c')) \cap e) \le \bar{\nu}\theta((c \triangle c') \cap e') + \bar{\nu}(e \setminus \theta e') \le \bar{\lambda}((c \triangle c') \cap e') + \frac{1}{2}\epsilon \le \epsilon.$$

By 3A4Cc, θ is uniformly continuous for the subspace uniformity on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$. **Q**

(d) Recall that $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is topologically dense in $\mathfrak{C}$ (325Ab), while $\mathfrak{B}$ is complete for its uniformity (323Gc). So there is a uniformly continuous function $\phi : \mathfrak{C} \to \mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c \mapsto \phi(1 \setminus c)$, $c \mapsto 1 \setminus \phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c : \phi(1 \setminus c) = 1 \setminus \phi(c)\}$ is closed; as it includes $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c, c') \mapsto \phi(c \cup c')$, $(c, c') \mapsto \phi(c) \cup \phi(c')$ are continuous, so $\{(c, c') : \phi(c \cup c') = \phi(c) \cup \phi(c')\}$ is closed in $\mathfrak{C} \times \mathfrak{C}$; as it includes $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \times (\mathfrak{A}_1 \otimes \mathfrak{A}_2)$, it must be the whole of $\mathfrak{C} \times \mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** Take any $e_1 \in \mathfrak{A}_1^f$, $e_2 \in \mathfrak{A}_2^f$. Then the functions $c \mapsto \bar{\lambda}(c \cap (e_1 \otimes e_2))$, $c \mapsto \bar{\nu}\phi(c \cap (e_1 \otimes e_2))$ are continuous and equal on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, so are equal on $\mathfrak{C}$. The argument of (b) shows that for any $b \in \mathfrak{B}$,

$$\begin{aligned} \bar{\nu}b &= \sup\{\bar{\nu}(b \cap e) : e \in \mathfrak{B}^f\} \\ &= \sup\{\bar{\nu}(b \cap \phi(e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\}, \end{aligned}$$

so that

$$\begin{aligned} \bar{\nu}\phi(c) &= \sup\{\bar{\nu}\phi(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\} \\ &= \sup\{\bar{\lambda}(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\} = \bar{\lambda}c \end{aligned}$$

for every $c \in \mathfrak{C}$. **Q**

(g) To see that ϕ is order-continuous, take any non-empty downwards-directed set $C \subseteq \mathfrak{C}$ with infimum 0. **?** If $\phi[C]$ has a non-zero lower bound b in $\mathfrak{B}$, let $e \subseteq b$ be such that $0 < \bar{\nu}e < \infty$. Let $e' \in \mathfrak{C}$ be such that $\bar{\lambda}e' < \infty$ and $\bar{\nu}(e \setminus \phi(e')) < \bar{\nu}e$, as in (b) above, so that $\bar{\nu}(e \cap \phi(e')) > 0$. Now, because $\inf C = 0$, there is a $c \in C$ such that $\bar{\lambda}(c \cap e') < \bar{\nu}(e \cap \phi(e'))$. But this means that

$$\bar{\nu}(b \cap \phi(e')) \le \bar{\nu}\phi(c \cap e') = \bar{\lambda}(c \cap e') < \bar{\nu}(e \cap \phi(e')) \le \bar{\nu}(b \cap \phi(e')),$$

which is absurd. **X** Thus $\inf \phi[C] = 0$ in $\mathfrak{B}$. As C is arbitrary, ϕ is order-continuous.

(h) Finally, to see that ϕ is unique, observe that any order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous (324Fc); so that if it agrees with ϕ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ it must agree with ϕ on $\mathfrak{C}$.

325D Theorem Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be semi-finite measure algebras.

(a) There is a localizable measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with order-continuous Boolean homomorphisms $\psi_1 : \mathfrak{A}_1 \to \mathfrak{C}$, $\psi_2 : \mathfrak{A}_2 \to \mathfrak{C}$ such that whenever $(\mathfrak{B}, \bar{\nu})$ is a localizable measure algebra, and $\phi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\phi_2 : \mathfrak{A}_2 \to \mathfrak{B}$ are order-continuous Boolean homomorphisms and $\bar{\nu}(\phi_1(a_1) \cap \phi_2(a_2)) = \bar{\mu}_1 a_1 \cdot \bar{\mu}_2 a_2$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\psi_j = \phi_j$ for both j.

(b) The structure $(\mathfrak{C}, \bar{\lambda}, \psi_1, \psi_2)$ is determined up to isomorphism by this property.

(c)(i) The Boolean homomorphism $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ defined from ψ_1 and ψ_2 is injective, and $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is topologically dense in $\mathfrak{C}$.

(ii) The order-closed subalgebra of $\mathfrak{C}$ generated by $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is the whole of $\mathfrak{C}$.

(d) If $j \in \{1, 2\}$ and $(\mathfrak{A}_j, \bar{\mu}_j)$ is localizable, then $\psi_j[\mathfrak{A}_j]$ is a closed subalgebra of $(\mathfrak{C}, \bar{\lambda})$.

proof (a)(i) We may regard $(\mathfrak{A}_1, \bar{\mu}_1)$ as the measure algebra of (Z_1, Σ_1, μ_1) where Z_1 is the Stone space of $\mathfrak{A}_1$, Σ_1 is the algebra of subsets of Z_1 differing from an open-and-closed set by a meager set, and μ_1 is an appropriate measure (321K). Note that in this representation, each $a \in \mathfrak{A}_1$ becomes identified with $\widehat{a}^\bullet$, where $\widehat{a}$ is the open-and-closed subset of Z_1 corresponding to a. Similarly, we may think of $(\mathfrak{A}_2, \bar{\mu}_2)$ as the measure algebra of (Z_2, Σ_2, μ_2), where Z_2 is the Stone space of $\mathfrak{A}_2$.

(ii) Let λ be the c.l.d. product measure on $Z_1 \times Z_2$. The point is that λ is strictly localizable. **P** By 322E, both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have partitions of unity consisting of elements of finite measure; let $\langle c_i \rangle_{i \in I}$, $\langle d_j \rangle_{j \in J}$ be such partitions. Then $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i \in I, j \in J}$ is a disjoint family of sets of finite measure in $Z_1 \times Z_2$. If $W \subseteq Z_1 \times Z_2$ is such that $\lambda W > 0$, there must be sets E_1, E_2 of finite measure such that $\lambda(W \cap (E_1 \times E_2)) > 0$. Because $E_1^\bullet = \sup_{i \in I} E_1^\bullet \cap c_i$, we must have

$$\mu_1 E_1 = \bar{\mu}_1 E_1^\bullet = \sum_{i \in I} \bar{\mu}_1(E_1^\bullet \cap c_i) = \sum_{i \in I} \mu_1(E_1 \cap \widehat{c}_i).$$

Similarly, $\mu_2 E_2 = \sum_{i \in J} \mu_2(E_2 \cap \widehat{d}_j)$. But this means that there must be finite $I' \subseteq I$, $J' \subseteq J$ such that

$$\sum_{i \in I', j \in J'} \mu_1(E_1 \cap \widehat{c}_i)\mu_2(E_2 \cap \widehat{d}_j) > \mu_1 E_1 \cdot \mu_2 E_2 - \lambda(W \cap (E_1 \times E_2)),$$

so that there have to be $i \in I'$, $j \in J'$ such that $\lambda(W \cap (\widehat{c}_i \times \widehat{d}_j)) > 0$.

Now this means that $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i \in I, j \in J}$ satisfies the conditions of 213O. Because λ is surely complete and locally determined, it is strictly localizable. **Q**

(iii) We may therefore take $(\mathfrak{C}, \bar{\lambda})$ to be just the measure algebra of λ. The maps ψ_1, ψ_2 will be the canonical maps described in 325Aa, inducing the map $\psi : \mathfrak{A}_1 \otimes \mathfrak{B}_1 \to \mathfrak{C}$ referred to in 325C; and 325C now gives the result.

(b) This is nearly obvious. Suppose we had an alternative structure $(\mathfrak{C}', \bar{\lambda}', \psi_1', \psi_2')$ with the same property. Then we must have an order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{C}'$ such that $\phi\psi_j = \psi_j'$ for both j; and similarly we have an order-continuous measure-preserving Boolean homomorphism $\phi' : \mathfrak{C}' \to \mathfrak{C}$ such that $\phi'\psi_j' = \psi_j$ for both j. Now $\phi'\phi : \mathfrak{C} \to \mathfrak{C}$ is an order-continuous measure-preserving Boolean homomorphism such that $\phi'\psi_j = \psi_j$ for both j. By the uniqueness assertion in (a), applied with $\mathfrak{B} = \mathfrak{C}$, $\phi'\phi$ must be the identity on $\mathfrak{C}$. In the same way, $\phi\phi'$ is the identity on $\mathfrak{C}'$. So ϕ and ϕ' are the two halves of the required isomorphism.

(c) In view of the construction for $\mathfrak{C}$ offered in part (a) of the proof, (i) is just a consequence of 325Ac and 325Ae. Now (ii) follows by 323J.

(d) If $\mathfrak{A}_j$ is Dedekind complete then $\psi_j[\mathfrak{A}_j]$ is order-closed in $\mathfrak{C}$ because ψ_j is order-continuous (314F(a-i)).

325E Remarks (a) We could say that a measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with embeddings ψ_1 and ψ_2, as described in 325D, is a **localizable measure algebra free product** of $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$; and its uniqueness up to isomorphism makes it safe, most of the time, to call it 'the' localizable measure algebra free product. Observe that it can equally well be regarded as the uniform space completion of the algebraic free product; see 325Yb.

(b) As the example in 325B shows, the localizable measure algebra free product of the measure algebras of given measure spaces need not appear directly as the measure algebra of their product. But there is one context in which it must so appear: if the product measure is localizable, 325C tells us at once that it has the right measure algebra. For σ-finite measure algebras, of course, any corresponding measure spaces have to be strictly localizable, so again we can use the product measure directly.

325F I ought not to proceed to the next topic without giving another pair of examples to show the subtlety of the concept of 'measure algebra free product'.

Example Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure μ on $[0, 1]$, and $(\mathfrak{C}, \bar{\lambda})$ the measure algebra of Lebesgue measure λ on $[0, 1]^2$. Then $(\mathfrak{C}, \bar{\lambda})$ can be regarded as the localizable measure algebra free product of $(\mathfrak{A}, \bar{\mu})$ with itself, by 251M and 325Eb. Let $\psi : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{C}$ be the canonical map, as described in 325A. Then $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ is not order-dense in $\mathfrak{C}$, and ψ is not order-continuous.

proof (a) Let $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \epsilon_n = \infty$, but $\sum_{n=0}^{\infty} \epsilon_n^2 < 1$; for instance, we could take $\epsilon_n = \frac{1}{n+2}$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a stochastically independent sequence of measurable subsets of $[0, 1]$ such that $\mu E_n = \epsilon_n$ for each n. In $\mathfrak{A}$ set $a_n = E_n^\bullet$, and consider $c_n = \sup_{i\le n} a_i \otimes a_i \in \mathfrak{A} \otimes \mathfrak{A}$ for each n.

(b) We have $\sup_{n\in\mathbb{N}} c_n = 1$ in $\mathfrak{A} \otimes \mathfrak{A}$. **P?** Otherwise, there is a non-zero $a \in \mathfrak{A} \otimes \mathfrak{A}$ such that $a \cap a_n = 0$ for every n, and now there are non-zero b, $b' \in \mathfrak{A}$ such that $b \otimes b' \subseteq a$. Set $I = \{n : a_n \cap b = 0\}$, $J = \{n : a_n \cap b'\} = 0$. Then $\langle E_n\rangle_{n\in I}$ is an independent family and $\mu(\bigcup_{n\in I} E_i) \le 1 - \bar{\mu}b < 1$, so $\sum_{n\in I} \mu E_n < \infty$, by the Borel-Cantelli lemma (273K). Similarly $\sum_{n\in J} \mu E_n < \infty$. Because $\sum_{n\in\mathbb{N}} \mu E_n = \infty$, there must be some $n \in \mathbb{N} \setminus (I \cup J)$. Now $a_n \cap b$ and $a_n \cap b'$ are both non-zero, so

$$0 \ne (a_n \cap b) \otimes (a_n \cap b') = (a_n \otimes a_n) \cap (b \otimes b') = 0,$$

which is absurd. **XQ**

(c) On the other hand,

$$\sum_{n=0}^{\infty} \bar{\lambda}\psi(c_n) \le \sum_{n=0}^{\infty} (\bar{\mu}a_n)^2 = \sum_{n=0}^{\infty} \epsilon_n^2 < \infty,$$

by the choice of the ϵ_n. So $\sup_{n\in\mathbb{N}} \psi(c_n)$ cannot be 1 in $\mathfrak{C}$.

Thus ψ is not order-continuous.

(d) By 313P(a-ii) and 313O, $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ cannot be order-dense in $\mathfrak{C}$; alternatively, (b) shows that there can be no non-zero member of $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ included in $1 \setminus \sup_{n\in\mathbb{N}} \psi(c_n)$. (Both these arguments rely tacitly on the fact that ψ is injective, as noted in 325Ae.)

325G Since 325F shows that the free product and the localizable measure algebra free product are very different constructions, I had better repeat an idea from §315 in the new context.

Example Again, let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$, and $(\mathfrak{C}, \bar{\lambda})$ the measure algebra of Lebesgue measure on $[0, 1]^2$. Then there is no order-continuous Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{A}$ such that $\phi(a \otimes b) = a \cap b$ for all a, $b \in \mathfrak{A}$. **P** Let $\phi : \mathfrak{C} \to \mathfrak{A}$ be a Boolean homomorphism such that $\phi(a\otimes b) = a \cap b$ for all a, $b \in \mathfrak{A}$. For $i < 2^n$ let a_{ni} be the equivalence class in $\mathfrak{A}$ of the interval $[2^{-n}i, 2^{-n}(i+1)]$, and set $c_n = \sup_{i<2^n} a_{ni} \otimes a_{ni}$. Then $\phi c_n = 1$ for every n, but $\bar{\lambda}c_n = 2^{-n}$ for each n, so $\inf_{n\in\mathbb{N}} c_n = 0$ in $\mathfrak{C}$; thus ϕ cannot be order-continuous. **Q** (Compare 315P.)

***325H Products of more than two factors** We can of course extend the ideas of 325A, 325C and 325D to products of any finite number of factors. No new ideas are needed, so I spell the results out without proofs.

(a) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a finite family of semi-finite measure algebras. Then there is a localizable measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with order-continuous Boolean homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ for $i \in I$, such that whenever $(\mathfrak{B}, \bar{\nu})$ is a localizable measure algebra, and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ are order-continuous Boolean homomorphisms such that $\bar{\nu}(\inf_{i\in I} \phi_i(a_i)) = \prod_{i\in I} \bar{\mu}_i a_i$ whenever $a_i \in \mathfrak{A}_i$ for each i, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\psi_i = \phi_i$ for every i.

(b) The structure $(\mathfrak{C}, \bar{\lambda}, \langle\psi_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi : \bigotimes_{i\in I}\mathfrak{A}_i \to \mathfrak{C}$ defined from the ψ_i is injective, and $\psi[\bigotimes_{i\in I}\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

(d) Write $\widehat{\bigotimes}^{\text{loc}}_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)$ for (a particular version of) the localizable measure algebra free product described in (a). If $\langle(A_i, \bar{\mu}_i)\rangle_{i\in I}$ is a finite family of semi-finite measure algebras and $\langle I(k)\rangle_{k\in K}$ is a partition of I, then $\widehat{\bigotimes}^{\text{loc}}_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)$ is isomorphic, in a canonical way, to $\widehat{\bigotimes}^{\text{loc}}_{k\in K}\bigl(\widehat{\bigotimes}^{\text{loc}}_{i\in I(k)}(\mathfrak{A}_i, \bar{\mu}_i)\bigr)$.

(e) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a finite family of semi-finite measure spaces, and write $(\mathfrak{A}_i, \bar{\mu}_i)$ for the measure algebra of (X_i, Σ_i, μ_i). Let λ be the c.l.d. product measure on $\prod_{i\in I} X_i$ (251W), and $(\mathfrak{C}, \bar{\lambda})$ the corresponding measure algebra. Then there is a canonical order-continuous measure-preserving embedding of $(\mathfrak{C}, \bar{\lambda})$ into the localizable measure algebra free product of the $(\mathfrak{A}_i, \bar{\mu}_i)$. If each μ_i is strictly localizable, this embedding is an isomorphism.

325I Infinite products Just as in §254, we can now turn to products of infinite families of probability algebras.

Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any family of probability spaces, with measure algebras $(\mathfrak{A}_i, \bar{\mu}_i)$. Let λ be the product measure on $\prod_{i\in I} X_i$, and $(\mathfrak{C}, \bar{\lambda})$ the corresponding measure algebra. For each $i \in I$, we have a measure-preserving homomorphism $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ corresponding to the inverse-measure-preserving function $x \mapsto x(i) : X \to X_i$. Let $(\mathfrak{B}, \bar{\nu})$ be a probability algebra, and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ Boolean homomorphisms such that $\bar{\nu}(\inf_{i\in J}\phi_i(a_i)) = \prod_{i\in J}\bar{\mu}_i a_i$ whenever $J \subseteq I$ is a finite set and $a_i \in \mathfrak{A}_i$ for every i. Then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\psi_i = \phi_i$ for every $i \in I$.

proof (a) As remarked in 254Fb, all the maps $x \mapsto x(i)$ are inverse-measure-preserving, so correspond to measure-preserving homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ (324M). It will be helpful to use some notation from §254. Write $\mathcal{C}$ for the family of subsets of X expressible in the form

$$E = \{x : x \in X,\ x(i) \in E_i \text{ for every } i \in J\},$$

where $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for every $i \in J$. Note that in this case

$$E^\bullet = \inf_{i\in J}\psi_i(E_i^\bullet).$$

Set

$$C = \{E^\bullet : E \in \mathcal{C}\} \subseteq \mathfrak{C},$$

so that C is precisely the family of elements of $\mathfrak{C}$ expressible in the form $\inf_{i\in J}\phi_i(a_i)$ where $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for each i.

The homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ define a Boolean homomorphism $\psi : \bigotimes_{i\in I}\mathfrak{A}_i \to \mathfrak{C}$ (315I), which is injective. **P** If $c \in \bigotimes_{i\in I}\mathfrak{A}_i$ is non-zero, there must be a finite set $J \subseteq I$ and a family $\langle a_i\rangle_{i\in J}$ such that $a_i \in \mathfrak{A}_i \setminus \{0\}$ for each i and $c \supseteq \inf_{i\in J}\varepsilon_i(a_i)$ (315Jb). Express each a_i as $E_i^\bullet$, where $E_i \in \Sigma_i$. Then

$$E = \{x : x \in X,\ x(i) \in E_i \text{ for each } i \in J\}$$

has measure

$$\lambda E = \prod_{i\in J}\mu E_i = \prod_{i\in J}\bar{\mu} a_i \neq 0,$$

while

$$E^\bullet = \psi(\inf_{i\in J}\varepsilon_i(a_i)) \subseteq \psi(c),$$

so $\psi(c) \neq 0$. As c is arbitrary, ψ is injective. **Q**

(b) Because ψ is injective, it is an isomorphism between $\bigotimes_{i\in I}\mathfrak{A}_i$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\bigotimes_{i\in I}\mathfrak{A}_i$ as actually a subalgebra of $\mathfrak{C}$, so that $\psi_j : \mathfrak{A}_j \to \mathfrak{C}$ becomes identified with the canonical map $\varepsilon_j : \mathfrak{A}_j \to \bigotimes_{i\in I}\mathfrak{A}_i$. Now the Boolean homomorphisms $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ correspond to a Boolean homomorphism $\theta : \bigotimes_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$. The point is that $\bar{\nu}\theta(c) = \bar{\lambda}c$ for every

$c \in \bigotimes_{i \in I} \mathfrak{A}_i$. **P** Suppose to begin with that $c \in C$. Then we have $c = E^\bullet$, where $E = \{x : x(i) \in E_i \ \forall \ i \in J\}$ and $E_i \in \Sigma_i$ for each $i \in J$. So

$$\bar{\lambda}c = \lambda E = \prod_{i \in J} \mu E_i = \prod_{i \in J} \bar{\mu}_i E_i^\bullet = \bar{\nu}(\inf_{i \in J} \phi a_i)$$
$$= \bar{\nu}(\inf_{i \in J} \theta\psi_i(a_i)) = \bar{\nu}\theta(\inf_{i \in J} \psi_i(a_i)) = \bar{\nu}\theta(c).$$

Next, any $c \in \mathfrak{C}$ is expressible as the supremum of a finite disjoint family $\langle c_k \rangle_{k \in K}$ in C (315Jb), so

$$\bar{\nu}\theta(c) = \sum_{k \in K} \bar{\nu}\theta(c_k) = \sum_{k \in K} \bar{\lambda}(c_k) = \bar{\lambda}c. \quad \mathbf{Q}$$

(c) It follows that θ is uniformly continuous for the metrics defined by $\bar{\nu}$, $\bar{\lambda}$, since

$$\bar{\nu}(\theta(c) \triangle \theta(c')) = \bar{\nu}\theta(c \triangle c') = \bar{\lambda}(c \triangle c')$$

for all c, $c' \in \bigotimes_{i \in I} \mathfrak{A}_i$.

(d) Next, $\bigotimes_{i \in I} \mathfrak{A}_i$ is topologically dense in $\mathfrak{C}$. **P** Let $c \in \mathfrak{C}$, $\epsilon > 0$. Express c as $W^\bullet$. Then by 254Fe there are $H_0, \dots, H_k \in \mathcal{C}$ such that $\lambda(W \triangle \bigcup_{j \le k} H_j) \le \epsilon$. Now $c_j = H_j^\bullet \in C$ for each j, so

$$c' = \sup_{j \le k} c_j = (\bigcup_{j \le k} H_j)^\bullet \in \bigotimes_{i \in I} \mathfrak{A}_i,$$

and $\bar{\lambda}(c \triangle c') \le \epsilon$. **Q**

Since $\mathfrak{B}$ is complete for its uniformity (323Gc), there is a uniformly continuous function $\phi : \mathfrak{C} \to \mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c \mapsto \phi(1 \setminus c)$, $1 \setminus \phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c : \phi(1 \setminus c) = 1 \setminus \phi(c)\}$ is closed; as it includes $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c, c') \mapsto \phi(c \cup c')$, $(c, c') \mapsto \phi(c) \cup \phi(c')$ are continuous, so $\{(c, c') : \phi(c \cup c') = \phi(c) \cup \phi(c')\}$ is closed in $\mathfrak{C} \times \mathfrak{C}$; as it includes $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \times (\mathfrak{A}_1 \otimes \mathfrak{A}_2)$, it must be the whole of $\mathfrak{C} \times \mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** The functions $c \mapsto \bar{\lambda}c$, $c \mapsto \bar{\nu}\phi(c)$ are continuous and equal on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, so are equal on $\mathfrak{C}$. **Q**

(g) Finally, to see that ϕ is unique, observe that any measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous, so that if it agrees with ϕ on $\bigotimes_{i \in I} \mathfrak{A}_i$ it must agree with ϕ on $\mathfrak{C}$.

325J Of course this leads at once to a result corresponding to 325D.

Theorem Let $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$ be a family of probability algebras.

(a) There is a probability algebra $(\mathfrak{C}, \bar{\lambda})$, together with order-continuous Boolean homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ such that whenever $(\mathfrak{B}, \bar{\nu})$ is a probability algebra, and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ are Boolean homomorphisms such that $\bar{\nu}(\inf_{i \in J} \phi_i(a_i)) = \prod_{i \in J} \bar{\mu}_i a_i$ whenever $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for each $i \in J$, then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\psi_j = \phi_j$ for every j.

(b) The structure $(\mathfrak{C}, \bar{\lambda}, \langle \psi_i \rangle_{i \in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi : \bigotimes_{i \in I} \mathfrak{A}_i \to \mathfrak{C}$ defined from the ψ_i is injective, and $\psi[\bigotimes_{i \in I} \mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

proof For (a) and (c), all we have to do is represent each $(\mathfrak{A}_i, \bar{\mu}_i)$ as the measure algebra of a probability space, and apply 325I. The uniqueness of $\mathfrak{C}$ and the ψ_i follows from the uniqueness of the homomorphisms ϕ, as in 325D.

325K Definition As in 325Ea, we can say that $(\mathfrak{C}, \bar{\lambda}, \langle \psi_i \rangle_{i \in I})$ is a, or the, **probability algebra free product** of $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$.

325L Independent subalgebras If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, we say that a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is **(stochastically) independent** if $\bar{\mu}(\inf_{i \in J} b_i) = \prod_{i \in J} \bar{\mu} b_i$ whenever $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i$ for each i. (Compare 272Ab.) In this case the embeddings $\mathfrak{B}_i \subseteq \mathfrak{A}$ give rise to an embedding of the probability algebra free product of $\langle (\mathfrak{B}_i, \bar{\mu} \restriction \mathfrak{B}_i) \rangle_{i \in I}$ into $\mathfrak{A}$. (Compare 272J, 315Xn.)

325M We can now make a general trawl through Chapters 25 and 27 seeking results which can be expressed in the language of this section. I give some in 325Xe-325Xh. Some ideas from §254 which are thrown into sharper relief by a reformulation are in the following theorem.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\lambda,\langle\psi_i\rangle_{i\in I})$ their probability algebra free product. For $J\subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\psi_i[\mathfrak{A}_i]$.

(a) For any $J\subseteq I$, $(\mathfrak{C}_J,\bar\lambda\restriction\mathfrak{C}_J,\langle\psi_i\rangle_{i\in J})$ is a probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$.

(b) For any $c\in\mathfrak{C}$, there is a unique smallest $J\subseteq I$ such that $c\in\mathfrak{C}_J$, and this J is countable.

(c) For any non-empty family $\mathcal{J}\subseteq\mathcal{P}I$, $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J=\mathfrak{C}_{\bigcap\mathcal{J}}$.

proof (a) If $(\mathfrak{B},\bar\nu,\langle\phi_i\rangle_{i\in J})$ is any probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$, then we have a measure-preserving homomorphism $\psi:\mathfrak{B}\to\mathfrak{C}$ such that $\psi\phi_i=\psi_i$ for every $i\in J$. Because the subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ generated by $\bigcup_{i\in J}\phi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{B}$ (325Jc), and ψ is continuous (324Kb), $\bigcup_{i\in J}\psi_i[\mathfrak{A}_i]$ is topologically dense in $\psi[\mathfrak{B}]$; also $\psi[\mathfrak{B}]$ is closed in $\mathfrak{C}$ (324Kb again). But this means that $\psi[\mathfrak{B}]$ is just the topological closure of $\bigcup_{i\in I}\psi_i[\mathfrak{A}_i]$ and must be $\mathfrak{C}_J$. Thus ψ is an isomorphism, and

$$(\mathfrak{C}_J,\bar\lambda\restriction\mathfrak{C}_J,\langle\psi_i\rangle_{i\in J})=(\psi[\mathfrak{B}],\bar\nu\psi^{-1},\langle\psi\phi_i\rangle_{i\in I})$$

is also a probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$.

(b) As in 325J, we may suppose that each $(\mathfrak{A}_i,\bar\mu_i)$ is the measure algebra of a probability space (X_i,Σ_i,μ_i), and that $\mathfrak{C}$ is the measure algebra of their product (X,Λ,λ). Let $W\in\Lambda$ be such that $c=W^\bullet$.

By 254Rd, there is a unique smallest $K\subseteq I$ such that $W\triangle U$ is negligible for some $U\in\Lambda_K$, where Λ_K is the set of members of Λ which are determined by coordinates in K; and K is countable. But if we look at any $J\subseteq I$, $\{x:x(i)\in E\}\in\Lambda_J$ for every $i\in J$, $E\in\Sigma_i$; so $\{U^\bullet:U\in\Lambda_J\}$ is a closed subalgebra of $\mathfrak{C}$ including $\psi_i[\mathfrak{A}_i]$ for every $i\in J$, and therefore including $\mathfrak{C}_J$. On the other hand, as observed in 254Ob, any member of Λ_J is approximated, in measure, by sets in the σ-algebra T_J generated by sets of the form $\{x:x(i)\in E\}$ where $i\in J$, $E\in\Sigma_i$. Of course $\mathrm{T}_J\subseteq\Lambda_J$, so $\{W^\bullet:W\in\Lambda_J\}=\{W^\bullet:W\in\mathrm{T}_J\}$ is the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in K}\psi_i[\mathfrak{A}_i]$, which is $\mathfrak{C}_J$. Thus K is also the unique smallest subset of I such that $c\in\mathfrak{C}_K$.

(c) Of course $\mathfrak{C}_K\subseteq\mathfrak{C}_J$ whenever $K\subseteq J\subseteq I$, so $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J\supseteq\mathfrak{C}_{\bigcap\mathcal{J}}$. On the other hand, suppose that $c\in\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J$; then by (b) there is some $K\subseteq\bigcap\mathcal{J}$ such that $c\in\mathfrak{C}_K\subseteq\mathfrak{C}_{\bigcap\mathcal{J}}$. As c is arbitrary, $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J=\mathfrak{C}_{\bigcap\mathcal{J}}$.

***325N Notation** In this context, I will say that an element c of $\mathfrak{C}$ is **determined by coordinates in** J if $c\in\mathfrak{C}_J$.

325X Basic exercises (a) Let $(\mathfrak{A}_1,\bar\mu_1)$, $(\mathfrak{A}_2,\bar\mu_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a closed subalgebra $\mathfrak{B}_j$ of $\mathfrak{A}_j$ such that $(\mathfrak{B}_j,\bar\nu_j)$ is also semi-finite, where $\bar\nu_j=\bar\mu_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product of $(\mathfrak{B}_1,\bar\nu_1)$ and $(\mathfrak{B}_2,\bar\nu_2)$ can be thought of as a closed subalgebra of the localizable measure algebra free product of $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$.

(b) Let $(\mathfrak{A}_1,\bar\mu_1)$, $(\mathfrak{A}_2,\bar\mu_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a principal ideal $\mathfrak{B}_j$ of $\mathfrak{A}_j$. Set $\bar\nu_j=\bar\mu_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product of $(\mathfrak{B}_1,\bar\nu_1)$ and $(\mathfrak{B}_2,\bar\nu_2)$ can be thought of as a principal ideal of the localizable measure algebra free product of $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$.

>(c) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ and $\langle(\mathfrak{B}_j,\bar\nu_j)\rangle_{j\in J}$ be families of semi-finite measure algebras, with simple products $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ (322K). Show that the localizable measure algebra free product $(\mathfrak{A},\bar\mu)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B},\bar\nu)$ can be identified with the simple product of the family $\langle(\mathfrak{A}_i,\bar\mu_i)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}_j,\bar\nu_j)\rangle_{i\in I,j\in J}$.

>(d) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar\lambda,\langle\psi_i\rangle_{i\in I})$ their probability algebra free product. Suppose that for each $i\in I$ we are given a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$. Show that there is a unique measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{C}$ such that $\pi\psi_i=\psi_i\pi_i$ for every $i\in I$.

>(e) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. We say that a family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **(stochastically) independent** if $\bar{\mu}(\inf_{i \in j} a_j) = \prod_{i \in J} \bar{\mu} a_i$ for every non-empty finite $J \subseteq I$. Show that this is so iff $\langle \mathfrak{A}_i \rangle_{i \in I}$ is independent, where $\mathfrak{A}_i = \{0, a_i, 1 \setminus a_i, 1\}$ for each i. (Compare 272F.)

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{A}_i \rangle_{i \in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Let $\langle J(k) \rangle_{k \in K}$ be a disjoint family of subsets of I, and for each $k \in K$ let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in J(k)} \mathfrak{A}_i$. Show that $\langle \mathfrak{B}_k \rangle_{k \in K}$ is independent. (Compare 272K.)

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{A}_i \rangle_{i \in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. For $J \subseteq I$ let $\mathfrak{B}_J$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in J} \mathfrak{A}_i$. Show that $\bigcap \{\mathfrak{B}_{I \setminus J} : J$ is a finite subset of $I\} = \{0, 1\}$. (*Hint*: For $J \subseteq I$, show that $\bar{\mu}(b \cap c) = \bar{\mu} b \cdot \bar{\mu} c$ for every $b \in \mathfrak{B}_{I \setminus J}$, $c \in \mathfrak{B}_J$. Compare 272O, 325M.)

(h) Let $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$ be a family of probability algebras with probability algebra free product $(\mathfrak{C}, \bar{\lambda}, \langle \psi_i \rangle_{i \in I})$. For $c \in \mathfrak{C}$ let J_c be the smallest subset of I such that c belongs to the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i \in J_c} \psi_i[\mathfrak{A}_i]$. Show that if $c \subseteq d$ in $\mathfrak{C}$, then there is an $e \in \mathfrak{C}$ such that $c \subseteq e \subseteq d$ and $J_e \subseteq J_c \cap J_d$. (*Hint*: 254R.)

325Y Further exercises (a) Let μ be counting measure on $X = \{0\}$, μ' the countable-cocountable measure on $X' = \omega_1$, and ν counting measure on $Y = \omega_1$. Show that the measure algebras of the primitive product measures on $X \times Y$, $X' \times Y$ are not isomorphic.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and $\mu : \mathfrak{A} \to [0, \infty]$ a function such that $\mu 0 = 0$ and $\mu(a \cup b) = \mu a + \mu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$; suppose that $\mathfrak{A}^f = \{a : \mu a < \infty\}$ is order-dense in $\mathfrak{A}$. For $e \in \mathfrak{A}^f$, a, $b \in \mathfrak{A}$ set $\rho_e(a, b) = \mu(e \cap (a \triangle b))$. Give $\mathfrak{A}$ the uniformity defined by $\{\rho_e : \mu e < \infty\}$. (i) Show that the completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ under this uniformity has a measure $\hat{\mu}$, extending μ, under which it is a localizable measure algebra. (ii) Show that if $a \in \widehat{\mathfrak{A}}$, $\hat{\mu} a < \infty$ and $\epsilon > 0$, there is a $b \in \mathfrak{A}$ such that $\hat{\mu}(a \triangle b) \le \epsilon$. (iii) Show that for every $a \in \widehat{\mathfrak{A}}$ there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$ such that $a \supseteq \sup_{n \in \mathbb{N}} \inf_{m \ge n} a_m$ and $\hat{\mu} a = \hat{\mu}(\sup_{n \in \mathbb{N}} \inf_{m \ge n} a_m)$. (iv) In particular, the set of infima in $\widehat{\mathfrak{A}}$ of sequences in $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$. (v) Explain the relevance of this construction to the embedding $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \subseteq \mathfrak{C}$ in 325D.

(c) In 325F, set $W = \bigcup_{n \in \mathbb{N}} E_n \times E_n$. Show that if A, B are any non-negligible subsets of $[0, 1]$, then $W \cap (A \times B)$ is not negligible.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$. Show that $\mathfrak{A} \otimes \mathfrak{A}$ is ccc but not weakly (σ, ∞)-distributive. (*Hint*: (i) $\mathfrak{A} \otimes \mathfrak{A}$ is embeddable as a subalgebra of a probability algebra (ii) in the notation of 325F, look at $c_{mn} = \sup_{m \le i \le n} e_i \otimes e_i$.)

(e) Repeat 325F-325G and 325Yc-325Yd with an arbitrary atomless probability space in place of $[0, 1]$.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle a_i \rangle_{i \in I}$ a (stochastically) independent family in $\mathfrak{A}$. Show that for any $a \in \mathfrak{A}$, $\epsilon > 0$ the set $\{i : i \in I, |\bar{\mu}(a \cap a_i) - \bar{\mu} a \cdot \bar{\mu} a_i| \ge \epsilon\}$ is finite, so that $\{i : \bar{\mu}(a \cap a_i) \ne \bar{\mu} a \cdot \bar{\mu} a_i\}$ is countable. (*Hint*: 272Yd.)

325 Notes and comments 325B shows that the measure algebra of a product measure may be irregular if we have factor measures which are not strictly localizable. But two facts lead the way to the 'localizable measure algebra free product' in 325D-325E. The first is that every semi-finite measure algebra is embeddable, in a canonical way, in a localizable measure algebra (322N); and the second is that the Stone representation of a localizable measure algebra is strictly localizable (322M). It is a happy coincidence that we can collapse these two facts together in the construction of 325D. Another way of looking at the localizable measure algebra free product of two localizable measure algebras is to express it as the simple product of measure algebra free products of totally finite measure algebras, using 325Xc and the fact that for σ-finite measure algebras there is only one reasonable measure algebra free product, being that provided by any representation of them as measure algebras of measure spaces (325Eb).

Yet a third way of approaching measure algebra free products is as the uniform space completions of algebraic free products, using 325Yb. This gives the same result as the construction of 325D because the algebraic free product appears as a topologically dense subalgebra of the localizable measure algebra free product (325Dc) which is complete as uniform space (325Dc). (I have to repeat such phrases as 'topologically dense' because the algebraic free product is emphatically *not* order-dense in the measure algebra free product (325F).) The results in 251I on approximating measurable sets for a c.l.d. product measure by combinations of measurable rectangles correspond to general facts about completions of finitely-additive measures (325Yb(ii), 325Yb(iii)). It is worth noting that the completion process can be regarded as made up of two steps; first take infima of sequences of sets of finite measure, and then take arbitrary suprema (325Yb(iv)).

The idea of 325F appears in many guises, and this is only the first time that I shall wish to call on it. The point of the set $W = \bigcup_{n\in\mathbb{N}} E_n \times E_n$ is that it is a measurable subset of the square (indeed, by taking the E_n to be open sets we can arrange that W should be open), of measure strictly less than 1 (in fact, as small as we wish), such that its complement does not include any non-negligible 'measurable rectangle' $G \times H$; indeed, $W \cap (A \times B)$ is non-negligible for any non-negligible sets A, $B \subseteq [0,1]$ (325Yc). I believe that the first published example of such a set was by Erdös & Oxtoby 55; I learnt the method of 325F from R.O.Davies.

I include 325G as a kind of guard-rail. The relationship between preservation of measure and order-continuity is a subtle one, as I have already tried to show in 324K, and it is often worth considering the possibility that a result involving order-continuous measure-preserving homomorphisms has a form applying to all order-continuous homomorphisms. However, there is no simple expression of such an idea in the present context.

In the context of infinite free products of probability algebras, there is a degree of simplification, since there is only one algebra which can plausibly be called the probability algebra free product, and this is produced by any realization of the algebras as measure algebras of probability spaces (325I-325K). The examples 325F-325G apply equally, of course, to this context. At this point I mention the concept of '(stochastically) independent' family (325L, 325Xe) because we have the machinery to translate several results from §272 into the language of measure algebras (325Xe-325Xg). I feel that I have to use the phrase 'stochastically independent' here because there is the much weaker alternative concept of 'Boolean independence' (315Xn) also present. But I leave most of this as exercises, because the language of measure algebras offers few ideas to the probability theory already covered in Chapter 27. All it can do is formalise the ever-present principle that negligible sets often can and should be ignored.

326 Additive functionals on Boolean algebras

I devote two sections to the general theory of additive functionals on measure algebras. As many readers will rightly be in a hurry to get on to the next two chapters, I remark that the only significant result needed for §§331-332 is the Hahn decomposition of a countably additive functional (326I), and that this is no more than a translation into the language of measure algebras of a theorem already given in Chapter 23. The concept of 'standard extension' of a countably additive functional from a subalgebra (327F-327G) will be used for a theorem in §333, and as preparation for Chapter 36.

I begin with notes on the space of additive functionals on an arbitrary Boolean algebra (326A-326D), corresponding to 231A-231B, but adding a more general form of the Jordan decomposition of a bounded additive functional into positive and negative parts (326D). The next subsection (326E-326I) deals with countably additive functionals, corresponding to 231C-231F. In 326J-326P I develop a new idea, that of 'completely additive' functional, which does not match anything in the previous treatment. In 326Q I return to additive functionals, giving a fundamental result on the construction of additive functionals on free products.

326A Additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

A non-negative additive functional is sometimes called a **finitely additive measure** or **charge**.

326B Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. The following will I hope be obvious.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is additive (because $(a \cap c) \cup (b \cap c) = (a \cup b) \cap c$).

(c) $\alpha\nu$ is an additive functional for any $\alpha \in \mathbb{R}$. If ν' is another finitely additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is additive.

(d) If $\langle \nu_i \rangle_{i \in I}$ is any family of finitely additive functionals such that $\nu' a = \sum_{i \in I} \nu_i a$ is defined in $\mathbb{R}$ for for every $a \in \mathfrak{A}$, then ν' is additive.

(e) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, then $\nu\pi : \mathfrak{B} \to \mathbb{R}$ is additive. In particular, if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\nu \restriction \mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is additive.

(f) ν is non-negative iff it is order-preserving – that is,

$$\nu a \ge 0 \text{ for every } a \in \mathfrak{A} \iff \nu b \le \nu c \text{ whenever } b \subseteq c$$

(because $\nu c = \nu b + \nu(c \setminus b)$ if $b \subseteq c$).

326C The space of additive functionals Let $\mathfrak{A}$ be any Boolean algebra. From 326Bc we see that the set M of all finitely additive real-valued functionals on $\mathfrak{A}$ is a linear space (a linear subspace of $\mathbb{R}^{\mathfrak{A}}$). We give it the ordering induced by that of $\mathbb{R}^{\mathfrak{A}}$, so that $\nu \le \nu'$ iff $\nu a \le \nu' a$ for every $a \in \mathfrak{A}$. This renders it a partially ordered linear space (because $\mathbb{R}^{\mathfrak{A}}$ is).

326D The Jordan decomposition (I): Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive real-valued functional on $\mathfrak{A}$. Then the following are equiveridical:

(i) ν is bounded;
(ii) $\sup_{n \in \mathbb{N}} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(iii) $\lim_{n \to \infty} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(iv) $\sum_{n=0}^{\infty} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(v) ν is expressible as the difference of two non-negative additive functionals.

proof (a)(i)⇒(v) Assume that ν is bounded. For each $a \in \mathfrak{A}$, set

$$\nu^+ a = \sup\{\nu b : b \subseteq a\}.$$

Because ν is bounded, ν^+ is real-valued. Now ν^+ is additive. **P** If a, $b \in \mathfrak{A}$ and $a \cap b = 0$, then

$$\nu^+(a \cup b) = \sup_{c \subseteq a \cup b} \nu c = \sup_{d \subseteq a, e \subseteq b} \nu(d \cup e) = \sup_{d \subseteq a, e \subseteq b} \nu d + \nu e$$

(because $d \cap e \subseteq a \cap b = 0$ whenever $d \subseteq a$, $e \subseteq b$)

$$= \sup_{d \subseteq a} \nu d + \sup_{e \subseteq b} \nu e = \nu^+ a + \nu^+ b. \quad \mathbf{Q}$$

Consequently $\nu^- = \nu^+ - \nu$ is also additive (326Bc).

Since

$$0 = \nu 0 \le \nu^+ a, \quad \nu a \le \nu^+ a$$

for every $a \in \mathfrak{A}$, $\nu^+ \ge 0$ and $\nu^- \ge 0$. Thus $\nu = \nu^+ - \nu^-$ is the difference of two non-negative additive functionals.

(b)(v)⇒(iv) If ν is expressible as $\nu_1 - \nu_2$, where ν_1 and ν_2 are non-negative additive functionals, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is disjoint, then

$$\textstyle\sum_{i=0}^{n} \nu_j a_i = \nu_j(\sup_{i \le n} a_i) \le \nu_j 1$$

for every n, both j, so that

$$\sum_{i=0}^{\infty} |\nu a_i| \le \sum_{i=0}^{\infty} \nu_1 a_i + \sum_{i=0}^{\infty} \nu_2 a_i \le \nu_1 1 + \nu_2 1 < \infty.$$

(c)(iv)⇒(iii)⇒(ii) are trivial.

(d) not-(i)⇒not-(ii) Suppose that ν is unbounded. Choose sequences $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $b_0 = 1$. Given that $\sup_{a\subseteq b_n} |\nu a| = \infty$, choose $c_n \subseteq b_n$ such that $|\nu c_n| \ge |\nu b_n| + n$; then $|\nu c_n| \ge n$ and

$$|\nu(b_n \setminus c_n)| = |\nu b_n - \nu c_n| \ge |\nu c_n| - |\nu b_n| \ge n.$$

We have

$$\begin{aligned}\infty = \sup_{a\subseteq b_n} |\nu a| &= \sup_{a\subseteq b_n} |\nu(a\cap c_n) + \nu(a\setminus c_n)| \\ &\le \sup_{a\subseteq b_n} |\nu(a\cap c_n)| + |\nu(a\setminus c_n)| \le \sup_{a\subseteq b_n\cap c_n} |\nu a| + \sup_{a\subseteq b_n\setminus c_n} |\nu a|,\end{aligned}$$

so at least one of $\sup_{a\subseteq b_n\cap c_n} |\nu a|$, $\sup_{a\subseteq b_n\setminus c_n} |\nu a|$ must be infinite; take b_{n+1} to be one of c_n, $b_n \setminus c_n$ such that $\sup_{a\subseteq b_{n+1}} |\nu a| = \infty$, and set $a_n = b_n \setminus b_{n+1}$, so that $|\nu a_n| \ge n$. Continue.

On completing the induction, we have a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $|\nu a_n| \ge n$ for every n, so that (ii) is false.

Remark I hope that this reminds you of the decomposition of a function of bounded variation as the difference of monotonic functions (224D).

326E Countably additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^{\infty} \nu a_n$ is defined and equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$.

A warning is perhaps in order. It can happen that $\mathfrak{A}$ is presented to us as a subalgebra of a larger algebra $\mathfrak{B}$; for instance, $\mathfrak{A}$ might be an algebra of sets, a subalgebra of some σ-algebra $\Sigma \subseteq \mathcal{P}X$. In this case, there may be sequences in $\mathfrak{A}$ which have a supremum in $\mathfrak{A}$ which is not a supremum in $\mathfrak{B}$ (indeed, this will happen just when the embedding is not sequentially order-continuous). So we can have a countably additive functional $\nu : \mathfrak{B} \to \mathbb{R}$ such that $\nu\restriction\mathfrak{A}$ is not countably additive in the sense used here. A similar phenomenon will arise when we come to the Daniell integral in Volume 4 (§434).

326F Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional.

(a) ν is finitely additive. (Setting $a_n = 0$ for every n, we see from the definition in 326E that $\nu 0 = 0$. Now, given $a \cap b = 0$, set $a_0 = a$, $a_1 = b$, $a_n = 0$ for $n \ge 2$ to see that $\nu(a \cup b) = \nu a + \nu b$.)

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with a supremum $a \in \mathfrak{A}$, then

$$\nu a = \nu a_0 + \sum_{n=0}^{\infty} \nu(a_{n+1} \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(c) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with an infimum $a \in \mathfrak{A}$, then

$$\nu a = \nu a_0 - \nu(a_0 \setminus a) = \nu a_0 - \lim_{n\to\infty} \nu(a_0 \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is countably additive. (For $\sup_{n\in\mathbb{N}} a_n \cap c = c \cap \sup_{n\in\mathbb{N}} a_n$ whenever the right-hand-side is defined, by 313Ba.)

(e) $\alpha\nu$ is a countably additive functional for any $\alpha \in \mathbb{R}$. If ν' is another countably additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is countably additive.

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then $\nu\pi$ is a countably additive functional on $\mathfrak{B}$. (For if $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$ with supremum b, then $\langle \pi b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence with supremum πb.)

(g) If $\mathfrak{A}$ is Dedekind σ-complete and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is countably additive. (For the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous, by 314Hb.)

326G Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive iff $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0 in $\mathfrak{A}$.

(b) If ν' is an additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, and ν is countably additive, then ν' is countably additive.

(c) If ν is non-negative, then ν is countably additive iff it is sequentially order-continuous.

proof (a)(i) If ν is countably additive and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty} \nu a_n = 0$ by 326Fc. **(ii)** If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n = a \setminus \sup_{i\le n} a_i$ for each $n \in \mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so

$$\nu a - \sum_{i=0}^{n} \nu a_i = \nu a - \nu(\sup_{i\le n} a_i) = \nu b_n \to 0$$

as $n \to \infty$, and $\nu a = \sum_{n=0}^{\infty} \nu a_n$; thus ν is countably additive.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n = \sup_{i\le n} a_i$ for each n; then $\nu a = \lim_{n\to\infty} \nu b_n$, so

$$\lim_{n\to\infty} |\nu' a - \nu' b_n| = \lim_{n\to\infty} |\nu'(a \setminus b_n)| \le \lim_{n\to\infty} \nu(a \setminus b_n) = 0,$$

and

$$\sum_{n=0}^{\infty} \nu' a_n = \lim_{n\to\infty} \nu' b_n = \nu' a.$$

(c) If ν is countably additive, then it is sequentially order-continuous by 326Fb-326Fc. If ν is sequentially order-continuous, then of course it satisfies the condition of (a), so is countably additive.

326H The Jordan decomposition (II): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a bounded countably additive real-valued functional on $\mathfrak{A}$. Then ν is expressible as the difference of two non-negative countably additive functionals.

proof Consider the functional $\nu^+ a = \sup_{b\subseteq a} \nu b$ defined in the proof of 326D. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, and $b \subseteq a$, then

$$\nu b = \sum_{n=0}^{\infty} \nu(b \cap a_n) \le \sum_{n=0}^{\infty} \nu^+ a_n.$$

As b is arbitrary, $\nu^+ a \le \sum_{n=0}^{\infty} \nu^+ a_n$. But of course

$$\nu^+ a \ge \nu^+(\sup_{i\le n} a_i) = \sum_{i=0}^{n} \nu^+ a_i$$

for every $n \in \mathbb{N}$, so $\nu^+ a = \sum_{n=0}^{\infty} \nu^+ a_n$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν^+ is countably additive.

Now $\nu^- = \nu^+ - \nu$ is also countably additive, and $\nu = \nu^+ - \nu^-$ is the difference of non-negative countably additive functionals.

326I The Hahn decomposition: Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Then ν is bounded and there is a $c \in \mathfrak{A}$ such that $\nu a \ge 0$ whenever $a \subseteq c$, while $\nu a \le 0$ whenever $a \cap c = 0$.

first proof By 314M, there are a set X and a σ-algebra Σ of subsets of X and a sequentially order-continuous Boolean homomorphism π from Σ onto $\mathfrak{A}$. Set $\nu_1 = \nu\pi : \Sigma \to \mathbb{R}$. Then ν_1 is countably additive (326Ff). So ν_1 is bounded and there is a set $H \in \Sigma$ such that $\nu_1 F \ge 0$ whenever $F \in \Sigma$ and $F \subseteq H$ and $\nu_1 F \le 0$ whenever $F \in \Sigma$ and $F \cap H = \emptyset$ (231E). Set $c = \pi H \in \mathfrak{A}$. If $a \subseteq c$, then there is an $F \in \Sigma$ such that $\pi F = a$; now $\pi(F \cap H) = a \cap c = a$, so $\nu a = \nu_1(F \cap H) \ge 0$. If $a \cap c = 0$, then there is an $F \in \Sigma$ such that $\pi F = a$; now $\pi(F \setminus H) = a \setminus c = a$, so $\nu a = \nu_1(F \setminus H) \le 0$.

second proof (a) Note first that ν is bounded. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, then $\sum_{n=0}^{\infty} \nu a_n$ must exist and be equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$; in particular, $\lim_{n\to\infty} \nu a_n = 0$. By 326D, ν is bounded. **Q**

(b)(i) We know that $\gamma = \sup\{\nu a : a \in \mathfrak{A}\} < \infty$. Choose a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu a_n \ge \gamma - 2^{-n}$ for every $n \in \mathbb{N}$. For $m \le n \in \mathbb{N}$, set $b_{mn} = \inf_{m\le i\le n} a_i$. Then $\nu b_{mn} \ge \gamma - 2\cdot 2^{-m} + 2^{-n}$ for every $n \ge m$. **P** Induce on n. For $n = m$, this is due to the choice of $a_m = b_{mm}$. For the inductive step, we have $b_{m,n+1} = b_{mn} \cap a_{n+1}$, while surely $\gamma \ge \nu(a_{n+1} \cup b_{mn})$, so

$$\gamma + \nu b_{m,n+1} \ge \nu(a_{n+1} \cup b_{mn}) + \nu(a_{n+1} \cap b_{mn})$$
$$= \nu a_{n+1} + \nu b_{mn} \ge \gamma - 2^{-n-1} + \gamma - 2\cdot 2^{-m} + 2^{-n}$$

(by the choice of a_{n+1} and the inductive hypothesis)

$$= 2\gamma - 2\cdot 2^{-m} + 2^{-n-1}.$$

Subtracting γ from both sides, $\nu b_{m,n+1} \ge \gamma - 2\cdot 2^{-m} + 2^{-n-1}$ and the induction proceeds. **Q**

(ii) Set

$$b_m = \inf_{n\ge m} b_{mn} = \inf_{n\ge m} a_n.$$

Then

$$\nu b_m = \lim_{n\to\infty} \nu b_{mn} \ge \gamma - 2\cdot 2^{-m},$$

by 326Fc. Next, $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, so setting $c = \sup_{n\in\mathbb{N}} b_n$ we have

$$\nu c = \lim_{n\to\infty} \nu b_n \ge \gamma;$$

since νc is surely less than or equal to γ, $\nu c = \gamma$.

If $b \in \mathfrak{A}$ and $b \subseteq c$, then

$$\nu c - \nu b = \nu(c \setminus b) \le \gamma = \nu c,$$

so $\nu b \ge 0$. If $b \in \mathfrak{A}$ and $b \cap c = 0$ then

$$\nu c + \nu b = \nu(c \cup b) \le \gamma = \nu c$$

so $\nu b \le 0$. This completes the proof.

326J Completely additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **completely additive** or **τ-additive** if it is finitely additive and $\inf_{a\in A} |\nu a| = 0$ whenever A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0.

326K Basic facts Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive. **P** If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then for any infinite $I \subseteq \mathbb{N}$ the set $\{a_i : i \in I\}$ is downwards-directed and has infimum 0, so $\inf_{i\in I} |\nu a_i| = 0$; which means that $\lim_{n\to\infty} \nu a_n$ must be zero. By 326Ga, ν is countably additive. **Q**

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$. **P?** Suppose, if possible, otherwise. Set

$$B = \{b : |\nu b| \ge \epsilon,\ \exists\, a \in A,\ b \supseteq a\}.$$

If $a \in A$ there is a $b' \subseteq a$ such that $|\nu b'| > \epsilon$. Now $\{a' \setminus b' : a' \in A,\ a' \subseteq a\}$ is downwards-directed and has infimum 0, so there is an $a' \in A$ such that $a' \subseteq a$ and $|\nu(a' \setminus b')| \le |\nu b'| - \epsilon$. Set $b = b' \cup a'$; then $a' \subseteq b$ and

$$|\nu b| = |\nu b' + \nu(a' \setminus b')| \ge |\nu b'| - |\nu(a' \setminus b')| \ge \epsilon,$$

so $b \in B$. But also $b \subseteq a$. Thus every member of A includes some member of B. Since every member of B includes a member of A, B is downwards-directed and has infimum 0; but this is impossible, since $\inf_{b\in B} |\nu b| \ge \epsilon$. **XQ**

(c) If ν is non-negative, it is order-continuous. **P** (i) If A is a non-empty upwards-directed set with supremum a_0, then $\{a_0 \setminus a : a \in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\sup_{a\in A}\nu a=\nu a_0-\inf_{a\in A}\nu(a_0\setminus a)=\nu a_0.$$

(ii) If A is a non-empty downwards-directed set with infimum a_0, then $\{a\setminus a_0 : a\in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\inf_{a\in A}\nu a=\nu a_0+\inf_{a\in A}\nu(a\setminus a_0)=\nu a_0.\ \mathbf{Q}$$

(d) If $c\in\mathfrak{A}$, then $a\mapsto\nu(a\cap c)$ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, so is $\{a\cap c : a\in A\}$, so $\inf_{a\in A}|\nu(a\cap c)|=0$. **Q**

(e) $\alpha\nu$ is a completely additive functional for any $\alpha\in\mathbb{R}$. If ν' is another completely additive functional on $\mathfrak{A}$, then $\nu+\nu'$ is completely additive. **P** We know from 326Bc that $\nu+\nu'$ is additive. Let A be a non-empty downwards-directed set with infimum 0. For any $\epsilon>0$, (b) tells us that there are a, $a'\in A$ such that $|\nu b|\le\epsilon$ whenever $b\subseteq a'$ and $|\nu' b|\le\epsilon$ whenever $b\subseteq a'$. But now, because A is downwards-directed, there is a $b\in A$ such that $b\subseteq a\cap a'$, which means that $|\nu b+\nu' b|\le|\nu b|+|\nu' b|$ is at most 2ϵ. As ϵ is arbitrary, $\inf_{a\in A}|(\nu+\nu')(a)|=0$, and $\nu+\nu'$ is completely additive. **Q**

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi:\mathfrak{B}\to\mathfrak{A}$ is an order-continuous Boolean homomorphism, then $\nu\pi$ is a completely additive functional on $\mathfrak{B}$. **P** By 326Be, $\nu\pi$ is additive. If $B\subseteq\mathfrak{B}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{B}$, then $\pi[B]$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, because π is order-continuous, so $\inf_{b\in B}|\nu\pi b|=0$. **Q** In particular, if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B}$ is completely additive.

(g) If ν' is another additive functional on $\mathfrak{A}$ and $|\nu' a|\le\nu a$ for every $a\in\mathfrak{A}$, then ν' is completely additive. **P** If $A\subseteq\mathfrak{A}$ is non-empty and downwards-directed and $\inf A=0$, then $\inf_{a\in A}|\nu' a|\le\inf_{a\in A}\nu a=0$. **Q**

326L I squeeze a useful fact in here.

Proposition If $\mathfrak{A}$ is a ccc Boolean algebra, a functional $\nu:\mathfrak{A}\to\mathbb{R}$ is countably additive iff it is completely additive.

proof If ν is completely additive it is countably additive, by 326Ka. If ν is countably additive and A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0, then there is a (non-empty) countable subset B of A also with infimum 0 (316E). Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over B, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $a_0=b_0$, $a_{n+1}\subseteq a_n\cap b_n$ for every $n\in\mathbb{N}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty}\nu a_n=0$ (326Fc) and $\inf_{a\in A}|\nu a|=0$. As A is arbitrary, ν is completely additive.

326M The Jordan decomposition (III): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$. Then ν is bounded and expressible as the difference of two non-negative completely additive functionals.

proof (a) I must first check that ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Set

$$A=\{a : a\in\mathfrak{A},\text{ there is an } n\in\mathbb{N}\text{ such that } a_i\subseteq a\text{ for every } i\ge n\}.$$

Then A is closed under $\cap$, and if b is any lower bound for A then $b\subseteq 1\setminus a_n\in A$, so $b\cap a_n=0$, for every $n\in\mathbb{N}$; but this means that $1\setminus b\in A$, so that $b\subseteq 1\setminus b$ and $b=0$. Thus $\inf A=0$. By 326Kb, there is an $a\in A$ such that $|\nu b|\le 1$ whenever $b\subseteq a$. By the definition of A, there must be an $n\in\mathbb{N}$ such that $|\nu a_i|\le 1$ for every $i\ge n$. But this means that $\sup_{n\in\mathbb{N}}|\nu a_n|$ is finite. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν is bounded, by 326D(ii). **Q**

(b) As in 326D and 326H, set $\nu^+a=\sup_{b\subseteq a}\nu b$ for every $a\in\mathfrak{A}$. Then ν^+ is completely additive. **P** We know that ν^+ is additive. If A is a non-empty downwards-directed subset of $\mathfrak{A}$ with infimum 0, then for every $\epsilon>0$ there is an $a\in A$ such that $|\nu b|\le\epsilon$ whenever $b\subseteq a$; in particular, $\nu^+a\le\epsilon$. As ϵ is arbitrary, $\inf_{a\in A}\nu^+a=0$; as A is arbitrary, ν^+ is completely additive. **Q**

Consequently $\nu^-=\nu^+-\nu$ is completely additive (326Ke) and $\nu=\nu^+-\nu^-$ is the difference of non-negative completely additive functionals.

326N I give an alternative definition of 'completely additive' which you may feel clarifies the concept.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. Then the following are equiveridical:
(i) ν is completely additive;
(ii) $\nu 1 = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$;
(iii) $\nu a = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a.

proof (For notes on sums $\sum_{i\in I}$, see 226A.)

(a)(i)⇒(ii) If ν is completely additive and $\langle a_i\rangle_{i\in I}$ is a partition of unity in A, then (inducing on $\#(J)$) $\nu(\sup_{i\in J} a_i) = \sum_{i\in J} \nu a_i$ for every finite $J \subseteq I$. Consider

$$A = \{1 \setminus \sup_{i\in J} a_i : J \subseteq I \text{ is finite}\}.$$

Then A is non-empty and downwards-directed and has infimum 0, so for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$. Express a as $1 \setminus \sup_{i\in J} a_i$ where $J \subseteq I$ is finite. If now K is another finite subset of I including J,

$$|\nu 1 - \textstyle\sum_{i\in K} a_i| = |\nu(1 \setminus \sup_{i\in K} a_i)| \le \epsilon.$$

As remarked in 226Ad, this means that $\nu 1 = \sum_{i\in I} \nu a_i$, as claimed.

(b)(ii)⇒(iii) Suppose that ν satisfies the condition (ii), and that $\langle a_i\rangle_{i\in I}$ is a disjoint family with supremum a. Take any $j \notin I$, set $J = I \cup \{j\}$ and $a_j = 1 \setminus a$; then $\langle a_i\rangle_{i\in J}$, $(a, 1 \setminus a)$ are both partitions of unity, so

$$\nu(1 \setminus a) + \nu a = \nu 1 = \textstyle\sum_{i\in J} \nu a_i = \nu(1 \setminus a) + \sum_{i\in I} \nu a_i,$$

and $\nu a = \sum_{i\in I} \nu a_i$.

(c)(iii)⇒(i) Suppose that ν satisfies (iii). Then ν is additive.

(α) ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Applying Zorn's Lemma to the set $\mathcal{C}$ of all disjoint families $C \subseteq \mathfrak{A}$ including $\{a_n : n \in \mathbb{N}\}$, we find a partition of unity $C \supseteq \{a_n : n \in \mathbb{N}\}$. Now $\sum_{c\in C} \nu c$ is defined in $\mathbb{R}$, so $\sup_{n\in\mathbb{N}} |\nu a_n| \le \sup_{c\in C} |\nu c|$ is finite. By 326D, ν is bounded. **Q**

(β) Define ν^+ from ν as in 326D. Then ν^+ satisfies the same condition as ν. **P** Let $\langle a_i\rangle_{i\in I}$ be a disjoint family in $\mathfrak{A}$ with supremum a. Then for any $b \subseteq a$, we have $b = \sup_{i\in I} b \cap a_i$, so

$$\nu b = \textstyle\sum_{i\in I} \nu(b \cap a_i) \le \sum_{i\in I} \nu^+ a_i.$$

Thus $\nu^+ a \le \sum_{i\in I} \nu^+ a_i$. But of course

$$\begin{aligned}\sum_{i\in I} \nu^+ a_i &= \sup\{\sum_{i\in J} \nu^+ a_i : J \subseteq I \text{ is finite}\}\\ &= \sup\{\nu^+(\sup_{i\in J} a_i) : J \subseteq I \text{ is finite}\} \le \nu^+ a,\end{aligned}$$

so $\nu^+ a = \sum_{i\in I} \nu^+ a_i$. **Q**

(γ) It follows that ν^+ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, then $B = \{b : \exists\ a \in A, b \cap a = 0\}$ is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle b_i\rangle_{i\in I}$ lying in B (313K). Now if $J \subseteq I$ is finite, there is an $a \in A$ such that $a \cap \sup_{i\in J} b_i = 0$ (because A is downwards-directed), and

$$\nu^+ a + \textstyle\sum_{i\in J} \nu^+ b_i \le \nu^+ 1.$$

Since $\nu^+ 1 = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \nu^+ b_i$, $\inf_{a\in A} \nu^+ a = 0$. As A is arbitrary, ν^+ is completely additive. **Q**

(δ) Now consider $\nu^- = \nu^+ - \nu$. Of course

$$\nu^- a = \nu^+ a - \nu a = \textstyle\sum_{i\in I} \nu^+ a_i - \sum_{i\in I} \nu a_i = \sum_{i\in I} \nu^- a_i$$

whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a. Because ν^- is non-negative, the argument of (γ) shows that $\nu^- = (\nu^-)^+$ is completely additive. So $\nu = \nu^+ - \nu^-$ is completely additive, as required.

326O For completely additive functionals, we have a useful refinement of the Hahn decomposition. I give it in a form adapted to the applications I have in mind.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a completely additive functional. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\nu > 0]\!]$, 'the region where $\nu > 0$', such that $\nu a > 0$ whenever $0 \ne a \subseteq [\![\nu > 0]\!]$, while $\nu a \le 0$ whenever $a \cap [\![\nu > 0]\!] = 0$.

proof Set

$$C_1 = \{c : c \in \mathfrak{A} \setminus \{0\},\ \nu a > 0 \text{ whenever } 0 \ne a \subseteq c\},$$

$$C_2 = \{c : c \in \mathfrak{A},\ \nu a \le 0 \text{ whenever } a \subseteq c\}.$$

Then $C_1 \cup C_2$ is order-dense in $\mathfrak{A}$. **P** There is a $c_0 \in \mathfrak{A}$ such that $\nu a \ge 0$ for every $a \subseteq c$, $\nu a \le 0$ whenever $a \cap c = 0$ (326I). Given $b \in \mathfrak{A} \setminus \{0\}$, then $b \setminus c_0 \in C_2$, so if $b \setminus c_0 \ne 0$ we can stop. Otherwise, $b \subseteq c_0$. If $b \in C_1$ we can stop. Otherwise, there is a non-zero $c \subseteq b$ such that $\nu c \le 0$; but in this case $\nu a \ge 0$, $\nu(c \setminus a) \ge 0$ so $\nu a = 0$ for every $a \subseteq c$, and $c \in C_2$. **Q**

There is therefore a partition of unity $D \subseteq C_1 \cup C_2$. Now $D \cap C_1$ is countable. **P** If $d \in D \cap C_1$, $\nu d > 0$. Also

$$\#(\{d : d \in D,\ \nu d \ge 2^{-n}\}) \le 2^n \sup_{a \in \mathfrak{A}} \nu a$$

is finite for each n, so $D \cap C_1$ is the union of a sequence of finite sets, and is countable. **Q**

Accordingly $D \cap C_1$ has a supremum e. If $0 \ne a \subseteq e$ then

$$\nu a = \sum_{c \in D} \nu(a \cap c) = \sum_{c \in D \cap C_1} \nu(a \cap c) \ge 0$$

by 326N. Also there must be some $c \in D \cap C_1$ such that $a \cap c \ne 0$, in which case $\nu(a \cap c) > 0$, so that $\nu a > 0$. If $a \cap e = 0$, then

$$\nu a = \sum_{c \in D} \nu(a \cap c) = \sum_{c \in D \cap C_2} \nu(a \cap c) \le 0.$$

Thus e has the properties demanded of $[\![\nu > 0]\!]$. To see that e is unique, we need observe only that if e' has the same properties then $\nu(e \setminus e') \le 0$ (because $(e \setminus e') \cap e' = 0$), so $e \setminus e' = 0$ (because $e \setminus e' \subseteq e$). Similarly, $e' \setminus e = 0$ and $e = e'$. Thus we may properly denote e by the formula $[\![\nu > 0]\!]$.

326P Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and μ, ν two completely additive functionals on $\mathfrak{A}$. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\mu > \nu]\!]$, 'the region where $\mu > \nu$', such that

$$\mu a > \nu a \text{ whenever } 0 \ne a \subseteq [\![\mu > \nu]\!],$$

$$\mu a \le \nu a \text{ whenever } a \cap [\![\mu > \nu]\!] = 0.$$

proof Apply 326O to the functional $\mu - \nu$, and set $[\![\mu > \nu]\!] = [\![\mu - \nu > 0]\!]$.

***326Q Additive functionals on free products** In Volume 4, when we return to the construction of measures on product spaces, the following fundamental fact will be useful.

Theorem Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j \in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite},\ a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that $\theta : C \to \mathbb{R}$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Then there is a unique finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending θ.

proof (a) It will help if I note at once that $\theta 0 = 0$. **P**

$$\theta 0 = \theta(0 \cap \varepsilon_i(0)) + \theta(0 \cap \varepsilon_i(1)) = 2\theta 0$$

for any $i \in I$. **Q**

(b) The key is of course the following fact: if $\langle c_r\rangle_{r\le m}$ and $\langle d_s\rangle_{s\le n}$ are two disjoint families in C with the same supremum in $\mathfrak{A}$, then $\sum_{r=0}^m \theta c_r = \sum_{s=0}^n \theta d_s$. **P** Let $J \subseteq I$ be a finite set and $\mathfrak{B}_i \subseteq \mathfrak{A}_i$ a finite subalgebra, for each $i \in J$, such that every c_r and every d_s belongs to the subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ generated by $\{\varepsilon_j(b) : j \in J, b \in \mathfrak{B}_j\}$. Next, if $j \in J$ and $b \in \mathfrak{B}_j$, then

$$\sum_{r=0}^m \theta c_r = \sum_{r=0}^m \theta(c_r \cap \varepsilon_j(b)) + \sum_{r=0}^m \theta(c_r \setminus \varepsilon_j(b)).$$

We can therefore find a disjoint family $\langle c'_r\rangle_{r\le m'}$ in $C \cap \mathfrak{A}_0$ such that

$$\sup_{r\le m'} c'_r = \sup_{r\le m} c_r, \quad \sum_{r=0}^{m'} \theta c'_r = \sum_{r=0}^m \theta c_r,$$

and whenever $r \le m'$, $j \in J$ and $b \in \mathfrak{B}_j$ then either $c'_r \subseteq \varepsilon_j(b)$ or $c'_r \cap \varepsilon_j(b)b = 0$; that is, every c'_r is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. Similarly, we can find $\langle d'_s\rangle_{s\le n'}$ such that

$$\sup_{s\le n'} d'_s = \sup_{s\le n} d_s, \quad \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^n \theta d_s,$$

and whenever $s \le n'$, $j \in J$ and $b \in \mathfrak{B}_j$ then d'_s is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. But we now have $\sup_{r\le m'} c'_r = \sup_{s\le n'} d'_s$ while for any $r \le m'$, $s \le n'$ either $c'_r = d'_s$ or $c'_r \cap d'_s = 0$. It follows that the non-zero terms in the finite sequence $\langle c'_r\rangle_{r\le m'}$ are just a rearrangement of the non-zero terms in $\langle d'_s\rangle_{s\le n'}$, so that

$$\sum_{r=0}^m \theta c_r = \sum_{r=0}^{m'} \theta c'_r = \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^n \theta d_s,$$

as required. **Q**

(c) By 315Jb, this means that we have a functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu(\sup_{r\le m} c_r) = \sum_{r=0}^m \theta c_r$ whenever $\langle c_r\rangle_{r\le m}$ is a disjoint family in C. It is now elementary to check that ν is additive, and it is clearly the only additive functional on $\mathfrak{A}$ extending θ.

326X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that (i) $\nu(a \cup b) = \nu a + \nu b - \nu(a \cap b)$ (ii) $\nu(a \cup b \cup c) = \nu a + \nu b + \nu c - \nu(a \cap b) - \nu(a \cap c) - \nu(b \cap c) + \nu(a \cap b \cap c)$ for all a, b, $c \in \mathfrak{A}$. Generalize these results to longer sequences in $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m = 0$; (iii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m$. (*Hint*: for (i)⇒(iii), consider non-negative ν first.)

(d) Let X be any uncountable set, and J an infinite subset of X. Let $\mathfrak{A}$ be the finite-cofinite algebra of X (316Yk), and for $a \in A$ set $\nu a = \#(a \cap J)$ if a is finite, $-\#(J \setminus a)$ if a is cofinite. Show that ν is countably additive and unbounded.

>(e) Let $\mathfrak{A}$ be the algebra of subsets of $[0, 1]$ generated by the family of (closed) intervals. Show that there is a unique additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu[\alpha, \beta] = \beta - \alpha$ whenever $0 \le \alpha \le \beta \le 1$. Show that ν is countably additive but not completely additive.

(f) (i) Let (X, Σ, μ) be any atomless probability space. Show that $\mu : \Sigma \to \mathbb{R}$ is a countably additive functional which is not completely additive. (ii) Let X be any uncountable set and μ the countable-cocountable measure on X (211R). Show that μ is countably additive but not completely additive.

(g) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. (i) Show that ν is finitely additive iff $\sum_{i\in I} \nu a_i = \nu 1$ for every finite partition of unity $\langle a_i\rangle_{i\in I}$. (ii) Show that ν is countably additive iff $\sum_{i\in I} \nu a_i = \nu 1$ for every countable partition of unity $\langle a_i\rangle_{i\in I}$.

(h) Show that 326O can fail if ν is only countably additive, rather than completely additive. (*Hint*: 326Xf.)

(i) Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$. Let us say that $a \in \mathfrak{A}$ is a **support** of ν if (α) $\nu b = 0$ whenever $b \cap a = 0$ (β) for every non-zero $b \subseteq a$ there is a $c \subseteq b$ such that $\nu c \neq 0$. (i) Check that ν can have at most one support. (ii) Show that if a is a support for ν and ν is bounded, then the principal ideal $\mathfrak{A}_a$ generated by a is ccc. (iii) Show that if $\mathfrak{A}$ is Dedekind σ-complete and ν is countably additive, then ν is completely additive iff it has a support, and that in the language of 326O this is $[\![\nu > 0]\!] \cup [\![-\nu > 0]\!]$. (iv) Taking $J = X$ in 326Xd, show that X is the support of the functional ν there.

326Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Show that the following are equiveridical: (i) for every $\epsilon > 0$ there is a finite partition $\langle a_i \rangle_{i \in I}$ of unity in $\mathfrak{A}$ such that $\nu a_i \leq \epsilon$ for every $i \in I$; (ii) whenever ν' is a non-zero finitely additive functional such that $0 \leq \nu' \leq \nu$ there is an $a \in \mathfrak{A}$ such that $\nu' a$ and $\nu'(1 \setminus a)$ are both non-zero. (Such functionals are called **atomless**.)

(b) Let $\mathfrak{A}$ be a Boolean algebra and ν_1, ν_2 atomless non-negative additive functionals on $\mathfrak{A}$. Show that $\nu_1 + \nu_2$, $\alpha\nu_1$ are atomless for every $\alpha \geq 0$, and that ν is atomless whenever ν is additive and $0 \leq \nu \leq \nu_1$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless non-negative finitely additive functional on $\mathfrak{A}$. Show that there is a family $\langle a_t \rangle_{t \in [0,1]}$ in $\mathfrak{A}$ such that $a_s \subseteq a_t$ and $\nu a_t = t\nu 1$ whenever $s \leq t \in [0, 1]$.

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu_0, \dots , \nu_n$ atomless non-negative finitely additive functionals on $\mathfrak{A}$. Show that there is an $a \in \mathfrak{A}$ such that $\nu_i a = \frac{1}{2}\nu_i 1$ for every $i \leq n$. (*Hint*: it is enough to consider the case $\nu_0 \geq \nu_1 \ldots \geq \nu_n$. For the inductive step, use the inductive hypothesis to construct $\langle a_t \rangle_{t \in [0,1]}$ such that $a_s \subseteq a_t$, $\nu_i a_t = t\nu_i 1$ if $i < n$, $0 \leq s \leq t \leq 1$. Now show that $t \mapsto \nu_n(a_{t+\frac{1}{2}} \setminus a_t)$ is continuous on $[0, \frac{1}{2}]$.)

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Let $\nu : \mathfrak{A} \to \mathbb{R}^r$, where $r \geq 1$, be additive (in the sense that $\nu(a \cup b) = \nu a + \nu b$ whenever $a \cap b = 0$) and atomless (in the sense that for every $\epsilon > 0$ there is a finite partition of unity $\langle a_i \rangle_{i \in I}$ such that $\|\nu a\| \leq \epsilon$ whenever $i \in I$ and $a \subseteq a_i$). Show that $\{\nu a : a \in \mathfrak{A}\}$ is a convex set in $\mathbb{R}^r$. (This is a version of **Liapounoff's theorem**. I am grateful to K.P.S.Bhaskara Rao for showing it to me.)

(f) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ a countably additive functional. Show that ν is atomless iff whenever $a \in \mathfrak{A}$ and $\nu a \neq 0$ there is a $b \subseteq a$ such that $0 < \nu b < \nu a$.

(g) Show that there is a finitely additive functional $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ such that $\nu\{n\} = 1$ for every $n \in \mathbb{N}$, so that ν is not bounded. (*Hint*: Use Zorn's Lemma to construct a maximal linearly independent subset of ℓ^∞ including $\{\chi\{n\} : n \in \mathbb{N}\}$, and hence to construct a linear map $f : \ell^\infty \to \mathbb{R}$ such that $f(\chi\{n\}) = 1$ for every n.)

(h) Let $\mathfrak{A}$ be any infinite Boolean algebra. Show that there is an unbounded finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$. (*Hint*: let $\langle t_n \rangle_{n \in \mathbb{N}}$ be a sequence of distinct points in the Stone space of $\mathfrak{A}$, and set $\nu a = \nu'\{n : t_n \in \widehat{a}\}$ for a suitable ν'.)

(i) Let $\mathfrak{A}$ be a Boolean algebra, and give $\mathbb{R}^{\mathfrak{A}}$ its product topology. Show that the space of finitely additive functionals on $\mathfrak{A}$ is a closed subset of $\mathbb{R}^{\mathfrak{A}}$, but that the space of bounded finitely additive functionals is closed only when $\mathfrak{A}$ is finite.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and M the linear space of all bounded finitely additive real-valued functionals on $\mathfrak{A}$. For ν, $\nu' \in M$ say that $\nu \leq \nu'$ if $\nu a \leq \nu' a$ for every $a \in \mathfrak{A}$. Show that

(i) ν^+, as defined in the proof of 326D, is just $\sup\{0, \nu\}$ in M;

(ii) M is a Dedekind complete Riesz space (241E-241F, 353G);

(iii) for ν, $\nu' \in M$, $|\nu| = \nu \vee (-\nu)$, $\nu \vee \nu'$, $\nu \wedge \nu'$ are given by the formulae

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b), \quad (\nu \vee \nu')(a) = \sup_{b \subseteq a} \nu b + \nu'(a \setminus b),$$

$$(\nu \wedge \nu')(a) = \inf_{b \subseteq a} \nu b + \nu'(a \setminus b);$$

(iv) for any non-empty $A \subseteq M$, A is bounded above in M iff

$$\sup\{\sum_{i=0}^{n} \nu_i a_i : \nu_i \in A \text{ for each } i \le n, \langle a_i \rangle_{i \le n} \text{ is disjoint}\}$$

is finite, and then $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup\{\sum_{i=0}^{n} \nu_i a_i : \nu_i \in A \text{ for each } i \le n, \langle a_i \rangle_{i \le n} \text{ is disjoint}, \sup_{i \le n} a_i = a\}$$

for every $a \in \mathfrak{A}$;

(v) setting $\|\nu\| = |\nu|(1)$, $\| \; \|$ is an order-continuous norm on M under which M is a Banach lattice.

(k) Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{C}$ is **finitely additive** if its real and imaginary parts are. Show that the space of bounded finitely additive functionals from $\mathfrak{A}$ to $\mathbb{C}$ is a Banach space under the norm $\|\nu\| = \sup\{\sum_{i=0}^{n} |\nu a_i| : \langle a_i \rangle_{i \le n}$ is a partition of unity in $\mathfrak{A}\}$.

(l) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}_\sigma$ for which the closed sets are the sequentially order-closed sets. Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive iff it is continuous for $\mathfrak{T}_\sigma$.

(m) Let $\mathfrak{A}$ be a Boolean algebra, and M_σ the set of all bounded countably additive real-valued functionals on $\mathfrak{A}$. Show that M_σ is a closed and order-closed linear subspace of the normed space M of all additive functionals on $\mathfrak{A}$ (326Yj), and that $|\nu| \in M_\sigma$ whenever $\nu \in M_\sigma$.

(n) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\sigma a = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } a\}$$

for every $a \in \mathfrak{A}$. Show that ν_σ is countably additive, and is $\sup\{\nu' : \nu' \le \nu$ is countably additive$\}$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$ such that $\nu a = \lim_{n \to \infty} \nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive. (*Hint*: use arguments from part (a) of the proof of 247C to see that $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} |\nu_k a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$, and therefore that $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} |\nu_k a_n| = 0$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence with infimum 0.)

(p) Let $\mathfrak{A}$ be a Boolean algebra, and M_τ the set of all completely additive real-valued functionals on $\mathfrak{A}$. Show that M_τ is a closed and order-closed linear subspace of the normed space M of all additive functionals, and that $|\nu| \in M_\tau$ whenever $\nu \in M_\tau$.

(q) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\tau b = \inf\{\sup_{a \in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } b\}$$

for every $b \in \mathfrak{A}$. Show that ν_τ is completely additive, and is $\sup\{\nu' : \nu' \le \nu$ is completely additive$\}$.

(r) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}$ for which the closed sets are the order-closed sets (313Xb). Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive iff it is continuous for $\mathfrak{T}$.

(s) Let X be a set, Σ any σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a functional. Show that ν is completely additive iff there are sequences $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\nu E = \sum_{n=0}^{\infty} \alpha_n \chi E(x_n)$ for every $E \in \Sigma$.

(t) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and μ, ν finitely additive functionals on $\mathfrak{A}$, $\mathfrak{B}$ respectively. Show that there is a unique finitely additive functional $\lambda : \mathfrak{A} \otimes \mathfrak{B} \to \mathbb{R}$ such that $\lambda(a \otimes b) = \mu a \cdot \nu b$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

(u) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $(\bigotimes_{i\in I}\mathfrak{A}_i,\langle\varepsilon_i\rangle_{i\in I})$, and for each $i\in I$ let ν_i be a finitely additive functional on $\mathfrak{A}_i$ such that $\nu_i 1=1$. Show that there is a unique finitely additive functional $\nu:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathbb{R}$ such that $\nu(\inf_{i\in J}\varepsilon_i(a_i))=\prod_{i\in J}\nu_i a_i$ whenever $J\subseteq I$ is non-empty and finite and $a_i\in\mathfrak{A}_i$ for each $i\in J$.

326 Notes and comments I have not mentioned the phrase 'measure algebra' anywhere in this section, and in principle this material could have been part of Chapter 31; but countably additive functionals are kissing cousins of measures, and most of the ideas here surely belong to 'measure theory' rather than to 'Boolean algebra', in so far as such divisions are meaningful at all. I have given as much as possible of the theory in a general form because the simplifications which are possible when we look only at measure algebras are seriously confusing if they are allowed too much prominence. In particular, it is important to understand that the principal properties of completely additive functionals do not depend on Dedekind completeness of the algebra, provided we take care over the definitions. Similarly, the definition of 'countably additive' functional for algebras which are not Dedekind σ-complete needs a moment's attention to the phrase 'and $\sup_{n\in\mathbb{N}}a_n$ is defined in $\mathfrak{A}$'. It can happen that a functional is countably additive mostly because there are too few such sequences (326Xd).

The formulations I have chosen as principal definitions (326A, 326E, 326J) are those which I find closest to my own intuitions of the concepts, but you may feel that 326G(i), 326Xc(iii) and 326N, or 326Yl and 326Yr, provide useful alternative patterns. The point is that countable additivity corresponds to sequential order-continuity (326Fb, 326Fc, 326Ff), while complete additivity corresponds to order-continuity (326Kc, 326Kf); the difficulty is that we must consider functionals which are not order-preserving, so that the simple definitions in 313H cannot be applied directly. It is fair to say that all the additive functionals ν we need to understand are bounded, and therefore may be studied in terms of their positive and negative parts ν^+, ν^-, which are order-preserving (326Bf); but many of the most important applications of these ideas depend precisely on using facts about ν to deduce facts about ν^+ and ν^-.

It is in 326D that we seem to start getting more out of the theory than we have put in. The ideas here have vast ramifications. What it amounts to is that we can discover much more than we might expect by looking at disjoint sequences. To begin with, the conditions here lead directly to 326I and 326M: every completely additive functional is bounded, and every countably additive functional on a Dedekind σ-complete Boolean algebra is bounded. (But note 326Yg-326Yh.)

Naturally enough, the theory of countably additive functionals on general Boolean algebras corresponds closely to the special case of countably additive functionals on σ-algebras of sets, already treated in §§231-232 for the sake of the Radon-Nikodým theorem. This should make 326E-326I very straightforward. When we come to completely additive functionals, however, there is room for many surprises. The natural map from a σ-algebra of measurable sets to the corresponding measure algebra is sequentially order-continuous but rarely order-continuous, so that there can be completely additive functionals on the measure algebra which do not correspond to completely additive functionals on the σ-algebra. Indeed there are very few completely additive functionals on σ-algebras of sets (326Ys). Of course these surprises can arise only when there is a difference between completely additive and countably additive functionals, that is, when the algebra involved is not ccc (326L). But I think that neither 326M nor 326N is obvious.

I find myself generally using the phrase 'countably additive' in preference to 'completely additive' in the context of ccc algebras, where there is no difference between them. This is an attempt at user-friendliness; the phrase 'countably additive' is the commoner one in ordinary use. But I must say that my personal inclination is to the other side. The reason why so many theorems apply to countably additive functionals in these contexts is just that they are completely additive.

I have given two proofs of 326I. I certainly assume that if you have got this far you are acquainted with the Radon-Nikodým theorem and the associated basic facts about countably additive functionals on σ-algebras of sets; so that the 'first proof' should be easy and natural. On the other hand, there are purist objections on two fronts. First, it relies on the Stone representation, which involves a much stronger form of the axiom of choice than is actually necessary. Second, the classical Hahn decomposition in 231E is evidently a special case of 326I, and if we need both (as we certainly do) then one expects the ideas to stand out more clearly if they are applied directly to the general case. In fact the two versions of the argument are so nearly identical that (as you will observe, if you have Volume 2 to hand) they can share nearly every word. You can take the 'second proof', therefore, as a worked example in the translation of ideas from the context of σ-algebras of sets to the context of Dedekind σ-complete Boolean algebras. What makes it possible is the fact that the only limit operations referred to involve countable families.

Arguments not involving limit operations can generally, of course, be applied to all Boolean algebras; I have lifted some exercises (326Yj, 326Yn) from §231 to give you some practice in such generalizations.

Almost any non-trivial measure provides an example of a countably additive functional on a Dedekind σ-complete algebra which is not completely additive (326Xf). The question of whether such a functional can exist on a Dedekind complete algebra is the 'Banach-Ulam problem', to which I will return in 363S.

In this section I have looked only at questions which can be adequately treated in terms of the underlying algebras $\mathfrak{A}$, without using any auxiliary structure. To go much farther we shall need to study the 'function spaces' $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ of Chapter 36. In particular, the ideas of 326Yg, 326Yj-326Yk and 326Ym-326Yq will make better sense when redeveloped in §362.

327 Additive functionals on measure algebras

When we turn to measure algebras, we have a simplification, relative to the general context of §326, because the algebras are always Dedekind σ-complete; but there are also elaborations, because we can ask how the additive functionals we examine are related to the measure. In 327A-327C I work through the relationships between the concepts of 'absolute continuity', '(true) continuity' and 'countable additivity', following 232A-232B, and adding 'complete additivity' from §326. These ideas provide a new interpretation of the Radon-Nikodým theorem (327D). I then use this theorem to develop some machinery (the 'standard extension' of an additive functional from a closed subalgebra to the whole algebra, 327F-327G) which will be used in §333.

327A I start with the following definition and theorem corresponding to 232A-232B.

Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a finitely additive functional. Then ν is **absolutely continuous** with respect to $\bar\mu$ if for every $\epsilon>0$ there is a $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $\bar\mu a\le\delta$.

327B Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\nu:\mathfrak{A}\to\mathbb{R}$ a finitely additive functional. Give $\mathfrak{A}$ its measure-algebra topology and uniformity (§323).

(a) If ν is continuous, it is completely additive.

(b) If ν is countably additive, it is absolutely continuous with respect to $\bar\mu$.

(c) The following are equiveridical:

(i) ν is continuous at 0;

(ii) ν is countably additive and whenever $a\in\mathfrak{A}$ and $\nu a\ne 0$ there is a $b\in\mathfrak{A}$ such that $\bar\mu b<\infty$ and $\nu(a\cap b)\ne 0$;

(iii) ν is continuous everywhere on $\mathfrak{A}$;

(iv) ν is uniformly continuous.

(d) If $(\mathfrak{A},\bar\mu)$ is semi-finite, then ν is continuous iff it is completely additive.

(e) If $(\mathfrak{A},\bar\mu)$ is σ-finite, then ν is continuous iff it is countably additive iff it is completely additive.

(f) If $(\mathfrak{A},\bar\mu)$ is totally finite, then ν is continuous iff it is absolutely continuous with respect to $\bar\mu$ iff it is countably additive iff it is completely additive.

proof (a) If ν is continuous, and $A\subseteq\mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $0\in\overline{A}$ (323D(b-ii)), so $\inf_{a\in A}|\nu a|=0$.

(b) ? Suppose, if possible, that ν is countably additive but not absolutely continuous. Then there is an $\epsilon>0$ such that for every $\delta>0$ there is an $a\in\mathfrak{A}$ such that $\bar\mu a\le\delta$ but $|\nu a|\ge\epsilon$. For each $n\in\mathbb{N}$ we may choose a $b_n\in\mathfrak{A}$ such that $\bar\mu b_n\le 2^{-n}$ and $|\nu b_n|\ge\epsilon$. Consider $b_n^*=\sup_{k\ge n}b_k$, $b=\inf_{n\in\mathbb{N}}b_n^*$. Then we have

$$\bar\mu b\le\inf_{n\in\mathbb{N}}\bar\mu(\sup_{k\ge n}b_k)\le\inf_{n\in\mathbb{N}}\sum_{k=n}^\infty 2^{-k}=0,$$

so $\bar\mu b=0$ and $b=0$. On the other hand, ν is expressible as a difference $\nu^+-\nu^-$ of non-negative countably additive functionals (326H), each of which is sequentially order-continuous (326Gc), and

$$0 = \lim_{n\to\infty}(\nu^+ + \nu^-)b_n^* \ge \inf_{n\in\mathbb{N}}(\nu^+ + \nu^-)b_n \ge \inf_{n\in\mathbb{N}} |\nu b_n| \ge \epsilon,$$

which is absurd. **X**

(c)(i)⇒(ii) Suppose that ν is continuous. Then it is completely additive, by (a), therefore countably additive. If $\nu a \ne 0$, there must be an b of finite measure such that $|\nu d| < |\nu a|$ whenever $d \cap b = \emptyset$, so that $|\nu(a \setminus b)| < |\nu a|$ and $\nu(a \cap b) \ne 0$. Thus the conditions are satisfied.

(ii)⇒(iv) Now suppose that ν satisfies the two conditions in (ii). Because $\mathfrak{A}$ is Dedekind σ-complete, ν must be bounded (326I), therefore expressible as the difference $\nu^+ - \nu^-$ of countably additive functionals. Set $\nu_1 = \nu^+ + \nu^-$. Set

$$\gamma = \sup\{\nu_1 b : b \in \mathfrak{A},\, \bar\mu b < \infty\},$$

and choose a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ of elements of $\mathfrak{A}$ of finite measure such that $\lim_{n\to\infty}\nu_1 b_n = \gamma$; set $b^* = \sup_{n\in\mathbb{N}} b_n$. If $d \in \mathfrak{A}$ and $d \cap b^* = \emptyset$ then $\nu d = 0$. **P** If $b \in \mathfrak{A}$ and $\bar\mu b < \infty$, then

$$|\nu(d \cap b)| \le \nu_1(d \cap b) \le \nu_1(b \setminus b_n) = \nu_1(b \cup b_n) - \nu_1 b_n \le \gamma - \nu_1 b_n$$

for every $n \in \mathbb{N}$, so $\nu(d \cap b) = 0$. As b is arbitrary, the second condition tells us that $\nu d = 0$. **Q**

Setting $b_n^* = \sup_{k\le n} b_k$ for each n, we have $\lim_{n\to\infty}\nu_1(b^* \setminus b_n^*) = 0$. Take any $\epsilon > 0$, and (using (b) above) let $\delta > 0$ be such that $|\nu a| \le \epsilon$ whenever $\bar\mu a \le \delta$. Let n be such that $\nu_1(b^* \setminus b_n^*) \le \epsilon$. Then

$$\begin{aligned}|\nu a| &\le |\nu(a \cap b_n^*)| + |\nu(a \cap (b^* \setminus b_n^*))| + |\nu(a \setminus b^*)| \\ &\le |\nu(a \cap b_n^*)| + \nu_1(b^* \setminus b_n^*) \le |\nu(a \cap b_n^*)| + \epsilon\end{aligned}$$

for any $a \in \mathfrak{A}$.

Now if b, $c \in \mathfrak{A}$ and $\bar\mu((b \,\triangle\, c) \cap b_n^*) \le \delta$ then

$$\begin{aligned}|\nu b - \nu c| &\le |\nu(b \setminus c)| + |\nu(c \setminus b)| \\ &\le |\nu((b \setminus c) \cap b^*)| + |\nu((c \setminus b) \cap b^*)| + 2\epsilon \le \epsilon + \epsilon + 2\epsilon = 4\epsilon\end{aligned}$$

because $\bar\mu((b \setminus c) \cap b_n^*)$, $\bar\mu((c \setminus b) \cap b_n^*)$ are both less than or equal to δ. As ϵ is arbitrary, ν is uniformly continuous.

(iv)⇒(iii)⇒(i) are trivial.

(d) One implication is covered by (a). For the other, suppose that ν is completely additive. Then it is countably additive. On the other hand, if $\nu a \ne 0$, consider $B = \{b : b \subseteq a,\, \bar\mu b < \infty\}$. Then B is upwards-directed and $\sup B = a$, because $\bar\mu$ is semi-finite (322Eb), so $\{a \setminus b : b \in B\}$ is downwards-directed and has infimum 0. Accordingly $\inf_{b\in B} |\nu(a \setminus b)| = 0$, and there must be a $b \in B$ such that $\nu b \ne 0$. But this means that condition (ii) of (c) is satisfied, so that ν is continuous.

(e) Now suppose that $(\mathfrak{A}, \bar\mu)$ is σ-finite. In this case $\mathfrak{A}$ is ccc (322G) so complete additivity and countable additivity are the same (326L) and we have a special case of (d).

(f) Finally, suppose that $\bar\mu 1 < \infty$ and that ν is absolutely continuous with respect to $\bar\mu$. If $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and has infimum 0, then $\inf_{a\in A} \bar\mu a = 0$ (321F), so $\inf_{a\in A} |\nu a|$ must be 0; thus ν is completely additive. With (b) and (e) this shows that all four conditions are equiveridical.

327C Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra.

(a) There is a one-to-one correspondence between finitely additive functionals $\bar\nu$ on $\mathfrak{A}$ and finitely additive functionals ν on Σ such that $\nu E = 0$ whenever $\mu E = 0$, given by the formula $\bar\nu E^\bullet = \nu E$ for every $E \in \Sigma$.

(b) In (a), $\bar\nu$ is absolutely continuous with respect to $\bar\mu$ iff ν is absolutely continuous with respect to μ.

(c) In (a), $\bar\nu$ is countably additive iff ν is countably additive; so that we have a one-to-one correspondence between the countably additive functionals on $\mathfrak{A}$ and the absolutely continuous countably additive functionals on Σ.

(d) In (a), $\bar\nu$ is continuous for the measure-algebra topology on $\mathfrak{A}$ iff ν is truly continuous in the sense of 232Ab.

(e) Suppose that μ is semi-finite. Then, in (a), $\bar\nu$ is completely additive iff ν is truly continuous.

proof (a) This should be nearly obvious. If $\bar\nu : \mathfrak{A} \to \mathbb{R}$ is additive, then the formula defines a functional $\nu : \Sigma \to \mathbb{R}$ which is additive by 326Be. Also, of course,

$$\mu E = 0 \implies E^\bullet = 0 \implies \nu E = 0.$$

On the other hand, if ν is an additive functional on Σ which is zero on negligible sets, then, for E, $F \in \Sigma$,

$$\begin{aligned} E^\bullet = F^\bullet &\implies \mu(E \setminus F) = \mu(F \setminus E) = 0 \\ &\implies \nu(E \setminus F) = \nu(F \setminus E) = 0 \\ &\implies \nu F = \nu E - \nu(E \setminus F) + \nu(F \setminus E) = \nu E, \end{aligned}$$

so we have a function $\bar\nu : \mathfrak{A} \to \mathbb{R}$ defined by the given formula. If E, $F \in \Sigma$ and $E^\bullet \cap F^\bullet = 0$, then

$$\begin{aligned} \bar\nu(E^\bullet \cup F^\bullet) &= \bar\nu(E \cup F)^\bullet = \nu(E \cup F) \\ &= \nu(E \setminus F) + \nu F = \bar\nu E^\bullet + \bar\nu F^\bullet \end{aligned}$$

because $(E \setminus F)^\bullet = E^\bullet \setminus F^\bullet = E^\bullet$. Thus $\bar\nu$ is additive, and the correspondence is complete.

(b) This is immediate from the definitions.

(c)(i) If ν is countably additive, and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, we can express it as $\langle E_n \rangle_{n\in\mathbb{N}}$ where $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ. Setting $F_n = E_n \setminus \bigcup_{i<n} E_i$, $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ and

$$F_n^\bullet = a_n \setminus \sup_{i<n} a_i = a_n$$

for each n. So

$$\bar\nu(\sup_{n\in\mathbb{N}} a_n) = \nu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n=0}^\infty \nu F_n = \sum_{n=0}^\infty \bar\nu a_n.$$

As $\langle a_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\bar\nu$ is countably additive.

(ii) If $\bar\nu$ is countably additive, then ν is countably additive by 326Ff.

(iii) For the last remark, note that by 232Ba a countably additive functional on Σ is absolutely continuous with respect to μ iff it is zero on the μ-negligible sets.

(d) The definition of 'truly continuous' functional translates directly to continuity at 0 in the measure algebra. But by 327Bc this is the same thing as continuity.

(e) Put (d) and 327Bd together.

327D The Radon-Nikodým theorem We are now ready for another look at this theorem.

Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar\mu)$. Let L^1 be the space of equivalence classes of real-valued integrable functions on X (§242), and write M_τ for the set of completely additive real-valued functionals on $\mathfrak{A}$. Then there is an ordered linear space bijection between M_τ and L^1 defined by saying that $\bar\nu \in M_\tau$ corresponds to $u \in L^1$ if

$$\bar\nu a = \int_E f \text{ whenever } a = E^\bullet \text{ in } \mathfrak{A} \text{ and } f^\bullet = u \text{ in } L^1.$$

proof (a) Given $\bar\nu \in M_\tau$, we have a truly continuous $\nu : \Sigma \to \mathbb{R}$ given by setting $\nu E = \nu E^\bullet$ for every $E \in \Sigma$ (327Ce). Now there is an integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$ (232E). There is likely to be more than one such function, but any two must be equal almost everywhere (232Hd), so the corresponding equivalence class $u_{\bar\nu} = f^\bullet$ is uniquely defined.

(b) Conversely, given $u \in L^1$, we have a well-defined functional ν_u on Σ given by setting

$$\nu_u E = \int_E u = \int_E f \text{ whenever } f^\bullet = u$$

for every $E \in \Sigma$ (242Ac). By 232E, ν_u is additive and truly continuous, and of course it is zero when μ is zero, so corresponds to a completely additive functional $\bar\nu_u$ on $\mathfrak{A}$ (327Ce).

(c) Clearly the maps $u \mapsto \bar\nu_u$ and $\bar\nu \mapsto u_{\bar\nu}$ are now the two halves of a one-to-one correspondence. To see that it is linear, we need note only that

$$(\bar\nu_u+\bar\nu_v)E^\bullet=\bar\nu_uE^\bullet+\bar\nu_vE^\bullet=\int_E u+\int_E v=\int_E u+v=\bar\nu_{u+v}E^\bullet$$

for every $E\in\Sigma$, so $\bar\nu_u+\bar\nu_v=\bar\nu_{u+v}$ for all u, $v\in L^1$; and similarly $\bar\nu_{\alpha u}=\alpha\bar\nu_u$ for $u\in L^1$, $\alpha\in\mathbb{R}$. As for the ordering, given u and $v\in L^1$, take integrable f, g such that $u=f^\bullet$, $v=g^\bullet$; then

$$\begin{aligned}\bar\nu_u\le\bar\nu_v &\iff \bar\nu_uE^\bullet\le\bar\nu_vE^\bullet\text{ for every }E\in\Sigma\\ &\iff \int_E u\le\int_E v\text{ for every }E\in\Sigma\\ &\iff \int_E f\le\int_E g\text{ for every }E\in\Sigma\\ &\iff f\le g\text{ a.e.}\iff u\le v,\end{aligned}$$

using 131Ha.

327E I slip in an elementary fact.

Proposition If $(\mathfrak{A},\bar\mu)$ is a measure algebra, then the functional $a\mapsto\mu_ca=\bar\mu(a\cap c)$ is completely additive whenever $c\in\mathfrak{A}$ and $\bar\mu c<\infty$.

proof μ_c is additive because $\bar\mu$ is additive, and by 321F $\inf_{a\in A}\mu_ca=0$ whenever A is non-empty, downwards-directed and has infimum 0.

327F Standard extensions The machinery of 327D provides the basis of a canonical method for extending countably additive functionals from closed subalgebras, which we shall need in §333.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}\subseteq\mathfrak{A}$ a closed subalgebra. Write $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ for the spaces of countably additive real-valued functionals on $\mathfrak{A}$, $\mathfrak{C}$ respectively.

(a) There is an operator $R:M_\sigma(\mathfrak{C})\to M_\sigma(\mathfrak{A})$ defined by saying that, for every $\nu\in M_\sigma(\mathfrak{C})$, $R\nu$ is the unique member of $M_\sigma(\mathfrak{A})$ such that $[\![R\nu>\alpha\bar\mu]\!]=[\![\nu>\alpha\bar\mu\restriction\mathfrak{C}]\!]$ for every $\alpha\in\mathbb{R}$.

(b)(i) $R\nu$ extends ν for every $\nu\in M_\sigma(\mathfrak{C})$.

(ii) R is linear and order-preserving.

(iii) $R(\bar\mu\restriction\mathfrak{C})=\bar\mu$.

(iv) If $\langle\nu_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functionals in $M_\sigma(\mathfrak{C})$ such that $\sum_{n=0}^\infty\nu_nc=\bar\mu c$ for every $c\in\mathfrak{C}$, then $\sum_{n=0}^\infty(R\nu_n)(a)=\bar\mu a$ for every $a\in\mathfrak{A}$.

Remarks When saying that $\mathfrak{C}$ is 'closed', I mean, indifferently, 'topologically closed' or 'order-closed'; see 323H-323I.

For the notation '$[\![\nu>\alpha\bar\mu]\!]$' see 326O-326P.

proof (a)(i) By 321J-321K, we may represent $(\mathfrak{A},\bar\mu)$ as the measure algebra of a measure space (X,Σ,μ); write π for the canonical map from Σ to $\mathfrak{A}$. Write T for $\{E:E\in\Sigma,\ \pi E\in\mathfrak{C}\}$. Because $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{C}$ and π is a sequentially order-continuous Boolean homomorphism, T is a σ-subalgebra of Σ.

(ii) For each $\nu\in M_\sigma(\mathfrak{C})$, $\nu\pi:\mathrm{T}\to\mathbb{R}$ is countably additive and zero on $\{F:F\in\mathrm{T},\ \mu F=0\}$, so we can choose a T-measurable function $f_\nu:X\to\mathbb{R}$ such that $\int_F f_\nu d(\mu\restriction\mathrm{T})=\nu\pi F$ for every $F\in\mathrm{T}$. Of course we can now think of f_ν as a μ-integrable function (233B), so we get a corresponding countably additive functional $R\nu:\mathfrak{A}\to\mathbb{R}$ defined by setting $(R\nu)(\pi E)=\int_E f_\nu$ for every $E\in\Sigma$ (327D). (In this context, of course, countably additive functionals are completely additive, by 327Bf.)

For $\alpha\in\mathbb{R}$, set $H_\alpha=\{x:f_\nu(x)>\alpha\}\in\mathrm{T}$. Then for any $E\in\Sigma$,

$$E\subseteq H_\alpha,\ \mu E>0\Longrightarrow\int_E f_\nu>\alpha\mu E,$$

$$E\cap H_\alpha=\emptyset\Longrightarrow\int_E f_\nu\le\alpha\mu E.$$

Translating into terms of elements of $\mathfrak{A}$, and setting $c_\alpha=\pi H_\alpha\in\mathfrak{C}$, we have

$$0\ne a\subseteq c_\alpha\Longrightarrow(R\nu)(a)>\alpha\bar\mu a,$$

$$a\cap c_\alpha=0\Longrightarrow(R\nu)(a)\le\alpha\bar\mu a.$$

So $[\![R\nu > \alpha\bar{\mu}]\!] = c_\alpha \in \mathfrak{C}$. Of course we now have

$$\nu c = (R\nu)(c) > \alpha\bar{\mu}c \text{ when } c \in \mathfrak{C},\ 0 \neq c \subseteq c_\alpha,$$

$$\nu c \le \alpha\bar{\mu}c \text{ when } c \in \mathfrak{C},\ c \cap c_\alpha = 0,$$

so that c_α is also equal to $[\![\nu > \bar{\mu}\restriction\mathfrak{C}]\!]$.

Thus the functional $R\nu$ satisfies the declared formula.

(iii) To see that $R\nu$ is uniquely defined, observe that if $\lambda \in M_\sigma(\mathfrak{A})$ and $[\![\lambda > \alpha\bar{\mu}]\!] = [\![R\nu > \alpha\bar{\mu}]\!]$ for every α, then there is a Σ-measurable function $g : X \to \mathbb{R}$ such that $\int_E g\,d\mu = \lambda\pi E$ for every $E \in \Sigma$; but in this case (just as in (ii)) $[\![\lambda > \alpha\bar{\mu}]\!] = \pi G_\alpha$, where $G_\alpha = \{x : g(x) > \alpha\}$, for each α. So we must have $\pi G_\alpha = \pi H_\alpha$, that is, $\mu(G_\alpha \triangle H_\alpha) = 0$, for every α. Accordingly

$$\{x : f_\nu(x) \neq g(x)\} = \bigcup_{q\in\mathbb{Q}} G_q \triangle H_q$$

is negligible; $f_\nu = g$ a.e., $\int_E f_\nu d\mu = \int_E g\,d\mu$ for every $E \in \Sigma$ and $\lambda = R\nu$.

(b)(i) If $\nu \in M_\sigma(\mathfrak{C})$,

$$(R\nu)(\pi F) = \int_F f_\nu d\mu = \int_F f_\nu d(\mu\restriction\mathrm{T}) = \nu\pi F$$

for every $F \in \mathrm{T}$, so $R\nu$ extends ν.

(ii) If $\nu = \nu_1 + \nu_2$, we must have

$$\int_F f_\nu = \nu\pi F = \nu_1\pi F + \nu_2\pi F = \int_F f_{\nu_1} + \int_F f_{\nu_2} = \int_F f_{\nu_1} + f_{\nu_2}$$

for every $F \in \mathrm{T}$, so $f_\nu = f_{\nu_1} + f_{\nu_2}$ a.e., and we can repeat the formulae

$$(R\nu)(\pi E) = \int_E f_\nu = \int_E f_{\nu_1} + f_{\nu_2} = \int_E f_{\nu_1} + \int_E f_{\nu_2} = (R\nu_1)(\pi E) + (R\nu_2)(\pi E),$$

in a different order, for every $E \in \Sigma$, to see that $R\nu = R\nu_1 + R\nu_2$. Similarly, if $\nu \in M_\sigma(\mathfrak{C})$ and $\gamma \in \mathbb{R}$, $f_{\gamma\nu} = \gamma f_\nu$ a.e. and $R(\gamma\nu) = \gamma R\nu$. If $\nu_1 \le \nu_2$ in $M_\sigma(\mathfrak{C})$, then

$$\int_F f_{\nu_1} = \nu_1\pi F \le \nu_2\pi F = \int_F f_{\nu_2}$$

for every $F \in \mathrm{T}$, so $f_{\nu_1} \le f_{\nu_2}$ a.e. (131Ha), and $R\nu_1 \le R\nu_2$.

Thus R is linear and order-preserving.

(iii) If $\nu = \bar{\mu}\restriction\mathfrak{C}$ then

$$\int_F f_\nu = \nu\pi F = \mu F = \int_F \mathbf{1}$$

for every $F \in \mathrm{T}$, so $f_\nu = \mathbf{1}$ a.e. and $R\nu = \bar{\mu}$.

(iv) Now suppose that $\langle\nu_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_\sigma(\mathfrak{C})$ such that, for every $c \in \mathfrak{C}$, $\nu_n c \ge 0$ for every n and $\sum_{n=0}^\infty \nu_n c = \bar{\mu}c$. Set $g_n = \sum_{i=0}^n f_{\nu_i}$ for each n; then $0 \le g_n \le g_{n+1} \le \mathbf{1}$ a.e. for every n, and

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty}\sum_{i=0}^n \nu_i 1 = \bar{\mu}1.$$

But this means that, setting $g = \lim_{n\to\infty} g_n$, $g \le \mathbf{1}$ a.e. and $\int g = \int \mathbf{1}$, so that $g = \mathbf{1}$ a.e. and

$$\sum_{n=0}^\infty (R\nu_i)(\pi E) = \lim_{n\to\infty}\int_E g_n = \mu E$$

for every $E \in \Sigma$, so that $\sum_{n=0}^\infty (R\nu_i)(a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

327G Definition In the context of 327F, I will call $R\nu$ the **standard extension** of ν to $\mathfrak{A}$.

Remark The point of my insistence on the uniqueness of R, and on the formula in 327Fa, is that $R\nu$ really is defined by the abstract structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C}, \nu)$, even though I have used a proof which runs through the representation of $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a measure space (X, Σ, μ).

327X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\mu}')$ be totally finite measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be a closed subalgebra of $\mathfrak{A}$, and ν a countably additive functional on the closed subalgebra $\pi[\mathfrak{C}]$ of $\mathfrak{B}$. (i) Show that $\nu\pi$ is a countably additive functional on $\mathfrak{C}$. (ii) Show that if $\tilde{\nu}$ is the standard extension of ν to $\mathfrak{B}$, then $\tilde{\nu}\pi$ is the standard extension of $\nu\pi$ to $\mathfrak{A}$.

(b) Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of (X,Σ,μ). Show that $\mathfrak{C}=\{F^\bullet : F\in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. Identify the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals with $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$, as in 327D. Show that the conditional expectation operator $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ (242Jd) corresponds to the map $\nu\mapsto\nu\restriction\mathfrak{C}:M_\sigma(\mathfrak{A})\to M_\sigma(\mathfrak{C})$.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a countably additive functional. Show that, for any $a\in\mathfrak{A}$,

$$\nu a=\int_0^\infty\bar{\mu}(a\cap[\![\nu>\alpha\bar{\mu}]\!])d\alpha-\int_{-\infty}^0\bar{\mu}(a\setminus[\![\nu>\alpha\bar{\mu}]\!])d\alpha,$$

the integrals being taken with respect to Lebesgue measure. (*Hint*: take $(\mathfrak{A},\bar{\mu})$ to be the measure algebra of (X,Σ,μ); represent ν by a μ-integrable function f; apply Fubini's theorem to the sets $\{(x,t):x\in E,\ 0\le t<f(x)\}$, $\{(x,t):x\in E,\ f(x)\le t\le 0\}$ in $X\times\mathbb{R}$, where $a=E^\bullet$.)

(d) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and $\nu:\mathfrak{C}\to\mathbb{R}$ a countably additive functional with standard extension $\tilde{\nu}:\mathfrak{A}\to\mathbb{R}$. Show that, for any $a\in\mathfrak{A}$,

$$\tilde{\nu}a=\int_0^\infty\bar{\mu}(a\cap[\![\nu>\alpha\bar{\mu}\restriction\mathfrak{C}]\!])d\alpha-\int_{-\infty}^0\bar{\mu}(a\setminus[\![\nu>\alpha\bar{\mu}\restriction\mathfrak{C}]\!])d\alpha.$$

(e) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\mathfrak{B}$, $\mathfrak{C}$ stochastically independent closed subalgebras of $\mathfrak{A}$ (definition: 325L). Let ν be a countably additive functional on $\mathfrak{C}$, and $\tilde{\nu}$ its standard extension to $\mathfrak{A}$. Show that $\tilde{\nu}(b\cap c)=\bar{\mu}b\cdot\nu c$ for every $b\in\mathfrak{B}$, $c\in\mathfrak{C}$.

(f) Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Let ν be a probability measure with domain T such that $\nu E=0$ whenever $E\in\mathrm{T}$ and $\mu E=0$. Show that there is a probability measure λ with domain Σ which extends ν.

327Y Further exercises (a) Let $(\mathfrak{A}_1,\bar{\mu}_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2)$ be localizable measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$. Show that if ν_1, ν_2 are completely additive functionals on $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively, there is a unique completely additive functional $\nu:\mathfrak{C}\to\mathbb{R}$ such that $\nu(a_1\otimes a_2)=\nu_1a_1\cdot\nu_2a_2$ for every $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$. (*Hint*: 253D.)

(b) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra; let $R:M_\sigma(\mathfrak{C})\to M_\sigma(\mathfrak{A})$ be the standard extension operator (327G). Show (i) that R is order-continuous (ii) that $R(\nu^+)=(R\nu)^+$, $\|R\nu\|=\|\nu\|$ for every $\nu\in M_\sigma(\mathfrak{C})$, defining ν^+ and $\|\nu\|$ as in 326Yj.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. For a countably additive functional ν on $\mathfrak{C}$ write $\tilde{\nu}$ for its standard extension to $\mathfrak{A}$. Show that if ν, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ are countably additive functionals on $\mathfrak{C}$ and $\lim_{n\to\infty}\nu_nc=\nu c$ for every $c\in\mathfrak{C}$, then $\lim_{n\to\infty}\tilde{\nu}_na=\tilde{\nu}a$ for every $a\in\mathfrak{A}$. (*Hint*: use ideas from §§246-247, as well as from 327G and 326Yo.)

327 Notes and comments When we come to measure algebras, it is the completely additive functionals which fit most naturally into the topological theory (327Bd); they correspond to the 'truly continuous' functionals which I discussed in §232 (327Cc), and therefore to the Radon-Nikodým theorem (327D). I will return to some of these questions in Chapter 36. I myself regard the form here as the best expression of the essence of the Radon-Nikodým theorem, if not the one most commonly applied.

The concept of 'standard extension' of a countably additive functional (or, as we could equally well say, of a completely additive functional, since in the context of 327F the two coincide) is in a sense dual to the concept of 'conditional expectation'. If (X,Σ,μ) is a probability space and T is a σ-subalgebra of Σ, then we have a corresponding closed subalgebra $\mathfrak{C}$ of the measure algebra $(\mathfrak{A},\bar{\mu})$ of μ, and identifications between the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals and the spaces $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$. Now we have a natural embedding S of $L^1(\mu\restriction\mathrm{T})$ as a subspace of $L^1(\mu)$ (242Jb), and a natural restriction map from $M_\sigma(\mathfrak{A})$ to $M_\sigma(\mathfrak{C})$. These give rise to corresponding operators between the opposite members of each pair; the standard extension operator R of 327G, and the conditional expectation operator P of 242Jd. (See 327Xb.) The fundamental fact

$$PSv = v \text{ for every } v \in L^1(\mu\restriction\mathrm{T})$$

(242Jg) is matched by the fact that

$$R\nu\restriction\mathfrak{C} = \nu \text{ for every } \nu \in M_\sigma(\mathfrak{C}).$$

The further identification of $R\nu$ in terms of integrals $\int \bar{\mu}(a \cap [\![\nu > \alpha\bar{\mu}]\!])d\alpha$ (327Xc) is relatively inessential, but is striking, and perhaps makes it easier to believe that R is truly 'standard' in the abstract contexts which will arise in §333 below. It is also useful in such calculations as 327Xe.

The isomorphisms between M_τ spaces and L^1 spaces described here mean that any of the concepts involving L^1 spaces discussed in Chapter 24 can be applied to M_τ spaces, at least in the case of measure algebras. In fact, as I will show in Chapter 36, there is much more to be said here; the space of bounded additive functionals on a Boolean algebra is already an L^1 space in an abstract sense, and ideas such as 'uniform integrability' are relevant and significant there, as well as in the spaces of countably additive and completely additive functionals. I hope that 326Yj, 326Ym-326Yn, 326Yp-326Yq and 327Yb will provide some hints to be going on with for the moment.

Chapter 33

Maharam's theorem

We are now ready for the astonishing central fact about measure algebras: there are very few of them. Any localizable measure algebra has a canonical expression as a simple product of measure algebras of easily described types. This complete classification necessarily dominates all further discussion of measure algebras; to the point that all the results of Chapter 32 have to be regarded as 'elementary', since however complex their formulation they have been proved by techniques not involving, nor providing, any particular insight into the special nature of measure algebras. The proof depends, of course, on developing methods of defining measure-preserving homomorphisms and isomorphisms; I give a number of results, progressively more elaborate, but all based on the same idea. These techniques are of great power, leading, for instance, to an effective classification of closed subalgebras and their embeddings.

'Maharam's theorem' itself, the classification of localizable measure algebras, is in §332. I devote §331 to the definition and description of 'homogeneous' probability algebras. In §333 I turn to the problem of describing pairs $(\mathfrak{A},\mathfrak{C})$ where $\mathfrak{A}$ is a probability algebra and $\mathfrak{C}$ is a closed subalgebra. Finally, in §334, I give some straightforward results on the classification of free products of probability algebras.

331 Classification of homogeneous measure algebras

I embark directly on the principal theorem of this chapter (331I), split between 331B, 331D and 331I; 331B and 331D will be the basis of many of the results in later sections of this chapter. In 331E-331H I introduce the concepts of 'Maharam type' and 'Maharam homogeneity'. I discuss the measure algebras of products $\{0,1\}^\kappa$, showing that these provide a complete set of examples of Maharam-type-homogeneous probability algebras (331J-331L). I end the section with a brief comment on 'homogeneous' Boolean algebras (331M-331N).

331A Definition The following idea is almost the key to the whole chapter. Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. A non-zero element a of $\mathfrak{A}$ is a **relative atom** over $\mathfrak{B}$ if every $c \subseteq a$ is of the form $a\cap b$ for some $b \in \mathfrak{B}$; that is, $\{a\cap b : b\in\mathfrak{B}\}$ is the principal ideal generated by a. We say that $\mathfrak{A}$ is **relatively atomless** over $\mathfrak{B}$ if there are no relative atoms in $\mathfrak{A}$ over $\mathfrak{B}$.

(I'm afraid the phrases 'relative atom', 'relatively atomless' are bound to seem opaque at this stage. I hope that after the structure theory of §333 they will seem more natural. For the moment, note only that a is an atom in $\mathfrak{A}$ iff it is a relative atom over the smallest subalgebra $\{0,1\}$, and every element of $\mathfrak{A}$ is a relative atom over the largest subalgebra $\mathfrak{A}$. In a way, a is a relative atom over $\mathfrak{B}$ if its image is an atom in a kind of quotient $\mathfrak{A}/\mathfrak{B}$. But we are two volumes away from any prospect of making sense of this kind of quotient.)

331B The first lemma is the heart of Maharam's theorem.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$. Let $\nu:\mathfrak{B}\to\mathbb{R}$ be an additive functional such that $0\le\nu b\le\bar\mu b$ for every $b\in\mathfrak{B}$. Then there is a $c\in\mathfrak{A}$ such that $\nu b=\bar\mu(b\cap c)$ for every $b\in\mathfrak{B}$.

Remark Recall that by 323H we need not distinguish between 'order-closed' and 'topologically closed' subalgebras.

proof (a) It is worth noting straight away that ν is necessarily countably additive. This is easy to check from first principles, but if you want to trace the underlying ideas they are in 313O (the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous), 326Ff (so $\mu\restriction\mathfrak{B}:\mathfrak{B}\to\mathbb{R}$ is countably additive) and 326Gb (therefore ν is countably additive).

(b) For each $a\in\mathfrak{A}$ set $\nu_a b=\bar\mu(b\cap a)$ for every $b\in\mathfrak{B}$; then ν_a is countably additive (326Fd). The key idea is the following fact: for every non-zero $a\in\mathfrak{A}$ there is a non-zero $d\subseteq a$ such that $\nu_d\le\frac{1}{2}\nu_a$. **P** Because $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, there is an $e\subseteq a$ such that $e\ne a\cap b$ for any $b\in\mathfrak{B}$. Consider the countably additive functional $\lambda = \nu_a - 2\nu_e : \mathfrak{B} \to \mathbb{R}$. By 326I, there is a $b_0 \in \mathfrak{B}$ such that $\lambda b \ge 0$ whenever $b \in \mathfrak{B}$, $b \subseteq b_0$, while $\lambda b \le 0$ whenever $b \in \mathfrak{B}$, $b \cap b_0 = 0$.

If $e \cap b_0 \ne 0$, try $d = e \cap b_0$. Then $0 \ne d \subseteq a$, and for every $b \in \mathfrak{B}$

$$\nu_d b = \nu_e(b \cap b_0) = \frac{1}{2}(\nu_a(b \cap b_0) - \lambda(b \cap b_0)) \le \frac{1}{2}\nu_a b$$

(because $\lambda(b \cap b_0) \ge 0$) so $\nu_d \le \frac{1}{2}\nu_a$.

If $e \cap b_0 = 0$, then (by the choice of e) $e \ne a \cap (1 \setminus b_0)$, so $d = a \setminus (e \cup b_0) \ne 0$, and of course $d \subseteq a$. In this case, for every $b \in \mathfrak{B}$,

$$\nu_d b = \nu_a(b \setminus b_0) - \nu_e(b \setminus b_0) = \frac{1}{2}(\lambda(b \setminus b_0) + \nu_a(b \setminus b_0)) \le \frac{1}{2}\nu_a b$$

(because $\lambda(b \setminus b_0) \le 0$), so once again $\nu_d \le \frac{1}{2}\nu_a$.

Thus in either case we have a suitable d. **Q**

(c) It follows at once, by induction on n, that if a is any non-zero element of $\mathfrak{A}$ and $n \in \mathbb{N}$ then there is a non-zero $d \subseteq a$ such that $\nu_d \le 2^{-n}\nu_a$.

(d) Now let C be the set

$$\{a : a \in \mathfrak{A},\ \nu_a \le \nu\}.$$

Then $0 \in C$, so $C \ne \emptyset$. If $D \subseteq C$ is upwards-directed and not empty, then $a = \sup D$ is defined in $\mathfrak{A}$, and

$$\nu_{\sup D} b = \bar\mu(b \cap \sup D) = \bar\mu(\sup_{d \in D} b \cap d) = \sup_{d \in D} \bar\mu(b \cap d) = \sup_{d \in D} \nu_d b \le \nu b$$

using 313Ba and 321C. So $a \in C$ and is an upper bound for D in C. In particular, any non-empty totally ordered subset of C has an upper bound in C. By Zorn's Lemma, C has a maximal element c say.

(e) ? Suppose, if possible, that $\nu_c \ne \nu$. Then there is some $b^* \in \mathfrak{B}$ such that $\nu_c b^* \ne \nu b^*$; since $\nu_c \le \nu$, $\nu_c b^* < \nu b^*$. Let $n \in \mathbb{N}$ be such that $\nu b^* > \nu_c b^* + 2^{-n}\bar\mu b^*$, and set

$$\lambda b = \nu b - \nu_c b - 2^{-n}\bar\mu b$$

for every $b \in \mathfrak{B}$. By 326I (for the second time), there is a $b_0 \in \mathfrak{B}$ such that $\lambda b \ge 0$ for $b \in \mathfrak{B}$, $b \subseteq b_0$, while $\lambda b \le 0$ when $b \in \mathfrak{B}$ and $b \cap b_0 = 0$. We have

$$\begin{aligned}\bar\mu(b_0 \setminus c) &= \bar\mu b_0 - \bar\mu(b_0 \cap c) \ge \nu b_0 - \nu_c b_0\\ &\ge \lambda b_0 = \lambda b^* + \lambda(b_0 \setminus b^*) - \lambda(b^* \setminus b_0) \ge \lambda b^* > 0,\end{aligned}$$

so $b_0 \setminus c \ne 0$. (This is where I use the hypothesis that $\nu \le \bar\mu\restriction\mathfrak{B}$.) By (c), there is a non-zero $d \subseteq b_0 \setminus c$ such that

$$\nu_d \le 2^{-n}\nu_{b_0 \setminus c} \le 2^{-n}\nu_{b_0}.$$

Now $d \cap c = 0$ so $c \subset d \cup c$. Also, for any $b \in \mathfrak{B}$,

$$\begin{aligned}\nu_{d \cup c} b &= \nu_d b + \nu_c(b \cap b_0) + \nu_c(b \setminus b_0)\\ &\le 2^{-n}\nu_{b_0 \setminus c} b + \nu(b \cap b_0) - 2^{-n}\bar\mu(b \cap b_0) - \lambda(b \cap b_0) + \nu(b \setminus b_0)\\ &\le 2^{-n}\bar\mu(b \cap b_0 \setminus c) + \nu(b \cap b_0) - 2^{-n}\bar\mu(b \cap b_0) + \nu(b \setminus b_0)\\ &\le \nu b.\end{aligned}$$

But this means that $d \cup c \in C$ and c is not maximal in C. **X**

Thus c is the required element of $\mathfrak{A}$ giving a representation of ν.

331C Corollary Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra, and $a \in \mathfrak{A}$. Suppose that $0 \le \gamma \le \bar\mu a$. Then there is a $c \in \mathfrak{A}$ such that $c \subseteq a$ and $\bar\mu c = \gamma$.

proof If $\gamma = \bar\mu a$, take $c = a$. If $\gamma < \bar\mu a$, there is a $d \in \mathfrak{A}$ such that $d \subseteq a$ and $\gamma \le \bar\mu d < \infty$ (322Eb). Apply 331B to the principal ideal $\mathfrak{A}_d$ generated by d, with $\mathfrak{B} = \{0, d\}$ and $\nu d = \gamma$. (The point is that because $\mathfrak{A}$ is atomless, no non-trivial principal ideal of $\mathfrak{A}_d$ can be of the form $\{c \cap b : b \in \mathfrak{B}\} = \{0, c\}$.)

Remark Of course this is also an easy consequence of 215D.

331D Lemma Let $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ be totally finite measure algebras and $\mathfrak{C}\subseteq\mathfrak{A}$ a closed subalgebra. Suppose that $\pi:\mathfrak{C}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism such that $\mathfrak{B}$ is relatively atomless over $\pi[\mathfrak{C}]$. Take any $a\in\mathfrak{A}$, and let $\mathfrak{C}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{a\}$. Then there is a measure-preserving homomorphism from $\mathfrak{C}_1$ to $\mathfrak{B}$ extending π.

proof We know that $\pi[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{B}$ (324Kb), and that π is a Boolean isomorphism between $\mathfrak{C}$ and $\pi[\mathfrak{C}]$. Consequently the countably additive functional $c\mapsto\bar{\mu}(c\cap a):\mathfrak{C}\to\mathbb{R}$ is transferred to a countably additive functional $\lambda:\pi[\mathfrak{C}]\to\mathbb{R}$, writing $\lambda(\pi c)=\bar{\mu}(c\cap a)$ for every $c\in\mathfrak{C}$. Of course $\lambda(\pi c)\le\bar{\mu}c=\bar{\nu}(\pi c)$ for every $c\in\mathfrak{C}$. So by 331B there is a $b\in\mathfrak{B}$ such that $\lambda(\pi c)=\bar{\nu}(b\cap\pi c)$ for every $c\in\mathfrak{C}$.

If $c\in\mathfrak{C}$, $c\subseteq a$ then

$$\bar{\nu}(b\cap\pi c)=\lambda(\pi c)=\bar{\mu}(a\cap c)=\bar{\mu}c=\bar{\nu}(\pi c),$$

so $\pi c\subseteq b$. Similarly, if $a\subseteq c\in\mathfrak{C}$, then

$$\bar{\nu}(b\cap\pi c)=\bar{\mu}(a\cap c)=\bar{\mu}(a\cap 1)=\bar{\nu}(b\cap\pi 1)=\bar{\nu}b,$$

so $b\subseteq\pi c$. It follows from 312N that there is a Boolean homomorphism $\pi_1:\mathfrak{C}_1\to\mathfrak{B}$, extending π, such that $\pi_1 a=b$.

To see that π_1 is measure-preserving, take any member of $\mathfrak{C}_1$. By 312M, this is expressible as $e=(c_1\cap a)\cup(c_2\setminus a)$, where c_1, $c_2\in\mathfrak{C}$. Now

$$\begin{aligned}\bar{\nu}(\pi_1 e)&=\bar{\nu}((\pi c_1\cap b)\cup(\pi c_2\setminus b))=\bar{\nu}(\pi c_1\cap b)+\bar{\nu}(\pi c_2)-\bar{\nu}(\pi c_2\cap b)\\&=\bar{\mu}(c_1\cap a)+\bar{\mu}c_2-\bar{\mu}(c_2\cap a)=\bar{\mu}e.\end{aligned}$$

As e is arbitrary, π_1 is measure-preserving.

331E Generating sets For the sake of the next definition, we need a language a little more precise than I have felt the need to use so far. The point is that if $\mathfrak{A}$ is a Boolean algebra and B is a subset of $\mathfrak{A}$, there is more than one subalgebra of $\mathfrak{A}$ which can be said to be 'generated' by B, because we can look at any of the three algebras

– $\mathfrak{B}$, the smallest subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\sigma$, the smallest σ-subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\tau$, the smallest order-closed subalgebra of $\mathfrak{A}$ including B.

(See 313Fb.) Now I will say henceforth, in this context, that

– $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by B, and B **generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}$;

– $\mathfrak{B}_\sigma$ is the σ-subalgebra of $\mathfrak{A}$ generated by B, and B **σ-generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}_\sigma$;

– $\mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by B, and B **τ-generates** or **completely generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}_\tau$.

There is a danger inherent in these phrases, because if we have $B\subseteq\mathfrak{A}'$, where $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$, it is possible that the smallest order-closed subalgebra of $\mathfrak{A}'$ including B might not be recoverable from the smallest order-closed subalgebra of $\mathfrak{A}$ including B. (See 331Yb-331Yc.) This problem will not seriously interfere with the ideas below; but for definiteness let me say that the phrases 'B σ-generates $\mathfrak{A}$', 'B τ-generates $\mathfrak{A}$' will always refer to suprema and infima taken in $\mathfrak{A}$ itself, not in any larger algebra in which it may be embedded.

331F Maharam types (a) With the language of 331E established, I can now define the **Maharam type** or **complete generation** $\tau(\mathfrak{A})$ of any Boolean algebra $\mathfrak{A}$; it is the smallest cardinal of any subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$.

(I think that this is the first 'cardinal function' which I have mentioned in this treatise. All you need to know, to confirm that the definition is well-conceived, is that there is *some* set which τ-generates $\mathfrak{A}$; and obviously $\mathfrak{A}$ τ-generates itself. For this means that the set $A=\{\#(B):B\subseteq\mathfrak{A}\ \tau\text{-generates }\mathfrak{A}\}$ is a non-empty class of cardinals, and therefore, assuming the axiom of choice, has a least member (2A1Lf). In 331Ye-331Yf I mention a further function, the 'density' of a topological space, which is closely related to Maharam type.)

(b) A Boolean algebra $\mathfrak{A}$ is **Maharam-type-homogeneous** if $\tau(\mathfrak{A}_a)=\tau(\mathfrak{A})$ for every non-zero $a\in\mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(c) Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar{\mu})$. Then the **Maharam type** of (X,Σ,μ), or of μ, is the Maharam type of $\mathfrak{A}$; and (X,Σ,μ), or μ, is **Maharam-type-homogeneous** if $\mathfrak{A}$ is.

Remark I should perhaps remark that the phrases 'Maharam type' and 'Maharam-type-homogeneous', while well established in the context of probability algebras, are not in common use for general Boolean algebras. But the cardinal $\tau(\mathfrak{A})$ is important in the general context, and is such an obvious extension of Maharam's idea (MAHARAM 42) that I am happy to propose this extension of terminology.

331G For the sake of those who have not mixed set theory and algebra before, I had better spell out some basic facts.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B, $\mathfrak{B}_\sigma$ the σ-subalgebra of $\mathfrak{A}$ generated by B, and $\mathfrak{B}_\tau$ the order-closed subalgebra of $\mathfrak{A}$ generated by B.

(a) $\mathfrak{B}\subseteq\mathfrak{B}_\sigma\subseteq\mathfrak{B}_\tau$.

(b) If B is finite, so is $\mathfrak{B}$, and in this case $\mathfrak{B}=\mathfrak{B}_\sigma=\mathfrak{B}_\tau$.

(c) For every $a\in\mathfrak{B}$, there is a finite $B'\subseteq B$ such that a belongs to the subalgebra of $\mathfrak{A}$ generated by B'. Consequently $\#(\mathfrak{B})\le\max(\omega,\#(B))$.

(d) For every $a\in\mathfrak{B}_\sigma$, there is a countable $B'\subseteq B$ such that a belongs to the σ-subalgebra of $\mathfrak{A}$ generated by B'.

(e) If $\mathfrak{A}$ is ccc, then $\mathfrak{B}_\sigma=\mathfrak{B}_\tau$.

proof (a) All we need to know is that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$ including B, and that $\mathfrak{B}_\tau$ is a σ-subalgebra of $\mathfrak{A}$ including B.

(b) Induce on $\#(B)$, using 312M for the inductive step, to see that $\mathfrak{B}$ is finite. In this case it must be order-closed, so is equal to $\mathfrak{B}_\tau$.

(c)(i) For $I\subseteq B$, let $\mathfrak{C}_I$ be the subalgebra of $\mathfrak{A}$ generated by I. If I, $J\subseteq B$ then $\mathfrak{C}_I\cup\mathfrak{C}_J\subseteq\mathfrak{C}_{I\cup J}$. So $\bigcup\{\mathfrak{C}_I:I\subseteq B\text{ is finite}\}$ is a subalgebra of $\mathfrak{A}$, and must be equal to $\mathfrak{B}$, as claimed.

(ii) To estimate the size of $\mathfrak{B}$, recall that the set $[B]^{<\omega}$ of all finite subsets of B has cardinal at most $\max(\omega,\#(B))$ (3A1Cd). For each $I\in[B]^{<\omega}$, $\mathfrak{C}_I$ is finite, so

$$\#(\mathfrak{B})=\#(\bigcup_{I\in[B]^{<\omega}}\mathfrak{C}_I)\le\max(\omega,\#(I),\sup_{I\in[B]^{<\omega}}\#(\mathfrak{C}_I))\le\max(\omega,\#(B))$$

by 3A1Cc.

(d) For $I\subseteq B$, let $\mathfrak{D}_I\subseteq\mathfrak{B}_\sigma$ be the σ-subalgebra of $\mathfrak{A}$ generated by I. If I, $J\subseteq B$ then $\mathfrak{D}_I\cup\mathfrak{D}_J\subseteq\mathfrak{D}_{I\cup J}$, so $\mathfrak{B}'_\sigma=\bigcup\{\mathfrak{D}_I:I\subseteq B\text{ is countable}\}$ is a subalgebra of $\mathfrak{A}$. But also it is sequentially order-closed in $\mathfrak{A}$. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}'_\sigma$ with supremum a in $\mathfrak{A}$. For each $n\in\mathbb{N}$ there is a countable $I(n)\subseteq B$ such that $a_n\in\mathfrak{C}_{I(n)}$. Set $K=\bigcup_{n\in\mathbb{N}}I(n)$; then K is a countable subset of B and every a_n belongs to $\mathfrak{D}_K$, so $a\in\mathfrak{D}_K\subseteq\mathfrak{B}'_\sigma$. **Q** So $\mathfrak{B}'_\sigma$ is a σ-subalgebra of $\mathfrak{A}$ including B and must be the whole of $\mathfrak{B}_\sigma$.

(e) By 316Fb, $\mathfrak{B}_\sigma$ is order-closed in $\mathfrak{A}$, so must be equal to $\mathfrak{B}_\tau$.

331H Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) $\tau(\mathfrak{A})=0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0,1\}$.

(ii) $\tau(\mathfrak{A})$ is finite iff $\mathfrak{A}$ is finite.

(b) If $\mathfrak{B}$ is another Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\tau(\mathfrak{B})\le\tau(\mathfrak{A})$.

(c) If $a\in\mathfrak{A}$ then $\tau(\mathfrak{A}_a)\le\tau(\mathfrak{A})$, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

(d) If $\mathfrak{A}$ has an atom and is Maharam-type-homogeneous, then $\mathfrak{A}=\{0,1\}$.

proof (a)(i) $\tau(\mathfrak{A})=0$ iff $\mathfrak{A}$ has no proper subalgebras. **(ii)** If $\mathfrak{A}$ is finite, then $\tau(\mathfrak{A})\le\#(\mathfrak{A})$ is finite. If $\tau(\mathfrak{A})$ is finite, then there is a finite set $B\subseteq\mathfrak{A}$ which τ-generates $\mathfrak{A}$; by 331Gb, $\mathfrak{A}$ is finite.

(b) We know that there is a set $A\subseteq\mathfrak{A}$, τ-generating $\mathfrak{A}$, with $\#(A)=\tau(\mathfrak{A})$. Now $\pi[A]$ τ-generates $\pi[\mathfrak{A}]=\mathfrak{B}$ (313Mb), so

$$\tau(\mathfrak{B})\le\#(\pi[A])\le\#(A)=\tau(\mathfrak{A}).$$

(c) Apply (b) to the map $b \mapsto a \cap b : \mathfrak{A} \to \mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is an atom, then $\tau(\mathfrak{A}_a) = 0$, so if $\mathfrak{A}$ is Maharam-type-homogeneous then $\tau(\mathfrak{A}) = 0$ and $\mathfrak{A} = \{0, a\} = \{0, 1\}$.

331I We are now ready for the theorem.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be Maharam-type-homogeneous measure algebras of the same Maharam type, with $\bar{\mu}1 = \bar{\nu}1 < \infty$. Then they are isomorphic as measure algebras.

proof (a) If $\tau(\mathfrak{A}) = \tau(\mathfrak{B}) = 0$, this is trivial. So let us take $\kappa = \tau(\mathfrak{A}) = \tau(\mathfrak{B}) > 0$. In this case, because $\mathfrak{A}$ and $\mathfrak{B}$ are Maharam-type-homogeneous, they can have no atoms and must be infinite, so κ is infinite (331H). Let $\langle a_\xi \rangle_{\xi<\kappa}$, $\langle b_\xi \rangle_{\xi<\kappa}$ enumerate τ-generating subsets of $\mathfrak{A}$, $\mathfrak{B}$ respectively.

The strategy of the proof is to define a measure-preserving isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ as the last of an increasing family $\langle \pi_\xi \rangle_{\xi\le\kappa}$ of isomorphisms between closed subalgebras $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ of $\mathfrak{A}$ and $\mathfrak{B}$. The inductive hypothesis will be that, for some families $\langle a'_\xi \rangle_{\xi<\kappa}$, $\langle b'_\xi \rangle_{\xi<\kappa}$ to be determined,

$\mathfrak{C}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta < \xi\} \cup \{a'_\eta : \eta < \xi\}$,

$\mathfrak{D}_\xi$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}$,

$\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ is a measure-preserving isomorphism,

π_ξ extends π_η whenever $\eta < \xi$.

(Formally speaking, this will be a transfinite recursion, defining a function $\xi \mapsto f(\xi) = (\mathfrak{C}_\xi, \mathfrak{D}_\xi, \pi_\xi, a'_\xi, b'_\xi)$ on the ordinal $\kappa + 1$ by a rule which chooses $f(\xi)$ in terms of $f{\restriction}\xi$, as described in 2A1B. The construction of an actual function F for which $f(\xi) = F(f{\restriction}\xi)$ will necessitate the axiom of choice.)

(b) The induction starts with $\mathfrak{C}_0 = \{0, 1\}$, $\mathfrak{D}_0 = \{0, 1\}$, $\pi_0(0) = 0$, $\pi_0(1) = 1$. (The hypothesis $\bar{\mu}1 = \bar{\nu}1$ is what we need to ensure that π_0 is measure-preserving.)

(c) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

(i) For any non-zero $b \in \mathfrak{B}$, the principal ideal $\mathfrak{B}_b$ of $\mathfrak{B}$ generated by b has Maharam type κ, because $\mathfrak{B}$ is Maharam-type-homogeneous. On the other hand, the Maharam type of $\mathfrak{D}_\xi$ is at most

$$\#(\{b_\eta : \eta \le \xi\} \cup \{b'_\eta : \eta < \xi\}) \le \#(\xi \times \{0, 1\}) < \kappa,$$

because if ξ is finite so is $\xi \times \{0, 1\}$, while if ξ is infinite then $\#(\xi \times \{0, 1\}) = \#(\xi) \le \xi < \kappa$. Consequently $\mathfrak{B}_b$ cannot be an order-continuous image of $\mathfrak{D}_\xi$ (331Hb). Now the map $c \mapsto c \cap b : \mathfrak{D}_\xi \to \mathfrak{B}_b$ is order-continuous, because $\mathfrak{D}_\xi$ is closed, so that the embedding $\mathfrak{D}_\xi \subseteq \mathfrak{B}$ is order-continuous. It therefore cannot be surjective, and

$$\{b \cap \pi_\xi a : a \in \mathfrak{C}_\xi\} = \{b \cap d : d \in \mathfrak{D}_\xi\} \ne \mathfrak{B}_b.$$

This means that $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ satisfies the conditions of 331D, and must have an extension ϕ_ξ to a measure-preserving homomorphism from the subalgebra $\mathfrak{C}'_\xi$ of $\mathfrak{A}$ generated by $\mathfrak{C}_\xi \cup \{a_\xi\}$ to $\mathfrak{B}$. We know that $\mathfrak{C}'_\xi$ is a closed subalgebra of $\mathfrak{A}$ (314Ja), so it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta < \xi\}$. Also $\mathfrak{D}'_\xi = \phi_\xi[\mathfrak{C}'_\xi]$ will be the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}_\xi \cup \{b'_\xi\}$, where $b'_\xi = \phi_\xi(a_\xi)$, so is closed in $\mathfrak{B}$, and is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta \le \xi\}$.

(ii) The next step is to repeat the whole of the argument above, but applying it to $\phi_\xi^{-1} : \mathfrak{D}'_\xi \to \mathfrak{C}_\xi$, b_ξ in place of $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ and a_ξ. Once again, we have $\tau(\mathfrak{D}'_\xi) < \kappa = \tau(\mathfrak{A}_a)$ for every $a \in \mathfrak{A}$, so we can use Lemma 331D to find a measure-preserving isomorphism $\psi_\xi : \mathfrak{D}_{\xi+1} \to \mathfrak{C}_{\xi+1}$ extending ϕ_ξ^{-1}, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi \cup \{b_\xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi \cup \{a'_\xi\}$, setting $a'_\xi = \psi_\xi(b_\xi)$. As in (i), we find that $\mathfrak{C}_{\xi+1}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta \le \xi\}$, while $\mathfrak{D}_{\xi+1}$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta \le \xi\} \cup \{b'_\eta : \eta \le \xi\}$.

(iii) We can therefore take $\pi_{\xi+1} = \psi_\xi^{-1} : \mathfrak{C}_{\xi+1} \to \mathfrak{D}_{\xi+1}$, and see that $\pi_{\xi+1}$ is a measure-preserving isomorphism, extending π_ξ, such that $\pi_{\xi+1}(a_\xi) = b'_\xi$, $\pi_{\xi+1}(a'_\xi) = b_\xi$. Evidently $\pi_{\xi+1}$ extends π_η for every $\eta \le \xi$ because it extends π_ξ and (by the inductive hypothesis) π_ξ extends π_η for every $\eta < \xi$.

(d) For the inductive step to a limit ordinal ξ, where $0<\xi\le\kappa$, suppose that $\mathfrak{C}_\eta$, $\mathfrak{D}_\eta$, a'_η, b'_η, π_η have been defined for $\eta<\xi$. Set $\mathfrak{C}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{C}_\xi$. Then $\mathfrak{C}^*_\xi$ is a subalgebra of $\mathfrak{A}$, because it is the union of an upwards-directed family of subalgebras; similarly, $\mathfrak{D}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{D}_\xi$ is a subalgebra of $\mathfrak{B}$. Next, we have a function $\pi^*_\xi:\mathfrak{C}^*_\xi\to\mathfrak{D}^*_\xi$ defined by setting $\pi^*_\xi a=\pi_\eta a$ whenever $\eta<\xi$ and $a\in\mathfrak{C}_\eta$; for if η, $\zeta<\xi$ and $a\in\mathfrak{C}_\eta\cap\mathfrak{C}_\zeta$, then $\pi_\eta a=\pi_{\max(\eta,\zeta)}a=\pi_\zeta a$. Clearly

$$\pi^*_\xi[\mathfrak{C}^*_\xi]=\bigcup_{\eta<\xi}\pi_\eta[\mathfrak{C}_\eta]=\mathfrak{D}^*_\xi.$$

Moreover, $\bar\nu\pi^*_\xi a=\bar\mu a$ for every $a\in\mathfrak{C}^*_\xi$, since $\bar\nu\pi_\eta a=\bar\mu a$ whenever $\eta<\xi$ and $a\in\mathfrak{C}_\eta$.

Now let $\mathfrak{C}_\xi$ be the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}^*_\xi$, that is, the metric closure of $\mathfrak{C}^*_\xi$ in $\mathfrak{A}$ (323J). Since $\mathfrak{C}_\xi$ is the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}_\eta$ for every $\eta<\xi$, it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}\cup\{a'_\eta:\eta<\xi\}$. By 324O, π^*_ξ has an extension to a measure-preserving homomorphism $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{B}$. Set $\mathfrak{D}_\xi=\pi_\xi[\mathfrak{C}_\xi]$; by 324Kb, $\mathfrak{D}_\xi$ is a closed subalgebra of $\mathfrak{B}$. Because $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{B}$ is continuous (also noted in 324Kb),

$$\mathfrak{D}^*_\xi=\pi^*_\xi[\mathfrak{C}^*_\xi]=\pi_\xi[\mathfrak{C}^*_\xi]$$

is topologically dense in $\mathfrak{D}_\xi$ (3A3Eb), and $\mathfrak{D}_\xi=\overline{\mathfrak{D}^*_\xi}$ is the closed subalgebra of $\mathfrak{B}$ τ-generated by $\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\}$. Finally, if $\eta<\xi$, π_ξ extends π_η because π^*_ξ extends π_η. Thus the induction continues.

(e) The induction ends with $\xi=\kappa$, $\mathfrak{C}_\kappa=\mathfrak{A}$, $\mathfrak{D}_\kappa=\mathfrak{B}$ and $\pi=\pi_\kappa:\mathfrak{A}\to\mathfrak{B}$ the required measure algebra isomorphism.

331J Lemma Let κ be any infinite cardinal. Let μ be the usual measure on $\{0,1\}^\kappa$ (254J) and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then if $(\mathfrak{B},\bar\nu)$ is any non-zero totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous Boolean homomorphism, $\tau(\mathfrak{B})\ge\kappa$.

proof Set $X=\{0,1\}^\kappa$ and write Σ for the domain of μ.

(a) Set $E_\xi=\{x:x\in X,\ x(\xi)=1\}$, $a_\xi=E^\bullet_\xi$ for each $\xi<\kappa$. If $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is any sequence of distinct elements of κ,

$$\mu(\bigcap_{n\in\mathbb{N}}E_{\xi_n})=\lim_{n\to\infty}\mu(\bigcap_{i\le n}E_{\xi_n})=\lim_{n\to\infty}2^{-n-1}=0,$$

so that $\bar\mu(\inf_{n\in\mathbb{N}}a_{\xi_n})=0$ and $\inf_{n\in\mathbb{N}}a_{\xi_n}=0$. Because π is order-continuous, $\inf_{n\in\mathbb{N}}\pi(a_{\xi_n})=0$ in $\mathfrak{B}$. Similarly, $\mu(\bigcup_{n\in\mathbb{N}}E_{\xi_n})=1$ and $\sup_{n\in\mathbb{N}}\pi(a_{\xi_n})=1$.

(b) For $b\in\mathfrak{B}$, $\delta>0$ set $U(b,\delta)=\{b':\bar\nu(b'\triangle b)<\delta\}$, the ordinary open δ-neighbourhood of b. If $b\in\mathfrak{B}$, then there is a $\delta>0$ such that $\{\xi:\xi<\kappa,\ a_\xi\in U(b,\delta)\}$ is finite. **P?** Suppose, if possible, otherwise. Then there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ of distinct elements of κ such that $\bar\nu(b\triangle\pi a_{\xi_n})\le2^{-n-2}\bar\nu1$ for every $n\in\mathbb{N}$. Now $\inf_{n\in\mathbb{N}}\pi a_{\xi_n}=0$, so

$$\bar\nu b=\bar\nu(b\setminus\inf_{n\in\mathbb{N}}\pi a_{\xi_n})\le\sum_{n=0}^\infty\bar\nu(b\setminus\pi a_{\xi_n}).$$

Similarly

$$\bar\nu(1\setminus b)=\bar\nu(\sup_{n\in\mathbb{N}}\pi a_{\xi_n}\setminus b)\le\sum_{n=0}^\infty\bar\nu(\pi a_{\xi_n}\setminus b).$$

So

$$\begin{aligned}\bar\nu1=\bar\nu b+\bar\nu(1\setminus b)&\le\sum_{n=0}^\infty\bar\nu(b\setminus\pi a_{\xi_n})+\sum_{n=0}^\infty\bar\nu(\pi a_{\xi_n}\setminus b)\\&=\sum_{n=0}^\infty\bar\nu(b\triangle\pi a_{\xi_n})\le\sum_{n=0}^\infty2^{-n-2}\bar\nu1<\bar\nu1,\end{aligned}$$

which is impossible. **XQ**

(c) Note that $\mathfrak{B}$ is infinite; for if $b\in\mathfrak{B}$ the set $\{\xi:\pi a_\xi=b\}$ must be finite, and κ is supposed to be infinite. So $\tau(\mathfrak{B})$ must be infinite.

(d) Now take a set $B\subseteq\mathfrak{B}$, of cardinal $\tau(\mathfrak{B})$, which τ-generates $\mathfrak{B}$. By (c), B is infinite. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}$ generated by B; then $\#(\mathfrak{C})=\#(B)=\tau(\mathfrak{B})$, by 331Gc, and $\mathfrak{C}$ is topologically dense in $\mathfrak{B}$.

If $b \in \mathfrak{B}$, there are $c \in \mathfrak{C}$, $k \in \mathbb{N}$ such that $b \in U(c, 2^{-k})$ and $\{\xi : \pi a_\xi \in U(c, 2^{-k})\}$ is finite. **P** By (b), there is a $\delta > 0$ such that $\{\xi : \pi a_\xi \in U(b, \delta)\}$ is finite. Take $k \in \mathbb{N}$ such that $2 \cdot 2^{-k} \le \delta$, and $c \in \mathfrak{C} \cap U(b, 2^{-k})$; then $U(c, 2^{-k}) \subseteq U(b, \delta)$ can contain only finitely many πa_ξ, so these c, k serve. **Q**

Consider

$$\mathcal{U} = \{U(c, 2^{-k}) : c \in \mathfrak{C},\ k \in \mathbb{N},\ \{\xi : \pi a_\xi \in U(c, 2^{-k})\} \text{ is finite}\}.$$

Then $\#(\mathcal{U}) \le \max(\#(\mathfrak{C}), \omega) = \tau(\mathfrak{B})$. Also $\mathcal{U}$ is a cover of $\mathfrak{B}$. In particular, $\kappa = \bigcup_{U \in \mathcal{U}} J_U$, where $J_U = \{\xi : \pi a_\xi \in U\}$. But this means that

$$\kappa = \#(\kappa) \le \max(\omega, \#(\mathcal{U}), \sup_{U \in \mathcal{U}} \#(J_U)) = \tau(\mathfrak{B}),$$

as claimed.

331K Theorem Let κ be any infinite cardinal. Let μ be the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{A}, \bar\mu)$ its measure algebra. Then $\mathfrak{A}$ is Maharam-type-homogeneous, of Maharam type κ.

proof Set $X = \{0,1\}^\kappa$ and write Σ for the domain of μ.

(a) To see that $\tau(\mathfrak{A}) \le \kappa$, set $E_\xi = \{x : x \in X,\ x(\xi) = 1\}$, $a_\xi = E_\xi^\bullet$ for each $\xi < \kappa$. Writing $\mathcal{E}$ for the algebra of subsets of X generated by $\{E_\xi : \xi < \kappa\}$, we see that every measurable cylinder in X, as defined in 254A, belongs to $\mathcal{E}$, so that every member of Σ is approximated, in measure, by members of $\mathcal{E}$ (254Fe), that is, $\{E^\bullet : E \in \mathcal{E}\}$ is topologically dense in $\mathfrak{A}$. But this means just that the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \kappa\}$ is topologically dense in $\mathfrak{A}$, so that $\{a_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{A}$, and $\tau(\mathfrak{A}) \le \kappa$.

(b) Next, if $c \in \mathfrak{A} \setminus \{0\}$ and $\mathfrak{A}_c$ is the principal ideal of $\mathfrak{A}$ generated by c, the map $a \mapsto a \cap c$ is an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{A}_c$, so by 331J we must have $\tau(\mathfrak{A}_c) \ge \kappa$. Thus

$$\kappa \le \tau(\mathfrak{A}_c) \le \tau(\mathfrak{A}) \le \kappa.$$

As c is arbitrary, $\mathfrak{A}$ is Maharam-type-homogeneous with Maharam type κ.

331L Theorem Let $(\mathfrak{A}, \bar\mu)$ be a Maharam-type-homogeneous probability algebra. Then there is exactly one κ, either 0 or an infinite cardinal, such that $(\mathfrak{A}, \bar\mu)$ is isomorphic, as measure algebra, to the measure algebra $(\mathfrak{A}_\kappa, \bar\mu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

proof If $\tau(\mathfrak{A})$ is finite, it is zero, and $\mathfrak{A} = \{0, 1\}$ (331Ha, 331He) so that (interpreting $\{0,1\}^0$ as $\{\emptyset\}$) we have the case $\kappa = 0$. If $\kappa = \tau(\mathfrak{A})$ is infinite, then by 331K we know that $(\mathfrak{A}_\kappa, \bar\mu_\kappa)$ is also Maharam-type-homogeneous of Maharam type κ, so 331I gives the required isomorphism. Of course κ is uniquely defined by $\mathfrak{A}$.

331M Homogeneous Boolean algebras Having introduced the word 'homogeneous', I think I ought not to leave you without mentioning its standard meaning in the context of Boolean algebras, which is connected with one of the most striking and significant consequences of Theorem 331I.

Definition A Boolean algebra $\mathfrak{A}$ is **homogeneous** if $\mathfrak{A}$ is isomorphic, as Boolean algebra, to every non-trivial principal ideal of $\mathfrak{A}$.

Remark Of course a homogeneous Boolean algebra must be Maharam-type-homogeneous, since $\tau(\mathfrak{A}) = \tau(\mathfrak{A}_c)$ whenever $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_c$. In general, a Boolean algebra can be Maharam-type-homogeneous without being homogeneous (331Xj, 331Yj). But for σ-finite measure algebras this doesn't happen.

331N Proposition Let $(\mathfrak{A}, \bar\mu)$ be a Maharam-type-homogeneous σ-finite measure algebra. Then it is homogeneous as a Boolean algebra.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so suppose that $\mathfrak{A} \ne \{0\}$. By 322G, there is a measure $\bar\nu$ on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra. Now let c be any non-zero member of $\mathfrak{A}$, and set $\gamma = \bar\nu c$, $\bar\nu_c' = \gamma^{-1}\bar\nu_c$, where $\bar\nu_c$ is the restriction of $\bar\nu$ to the principal ideal $\mathfrak{A}_c$ of $\mathfrak{A}$ generated by c. Then $(\mathfrak{A}, \bar\nu)$ and $(\mathfrak{A}_c, \bar\nu_c')$ are Maharam-type-homogeneous probability algebras of the same Maharam type, so are isomorphic as measure algebras, and *a fortiori* as Boolean algebras.

331X Basic exercises (a) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ such that for any non-negligible $E\in\Sigma$ there in an $F\in\Sigma$ such that $F\subseteq E$ and $\mu(F\triangle(E\cap H))>0$ for every $H\in\mathrm{T}$. Suppose that $f:X\to[0,1]$ is a measurable function. Show that there is an $F\in\Sigma$ such that $\int_H f=\mu(H\cap F)$ for every $H\in\mathrm{T}$.

>(b) Write out a direct proof of 331C not relying on 331B.

(c) Let $\mathfrak{A}$ be a finite Boolean algebra with n atoms. Show that $\tau(\mathfrak{A})$ is the least k such that $n\le 2^k$.

>(d) Show that the measure algebra of Lebesgue measure on $\mathbb{R}$ is Maharam-type-homogeneous and of Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet:q\in\mathbb{Q}\}$.)

(e) Show that the measure algebra of Lebesgue measure on $\mathbb{R}^r$ is Maharam-type-homogeneous, of Maharam type ω, for any $r\ge 1$. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet:q\in\mathbb{Q}^r\}$.)

(f) Show that the measure algebra of any Radon measure on $\mathbb{R}^r$ (256A) has countable Maharam type. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet:q\in\mathbb{Q}^r\}$.)

>(g) Show that $\mathcal{P}\mathbb{R}$ has Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q]:q\in\mathbb{Q}\}$.)

>(h) Show that the regular open algebra of $\mathbb{R}$ is Maharam-type-homogeneous, of Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet:q\in\mathbb{Q}\}$.)

(i) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal. Suppose that there is a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I}a_\xi=0$, $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\kappa$. Show that $\tau(\mathfrak{A}_a)\ge\kappa$ for every non-zero principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $\mathbb{R}$, and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. Show that the simple product $\mathfrak{A}\times\mathfrak{G}$ is Maharam-type-homogeneous, with Maharam type ω, but is not homogeneous. (*Hint*: $\mathfrak{A}$ is weakly (σ,∞)-distributive, but $\mathfrak{G}$ is not, so they are not isomorphic.)

331Y Further exercises (a) Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{B}$. Show that if $a\in\mathfrak{A}$ is a relative atom in $\mathfrak{A}$ over $\mathfrak{C}$, then $\operatorname{upr}(a,\mathfrak{B})$ is a relative atom in $\mathfrak{B}$ over $\mathfrak{C}$. So if $\mathfrak{B}$ is relatively atomless over $\mathfrak{C}$, then $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$.

(b) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}'$, but τ-generates $\mathfrak{A}$. (*Hint*: take $\mathfrak{A}$ to be the measure algebra $\mathfrak{A}_L$ of Lebesgue measure on $\mathbb{R}$ and $\mathfrak{B}$ the subalgebra $\mathfrak{B}_{\mathbb{Q}}$ of $\mathfrak{A}$ generated by $\{[a,b]^\bullet:a,\,b\in\mathbb{Q}\}$. Take $E\subseteq\mathbb{R}$ such that $I\cap E$, $I\setminus E$ have non-zero measure for every non-trivial interval $I\subseteq\mathbb{R}$ (134Jb), and let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup\{E^\bullet\}$.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}$, but τ-generates $\mathfrak{A}'$. (*Hint*: in the notation of 331Yb, take Z to be the Stone space of $\mathfrak{A}_L$, and set $\mathfrak{A}'=\{\widehat{a}:a\in\mathfrak{A}_L\}$, $\mathfrak{B}=\{\widehat{a}:a\in\mathfrak{B}_{\mathbb{Q}}\}$; let $\mathfrak{A}$ be the subalgebra of $\mathcal{P}Z$ generated by $\mathfrak{A}'\cup\{\{z\}:z\in Z\}$.)

(d) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra, and A the set of its atoms. Show that $\tau(\mathfrak{A})$ is the least cardinal κ such that $\#(A)\le 2^\kappa$.

(e) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write $d(\mathfrak{A})$ for the smallest cardinal of any subset of $\mathfrak{A}$ which is dense for the measure-algebra topology. Show that $d(\mathfrak{A})\le\max(\omega,\tau(\mathfrak{A}))$. Show that if $(\mathfrak{A},\bar\mu)$ is localizable, then $\tau(\mathfrak{A})\le d(\mathfrak{A})$.

(f) Let (X,ρ) be a metric space. Write $d(X)$ for the **density** of X, the smallest cardinal of any dense subset of X. (i) Show that if $\mathcal{G}$ is any family of open subsets of X, there is a family $\mathcal{H}\subseteq\mathcal{G}$ such that $\bigcup\mathcal{H}=\bigcup\mathcal{G}$ and $\#(\mathcal{H})\le\max(\omega,d(X))$. (ii) Show that if $\kappa>\max(\omega,d(X))$ and $\langle x_\xi\rangle_{\xi<\kappa}$ is any family in X, then there is an $x\in X$ such that $\#(\{\xi:x_\xi\in G\})>\max(\omega,d(X))$ for every open set G containing x, and that there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ of distinct members of κ such that $x=\lim_{n\to\infty}x_{\xi_n}$.

(g) Show that the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ is homogeneous.

(h) Show that if $\mathfrak{A}$ is a homogeneous Boolean algebra so is its Dedekind completion $\widehat{\mathfrak{A}}$ (314U).

(i) Show that the regular open algebra of $\mathbb{R}$ is a homogeneous Boolean algebra.

(j) Let $(\mathfrak{A},\bar{\mu})$ be the simple product (322K) of $\mathfrak{c}$ copies of the measure algebra of the usual measure on $\{0,1\}^{\mathfrak{c}}$. Show that $\mathfrak{A}$ is Maharam-type-homogeneous but not homogeneous.

331 Notes and comments Maharam's theorem belongs with the Radon-Nikodým theorem, Fubini's theorem and the strong law of large numbers as one of the theorems which make measure theory what it is. Once you have this theorem and its consequences in the next section properly absorbed, you will never again look at a measure space without classifying its measure algebra in terms of the types of its homogeneous principal ideals. As one might expect, a very large proportion of the important measure spaces of analysis are homogeneous, and indeed a great many are homogeneous with Maharam type ω.

In this section I have contented myself with the basic statement of Theorem 331I on the isomorphism of Maharam-type-homogeneous measure algebras and the identification of representative homogeneous probability algebras (331K). The same techniques lead to an enormous number of further facts, some of which I will describe in the rest of the chapter. For the moment, it gives us a complete description of Maharam-type-homogeneous probability algebras (331L). There is the atomic algebra $\{0,1\}$, with Maharam type 0, and for each infinite cardinal κ there is the measure algebra of $\{0,1\}^{\kappa}$, with Maharam type κ; these are all non-isomorphic, and every Maharam-type-homogeneous probability algebra is isomorphic to exactly one of them. The isomorphisms here are not unique; indeed, it is characteristic of measure algebras that they have very large automorphism groups (see Chapter 38 below), and there are correspondingly large numbers of isomorphisms between any isomorphic pair. The proof of 331I already suggests this, since we have such a vast amount of choice concerning the lists $\langle a_\xi\rangle_{\xi<\kappa}$, $\langle b_\xi\rangle_{\xi<\kappa}$, and even with these fixed there remains a good deal of scope in the choice of $\langle a'_\xi\rangle_{\xi<\kappa}$ and $\langle b'_\xi\rangle_{\xi<\kappa}$.

The isomorphisms described in Theorem 331I are measure algebra isomorphisms, that is, measure-preserving Boolean isomorphisms. Obvious questions arise concerning Boolean isomorphisms which are not necessarily measure-preserving; the theorem also helps us to settle many of these (see 331M-331N). But we can observe straight away the remarkable fact that two homogeneous probability algebras which are isomorphic as Boolean algebras are also isomorphic as probability algebras, since they must have the same Maharam type.

I have already mentioned certain measure space isomorphisms (254K, 255A). Of course any isomorphism between measure spaces must induce an isomorphism between their measure algebras (see 324M), and any isomorphism between measure algebras corresponds to an isomorphism between their Stone spaces (see 324N). But there are many important examples of isomorphisms between measure algebras which do not correspond to isomorphisms between the measure spaces most naturally involved. (I describe one in 343J.) Maharam's theorem really is a theorem about measure *algebras* rather than measure *spaces*.

The particular method I use to show that the measure algebra of the usual measure on $\{0,1\}^{\kappa}$ is homogeneous for infinite κ (331J-331K) is chosen with a view to a question in the next section (332O). There are other ways of doing it. But I recommend study of this particular one because of the way in which it involves the topological, algebraic and order properties of the algebra $\mathfrak{B}$. I have extracted some of the elements of the argument in 331Xi and 331Ye-331Yf. These use the concept of 'density' of a topological space. This does not seem the moment to go farther along this road, but I hope you can see that there are likely to be many further 'cardinal functions' to provide useful measures of complexity in both algebraic and topological structures.

332 Classification of localizable measure algebras

In this section I present what I call 'Maharam's theorem', that every localizable measure algebra is expressible as a weighted simple product of measure algebras of spaces of the form $\{0,1\}^\kappa$ (332B). Among its many consequences is a complete description of the isomorphism classes of localizable measure algebras (332J). This description needs the concepts of 'cellularity' of a Boolean algebra (332D) and its refinement, the 'magnitude' of a measure algebra (332G). I end this section with a discussion of those pairs of measure algebras for which there is a measure-preserving homomorphism from one to the other (332P-332Q), and a general formula for the Maharam type of a localizable measure algebra (332S).

332A Lemma Let $\mathfrak{A}$ be any Boolean algebra. Writing $\mathfrak{A}_a$ for the principal ideal generated by $a \in \mathfrak{A}$, the set $\{a : a \in \mathfrak{A},\ \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$ is order-dense in $\mathfrak{A}$.

proof Take any $a \in \mathfrak{A} \setminus \{0\}$. Then $A = \{\tau(\mathfrak{A}_b) : 0 \neq b \subseteq a\}$ has a least member; take $c \subseteq a$ such that $c \neq 0$ and $\tau(\mathfrak{A}_c) = \min A$. If $0 \neq b \subseteq c$, then $\tau(\mathfrak{A}_b) \leq \tau(\mathfrak{A}_c)$, by 331Hc, while $\tau(\mathfrak{A}_b) \in A$, so $\tau(\mathfrak{A}_c) \leq \tau(\mathfrak{A}_b)$. Thus $\tau(\mathfrak{A}_b) = \tau(\mathfrak{A}_c)$ for every non-zero $b \subseteq c$, and $\mathfrak{A}_c$ is Maharam-type-homogeneous.

332B Maharam's Theorem Let $(\mathfrak{A}, \bar\mu)$ be any localizable measure algebra. Then it is isomorphic to the simple product of a family $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i \in I}$ of measure algebras, where for each $i \in I$ $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a re-normalization of the measure, to the measure algebra of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal.

proof (a) For $a \in \mathfrak{A}$, let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then

$$D = \{a : a \in \mathfrak{A},\ 0 < \bar\mu a < \infty,\ \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$$

is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite) there is a $b \subseteq a$ such that $0 < \bar\mu b < \infty$; now by 332A there is a non-zero $d \subseteq b$ such that $\mathfrak{A}_d$ is Maharam-type-homogeneous. By 331N, $\mathfrak{A}_d$ is Maharam-type-homogeneous, and $d \in D$. **Q**

(b) By 313K, there is a partition of unity $\langle e_i\rangle_{i \in I}$ consisting of members of D; by 322Kd, $(\mathfrak{A}, \bar\mu)$ is isomorphic, as measure algebra, to the simple product of the principal ideals $\mathfrak{A}_i = \mathfrak{A}_{e_i}$.

(c) For each $i \in I$, $(\mathfrak{A}_i, \bar\mu_i)$ is a non-trivial totally finite Maharam-type-homogeneous measure algebra, writing $\bar\mu_i = \bar\mu \restriction \mathfrak{A}_i$. Take $\gamma_i = \bar\mu_i(1_{\mathfrak{A}_i}) = \bar\mu e_i$, and set $\bar\mu_i' = \gamma_i^{-1}\bar\mu_i$. Then $(\mathfrak{A}_i, \bar\mu_i')$ is a Maharam-type-homogeneous probability algebra, so by 331L is isomorphic to the measure algebra $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$ of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal. Thus $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$.

Remark For the case of totally finite measure algebras, this is Theorem 2 of Maharam 42.

332C Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. For any cardinal κ, write ν_κ for the usual measure on $\{0,1\}^\kappa$, and Σ_κ for its domain. Then we can find families $\langle\kappa_i\rangle_{i \in I}$, $\langle\gamma_i\rangle_{i \in I}$ such that every κ_i is either 0 or an infinite cardinal, every γ_i is a strictly positive real number, and $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra of (X, Σ, ν), where

$$X = \{(x, i) : i \in I,\ x \in \{0,1\}^{\kappa_i}\},$$

$$\Sigma = \{E : E \subseteq X,\ \{x : (x, i) \in E\} \in \Sigma_{\kappa_i} \text{ for every } i \in I\},$$

$$\nu E = \sum_{i \in I} \gamma_i \nu_{\kappa_i}\{x : (x, i) \in E\}$$

for every $E \in \Sigma$.

proof Take the family $\langle\kappa_i\rangle_{i \in I}$ from the last theorem, take the $\gamma_i = \bar\mu e_i$ to be the normalizing factors of the proof there, and apply 322Kb to identify the simple product of the measure algebras of $(\{0,1\}^{\kappa_i}, \Sigma_{\kappa_i}, \gamma_i\nu_{\kappa_i})$ with the measure algebra of their direct sum (X, Σ, ν).

332D The cellularity of a Boolean algebra In order to properly describe non-sigma-finite measure algebras, we need the following concept. If $\mathfrak{A}$ is any Boolean algebra, write

$$c(\mathfrak{A}) = \sup\{\#(C) : C \subseteq \mathfrak{A} \setminus \{0\} \text{ is disjoint}\},$$

the **cellularity** of $\mathfrak{A}$. (If $\mathfrak{A} = \{0\}$, take $c(\mathfrak{A}) = 0$.) Thus $\mathfrak{A}$ is ccc (316A) iff $c(\mathfrak{A}) \le \omega$.

332E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra, and C any partition of unity in $\mathfrak{A}$ consisting of elements of finite measure. Then $\max(\omega, \#(C)) = \max(\omega, c(\mathfrak{A}))$.

proof Of course $\#(C \setminus \{0\}) \le c(\mathfrak{A})$, because $C \setminus \{0\}$ is disjoint, so

$$\max(\omega, \#(C)) = \max(\omega, \#(C \setminus \{0\}) \le \max(\omega, c(\mathfrak{A})).$$

Now suppose that D is any disjoint set in $\mathfrak{A} \setminus \{0\}$. For $c \in C$, $\{d \cap c : d \in D\}$ is a disjoint set in the principal ideal $\mathfrak{A}_c$ generated by c. But $\mathfrak{A}_c$ is ccc (322G), so $\{d \cap c : d \in D\}$ must be countable, and $D_c = \{d : d \in D,\, d \cap c \ne 0\}$ is countable. Because $\sup C = 1$, $D = \bigcup_{c \in C} D_c$, so

$$\#(D) \le \max(\omega, \#(C), \sup_{c \in C} \#(D_c)) = \max(\omega, \#(C)).$$

As D is arbitrary, $c(\mathfrak{A}) \le \max(\omega, \#(C))$ and $\max(\omega, c(\mathfrak{A})) = \max(\omega, \#(C))$.

332F Corollary Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra. Then there is a disjoint set in $\mathfrak{A} \setminus \{0\}$ of cardinal $c(\mathfrak{A})$.

proof Start by taking any partition of unity C consisting of non-zero elements of finite measure. If $\#(C) = c(\mathfrak{A})$ we can stop, because C is a disjoint set in $\mathfrak{A} \setminus \{0\}$. Otherwise, by 332E, we must have C finite and $c(\mathfrak{A}) = \omega$. Let A be the set of atoms in $\mathfrak{A}$. If A is infinite, it is a disjoint set of cardinal ω, so we can stop. Otherwise, since there is certainly a disjoint set $D \subseteq \mathfrak{A} \setminus \{0\}$ of cardinal greater than $\#(A)$, and since each member of A can meet at most one member of D, there must be a member d of D which does not include any atom. Accordingly we can choose inductively a sequence $\langle d_n \rangle_{n \in \mathbb{N}}$ such that $d_0 = d$, $0 \ne d_{n+1} \subset d_n$ for every n. Now $\{d_n \setminus d_{n+1} : n \in \mathbb{N}\}$ is a disjoint set in $\mathfrak{A} \setminus \{0\}$ of cardinal $\omega = c(\mathfrak{A})$.

332G Definitions For the next theorem, it will be convenient to have some special terminology.

(a) The first word I wish to introduce is a variant of the idea of 'cellularity', adapted to measure algebras. If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, let us say that the **magnitude** of an $a \in \mathfrak{A}$ is $\bar{\mu}a$ if $\bar{\mu}a$ is finite, and otherwise is the cellularity of the principal ideal $\mathfrak{A}_a$ generated by a. (This is necessarily infinite, since any partition of a into sets of finite measure must be infinite.) If we take it that any real number is less than any infinite cardinal, then the class of possible magnitudes is totally ordered.

I shall sometimes speak of the **magnitude** of the measure algebra $(\mathfrak{A}, \bar{\mu})$ itself, meaning the magnitude of $1_{\mathfrak{A}}$. Similarly, if (X, Σ, μ) is a semi-finite measure space, the **magnitude** of (X, Σ, μ), or of μ, is the magnitude of its measure algebra.

(b) Next, for any Dedekind complete Boolean algebra $\mathfrak{A}$, and any cardinal κ, we can look at the element

$$e_\kappa = \sup\{a : a \in \mathfrak{A} \setminus \{0\},\ \mathfrak{A}_a \text{ is Maharam-type-homogeneous of Maharam type } \kappa\},$$

writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, as usual. I will call this the **Maharam-type-κ component** of $\mathfrak{A}$. Of course $e_\kappa \cap e_\lambda = 0$ whenever λ, κ are distinct cardinals. **P** $a \cap b = 0$ whenever $\mathfrak{A}_a$, $\mathfrak{A}_b$ are Maharam-type-homogeneous of different Maharam types, since $\tau(\mathfrak{A}_{a \cap b})$ cannot be equal simultaneously to $\tau(\mathfrak{A}_a)$ and $\tau(\mathfrak{A}_b)$. **Q**

Also $\{e_\kappa : \kappa \text{ is a cardinal}\}$ is a partition of unity in $\mathfrak{A}$, because

$$\sup\{e_\kappa : \kappa \text{ is a cardinal}\} = \sup\{a : \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\} = 1$$

by 332A. Note that there is no claim that $\mathfrak{A}_{e_\kappa}$ itself is homogeneous; but we do have a useful result in this direction.

332H Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and κ an infinite cardinal. Let e be the Maharam-type-κ component of $\mathfrak{A}$. If $0 \ne d \subseteq e$ and the principal ideal $\mathfrak{A}_d$ generated by d is ccc, then it is Maharam-type-homogeneous with Maharam type κ.

proof (a) The point is that $\tau(\mathfrak{A}_d)\le\kappa$. **P** Set

$$A=\{a:a\in\mathfrak{A}\setminus\{0\},\ \mathfrak{A}_a\text{ is Maharam-type-homogeneous of Maharam type }\kappa\}.$$

Then $d=\sup\{a\cap d:a\in A\}$. Because $\mathfrak{A}_d$ is ccc, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $d=\sup_{n\in\mathbb{N}}d\cap a_n$ (316E); set $b_n=d\cap a_n$. We have $\tau(\mathfrak{A}_{b_n})\le\tau(\mathfrak{A}_{a_n})=\kappa$ for each n; let D_n be a subset of $\mathfrak{A}_{b_n}$, of cardinal at most κ, which τ-generates $\mathfrak{A}_{b_n}$. Set

$$D=\bigcup_{n\in\mathbb{N}}D_n\cup\{b_n:n\in\mathbb{N}\}\subseteq\mathfrak{A}_d.$$

If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}_d$ generated by D, then $\mathfrak{C}\cap\mathfrak{A}_{b_n}$ is an order-closed subalgebra of $\mathfrak{A}_{b_n}$ including D_n, so is equal to $\mathfrak{A}_{b_n}$, for every n. But $a=\sup_{n\in\mathbb{N}}a\cap b_n$ for every $a\in\mathfrak{A}_d$, so $\mathfrak{C}=\mathfrak{A}_d$. Thus D τ-generates $\mathfrak{A}_d$, and

$$\tau(\mathfrak{A}_d)\le\#(D)\le\max(\omega,\sup_{n\in\mathbb{N}}\#(D_n))=\kappa.\ \mathbf{Q}$$

(b) If now b is any non-zero member of $\mathfrak{A}_d$, there is some $a\in A$ such that $b\cap a\ne0$, so that

$$\kappa=\tau(\mathfrak{A}_{b\cap a})\le\tau(\mathfrak{A}_b)\le\tau(\mathfrak{A}_d)\le\kappa.$$

Thus we must have $\tau(\mathfrak{A}_b)=\kappa$ for every non-zero $b\in\mathfrak{A}_d$, and $\mathfrak{A}_d$ is Maharam-type-homogeneous of type κ, as claimed.

332I Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra which is not totally finite. Then it has a partition of unity consisting of elements of measure 1.

proof Let A be the set $\{a:\bar\mu a=1\}$, and $\mathcal{C}$ the family of disjoint subsets of A. By Zorn's lemma, $\mathcal{C}$ has a maximal member C_0 (compare the proof of 313K). Set $D=\{d:d\in\mathfrak{A},\ d\cap c=0\text{ for every }c\in C_0\}$. Then D is upwards-directed. If $d\in D$, then $\bar\mu a\ne1$ for every $a\subseteq d$, so $\bar\mu d<1$, by 331C. So $d_0=\sup D$ is defined in $\mathfrak{A}$ (321C); of course $d_0\in D$, so $\bar\mu d_0<1$. Observe that $\sup C_0=1\setminus d_0$.

Because $\bar\mu1=\infty$, C_0 must be infinite; let $\langle a_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct elements of C_0. For each $n\in\mathbb{N}$, use 331C again to choose an $a'_n\subseteq a_n$ such that $\bar\mu a'_n=\bar\mu d_0$. Set

$$b_0=d_0\cup(a_0\setminus a'_0),\qquad b_n=a'_{n-1}\cup(a_n\setminus a'_n)$$

for every $n\ge1$. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of elements of measure 1 and $\sup_{n\in\mathbb{N}}b_n=\sup_{n\in\mathbb{N}}a_n\cup d_0$. Now

$$(C_0\setminus\{a_n:n\in\mathbb{N}\})\cup\{b_n:n\in\mathbb{N}\}$$

is a partition of unity consisting of elements of measure 1.

332J Now I can formulate a complete classification theorem for localizable measure algebras, refining the expression in 332B.

Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. For each cardinal κ, let e_κ, f_κ be the Maharam-type-κ components of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Then $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, as measure algebras, iff (i) e_κ and f_κ have the same magnitude for every infinite cardinal κ (ii) for every $\gamma\in\,]0,\infty[$, $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ have the same number of atoms of measure γ.

proof Throughout the proof, write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, and $\bar\mu_a$ for the restriction of $\bar\mu$ to $\mathfrak{A}_a$; and define $\mathfrak{B}_b$, $\bar\nu_b$ similarly for $b\in\mathfrak{B}$.

(a) If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, then of course the isomorphism matches their Maharam-type components together and retains their magnitudes, and matches atoms of the same measure together; so the conditions are surely satisfied.

(b) Now suppose that the conditions are satisfied. Set

$$K=\{\kappa:\kappa\text{ is an infinite cardinal},\ e_\kappa\ne0\}=\{\kappa:\kappa\text{ is an infinite cardinal},\ f_\kappa\ne0\}.$$

For $\gamma\in\,]0,\infty[$, let A_γ be the set of atoms of measure γ in $\mathfrak{A}$, and set $e_\gamma=\sup A_\gamma$. Write $I=K\cup\,]0,\infty[$. Then $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, so $(\mathfrak{A},\bar\mu)$ is isomorphic to the simple product of $\langle(\mathfrak{A}_{e_i},\bar\mu_{e_i})\rangle_{i\in I}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal generated by e_i and $\bar\mu_{e_i}$ for the restriction $\bar\mu{\restriction}\mathfrak{A}_{e_i}$.

In the same way, writing B_γ for the set of atoms of measure γ in $\mathfrak{B}$, f_γ for $\sup B_\gamma$, $\mathfrak{B}_{f_i}$ for the principal ideal generated by f_i and $\bar\nu_{f_i}$ for the restriction of $\bar\nu$ fo $\mathfrak{B}_{f_i}$, we have $(\mathfrak{B},\bar\nu)$ isomorphic to the simple product of $\langle(\mathfrak{B}_{f_i},\bar\nu_{f_i})\rangle_{i\in I}$.

(c) It will therefore be enough if I can show that $(\mathfrak{A}_{e_i},\bar\mu_{e_i})\cong(\mathfrak{B}_{f_i},\bar\nu_{f_i})$ for every $i\in I$.

(i) For $\kappa\in K$, the hypothesis is that e_κ and f_κ have the same magnitude. If they are both of finite magnitude, that is, $\bar\mu e_\kappa=\bar\nu f_\kappa<\infty$, then both $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ and $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ are homogeneous and of Maharam type κ, by 332H. So 331I tells us that they are isomorphic. If they are both of infinite magnitude λ, then 332I tells us that both $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ have partitions of unity C, D consisting of sets of measure 1. So $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_c,\bar\mu_c)\rangle_{c\in C}$, while $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_d,\bar\nu_d)\rangle_{d\in D}$. But we know also that every $(\mathfrak{A}_c,\bar\mu_c)$, $(\mathfrak{B}_d,\bar\nu_d)$ is a homogeneous probability algebra of Maharam type κ, by 332H again, so by Maharam's theorem again they are all isomorphic. Since C, D and λ are all infinite,

$$\#(C)=c(\mathfrak{A}_{e_\kappa})=\lambda=c(\mathfrak{B}_{f_\kappa})=\#(D)$$

by 332E. So we are taking the same number of factors in each product and $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ must be isomorphic to $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$.

(ii) For $\gamma\in\,]0,\infty[$, our hypothesis is that $\#(A_\gamma)=\#(B_\gamma)$. Now A_γ is a partition of unity in $\mathfrak{A}_{e_\gamma}$, so $(\mathfrak{A}_{e_\gamma},\bar\mu_{e_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_a,\bar\mu_a)\rangle_{a\in A_\gamma}$. Similarly, $(\mathfrak{B}_{f_\gamma},\bar\nu_{f_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_b,\bar\nu_b)\rangle_{b\in B_\gamma}$. Since every $(\mathfrak{A}_a,\bar\mu_a)$, $(\mathfrak{B}_b,\bar\nu_b)$ is just a simple atom of measure γ, these are all isomorphic; since we are taking the same number of factors in each product, $(\mathfrak{A}_{e_\gamma},\bar\mu_{e_\gamma})$ must be isomorphic to $(\mathfrak{B}_{f_\gamma},\bar\nu_{f_\gamma})$.

(iii) Thus we have the full set of required isomorphisms, and $(\mathfrak{A},\bar\mu)$ is isomorphic to $(\mathfrak{B},\bar\nu)$.

332K Remarks (a) The partition of unity $\{e_i:i\in I\}$ of $\mathfrak{A}$ used in the above theorem is in some sense canonical. (You might feel it more economical to replace I by $K\cup\{\gamma:A_\gamma\ne\emptyset\}$.) The further partition of the atomic part into individual atoms (part (c-ii) of the proof) is also canonical. But of course the partition of the e_κ of infinite magnitude into elements of measure 1 requires a degree of arbitrary choice.

The value of the expressions in 332C is that the parameters κ_i, γ_i there are sufficient to identify the measure algebra up to isomorphism. For, amalgamating the language of 332C and 332J, we see that the magnitude of e_κ in 332J is just $\sum_{\kappa_i=\kappa}\gamma_i$ if this is finite, $\#(\{i:\kappa_i=\kappa\})$ otherwise (using 332E, as usual); while the number of atoms of measure γ is $\#(\{i:\kappa_i=0,\ \gamma_i=\gamma\})$.

(b) The classification which Maharam's theorem gives us is not merely a listing. It involves a real insight into the nature of the algebras, enabling us to answer a very wide variety of natural questions. I give the next couple of results as a sample of what we can expect these methods to do for us.

332L Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and a, $b\in\mathfrak{A}$ two elements of finite measure. Suppose that $\pi:\mathfrak{A}_a\to\mathfrak{A}_b$ is a measure-preserving isomorphism, where $\mathfrak{A}_a$, $\mathfrak{A}_b$ are the principal ideals generated by a and b. Then there is a measure-preserving automorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ which extends π.

proof The point is that $\mathfrak{A}_{b\setminus a}$ is isomorphic, as measure algebra, to $\mathfrak{A}_{a\setminus b}$. **P** Set $c=a\cup b$. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}_c$. Then $e_\kappa\cap a$ is the Maharam-type-κ component of $\mathfrak{A}_a$, because if $d\subseteq c$ and $\mathfrak{A}_d$ is Maharam homogeneous with Maharam type κ, then $\mathfrak{A}_{d\cap a}$ is either $\{0\}$ or again Maharam-type-homogeneous with Maharam type κ. Similarly, $e_\kappa\setminus a$ is the Maharam-type-κ component of $\mathfrak{A}_{c\setminus a}=\mathfrak{A}_{b\setminus a}$, $e_\kappa\cap b$ is the Maharam-type-κ component of $\mathfrak{A}_b$ and $e_\kappa\setminus b$ is the Maharam-type-κ component of $\mathfrak{A}_{a\setminus b}$. Now $\pi:\mathfrak{A}_a\to\mathfrak{A}_b$ is an isomorphism, so $\pi(e_\kappa\cap a)$ must be $e_\kappa\cap b$, and

$$\begin{aligned}\bar\mu(e_\kappa\setminus a)&=\bar\mu e_\kappa-\bar\mu(e_\kappa\cap a)=\bar\mu e_\kappa-\bar\mu\pi(e_\kappa\cap a)\\&=\bar\mu e_\kappa-\bar\mu(e_\kappa\cap b)=\bar\mu(e_\kappa\setminus b).\end{aligned}$$

In the same way, if we write $n_\gamma(d)$ for the number of atoms of measure γ in $\mathfrak{A}_d$, then

$$n_\gamma(b\setminus a)=n_\gamma(c)-n_\gamma(a)=n_\gamma(c)-n_\gamma(b)=n_\gamma(a\setminus b)$$

for every $\gamma \in]0,\infty[$. By 332J, there is a measure-preserving isomorphism $\pi_1 : \mathfrak{A}_{b\setminus a} \to \mathfrak{A}_{a\setminus b}$. **Q**

If we now set

$$\phi d = \pi(d \cap a) \cup \pi_1(d \cap b \setminus a) \cup (d \setminus c)$$

for every $d \in \mathfrak{A}$, $\phi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving isomorphism which agrees with π on $\mathfrak{A}_a$.

332M Lemma Suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are homogeneous measure algebras, with $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$ and $\bar\mu 1 = \bar\nu 1 < \infty$. Then there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof The case $\tau(\mathfrak{A}) = 0$ is trivial. Otherwise, considering normalized versions of the measures, we are reduced to the case $\bar\mu 1 = \bar\nu 1 = 1$, $\tau(\mathfrak{A}) = \kappa \ge \omega$, $\tau(\mathfrak{B}) = \lambda \ge \kappa$, so that $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{A}_\kappa,\bar\mu_\kappa)$ of the usual measure $\bar\mu_\kappa$ on $\{0,1\}^\kappa$; and similarly $(\mathfrak{B},\bar\nu)$ is isomorphic to the measure algebra of the usual measure on $\{0,1\}^\lambda$. Now (identifying the cardinals κ, λ with von Neumann ordinals, as usual), $\kappa \subseteq \lambda$, so we have an inverse-measure-preserving map $x \mapsto x\restriction\kappa : \{0,1\}^\lambda \to \{0,1\}^\kappa$ (254Oa), which induces a measure-preserving Boolean homomorphism from $\mathfrak{A}_\kappa$ to $\mathfrak{A}_\lambda$ (324M), and hence a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

332N Lemma If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\kappa \ge \max(\omega,\tau(\mathfrak{A}))$, then there is a measure-preserving homomorphism from $(\mathfrak{A},\bar\mu)$ to the measure algebra $(\mathfrak{B},\bar\nu)$ of the usual measure ν on $\{0,1\}^\kappa$; that is, $(\mathfrak{A},\bar\mu)$ is isomorphic to a closed subalgebra of $(\mathfrak{B},\bar\nu)$.

proof Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that every principal ideal $\mathfrak{A}_{c_i}$ is homogeneous and no c_i is zero. Then I is countable and $\sum_{i\in I}\bar\mu c_i = 1$. Accordingly there is a partition of unity $\langle d_i\rangle_{i\in I}$ in $\mathfrak{B}_\kappa$ such that $\bar\nu d_i = \bar\mu c_i$ for every i. **P** Because I is countable, we may suppose that it is either $\mathbb{N}$ or an initial segment of $\mathbb{N}$. In this case, choose $\langle d_i\rangle_{i\in I}$ inductively such that $d_i \subseteq 1 \setminus \sup_{j<i} d_j$ and $\bar\nu d_i = \bar\mu d_i$ for each $i \in I$, using 331C. **Q**

If $i \in I$, then $\tau(\mathfrak{A}_{c_i}) \le \kappa = \tau(\mathfrak{B}_{d_i})$, so there is a measure-preserving Boolean homomorphism $\pi_i : \mathfrak{A}_{c_i} \to \mathfrak{B}_{d_i}$. Setting $\pi a = \sup_{i\in I}\pi_i(a \cap c_i)$ for $a \in \mathfrak{A}$, we have a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. By 324Kb, $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$, and of course $(\pi[\mathfrak{A}],\bar\nu\restriction\pi[\mathfrak{A}])$ is isomorphic to $(\mathfrak{A},\bar\mu)$.

332O Lemma Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and for $\gamma \in]0,\infty[$ let e_γ, f_γ be the suprema of the atoms of measure γ in $\mathfrak{A}$, $\mathfrak{B}$ respectively. If there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the magnitude of $\sup_{\kappa\ge\lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\ge\lambda} f_\kappa$ whenever λ is an infinite cardinal, while the magnitude of $\sup_{\kappa\ge\omega} e_\kappa \cup \sup_{\gamma\le\delta} e_\gamma$ is not greater than the magnitude of $\sup_{\kappa\ge\omega} f_\kappa \cup \sup_{\gamma\le\delta} f_\gamma$ for any $\delta \in]0,\infty[$.

proof Suppose that $\pi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism. For infinite cardinals λ, set $e^*_\lambda = \sup_{\kappa\ge\lambda} e_\kappa$, $f^*_\lambda = \sup_{\kappa\ge\lambda} f_\kappa$, while for $\delta \in]0,\infty[$ set $e^*_\delta = \sup_{\kappa\ge\omega} e_\kappa \cup \sup_{\gamma\le\delta} e_\gamma$, $f^*_\delta = \sup_{\kappa\ge\omega} f_\kappa \cup \sup_{\gamma\le\delta} f_\gamma$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_{c_i}$ are totally finite and homogeneous, as in 332B. Then $c_i \subseteq e_\kappa$ whenever $\kappa = \tau(\mathfrak{A}_{c_i})$ is infinite, and $c_i \subseteq e_\gamma$ if c_i is an atom of measure γ. Take v to be either an infinite cardinal or a strictly positive real number. Set

$$J = \{i : i \in I,\ c_i \subseteq e^*_v\};$$

then $e^*_v = \sup_{i\in J} c_i$.

Now the point is that if $i \in J$ then $\pi c_i \subseteq f^*_v$. **P** We need to consider two cases. (i) If c_i is an atom, then $v \in]0,\infty[$ and $\bar\mu c_i \le v$. So we need only observe that $1 \setminus f^*_v$ is just the supremum in $\mathfrak{B}$ of the atoms of measure greater than v, none of which can meet πc_i, since this has measure at most v. (ii) Now suppose that $\mathfrak{A}_{c_i}$ is atomless, with $\tau(\mathfrak{A}_{c_i}) = \kappa \ge v$. If $0 \ne b \subseteq \pi c_i$, then $a \mapsto b \cap \pi a : \mathfrak{A}_{c_i} \to \mathfrak{B}_b$ is an order-continuous Boolean homomorphism, while $\mathfrak{A}_{c_i}$ is isomorphic (as Boolean algebra) to the measure algebra of $\{0,1\}^\kappa$, so 331J tells us that $\tau(\mathfrak{B}_b) \ge \kappa$. This means, first, that b cannot be an atom, so that πc_i cannot meet $\sup_{\gamma\in]0,\infty[} f_\gamma$; and also that b cannot be included in $f_{\kappa'}$ for any infinite $\kappa' < \kappa$, so that πc_i cannot meet $\sup_{\omega\le\kappa'<\kappa} f_\kappa$. Thus πc_i must be included in $\sup_{\kappa'\ge\kappa} f_\kappa = f^*_v$. **Q**

Of course $\langle\pi c_i\rangle_{i\in J}$ is disjoint. So if e^*_v has finite magnitude, the magnitude of f^*_v is at least

$$\sum_{i\in J}\bar\nu\pi c_i=\sum_{i\in J}\bar\mu c_i=\bar\mu e^*_v,$$

the magnitude of e^*_v. While if e^*_v has infinite magnitude, this is $\#(J)$, by 332E, which is not greater than the magnitude of f^*_v.

332P Proposition Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be atomless totally finite measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B},\bar\nu)$;

(ii) for every cardinal λ,

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa).$$

proof (a)(i)⇒(ii) Suppose that $\pi:\mathfrak{A}\to\mathfrak{B}_d$ is a measure-preserving isomorphism between $\mathfrak{A}$ and a closed subalgebra of a principal ideal $\mathfrak{B}_d$ of $\mathfrak{B}$. The Maharam-type-κ component of $\mathfrak{B}_d$ is just $d\cap f_\kappa$, so 332O tells us that

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}d\cap f_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa)$$

for every λ.

(b)(ii)⇒(i) Now suppose that the condition is satisfied.

(α) Let P be the set of all measure-preserving Boolean homomorphisms π from principal ideals $\mathfrak{A}_{c_\pi}$ of $\mathfrak{A}$ to principal ideals $\mathfrak{B}_{d_\pi}$ of $\mathfrak{B}$ such that

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c_\pi)\le\bar\nu(\sup_{\kappa\ge\lambda}\bar\nu f_\kappa\setminus d_\pi)$$

for every cardinal $\lambda\ge\omega$. Then the trivial homomorphism from $\mathfrak{A}_0$ to $\mathfrak{B}_0$ belongs to P, so P is not empty. Order P by saying that $\pi\le\pi'$ if π' extends π, that is, if $c_\pi\subseteq c_{\pi'}$ and $\pi'a=\pi a$ for every $a\in\mathfrak{A}_{c_\pi}$. Then P is a partially ordered set.

(β) If $Q\subseteq P$ is non-empty and totally ordered, it is bounded above in P. **P** Set $c^*=\sup_{\pi\in Q}c_\pi$, $d^*=\sup_{\pi\in Q}d_\pi$. For $a\subseteq c^*$ set $\pi^*a=\sup_{\pi\in Q}\pi(a\cap c_\pi)$. Because Q is totally ordered, π^* extends all the functions in Q. It is also easy to check that $\pi^*0=0$, $\pi^*(a\cap a')=\pi^*a\cap\pi^*a'$ and $\pi^*(a\cup a')=\pi^*a\cup\pi^*a'$ for all a, $a'\in\mathfrak{A}_{c^*}$, $\pi^*c^*=d^*$ and that $\bar\nu\pi^*a=\bar\mu a$ for every $a\in\mathfrak{A}_{c^*}$; so that π^* is a measure-preserving Boolean homomorphism from $\mathfrak{A}_{c^*}$ to $\mathfrak{B}_{d^*}$.

Now suppose that λ is any cardinal; then

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c^*)=\inf_{\pi\in Q}\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c_\pi)\le\inf_{\pi\in Q}\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa\setminus d_\pi)=\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa\setminus d^*).$$

So $\pi^*\in P$ and is the required upper bound of Q. **Q**

(γ) By Zorn's Lemma, P has a maximal element $\tilde\pi$ say. Now $c_{\tilde\pi}=1$. **P?** If not, then let κ_0 be the least cardinal such that $e_{\kappa_0}\setminus c_{\tilde\pi}\ne0$. Then

$$0<\bar\mu(\sup_{\kappa\ge\kappa_0}e_\kappa\setminus c_{\tilde\pi})\le\bar\nu(\sup_{\kappa\ge\kappa_0}\bar\nu f_\kappa\setminus d_{\tilde\pi}),$$

so there is a least $\kappa_1\ge\kappa_0$ such that $f_{\kappa_1}\setminus d_{\tilde\pi}\ne0$. Set $\delta=\min(\bar\mu(e_{\kappa_0}\setminus c_{\tilde\pi}),\bar\nu(f_{\kappa_1}\setminus d_{\tilde\pi}))>0$. Because $\mathfrak{A}$ and $\mathfrak{B}$ are atomless, there are $a\subseteq e_{\kappa_0}\setminus c_{\tilde\pi}$, $b\subseteq f_{\kappa_1}\setminus d_{\tilde\pi}$ such that $\bar\mu a=\bar\nu b=\delta$ (331C). Now $\mathfrak{A}_a$ is homogeneous with Maharam type κ_0, while $\mathfrak{B}_b$ is homogeneous with Maharam type κ_1 (332H), so there is a measure-preserving Boolean homomorphism $\phi:\mathfrak{A}_a\to\mathfrak{B}_b$ (332M). Set

$$c^*=c_{\tilde\pi}\cup a,\quad d^*=d_{\tilde\pi}\cup b,$$

and define $\pi^*:\mathfrak{A}_{c^*}\to\mathfrak{B}_{d^*}$ by setting $\pi^*(g)=\tilde\pi(g\cap c_{\tilde\pi})\cup\phi(g\cap a)$ for every $g\subseteq c^*$. It is easy to check that π^* is a measure-preserving Boolean homomorphism.

If λ is a cardinal and $\lambda\le\kappa_0$,

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c^*)=\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c_{\tilde\pi})-\delta\le\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa\setminus d_{\tilde\pi})-\delta=\bar\nu(\sup_{\kappa\ge\lambda}\bar\nu f_\kappa\setminus d^*).$$

If $\kappa_0<\lambda\le\kappa_1$,

$$\begin{aligned}\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c^*) &= \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_{\tilde\pi}) \le \bar\mu(\sup_{\kappa\ge\kappa_0} e_\kappa\setminus c_{\tilde\pi}) - \bar\mu(e_{\kappa_0}\setminus c_\pi)\\ &\le \bar\mu(\sup_{\kappa\ge\kappa_0} e_\kappa\setminus c_{\tilde\pi}) - \delta \le \bar\nu(\sup_{\kappa\ge\kappa_0} f_\kappa\setminus d_{\tilde\pi}) - \delta\\ &= \bar\nu(\sup_{\kappa\ge\kappa_1} f_\kappa\setminus d_{\tilde\pi}) - \delta\end{aligned}$$

(by the choice of κ_1)

$$= \bar\nu(\sup_{\kappa\ge\kappa_1} f_\kappa\setminus d^*) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d^*).$$

If $\lambda > \kappa_1$,

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c^*) = \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_{\tilde\pi}) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d_{\tilde\pi}) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d^*).$$

But this means that $\pi^*\in P$, and evidently it is a proper extension of $\tilde\pi$, which is supposed to be impossible. **XQ**

(δ) Thus $\tilde\pi$ has domain $\mathfrak{A}$ and is the required measure-preserving homomorphism from $\mathfrak{A}$ to the principal ideal $\mathfrak{B}_{d_{\tilde\pi}}$ of $\mathfrak{B}$.

332Q Proposition Let $(\mathfrak{A},\bar\mu)$ and $\mathfrak{B},\bar\nu)$ be totally finite measure algebras, and suppose that there are measure-preserving Boolean homomorphisms $\pi_1:\mathfrak{A}\to\mathfrak{B}$ and $\pi_2:\mathfrak{B}\to\mathfrak{A}$. Then $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic.

proof Writing e_κ, f_κ for their Maharam-type-κ components, 332O (applied to both π_1 and π_2) tells us that

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa)$$

for every λ. Because all these measures are finite,

$$\begin{aligned}\bar\mu e_\lambda &= \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) - \bar\mu(\sup_{\kappa>\lambda} e_\kappa)\\ &= \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa - \bar\nu(\sup_{\kappa>\lambda} f_\kappa) = \bar\nu f_\lambda\end{aligned}$$

for every λ.

Similarly, writing e_γ, f_γ for the suprema in $\mathfrak{A}$, $\mathfrak{B}$ of the atoms of measure γ, 332O tells us that

$$\bar\mu(\sup_{\gamma\le\delta} e_\gamma) = \bar\nu(\sup_{\gamma\le\delta} f_\gamma)$$

for every $\delta\in\,]0,\infty[$, and hence that $\bar\mu e_\gamma = \bar\nu f_\gamma$ for every γ, that is, that $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of atoms of measure γ.

So $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, by 332J.

332R 332J tells us that if we know the magnitudes of the Maharam-type-κ components of a localizable measure algebra, we shall have specified the algebra completely, so that all its properties are determined. The calculation of its Maharam type is straightforward and useful, so I give the details.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then $c(\mathfrak{A})\le 2^{\tau(\mathfrak{A})}$.

proof Let $C\subseteq\mathfrak{A}\setminus\{0\}$ be a disjoint set, and $B\subseteq\mathfrak{A}$ a τ-generating set of size $\tau(\mathfrak{A})$.

(a) If $\mathfrak{A}$ is purely atomic, then for each $c\in C$ choose an atom $c'\subseteq c$, and set $f(c)=\{b: b\in B,\ c'\subseteq b\}$. If c_1, c_2 are distinct members of C, the set

$$\{a: a\in\mathfrak{A},\ c_1'\subseteq a \iff c_2'\subseteq a\}$$

is an order-closed subalgebra of $\mathfrak{A}$ not containing either c_1' or c_2', so cannot include B, and $f(c_1)\ne f(c_2)$. Thus f is injective, and

$$\#(C)\le\#(\mathcal{P}B) = 2^{\tau(\mathfrak{A})}.$$

(b) Now suppose that $\mathfrak{A}$ is not purely atomic; in this case $\tau(\mathfrak{A})$ is infinite. For each $c \in C$ choose an element $c' \subseteq c$ of non-zero finite measure. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B. Then the topological closure of $\mathfrak{B}$ is $\mathfrak{A}$ itself (323J), and $\#(\mathfrak{B}) = \tau(\mathfrak{A})$ (331Gc). For $c \in C$ set

$$f(c) = \{b : b \in \mathfrak{B},\, \bar{\mu}(b \cap c') \ge \frac{1}{2}\bar{\mu}c'\}.$$

Then $f : C \to \mathcal{P}\mathfrak{B}$ is injective. **P** If c_1, c_2 are distinct members of C, then (because $\mathfrak{B}$ is topologically dense in $\mathfrak{A}$) there is a $b \in \mathfrak{B}$ such that

$$\bar{\mu}((c_1' \cup c_2') \cap (c_1' \triangle b)) \le \frac{1}{3}\min(\bar{\mu}c_1', \bar{\mu}c_2').$$

But in this case

$$\bar{\mu}(c_1' \setminus b) \le \frac{1}{3}\bar{\mu}c_1', \quad \bar{\mu}(c_2' \cap b) \le \frac{1}{3}\bar{\mu}c_2',$$

and $b \in f(c_1) \triangle f(c_2)$, so $f(c_1) \ne f(c_2)$. **Q** Accordingly $\#(C) \le 2^{\#(\mathfrak{B})} = 2^{\tau(\mathfrak{A})}$ in this case also.

As C is arbitrary, $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

332S Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Then $\tau(\mathfrak{A})$ is the least cardinal λ such that (α) $c(\mathfrak{A}) \le 2^\lambda$ (β) $\tau(\mathfrak{A}_a) \le \lambda$ for every Maharam-type-homogeneous principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

proof Fix λ as the least cardinal satisfying (α) and (β).

(a) By 331Hc, $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$ for every $a \in \mathfrak{A}$, while $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$ by 332R; so $\lambda \le \tau(\mathfrak{A})$.

(b) Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of non-zero finite measure generating Maharam-type-homogeneous principal ideals (as in the proof of 332B); then $\#(C) \le c(\mathfrak{A}) \le 2^\lambda$, and there is an injective function $f : C \to \mathcal{P}\lambda$. For each $c \in \mathfrak{C}$, let $B_c \subseteq \mathfrak{A}_c$ be a τ-generating set of cardinal $\tau(\mathfrak{A}_c)$, and $f_c : B_c \to \lambda$ an injection. Set

$$b_\xi = \sup\{c : c \in C,\, \xi \in f(c)\},$$

$$b_\xi' = \sup\{b : \text{there is some } c \in C \text{ such that } b \in B_c,\, f_c(b) = \xi\}$$

for $\xi < \lambda$. Set $B = \{b_\xi : \xi < \lambda\} \cup \{b_\xi' : \xi < \lambda\}$ if λ is infinite, $\{b_\xi : \xi < \lambda\}$ if λ is finite; then $\#(B) \le \lambda$. Note that if $c \in C$ and $b \in B_c$ there is a $b' \in B$ such that $b = b' \cap c$. **P** Since $B_c \ne \emptyset$, $\tau(\mathfrak{A}_c) > 0$; but this means that $\tau(\mathfrak{A}_c)$ is infinite (see 331H) so λ is infinite and $b_\xi' \in B$, where $\xi = f_c(b)$; now $b = b_\xi' \cap c$. **Q**

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by B. Then $C \subseteq \mathfrak{B}$. **P** For $c \in C$, we surely have $c \subseteq b_\xi$ if $\xi \in f(c)$; but also, because C is disjoint, $c \cap b_\xi = 0$ if $\xi \in \lambda \setminus f(c)$. Consequently

$$c^* = \inf_{\xi \in f(c)} b_\xi \cap \inf_{\xi \in \lambda \setminus f(c)} (1 \setminus b_\xi)$$

includes c. On the other hand, if d is any other member of C, there is some $\xi \in f(c) \triangle f(d)$, so that

$$d^* \cap c^* \subseteq b_\xi \cap (1 \setminus b_\xi) = 0.$$

Since $\sup C = 1$, it follows that $c = c^*$; but $c^* \in \mathfrak{B}$, so $c \in \mathfrak{B}$. **Q**

For any $c \in C$, look at $\{b \cap c : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$. This is a closed subalgebra of $\mathfrak{A}_c$ (314F(a-i)) including B_c, so must be the whole of $\mathfrak{A}_c$. Thus $\mathfrak{A}_c \subseteq \mathfrak{B}$ for every $c \in C$. But $\sup C = 1$, so $a = \sup_{c \in C} a \cap c \in \mathfrak{B}$ for every $a \in \mathfrak{A}$, and $\mathfrak{A} = \mathfrak{B}$. Consequently $\tau(\mathfrak{A}) \le \#(B) \le \lambda$, and $\tau(\mathfrak{A}) = \lambda$.

332T Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. Then

(a) there is a function $\bar{\nu} : \mathfrak{B} \to [0, \infty]$ such that $(\mathfrak{B}, \bar{\nu})$ is a localizable measure algebra;

(b) $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

proof (a) Let D be the set of those $b \in \mathfrak{B}$ such that the principal ideal $\mathfrak{B}_b$ has Maharam type at most $\tau(\mathfrak{A})$ and is a totally finite measure algebra when endowed with an appropriate measure. Then D is order-dense in $\mathfrak{B}$. **P** Take any non-zero $b_0 \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a \subseteq b_0$ and $0 < \bar{\mu}a < \infty$. Set $c = \inf\{b : b \in \mathfrak{B},\, a \subseteq b\}$; then $c \in \mathfrak{B}$ and $a \subseteq c \subseteq b_0$. If $0 \ne b \in \mathfrak{B}_c$, then $c \setminus b$ belongs to $\mathfrak{B}$ and is properly included in c, so cannot include a; accordingly $a \cap b \ne 0$. For $b \in \mathfrak{B}_c$, set $\bar{\nu}b = \bar{\mu}(a \cap b)$. Because the map $b \mapsto a \cap b$ is an injective order-continuous Boolean homomorphism, $\bar{\nu}$ is countably additive and strictly positive, that is, $(\mathfrak{B}_c, \bar{\nu})$ is a measure algebra. It is totally finite because $\bar{\nu}c = \bar{\mu}a < \infty$.

Let $d \in \mathfrak{B}_c \setminus \{0\}$ be such that $\mathfrak{B}_d$ is Maharam-type-homogeneous; suppose that its Maharam type is κ. The map $b \mapsto b \cap a$ is a measure-preserving Boolean homomorphism from $\mathfrak{B}_d$ to $\mathfrak{A}_{a \cap d}$, so by 332O $\mathfrak{A}_{a \cap d}$ must have a non-zero Maharam-type-κ' component for some $\kappa' \ge \kappa$; but this means that

$$\tau(\mathfrak{B}_d) \le \kappa \le \kappa' \le \tau(\mathfrak{A}_{a \cap d}) \le \tau(\mathfrak{A}).$$

Thus $d \in D$, while $0 \ne d \subseteq c \subseteq b_0$. As b_0 is arbitrary, D is order-dense. **Q**

Accordingly there is a partition of unity C in $\mathfrak{B}$ such that $C \subseteq D$. For each $c \in C$ we have a functional $\bar{\nu}_c$ such that $(\mathfrak{B}_c, \bar{\nu}_c)$ is a totally finite measure algebra of Maharam type at most $\tau(\mathfrak{A})$; define $\bar{\nu} : \mathfrak{B} \to [0, \infty]$ by setting $\bar{\nu}b = \sum_{c \in C} \bar{\nu}_c(b \cap c)$ for every $b \in \mathfrak{B}$. It is easy to check that $(\mathfrak{B}, \bar{\nu})$ is a measure algebra (compare 322Ka); it is localizable because $\mathfrak{B}$ (being order-closed in a Dedekind complete partially ordered set) is Dedekind complete.

(b) The construction above ensures that every homogeneous principal ideal of $\mathfrak{B}$ can have Maharam type at most $\tau(\mathfrak{A})$, since it must share a principal ideal with some $\mathfrak{B}_c$ for $c \in C$. Moreover, any disjoint set in $\mathfrak{B}$ is also a disjoint set in $\mathfrak{A}$, so $c(\mathfrak{B}) \le c(\mathfrak{A})$. So 332S tells us that $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

Remark I think the only direct appeal I shall make to this result will be when $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, in which case (a) above becomes trivial, and the proof of (b) can be shortened to some extent, though I think we still need some of the ideas of 332S.

332X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that it is isomorphic to a simple product of Maharam-type-homogeneous Boolean algebras.

(b) Let $\mathfrak{A}$ be a Boolean algebra of finite cellularity. Show that $\mathfrak{A}$ is purely atomic.

(c) Let $\mathfrak{A}$ be a purely atomic Boolean algebra. Show that $c(\mathfrak{A})$ is the number of atoms in $\mathfrak{A}$.

(d) Let $\mathfrak{A}$ be any Boolean algebra, and Z its Stone space. Show that $c(\mathfrak{A})$ is equal to

$$c(Z) = \sup\{\#(\mathcal{G}) : G \text{ is a disjoint family of non-empty open subsets of } Z\},$$

the **cellularity** of the topological space Z.

(e) Let X be a topological space, and $\mathfrak{G}$ its regular open algebra. Show that $c(\mathfrak{G}) = c(X)$ as defined in 332Xd.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$. Show that $c(\mathfrak{B}) \le c(\mathfrak{A})$.

(g) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$. Show that the cellularity of $\mathfrak{A}$ is at most $\max(\omega, \#(I), \sup_{i \in I} c(\mathfrak{A}_i))$. Devise an elegant expression of a necessary and sufficient condition for equality.

(h) Let $\mathfrak{A}$ be any Boolean algebra, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal generated by a. Show that $c(\mathfrak{A}_a) \le c(\mathfrak{A})$.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Show that it has a partition of unity of cardinal $c(\mathfrak{A})$.

(j) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras. For each cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ the corresponding principal ideals. Show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, as Boolean algebras, iff $c(\mathfrak{A}_{e_\kappa}) = c(\mathfrak{B}_{f_\kappa})$ for every κ.

(k) Let ζ be an ordinal, and $\langle \alpha_\xi \rangle_{\xi < \zeta}$, $\langle \beta_\xi \rangle_{\xi < \zeta}$ two families of non-negative real numbers such that $\sum_{\theta \le \xi < \zeta} \alpha_\xi \le \sum_{\theta \le \eta < \zeta} \beta_\eta < \infty$ for every $\theta \le \zeta$. Show that there is a family $\langle \gamma_{\xi\eta} \rangle_{\xi \le \eta < \zeta}$ of non-negative real numbers such that $\alpha_\xi = \sum_{\xi \le \eta < \zeta} \gamma_{\xi\eta}$ for every $\xi < \zeta$, $\beta_\eta \ge \sum_{\xi \le \eta} \gamma_{\xi\eta}$ for every $\eta < \zeta$. (If only finitely many of the α_ξ, β_ξ are non-zero, this is an easy special case of the max-flow min-cut theorem; see Bollobás 79, §III.1 or Anderson 87, 12.3.1.); there is a statement of the theorem in 4A3M in the next volume.) Show that $\gamma_{\xi\eta}$ can be chosen in such a way that if $\xi < \xi'$, $\eta' < \eta$ then at least one of $\gamma_{\xi\eta}$, $\gamma_{\xi'\eta'}$ is zero.

(l) Use 332Xk and 332M to give another proof of 332P.

(m) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar{\mu})$ be the simple product of $\langle(\mathfrak{B}_{\omega_n}, \bar{\nu}_{\omega_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar{\nu})$ the simple product of $(\mathfrak{A}, \bar{\mu})$ with $(\mathfrak{B}_{\omega_\omega}, \bar{\nu}_{\omega_\omega})$. (See 3A1E for the notation ω_n, ω_ω.) Show that there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, but that no such homomorphism can be order-continuous.

(n) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar{\mu})$ be the simple product of $\langle(\mathfrak{B}_{\kappa_n}, \bar{\nu}_{\kappa_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar{\nu})$ the simple product of $\langle(\mathfrak{B}_{\lambda_n}, \bar{\nu}_{\lambda_n})\rangle_{n\in\mathbb{N}}$, where $\kappa_n = \omega$ for even n, ω_n for odd n, while $\lambda_n = \omega$ for odd n, ω_n for even n. Show that there are order-continuous measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{A}$, but that these two measure algebras are not isomorphic.

(o) Let $\mathfrak{C}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{C}$ is isomorphic (as Boolean algebra) to a closed subalgebra of a localizable measure algebra; (ii) there is a $\bar{\mu}$ such that $(\mathfrak{C}, \bar{\mu})$ is itself a localizable measure algebra; (iii) $\mathfrak{C}$ is Dedekind complete and for every non-zero $c \in \mathfrak{C}$ there is a completely additive real-valued functional ν on $\mathfrak{C}$ such that $\nu c \neq 0$. (*Hint for (iii)⇒(ii)*: show that the set of supports of non-negative completely additive functionals is order-dense in $\mathfrak{C}$, so includes a partition of unity.)

332Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ be atomless localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $(\mathfrak{A}, \bar{\mu})$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B}, \bar{\nu})$; (ii) for every cardinal λ, the magnitude of $\sup_{\kappa\geq\lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\geq\lambda} f_\kappa$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be any semi-finite measure algebras, and $(\widehat{\mathfrak{A}}, \hat{\mu})$, $(\widehat{\mathfrak{B}}, \hat{\nu})$ their localizations (322O-322P). Let $\langle e_i\rangle_{i\in I}$, $\langle f_j\rangle_{j\in J}$ be partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively into elements of finite measure generating homogeneous principal ideals $\mathfrak{A}_{e_i}$, $\mathfrak{B}_{f_j}$. For each infinite cardinal κ set $I_\kappa = \{i : \tau(\mathfrak{A}_{e_i}) = \kappa\}$, $J_\kappa = \{j : \tau(\mathfrak{B}_{f_j}) = \kappa\}$; for $\gamma \in]0, \infty[$, set $I_\gamma = \{i : e_i$ is an atom, $\bar{\mu}e_i = \gamma\}$, $J_\gamma = \{j : f_j$ is an atom, $\bar{\nu}f_j = \gamma\}$. Show that $(\widehat{\mathfrak{A}}, \hat{\mu})$ and $(\widehat{\mathfrak{B}}, \hat{\nu})$ are isomorphic iff for each u, either $\sum_{i\in I_u} \bar{\mu}e_i = \sum_{j\in J_u} \bar{\nu}f_j < \infty$ or $\sum_{i\in I_u} \bar{\mu}e_i = \sum_{j\in J_u} \bar{\nu}f_j = \infty$ and $\#(I_u) = \#(J_u)$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be non-zero localizable measure algebras; let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $\mathfrak{A}$ is isomorphic, as Boolean algebra, to an order-closed subalgebra of a principal ideal of $\mathfrak{B}$; (ii) $c(\mathfrak{A}^*_\lambda) \leq c(\mathfrak{B}^*_\lambda)$ for every cardinal λ, where $\mathfrak{A}^*_\lambda$, $\mathfrak{B}^*_\lambda$ are the principal ideals generated by $\sup_{\kappa\geq\lambda} e_\kappa$ and $\sup_{\kappa\geq\lambda} f_\kappa$ respectively.

332 Notes and comments Maharam's theorem tells us that all localizable measure algebras – in particular, all σ-finite measure algebras – can be obtained from the basic algebra $\mathfrak{A} = \{0, a, 1 \setminus a, 1\}$, with $\bar{\mu}a = \bar{\mu}(1 \setminus a) = \frac{1}{2}$, by combining the constructions of probability algebra free products, scalar multiples of measures and simple products. But what is much more important is the fact that we get a description of our measure algebras in terms sufficiently explicit to make a very wide variety of questions resolvable. The description I offer in 332J hinges on the complementary concepts of 'Maharam type' and 'magnitude'. If you like, the magnitude of a measure algebra is a measure of its width, while its Maharam type is a measure of its depth. The latter is more important just because, for localizable algebras, we have such a simple decomposition into algebras of finite magnitude. Of course there is a good deal of scope for further complications if we seek to consider non-localizable semi-finite algebras. For these, the natural starting point is a proper description of their localizations, which is not difficult (332Yb).

Observe that 332C gives a representation of a localizable measure algebra as the measure algebra of a measure space which is completely different from the Stone representation in 321K. It is less canonical (since there is a degree of choice about the partition $\langle e_i\rangle_{i\in I}$) but very much more informative, since the κ_i, γ_i carry enough information to identify the measure algebra up to isomorphism (332K).

'Cellularity' is the second cardinal function I have introduced in this chapter. It refers as much to topological spaces as to Boolean algebras (see 332Xd-332Xe). There is an interesting question in this context. If $\mathfrak{A}$ is an arbitrary Boolean algebra, is there necessarily a disjoint set in $\mathfrak{A}$ of cardinal $c(\mathfrak{A})$? This is believed to be undecidable from the ordinary axioms of set theory (including the axiom of choice); see Juhász 71, 3.1 and 6.5. But for semi-finite measure algebras we have a definite answer (332F).

Maharam's classification not only describes the isomorphism classes of localizable measure algebras, but also tells us when to expect Boolean homomorphisms between them (332P, 332Yc). I have given 332P only for atomless totally finite measure algebras because the non-totally-finite case (332Ya, 332Yc) seems to require a new idea, while atoms introduce acute combinatorial complications.

I offer 332T as an example of the kind of result which these methods make very simple. It fails for general Boolean algebras; in fact, there is for any κ a countably τ-generated Dedekind complete Boolean algebra $\mathfrak{A}$ with cellularity κ (Koppelberg 89, 13.1), so that $\mathcal{P}\kappa$ is isomorphic to an order-closed subalgebra of $\mathfrak{A}$, and if $\kappa > \mathfrak{c}$ then $\tau(\mathcal{P}\kappa) > \omega$ (332R).

For totally finite measure algebras we have a kind of weak Schröder-Bernstein theorem: if we have two of them, each isomorphic to a closed subalgebra of the other, they are isomorphic (332Q). This fails for σ-finite algebras (332Xn). I call it a 'weak' Schröder-Bernstein theorem because it is not clear how to build the isomorphism from the two injections; 'strong' Schröder-Bernstein theorems include definite recipes for constructing the isomorphisms declared to exist (see, for instance, 344D below).

333 Closed subalgebras

Proposition 332P tells us, in effect, which totally finite measure algebras can be embedded as closed subalgebras of each other. Similar techniques make it possible to describe the possible forms of such embeddings. In this section I give the fundamental theorems on extension of measure-preserving homomorphisms from closed subalgebras (333C, 333D); these rely on the concept of 'relative Maharam type' (333A). I go on to describe possible canonical forms for structures $(\mathfrak{A},\bar\mu,\mathfrak{C})$, where $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (333K, 333N). I end the section with a description of fixed-point subalgebras (333R).

333A Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. The **relative Maharam type of $\mathfrak{A}$ over $\mathfrak{C}$**, $\tau_{\mathfrak{C}}(\mathfrak{A})$, is the smallest cardinal of any set $A\subseteq\mathfrak{A}$ such that $A\cup\mathfrak{C}$ τ-generates $\mathfrak{A}$.

(b) In this section, I will regularly use the following notation: if $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, and $a\in\mathfrak{A}$, then I will write $\mathfrak{C}_a$ for $\{c\cap a : c\in\mathfrak{C}\}$. Observe that $\mathfrak{C}_a$ is a subalgebra of the principal ideal $\mathfrak{A}_a$ (because $c\mapsto c\cap a:\mathfrak{C}\to\mathfrak{A}_a$ is a Boolean homomorphism); it is included in $\mathfrak{C}$ iff $a\in\mathfrak{C}$..

(c) Still taking $\mathfrak{A}$ to be a Boolean algebra and $\mathfrak{C}$ to be a subalgebra of $\mathfrak{A}$, I will say that an element a of $\mathfrak{A}$ is **relatively Maharam-type-homogeneous over** $\mathfrak{C}$ if $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)=\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $b\subseteq a$.

333B Evidently this is a generalization of the ordinary concept of Maharam type as used in §§331-332; if $\mathfrak{C}=\{0,1\}$ then $\tau_{\mathfrak{C}}(\mathfrak{A})=\tau(\mathfrak{A})$. The first step is naturally to check the results corresponding to 331H.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$.

(a) If $a\subseteq b$ in $\mathfrak{A}$, then $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\le\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$. In particular, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\le\tau_{\mathfrak{C}}(\mathfrak{A})$ for every $a\in\mathfrak{A}$.

(b) The set $\{a : a\in\mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}\}$ is order-dense in $\mathfrak{A}$.

(c) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}$ is order-closed in $\mathfrak{A}$, then $\mathfrak{C}_a$ is order-closed in $\mathfrak{A}_a$.

(d) If $a\in\mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$ then either $\mathfrak{A}_a=\mathfrak{C}_a$, so that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=0$ and a is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$ (definition: 331A), or $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\ge\omega$.

(e) If $\mathfrak{D}$ is another subalgebra of $\mathfrak{A}$ and $\mathfrak{D}\subseteq\mathfrak{C}$, then

$$\tau(\mathfrak{A}_a)=\tau_{\{0,a\}}(\mathfrak{A}_a)\ge\tau_{\mathfrak{D}_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{A}_a}(\mathfrak{A}_a)=0$$

for every $a\in\mathfrak{A}$.

proof (a) Let $D\subseteq\mathfrak{A}_b$ be a set of cardinal $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$ such that $D\cup\mathfrak{C}_b$ τ-generates $\mathfrak{A}_b$. Set $D'=\{d\cap a : d\in D\}$. Then $D'\cup\mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. **P** Apply 313Mc to the map $d\mapsto d\cap a:\mathfrak{A}_b\to\mathfrak{A}_a$, as in 331Hc. **Q** Consequently

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\le\#(D')\le\#(D)=\tau_{\mathfrak{C}_b}(\mathfrak{A}_b),$$

as claimed. Setting $b=1$ we get $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\le\tau_{\mathfrak{C}}(\mathfrak{A})$.

(b) Just as in the proof of 332A, given $b\in\mathfrak{A}\setminus\{0\}$, there is an $a\in\mathfrak{A}_b\setminus\{0\}$ minimising $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$, and this a must be relatively Maharam-type-homogeneous over $\mathfrak{C}$.

(c) $\mathfrak{C}_a$ is the image of the Dedekind complete Boolean algebra $\mathfrak{C}$ under the order-continuous Boolean homomorphism $c\mapsto c\cap a$, so must be order-closed (314Fa).

(d) Suppose that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ is finite. Let $D\subseteq\mathfrak{A}_a$ be a finite set such that $D\cup\mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. Then there is a non-zero $b\in\mathfrak{A}_a$ such that $b\cap d$ is either 0 or b for every $d\in D$. But this means that $\mathfrak{C}_b=\{d\cap b: d\in D\cup\mathfrak{C}_a\}$, which τ-generates $\mathfrak{A}_b$; so that $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)=0$. Since a is relatively Maharam-type-homogeneous over $\mathfrak{C}$, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ must be zero, that is, $\mathfrak{A}_a=\mathfrak{C}_a$.

(e) The middle inequality is true just because $\mathfrak{A}_a$ will be τ-generated by $D\cup\mathfrak{C}_a$ whenever it is τ-generated by $D\cup\mathfrak{D}_a$. The neighbouring inequalities are special cases of the middle one, and the outer equalities are elementary.

333C Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\phi:\mathfrak{C}\to\mathfrak{B}$ be a measure-preserving Boolean homomorphism.

(a) If, in the notation of 333A, $\tau_{\mathfrak{C}}(\mathfrak{A})\le\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $b\in\mathfrak{B}$, there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ extending ϕ.

(b) If $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$, then there is a measure algebra isomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ extending ϕ.

proof In both parts, the idea is to use the technique of the proof of 331I to construct π as the last of an increasing family $\langle\pi_\xi\rangle_{\xi\le\kappa}$ of measure-preserving homomorphisms from closed subalgebras $\mathfrak{C}_\xi$ of $\mathfrak{A}$, where $\kappa=\tau_{\mathfrak{C}}(\mathfrak{A})$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ such that $\mathfrak{C}\cup\{a_\xi:\xi<\kappa\}$ τ-generates $\mathfrak{A}$. Write $\mathfrak{D}$ for $\phi[\mathfrak{C}]$; remember that $\mathfrak{D}$ is a closed subalgebra of $\mathfrak{B}$ (324L).

(a)(i) In this case, we can describe the $\mathfrak{C}_\xi$ immediately; $\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{a_\eta:\eta<\xi\}$. The induction starts with $\mathfrak{C}_0=\mathfrak{C}$, $\pi_0=\phi$.

(ii) For the inductive step to a successor ordinal $\xi+1$, where $\xi<\kappa$, suppose that $\mathfrak{C}_\xi$ and π_ξ have been defined. Take any non-zero $b\in\mathfrak{B}$. We are supposing that $\tau_{\mathfrak{D}_b}(\mathfrak{B}_b)\ge\kappa>\#(\xi)$, so $\mathfrak{B}_b$ cannot be τ-generated by

$$D=\mathfrak{D}_b\cup\{b\cap\pi_\xi a_\eta:\eta<\xi\}=\pi_\xi[\mathfrak{C}]_b\cup\{b\cap\pi_\xi a_\eta:\eta<\xi\}=\psi[\mathfrak{C}\cup\{a_\eta:\eta<\xi\}],$$

writing $\psi c=b\cap\pi_\xi c$ for $c\in\mathfrak{C}_\xi$. As ψ is order-continuous, $\psi[\mathfrak{C}_\xi]$ is precisely the closed subalgebra of $\mathfrak{B}_b$ generated by D (314Gb), and is therefore not the whole of $\mathfrak{B}_b$.

But this means that $\mathfrak{B}_b\ne\{b\cap\pi_\xi c:c\in\mathfrak{C}_\xi\}$. As b is arbitrary, π_ξ satisfies the conditions of 331D, and has an extension to a measure-preserving Boolean homomorphism $\pi_{\xi+1}:\mathfrak{C}_{\xi+1}\to\mathfrak{B}$, since $\mathfrak{C}_{\xi+1}$ is just the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{a_\xi\}$.

(iii) For the inductive step to a non-zero limit ordinal $\xi\le\kappa$, we can argue exactly as in part (d) of the proof of 331I; $\mathfrak{C}_\xi$ will be the metric closure of $\mathfrak{C}_\xi^*=\bigcup_{\eta<\xi}\mathfrak{C}_\eta$, so we can take $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{B}$ to be the unique measure-preserving homomorphism extending $\pi_\xi^*=\bigcup_{\eta<\xi}\pi_\eta$.

Thus the induction proceeds, and evidently $\pi=\pi_\kappa$ will be a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ extending ϕ.

(b) (This is rather closer to the proof of 331I, being indeed a direct generalization of it.) Observe that the hypothesis (b) implies that $1_{\mathfrak{A}}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$; so either $\kappa=0$, in which case $\mathfrak{A}=\mathfrak{C}$, $\mathfrak{B}=\phi[\mathfrak{C}]$ and the result is trivial, or $\kappa\ge\omega$, by 333Bd. Let us therefore take it that κ is infinite.

We are supposing, among other things, that $\tau_{\mathfrak{D}}(\mathfrak{B})=\kappa$; let $\langle b_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}$ such that $\mathfrak{B}$ is τ-generated by $\mathfrak{D}\cup\{b_\xi:\xi<\kappa\}$. This time, as in 331I, we shall have to choose further families $\langle a'_\xi\rangle_{\xi<\kappa}$ and $\langle b'_\xi\rangle_{\xi<\kappa}$, and

$\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by

$$\mathfrak{C}\cup\{a_\eta:\eta<\xi\}\cup\{a'_\eta:\eta<\xi\},$$

$\mathfrak{D}_\xi$ will be the closed subalgebra of $\mathfrak{B}$ generated by

$$\mathfrak{D}\cup\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\},$$

$\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{D}_\xi$ will be a measure-preserving homomorphism.
The induction will start with $\mathfrak{C}_0=\mathfrak{C}$, $\mathfrak{D}_0=\mathfrak{D}$ and $\pi_0=\phi$, as in (a).

(i) For the inductive step to a successor ordinal $\xi+1$, where $\xi<\kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

(α) Let $b\in\mathfrak{B}\setminus\{0\}$. Because

$$\tau_{\mathfrak{D}_b}(\mathfrak{B}_b)=\kappa>\#(\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\}),$$

$\mathfrak{B}_b$ cannot be τ-generated by $\mathfrak{D}_b\cup\{b\cap b_\eta:\eta<\xi\}\cup\{b\cap b'_\eta:\eta<\xi\}$, and cannot be equal to $\{b\cap d:d\in\mathfrak{D}_\xi\}$. As b is arbitrary, there is an extension of π_ξ to a measure-preserving homomorphism ϕ_ξ from $\mathfrak{C}'_\xi$ to $\mathfrak{B}$, where $\mathfrak{C}'_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{a_\eta:\eta\le\xi\}\cup\{a'_\eta:\eta<\xi\}$. Setting $b'_\xi=\phi_\xi(a_\xi)$, the image $\mathfrak{D}'_\xi=\phi_\xi[\mathfrak{C}_\xi]$ will be the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}\cup\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta\le\xi\}$.

(β) Now, as in 331I, we must repeat the argument of (α), applying it now to $\phi_\xi^{-1}:\mathfrak{D}_\xi\to\mathfrak{A}$. If $a\in\mathfrak{A}\setminus\{0\}$,

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\kappa>\#(\{a_\eta:\eta\le\xi\}\cup\{a'_\eta:\eta<\xi\}),$$

so that $\mathfrak{A}_a$ cannot be $\{a\cap c:c\in\mathfrak{C}'_\xi\}$. As a is arbitrary, ϕ_ξ^{-1} has an extension to a measure-preserving homomorphism $\psi_\xi:\mathfrak{D}_{\xi+1}\to\mathfrak{C}_{\xi+1}$, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi\cup\{b_\xi\}$, that is, the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}\cup\{b_\eta:\eta\le\xi\}\cup\{b'_\eta:\eta<\xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi\cup\{a'_\xi\}$, setting $a'_\xi=\psi_\xi(b_\xi)$.

We can therefore take $\pi_{\xi+1}=\psi_\xi^{-1}:\mathfrak{C}_{\xi+1}\to\mathfrak{D}_{\xi+1}$, as in 331I.

(ii) The inductive step to a non-zero limit ordinal $\xi\le\kappa$ is exactly the same as in (a) above or in 331I; $\mathfrak{C}_\xi$ is the metric closure of $\mathfrak{C}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{C}_\eta$, $\mathfrak{D}_\xi$ is the metric closure of $\mathfrak{D}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{D}_\eta$, and π_ξ is the unique measure-preserving homomorphism from $\mathfrak{C}_\xi$ to $\mathfrak{D}_\xi$ extending every π_η for $\eta<\xi$.

(iii) The induction stops, as before, with $\pi=\pi_\kappa:\mathfrak{C}_\kappa\to\mathfrak{D}_\kappa$, where $\mathfrak{C}_\kappa=\mathfrak{A}$, $\mathfrak{D}_\kappa=\mathfrak{B}$.

333D Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Suppose that

$$\tau(\mathfrak{C})<\max(\omega,\tau(\mathfrak{A}))\le\min\{\tau(\mathfrak{B}_b):b\in\mathfrak{B}\setminus\{0\}\}.$$

Then any measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ can be extended to a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$.

proof Set $\kappa=\min\{\tau(\mathfrak{B}_b):b\in\mathfrak{B}\setminus\{0\}\}$. Then for any non-zero $b\in\mathfrak{B}$,

$$\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)\ge\kappa.$$

P There is a set $C\subseteq\mathfrak{C}$, of cardinal $\tau(\mathfrak{C})$, which τ-generates $\mathfrak{C}$, so that $C'=\{b\cap\phi c:c\in C\}$ τ-generates $\phi[\mathfrak{C}]_b$. Now there is a set $D\subseteq\mathfrak{B}_b$, of cardinal $\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$, such that $\phi[\mathfrak{C}]_b\cup D$ τ-generates $\mathfrak{B}_b$. In this case $C'\cup D$ must τ-generate $\mathfrak{B}_b$, so $\kappa\le\#(C'\cup D)$. But $\#(C')\le\#(C)<\kappa$ and κ is infinite, so we must have $\#(D)\ge\kappa$, as claimed. **Q**

On the other hand, $\tau_{\mathfrak{C}}(\mathfrak{A})\le\tau(\mathfrak{A})\le\kappa$. So we can apply 333C to give the result.

333E Theorem Let $(\mathfrak{C},\bar\mu)$ be a totally finite measure algebra, κ an infinite cardinal, and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A},\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu)$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, and $\psi:\mathfrak{C}\to\mathfrak{A}$ the corresponding homomorphism. Then for any non-zero $a\in\mathfrak{A}$,

$$\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a)=\kappa,$$

in the notation of 333A above.

proof Recall from 325Dd that $\psi[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{A}$.

(a) Let $\langle b_\xi\rangle_{\xi<\kappa}$ be the canonical independent family in $\mathfrak{B}_\kappa$ of elements of measure $\frac{1}{2}$. Let $\psi':\mathfrak{B}_\kappa\to\mathfrak{A}$ be the canonical map, and set $b'_\xi=\psi'b_\xi$ for each ξ.

We know that $\{b_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{B}_\kappa$ (see part (a) of the proof of 331K). Consequently $\psi[\mathfrak{C}] \cup \{b'_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{A}$. **P** Let $\mathfrak{A}_1$ be the closed subalgebra of $\mathfrak{A}$ generated by $\psi[\mathfrak{C}] \cup \{b'_\xi : \xi < \kappa\}$. Because $\psi' : \mathfrak{B}_\kappa \to \mathfrak{A}$ is order-continuous (325Da), $\psi'[\mathfrak{B}_\kappa] \subseteq \mathfrak{A}_1$ (313Mb). But this means that $\mathfrak{A}_1$ includes $\psi[\mathfrak{C}] \cup \psi'[\mathfrak{B}_\kappa]$ and therefore includes the image of $\mathfrak{C} \otimes \mathfrak{B}_\kappa$ in $\mathfrak{A}$; because this is topologically dense in $\mathfrak{A}$ (325Dc), $\mathfrak{A}_1 = \mathfrak{A}$, as claimed. **Q**

(b) It follows that

$$\tau_{\psi[\mathfrak{C}]_a}\mathfrak{A}_a \le \tau_{\psi[\mathfrak{C}]}\mathfrak{A} \le \kappa.$$

(c) We need to know that if $\xi < \kappa$ and e belongs to the closed subalgebra $\mathfrak{C}_\xi$ of $\mathfrak{A}$ generated by $\psi[\mathfrak{C}] \cup \{b'_\eta : \eta \ne \xi\}$, then $\bar\lambda(e \cap b'_\xi) = \frac{1}{2}\bar\lambda e$. **P** Set

$$E = \psi[\mathfrak{C}] \cup \{b'_\eta : \eta \ne \xi\}, \quad F = \{e_0 \cap \ldots \cap e_n : e_0, \ldots, e_n \in E\}.$$

Then every member of F is expressible in the form

$$d = \psi a \cap \inf_{\eta\in J} b'_\eta,$$

where $a \in \mathfrak{C}$ and $J \subseteq \kappa \setminus \{\xi\}$ is finite. Now

$$\bar\lambda d = \bar\mu a \cdot \bar\nu(\inf_{\eta\in J} b_\eta) = 2^{-\#(J)}\bar\mu a,$$

$$\bar\lambda(b'_\xi \cap d) = \bar\mu a \cdot \bar\nu(b_\xi \cap \inf_{\eta\in J} b_\eta) = 2^{-\#(J\cup\{\xi\})}\bar\mu a = \frac{1}{2}\bar\lambda d.$$

Now consider the set

$$G = \{d : d \in \mathfrak{A},\ \bar\lambda(b_\xi \cap d) = \frac{1}{2}\bar\lambda d\}.$$

We have $1_{\mathfrak{A}} \in F \subseteq G$, and F is closed under $\cap$. Secondly, if d, $d' \in G$ and $d \subseteq d'$, then

$$\bar\lambda(b_\xi \cap (d' \setminus d)) = \bar\lambda(b_\xi \cap d') - \bar\lambda(b_\xi \cap d) = \frac{1}{2}\bar\lambda d' - \frac{1}{2}\bar\lambda d = \frac{1}{2}\bar\lambda(d' \setminus d),$$

so $d' \setminus d \in G$. Also, if $H \subseteq G$ is non-empty and upwards-directed,

$$\bar\lambda(b_\xi \cap \sup H) = \bar\lambda(\sup_{d\in H} b_\xi \cap d) = \sup_{d\in H} \bar\lambda(b_\xi \cap d) = \sup_{d\in H} \frac{1}{2}\bar\lambda d = \frac{1}{2}\bar\lambda(\sup H),$$

so $\sup H \in G$. By the Monotone Class Theorem (313Gc), G includes the order-closed subalgebra of $\mathfrak{D}$ generated by F. But this is just $\mathfrak{C}_\xi$. **Q**

(d) The next step is to see that $\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a) > 0$. **P** By (a) and 323J, $\mathfrak{A}$ is the metric closure of the subalgebra $\mathfrak{A}_0$ generated by $\psi[\mathfrak{C}] \cup \{b'_\eta : \eta < \kappa\}$, so there must be an $a_0 \in \mathfrak{A}_0$ such that $\bar\lambda(a_0 \triangle a) \le \frac{1}{4}\bar\lambda a$. Now there is a finite $J \subseteq \kappa$ such that a_0 belongs to the subalgebra $\mathfrak{A}_1$ generated by $\psi[\mathfrak{C}] \cup \{b'_\eta : \eta \in J\}$. Take any $\xi \in \kappa \setminus J$ (this is where I use the hypothesis that κ is infinite). If $c \in \mathfrak{C}$, then by (c) we have

$$\begin{aligned}\bar\lambda((a \cap \psi c) \triangle (a \cap b'_\xi)) &= \bar\lambda(a \cap (\psi c \triangle b'_\xi)) \ge \bar\lambda(a_0 \cap (\psi c \triangle b'_\xi)) - \bar\lambda(a \triangle a_0)\\ &= \bar\lambda(a_0 \cap b'_\xi) + \bar\lambda(a_0 \cap \psi c) - 2\bar\lambda(a_0 \cap \psi c \cap b'_\xi) - \bar\lambda(a \triangle a_0)\\ &= \frac{1}{2}\bar\lambda a_0 - \bar\lambda(a \triangle a_0)\end{aligned}$$

(because both a_0 and $a_0 \cap \psi c$ belong to $\mathfrak{C}_\xi$)

$$\ge \frac{1}{2}\bar\lambda a - \frac{3}{2}\bar\lambda(a \triangle a_0) > 0.$$

Thus $a \cap b'_\xi$ is not of the form $a \cap \psi c$ for any $c \in \mathfrak{C}$, and $\mathfrak{A}_a \ne \psi[\mathfrak{C}]_a$, so that $\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a) > 0$. **Q**

(e) It follows that $\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **P** There is a non-zero $d \subseteq a$ which is relatively Maharam-type-homogeneous over $\psi[\mathfrak{C}]$. By (d), applied to d, $\tau_{\psi[\mathfrak{C}]_d}(\mathfrak{A}_d) > 0$; but now 333Bd tells us that $\tau_{\psi[\mathfrak{C}]_d}(\mathfrak{A}_d)$ must be infinite, so $\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **Q**

(f) If $\kappa=\omega$, we can stop here. If $\kappa>\omega$, we continue, as follows. Let $D\subseteq\mathfrak{A}_a$ be any set of cardinal less than κ. Each $d\in D\cup\{a\}$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $C=\psi[\mathfrak{C}]\cup\{b'_\xi:\xi<\kappa\}$. But because $\mathfrak{A}$ is ccc, this is just the σ-subalgebra of $\mathfrak{A}$ generated by C (331Ge). So d belongs to the closed subalgebra of $\mathfrak{A}$ generated by some countable subset C_d of C, by 331Gd. Now $J_d=\{\eta:b'_\eta\in C_d\}$ is countable. Set $J=\bigcup_{d\in D\cup\{a\}}J_d$; then

$$\#(J)\le\max(\omega,\#(D\cup\{a\}))=\max(\omega,\#(D))<\kappa,$$

so $J\ne\kappa$, and there is a $\xi\in\kappa\setminus J$. Accordingly $\psi[\mathfrak{C}]\cup D\cup\{a\}$ is included in $\mathfrak{C}_\xi$, as defined in (c) above, and $\psi[\mathfrak{C}]_a\cup D\subseteq\mathfrak{C}_\xi$. As $\mathfrak{A}_a\cap\mathfrak{C}_\xi$ is a closed subalgebra of $\mathfrak{A}_a$, it includes the closed subalgebra generated by $\psi[\mathfrak{C}]_a\cup D$. But $a\cap b'_\xi$ surely does not belong to $\mathfrak{C}_\xi$, since

$$\bar\lambda(a\cap b'_\xi\cap b'_\xi)=\bar\lambda(a\cap b'_\xi)=\frac{1}{2}\bar\lambda a>0,$$

and $\bar\lambda(a\cap b'_\xi\cap b'_\xi)\ne\frac{1}{2}\bar\lambda(a\cap b'_\xi)$. Thus $a\cap b'_\xi$ cannot belong to the closed subalgebra of $\mathfrak{A}_a$ generated by $\psi[\mathfrak{C}]_a\cup D$, and $\psi[\mathfrak{C}]_a\cup D$ does not τ-generate $\mathfrak{A}_a$. As D is arbitrary, $\tau_{\phi[\mathfrak{C}]_a}(\mathfrak{A}_a)\ge\kappa$.

This completes the proof.

333F Corollary Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and κ an infinite cardinal. Let $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$.

(i) Suppose that $\kappa\ge\tau_{\mathfrak{C}}(\mathfrak{A})$. Let $(\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, and $\psi:\mathfrak{C}\to\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ the corresponding homomorphism. Then there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ extending ψ.

(ii) Suppose further that $\kappa=\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $a\in\mathfrak{A}$. Then π can be taken to be an isomorphism.

proof All we have to do is apply 333C with $\mathfrak{B}=\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$, using 333E to see that the hypothesis

$$\tau_{\psi[\mathfrak{C}]_b}(\mathfrak{B}_b)=\kappa\text{ for every non-zero }b\in\mathfrak{B}$$

is satisfied.

333G Corollary Let $(\mathfrak{C},\bar\mu)$ be a totally finite measure algebra. Suppose that $\kappa\ge\max(\omega,\tau(\mathfrak{C}))$ is a cardinal, and write $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu)$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$. Then

(a) $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ;

(b) for every measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{C}$ there is a measure-preserving automorphism $\pi:\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa\to\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ such that $\pi(c\otimes 1)=\phi c\otimes 1$ for every $c\in\mathfrak{C}$, writing $c\otimes 1$ for the canonical image in $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ of any $c\in\mathfrak{C}$.

proof Write $\mathfrak{A}$ for $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$, as in 333E.

(a) If $C\subseteq\mathfrak{C}$ is a set of cardinal $\tau(\mathfrak{C})$ which τ-generates $\mathfrak{C}$, and $B\subseteq\mathfrak{B}_\kappa$ a set of cardinal κ which τ-generates $\mathfrak{B}_\kappa$ (331K), then $\{c\otimes b:c\in C,\,b\in B\}$ is a set of cardinal at most $\max(\omega,\tau(\mathfrak{C}),\kappa)=\kappa$ which τ-generates $\mathfrak{A}$ (because the subalgebra it generates is topologically dense in $\mathfrak{A}$, by 325Dc). So $\tau(\mathfrak{A})\le\kappa$. On the other hand, if $a\in\mathfrak{A}$ is non-zero, then $\tau(\mathfrak{A}_a)\ge\tau_{\psi[\mathfrak{C}]_a}(\mathfrak{A}_a)\ge\kappa$, by 333E; so $\mathfrak{A}$ is Maharam-type-homogeneous, with Maharam type κ.

(b) Writing $\mathfrak{D}=\{c\otimes 1:c\in\mathfrak{C}\}$ for the canonical image of $\mathfrak{C}$ in $\mathfrak{A}$, we have a measure-preserving automorphism $\phi_1:\mathfrak{D}\to\mathfrak{D}$ defined by setting $\phi_1(c\otimes 1)=\phi c\otimes 1$ for every $c\in\mathfrak{C}$. Because $\phi_1[\mathfrak{D}]\subseteq\mathfrak{D}$, 333Be and 333E tell us that

$$\kappa=\tau(\mathfrak{A}_a)\ge\tau_{\phi_1[\mathfrak{D}]_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{D}_a}(\mathfrak{A}_a)=\kappa$$

for every non-zero $a\in\mathfrak{A}$, so we can use 333Cb, with $\mathfrak{B}=\mathfrak{A}$, to see that ϕ_1 can be extended to a measure-preserving automorphism on $\mathfrak{A}$.

333H I turn now to the classification of closed subalgebras.

Theorem Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are $\langle\mu_i\rangle_{i\in I}$, $\langle c_i\rangle_{i\in I}$, $\langle\kappa_i\rangle_{i\in I}$ such that

for each $i\in I$, μ_i is a non-negative completely additive functional on $\mathfrak{C}$,

$c_i=[\![\mu_i>0]\!]\in\mathfrak{C}$,

κ_i is 0 or an infinite cardinal,

$(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ is a totally finite measure algebra, writing $\mathfrak{C}_{c_i}$ for the principal ideal of $\mathfrak{C}$ generated by c_i,

$\sum_{i\in I}\mu_i c=\bar{\mu}c$ for every $c\in\mathfrak{C}$,

there is a measure-preserving isomorphism π from $\mathfrak{A}$ to the simple product $\prod_{i\in I}\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of the localizable measure algebra free products $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$, $(\mathfrak{B}_{\kappa_i},\bar{\nu}_{\kappa_i})$, writing $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$.

Moreover, π may be taken such that

for every $c\in\mathfrak{C}$, $\pi c=\langle(c\cap c_i)\otimes 1\rangle_{i\in I}$, writing $c\otimes 1$ for the image in $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $c\in\mathfrak{C}_{c_i}$.

Remark Recall that $[\![\mu_i>0]\!]$ is that element of $\mathfrak{C}$ such that $\mu_i c>0$ whenever $c\in\mathfrak{C}$ and $0\ne c\subseteq[\![\mu_i>0]\!]$, $\mu_i c\le 0$ whenever $c\in\mathfrak{C}$ and $c\cap[\![\mu_i>0]\!]=0$ (326O).

proof (a) Let A be the set of those elements of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$ (see 333Ac). By 333Bb, A is order-dense in $\mathfrak{A}$ (compare part (a) of the proof of 332B), and consequently $A'=\{a:a\in A,\bar{\mu}a<\infty\}$ is order-dense in $\mathfrak{A}$. So there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ consisting of members of A' (313K). For each $i\in I$, set $\mu_i c=\bar{\mu}(a_i\cap c)$ for every $c\in\mathfrak{C}$; then μ_i is non-negative, and it is completely additive by 327E. Because $\langle a_i\rangle_{i\in I}$ is a partition of the identity in $\mathfrak{A}$,

$$\bar{\mu}c=\sum_{i\in I}\bar{\mu}(c\cap a_i)=\sum_{i\in I}\mu_i c$$

for every $c\in\mathfrak{C}$. Next, $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ is a totally finite measure algebra. **P** $\mathfrak{C}_{c_i}$ is a Dedekind σ-complete Boolean algebra because $\mathfrak{C}$ is. $\mu_i\restriction\mathfrak{C}_{c_i}$ is a non-negative countably additive functional because μ_i is. If $c\in\mathfrak{C}_{c_i}$ and $\mu_i c=0$, then $c=0$ by the choice of c_i. **Q** Note also that

$$\bar{\mu}(a_i\setminus c_i)=\mu_i(1\setminus c_i)=0,$$

so that $a_i\subseteq c_i$.

(b) By 333Bd, any finite κ_i must actually be zero. The next element we need is the fact that, for each $i\in I$, we have a measure-preserving isomorphism $c\mapsto c\cap a_i$ from $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ to $(\mathfrak{C}_{a_i},\bar{\mu}\restriction\mathfrak{C}_{a_i})$. **P** Of course this is a ring homomorphism. Because $a_i\subseteq c_i$, it is a surjective Boolean homomorphism. It is measure-preserving by the definition of μ_i, and therefore injective. **Q**

(c) Still focusing on a particular $i\in I$, let $\mathfrak{A}_{a_i}$ be the principal ideal of $\mathfrak{A}$ generated by a_i. Then we have a measure-preserving isomorphism $\tilde{\pi}_i:\mathfrak{A}_{a_i}\to\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, extending the canonical homomorphism $c\mapsto c\otimes 1:\mathfrak{C}_{a_i}\to\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. **P** When κ_i is infinite, this is just 333F(ii). But the only other case is when $\kappa_i=0$, that is, $\mathfrak{C}_{a_i}=\mathfrak{A}_{a_i}$, while $\mathfrak{B}_{\kappa_i}=\{0,1\}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}\cong\mathfrak{C}_{c_i}$. **Q**

The isomorphism between $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ and $(\mathfrak{C}_{a_i},\bar{\mu}\restriction\mathfrak{C}_{a_i})$ induces an isomorphism between $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. So we have a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ such that $\pi_i(c\cap a_i)=c\otimes 1$ for every $c\in\mathfrak{C}_{c_i}$.

(d) By 322Kd, we have a measure-preserving isomorphism $a\mapsto\langle a\cap a_i\rangle_{i\in I}:\mathfrak{A}\to\prod_{i\in I}\mathfrak{A}_{a_i}$.

Putting this together with the isomorphisms of (c), we have a measure-preserving isomorphism $\pi:\mathfrak{A}\to\prod_{i\in I}\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, setting $\pi a=\langle\pi_i(a\cap a_i)\rangle_{i\in I}$ for $a\in\mathfrak{A}$. Observe that, for $c\in\mathfrak{C}$,

$$\pi c=\langle\pi_i(c\cap a_i)\rangle_{i\in I}=\langle(c\cap c_i)\otimes 1\rangle_{i\in I},$$

as required.

333I Remarks (a) I hope it is clear that whenever $(\mathfrak{C},\bar{\mu})$ is a Dedekind complete measure algebra, $\langle\mu_i\rangle_{i\in I}$ is a family of non-negative completely additive functionals on $\mathfrak{C}$ such that $\sum_{i\in I}\mu_i=\bar{\mu}$, and $\langle\kappa_i\rangle_{i\in I}$

is a family of cardinals all infinite or zero, then the construction above can be applied to give a measure algebra $(\mathfrak{A},\tilde{\mu})$, the product of the family $\langle\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}\rangle_{i\in I}$, together with an order-continuous measure-preserving homomorphism $\pi:\mathfrak{C}\to\mathfrak{A}$; and that the partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ corresponding to this product (315E) has $\mu_i c=\tilde{\mu}(a_i\cap\pi c)$ for every $c\in\mathfrak{C}$ and $i\in I$, while each principal ideal $\mathfrak{A}_{a_i}$ can be identified with $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, so that a_i is relatively Maharam-type-homogeneous over $\pi[\mathfrak{C}]$. Thus any structure $(\mathfrak{C},\bar{\mu},\langle\mu_i\rangle_{i\in I},\langle\kappa_i\rangle_{i\in I})$ of the type described here corresponds to an embedding of $\mathfrak{C}$ as a closed subalgebra of a localizable measure algebra.

(b) The obvious next step is to seek a complete classification of objects $(\mathfrak{A},\bar{\mu},\mathfrak{C})$, where $(\mathfrak{A},\bar{\mu})$ is a localizable measure algebra and $\mathfrak{C}$ is a closed subalgebra, corresponding to the classification of localizable measure algebras in terms of the magnitudes of their Maharam-type-κ components in 332J. The general case seems to be complex. But I can deal with the special case in which $(\mathfrak{A},\bar{\mu})$ is totally finite. In this case, we have the following facts.

333J Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{C}$ a closed subalgebra. Let A be the set of relative atoms of $\mathfrak{A}$ over $\mathfrak{C}$. Then there is a unique sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of additive functionals on $\mathfrak{C}$ such that (i) $\mu_{n+1}\le\mu_n$ for every n (ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $\sup_{n\in\mathbb{N}}a_n=\sup A$ and $\mu_n c=\bar{\mu}(a_n\cap c)$ for every $n\in\mathbb{N}$, $c\in\mathfrak{C}$.

Remark I hope it is plain from my wording that it is the μ_n which are unique, not the a_n.

proof (a) For each $a\in\mathfrak{A}$ set $\nu_a(c)=\bar{\mu}(c\cap a)$ for $c\in\mathfrak{C}$. Then ν_a is a non-negative completely additive real-valued functional on $\mathfrak{C}$ (see 326Kd).

The key step is I suppose in (c) below; I approach by a two-stage argument. For each $b\in\mathfrak{A}$ write A_b for $\{a:a\in A,\ a\cap b=0\}$.

(b) For every $b\in\mathfrak{A}$, non-zero $c\in\mathfrak{C}$ there are $a\in A_b$, $c'\in\mathfrak{C}$ such that $0\ne c'\subseteq c$ and $\nu_a(d)\ge\nu_e(d)$ whenever $d\in\mathfrak{C}$, $e\in A_b$ and $d\subseteq c'$. **P?** Otherwise, choose $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ as follows. Since 0, c won't serve for a, c', there must be an $a_0\in A_b$ such that $\nu_{a_0}(c)>0$. Let $\delta>0$ be such that $\nu_{a_0}(c)>\delta\bar{\mu}c$ and set $c_0=c\cap[\![\nu_{a_0}>\delta\bar{\mu}\restriction\mathfrak{C}]\!]$; then $c_0\in\mathfrak{C}$ and $0\ne c_0\subseteq c$. Given that $a_n\in A_b$, $c_n\in\mathfrak{C}$ and $0\ne c_n\subseteq c$, then there must be $a_{n+1}\in A_b$, $d_n\in\mathfrak{C}$ such that $d_n\subseteq c_n$ and $\nu_{a_{n+1}}(d_n)>\nu_{a_n}(d_n)$. Set $c_{n+1}=d_n\cap[\![\nu_{a_{n+1}}>\nu_{a_n}]\!]$, so that $c_{n+1}\in\mathfrak{C}$ and $0\ne c_{n+1}\subseteq c_n$, and continue.

There is some $n\in\mathbb{N}$ such that $n\delta\ge 1$. For any $i<n$, the construction ensures that

$$0\ne c_{n+1}\subseteq c_{i+1}\subseteq[\![\nu_{a_{i+1}}>\nu_{a_i}]\!],$$

so $\nu_{a_i}(c_{n+1})<\nu_{a_{i+1}}(c_{n+1})$; also $c_{n+1}\subseteq c_0$ so

$$\bar{\mu}(a_i\cap c_{n+1})=\nu_{a_i}(c_{n+1})\ge\nu_{a_0}(c_{n+1})>\delta\bar{\mu}c_{n+1}.$$

But this means that $\sum_{i=0}^{n-1}\bar{\mu}(a_i\cap c_{n+1})>\bar{\mu}c_{n+1}$ and there must be distinct j, $k<n$ such that $a_j\cap a_k\cap c_{n+1}$ is non-zero. Because a_j, $a_k\in A$ there are d', $d''\in\mathfrak{C}$ such that $a_j\cap a_k=a_j\cap d'=a_k\cap d''$; set $d=c_{n+1}\cap d'\cap d''$, so that $d\in\mathfrak{C}$ and

$$a_j\cap d=a_j\cap a_k\cap c_{n+1}=a_k\cap d,\quad \nu_{a_j}(d)=\bar{\mu}(a_j\cap a_k\cap c_{n+1})=\nu_{a_k}(d).$$

But as $0\ne d\subseteq[\![\nu_{a_{i+1}}>\nu_{a_i}]\!]$ for every $i<n$, $\nu_{a_0}(d)<\nu_{a_1}(d)<\ldots<\nu_{a_n}(d)$, so this is impossible. **XQ**

(c) Now for a global, rather than local, version of the same idea. For every $b\in\mathfrak{A}$ there is an $a\in A_b$ such that and $\nu_a\ge\nu_e$ whenever $e\in A_b$. **P** (i) By (b), the set C of those $c\in\mathfrak{C}$ such that there is an $a\in A_b$ such that $\nu_a\restriction\mathfrak{C}_c\ge\nu_e\restriction\mathfrak{C}_c$ for every $e\in A_b$ is order-dense in $\mathfrak{C}$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{C}$ consisting of members of C, and for each $i\in I$ choose $a_i\in A_b$ such that $\nu_{a_i}\restriction\mathfrak{C}_{c_i}\ge\nu_e\restriction\mathfrak{C}_{c_i}$ for every $e\in A_b$. Consider $a=\sup_{i\in I}a_i\cap c_i$. (ii) If $a'\in\mathfrak{A}$ and $a'\subseteq a$, then for each $i\in I$ there is a $d_i\in\mathfrak{C}$ such that $a_i\cap a'=a_i\cap d_i$. Set $d'=\sup_{i\in I}c_i\cap d_i$; then (because $\langle c_i\rangle_{i\in I}$ is disjoint)

$$a\cap d'=\sup\nolimits_{i\in I}a_i\cap c_i\cap d_i=\sup\nolimits_{i\in I}a_i\cap c_i\cap a'=a\cap a'=a'.$$

As a' is arbitrary, this shows that $a\in A$. (iii) Of course $a\cap b=0$, so $a\in A_b$. Now take any $e\in A_b$, $d\in\mathfrak{C}$. Then

$$\nu_a(d)=\sum\nolimits_{i\in I}\nu_{a_i}(c_i\cap d)\ge\sum\nolimits_{i\in I}\nu_e(c_i\cap d)=\nu_e(d).$$

So this a has the required property. **Q**

(d) Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively in A so that, for each n, $a_n\cap\sup_{i<n}a_i=0$ and $\nu_{a_n}\ge\nu_e$ whenever $e\in A$ and $e\cap\sup_{i<n}a_i=0$. Set $\mu_n=\nu_{a_n}$. Because $a_{n+1}\cap\sup_{i<n}a_i=0$, $\mu_{n+1}\le\mu_n$ for each n. Also $\sup_{n\in\mathbb{N}}a_n=\sup A$. **P** Take any $a\in A$ and set $e=a\setminus\sup_{n\in\mathbb{N}}a_n$. Then $e\in A$ and, for any $n\in\mathbb{N}$, $e\cap\sup_{i<n}a_i=0$, so $\nu_e\le\nu_{a_n}$ and

$$\bar\mu e=\nu_e(1)\le\nu_{a_n}(1)=\bar\mu a_n.$$

But as $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint, this means that $e=0$, that is, $a\subseteq\sup_{n\in\mathbb{N}}a_n$. As a is arbitrary, $\sup A\subseteq\sup_{n\in\mathbb{N}}a_n$. **Q**

(e) Thus we have a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of the required type, witnessed by $\langle a_n\rangle_{n\in\mathbb{N}}$. To see that it is unique, suppose that $\langle\mu'_n\rangle_{n\in\mathbb{N}}$, $\langle a'_n\rangle_{n\in\mathbb{N}}$ are another pair of sequences with the same properties. Note first that if $c\in\mathfrak{C}$ and $0\ne c\subseteq[\![\mu'_i>0]\!]$ there is some $k\in\mathbb{N}$ such that $c\cap a'_i\cap a_k\ne0$; this is because $\bar\mu(a'_i\cap c)=\mu'_i(c)>0$, so that $a'_i\cap c\ne0$, while $a'_i\subseteq\sup A=\sup_{k\in\mathbb{N}}a_k$. **?** Suppose, if possible, that there is some n such that $\mu_n\ne\mu'_n$; since the situation is symmetric, there is no loss of generality in supposing that $\mu'_n\not\le\mu_n$, that is, that $c=[\![\mu'_n>\mu_n]\!]\ne0$. For any $i\le n$, $\mu'_i\ge\mu'_n$ so $c\subseteq[\![\mu'_i>0]\!]$. We may therefore choose $c_0,\dots,c_{n+1}\in\mathfrak{C}_c\setminus\{0\}$ and $k(0),\dots,k(n)\in\mathbb{N}$ such that $c_0=c$ and, for $i\le n$,

$$c_i\cap a'_i\cap a_{k(i)}\ne0$$

(choosing $k(i)$, recalling that $0\ne c_i\subseteq c\subseteq[\![\mu'_i>0]\!]$),

$$c_{i+1}\in\mathfrak{C},\ c_{i+1}\subseteq c_i,\ c_{i+1}\cap a'_i=c_{i+1}\cap a_{k(i)}=c_i\cap a'_i\cap a_{k(i)}$$

(choosing c_{i+1}, using the fact that a'_i and $a_{k(i)}$ both belong to A – see the penultimate sentence in part (b) of the proof.) On reaching c_{n+1}, we have $0\ne c_{n+1}\subseteq c$ so $\mu_n(c_{n+1})<\mu'_n(c_{n+1})$. On the other hand, for each $i\le n$,

$$c_{n+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap c_{i+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)},$$

so

$$\mu_n(c_{n+1})<\mu'_n(c_{n+1})\le\mu'_i(c_{n+1})=\bar\mu(c_{n+1}\cap a'_i)=\bar\mu(c_{n+1}\cap a_{k(i)})=\mu_{k(i)}(c_{n+1}),$$

and $k(i)$ must be less than n. There are therefore distinct i, $j\le n$ such that $k(i)=k(j)$. But in this case

$$c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)}=c_{n+1}\cap a_{k(j)}=c_{n+1}\cap a'_j\ne0$$

because $0\ne c_{n+1}\subseteq[\![\mu'_j>0]\!]$. So a'_i, a'_j cannot be disjoint, breaking one of the rules of the construction. **X** Thus $\mu_n=\mu'_n$ for every $n\in\mathbb{N}$.

This completes the proof.

333K Theorem Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are unique families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ such that

K is a countable set of infinite cardinals,

for $i\in\mathbb{N}\cup K$, μ_i is a non-negative countably additive functional on $\mathfrak{C}$, and $\sum_{i\in\mathbb{N}\cup K}\mu_ic=\bar\mu c$ for every $c\in\mathfrak{C}$,

$\mu_{n+1}\le\mu_n$ for every $n\in\mathbb{N}$, and $\mu_\kappa\ne0$ for $\kappa\in K$,

setting $e_i=[\![\mu_i>0]\!]\in\mathfrak{C}$, and giving the principal ideal $\mathfrak{C}_{e_i}$ generated by e_i the measure $\mu_i\restriction\mathfrak{C}_{e_i}$ for each $i\in\mathbb{N}\cup K$, and writing $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$ for $\kappa\in K$, we have a measure algebra isomorphism

$$\pi:\mathfrak{A}\to\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$$

such that

$$\pi c=(\langle c\cap e_n\rangle_{n\in\mathbb{N}},\langle(c\cap e_\kappa)\otimes1\rangle_{\kappa\in K})$$

for each $c\in C$, writing $c\otimes1$ for the canonical image in $C_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ of $c\in\mathfrak{C}_{e_\kappa}$.

proof (a) I aim to use the construction of 333H, but taking much more care over the choice of $\langle a_i\rangle_{i\in I}$ in part (a) of the proof there. We start by taking $\langle a_n\rangle_{n\in\mathbb{N}}$ as in 333J, and setting $\mu_nc=\bar\mu(a_n\cap c)$ for every $n\in\mathbb{N}$, $c\in\mathfrak{C}$; then these a_n will deal with the part in $\sup A$, as defined in the proof of 333J.

(b) The further idea required here concerns the treatment of infinite κ. Let $\langle b_i\rangle_{i\in I}$ be any partition of unity in $\mathfrak{A}$ consisting of non-zero members of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$, and $\langle\kappa_i\rangle_{i\in I}$ the corresponding cardinals, so that $\kappa_i=0$ iff $b_i\in A$. Set $I_1=\{i:i\in I,\,\kappa_i\ge\omega\}$. Set $K=\{\kappa_i:i\in I_1\}$, so that K is a countable set of infinite cardinals, and for $\kappa\in K$ set $J_\kappa=\{i:\kappa_i=\kappa\}$, $a_\kappa=\sup_{i\in J_\kappa}b_i$ for $\kappa\in K$. Now every a_κ is relatively Maharam-type-homogeneous over $\mathfrak{C}$. **P** (Compare 332H.) J_κ must be countable, because $\mathfrak{A}$ is ccc. If $0\ne a\subseteq a_\kappa$, there is some $i\in J_\kappa$ such that $a\cap b_i\ne 0$; now

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{C}_{a\cap b_i}}(\mathfrak{A}_{a\cap b_i})=\kappa_i=\kappa.$$

At the same time, for each $i\in J_\kappa$, there is a set $D_i\subseteq\mathfrak{A}_{b_i}$ such that $\#(D_i)=\kappa$ and $\mathfrak{C}_{b_i}\cup D_i$ τ-generates $\mathfrak{A}_{b_i}$. Set $D=\bigcup_{i\in J_\kappa}D_i\cup\{b_i:i\in J_\kappa\}$; then

$$\#(D)\le\max(\omega,\#(J_\kappa),\sup_{i\in K}\#(D_i))=\kappa.$$

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}_{a_\kappa}$ generated by $\mathfrak{C}_{a_\kappa}\cup D$. Then

$$\mathfrak{C}_{b_i}\cup D_i\subseteq\{b\cap b_i:b\in\mathfrak{B}\}=\mathfrak{B}\cap\mathfrak{A}_{b_i}$$

so $\mathfrak{B}\supseteq\mathfrak{A}_{b_i}$ for each $i\in J_\kappa$, and $\mathfrak{B}=\mathfrak{A}_{a_\kappa}$. Thus $\mathfrak{C}_{a_\kappa}\cup D$ τ-generates $\mathfrak{A}_{a_\kappa}$, and

$$\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa})\le\kappa\le\min_{0\ne a\subseteq a_\kappa}\tau_{\mathfrak{C}_a}(\mathfrak{A}_a).$$

This shows that a_κ is relatively Maharam-type-homogeneousover $\mathfrak{C}$, with $\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa})=\kappa$. **Q**

Since evidently $\langle J_\kappa\rangle_{\kappa\in K}$ and $\langle a_\kappa\rangle_{\kappa\in K}$ are disjoint, and $\sup_{\kappa\in K}a_\kappa=\sup_{i\in I_1}b_i$, this process yields a partition $\langle a_i\rangle_{i\in\mathbb{N}\cup K}$ of unity in $\mathfrak{A}$. Now the arguments of 333H show that we get an isomorphism π of the kind described.

(c) To see that the families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ (and therefore the e_i and the $(\mathfrak{C}_{e_i},\mu_i\restriction\mathfrak{C}_{e_i})$, but not π) are uniquely defined, argue as follows. Take families $\langle\tilde\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\tilde\mu_\kappa\rangle_{\kappa\in\tilde K}$ which correspond to an isomorphism

$$\tilde\pi:\mathfrak{A}\to\mathfrak{D}=\prod_{n\in\mathbb{N}}\mathfrak{C}_{\tilde e_n}\times\prod_{\kappa\in\tilde K}\mathfrak{C}_{\tilde e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa,$$

writing $\tilde e_i=[\![\tilde\mu_i>0]\!]$ for $i\in\mathbb{N}\cup\tilde K$. In the simple product $\prod_{n\in\mathbb{N}}\mathfrak{C}_{\tilde e_n}\times\prod_{\kappa\in\tilde K}\mathfrak{C}_{\tilde e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$, we have a partition of unity $\langle e_i^*\rangle_{i\in\mathbb{N}\cup\tilde K}$ corresponding to the product structure. Now for $d\subseteq e_i^*$, we have

$$\begin{aligned}\tau_{\tilde\pi[\mathfrak{C}]_d}(\mathfrak{D}_d)&=0\text{ if }i\in\mathbb{N},\\&=\kappa\text{ if }i=\kappa\in\tilde K.\end{aligned}$$

So $\tilde K$ must be

$$\{\kappa:\kappa\ge\omega,\,\exists\,a\in\mathfrak{A},\,\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\kappa\}=K,$$

and for $\kappa\in\tilde K$,

$$\tilde\pi^{-1}e_\kappa^*=\sup\{a:a\in A,\,\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\kappa\}=a_\kappa,$$

so that $\tilde\mu_\kappa=\mu_\kappa$. On the other hand, $\langle\tilde\pi^{-1}e_n^*\rangle_{n\in\mathbb{N}}$ must be a disjoint sequence with supremum $\sup A$, and the corresponding functionals $\tilde\mu_n$ are supposed to form a non-increasing sequence, so must be equal to the μ_n by 333J.

333L Remark Thus for the classification of structures $(\mathfrak{A},\bar\mu,\mathfrak{C})$, where $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra, it will be enough to classify objects $(\mathfrak{C},\bar\mu,\langle\mu_n\rangle_{n\in\mathbb{N}},\langle\mu_\kappa\rangle_{\kappa\in K})$, where

$(\mathfrak{C},\bar\mu)$ is a totally finite measure algebra,

$\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$,

K is a countable set of infinite cardinals (possibly empty),

$\langle\mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$,

$\sum_{n=0}^\infty\mu_n+\sum_{\kappa\in K}\mu_\kappa=\bar\mu$.

To do this we need the concept of 'standard extension' of a countably additive functional on a closed subalgebra of a measure algebra, treated in 327F-327G, together with the following idea.

333M Lemma Let $(\mathfrak{C},\bar{\mu})$ be a totally finite measure algebra and $\langle\mu_i\rangle_{i\in I}$ a family of countably additive functionals on $\mathfrak{C}$. For $i \in I$, $\alpha \in \mathbb{R}$ set $e_{i\alpha} = [\![\mu_i > \alpha\bar{\mu}]\!]$ (326P), and let $\mathfrak{C}_0$ be the closed subalgebra of $\mathfrak{C}$ generated by $\{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$. Write Σ for the σ-algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $E_{i\alpha} = \{x : x(i) > \alpha\}$ as i runs through I, α runs over $\mathbb{R}$. Then

(a) there is a measure μ, with domain Σ, such that there is a measure-preserving isomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}_0$ for which $\pi E^\bullet_{i\alpha} = e_{i\alpha}$ for every $i \in I$, $\alpha \in \mathbb{R}$, writing $\mathcal{N}_\mu$ for $\mu^{-1}[\{0\}]$;

(b) this formula determines both μ and π;

(c) for every $E \in \Sigma$, $i \in I$, we have

$$\mu_i\pi E^\bullet = \int_E x(i)\mu(dx);$$

(d) for every $i \in I$, μ_i is the standard extension of $\mu_i{\restriction}\mathfrak{C}_0$ to $\mathfrak{C}$;

(e) for every $i \in I$, $\mu_i \ge 0$ iff $x(i) \ge 0$ for μ-almost every x;

(f) for every i, $j \in I$, $\mu_i \ge \mu_j$ iff $x(i) \ge x(j)$ for μ-almost every x;

(g) for every $i \in I$, $\mu_i = 0$ iff $x(i) = 0$ for μ-almost every x.

proof (a) Express $(\mathfrak{C},\bar{\mu})$ as the measure algebra of a measure space (Y, T, ν); write $\phi : \mathrm{T} \to \mathfrak{C}$ for the corresponding homomorphism. For each $i \in I$ let $f_i : Y \to \mathbb{R}$ be a T-measurable, ν-integrable function such that $\int_H f_i = \mu_i\phi H$ for every $H \in \mathrm{T}$. Define $\psi : Y \to \mathbb{R}^I$ by setting $\psi(y) = \langle f_i(y)\rangle_{i\in I}$; then $\psi^{-1}[E_{i\alpha}] \in \Sigma$, and $e_{i\alpha} = \phi(\psi^{-1}[E_{i\alpha}])$ for every $i \in I$, $\alpha \in \mathbb{R}$. (See part (a) of the proof of 327F.) So $\{E : E \subseteq \mathbb{R}^I, \psi^{-1}[E] \in \mathrm{T}\}$, which is a σ-algebra of subsets of $\mathbb{R}^I$, contains every $E_{i\alpha}$, and therefore includes Σ; that is, $\psi^{-1}[E] \in \mathrm{T}$ for every $E \in \Sigma$. Accordingly we may define μ by setting $\mu E = \nu\psi^{-1}[E]$ for every $E \in \Sigma$, and μ will be a measure on $\mathbb{R}^I$ with domain Σ. The Boolean homomorphism $E \mapsto \phi\psi^{-1}[E] : \Sigma \to \mathfrak{C}$ has kernel $\mathcal{N}_\mu$, so descends to a homomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}$, which is measure-preserving. To see that $\pi[\Sigma/\mathcal{N}_\mu] = \mathfrak{C}_0$, observe that because Σ is the σ-algebra generated by $\{E_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$, $\pi[\Sigma/\mathcal{N}_\mu]$ must be the closed subalgebra of $\mathfrak{C}$ generated by $\{\pi E^\bullet_{i\alpha} : i \in I, \alpha \in \mathbb{R}\} = \{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$, which is $\mathfrak{C}_0$.

(b) Now suppose that μ', π' have the same properties. Consider

$$\mathcal{A} = \{E : E \in \Sigma, \pi E^\bullet = \pi' E^\circ\},$$

where I write $E^\bullet$ for the equivalence class of E in $\Sigma/\mathcal{N}_\mu$, and E° for the equivalence class of E in $\Sigma/\mathcal{N}_{\mu'}$. Then $\mathcal{A}$ is a σ-subalgebra of Σ, because $E \mapsto \pi E^\bullet$, $E \mapsto \pi' E^\circ$ are both sequentially order-continuous Boolean homomorphisms, and contains every $E_{i\alpha}$, so must be the whole of Σ. Consequently

$$\mu E = \bar{\mu}\pi E^\bullet = \bar{\mu}\pi' E^\circ = \mu' E$$

for every $E \in \Sigma$, and $\mu' = \mu$; it follows at once that $\pi' = \pi$. So μ and π are uniquely determined.

(c) If $E \in \Sigma$ and $i \in I$,

$$\int_E x(i)\mu(dx) = \int x(i)\chi E(x)\mu(dx) = \int \psi(y)(i)\chi E(\psi(y))\nu(dy)$$

(applying 235I to the inverse-measure-preserving function $\psi : Y \to \mathbb{R}^I$)

$$= \int_{\psi^{-1}[E]} f_i(y)\nu(dy)$$

(by the definition of ψ)

$$= \mu_i\phi(\psi^{-1}[E])$$

(by the choice of f_i)

$$= \mu_i\pi E^\bullet$$

by the definition of π.

(d) Because $[\![\mu_i > \alpha\bar{\mu}]\!] \in \mathfrak{C}_0$, it is equal to $[\![\mu_i{\restriction}\mathfrak{C}_0 > \alpha\bar{\mu}{\restriction}\mathfrak{C}_0]\!]$, for every $\alpha \in \mathbb{R}$. So μ_i must be the standard extension of $\mu_i{\restriction}\mathfrak{C}_0$ (327F).

(e)-(g) The point is that, because the standard-extension operator is order-preserving (327F(b-ii)),

$$\begin{aligned}
\mu_i \ge 0 &\iff \mu_i\restriction\mathfrak{C}_0 \ge 0\\
&\iff \int_E x(i)\mu(dx) \ge 0 \text{ for every } E \in \Sigma\\
&\iff x(i) \ge 0 \ \mu\text{-a.e.},\\
\mu_i \ge \mu_j &\iff \mu_i\restriction\mathfrak{C}_0 \ge \mu_j\restriction\mathfrak{C}_0\\
&\iff \int_E x(i)\mu(dx) \ge \int_E x(j)\mu(dx) \text{ for every } E \in \Sigma\\
&\iff x(i) \ge x(j) \ \mu\text{-a.e.},\\
\mu_i = 0 &\iff \mu_i\restriction\mathfrak{C}_0 = 0\\
&\iff \int_E x(i)\mu(dx) = 0 \text{ for every } E \in \Sigma\\
&\iff x(i) = 0 \ \mu\text{-a.e.}.
\end{aligned}$$

333N A canonical form for closed subalgebras We now have all the elements required to describe a canonical form for structures

$$(\mathfrak{A}, \bar\mu, \mathfrak{C}),$$

where $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. The first step is the matching of such structures with structures

$$(\mathfrak{C}, \bar\mu, \langle\mu_n\rangle_{n\in\mathbb{N}}, \langle\mu_\kappa\rangle_{\kappa\in K}),$$

where $(\mathfrak{C}, \bar\mu)$ is a totally finite measure algebra, $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$, K is a countable set of infinite cardinals, $\langle\mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$, and $\sum_{n=0}^\infty \mu_n + \sum_{\kappa\in K}\mu_\kappa = \bar\mu$; this is the burden of 333K.

Next, given any structure of this second kind, we have a corresponding closed subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a measure μ on $\mathbb{R}^I$, where $I = \mathbb{N} \cup K$, and an isomorphism π from the measure algebra $\mathfrak{C}_0^*$ of μ to $\mathfrak{C}_0$, all uniquely defined from the family $\langle\mu_i\rangle_{i\in I}$ by the process of 333M. For any E belonging to the domain Σ of μ, and $i \in I$, we have

$$\mu_i\pi E^\bullet = \int_E x(i)\mu(dx)$$

(333Mc), so that $\mu_i\restriction\mathfrak{C}_0$ is fixed by π and μ. Moreover, the functionals μ_i can be recovered from their restrictions to $\mathfrak{C}_0$ by the formulae of 327F (333Md). Thus from $(\mathfrak{C}, \bar\mu, \langle\mu_i\rangle_{i\in I})$ we are led, by a canonical and reversible process, to the structure

$$(\mathfrak{C}, \bar\mu, \mathfrak{C}_0, I, \mu, \pi).$$

But the extension $\mathfrak{C}$ of $\mathfrak{C}_0 = \pi[\mathfrak{C}_0^*]$ can be described, up to isomorphism, by the same process as before; that is, it corresponds to a sequence $\langle\bar\nu_n\rangle_{n\in\mathbb{N}}$ and a family $\langle\bar\nu_\kappa\rangle_{\kappa\in L}$ of countably additive functionals on $\mathfrak{C}_0$ satisfying the conditions of 333K. We can transfer these to $\mathfrak{C}_0^*$, where they correspond to families $\langle\nu_n\rangle_{n\in\mathbb{N}}$, $\langle\nu_\kappa\rangle_{\kappa\in L}$ of absolutely continuous countably additive functionals defined on Σ, setting

$$\nu_j E = \bar\nu_j\pi E^\bullet$$

for $E \in \Sigma$, $j \in \mathbb{N}\cup L$. This process too is reversible; every absolutely continuous countably additive functional ν on Σ corresponds to countably additive functionals on $\mathfrak{C}_0^*$ and $\mathfrak{C}_0$. Let me repeat that the results of 327F mean that the whole structure $(\mathfrak{C}, \bar\mu, \langle\mu_i\rangle_{i\in I})$ can be recovered from $(\mathfrak{C}_0, \bar\mu\restriction\mathfrak{C}_0, \langle\mu_i\restriction\mathfrak{C}_0\rangle_{i\in I})$ if we can get the description of $(\mathfrak{C}, \bar\mu)$ right, and that the requirements $\mu_i \ge 0$, $\mu_n \ge \mu_{n+1}$, $\mu_\kappa \ne 0$, $\sum_{i\in I}\mu_i = \bar\mu$ imposed in 333K will survive the process (327F(b-iv)).

Putting all this together, a structure $(\mathfrak{A}, \bar\mu, \mathfrak{C})$ leads, in a canonical and (up to isomorphism) reversible way, to a structure

$$(K, \mu, L, \langle\nu_\kappa\rangle_{\kappa\in\mathbb{N}\cup L})$$

such that

K and L are countable sets of infinite cardinals,

μ is a totally finite measure on $\mathbb{R}^I$, where $I = \mathbb{N} \cup K$, and its domain Σ is precisely the σ-algebra of subsets of $\mathbb{R}^I$ defined by the coordinate functionals,

for μ-almost every $x \in \mathbb{R}^I$ we have $x(i) \geq 0$ for every $i \in I$, $x(n) \geq x(n+1)$ for every $n \in \mathbb{N}$ and $\sum_{i\in I} x(i) = 1$,

for $\kappa \in K$, $\mu\{x : x(\kappa) > 0\} > 0$,

(these two sections corresponding to the requirements $\mu_i \geq 0$, $\mu_n \geq \mu_{n+1}$, $\sum_{i\in I} \mu_i = \bar{\mu}$, $\mu_\kappa \neq 0$ – see 333M(e)-(g))

for $j \in J = \mathbb{N} \cup L$, ν_j is a non-negative countably additive functional on Σ,

$\nu_n \geq \nu_{n+1}$ for every $n \in \mathbb{N}$, $\nu_\kappa \neq 0$ for every $\kappa \in L$, $\sum_{j\in J} \nu_j = \mu$.

333O Remark I do not envisage quoting the result above very often. Indeed I do not claim that its final form adds anything to the constituent results 333K, 327F and 333M. I have taken the trouble to spell it out, however, because it does not seem to me obvious that the trail is going to end quite as quickly as it does. We need to use 333K twice, but only twice. The most important use of the ideas expressed here, I suppose, is in constructing examples to strengthen our intuition for the structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ under consideration, and I hope that you will experiment in this direction.

333P At the risk of trespassing on the province of Chapter 38, I turn now to a special type of closed subalgebra, in which there is a particularly elegant alternative form for a canonical description. The first step is an important result concerning automorphisms of homogeneous probability algebras.

Proposition Let $(\mathfrak{B}, \bar{\nu})$ be a homogeneous probability algebra. Then there is a measure-preserving automorphism $\phi : \mathfrak{B} \to \mathfrak{B}$ such that

$$\lim_{n\to\infty} \bar{\nu}(c \cap \phi^n(b)) = \bar{\nu}c \cdot \bar{\nu}b$$

for all b, $c \in \mathfrak{B}$.

proof (a) The case $\mathfrak{B} = \{0, 1\}$ is trivial (ϕ is, and must be, the identity map) so we may take it that $\mathfrak{B}$ is the measure algebra of $\{0, 1\}^\kappa$ with its usual measure ν_κ, where κ is an infinite cardinal. Because $\#(\kappa \times \mathbb{Z}) = \max(\omega, \kappa) = \kappa$, there must be a bijection $\theta : \kappa \to \kappa$ such that every orbit of θ in κ is infinite (take θ to correspond to the bijection $(\xi, n) \mapsto (\xi, n + 1) : \kappa \times \mathbb{Z} \to \kappa \times \mathbb{Z}$). This induces a bijection $\hat{\theta} : \{0, 1\}^\kappa \to \{0, 1\}^\kappa$ through the formula $\hat{\theta}(x) = x\theta$ for every $x \in \{0, 1\}^\kappa$, and of course $\hat{\theta}$ is an automorphism of the measure space $(\{0, 1\}^\kappa, \nu_\kappa)$. It therefore induces a corresponding automorphism ϕ of $\mathfrak{B}$, setting $\phi E^\bullet = (\hat{\theta}^{-1}[E])^\bullet$ for every E in the domain Σ of ν_κ.

(b) Let Σ_0 be the family of subsets E of $\{0, 1\}^\kappa$ which are determined by coordinates in finite sets, that is, are expressible in the form $E = \{x : x\restriction J \in \tilde{E}\}$ for some finite set $J \subseteq \kappa$ and some $\tilde{E} \subseteq \{0, 1\}^J$; equivalently, expressible as a finite union of basic cylinder sets $\{x : x\restriction J = y\}$. Then Σ_0 is a subalgebra of Σ, so $\mathfrak{C} = \{E^\bullet : E \in \Sigma_0\}$ is a subalgebra of $\mathfrak{B}$.

(c) Now if b, $c \in \mathfrak{C}$, there is an $n \in \mathbb{N}$ such that $\bar{\nu}(c \cap \phi^m(b)) = \bar{\nu}c \cdot \bar{\nu}b$ for every $m \geq n$. **P** Express b, c as $E^\bullet$, $F^\bullet$ where $E = \{x : x\restriction J \in \tilde{E}\}$, $F = \{x : x\restriction K \in \tilde{F}\}$ and J, K are finite subsets of κ. For $\xi \in K$, all the $\theta^n(\xi)$ are distinct, so only finitely many of them can belong to J; as K is also finite, there is an n such that $\theta^m[J] \cap K = \emptyset$ for every $m \geq n$. Fix $m \geq n$. Then $\phi^m(b) = H^\bullet$ where

$$H = \{x : x\theta^m \in E\} = \{x : x\theta^m\restriction J \in \tilde{E}\} = \{x : x\restriction L \in \tilde{H}\},$$

where $L = \theta^m[J]$ and $\tilde{H} = \{z\theta^{-m} : z \in \tilde{E}\}$. So $\bar{\nu}(c \cap \phi^m(b)) = \nu(F \cap H)$. But L and K are disjoint, because $m \geq n$, so F and H must be independent (cf. 272K), and

$$\bar{\nu}(c \cap \phi^m(b)) = \nu F \cdot \nu H = \nu F \cdot \nu E = \bar{\nu}c \cdot \bar{\nu}b,$$

as claimed. **Q**

(d) Now recall that for every $E \in \Sigma$, $\epsilon > 0$ there is an $E' \in \Sigma_0$ such that $\nu(E \triangle E') \leq \epsilon$ (254Fe). So, given b, $c \in \mathfrak{B}$ and $\epsilon > 0$, we can find b', $c' \in \mathfrak{C}$ such that $\bar{\nu}(b \triangle b') \leq \epsilon$ and $\bar{\nu}(c \triangle c') \leq \epsilon$, and in this case

$$\begin{aligned}
\limsup_{n\to\infty}&|\bar\nu(c\cap\phi^n(b))-\bar\nu c\cdot\bar\nu b|\\
&\le\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|\\
&\qquad\qquad+|\bar\nu(c'\cap\phi^n(b'))-\bar\nu c'\cdot\bar\nu b'|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&=\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&\le\limsup_{n\to\infty}\bar\nu(c\triangle c')+\bar\nu(\phi^n(b)\triangle\phi^n(b'))\\
&\qquad\qquad+\bar\nu c|\bar\nu b-\bar\nu b'|+|\bar\nu c-\bar\nu c'|\bar\nu b'\\
&\le\bar\nu(c\triangle c')+\bar\nu(b\triangle b')+\bar\nu c\cdot\bar\nu(b\triangle b')+\bar\nu(c\triangle c')\bar\nu b'\le 4\epsilon.
\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{n\to\infty}\bar\nu(c\cap\phi^n(b))=\bar\nu c\cdot\bar\nu b,$$

as required.

Remark Automorphisms of this type are called **mixing** (see 372P below).

333Q Corollary Let $(\mathfrak C,\bar\mu_0)$ be a totally finite measure algebra and $(\mathfrak B,\bar\nu)$ a probability algebra which is *either* homogeneous *or* purely atomic with finitely many atoms all of the same measure. Let $(\mathfrak A,\bar\mu)$ be the localizable measure algebra free product of $(\mathfrak C,\bar\mu_0)$ and $(\mathfrak B,\bar\nu)$. Then there is a measure-preserving automorphism $\pi:\mathfrak A\to\mathfrak A$ such that

$$\{a:a\in\mathfrak A,\ \pi a=a\}=\{c\otimes 1:c\in\mathfrak C\}.$$

Remark I am following 315M in using the notation $c\otimes b$ for the intersection in $\mathfrak A$ of the canonical images of $c\in\mathfrak C$ and $b\in\mathfrak B$. By 325Dc I need not distinguish between the free product $\mathfrak C\otimes\mathfrak B$ and its image in $\mathfrak A$.

proof (a) Let me deal with the case of atomic $\mathfrak B$ first. In this case, if $\mathfrak B$ has n atoms $b_0,\dots,b_{n-1}$, let $\phi:\mathfrak B\to\mathfrak B$ be the measure-preserving homomorphism cyclically permuting these atoms, so that $\phi b_0=b_1,\dots,\phi b_{n-1}=b_0$. Because ϕ is an automorphism of $(\mathfrak B,\bar\nu)$, it induces an automorphism π of $(\mathfrak A,\bar\mu)$; any member of $\mathfrak A$ is uniquely expressible as $a=\sup_{i<n}c_i\otimes b_i$, and now $\pi a=\sup_{i<n}c_i\otimes b_{i+1}$, if we set $b_n=b_0$. So $\pi a=a$ iff $c_i=c_{i+1}$ for $i<n-1$ and $c_{n-1}=c_0$, that is, iff all the c_i are the same and $a=\sup_{i<n}c\otimes b_i=c\otimes 1$ for some $c\in\mathfrak C$.

(b) If $\mathfrak B$ is homogeneous, then take a mixing measure-preserving automorphism $\phi:\mathfrak B\to\mathfrak B$ as described in 333P. As in (a), this corresponds to an automorphism π of $\mathfrak A$, defined by saying that $\pi(c\otimes b)=c\otimes\phi(b)$ for every $c\in\mathfrak C$, $b\in\mathfrak A$. Of course $\pi(c\otimes 1)=c\otimes 1$ for every $c\in\mathfrak C$.

Now suppose that $a\in\mathfrak A$ and $\pi a=a$; I need to show that $a\in\mathfrak C_1=\{c\otimes 1:c\in\mathfrak C\}$. Take any $\epsilon>0$. We know that $\mathfrak C\otimes\mathfrak B$ is topologically dense in $\mathfrak A$ (325Dc), so there is an $a'\in\mathfrak C\otimes\mathfrak B$ such that $\bar\mu(a\triangle a')\le\epsilon^2$. Express a' as $\sup_{i\in I}c_i\otimes b_i$, where $\langle c_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak C$ (315Na). Then

$$\pi a'=\sup_{i\in I}c_i\otimes\phi(b_i),\quad \pi^n(a')=\sup_{i\in I}c_i\otimes\phi^n(b_i)\text{ for every }n\in\mathbb N.$$

So we can get a formula for

$$\begin{aligned}
\lim_{n\to\infty}\bar\mu(a'\cap\pi^n(a'))&=\lim_{n\to\infty}\bar\mu(\sup_{i\in I}c_i\otimes(b_i\cap\phi^n(b_i)))\\
&=\lim_{n\to\infty}\sum_{i\in I}\bar\mu_0c_i\cdot\bar\nu(b_i\cap\phi^n(b_i))=\sum_{i\in I}\bar\mu_0c_i(\bar\nu b_i)^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{i\in I}\bar\mu_0c_i(\bar\nu b_i)^2&=\lim_{n\to\infty}\bar\mu(a'\cap\pi^n(a'))\\
&\ge\limsup_{n\to\infty}\bar\mu(a\cap\pi^n(a))-\bar\mu(a\triangle a')-\bar\mu(\pi^n(a)\triangle\pi^n(a'))\\
&=\bar\mu a-2\bar\mu(a\triangle a')\ge\bar\mu a'-3\bar\mu(a\triangle a')\ge\sum_{i\in I}\bar\mu_0c_i\cdot\bar\nu b_i-3\epsilon^2,
\end{aligned}$$

that is,

$$\textstyle\sum_{i\in I}\bar\mu_0 c_i\cdot\bar\nu b_i\cdot(1-\bar\nu b_i)\le 3\epsilon^2.$$

But this means that, setting $J=\{i:i\in I,\,\bar\nu b_i(1-\bar\nu b_i)\ge\epsilon\}$, we must have $\sum_{i\in J}\bar\mu_0 c_i\le 3\epsilon$. Set

$$K=\{i:i\in I,\,\bar\nu b_i\ge 1-2\epsilon\},\quad L=\{i:i\in I\setminus K,\,\bar\nu b_i\le 2\epsilon\},\quad c=\sup_{i\in K}c_i.$$

Then $I\setminus(K\cup L)\subseteq J$, so

$$\begin{aligned}\bar\mu(a'\mathbin{\triangle}(c\otimes 1))&=\sum_{i\in I\setminus K}\bar\mu_0 c_i\cdot\bar\nu b_i+\sum_{i\in K}\bar\mu_0 c_i\cdot(1-\bar\nu b_i)\\&\le\sum_{i\in J}\bar\mu_0 c_i\cdot\bar\nu b_i+\sum_{i\in L}\bar\mu_0 c_i\cdot\bar\nu b_i+\sum_{i\in K}\bar\mu_0 c_i\cdot(1-\bar\nu b_i)\\&\le\sum_{i\in J}\bar\mu_0 c_i+2\epsilon\sum_{i\in L}\bar\mu_0 c_i+2\epsilon\sum_{i\in K}\bar\mu_0 c_i\le 3\epsilon+2\epsilon=5\epsilon,\end{aligned}$$

and

$$\bar\mu(a\mathbin{\triangle}(c\otimes 1))\le\epsilon^2+5\epsilon.$$

As ϵ is arbitrary, a belongs to the topological closure of $\mathfrak{C}_1$. But of course $\mathfrak{C}_1$ is a closed subalgebra of $\mathfrak{A}$ (325Dd), so must actually contain a.

As a is arbitrary, π has the required property.

333R Now for the promised special type of closed subalgebra. It will be convenient to have the following temporary notation. Write Card* for the (proper) class of all non-zero cardinals. For infinite $\kappa\in$ Card*, let $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$. For finite $n\in$ Card*, let $\mathfrak{B}_n$ be the power set of $\{0,\dots,n-1\}$ and set $\bar\nu_n b=\frac{1}{n}\#(b)$ for $b\in\mathfrak{B}_n$.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a subset of $\mathfrak{A}$. Then the following are equiveridical:

(i) there is some set G of measure-preserving automorphisms of $\mathfrak{A}$ such that

$$\mathfrak{C}=\{c:c\in\mathfrak{A},\,\pi c=c\text{ for every }\pi\in G\};$$

(ii) $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ and there is a partition of unity $\langle e_i\rangle_{i\in I}$ of $\mathfrak{C}$, where I is a countable subset of Card*, such that $\mathfrak{A}$ is isomorphic to $\prod_{i\in I}\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$, writing $\mathfrak{C}_{e_i}$ for the principal ideal of $\mathfrak{C}$ generated by e_i and endowed with $\bar\mu\restriction\mathfrak{C}_{e_i}$, and $\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$ for the localizable measure algebra free product of $\mathfrak{C}_{e_i}$ and $\mathfrak{B}_i$ – the isomorphism being one which takes any $c\in\mathfrak{C}$ to $\langle(c\cap e_i)\otimes 1\rangle_{i\in I}$, as in 333H and 333K;

(iii) there is a single measure-preserving automorphism π of $\mathfrak{A}$ such that

$$\mathfrak{C}=\{c:c\in\mathfrak{A},\,\pi c=c\}.$$

proof (a)(i)⇒(ii)(α) $\mathfrak{C}$ is a subalgebra because every $\pi\in G$ is a Boolean homomorphism, and it is order-closed because every π is order-continuous (324Kb). (Or, if you prefer, $\mathfrak{C}$ is topologically closed because every π is continuous.)

(β) Because $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$, its embedding can be described in terms of families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ as in Theorem 333K. Set $I'=K\cup\mathbb{N}$. Recall that each μ_i is defined by setting $\mu_i c=\bar\mu(c\cap a_i)$, where $\langle a_i\rangle_{i\in I'}$ is a partition of unity in $\mathfrak{A}$ (see the proofs of 333H and 333K). Now for $\kappa\in K$, a_κ is the maximal element of $\mathfrak{A}$ which is relatively Maharam-type-homogeneous over $\mathfrak{C}$ with relative Maharam type κ (part (b) of the proof of 333K). Consequently we must have $\pi a_\kappa=a_\kappa$ for any measure algebra automorphism of $(\mathfrak{A},\bar\mu)$ which leaves $\mathfrak{C}$ invariant; in particular, for every $\pi\in G$. Thus $a_\kappa\in\mathfrak{C}$ for every $\kappa\in K$.

(γ) Now consider the relatively atomic part of $\mathfrak{A}$. The elements a_n, for $n\in\mathbb{N}$, are not uniquely defined. However, the functionals μ_n and their supports $e'_n=[\![\mu_n>0]\!]$ are uniquely defined from the structure $(\mathfrak{A},\bar\mu,\mathfrak{C})$ and therefore invariant under G. Observe also that because $\sup_{n\in\mathbb{N}}a_n=1\setminus\sup_{\kappa\in K}a_\kappa$ belongs to $\mathfrak{C}$, and $e'_n=\inf\{c:c\in\mathfrak{C},\,c\supseteq a_n\}$, while $e'_n\supseteq e'_{n+1}$ for every n, we must have $e'_0=\sup_{n\in\mathbb{N}}a_n$.

Let G^* be the set of all those automorphisms π of the measure algebra $(\mathfrak{A},\bar\mu)$ such that $\pi c=c$ for every $c\in\mathfrak{C}$. Then of course G^* is a group including G. Now $\sup_{\pi\in G^*}\pi a_n$ must be invariant under every member of G^*, so belongs to $\mathfrak{C}$; it includes a_n and is included in any member of $\mathfrak{C}$ including a_n, so must be e'_n.

(δ) I claim now that if $n\in\mathbb{N}$ then $e'_n\cap[\![\mu_0>\mu_n]\!]=0$. **P?** Otherwise, set $c=[\![\mu_0>\mu_n]\!]\cap e'_n$. Then $\mu_0 c>0$ so $c\cap a_0\ne 0$. By the last remark in (γ), there is a $\pi\in G^*$ such that $c\cap a_0\cap\pi a_n\ne 0$. Now there is a $c'\in\mathfrak{C}$ such that $c\cap a_0\cap\pi a_n=c'\cap a_0$, and of course we may suppose that $c'\subseteq c$. But this means that

$$\pi(c'\cap a_n)=c'\cap\pi a_n\supseteq c'\cap a_0\cap\pi a_n=c'\cap a_0,$$

so that

$$\mu_n c'=\bar\mu(c'\cap a_n)=\bar\mu\pi(c'\cap a_n)\ge\bar\mu(c'\cap a_0)=\mu_0 c',$$

which is impossible, because $0\ne c'\subseteq[\![\mu_0>\mu_n]\!]$. **XQ**

So $\mu_0 c\le\mu_n c$ whenever $c\in\mathfrak{C}$ and $c\subseteq e'_n$. Because the μ_k have been chosen to be a non-increasing sequence, we must have $\mu_0 c=\mu_1 c=\ldots=\mu_n c$ for every $c\subseteq e'_n$.

(ϵ) Recalling now that $\sum_{i\in I'}\mu_i=\bar\mu\restriction\mathfrak{C}$, we see that $\mu_0 c\le\frac{1}{n+1}\bar\mu c$ for every $c\subseteq e'_n$. It follows that if $e^*=\inf_{n\in\mathbb{N}}e'_n$, $\mu_0 e^*=0$; but this must mean that $e^*=0$. Consequently, setting $I=I'\setminus\{0\}$, $e_n=e'_{n-1}\setminus e'_n$ for $n\ge 1$, $e_\kappa=a_\kappa$ for $\kappa\in K$, we find that $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{C}$.

Moreover, for $n\ge 1$ and $c\subseteq e_n$, we must have

$$\bar\mu c=\sum_{i\in I'}\mu_i c=\sum_{k<n}\mu_k c=n\mu_0 c,$$

so that $\mu_k c=\frac{1}{n}\mu c$ for every $k<n$. But this means that we have a measure-preserving homomorphism $\psi_n:\mathfrak{A}_{e_n}\to\mathfrak{C}_{e_n}\widehat{\otimes}\mathfrak{B}_n$ given by setting

$$\psi_n(a_k\cap c)=c\otimes\{k\}$$

whenever $c\in\mathfrak{C}_{e_n}$ and $k<n$; this is well-defined because $e_n\subseteq e'_k$, so that $a_k\cap c\ne a_k\cap c'$ if c, c' are distinct members of $\mathfrak{C}_{e_n}$, and it is measure-preserving because

$$\bar\mu(a_k\cap c)=\mu_k c=\frac{1}{n}\bar\mu c=\bar\mu c\cdot\bar\nu_n\{k\}$$

for all relevant k and c. Because $\mathfrak{B}_n$ is finite, ψ_n is surjective.

(ζ) Just as in 333H, we now see that because $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ as well as in $\mathfrak{C}$, we can identify $\mathfrak{A}$ with $\prod_{i\in I}\mathfrak{A}_{e_i}$ and therefore with $\prod_{i\in I}\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$.

(b)(ii)$\Rightarrow$(iii) Let us work in $\mathfrak{D}=\prod_{i\in I}\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$, writing $\psi:\mathfrak{A}\to\mathfrak{D}$ for the canonical map. For each $i\in I$, we have a measure-preserving automorphism π_i of $\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$ with fixed-point subalgebra $\{c\otimes 1:c\in\mathfrak{C}_{e_i}\}$ (333Q). For $d=\langle d_i\rangle_{i\in I}\in\mathfrak{D}$, set

$$\pi d=\langle\pi_i d_i\rangle_{i\in I}.$$

Then π is a measure-preserving automorphism because every π_i is. If $\pi d=d$, then for every $i\in I$ there must be a $c_i\subseteq e_i$ such that $d_i=c_i\otimes 1$. But this means that $d=\psi c$, where $c=\sup_{i\in I}c_i\in\mathfrak{C}$. Thus the fixed-point subalgebra of π is just $\psi[\mathfrak{C}]$. Transferring the structure $(\mathfrak{D},\psi[\mathfrak{C}],\pi)$ back to $\mathfrak{A}$, we obtain a measure-preserving automorphism $\psi^{-1}\pi\psi$ of $\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, as required.

(c)(iii)$\Rightarrow$(i) is trivial.

333X Basic exercises (a) Show that, in the proof of 333H, $c_i=\operatorname{upr}(a_i,\mathfrak{C})$ (definition: 314V) for every $i\in I$.

(b) In the context of Lemma 333M, show that we have a one-to-one correspondence between atoms c of $\mathfrak{C}_0$ and points x of non-zero mass in $\mathbb{R}^I$, given by the formula $\pi\{x\}^\bullet=c$.

(c) Let $(\mathfrak{A},\bar\mu)$ be totally finite measure algebra and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\pi\phi\in G$ for all π, $\phi\in G$. (i) Show that $a\subseteq\sup_{\pi\in G}\pi a$ for every $a\in\mathfrak{A}$. (*Hint*: if $\pi c\subseteq c$, where $\pi\in G$ and $c\in\mathfrak{A}$, then $\pi c=c$; apply this to $c=\sup_{\pi\in G}\pi a$.) (ii) Set $\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi c=c$ for every $\pi\in G\}$. Show that $\sup_{\pi\in G}\pi a=\operatorname{upr}(a,\mathfrak{C})$ for every $a\in\mathfrak{A}$.

333Y Further exercises (a) Show that when $I=\mathbb{N}$ the algebra Σ of subsets of $\mathbb{R}^I$, used in 333M, is precisely the Borel σ-algebra as described in 271Ya.

(b) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and $\mathfrak{B}$, $\mathfrak{C}$ two closed subalgebras of $\mathfrak{A}$ with $\mathfrak{C}\subseteq\mathfrak{B}$. Show that $\tau_{\mathfrak{C}}(\mathfrak{B})\le\tau_{\mathfrak{C}}(\mathfrak{A})$. (*Hint*: use 333K and the ideas of 332T.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Show that $\mathfrak{A}$ is homogeneous iff there is a measure-preserving automorphism of $\mathfrak{A}$ which is mixing in the sense of 333P.

(d) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Set $\mathfrak{C}=\{c:c\in\mathfrak{A},\,\pi c=c$ for every $\pi\in G\}$. Show that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ of the type described in 333R. (*Hint*: in the language of part (a) of the proof of 333R, show that $\sup_{n\in\mathbb{N}}a_n$ still belongs to $\mathfrak{C}$.)

333 Notes and comments I have done my best, in the first part of this section, to follow the lines already laid out in §§331-332, using what should (once you have seen them) be natural generalizations of the former definitions and arguments. Thus the Maharam type $\tau(\mathfrak{A})$ of an algebra is just the relative Maharam type $\tau_{\{0,1\}}(\mathfrak{A})$, and $\mathfrak{A}$ is Maharam-type-homogeneous iff it is relatively Maharam-type-homogeneous over $\{0,1\}$. To help you trace through the correspondence, I list the code numbers: 331Fa→333Aa, 331Fb→333Ac, 331Hc→333Ba, 331Hd→333Bd, 332A→333Bb, 331I→333Cb, 331K→333E, 331L→333F(ii), 332B→333H, 332J→333K. 333D overlaps with 332P. Throughout, the principle is the same: everything can be built up from products and free products.

Theorem 333Ca does not generalize any explicitly stated result, but overlaps with Proposition 332P. In the proof of 333E I have used a new idea; the same method would of course have worked just as well for 331K, but I thought it worth while to give an example of an alternative technique, which displays a different facet of homogeneous algebras, and a different way in which the algebraic, topological and metric properties of homogeneous algebras interact. The argument of 331K-331L relies (without using the term) on the fact that measure algebras of Maharam type κ have topological density at most $\max(\kappa,\omega)$ (see 331Ye), while the the argument of 333E uses the rather more sophisticated concept of stochastic independence.

Corollary 333F(i) is cruder than the more complicated results which follow, but I think that it is invaluable as a first step in forming a picture of the possible embeddings of a given (totally finite) measure algebra $\mathfrak{C}$ in a larger algebra $\mathfrak{A}$. If we think of $\mathfrak{C}$ as the measure algebra of a measure space (X,Σ,μ), then we can be sure that $\mathfrak{A}$ is representable as a closed subalgebra of the measure algebra of $X\times\{0,1\}^\kappa$ for some κ, that is, the measure algebra of $\lambda\restriction\mathrm{T}$ where λ is the product measure on $X\times\{0,1\}^\kappa$ and T is some σ-subalgebra of the domain of λ; the embedding of $\mathfrak{C}$ in $\mathfrak{A}$ being defined by the formula $E^\bullet\to(E\times\{0,1\}^\kappa)^\bullet$ for $E\in\Sigma$ (325A, 325D). Identifying, in our imaginations, both X and $\{0,1\}^\kappa$ with the unit interval, we can try to picture everything in the unit square – and these pictures, although necessarily inadequate for algebras of uncountable Maharam type, already give a great deal of scope for invention.

I said above that everything can be constructed from simple products and free products, judiciously combined; of course some further ideas must be mixed with these. The difference between 332B and 333H, for instance, is partly in the need for the functionals μ_i in the latter, whereas in the former the decomposition involves only principal ideals with the induced measures. Because the μ_i are completely additive, they all have supports c_i (326Xi) and we get measure algebras $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ to use in the products. (I note that the c_i can be obtained directly from the a_i, without mentioning the functionals μ_i, by the process of 333Xa.) The fact that the c_i can overlap means that the 'relatively atomic' part of the larger algebra $\mathfrak{A}$ needs a much more careful description than before; this is the burden of 333J, and also the principal complication in the proof of 333R. The 'relatively atomless' part is (comparatively) straightforward, since we can use the same kind of amalgamation as before (part (c-i) of the proof of 332J, part (b) of the proof of 333K), simplified because I am no longer seeking to deal with algebras of infinite magnitude.

Theorem 333K gives a canonical form for superalgebras of a given totally finite measure algebra $(\mathfrak{C},\bar\mu)$, taking the structure $(\mathfrak{C},\bar\mu)$ itself for granted. I hope it is clear that while the μ_i amd e_i and the algebra $\widehat{\mathfrak{A}}=\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ and the embedding of $\mathfrak{C}$ in $\widehat{\mathfrak{A}}$ are uniquely defined, the rest of the isomorphism $\pi:\mathfrak{A}\to\widehat{\mathfrak{A}}$ generally is not. Even when the a_κ are uniquely defined the isomorphisms between $\mathfrak{A}_{a_\kappa}$ and $\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ depend on choosing generating families in the $\mathfrak{A}_{a_\kappa}$; see the proof of 333Cb.

To understand the possible structures $(\mathfrak{C}, \langle\mu_i\rangle_{i\in I})$ of that theorem, we have to go rather deeper. The route I have chosen is to pick out the subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ determined by $\langle\mu_i\rangle_{i\in I}$ and identify it with the measure algebra of a particular measure on $\mathbb{R}^I$. Perhaps I should apologise for not stating explicitly in the course of Lemma 333M that the measures μ here are 'Borel measures' (see 333Ya); but I am afraid of opening a door to an invasion of ideas which belong in Volume 4. Besides, if I were going to do anything more with these measures than observe that they are uniquely defined by the construction proposed, I would complete them and call them Radon measures. In order to validate this approach, I must show that the μ_i can be recovered from their restrictions to $\mathfrak{C}_0$; this is 333Md, and is the motive for the discussion of 'standard extensions' in §327. No doubt there are other ways of doing it. One temptation which I felt it right to resist was the idea of decomposing $\mathfrak{C}$ into its homogeneous principal ideals; this seemed merely an additional complication. Of course the subalgebra $\mathfrak{C}_0$ has countable Maharam type (being τ-generated by the elements e_{iq}, for $i \in I$ and $q \in \mathbb{Q}$, of 333M), so that its decomposition is relatively simple, being just a matter of picking out the atoms (333Xb).

In 333P I find myself presenting an important fact about homogeneous measure algebras, rather out of context; but I hope that it will help you to believe that I have by no means finished with the insights which Maharam's theorem provides. I give it here for the sake of 333R. For the moment, I invite you to think of 333R as just a demonstration of the power of the techniques I have developed in this chapter, and of the kind of simplification (in the equivalence of conditions (i) and (iii)) which seems to arise repeatedly in the theory of measure algebras. But you will see that the first step to understanding any automorphism will be a description of its fixed-point subalgebra, so 333R will also be basic to the theory of automorphisms of measure algebras. I note that the hypothesis (i) of 333R can in fact be relaxed (333Yd), but this seems to need an extra idea.

334 Products

I devote a short section to results on the Maharam classification of the measure algebras of product measures, or, if you prefer, of the free products of measure algebras. The complete classification, even for probability algebras, is complex (334Xe, 334Ya), so I content myself with a handful of the most useful results. I start with upper bounds for the Maharam type of the c.l.d. product of two measure spaces (334A) and the localizable measure algebra free product of two semi-finite measure algebras (334B), and go on to the corresponding results for products of probability spaces and algebras (334C-334D). Finally, I show that any infinite power of a probability space is Maharam-type-homogeneous (334E).

334A Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$, $\mathfrak{B}$. Let λ be the c.l.d. product measure on $X \times Y$, and $(\mathfrak{C}, \bar{\lambda})$ its measure algebra. Then the Maharam type $\tau(\mathfrak{C})$ is at most $\max(\omega, \tau(\mathfrak{A}), \tau(\mathfrak{B}))$.

proof Recall from 325A that we have order-continuous Boolean homomorphisms $\varepsilon_1 : \mathfrak{A} \to \mathfrak{C}$ and $\varepsilon_2 : \mathfrak{B} \to \mathfrak{C}$ defined by setting $\varepsilon_1 E^\bullet = (E \times Y)^\bullet$, $\varepsilon_2 F^\bullet = (X \times F)^\bullet$ for $E \in \Sigma$, $F \in \mathrm{T}$. Let $A \subseteq \mathfrak{A}$, $B \subseteq \mathfrak{B}$ be τ-generating sets with $\#(A) = \tau(\mathfrak{A})$, $\#(B) = \tau(\mathfrak{B})$; set $C = \varepsilon_1[A] \cup \varepsilon_2[B]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ε_1 is order-continuous, $\varepsilon_1^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}$, and it includes A, so must be the whole of $\mathfrak{A}$; thus $\varepsilon_1 a \in \mathfrak{C}_1$ for every $a \in \mathfrak{A}$. Similarly, $\varepsilon_2 b \in \mathfrak{C}_1$ for every $b \in \mathfrak{B}$.

This means that

$$\Lambda_1 = \{W : W \in \Lambda,\ W^\bullet \in \mathfrak{C}_1\}$$

contains $E \times F$ for every $E \in \Sigma$, $F \in \mathrm{T}$. Also Λ_1 is a σ-algebra of subsets of $X \times Y$, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1 \supseteq \Sigma\widehat{\otimes}\mathrm{T}$ (definition: 251D). But this means that if $W \in \Lambda$ there is a $V \in \Lambda_1$ such that $V \subseteq W$ and $\lambda V = \lambda W$ (251Ib); that is, if $c \in \mathfrak{C}$ there is a $d \in \mathfrak{C}_1$ such that $d \subseteq c$ and $\bar{\lambda} d = \bar{\lambda} c$. Thus $\mathfrak{C}_1$ is order-dense in $\mathfrak{C}$, and

$$c = \sup\{d : d \in \mathfrak{C}_1,\ d \subseteq c\} \in \mathfrak{C}_1$$

for every $c \in \mathfrak{C}$. So $\mathfrak{C}_1 = \mathfrak{C}$ and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\mathfrak{C}) \le \#(C) \le \max(\omega, \tau(\mathfrak{A}), \tau(\mathfrak{B})).$$

334B Corollary Let $(\mathfrak{A}_1, \bar{\mu}_1)$, $(\mathfrak{A}_2, \bar{\mu}_2)$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C}, \bar{\lambda})$ (325E). Then $\tau(\mathfrak{C}) \le \max(\omega, \tau(\mathfrak{A}), \tau(\mathfrak{B}))$.

proof By the construction of part (a) of the proof of 325D, $\mathfrak{C}$ can be regarded as the measure algebra of the c.l.d. product of the Stone representations of $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$; so the result follows at once from 334A.

334C Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Λ, λ). Let $\mathfrak{A}_i$, $\mathfrak{C}$ be the corresponding measure algebras. Then

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mathfrak{A}_i)).$$

proof Recall from 325I that we have order-continuous Boolean homomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{C}$ corresponding to the inverse-measure-preserving maps $x \mapsto \pi_i(x) = x(i) : X \to X_i$. For each $i \in I$, let $A_i \subseteq \mathfrak{A}_i$ be a set of cardinal $\tau(\mathfrak{A}_i)$ which τ-generates $\mathfrak{A}_i$. Set $C = \bigcup_{i\in I} \psi_i[A_i]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ψ_i is order-continuous, $\psi_i^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}_i$, and it includes A_i, so must be the whole of $\mathfrak{A}_i$; thus $\psi_i a \in \mathfrak{C}_1$ for every $a \in \mathfrak{A}_i$, $i \in I$.

This means that

$$\Lambda_1 = \{W : W \in \Lambda,\ W^\bullet \in \mathfrak{C}_1\}$$

contains $\pi_i^{-1}[E]$ for every $E \in \Sigma_i$, $i \in I$. Also Λ_1 is a σ-algebra of subsets of X, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1 \supseteq \widehat{\bigotimes}_{i\in I}\Sigma_i$. But this means that if $W \in \Lambda$ there is a $V \in \Lambda_1$ such that $V^\bullet = W^\bullet$ (254Ff), that is, that $\mathfrak{C}_1 = \mathfrak{C}$, and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\mathfrak{C}) \le \#(C) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mathfrak{A}_i)).$$

334D Corollary Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar{\lambda})$. Then

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mathfrak{A}_i)).$$

proof See 325J-325K.

334E I come now to the question of when a probability algebra free product is homogeneous. I give just one result in detail, leaving others to the exercises.

Theorem Let (X, Σ, μ) be a probability space, with measure algebra $\mathfrak{A}$, and I an infinite set; let $\mathfrak{C}$ be the measure algebra of the product measure on X^I. Then $\mathfrak{C}$ is homogeneous. If $\tau(\mathfrak{A}) = 0$ then $\tau(\mathfrak{C}) = 0$; otherwise $\tau(\mathfrak{C}) = \max(\tau(\mathfrak{A}), \#(I))$.

proof (a) As usual, write $\bar{\mu}$ for the measure of $\mathfrak{A}$; let λ be the measure on X^I, $\bar{\lambda}$ the measure on $\mathfrak{C}$. If $\tau(\mathfrak{A}) = 0$, that is, $\mathfrak{A} = \{0, 1\}$, then $\mathfrak{C} = \{0, 1\}$ (by 254Fe, or 325Jc, or otherwise), and in this case is surely homogeneous, with $\tau(\mathfrak{C}) = 0$. So let us suppose hencforth that $\tau(\mathfrak{A}) > 0$. We have

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \tau(\mathfrak{A})) = \max(\#(I), \tau(\mathfrak{A})),$$

by 334C.

(b) Fix on $b \in \mathfrak{A} \setminus \{0, 1\}$. For each $i \in I$, let $\psi_i : \mathfrak{A} \to \mathfrak{C}$ be the canonical measure-preserving homomorphism corresponding to the inverse-measure-preserving function $x \mapsto x(i) : X^I \to X$. For each $n \in \mathbb{N}$, there is a set $J \subseteq I$ of cardinal n, and now the finite subalgebra of $\mathfrak{C}$ generated by $\{\psi_i b : i \in J\}$ has atoms of measure at most δ^n, where $\delta = \max(\bar{\mu}b, 1 - \bar{\mu}b) < 1$. Consequently $\mathfrak{C}$ can have no atom of measure greater than δ^n, for any n, and is therefore atomless.

(c) Because I is infinite, there is a bijection between I and $I \times \mathbb{N}$; that is, there is a partition $\langle J_i \rangle_{i \in I}$ of I into countably infinite sets. Now (X^I, λ) can be identified with the product of the family $\langle (X^{J_i}, \lambda_i) \rangle_{i \in I}$, where λ_i is the product measure on X^{J_i} (254N). By (b), every λ_i is atomless, so there are sets $E_i \subseteq X^{J_i}$ of measure $\frac{1}{2}$. The sets $E'_i = \{x : x \restriction J_i \in E_i\}$ are now stochastically independent in X. Accordingly we have an inverse-measure-preserving function $f : X \to \{0,1\}^I$, endowed with its usual measure ν_I, defined by setting $f(x)(i) = 1$ if $x \in E'_i$, 0 otherwise, and therefore a measure-preserving Boolean homomorphism $\pi : \mathfrak{B}_I \to \mathfrak{C}$, writing $\mathfrak{B}_I$ for the measure algebra of ν_I.

Now if $c \in \mathfrak{C} \setminus \{0\}$ and $\mathfrak{C}_c$ is the corresponding ideal, $b \mapsto c \cap \pi b : \mathfrak{B}_I \to \mathfrak{C}_c$ is an order-continuous Boolean homomorphism. It follows that $\tau(\mathfrak{C}_c) \ge \#(I)$ (331J).

(d) Again take any non-zero $c \in \mathfrak{C}$. For each $i \in I$, set $a_i = \inf\{a : \psi_i a \supseteq c\}$. Writing $\mathfrak{A}_{a_i}$ for the corresponding principal ideal of $\mathfrak{A}$, we have an order-continuous Boolean homomorphism $\psi'_i : \mathfrak{A}_{a_i} \to \mathfrak{C}_c$, given by the formula

$$\psi'_i a = \psi_i a \cap c \text{ for every } a \in \mathfrak{A}_{a_i}.$$

Now ψ'_i is injective, so is a Boolean isomorphism between $\mathfrak{A}_{a_i}$ and its image $\psi'_i[\mathfrak{A}_{a_i}]$, which by 314F(a-i) is a closed subalgebra of $\mathfrak{C}_c$. So

$$\tau(\mathfrak{A}_{a_i}) = \tau(\psi'_i[\mathfrak{A}_{a_i}]) \le \tau(\mathfrak{C}_c)$$

by 332Tb.

For any finite $J \subseteq I$,

$$0 < \bar{\lambda} c \le \bar{\lambda}(\inf_{i \in J} \psi_i a_i) = \prod_{i \in J} \bar{\lambda}(\psi_i a_i) = \prod_{i \in J} \bar{\mu} a_i.$$

So for any $\delta < 1$, $\{i : \bar{\mu} a_i \le \delta\}$ must be finite, and $\sup_{i \in I} \bar{\mu} a_i = 1$. In particular, $\sup_{i \in I} a_i = 1$ in $\mathfrak{A}$. But this means that if ζ is any cardinal such that the Maharam-type-ζ component e_ζ of $\mathfrak{A}$ is non-zero, then $e_\zeta \cap a_i \ne 0$ for some $i \in I$, so that

$$\zeta \le \tau(\mathfrak{A}_{e_\zeta \cap a_i}) \le \tau(\mathfrak{A}_{a_i}) \le \tau(\mathfrak{C}_c).$$

As ζ is arbitrary, $\tau(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{C}_c))$ (332S).

(e) Putting (a)-(d) together, we have

$$\max(\tau(\mathfrak{A}), \#(I)) \le \max(\omega, \tau(\mathfrak{C}_c)) = \tau(\mathfrak{C}_c) \le \tau(\mathfrak{C}) \le \max(\tau(\mathfrak{A}), \#(I))$$

for every non-zero $c \in \mathfrak{C}$; so $\mathfrak{C}$ is homogeneous.

334X Basic exercises (a) Let (X, Σ, μ) and (Y, T, ν) be complete locally determined measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Let $\mathfrak{A}$, $\mathfrak{C}$ be the measure algebras of μ, λ respectively. Show that if $\nu Y > 0$ then $\tau(\mathfrak{A}) \le \tau(\mathfrak{C})$.

(b) Let X be a set and μ and ν two totally finite measures on X with the same domain Σ; then $\lambda = \mu + \nu$ is also a totally finite measure. Write $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ and $(\mathfrak{C}, \bar{\lambda})$ for the three measure algebras. Show that (i) there is a surjective order-continuous Boolean homomorphism from $\mathfrak{C}$ onto $\mathfrak{A}$; (ii) $\mathfrak{C}$ is isomorphic to a closed subalgebra of the localizable measure algebra free product of $\mathfrak{A}$ and $\mathfrak{B}$; (iii) $\tau(\mathfrak{A}) \le \tau(\mathfrak{C}) \le \max(\omega, \tau(\mathfrak{B}), \tau(\mathfrak{C}))$.

>(c) Let $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar{\lambda})$. Show that $\tau(\mathfrak{A}_i) \le \tau(\mathfrak{C})$ for every i, and that

$$\#(\{i : i \in I,\ \tau(\mathfrak{A}_i) > 0\}) \le \tau(\mathfrak{C}).$$

(d) Let X be a set and $\langle \mu_n \rangle_{n \in \mathbb{N}}$ a sequence of totally finite measures on X all with the same domain Σ. Let $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{n=0}^\infty \alpha_n \mu_n X < \infty$, and set $\lambda E = \sum_{n=0}^\infty \alpha_n \mu_n E$ for $E \in \Sigma$. Check that λ is a measure. Write $(\mathfrak{A}_n, \bar{\mu}_n)$ for the measure algebra of μ_n and $(\mathfrak{C}, \bar{\lambda})$ for the measure algebra of λ. Show that $\tau(\mathfrak{C}) \le \max(\omega, \sup_{n \in \mathbb{N}} \tau(\mathfrak{A}_n))$.

(e) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, and λ the product measure on $X \times Y$. Show that λ is Maharam-type-homogeneous iff one of μ, ν is Maharam-type-homogeneous with Maharam type at least as great as the Maharam type of the other.

(f) Show that the product of any family of Maharam-type-homogeneous probability spaces is again Maharam-type-homogeneous.

>(g) Let (X, Σ, μ) be a probability space of Maharam type κ, and I any set of cardinal at least $\max(\omega, \kappa)$. Show that the product measure on $X \times \{0,1\}^I$ is Maharam-type-homogeneous, with Maharam type $\#(I)$.

334Y Further exercises (a) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be an infinite family of probability spaces, with product (X, Λ, λ). Let κ_i be the Maharam type of μ_i for each i; set $\kappa = \max(\#(I), \sup_{i \in I} \kappa_i)$. Show that *either* λ is Maharam-type-homogeneous, with Maharam type κ, *or* there are $\kappa' < \kappa$, $X'_i \in \Sigma_i$ such that $\sum_{i \in I} \mu_i(X_i \setminus X'_i) < \infty$ and the Maharam type of the subspace measure on X'_i is at most κ' for every $i \in I$ and either $\kappa' = 0$ or $\#(I) < \kappa$.

334 Notes and comments The results above are all very natural ones; I have spelt them out partly for completeness and partly for the sake of an application in §346 below. But note the second alternative in 334Ya; it is possible, even in an infinite product, for a kernel of relatively small Maharam type to be preserved.

Chapter 34

The lifting theorem

Whenever we have a surjective homomorphism $\phi : P \to Q$, where P and Q are mathematical structures, we can ask whether there is a right inverse of ϕ, a homomorphism $\psi : Q \to P$ such that $\phi\psi$ is the identity on Q. As a general rule, we expect a negative answer; those categories in which epimorphisms always have right inverses (e.g., the category of linear spaces) are rather special, and elsewhere the phenomenon is relatively rare and almost always important. So it is notable that we have a case of this at the very heart of the theory of measure algebras: for any complete probability space (X, Σ, μ) (in fact, for any complete strictly localizable space of non-zero measure) the canonical homomorphism from Σ to the measure algebra of μ has a right inverse (341K). This is the von Neumann-Maharam lifting theorem. Its proof, together with some essentially elementary remarks, takes up the whole of of §341.

As a first application of the theorem (there will be others in Volume 4) I apply it to one of the central problems of measure theory: under what circumstances will a homomorphism between measure algebras be representable by a function between measure spaces? Variations on this question are addressed in §343. For a reasonably large proportion of the measure spaces arising naturally in analysis, homomorphisms are representable (343B). New difficulties arise if we ask for isomorphisms of measure algebras to be representable by isomorphisms of measure spaces, and here we have to work rather hard for rather narrowly applicable results; but in the case of Lebesgue measure and its closest relatives, a good deal can be done, as in 344H and 344I.

Returning to liftings, there are many difficult questions concerning the extent to which liftings can be required to have special properties, reflecting the natural symmetries of the standard measure spaces. For instance, Lebesgue measure is translation-invariant; if liftings were in any sense canonical, they could be expected to be automatically translation-invariant in some sense. It seems sure that there is no canonical lifting for Lebesgue measure – all constructions of liftings involve radical use of the axiom of choice – but even so we do have many translation-invariant liftings (§345). We have less luck with product spaces; here the construction of liftings which respect the product structure is fraught with difficulties. I give the currently known results in §346.

341 The lifting theorem

I embark directly on the principal theorem of this chapter (341K, 'every non-trivial complete strictly localizable measure space has a lifting'), using the minimum of advance preparation. 341A-341B give the definition of 'lifting'; the main argument is in 341F-341K, using the concept of 'lower density' (341C-341E) and a theorem on martingales from §275. In 341P I describe an alternative way of thinking about liftings in terms of the Stone space of the measure algebra.

341A Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lifting** of $\mathfrak{A}$ (or of (X, Σ, μ), or of μ) I shall mean

either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$

or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that (i) $\phi E = \emptyset$ whenever $\mu E = 0$ (ii) $\mu(E \triangle \phi E) = 0$ for every $E \in \Sigma$.

341B Remarks (a) I trust that the ambiguities permitted by this terminology will not cause any confusion. The point is that there is a natural one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and liftings $\phi : \Sigma \to \Sigma$ given by the formula

$$\theta E^\bullet = \phi E \text{ for every } E \in \Sigma.$$

P (i) Given a lifting $\theta : \mathfrak{A} \to \Sigma$, the formula defines a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that

$$\phi\emptyset = \theta 0 = \emptyset, \quad (E \triangle \phi E)^\bullet = E^\bullet \triangle (\theta E^\bullet)^\bullet = 0 \ \forall \ E \in \Sigma,$$

so that ϕ is a lifting. (ii) Given a lifting $\phi : \Sigma \to \Sigma$, the kernel of ϕ includes $\{E : \mu E = 0\}$, so there is a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $\theta E^\bullet = \phi E$ for every E (3A2G), and now

$$(\theta E^\bullet)^\bullet = (\phi E)^\bullet = E^\bullet$$

for every $E \in \Sigma$, so θ is a lifting. **Q**

I suppose that the word 'lifting' applies most naturally to functions from $\mathfrak{A}$ to Σ; but for applications in measure theory the other type of lifting is used at least equally often.

(b) Note that if $\phi : \Sigma \to \Sigma$ is a lifting then $\phi^2 = \phi$. **P** For any $E \in \Sigma$,

$$\phi^2 E \triangle \phi E = \phi(E \triangle \phi E) = \emptyset. \text{ **Q**}$$

If ϕ is associated with $\theta : \mathfrak{A} \to \Sigma$, then $\phi\theta a = \theta a$ for every $a \in \mathfrak{A}$. **P** $\phi\theta a = \theta((\theta a)^\bullet) = \theta a$. **Q**

341C Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lower density** of $\mathfrak{A}$ (or of (X, Σ, μ), or of μ) I shall mean

either a function $\underline{\theta} : \mathfrak{A} \to \Sigma$ such that (i) $(\underline{\theta} a)^\bullet = a$ for every $a \in \mathfrak{A}$ (ii) $\underline{\theta} 0 = \emptyset$ (iii) $\underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b$ for all a, $b \in \mathfrak{A}$

or a function $\underline{\phi} : \Sigma \to \Sigma$ such that (i) $\underline{\phi} E = \underline{\phi} F$ whenever E, $F \in \Sigma$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi} E) = 0$ for every $E \in \Sigma$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi} E \cap \underline{\phi} F$ for all E, $F \in \Sigma$.

341D Remarks (a) As in 341B, there is a natural one-to-one correspondence between lower densities $\underline{\theta} : \mathfrak{A} \to \Sigma$ and lower densities $\underline{\phi} : \Sigma \to \Sigma$ given by the formula

$$\underline{\theta} E^\bullet = \underline{\phi} E \text{ for every } E \in \Sigma.$$

(For the requirement $\underline{\phi} E = \underline{\phi} F$ whenever $E^\bullet = F^\bullet$ in $\mathfrak{A}$ means that every $\underline{\phi}$ corresponds to a function $\underline{\theta}$, and the other clauses match each other directly.)

(b) As before, if $\underline{\phi} : \Sigma \to \Sigma$ is a lower density then $\underline{\phi}^2 = \underline{\phi}$. If $\underline{\phi}$ is associated with $\underline{\theta} : \mathfrak{A} \to \Sigma$, then $\underline{\phi}\underline{\theta} = \underline{\theta}$.

(c) It will be convenient, in the course of the proofs of 341F-341H below, to have the following concept available. If (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, a **partial lower density** of $\mathfrak{A}$ is a function $\underline{\theta} : \mathfrak{B} \to \Sigma$ such that (i) the domain $\mathfrak{B}$ of $\underline{\theta}$ is a subalgebra of $\mathfrak{A}$ (ii) $(\underline{\theta} b)^\bullet = b$ for every $b \in \mathfrak{B}$ (iii) $\underline{\theta} 0 = \emptyset$ (iv) $\underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b$ for all a, $b \in \mathfrak{B}$.

Similarly, if T is a subalgebra of Σ, a function $\underline{\phi} : \mathrm{T} \to \Sigma$ is a **partial lower density** if (i) $\underline{\phi} E = \underline{\phi} F$ whenever E, $F \in \mathrm{T}$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi} E) = 0$ for every $E \in \mathrm{T}$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi} E \cap \underline{\phi} F$ for all E, $F \in \mathrm{T}$.

(d) Note that lower densities and partial lower densities are order-preserving; if $a \subseteq b$ in $\mathfrak{A}$, and $\underline{\theta}$ is a lower density of $\mathfrak{A}$, then

$$\underline{\theta} a = \underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b \subseteq \underline{\theta} b.$$

(e) Of course a Boolean homomorphism from $\mathfrak{A}$ to Σ, or from Σ to itself, is a lifting iff it is a lower density.

341E Example Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r \ge 1$, and Σ its domain. For $E \in \Sigma$ set

$$\underline{\phi} E = \{x : x \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}.$$

(Here $B(x, \delta)$ is the closed ball with centre x and radius δ.) Then $\underline{\phi}$ is a lower density for μ; we may call it **lower Lebesgue density**. **P** (You may prefer at first to suppose that $r = 1$, so that $B(x, \delta) = [x - \delta, x + \delta]$ and $\mu B(x, \delta) = 2\delta$.) By 261Db (or 223B, for the one-dimensional case) $\underline{\phi} E \triangle E$ is negligible for every E; in particular, $\underline{\phi} E \in \Sigma$ for every $E \in \Sigma$. If $E \triangle F$ is negligible, then $\mu(E \cap B(x, \delta)) = \mu(F \cap B(x, \delta))$ for every x and δ, so $\underline{\phi} E = \underline{\phi} F$. If $E \subseteq F$, then $\mu(E \cap B(x, \delta)) \le \mu(F \cap B(x, \delta))$ for every x, δ, so $\underline{\phi} E \subseteq \underline{\phi} F$; consequently $\underline{\phi}(E \cap F) \subseteq \underline{\phi} E \cap \underline{\phi} F$ for all E, $F \in \Sigma$. If E, $F \in \Sigma$ and $x \in \underline{\phi} E \cap \underline{\phi} F$, then

$$\mu(E \cap F \cap B(x,\delta)) = \mu(E \cap B(x,\delta)) + \mu(F \cap B(x,\delta)) - \mu((E \cup F) \cap B(x,\delta))$$
$$\geq \mu(E \cap B(x,\delta)) + \mu(F \cap B(x,\delta)) - \mu(B(x,\delta))$$

for every δ, so

$$\frac{\mu(E\cap F\cap B(x,\delta))}{\mu B(x,\delta)} \geq \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} + \frac{\mu(F\cap B(x,\delta))}{\mu B(x,\delta)} - 1 \to 1$$

as $\delta \downarrow 0$, and $x \in \underline{\phi}(E \cap F)$. Thus $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma$, and $\underline{\phi}$ is a lower density. **Q**

341F The hard work of this section is in the proof of 341H below. To make it a little more digestible, I extract two parts of the proof as separate lemmas.

Lemma Let (X, Σ, μ) be a probability space and $\mathfrak{A}$ its measure algebra. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $\underline{\theta} : \mathfrak{B} \to \Sigma$ a partial lower density. Then for any $e \in \mathfrak{A}$ there is a partial lower density $\underline{\theta}_1$, extending $\underline{\theta}$, defined on the subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{e\}$.

proof (a) Because $\mathfrak{B}$ is order-closed,

$$v = \operatorname{upr}(e, \mathfrak{B}) = \inf\{a : a \in \mathfrak{B},\, a \supseteq e\}, \quad w = \operatorname{upr}(1 \setminus e, \mathfrak{B})$$

are defined in $\mathfrak{B}$ (314V). Let $E \in \Sigma$ be such that $E^\bullet = e$.

(b) We have a function $\underline{\theta}_1 : \mathfrak{B}_1 \to \Sigma$ defined by writing

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr)$$

for a, $b \in \mathfrak{B}$. **P** By 312M, every element of $\mathfrak{B}_1$ is expressible as $(a \cap e) \cup (b \setminus e)$ for some a, $b \in \mathfrak{B}$. If a, a', b, $b' \in \mathfrak{B}$ are such that $(a \cap e) \cup (b \setminus e) = (a' \cap e) \cup (b' \setminus e)$, then $a \cap e = a' \cap e$ and $b \setminus e = b' \setminus e$, that is,

$$a \,\triangle\, a' \subseteq 1 \setminus e \subseteq w, \quad b \,\triangle\, b' \subseteq e \subseteq v.$$

This means that $e \subseteq 1 \setminus (a \,\triangle\, a') \in \mathfrak{B}$ and $1 \setminus e \subseteq 1 \setminus (b \,\triangle\, b') \in \mathfrak{B}$. So we also have $v \subseteq 1 \setminus (a \,\triangle\, a')$ and $w \subseteq 1 \setminus (b \,\triangle\, b')$. Accordingly

$$a \cap v = a' \cap v, \quad b \cap w = b' \cap w. \quad a \setminus w = a' \setminus w, \quad b \setminus v = b' \setminus v.$$

But this means that

$$\bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr)$$
$$= \bigl(\underline{\theta}((a' \cap v) \cup (b' \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a' \setminus w) \cup (b' \cap w)) \setminus E\bigr).$$

Thus the formula given defines $\underline{\theta}_1$ uniquely. **Q**

(c) Now $\underline{\theta}_1$ is a lower density.

P (i) If a, $b \in \mathfrak{B}$,

$$(\underline{\theta}_1((a \cap e) \cup (b \setminus e)))^\bullet = \Bigl(\bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr)\Bigr)^\bullet$$
$$= \bigl(((a \cap v) \cup (b \setminus v)) \cap e\bigr) \cup \bigl(((a \setminus w) \cup (b \cap w)) \setminus e\bigr)$$
$$= (a \cap e) \cup (b \setminus e).$$

So $(\underline{\theta}_1 c)^\bullet = c$ for every $c \in \mathfrak{B}_1$.

(ii)

$$\underline{\theta}_1(0) = \bigl(\underline{\theta}((0 \cap v) \cup (0 \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((0 \setminus w) \cup (0 \cap w)) \setminus E\bigr) = \emptyset.$$

(iii) If a, a', b, $b' \in \mathfrak{B}$, then

$$\begin{aligned}
\underline{\theta}_1(((a\cap e)\cup(b\setminus e))\cap((a'\cap e)\cup(b'\setminus e)))
&= \underline{\theta}_1((a\cap a'\cap e)\cup(b\cap b'\setminus e))\\
&= \bigl(\underline{\theta}((a\cap a'\cap v)\cup(b\cap b'\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\cap a'\setminus w)\cup(b\cap b'\cap w))\setminus E\bigr)\\
&= \bigl(\underline{\theta}(((a\cap v)\cup(b\setminus v))\cap((a'\cap v)\cup(b'\setminus v)))\cap E\bigr)\\
&\qquad\cup\bigl(\underline{\theta}(((a\setminus w)\cup(b\cap w))\cap((a'\setminus w)\cup(b'\cap w)))\setminus E\bigr)\\
&= \bigl(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\bigr)\\
&\qquad\cup\bigl(\underline{\theta}((a\setminus w)\cup(b\cap w))\cap\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\bigr)\\
&= \bigl(\bigl(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\setminus w)\cup(b\cap w))\setminus E\bigr)\bigr)\\
&\qquad\cap\bigl(\bigl(\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\bigr)\bigr)\\
&= \underline{\theta}_1((a\cap e)\cup(b\setminus e))\cap\underline{\theta}_1((a'\cap e)\cup(b'\setminus e)).
\end{aligned}$$

So $\underline{\theta}_1(c\cap c')=\underline{\theta}_1(c)\cap\underline{\theta}_1(c')$ for all c, $c'\in\mathfrak{B}_1$. **Q**

(d) If $a\in\mathfrak{B}$, then

$$\begin{aligned}
\underline{\theta}_1(a) &= \underline{\theta}_1((a\cap e)\cup(a\setminus e))\\
&= \bigl(\underline{\theta}((a\cap v)\cup(a\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\setminus w)\cup(a\cap w))\setminus E\bigr)\\
&= \bigl(\underline{\theta}(a)\cap E\bigr)\cup\bigl(\underline{\theta}(a)\setminus E\bigr)=\underline{\theta}a.
\end{aligned}$$

Thus $\underline{\theta}_1$ extends $\underline{\theta}$, as required.

341G Lemma Let (X,Σ,μ) be a probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Suppose we have a sequence $\langle\underline{\theta}_n\rangle_{n\in\mathbb{N}}$ of partial lower densities such that, for each n, (i) the domain $\mathfrak{B}_n$ of $\underline{\theta}_n$ is a closed subalgebra of $\mathfrak{A}$ (ii) $\mathfrak{B}_n\subseteq\mathfrak{B}_{n+1}$ and $\underline{\theta}_{n+1}$ extends $\underline{\theta}_n$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$. Then there is a partial lower density $\underline{\theta}$, with domain $\mathfrak{B}$, extending every $\underline{\theta}_n$.

proof (a) For each n, set

$$\Sigma_n=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{B}_n\},$$

and set

$$\Sigma_\infty=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{B}\}.$$

Then (because all the $\mathfrak{B}_n$, $\mathfrak{B}$ are σ-subalgebras of $\mathfrak{A}$, and $E\mapsto E^\bullet$ is sequentially order-continuous) all the Σ_n, Σ_∞ are σ-subalgebras of Σ. We need to know that Σ_∞ is just the σ-algebra Σ_∞^* of subsets of X generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. **P** Because Σ_∞ is a σ-algebra including $\bigcup_{n\in\mathbb{N}}\Sigma_n$, $\Sigma_\infty^*\subseteq\Sigma_\infty$. On the other hand, $\mathfrak{B}^*=\{E^\bullet:E\in\Sigma_\infty^*\}$ is a σ-subalgebra of $\mathfrak{A}$ including $\mathfrak{B}_n$ for every $n\in\mathbb{N}$. Because $\mathfrak{A}$ is ccc, $\mathfrak{B}^*$ is (order-)closed (316Fb), so includes $\mathfrak{B}$. This means that if $E\in\Sigma_\infty$ there must be an $F\in\Sigma_\infty^*$ such that $E^\bullet=F^\bullet$. But now $(E\triangle F)^\bullet=0\in\mathfrak{B}_0$, so $E\triangle F\in\Sigma_0\subseteq\Sigma_\infty^*$, and E also belongs to Σ_∞^*. This shows that $\Sigma_\infty\subseteq\Sigma_\infty^*$ and the two algebras are equal. **Q**

(b) For each $n\in\mathbb{N}$, we have the partial lower density $\underline{\theta}_n:\mathfrak{B}_n\to\Sigma$. Since $(\underline{\theta}_na)^\bullet=a\in\mathfrak{B}_n$ for every $a\in\mathfrak{B}_n$, $\underline{\theta}_n$ takes all its values in Σ_n. For $n\in\mathbb{N}$, let $\underline{\phi}_n:\Sigma_n\to\Sigma_n$ be the lower density corresponding to $\underline{\theta}_n$ (341Ba), that is, $\underline{\phi}_nE=\underline{\theta}_nE^\bullet$ for every $E\in\Sigma_n$.

(c) For $a\in\mathfrak{A}$, $n\in\mathbb{N}$ choose $G_a\in\Sigma$, g_{an} such that $G_a^\bullet=a$ and g_{an} is a conditional expectation of χG_a on Σ_n; that is,

$$\int_E g_{an}=\int_E\chi G_a=\mu(E\cap G_a)=\bar\mu(E^\bullet\cap a)$$

for every $E\in\Sigma_n$. As remarked in 233Db, such a function g_{an} can always be found, and moreover we may take it to be Σ_n-measurable and defined everywhere on X, by 232He. Now if $a\in\mathfrak{B}$, $\lim_{n\to\infty}g_{an}(x)$ exists and is equal to $\chi G_a(x)$ for almost every x. **P** By Lévy's martingale theorem (275I), $\lim_{n\to\infty}g_{an}$ is defined almost everywhere and is a conditional expectation of χG_a on the σ-algebra generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. As observed in (a), this is just Σ_∞; and as χG_a is itself Σ_∞-measurable, it is also a conditional expectation of itself on Σ_∞, and must be equal almost everywhere to $\lim_{n\to\infty} g_{an}$. **Q**

(d) For $a\in\mathfrak{B}$, $k\ge 1$, $n\in\mathbb{N}$ set

$$H_{kn}(a)=\{x:x\in X,\ g_{an}(x)\ge 1-2^{-k}\}\in\Sigma_n,\quad \tilde H_{kn}(a)=\underline{\phi}_n(H_{kn}(a)),$$

$$\underline{\theta}a=\bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde H_{km}(a).$$

The rest of the proof is devoted to showing that $\underline{\theta}:\mathfrak{B}\to\Sigma$ has the required properties.

(e) G_0 is negligible, so every g_{0n} is zero almost everywhere, every $H_{kn}(0)$ is negligible and every $\tilde H_{kn}(0)$ is empty; so $\underline{\theta}0=\emptyset$.

(f) If $a\subseteq b$ in $\mathfrak{B}$, then $\underline{\theta}a\subseteq\underline{\theta}b$. **P** $G_a\setminus G_b$ is negligible, $g_{an}\le g_{bn}$ almost everywhere for every n, every $H_{kn}(a)\setminus H_{kn}(b)$ is negligible, $\tilde H_{kn}(a)\subseteq\tilde H_{kn}(b)$ for every n and k, and $\underline{\theta}a\subseteq\underline{\theta}b$. **Q**

(g) If a, $b\in\mathfrak{B}$ then $\underline{\theta}(a\cap b)=\underline{\theta}a\cap\underline{\theta}b$. **P** $\chi G_{a\cap b}\ge\chi G_a+\chi G_b-1$ a.e. so $g_{a\cap b,n}\ge g_{an}+g_{bn}-1$ a.e. for every n. Accordingly

$$H_{k+1,n}(a)\cap H_{k+1,n}(b)\setminus H_{kn}(a\cap b)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde H_{kn}(a\cap b)\supseteq\underline{\phi}_n(H_{k+1,n}(a)\cap H_{k+1,n}(b))=\tilde H_{k+1,n}(a)\cap\tilde H_{k+1,n}(b)$$

for all $k\ge1$, $n\in\mathbb{N}$. Now, if $x\in\underline{\theta}a\cap\underline{\theta}b$, then, for any $k\ge1$, there are n_1, $n_2\in\mathbb{N}$ such that

$$x\in\bigcap_{m\ge n_1}\tilde H_{k+1,m}(a),\quad x\in\bigcap_{m\ge n_2}\tilde H_{k+1,m}(b).$$

But this means that

$$x\in\bigcap_{m\ge\max(n_1,n_2)}\tilde H_{km}(a\cap b).$$

As k is arbitrary, $x\in\underline{\theta}(a\cap b)$; as x is arbitrary, $\underline{\theta}a\cap\underline{\theta}b\subseteq\underline{\theta}(a\cap b)$. We know already from (f) that $\underline{\theta}(a\cap b)\subseteq\underline{\theta}a\cap\underline{\theta}b$, so $\underline{\theta}(a\cap b)=\underline{\theta}a\cap\underline{\theta}b$. **Q**

(h) If $a\in\mathfrak{B}$, then $\underline{\theta}a^\bullet=a$. **P** $\langle g_{an}\rangle_{n\in\mathbb{N}}\to\chi G_a$ a.e., so setting

$$V_a=\bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}H_{km}(a)=\{x:\liminf_{n\to\infty}g_{an}(x)\ge1\},$$

$V_a\triangle G_a$ is negligible, and $V_a^\bullet=a$; but

$$\underline{\theta}a\triangle V_a\subseteq\bigcup_{k\ge1,n\in\mathbb{N}}H_{kn}(a)\triangle\tilde H_{kn}(a)$$

is also negligible, so $\underline{\theta}a^\bullet$ is also equal to a. **Q** Thus $\underline{\theta}$ is a partial lower density with domain $\mathfrak{B}$.

(i) Finally, $\underline{\theta}$ extends $\underline{\theta}_n$ for every $n\in\mathbb{N}$. **P** If $a\in\mathfrak{B}_n$, then $G_a\in\Sigma_m$ for every $m\ge n$, so $g_{am}=\chi G_a$ a.e. for every $m\ge n$; $H_{km}(a)\triangle G_a$ is negligible for $k\ge1$, $m\ge n$;

$$\tilde H_{km}=\underline{\phi}_mG_a=\underline{\theta}_ma=\underline{\theta}_na$$

for $k\ge1$, $m\ge n$ (this is where I use the hypothesis that $\underline{\theta}_{m+1}$ extends $\underline{\theta}_m$ for every m); and

$$\begin{aligned}\underline{\theta}a&=\bigcap_{k\ge1}\bigcup_{r\in\mathbb{N}}\bigcap_{m\ge r}\tilde H_{km}(a)\\&=\bigcap_{k\ge1}\bigcup_{r\ge n}\bigcap_{m\ge r}\tilde H_{km}(a)=\bigcap_{k\ge1}\bigcup_{r\ge n}\underline{\theta}_na=\underline{\theta}_na.\ \textbf{Q}\end{aligned}$$

The proof is complete.

341H Now for the first main theorem.

Theorem Let (X,Σ,μ) be any strictly localizable measure space. Then it has a lower density $\underline{\phi}:\Sigma\to\Sigma$. If $\mu X>0$ we can take $\underline{\phi}X=X$.

proof : Part A I deal first with the case of probability spaces. Let (X,Σ,μ) be a probability space, and $(\mathfrak{A},\bar{\mu})$ its measure algebra.

(a) Set $\kappa=\#(\mathfrak{A})$ and enumerate $\mathfrak{A}$ as $\langle a_\xi\rangle_{\xi<\kappa}$. For $\xi\le\kappa$ let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}$. I seek to define a lower density $\underline{\theta}:\mathfrak{A}\to\Sigma$ as the last of a family $\langle\underline{\theta}_\xi\rangle_{\xi\le\kappa}$, where $\underline{\theta}_\xi:\mathfrak{A}_\xi\to\Sigma$ is a partial lower density for each ξ. The inductive hypothesis will be that $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$ whenever $\eta\le\xi\le\kappa$.

To start the induction, we have $\mathfrak{A}_0=\{0,1\}$, $\underline{\theta}_0 0=\emptyset$, $\underline{\theta}_0 1=X$.

(b) *Inductive step to a successor ordinal* ξ Given a succesor ordinal $\xi\le\kappa$, express it as $\zeta+1$; we are supposing that $\underline{\theta}_\zeta:\mathfrak{A}_\zeta\to\Sigma$ has been defined. Now $\mathfrak{A}_\xi$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\zeta\cup\{a_\zeta\}$ (because this is a closed subalgebra, by 323K). So 341F tells us that $\underline{\theta}_\zeta$ can be extended to a partial lower density $\underline{\theta}_\xi$ with domain $\mathfrak{A}_\xi$.

(c) *Inductive step to a non-zero limit ordinal* ξ *of countable cofinality* In this case, there is a strictly increasing sequence $\langle\zeta(n)\rangle_{n\in\mathbb{N}}$ with supremum ξ. Applying 341G with $\mathfrak{B}_n=\mathfrak{A}_{\zeta(n)}$, we see that there is a partial lower density $\underline{\theta}_\xi$, with domain the closed subalgebra $\mathfrak{B}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{A}_{\zeta(n)}$, extending every $\underline{\theta}_{\zeta(n)}$. Now $\mathfrak{A}_{\zeta(n)}\subseteq\mathfrak{A}_\xi$ for every ξ, so $\mathfrak{B}\subseteq\mathfrak{A}_\xi$; but also, if $\eta<\xi$, there is an $n\in\mathbb{N}$ such that $\eta<\zeta(n)$, so that $a_\eta\in\mathfrak{A}_{\zeta(n)}\subseteq\mathfrak{B}$; as η is arbitrary, $\mathfrak{A}_\xi\subseteq\mathfrak{B}$ and $\mathfrak{A}_\xi=\mathfrak{B}$. Again, if $\eta<\xi$, there is an n such that $\eta\le\zeta(n)$, so that $\underline{\theta}_{\zeta(n)}$ extends $\underline{\theta}_\eta$ and $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$. Thus the induction continues.

(d) *Inductive step to a limit ordinal* ξ *of uncountable cofinality* In this case, $\mathfrak{A}_\xi=\bigcup_{\eta<\xi}\mathfrak{A}_\eta$. **P** Because $\mathfrak{A}$ is ccc, every member a of $\mathfrak{A}_\xi$ must be in the closed subalgebra of $\mathfrak{A}$ generated by some countable subset A of $\{a_\eta:\eta<\xi\}$ (331Gd-e). Now A can be expressed as $\{a_\eta:\eta\in I\}$ for some countable $I\subseteq\xi$. As I cannot be cofinal with ξ, there is a $\zeta<\xi$ such that $\eta<\zeta$ for every $\eta\in I$, so that $A\subseteq\mathfrak{A}_\zeta$ and $a\in\mathfrak{A}_\zeta$. **Q**

But now, because $\underline{\theta}_\zeta$ extends $\underline{\theta}_\eta$ whenever $\eta\le\zeta<\xi$, we have a function $\underline{\theta}_\xi:\mathfrak{A}_\xi\to\Sigma$ defined by writing $\underline{\theta}_\xi a=\underline{\theta}_\eta a$ whenever $\eta<\xi$ and $a\in\mathfrak{A}_\eta$. Because the family $\{\mathfrak{A}_\eta:\eta<\xi\}$ is totally ordered and every $\underline{\theta}_\eta$ is a partial lower density, $\underline{\theta}_\xi$ is a partial lower density.

Thus the induction proceeds when ξ is a limit ordinal of uncountable cofinality.

(e) The induction stops when we reach $\underline{\theta}_\kappa:\mathfrak{A}\to\Sigma$, which is a lower density such that $\underline{\theta}_\kappa 1=X$. Setting $\underline{\phi}E=\underline{\theta}_\kappa E^\bullet$, $\underline{\phi}$ is a lower density such that $\underline{\phi}X=X$.

Part B The general case of a strictly localizable measure space follows easily. First, if $\mu X=0$, then $\mathfrak{A}=\{0\}$ and we can set $\underline{\phi}0=\emptyset$. Second, if μ is totally finite but not zero, we can replace it by ν, where $\nu E=\mu E/\mu X$ for every $E\in\Sigma$; a lower density for ν is also a lower density for μ. Third, if μ is not totally finite, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X (211E). There is surely some j such that $\mu X_j>0$; replacing X_j by $X_j\cup\bigcup\{X_i:i\in I,\,\mu X_i=0\}$, we may assume that $\mu X_i>0$ for every $i\in I$. For each $i\in I$, let $\underline{\phi}_i:\Sigma_i\to\Sigma_i$ be a lower density of μ_i, where $\Sigma_i=\Sigma\cap\mathcal{P}X_i$ and $\mu_i=\mu\restriction\Sigma_i$, such that $\underline{\phi}_iX_i=X_i$. Then it is easy to check that we have a lower density $\underline{\phi}:\Sigma\to\Sigma$ given by setting

$$\underline{\phi}E=\bigcup_{i\in I}\underline{\phi}_i(E\cap X_i)$$

for every $E\in\Sigma$, and that $\underline{\phi}X=X$.

341I The next step is to give a method of moving from lower densities to liftings. I start with an elementary remark on lower densities on complete measure spaces.

Lemma Let (X,Σ,μ) be a complete measure space with measure algebra $\mathfrak{A}$.

(a) Suppose that $\underline{\theta}:\mathfrak{A}\to\Sigma$ is a lower density and $\underline{\theta}_1:\mathfrak{A}\to\mathcal{P}X$ is a function such that $\underline{\theta}_1 0=\emptyset$, $\underline{\theta}_1(a\cap b)=\underline{\theta}_1 a\cap\underline{\theta}_1 b$ for all a, $b\in\mathfrak{A}$ and $\underline{\theta}_1 a\supseteq\underline{\theta}a$ for all $a\in\mathfrak{A}$. Then $\underline{\theta}_1$ is a lower density. If $\underline{\theta}_1$ is a Boolean homomorphism, it is a lifting.

(b) Suppose that $\underline{\phi}:\Sigma\to\Sigma$ is a lower density and $\underline{\phi}_1:\Sigma\to\mathcal{P}X$ is a function such that $\underline{\phi}_1E=\underline{\phi}_1F$ whenever $E\triangle F$ is negligible, $\underline{\phi}_1\emptyset=\emptyset$, $\underline{\phi}_1(E\cap F)=\underline{\phi}_1E\cap\underline{\phi}_1F$ for all E, $F\in\Sigma$ and $\underline{\phi}_1E\supseteq\underline{\phi}E$ for all $E\in\Sigma$. Then $\underline{\phi}_1$ is a lower density. If $\underline{\phi}_1$ is a Boolean homomorphism, it is a lifting.

proof (a) All I have to check is that $\underline{\theta}_1 a\in\Sigma$ and $(\underline{\theta}_1 a)^\bullet=a$ for every $a\in\mathfrak{A}$. But

$$\underline{\theta}a\subseteq\underline{\theta}_1 a,\quad \underline{\theta}(1\setminus a)\subseteq\underline{\theta}_1(1\setminus a),\quad \underline{\theta}_1 a\cap\underline{\theta}_1(1\setminus a)=\underline{\theta}_1 0=\emptyset.$$

So

$$\underline{\theta}a \subseteq \underline{\theta}_1 a \subseteq X \setminus \underline{\theta}(1 \setminus a).$$

Since

$$(\underline{\theta}a)^\bullet = a = (X \setminus \underline{\theta}(1 \setminus a))^\bullet,$$

and μ is complete, $\underline{\theta}_1$ is a lower density. If it is a Boolean homomorphism, then it is also a lifting (341De).

(b) This follows by the same argument, or by looking at the functions from $\mathfrak{A}$ to Σ defined by $\underline{\phi}$ and $\underline{\phi}_1$ and using (a).

341J Proposition Let (X, Σ, μ) be a complete measure space such that $\mu X > 0$, and $\mathfrak{A}$ its measure algebra.

(a) If $\underline{\theta} : \mathfrak{A} \to \Sigma$ is any lower density, there is a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\theta a \supseteq \underline{\theta}a$ for every $a \in \mathfrak{A}$.

(b) If $\underline{\phi} : \Sigma \to \Sigma$ is any lower density, there is a lifting $\phi : \Sigma \to \Sigma$ such that $\phi E \supseteq \underline{\phi}E$ for every $E \in \Sigma$.

proof (a) For each $x \in \underline{\theta}1$, set

$$I_x = \{a : a \in \mathfrak{A},\, x \in \underline{\theta}(1 \setminus a)\}.$$

Then I_x is a proper ideal of $\mathfrak{A}$. **P** We have

$0 \in I_x$, because $x \in \underline{\theta}1$,

if $b \subseteq a \in I_x$ then $b \in I_x$, because $x \in \underline{\theta}(1 \setminus a) \subseteq \underline{\theta}(1 \setminus b)$,

if $a, b \in I_x$ then $a \cup b \in I_x$, because $x \in \underline{\theta}(1 \setminus a) \cap \underline{\theta}(1 \setminus b) = \underline{\theta}(1 \setminus (a \cup b))$,

$1 \notin I_x$ because $x \notin \emptyset = \underline{\theta}0$. **Q**

For $x \in X \setminus \underline{\theta}1$, set $I_x = \{0\}$; this is also a proper ideal of $\mathfrak{A}$, because $\mathfrak{A} \neq \{0\}$. By 311D, there is a surjective Boolean homomorphism $\pi_x : \mathfrak{A} \to \{0, 1\}$ such that $\pi_x d = 0$ for every $d \in I_x$.

Define $\theta : \mathfrak{A} \to \mathcal{P}X$ by setting

$$\theta a = \{x : x \in X,\, \pi_x(a) = 1\}$$

for every $a \in \mathfrak{A}$. It is easy to check that, because every π_x is a surjective Boolean homomorphism, θ is a Boolean homomorphism. Now for any $a \in \mathfrak{A}$, $x \in X$,

$$x \in \underline{\theta}a \implies 1 \setminus a \in I_x \implies \pi_x(1 \setminus a) = 0 \implies \pi_x a = 1 \implies x \in \theta a.$$

Thus $\theta a \supseteq \underline{\theta}a$ for every $a \in \mathfrak{A}$. By 341I, θ is a lifting, as required.

(b) Repeat the argument above, or apply it, defining $\underline{\theta}$ by setting $\underline{\theta}(E^\bullet) = \underline{\phi}E$ for every $E \in \Sigma$, and ϕ by setting $\phi E = \theta(E^\bullet)$ for every E.

341K The Lifting Theorem Every complete strictly localizable measure space of non-zero measure has a lifting.

proof By 341H, it has a lower density, so by 341J it has a lifting.

341L Remarks If we count 341F-341K as a single argument, it may be the longest proof, after Carleson's theorem (§286), which I have yet presented in this treatise, and perhaps it will be helpful if I suggest ways of looking at its components.

(a) The first point is that the theorem should be thought of as one about probability spaces. The shift to general strictly localizable spaces (Part B of the proof of 341H) is purely a matter of technique. I would not have presented it if I did not think that it's worth doing, for a variety of reasons, but there is no significant idea needed, and if – for instance – the result were valid only for σ-finite spaces, it would still be one of the great theorems of mathematics. So the rest of these remarks will be directed to the ideas needed in probability spaces.

(b) All the proofs I know of the theorem depend in one way or another on an inductive construction. We do not, of course, need a transfinite induction written out in the way I have presented it in 341H above. Essentially the same proof can be presented as an application of Zorn's Lemma; if we take P to be the set of partial lower densities, then the arguments of 341G and part (A-d) of the proof of 341H can be adapted to prove that any totally ordered subset of P has an upper bound in P, while the argument of 341F shows that any maximal element of P must have domain $\mathfrak{A}$. I think it is purely a matter of taste which form one prefers. I suppose I have used the ordinal-indexed form largely because that seemed appropriate for Maharam's theorem in the last chapter.

(c) There are then three types of inductive step to examine, corresponding to 341F, 341G and (A-d) in 341H. The first and last are easier than the second. Seeking the one-step extension of $\underline{\theta} : \mathfrak{B} \to \Sigma$ to $\underline{\theta}_1 : \mathfrak{B}_1 \to \Sigma$, the natural model to use is the one-step extension of a Boolean homomorphism presented in 312N. The situation here is rather more complicated, as $\underline{\theta}_1$ is not fully specified by the value of $\underline{\theta}_1 e$, and we do in fact have more freedom at this point than is entirely welcome. The formula used in the proof of 341F is derived from Graf & Weizsäcker 76.

(d) At this point I must call attention to the way in which the whole proof is dominated by the choice of *closed* subalgebras as the domains of our partial liftings. This is what makes the inductive step to a limit ordinal ξ of countable cofinality difficult, because $\mathfrak{A}_\xi$ will ordinarily be larger than $\bigcup_{\eta<\xi} \mathfrak{A}_\eta$. But it is absolutely essential in the one-step extensions.

Because we are dealing with a ccc algebra $\mathfrak{A}$, the requirement that the $\mathfrak{A}_\xi$ should be closed is not a problem when $\mathrm{cf}(\xi)$ is uncountable, since in this case $\bigcup_{\eta<\xi} \mathfrak{A}_\eta$ is already a closed subalgebra; this is the only idea needed in (A-d) of 341H.

(e) So we are left with the inductive step to ξ when $\mathrm{cf}(\xi) = \omega$, which is 341G. Here we actually need some measure theory, and a particularly striking bit. (You will see that the *measure* μ, as opposed to the algebras Σ and $\mathfrak{A}$ and the homomorphism $E \mapsto E^\bullet$ and the ideal of negligible sets, is simply not mentioned anywhere else in the whole argument.)

(i) The central idea is to use the fact that bounded martingales converge to define $\underline{\theta} a$ in terms of a sequence of conditional expectations. Because I have chosen a fairly direct assault on the problem, some of the surrounding facts are not perhaps so clearly visible as they might have been if I had used a more leisurely route. For each $a \in \mathfrak{A}$, I start by choosing a representative $G_a \in \Sigma$; let me emphasize that this is a crude application of the axiom of choice, and that the different sets G_a are in no way coordinated. (The theorem we are proving is that they *can* be coordinated, but we have not reached that point yet.) Next, I choose, arbitrarily, a conditional expectation g_{an} of χG_a on Σ_n. Once again, the choices are not coordinated; but the martingale theorem assures us that $g_a = \lim_{n\to\infty} g_{an}$ is defined almost everywhere, and is equal almost everywhere to χG_a if $a \in \mathfrak{B}$. Of course I could have gone to the g_{an} without mentioning the G_a; they are set up as Radon-Nikodým derivatives of the countably additive functionals $E \mapsto \bar{\mu}(E^\bullet \cap a) : \Sigma_n \to \mathbb{R}$. Now the g_{an}, like the G_a, are not uniquely defined. But they are defined 'up to a negligible set', so that any alternative functions g'_{an} would have $g'_{an} = g_{an}$ a.e. This means that the sets $H_{kn}(a) = \{x : g_{an}(x) \ge 1 - 2^{-k}\}$ are also defined 'up to a negligible set', and consequently the sets $\tilde{H}_{kn}(a) = \underline{\phi}_n(H_{kn}(a))$ are uniquely defined. I point this out to show that it is not a complete miracle that we have formulae

$$\tilde{H}_{kn}(a) \subseteq \tilde{H}_{kn}(b) \text{ if } a \subseteq b,$$

$$\tilde{H}_{kn}(a \cap b) \supseteq \tilde{H}_{k+1,n}(a) \cap \tilde{H}_{k+1,n}(b) \text{ for all } a,\, b \in \mathfrak{A}$$

which do not ask us to turn a blind eye to any negligible sets. I note in passing that I could have defined the $\tilde{H}_{kn}(a)$ without mentioning the g_{an}; in fact

$$\tilde{H}_{kn}(a) = \underline{\theta}_n(\sup\{c : c \in \mathfrak{B}_n,\, \bar{\mu}(a \cap d) \ge 1 - 2^{-k}\bar{\mu}d \text{ whenever } d \in \mathfrak{B}_n,\, d \subseteq c\}).$$

(ii) Now, with the sets $\tilde{H}_{kn}(a)$ in hand, we can look at

$$\tilde{V}_a = \bigcap_{k\ge1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \tilde{H}_{kn}(a);$$

because $g_{an} \to \chi G_a$ a.e., $\tilde{V}_a \triangle G_a$ is negligible and $\tilde{V}_a^\bullet = a$ for every $a \in \mathfrak{A}_\xi$. The rest of the argument amounts to checking that $a \mapsto \tilde{V}_a$ will serve for $\underline{\theta}$.

(f) The arguments above apply to all probability spaces, and show that every probability space has a lower density. The next step is to convert a lower density into a lifting. It is here that we need to assume completeness. The point is that we can find a Boolean homomorphism $\theta : \mathfrak{A} \to \mathcal{P}X$ such that $\underline{\theta}a \subseteq \theta a$ for every a; this corresponds just to extending the ideals $I_x = \{a : x \in \underline{\theta}(1 \setminus a)\}$ to maximal ideals (and giving a moment's thought to $x \in X \setminus \underline{\theta}1$). In order to ensure that $\theta a \in \Sigma$ and $(\theta a)^\bullet = a$, we have to observe that θa is sandwiched between $\underline{\theta}a$ and $X \setminus \underline{\theta}(1 \setminus a)$, which differ by a negligible set; so that if μ is complete all will be well.

(g) The fact that completeness is needed at only one point in the argument makes it natural to wonder whether the theorem might be true for probability spaces in general. (I will come later, in 341M, to non-strictly-localizable spaces.) There is as yet no satisfactory answer to this. For Borel measure on $\mathbb{R}$, the question is known to be undecidable from the ordinary axioms of set theory (including the axiom of choice, but not the continuum hypothesis, as usual); I will give some of the arguments in Volume 5. (For the moment, I refer you to the discussion in Fremlin 89, §4, and to Burke 93.) But I conjecture that there is an counter-example under the ordinary axioms (see 341Z below).

(h) Quite apart from whether completeness is needed in the argument, it is not absolutely clear why measure theory is required. The general question of whether a lifting exists can be formulated for any triple $(X, \Sigma, \mathcal{I})$ where X is a set, Σ is a σ-algebra of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ. (See 341Ya below.) S.Shelah has given an example of such a triple without a lifting in which two of the basic properties of the measure-theoretic case are satisfied: $(X, \Sigma, \mathcal{I})$ is 'complete' in the sense that every subset of any member of $\mathcal{I}$ belongs to Σ (and therefore to $\mathcal{I}$), and $\mathcal{I}$ is ω_1-saturated in Σ in the sense of 316C (see Shelah Sh636, Burke n96). But many other cases are known (e.g., 341Yb) in which liftings do exist.

(i) It is of course possible to prove 341K without mentioning 'lower densities', and there are even some advantages in doing so. The idea is to follow the lines of 341H, but with 'liftings' instead of 'lower densities' throughout. The inductive step to a successor ordinal is actually easier, because we have a Boolean homomorphism θ in 341F to extend, and we can use 312N as it stands if we can choose the pair E, $F = X \setminus E$ correctly. The inductive step to an ordinal of uncountable cofinality remains straightforward. But in the inductive step to an ordinal of countable cofinality, we find that in 341G we get no help from assuming that the $\underline{\theta}_n$ are actually liftings; we are still led to to a lower density $\underline{\theta}$. So at this point we have to interpolate the argument of 341J to convert this lower density into a lifting.

I have chosen the more leisurely exposition, with the extra concept, partly in order to get as far as possible without assuming completeness of the measure and partly because lower densities are an important tool for further work (see §§345-346).

(j) For more light on the argument of 341G see also 363Xe and 363Yg below.

341M I remarked above that the shift from probability spaces to general strictly localizable spaces was simply a matter of technique. The question of which spaces have liftings is also primarily a matter concerning probability spaces, as the next result shows.

Proposition Let (X, Σ, μ) be a complete locally determined space with $\mu X > 0$. Then it has a lifting iff it has a lower density iff it is strictly localizable.

proof If (X, Σ, μ) is strictly localizable then it has a lifting, by 341K. A lifting is already a lower density, and if (X, Σ, μ) has a lower density it has a lifting, by 341J. So we have only to prove that if it has a lifting then it is strictly localizable.

Let $\theta : \mathfrak{A} \to \Sigma$ be a lifting, where $\mathfrak{A}$ is the measure algebra of (X, Σ, μ). Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{A} = \{\theta c : c \in C\}$. Because C is disjoint, so is $\mathcal{A}$. Because $\sup C = 1$ in $\mathfrak{A}$, every set of positive measure meets some member of $\mathcal{A}$ in a set of positive measure. So the conditions of 213O are satisfied, and (X, Σ, μ) is strictly localizable.

341N Extension of partial liftings The following facts are obvious from the proof of 341H, but it will be useful to have them out in the open.

Proposition Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ.

(a) Any partial lower density $\underline{\phi}_0 : \mathrm{T} \to \Sigma$ has an extension to a lower density $\underline{\phi} : \Sigma \to \Sigma$.

(b) Suppose now that μ is complete. If $\underline{\phi}_0$ is a Boolean homomorphism, it has an extension to a lifting ϕ for μ.

proof (a) In Part A of the proof of 341H, let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{E^\bullet : E \in \mathrm{T}\} \cup \{a_\eta : \eta < \xi\}$, and set $\underline{\theta}_0 E^\bullet = \underline{\phi}_0 E$ for every $E \in \mathrm{T}$. Proceed with the induction as before. The only difference is that we no longer have a guarantee that $\underline{\phi} X = X$.

(b) Suppose now that $\underline{\phi}_0$ is a Boolean homomorphism. 341J tells us that there is a lifting $\phi : \Sigma \to \Sigma$ such that $\phi E \supseteq \underline{\phi} E$ for every $E \in \Sigma$. But if $E \in \mathrm{T}$ we must have $\phi E \supseteq \phi_0 E$,

$$\phi E \setminus \phi_0 E = \phi E \cap \phi_0 (X \setminus E) \subseteq \phi E \cap \phi(X \setminus E) = \emptyset,$$

so that $\phi E = \phi_0 E$, and ϕ extends ϕ_0.

341O Liftings and Stone spaces The arguments of this section so far involve repeated use of the axiom of choice, and offer no suggestion that any liftings (or lower densities) are in any sense 'canonical'. There is however one context in which we have a distinguished lifting. Suppose that we have the Stone space (Z, T, ν) of a measure algebra $(\mathfrak{A}, \bar{\mu})$; as in 311E, I think of Z as being the set of surjective Boolean homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that each $a \in \mathfrak{A}$ corresponds to the open-and-closed set $\widehat{a} = \{z : z(a) = 1\}$. Then we have a lifting $\theta : \mathfrak{A} \to \mathrm{T}$ defined by setting $\theta a = \widehat{a}$ for each $a \in \mathfrak{A}$. (I am identifying $\mathfrak{A}$ with the measure algebra of ν, as in 321J.) The corresponding lifting $\phi : \mathrm{T} \to \mathrm{T}$ is defined by taking ϕE to be that unique open-and-closed set such that $E \triangle \phi E$ is negligible (or, if you prefer, meager).

Generally, liftings can be described in terms of Stone spaces, as follows.

341P Proposition Let (X, Σ, μ) be a measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra, and (Z, T, ν) the Stone space of $(\mathfrak{A}, \bar{\mu})$ with its canonical measure.

(a) There is a one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and functions $f : X \to Z$ such that $f^{-1}[\widehat{a}] \in \Sigma$ and $(f^{-1}[\widehat{a}])^\bullet = a$ for every $a \in \mathfrak{A}$, defined by the formula

$$\theta a = f^{-1}[\widehat{a}] \text{ for every } a \in \mathfrak{A}.$$

(b) If (X, Σ, μ) is complete and locally determined, then a function $f : X \to Z$ satisfies the conditions of (a) iff (α) it is inverse-measure-preserving (β) the homomorphism it induces between the measure algebras of μ and ν is the canonical isomorphism defined by the construction of Z.

proof Recall that T is just the set $\{\widehat{a} \triangle M : a \in \mathfrak{A},\ M \subseteq Z \text{ is meager}\}$, and that $\nu(\widehat{a} \triangle M) = \bar{\mu} a$ for all such a, M; while the canonical isomorphism π between $\mathfrak{A}$ and the measure algebra of ν is defined by the formula

$$\pi F^\bullet = a \text{ whenever } F \in \mathrm{T},\ a \in \mathfrak{A} \text{ and } F \triangle \widehat{a} \text{ is meager}$$

(341K).

(a) If $\theta : \mathfrak{A} \to \Sigma$ is any Boolean homomorphism, then for every $x \in X$ we have a surjective Boolean homomorphism $f_\theta(x) : \mathfrak{A} \to \mathbb{Z}_2$ defined by saying that $f_\theta(x)(a) = 1$ if $x \in \theta a$, 0 otherwise. f_θ is a function from X to Z. We can recover θ from f_θ by the formula

$$\theta a = \{x : f_\theta(x)(a) = 1\} = \{x : f_\theta(x) \in \widehat{a}\} = f_\theta^{-1}[\widehat{a}].$$

So $f_\theta^{-1}[\widehat{a}] \in \Sigma$ and, if θ is a lifting,

$$(f_\theta^{-1}[\widehat{a}])^\bullet = (\theta a)^\bullet = a.$$

for every $a \in \mathfrak{A}$.

Similarly, given a function $f : X \to Z$ with this property, then we can set $\theta a = f^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$ to obtain a lifting $\theta : \mathfrak{A} \to \Sigma$; and of course we now have

$$f(x)(a) = 1 \iff f(x) \in \widehat{a} \iff x \in \theta a,$$

so $f_\theta = f$.

(b) Assume now that (X,Σ,μ) is complete and locally determined.

(i) Let $f : X \to Z$ be the function associated with a lifting θ, as in (a). I show first that f is inverse-measure-preserving. **P** If $F \in \mathrm{T}$, express it as $\widehat{a}\triangle M$, where $a \in \mathfrak{A}$ and $M \subseteq Z$ is meager. By 322F, $\mathfrak{A}$ is weakly (σ,∞)-distributive, so M is nowhere dense (316I). Consider $f^{-1}[M]$. If $E \subseteq X$ is measurable and of finite measure, then $E \cap f^{-1}[M]$ has a measurable envelope H (132Ed). **?** If $\mu H > 0$, then $b = H^\bullet \neq 0$ and $\widehat{b}$ is a non-empty open set in Z. Because M is nowhere dense, there is a non-zero $a \in \mathfrak{A}$ such that $\widehat{a} \subseteq \widehat{b}\setminus M$. Now $\mu(f^{-1}[\widehat{b}]\triangle H) = 0$, so $f^{-1}[\widehat{a}]\setminus H$ is negligible, and $f^{-1}[\widehat{a}]\cap H$ is a non-negligible measurable set disjoint from $E \cap f^{-1}[M]$ and included in H; which is impossible. **X** Thus H and $E \cap f^{-1}[M]$ are negligible. This is true for every measurable set E of finite measure. Because μ is complete and locally determined, $f^{-1}[M] \in \Sigma$ and $\mu f^{-1}[M] = 0$. So $f^{-1}[F] = f^{-1}[\widehat{a}]\triangle f^{-1}[M]$ is measurable, and

$$\mu f^{-1}[F] = \mu f^{-1}[\widehat{a}] = \mu\theta a = \bar{\mu}a = \nu\widehat{a} = \nu F.$$

As F is arbitrary, f is inverse-measure-preserving. **Q**

It follows at once that for any $F \in \mathrm{T}$,

$$f^{-1}[F]^\bullet = a = \pi F^\bullet$$

where a is that element of $\mathfrak{A}$ such that $M = F\triangle a$ is meager, because in this case $f^{-1}[\widehat{a}]^\bullet = a$, by (a), while $f^{-1}[M]$ is negligible. So π is the homomorphism induced by f.

(ii) Now suppose that $f : X \to Z$ is an inverse-measure-preserving function such that $f^{-1}[F]^\bullet = \pi F^\bullet$ for every $F \in \mathrm{T}$. Then, in particular,

$$f^{-1}[\widehat{a}]^\bullet = \pi\widehat{a}^\bullet = a$$

for every $a \in \mathfrak{A}$, so that f satisfies the conditions of (a).

341Q Corollary Let (X,Σ,μ) be a strictly localizable measure space, $(\mathfrak{A},\bar{\mu})$ its measure algebra, and Z the Stone space of $\mathfrak{A}$; suppose that $\mu X > 0$. For $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to $E^\bullet \in \mathfrak{A}$. Then there is a function $f : X \to Z$ such that $E\triangle f^{-1}[E^*]$ is negligible for every $E \in \Sigma$. If μ is complete, then f is inverse-measure-preserving.

proof Let $\hat{\mu}$ be the completion of μ, and $\hat{\Sigma}$ its domain. Then we can identify $(\mathfrak{A},\bar{\mu})$ with the measure algebra of $\hat{\mu}$ (322Da). Let $\theta : \mathfrak{A} \to \hat{\Sigma}$ be a lifting, and $f : X \to Z$ the corresponding function. If $E \in \Sigma$ then $E^* = \widehat{a}$ where $a = E^\bullet$, so $E\triangle f^{-1}[E^*] = E\triangle\theta E^\bullet$ is negligible. If μ is itself complete, so that $\hat{\Sigma} = \Sigma$, then f is inverse-measure-preserving, by 341Pb.

341X Basic exercises (a) Let (X,Σ,μ) be a measure space and $\phi : \Sigma \to \Sigma$ a function. Show that ϕ is a lifting iff it is a lower density and $\phi E \cup \phi(X \setminus E) = X$ for every $E \in \Sigma$.

>(b) Let μ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, and Σ its domain. For $x \in X$ and $n \in \mathbb{N}$ set $U_n(x) = \{y : y \in X,\ y\restriction n = x\restriction n\}$. For $E \in \Sigma$ set $\underline{\phi}E = \{x : \lim_{n\to\infty} 2^n\mu(E \cap U_n(x)) = 1\}$. Show that $\underline{\phi}$ is a lower density of (X,Σ,μ).

>(c) Let P be the set of all lower densities of a complete measure space (X,Σ,μ), with measure algebra $\mathfrak{A}$, ordered by saying that $\underline{\theta} \le \underline{\theta}'$ if $\underline{\theta}a \subseteq \underline{\theta}'a$ for every $a \in \mathfrak{A}$. Show that any non-empty totally ordered subset of P has an upper bound in P. Show that if $\underline{\theta} \in P$ and $a \in \mathfrak{A}$ and $x \in X \setminus (\underline{\theta}a \cup \underline{\theta}(1\setminus a))$, then $\underline{\theta}' : \mathfrak{A} \to \Sigma$ is a lower density, where $\underline{\theta}'b = \underline{\theta}b \cup \{x\}$ if either $a \subseteq b$ or there is a $c \in \mathfrak{A}$ such that $x \in \underline{\theta}c$ and $a \cap c \subseteq b$, and $\underline{\theta}'b = \underline{\theta}b$ otherwise. Hence prove 341J.

(d) Let (X,Σ,μ) and (Y,T,ν) be measure spaces and suppose that there is an inverse-measure-preserving function $f : X \to Y$ such that the associated homomorphism from the measure algebra of ν to that of μ is an isomorphism. Show that for every lifting ϕ of (Y,T,ν) we have a corresponding lifting ψ of (X,Σ,μ) defined uniquely by the formula

$$\psi(f^{-1}[F]) = f^{-1}[\phi F] \text{ for every } F \in \mathrm{T}.$$

(e) Let (X,Σ,μ) be a measure space, and write $\mathcal{L}^\infty(\Sigma)$ for the linear space of all bounded Σ-measurable functions from X to $\mathbb{R}$. Show that for any lifting $\phi:\Sigma\to\Sigma$ of μ there is a unique linear operator $T:L^\infty(\mu)\to\mathcal{L}^\infty(\Sigma)$ such that $T(\chi E)^\bullet=\chi(\phi E)$ for every $E\in\Sigma$ and $Tu\ge 0$ in $\mathcal{L}^\infty(\Sigma)$ whenever $u\ge 0$ in $L^\infty(\mu)$. Show that (i) $(Tu)^\bullet=u$ and $\sup_{x\in X}|(Tu)(x)|=\|u\|_\infty$ for every $u\in L^\infty(\mu)$ (ii) $T(u\times v)=Tu\times Tv$ for all u, $v\in L^\infty(\mu)$.

341Y Further exercises (a) Let X be a set, Σ an algebra of subsets of X and $\mathcal{I}$ an ideal of Σ; let $\mathfrak{A}$ be the quotient Boolean algebra $\Sigma/\mathcal{I}$. We say that a function $\theta:\mathfrak{A}\to\Sigma$ is a **lifting** if it is a Boolean homomorphism and $(\theta a)^\bullet=a$ for every $a\in\mathfrak{A}$, and that $\underline{\theta}:\mathfrak{A}\to\Sigma$ is a **lower density** if $\underline{\theta}0=\emptyset$, $\underline{\theta}(a\cap b)=\underline{\theta}a\cap\underline{\theta}b$ for all a, $b\in\mathfrak{A}$, and $(\underline{\theta}a)^\bullet=a$ for every $a\in\mathfrak{A}$.

Show that if $(X,\Sigma,\mathcal{I})$ is 'complete' in the sense that $F\in\Sigma$ whenever $F\subseteq E\in\mathcal{I}$, and if $X\notin\mathcal{I}$, and $\underline{\theta}:\mathfrak{A}\to\Sigma$ is a lower density, then there is a lifting $\theta:\mathfrak{A}\to\Sigma$ such that $\underline{\theta}a\subseteq\theta a$ for every $a\in\mathfrak{A}$.

(b) Let X be a non-empty Baire space, $\widehat{\mathcal{B}}$ the σ-algebra of subsets of X with the Baire property (314Yd) and $\mathcal{M}$ the ideal of meager subsets of X. Show that there is a lifting θ from $\widehat{\mathcal{B}}/\mathcal{M}$ to $\widehat{\mathcal{B}}$ such that $\theta G^\bullet\supseteq G$ for every open $G\subseteq X$. (*Hint*: in 341Ya, set $\underline{\theta}(G^\bullet)=G$ for every regular open set G.)

(c) Let (X,Σ,μ) be a Maharam-type-homogeneous probability space with Maharam type $\kappa\ge\omega$. Let Σ be the Baire σ-algebra of $Y=\{0,1\}^\kappa$, that is, the σ-algebra of subsets of Y generated by the family $\{\{x:x(\xi)=1\}:\xi<\kappa\}$, and let ν be the restriction to Σ of the usual measure on $\{0,1\}^\kappa$. Show that there is an inverse-measure-preserving function $f:X\to Y$ which induces an isomorphism between their measure algebras.

(d) Let (X,Σ,μ) be a complete Maharam-type-homogeneous probability space with Maharam type $\kappa\ge\omega$, and let ν be the usual measure on $\{0,1\}^\kappa$. Show that there is an inverse-measure-preserving function $f:X\to Y$ which induces an isomorphism between their measure algebras.

(e) Let (X,Σ,μ) be a semi-finite measure space which is not purely atomic. Write $\mathcal{L}^1_{\text{strict}}$ for the linear space of integrable functions $f:X\to\mathbb{R}$. Show that there is no operator $T:L^1(\mu)\to\mathcal{L}^1_{\text{strict}}$ such that (i) $(Tu)^\bullet=u$ for every $u\in L^1(\mu)$ (ii) $Tu\ge Tv$ whenever $u\ge v$ in $L^1(\mu)$. (*Hint*: Suppose first that μ is the usual measure on $\{0,1\}^{\mathbb{N}}$. Let F be the countable set of continuous functions $f:\{0,1\}^{\mathbb{N}}\to\mathbb{N}$. Show that if T satisfies (i) then there is an $x\in\{0,1\}^{\mathbb{N}}$ such that $T(f^\bullet)(x)=f(x)$ for every $f\in F$; find a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in F such that $\{f_n^\bullet:n\in\mathbb{N}\}$ is bounded above in $L^1(\mu)$ but $\sup_{n\in\mathbb{N}}f_n(x)=\infty$. Now transfer this argument to some atomless fragment of X.)

341Z Problems (a) Can we construct, using the ordinary axioms of mathematics (including the axiom of choice, but not the continuum hypothesis), a probability space (X,Σ,μ) with no lifting?

(b) Set $\kappa=\omega_3$. (There is a reason for taking ω_3 here; see Volume 5, when it appears, or Fremlin 89.) Let Σ be the Baire σ-algebra of $X=\{0,1\}^\kappa$ (as in 341Yc), and let μ be the restriction to Σ of the usual measure on $\{0,1\}^\kappa$. Does (X,Σ,μ) have a lifting?

341 Notes and comments Innumerable variations of the proof of 341K have been devised, as each author has struggled with the technical complications. I have discussed the reasons for my own choices in 341L.

The theorem has a curious history. It was originally announced by von Neumann, but he seems never to have written his proof down, and the first published proof is that of Maharam 58. That argument is based on Maharam's theorem, 341Xd and 341Yd, which show that it is enough to find liftings for every $\{0,1\}^\kappa$; this requires most of the ideas presented above, but feels more concrete, and some of the details are slightly simpler. The argument as I have written it owes a great deal to Ionescu Tulcea & Ionescu Tulcea 69.

The lifting theorem and Maharam's theorem are the twin pillars of modern abstract measure theory. But there remains a degree of mystery about the lifting theorem which is absent from the other. The first point is that there is nothing canonical about the liftings we can construct, except in the quite exceptional case of Stone spaces (341O). Even when there is a more or less canonical lower density present (341E, 341Xb), the conversion of this into a lifting requires arbitrary choices, as in 341J. While we can distinguish some liftings as being somewhat more regular than others, I know of no criterion which marks out any particular lifting of Lebesgue measure, for instance, among the rest. Perhaps associated with this arbitrariness is the extreme difficulty of deciding whether liftings of any given type exist. Neither positive nor negative results are easily come by (I will present a few in the later sections of this chapter), and the nature of the obstacles remains quite unclear.

342 Compact measure spaces

The next three sections amount to an extended parenthesis, showing how the Lifting Theorem can be used to attack one of the fundamental problems of measure theory: the representation of Boolean homomorphisms between measure algebras by functions between appropriate measure spaces. This section prepares for the main idea by introducing the class of 'locally compact' measures (342Ad), with the associated concepts of 'compact' and 'perfect' measures (342Ac, 342K). These depend on the notions of 'inner regularity' (342Aa, 342B) and 'compact class' (342Ab, 342D). I list the basic permanence properties for compact and locally compact measures (342G-342I) and mention some of the compact measures which we have already seen (342J). Concerning perfect measures, I content myself with the proof that a locally compact measure is perfect (342L). I end the section with two examples (342M, 342N).

342A Definitions (a) Let (X, Σ, μ) be a measure space. If $\mathcal{K} \subseteq \mathcal{P}X$, I will say that μ is **inner regular** with respect to $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

for every $E \in \Sigma$.

Of course μ is inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cap \Sigma$.

(b) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property.

Note that any subset of a compact class is again a compact class. (In particular, it is convenient to allow the empty set as a compact class.)

(c) A measure space (X, Σ, μ), or a measure μ, is **compact** if μ is inner regular with respect to some compact class of subsets of X.

Allowing $\emptyset$ as a compact class, and interpreting $\sup \emptyset$ as 0 in (a) above, μ is a compact measure whenever $\mu X = 0$.

(d) A measure space (X, Σ, μ), or a measure μ, is **locally compact** if the subspace measure μ_E is compact whenever $E \in \Sigma$ and $\mu E < \infty$.

Remark I ought to point out that the original definitions of 'compact class' and 'compact measure' (Marczewski 53) correspond to what I call 'countably compact class' and 'countably compact measure' in Volume 4. For another variation on the concept of 'compact class' see condition (β) in 343B(ii)-(iii).

For examples of compact measure spaces see 342J.

342B I prepare the ground with some straightforward lemmas.

Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{K} \subseteq \Sigma$ a set such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. Let $E \in \Sigma$.

(a) There is a countable disjoint set $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $K \subseteq E$ for every $K \in \mathcal{K}_1$ and $\mu(\bigcup \mathcal{K}_1) = \mu E$.

(b) If $\mu E < \infty$ then $\mu(E \setminus \bigcup \mathcal{K}_1) = 0$.

(c) In any case, there is for any $\gamma < \mu E$ a finite disjoint $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $K \subseteq E$ for every $K \in \mathcal{K}_0$ and $\mu(\bigcup \mathcal{K}_0) \geq \gamma$.

proof Set $\mathcal{K}' = \{K : K \in \mathcal{K},\ K \subseteq E,\ \mu K > 0\}$. Let $\mathcal{K}^*$ be a maximal disjoint subfamily of $\mathcal{K}'$. If $\mathcal{K}^*$ is uncountable, then there is some $n \in \mathbb{N}$ such that $\{K : K \in \mathcal{K}^*,\ \mu K \geq 2^{-n}\}$ is infinite, so that there is a countable $\mathcal{K}_1 \subseteq \mathcal{K}^*$ such that $\mu(\bigcup \mathcal{K}_1) = \infty = \mu E$.

If $\mathcal{K}^*$ is countable, set $\mathcal{K}_1 = \mathcal{K}^*$. Then $F = \bigcup \mathcal{K}_1$ is measurable, and $F \subseteq E$. Moreover, there is no member of $\mathcal{K}'$ disjoint from F; but this means that $E \setminus F$ must be negligible. So $\mu F = \mu E$, and (a) is true. Now (b) and (c) follow at once, because

$$\mu(\bigcup \mathcal{K}_1) = \sup\{\mu(\bigcup \mathcal{K}_0) : \mathcal{K}_0 \subseteq \mathcal{K}_1 \text{ is finite}\}.$$

Remark This lemma can be thought of as two more versions of the principle of exhaustion; compare 215A.

342C Corollary Let (X, Σ, μ) be a measure space and $\mathcal{K} \subseteq \mathcal{P}X$ a family of sets such that (α) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$ and $K \cap K' = \emptyset$ (β) whenever $E \in \Sigma$ and $\mu E > 0$, there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof Apply 342Bc to $\mathcal{K} \cap \Sigma$.

342D Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X.

(a) The following are equiveridical:

(i) $\mathcal{K}$ is a compact class;

(ii) there is a topology $\mathfrak{T}$ on X such that X is compact and every member of $\mathcal{K}$ is a closed set for $\mathfrak{T}$.

(b) If $\mathcal{K}$ is a compact class, so are the families $\mathcal{K}_1 = \{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$ and $\mathcal{K}_2 = \{\bigcap \mathcal{K}' : \emptyset \neq \mathcal{K}' \subseteq \mathcal{K}\}$.

proof (a)(i)⇒(ii) Let $\mathfrak{T}$ be the topology generated by $\{X \setminus K : K \in \mathcal{K}\}$. Then of course every member of $\mathcal{K}$ is closed for $\mathfrak{T}$. Let $\mathcal{F}$ be an ultrafilter on X. Then $\mathcal{K} \cap \mathcal{F}$ has the finite intersection property; because $\mathcal{K}$ is a compact class, it has non-empty intersection; take $x \in X \bigcap(\mathcal{K} \cap \mathcal{F})$. The family

$$\{G : G \subseteq X, \text{ either } G \in \mathcal{F} \text{ or } x \notin G\}$$

is easily seen to be a topology on X, and contains $X \setminus K$ for every $K \in \mathcal{K}$ (because if $X \setminus K \notin \mathcal{F}$ then $K \in \mathcal{F}$ and $x \in K$), so includes $\mathfrak{T}$; but this just means that every $\mathfrak{T}$-open set containing x belongs to $\mathcal{F}$, that is, that $\mathcal{F} \to x$. As $\mathcal{F}$ is arbitrary, X is compact for $\mathfrak{T}$ (2A3R).

(ii)⇒(i) Use 3A3Da.

(b) Let $\mathfrak{T}$ be a topology on X such that X is compact and every member of $\mathcal{K}$ is closed for $\mathfrak{T}$; then the same is true of every member of $\mathcal{K}_1$ or $\mathcal{K}_2$.

342E Corollary Suppose that (X, Σ, μ) is a measure space and that $\mathcal{K}$ is a compact class such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is compact.

proof Set $\mathcal{K}_1 = \{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. By 342Db, $\mathcal{K}_1$ is a compact class, and by 342C μ is inner regular with respect to $\mathcal{K}_1$.

342F Corollary A measure space (X, Σ, μ) is compact iff there is a topology on X such that X is compact and μ is inner regular with respect to the closed sets.

proof (a) If μ is inner regular with respect to a compact class $\mathcal{K}$, then there is a compact topology on X such that every member of $\mathcal{K}$ is closed; now the family $\mathcal{F}$ of closed sets includes $\mathcal{K}$, so μ is also inner regular with respect to $\mathcal{F}$.

(b) If there is a compact topology on X such that μ is inner regular with respect to the family $\mathcal{K}$ of closed sets, then this is a compact class, so μ is a compact measure.

342G Now I look at the standard questions concerning preservation of the properties of 'compactness' or 'local compactness' under the usual manipulations.

Proposition (a) Any measurable subspace of a compact measure space is compact.

(b) The completion and c.l.d. version of a compact measure space are compact.

(c) A semi-finite measure space is compact iff its completion is compact iff its c.l.d. version is compact.

(d) The direct sum of a family of compact measure spaces is compact.

(e) The c.l.d. product of two compact measure spaces is compact.

(f) The product of any family of compact probability spaces is compact.

proof (a) Let (X,Σ,μ) be a compact measure space, and $E\in\Sigma$. If $\mathcal{K}$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$, then $\mathcal{K}_E=\mathcal{K}\cap\mathcal{P}E$ is a compact class (just because it is a subset of $\mathcal{K}$) and the subspace measure μ_E is inner regular with respect to $\mathcal{K}_E$.

(b) Let (X,Σ,μ) be a compact measure space. Write $(X,\check\Sigma,\check\mu)$ for *either* the completion *or* the c.l.d. version of (X,Σ,μ). Let $\mathcal{K}\subseteq\mathcal{P}X$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then $\check\mu$ is also inner regular with respect to $\mathcal{K}$. **P** If $E\in\check\Sigma$ and $\gamma<\check\mu E$ there is an $E'\in\Sigma$ such that $E'\subseteq E$ and $\mu E'>\gamma$; if $\check\mu$ is the c.l.d. version of μ, we may take $\mu E'$ to be finite. There is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E'$ and $\mu K\ge\gamma$. Now $\check\mu K=\mu K\ge\gamma$ and $K\subseteq E$ and $K\in\mathcal{K}\cap\check\Sigma$. **Q**

(c) Now suppose that (X,Σ,μ) is semi-finite; again write $(X,\check\Sigma,\check\mu)$ for *either* its completion *or* its c.l.d. version. We already know that if μ is compact, so is $\check\mu$. If $\check\mu$ is compact, let $\mathcal{K}\subseteq\mathcal{P}X$ be a compact class such that $\check\mu$ is inner regular with respect to $\mathcal{K}$. Set $\mathcal{K}^*=\{\bigcap\mathcal{K}':\emptyset\ne\mathcal{K}'\subseteq\mathcal{K}\}$; then $\mathcal{K}^*$ is a compact class (342Db). Now μ is inner regular with respect to $\mathcal{K}^*$. **P** Take $E\in\Sigma$ and $\gamma<\mu E$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\in\mathbb{N}}$ as follows. Because μ is semi-finite, there is an $E_0\subseteq E$ such that $E_0\in\Sigma$ and $\gamma<\mu E_0<\infty$. Given $E_n\in\Sigma$ such that $\mu E_n>\gamma$, there is a $K_n\in\mathcal{K}\cap\check\Sigma$ such that $K_n\subseteq E_n$ and $\check\mu K_n>\gamma$. Now there is an $E_{n+1}\in\Sigma$ such that $E_{n+1}\subseteq K_n$ and $\mu E_{n+1}>\gamma$. Continue. On completing the induction, set $K=\bigcap_{n\in\mathbb{N}}K_n=\bigcap_{n\in\mathbb{N}}E_n$, so that $K\in\mathcal{K}^*\cap\Sigma$ and $K\subseteq E$ and $\mu K=\lim_{n\to\infty}\mu E_n\ge\gamma$. As E, γ are arbitrary, μ is inner regular with respect to $\mathcal{K}^*$. **Q** As $\mathcal{K}^*$ is a compact class, μ is a compact measure.

(d) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of compact measure spaces, with direct sum (X,Σ,μ). We may suppose that each X_i is actually a subset of X, with μ_i the subspace measure. For each $i\in I$ let $\mathcal{K}_i\subseteq\mathcal{P}X_i$ be a compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K}=\bigcup_{i\in I}\mathcal{K}_i$ is a compact class, for if $\mathcal{K}'\subseteq\mathcal{K}$ has the finite intersection property, then $\mathcal{K}'\subseteq\mathcal{K}_i$ for some i, so has non-empty intersection. Now if $E\in\Sigma$, $\mu E>0$ there is some $i\in I$ such that $\mu_i(E\cap X_i)>0$, and we can find a $K\in\mathcal{K}_i\cap\Sigma_i\subseteq\mathcal{K}\cap\Sigma$ such that $K\subseteq E\cap X_i$ and $\mu_iK>0$, in which case $\mu K>0$. By 342E, μ is compact.

(e) Let (X,Σ,μ) and (Y,T,ν) be two compact measure spaces, with c.l.d. product measure $(X\times Y,\Lambda,\lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X, Y respectively such that X and Y are compact spaces and μ, ν are inner regular with respect to the closed sets. Then the product topology on $X\times Y$ is compact (3A3J).

The point is that λ is inner regular with respect to the family $\mathcal{K}$ of closed subsets of $X\times Y$. **P** Suppose that $W\in\Lambda$ and $\lambda W>\gamma$. Then there are $E\in\Sigma$, $F\in\mathrm{T}$ such that $\mu E<\infty$, $\nu F<\infty$ and $\lambda(W\cap(E\times F))>\gamma$ (251F). Now there are sequences $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ in Σ, T respectively such that

$$(E\times F)\setminus W\subseteq\bigcup_{n\in\mathbb{N}}E_n\times F_n,$$

$$\sum_{n=0}^\infty\mu E_n\cdot\nu F_n<\lambda((E\times F)\setminus W)+\lambda((E\times F)\cap W)-\gamma=\lambda(E\times F)-\gamma$$

(251C). Set

$$W'=(E\times F)\setminus\bigcup_{n\in\mathbb{N}}E_n\times F_n=\bigcap_{n\in\mathbb{N}}((E\times(F\setminus F_n))\cup((E\setminus E_n)\times F)).$$

Then $W'\subseteq W$, and

$$\lambda((E\times F)\setminus W')\le\lambda(\bigcup_{n\in\mathbb{N}}E_n\times F_n)\le\sum_{n=0}^\infty\mu E_n\cdot\nu F_n<\lambda(E\times F)-\gamma,$$

so $\lambda W'>\gamma$.

Set $\epsilon=\frac{1}{4}(\lambda W'-\gamma)/(1+\mu E+\mu F)$. For each n, we can find closed measurable sets K_n, $K'_n\subseteq X$ and L_n, $L'_n\subseteq Y$ such that

$$K_n\subseteq E,\quad \mu(E\setminus K_n)\le 2^{-n}\epsilon,$$

$$L'_n \subseteq F \setminus F_n, \quad \nu((F \setminus F_n) \setminus L'_n) \le 2^{-n}\epsilon,$$

$$K'_n \subseteq E \setminus E_n, \quad \mu((E \setminus E_n) \setminus K'_n) \le 2^{-n}\epsilon,$$

$$L_n \subseteq F, \quad \nu(F \setminus L_n) \le 2^{-n}\epsilon.$$

Set

$$V = \bigcap_{n\in\mathbb{N}}(K_n \times L'_n) \cup (K'_n \times L_n) \subseteq W' \subseteq W.$$

Now

$$W' \setminus V \subseteq \bigcup_{n\in\mathbb{N}}((E \setminus K_n) \times F) \cup (E \times ((F \setminus F_n) \setminus L'_n)) \cup (((E \setminus E_n) \setminus K'_n) \times F) \cup (E \times (F \setminus L_n)),$$

so

$$\begin{aligned}\lambda(W' \setminus V) &\le \sum_{n=0}^{\infty} \mu(E \setminus K_n) \cdot \nu F + \mu E \cdot \nu((F \setminus F_n) \setminus L'_n) \\ &\qquad + \mu((E \setminus E_n) \setminus K'_n) \cdot \nu F + \mu E \cdot \nu(F \setminus L_n) \\ &\le \sum_{n=0}^{\infty} 2^{-n}\epsilon(2\mu E + 2\mu F) \le \lambda W' - \gamma,\end{aligned}$$

and $\lambda V \ge \gamma$. But V is a countable intersection of finite unions of products of closed measurable sets, so is itself a closed measurable set, and belongs to $\mathcal{K} \cap \Lambda$. **Q**

Accordingly the product topology on $X \times Y$ witnesses that λ is a compact measure.

(f) The same method works. In detail: let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of compact probability spaces, with product (X, Λ, λ). For each i, let $\mathfrak{T}_i$ be a topology on X_i such that X_i is compact and μ_i is inner regular with respect to the closed sets. Give X the product topology; this is compact. If $W \in \Lambda$ and $\epsilon > 0$, let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable cylinder sets (in the sense of 254A) such that $X \setminus W \subseteq \bigcup_{n\in\mathbb{N}} C_n$ and $\sum_{n=0}^{\infty} \lambda C_n \le \lambda(X \setminus W) + \epsilon$. Express each C_n as $\prod_{i\in I} E_{ni}$ where $E_{ni} \in \Sigma_i$ for each i and $J_n = \{i : E_{ni} \ne X_i\}$ is finite. For $n \in \mathbb{N}$, $i \in J_n$ set $\epsilon_{ni} = 2^{-n}\epsilon/(1 + \#(J_n))$. Choose closed measurable sets $K_{ni} \subseteq X_i \setminus E_{ni}$ such that $\mu_i((X_i \setminus E_{ni}) \setminus K_{ni}) \le \epsilon_{ni}$ for $n \in \mathbb{N}$, $i \in J_n$. For each $n \in \mathbb{N}$, set

$$V_n = \bigcup_{i\in J_n}\{x : x \in X,\, x(i) \in K_{ni}\},$$

so that V_n is a closed measurable subset of X. Observe that

$$X \setminus V_n = \{x : x(i) \in X \setminus K_{ni} \text{ for } i \in J_n\}$$

includes C_n, and that

$$\lambda(X \setminus (V_n \cup C_n)) \le \sum_{i\in J_n} \lambda\{x : x(i) \in X_i \setminus (K_{ni} \cup E_{ni})\} \le \sum_{i\in J_n} \epsilon_{ni} \le 2^{-n}\epsilon.$$

Now set $V = \bigcap_{n\in\mathbb{N}} V_n$; then V is again a closed measurable set, and

$$X \setminus V \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup (X \setminus (C_n \cup V_n))$$

has measure at most

$$\sum_{n=0}^{\infty} \lambda C_n + 2^{-n}\epsilon \le 1 - \lambda W + \epsilon + 2\epsilon,$$

so $\lambda V \ge \lambda W - 3\epsilon$. As W and ϵ are arbitrary, λ is inner regular with respect to the closed sets, and is a compact measure.

342H Proposition (a) A compact measure space is locally compact.

(b) A strictly localizable locally compact measure space is compact.

(c) Let (X, Σ, μ) be a measure space. Suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F > 0$ and the subspace measure on F is compact. Then μ is locally compact.

proof (a) This is immediate from 342Ga and the definition of 'locally compact' measure space.

(b) Suppose that (X, Σ, μ) is a strictly localizable locally compact measure space. Let $\langle X_i \rangle_{i \in I}$ be a decomposition of X, and for each $i \in I$ let μ_i be the subspace measure on X_i. Then μ_i is compact. Now μ can be identified with the direct sum of the μ_i, so itself is compact, by 342Gd.

(c) Write $\mathcal{F}$ for the set of measurable sets $F \subseteq X$ such that the subspace measures μ_F are compact. Take $E \in \Sigma$ with $\mu E < \infty$. By 342Bb, there is a countable disjoint family $\langle F_i \rangle_{i \in I}$ in $\mathcal{F}$ such that $F_i \subseteq E$ for each i, and $F' = E \setminus \bigcup_{i \in I} F_i$ is negligible; now this means that $F' \in \mathcal{F}$ (342Ac), so we may take it that $E = \bigcup_{i \in I} F_i$. In this case μ_E is isomorphic to the direct sum of the measures μ_{F_i} and is compact. As E is arbitrary, μ is locally compact.

342I Proposition (a) Any measurable subspace of a locally compact measure space is locally compact.

(b) A measure space is locally compact iff its completion is locally compact iff its c.l.d. version is locally compact.

(c) The direct sum of a family of locally compact measure spaces is locally compact.

(d) The c.l.d. product of two locally compact measure spaces is locally compact.

proof (a) Trivial: if (X, Σ, μ) is locally compact, and $E \in \Sigma$, and $F \subseteq E$ is a measurable set of finite measure for the subspace measure on E, then $F \in \Sigma$ and $\mu F < \infty$, so the subspace measure on F is compact.

(b) Let (X, Σ, μ) be a measure space, and write $(X, \check{\Sigma}, \check{\mu})$ for *either* its completion *or* its c.l.d. version.

(i) Suppose that μ is locally compact, and that $\check{\mu} F < \infty$. Then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\mu E = \check{\mu} F$. Let μ_E be the subspace measure on E induced by the measure μ; then we are assuming that μ_E is compact. Let $\mathcal{K} \subseteq \mathcal{P}E$ be a compact class such that μ_E is inner regular with respect to $\mathcal{K}$. Then, as in the proof of 342Gb, the subspace measure $\check{\mu}_F$ on F induced by $\check{\mu}$ is also inner regular with respect to $\mathcal{K}$, so $\check{\mu}_F$ is compact; as F is arbitrary, $\check{\mu}$ is locally compact.

(ii) Now suppose that $\check{\mu}$ is locally compact, and that $\mu E < \infty$. Then the subspace measure $\check{\mu}_E$ is compact. But this is just the completion of the subspace measure μ_E, so μ_E is compact, by 342Gc; as E is arbitrary, μ is locally compact.

(c) Put (a) and 342Hc together.

(d) Let (X, Σ, μ) and (Y, T, ν) be locally compact measure spaces, with product $(X \times Y, \Lambda, \lambda)$. If $W \in \Lambda$ and $\lambda W > 0$, there are $E \in \Sigma$, $F \in \mathrm{T}$ such that $\mu E < \infty$, $\nu F < \infty$ and $\lambda(W \cap (E \times F)) > 0$. Now the subspace measure $\lambda_{E \times F}$ induced by λ on $E \times F$ is just the product of the subspace measures (251P(ii-α), so is compact, and the subspace measure $\lambda_{W \cap (E \times F)}$ is therefore again compact, by 342Ga. By 342Hc, this is enough to show that λ is locally compact.

342J Examples It is time I listed some examples of compact measure spaces.

(a) Lebesgue measure on $\mathbb{R}^r$ is compact. (Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}^r$ which are compact for the usual topology. By 134Fb, Lebesgue measure is inner regular with respect to $\mathcal{K}$.)

(b) Similarly, any Radon measure on $\mathbb{R}^r$ (256A) is compact.

(c) If $(\mathfrak{A}, \bar{\mu})$ is any semi-finite measure algebra, the standard measure ν on its Stone space Z is compact. (By 322Qa, ν is inner regular with respect to the family of open-and-closed subsets of Z, which are all compact for the standard topology of Z, so form a compact class.)

(d) The usual measure on $\{0,1\}^I$ is compact, for any set I. (It is obvious that the usual measure on $\{0,1\}$ is compact; now use 342Gf.)

Remark (a)-(c) above are special cases of the fact that all Radon measures are compact; I will return to this in §416.

342K One of the most important properties of (locally) compact measure spaces has been studied under the following name.

Definition Let (X, Σ, μ) be a measure space. Then (X, Σ, μ), or μ, is **perfect** if whenever $f : X \to \mathbb{R}$ is measurable, $E \in \Sigma$ and $\mu E > 0$, then there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$.

342L Theorem A semi-finite locally compact measure space is perfect.

proof Let (X, Σ, μ) be a semi-finite locally compact measure space, $f : X \to \mathbb{R}$ a measurable function, and $E \in \Sigma$ a set of non-zero measure. Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Now the subspace measure μ_F is compact; let $\mathfrak{T}$ be a topology on F such that F is compact and μ_F is inner regular with respect to the family $\mathcal{K}$ of closed sets for $\mathfrak{T}$.

Let $\langle \epsilon_q \rangle_{q \in \mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q \in \mathbb{Q}} \epsilon_q < \frac{1}{2}\mu F$. (For instance, you could set $\epsilon_{q(n)} = 2^{-n-3}\mu F$ where $\langle q(n) \rangle_{n \in \mathbb{N}}$ is an enumeration of $\mathbb{Q}$.) For each $q \in \mathbb{Q}$, set $E_q = \{x : x \in F,\, f(x) \le q\}$, $E'_q = \{x : x \in F,\, f(x) > q\}$, and choose K_q, $K'_q \in \mathcal{K} \cap \Sigma$ such that $K_q \subseteq E_q$, $K'_q \subseteq E'_q$, $\mu(E_q \setminus K_q) \le \epsilon_q$ and $\mu(E'_q \setminus K'_q) \le \epsilon_q$. Then $K = \bigcap_{q \in \mathbb{Q}} (K_q \cup K'_q) \in \mathcal{K} \cap \Sigma$, $K \subseteq F$ and

$$\mu(F \setminus K) \le \textstyle\sum_{q \in \mathbb{Q}} \mu(E_q \setminus K_q) + \mu(E'_q \setminus K'_q) < \mu F,$$

so $\mu K > 0$.

The point is that $f \restriction K$ is continuous. **P** For any $q \in \mathbb{Q}$, $\{x : x \in K,\, f(x) \le q\} = K \cap K_q$ and $\{x : x \in K,\, f(x) > q\} = K \cap K'_q$. If $H \subseteq \mathbb{R}$ is open and $x \in K \cap f^{-1}[H]$, take q, $q' \in \mathbb{Q}$ such that $f(x) \in \,]q, q'] \subseteq H$; then $G = K \setminus (K_q \cup K'_{q'})$ is a relatively open subset of K containing x and included in $f^{-1}[H]$. Thus $K \cap f^{-1}[H]$ is relatively open in K; as H is arbitrary, f is continuous. **Q**

Accordingly $f[K]$ is a continuous image of a compact set, therefore compact; it is a subset of $f[E]$, and $\mu f^{-1}[f[K]] \ge \mu K > 0$. As f and E are arbitrary, μ is perfect.

342M I ought to give examples to distinguish between the concepts introduced here, partly on general principles, but also because it is not obvious that the concept of 'locally compact' measure space is worth spending time on at all. It is easy to distinguish between 'perfect' and '(locally) compact'; 'locally compact' and 'compact' are harder to separate.

Example Let X be an uncountable set and μ the countable-cocountable measure on X (211R). Then μ is perfect but not compact or locally compact.

proof (a) If $f : X \to \mathbb{R}$ is measurable and $E \subseteq X$ is measurable, with measure greater than 0, set $A = \{\alpha : \alpha \in \mathbb{R}, \{x : x \in X,\, f(x) \le r\}$ is negligible$\}$. Then $\alpha \in A$ whenever $\alpha \le \beta \in A$. Since $X = \bigcup_{n \in \mathbb{N}} \{x : f(x) \le n\}$, there is some n such that $n \notin A$, in which case A is bounded above by n. Also there is some $m \in \mathbb{N}$ such that $\{x : f(x) > -m\}$ is non-negligible, in which case it must be conegligible, and $-m \in A$, so A is non-empty. Accordingly $\gamma = \sup A$ is defined in $\mathbb{R}$. Now for any $k \in \mathbb{N}$, $\{x : f(x) \le \gamma - 2^{-k}\}$ is negligible, so $\{x : f(x) < \gamma\}$ is negligible. Also, for any k, $\{x : f(x) \le \gamma + 2^{-k}\}$ is non-negligible, so $\{x : f(x) > \gamma_2^{-k}\}$ must be negligible; accordingly, $\{x : f(x) > \gamma\}$ is negligible. But this means that $\{x : f(x) = \gamma\}$ is conegligible and has measure 1. Thus we have a compact set $K = \{\gamma\}$ such that $\mu f^{-1}[K] = 1$, and γ must belong to $f[E]$. As f and E are arbitrary, μ is perfect.

(b) μ is not compact. **P?** Suppose, if possible, that $\mathcal{K} \subseteq \mathcal{P}X$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then for every $x \in X$ there is a measurable set $K_x \in \mathcal{K}$ such that $K_x \subseteq X \setminus \{x\}$ and $\mu K_x > 0$, that is, K_x is conegligible. But this means that $\{K_x : x \in X\}$ must have the finite intersection property; as it also has empty intersection, $\mathcal{K}$ cannot be a compact class. **XQ**

(c) Because μ is totally finite, it cannot be locally compact (342Hb).

***342N Example** There is a complete locally determined localizable locally compact measure space which is not compact.

proof (a) I refer to the example of 216E. In that construction, we have a set I and a family $\langle f_\gamma \rangle_{\gamma \in C}$ in $X = \{0,1\}^I$ such that for every $D \subseteq C$ there is an $i \in I$ such that $D = \{\gamma : f_\gamma(i) = 1\}$; moreover, $\#(C) > \mathfrak{c}$. The σ-algebra Σ is the family of sets $E \subseteq X$ such that for every γ there is a countable set $J \subseteq I$ such that $\{x : x \restriction J = f_\gamma \restriction J\}$ is a subset of either E or $X \setminus E$; and for $E \in \Sigma$, μE is $\#(\{\gamma : f_\gamma \in E\})$ if this is finite, ∞ otherwise. Note that any subset of X determined by a countable set of coordinates belongs to Σ.

For each $\gamma \in C$, let $i_\gamma \in I$ be such that $f_\gamma(i_\gamma) = 1$, $f_\delta(i_\gamma) = 0$ for $\delta \ne \gamma$. (In 216E I took I to be $\mathcal{P}C$, and i_γ would be $\{\gamma\}$.) Set

$$Y = \{x : x \in X,\, \{\gamma : \gamma \in C,\, x(i_\gamma) = 1\} \text{ is finite}\}.$$

Give Y its subspace measure μ_Y with domain Σ_Y. Then μ_Y is complete, locally determined and localizable (214Id). Note that $f_\gamma \in Y$ for every $\gamma \in C$.

(b) μ_Y is locally compact. **P** Suppose that $F \in \Sigma_Y$ and $\mu_Y F < \infty$. If $\mu_Y F = 0$ then surely the subspace measure μ_F is compact. Otherwise, we can express F as $E \cap Y$ where $E \in \Sigma$ and $\mu E = \mu_Y F$. Then $D = \{\gamma : f_\gamma \in E\} = \{\gamma : f_\gamma \in F\}$ is finite. For $\gamma \in D$ set

$$G'_\gamma = \{x : x \in X,\, x(i_\gamma) = 1,\, x(i_\delta) = 0 \text{ for every } \delta \in D \setminus \{\gamma\}\} \in \Sigma,$$

$$\mathcal{K}_\gamma = \{K : f_\gamma \in K \subseteq F \cap G'_\gamma\}.$$

Then each $\mathcal{K}_\gamma$ is a compact class, and members of different $\mathcal{K}_\gamma$'s are disjoint, so $\mathcal{K} = \bigcup_{\gamma \in D} \mathcal{K}_\gamma$ is a compact class.

Now suppose that H belongs to the subpsace σ-algebra Σ_F and $\mu_F H > 0$. Then there is a $\gamma \in D$ such that $f_\gamma \in H$, so that $H \cap G'_\gamma \in \mathcal{K} \cap \Sigma_F$ and $\mu_F(H \cap G'_\gamma) > 0$. By 342E, this is enough to show that μ_F is compact. As F is arbitrary, μ_Y is locally compact. **Q**

(c) μ_Y is not compact. **P?** Suppose, if possible, that μ_Y is inner regular with respect to a compact class $\mathcal{K} \subseteq \mathcal{P}Y$. For each $\gamma \in C$ set $G_\gamma = \{x : x \in X,\, x(i_\gamma) = 1\}$, so that $f_\gamma \in G_\gamma \in \Sigma$ and $\mu_Y(G_\gamma \cap Y) = 1$. There must therefore be a $K_\gamma \in \mathcal{K}$ such that $K_\gamma \subseteq G_\gamma \cap Y$ and $\mu_Y K_\gamma = 1$ (since μ_Y takes no value in $]0,1[$). Express K_γ as $Y \cap E_\gamma$, where $E_\gamma \in \Sigma$, and let $J_\gamma \subseteq I$ be a countable set such that

$$E_\gamma \supseteq \{x : x \in X,\, x{\restriction}J_\gamma = f_\gamma{\restriction}J_\gamma\}.$$

At this point I call on the full strength of 2A1P. There is a set $B \subseteq C$, of cardinal greater than $\mathfrak{c}$, such that $f_\gamma{\restriction}J_\gamma \cap J_\delta = f_\delta{\restriction}J_\gamma \cap J_\delta$ for all γ, $\delta \in B$. But this means that, for any finite set $D \subseteq B$, we can define $x \in X$ by setting

$$\begin{aligned} x(i) &= f_\alpha(i) \text{ if } \alpha \in D,\, i \in J_\alpha, \\ &= 0 \text{ if } i \in I \setminus \bigcup_{\alpha \in D} J_\alpha. \end{aligned}$$

It is easy to check that $\{\gamma : \gamma \in C,\, x(i_\gamma) = 1\} = D$, so that $x \in Y$; but now

$$x \in Y \cap \bigcap_{\alpha \in D} E_\alpha = \bigcap_{\alpha \in D} K_\alpha.$$

What this shows is that $\{K_\alpha : \alpha \in B\}$ has the finite intersection property. It must therefore have non-empty intersection; say

$$y \in \bigcap_{\alpha \in B} K_\alpha \subseteq \bigcap_{\alpha \in B} G_\alpha.$$

But now we have a member y of Y such that $\{\gamma : y(i_\gamma) = 1\} \supseteq B$ is infinite, contrary to the definition of Y. **XQ**

342X Basic exercises >(a) Show that a measure space (X, Σ, μ) is semi-finite iff μ is inner regular with respect to $\{E : \mu E < \infty\}$.

(b) Find a proof of 342B based on 215A.

(c) Let (X, Σ, μ) be a locally compact semi-finite measure space in which all singleton sets are negligible. Show that it is atomless.

(d) Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ (234B). Show that ν is compact, or locally compact, if μ is. (*Hint*: if $\mathcal{K}$ satisfies the conditions of 342E with respect to μ, then it satisfies them for ν.)

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be any non-decreasing function, and ν_f the corresponding Lebesgue-Stieltjes measure. Show that ν_f is compact. (*Hint*: 256Xg.)

(f) Let μ be Lebesgue measure on $[0,1]$, ν the countable-cocountable measure on $[0,1]$, and λ their c.l.d. product. Show that λ is a compact measure. (*Hint*: let $\mathcal{K}$ be the family of sets $K \times A$ where $A \subseteq [0,1]$ is cocountable and $K \subseteq A$ is compact.)

(g) (i) Give an example of a compact probability space (X,Σ,μ), a set Y and a function $f:X\to Y$ such that the image measure μf^{-1} is not compact. (ii) Give an example of a compact probability space (X,Σ,μ) and a σ-subalgebra T of Σ such that $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is not compact. (*Hint*: 342Xf.)

(h) Let (X,Σ,μ) be a perfect measure space, and $f:X\to\mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is inner regular with respect to the compact subsets of $\mathbb{R}$, so is a compact measure.

(i) Let (X,Σ,μ) be a σ-finite measure space. Show that it is perfect iff for every measurable $f:X\to\mathbb{R}$ there is a Borel set $H\subseteq f[X]$ such that $f^{-1}[H]$ is conegligible in X. (*Hint*: 342Xh for 'only if', 256C for 'if'.)

(j) Let (X,Σ,μ) be a complete totally finite perfect measure space and $f:X\to\mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is a Radon measure, and is the only Radon measure on $\mathbb{R}$ for which f is inverse-measure-preserving. (*Hint*: 256G.)

(k) Suppose that (X,Σ,μ) is a perfect measure space. (i) Show that if (Y,T,ν) is a measure space, and $f:X\to Y$ is a function such that $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $f^{-1}[F]$ is μ-negligible for every ν-negligible set F, then (Y,T,ν) is perfect. (ii) Show that if T is a σ-subalgebra of Σ then $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is perfect.

(l) Let (X,Σ,μ) be a perfect measure space such that Σ is the σ-algebra generated by a sequence of sets. Show that μ is compact. (*Hint*: if Σ is generated by $\{E_n:n\in\mathbb{N}\}$, set $f=\sum_{n=0}^{\infty}3^{-n}\chi E_n$ and consider $\{f^{-1}[K]:K\subseteq f[X]\text{ is compact}\}$.)

(m) Let (X,Σ,μ) be a semi-finite measure space. Show that μ is perfect iff $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ.

(n) Show that (i) a measurable subspace of a perfect measure space is perfect (ii) a semi-finite measure space is perfect iff all its totally finite subspaces are perfect (iii) the direct sum of any family of perfect measure spaces is perfect (iv) the c.l.d. product of two perfect measure spaces is perfect (*hint*: put 342Xm and 342Ge together) (v) the product of any family of perfect probability spaces is perfect (vi) a measure space is perfect iff its completion is perfect (vii) the c.l.d. version of a perfect measure space is perfect (viii) any purely atomic measure space is perfect (ix) an indefinite-integral measure over a perfect measure is perfect.

(o) Let μ be Lebesgue measure on $\mathbb{R}$, let A be a subset of $\mathbb{R}$, and let μ_A be the subspace measure. Show that μ_A is compact iff it is perfect iff A is Lebesgue measurable. (*Hint*: if μ_A is perfect, consider the image measure $h\mu_A^{-1}$ on $\mathbb{R}$, where $h(x)=x$ for $x\in A$.)

342Y Further exercises (a) Show that the space (X,Σ,μ) of 216E and 342N is a compact measure space. (*Hint*: use the usual topology on $X=\{0,1\}^I$.)

(b) Give an example of a compact complete locally determined measure space which is not localizable. (*Hint*: in 216D, add a point to each horizontal and vertical section of X, so that all the sections become compact measure spaces.)

342 Notes and comments The terminology I find myself using in this section – 'compact', 'locally compact', 'countably compact', 'perfect' – is not entirely satisfactory, in that it risks collision with the same words applied to topological spaces. For the moment, this is not a serious problem; but when in Volume 4 we come to the systematic analysis of spaces which have both topologies and measures present, it will be necessary to watch our language carefully. Of course there are cases in which a 'compact class' of the sort discussed here can be taken to be the family of compact sets for some familiar topology, as in 342Ja-342Jd, but in others this is not so (see 342Xf); and even when we have a familiar compact class, the topology constructed from it by the method of 342Da need not be one we might expect. (Consider, for instance, the topology on $\mathbb{R}$ for which the closed sets are just the sets which are compact for the usual topology.)

I suppose that 'compact' and 'perfect' measure spaces look reasonably natural objects to study; they offer to illuminate one of the basic properties of Radon measures, the fact that (at least for totally finite Radon measures on Euclidean space) the image measure of a Radon measure under a measurable function is again Radon (256G, 342Xj). Indeed this was the original impetus for the study of perfect measures (Gnedenko & Kolmogorov 54, Sazonov 66). It is not obvious that there is any need to examine 'locally compact' measure spaces, but actually they are the chief purpose of this section, since the main theorem of the next section is an alternative characterization of semi-finite locally compact measure spaces (343B). Of course you may feel that the fact that 'locally compact' and 'compact' coincide for strictly localizable spaces (342Hb) excuses you from troubling about the distinction at first reading.

As with any new classification of measure spaces, it is worth finding out how the classes of 'compact' and 'perfect' measure spaces behave with respect to the standard constructions. I run through the basic facts in 342G-342I, 342Xd, 342Xk and 342Xn. We can also look for relationships between the new properties and those already studied. Here, in fact, there is not much to be said; 342N and 342Yb show that 'compactness' is largely independent of the classification in §211. However there are interactions with the concept of 'atom' (342Xc, 342Xn(viii)).

I give examples to show that perfect measure spaces need not be locally compact, and that locally compact measure spaces need not be compact (342M, 342N). The standard examples of measure spaces which are not perfect are non-measurable subspaces (342Xo); I will return to these in the next section (343L-343M).

Something which is not important to us at the moment, but is perhaps worth taking note of, is the following observation. To determine whether a measure space (X,Σ,μ) is compact, we need only the structure $(X,\Sigma,\mathcal{N})$, where $\mathcal{N}$ is the σ-ideal of negligible sets, since that is all that is referred to in the criterion of 342E. The same is true of local compactness, by 342Hc, and of perfectness, by the definition in 342K. Compare 342Xd, 342Xk, 342Xn(ix).

Much of the material of this section will be repeated in Volume 4 as part of a more systematic analysis of inner regularity.

343 Realization of homomorphisms

We are now in a position to make progress in one of the basic questions of abstract measure theory. In §324 I have already described the way in which a function between two measure spaces can give rise to a homomorphism between their measure algebras. In this section I discuss some conditions under which we can be sure that a homomorphism can be represented by a function.

The principal theorem of the section is 343B. If a measure space (X,Σ,μ) is locally compact, then many homomorphisms from the measure algebra of μ to other measure algebras will be representable by functions into X; moreover, this characterizes locally compact spaces. In general, a homomorphism between measure algebras can be represented by widely different functions (343I, 343J). But in some of the most important cases (e.g., Lebesgue measure) representing functions are 'almost' uniquely defined; I introduce the concept of 'countably separated' measure space to describe these (343D-343H).

343A Preliminary remarks It will be helpful to establish some vocabulary and a couple of elementary facts.

(a) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, I will say that a function $f:X\to Y$ **represents** a homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ if $f^{-1}[F]\in\Sigma$ and $(f^{-1}[F])^\bullet=\pi(F^\bullet)$ for every $F\in\mathrm{T}$.

(Perhaps I should emphasize here that some homomorphisms are representable in this sense, and some are not; see 343M below for examples of non-representable homomorphisms.)

(b) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, $f:X\to Y$ is a function, and $\pi:\mathfrak{B}\to\mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then

$$\{F:F\in\mathrm{T},\ f^{-1}[F]\in\Sigma\text{ and }f^{-1}[F]^\bullet=\pi F^\bullet\}$$

is a σ-subalgebra of T. (The verification is elementary.)

(c) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, and $\pi:\mathfrak{B}\to\mathfrak{A}$ a Boolean homomorphism which is represented by a function $f:X\to Y$. Let $(X,\hat{\Sigma},\hat{\mu})$, $(Y,\hat{\mathrm{T}},\hat{\nu})$ be the completions of (X,Σ,μ), (Y,T,ν); then $\mathfrak{A}$ and $\mathfrak{B}$ can be identified with the measure algebras of $\hat{\mu}$ and $\hat{\nu}$ (322Da). Now f still represents π when regarded as a function from $(X,\hat{\Sigma},\hat{\mu})$ to $(Y,\hat{\mathrm{T}},\hat{\nu})$. **P** If G is ν-negligible, there is a negligible $F\in\mathrm{T}$ such that $G\subseteq F$; since

$$f^{-1}[F]^\bullet=\pi F^\bullet=0,$$

$f^{-1}[F]$ is μ-negligible, so $f^{-1}[E]$ is negligible, therefore belongs to $\hat{\Sigma}$. If G is any element of $\hat{\mathrm{T}}$, there is an $F\in\mathrm{T}$ such that $G\triangle F$ is negligible, so that

$$f^{-1}[G]=f^{-1}[F]\triangle f^{-1}[G\triangle F]\in\hat{\Sigma},$$

and

$$f^{-1}[G]^\bullet=f^{-1}[F]^\bullet=\pi F^\bullet=\pi G^\bullet.\ \mathbf{Q}$$

(d) In particular, if (X,Σ,μ) and (Y,T,ν) are measure spaces, and $f:X\to Y$ is inverse-measure-preserving, then f is still inverse-measure-preserving with respect to the completed measures $\hat{\mu}$ and $\hat{\nu}$ (apply (c) with $\pi:\mathfrak{B}\to\mathfrak{A}$ the homomorphism induced by f). (See 235Hc.)

343B Theorem Let (X,Σ,μ) be a non-empty semi-finite measure space, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Let (Z,Λ,λ) be the Stone space of $(\mathfrak{A},\bar{\mu})$; for $E\in\Sigma$ write E^* for the open-and-closed subset of Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Then the following are equiveridical:

(i) (X,Σ,μ) is locally compact in the sense of 342Ad.

(ii) There is a family $\mathcal{K}\subseteq\Sigma$ such that (α) whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$ (β) whenever $\mathcal{K}'\subseteq\mathcal{K}$ is such that $\mu(\bigcap\mathcal{K}_0)>0$ for every non-empty finite set $\mathcal{K}_0\subseteq\mathcal{K}'$, then $\bigcap\mathcal{K}'\ne\emptyset$.

(iii) There is a family $\mathcal{K}\subseteq\Sigma$ such that $(\alpha)'$ μ is inner regular with respect to $\mathcal{K}$ (β) whenever $\mathcal{K}'\subseteq\mathcal{K}$ is such that $\mu(\bigcap\mathcal{K}_0)>0$ for every non-empty finite set $\mathcal{K}_0\subseteq\mathcal{K}'$, then $\bigcap\mathcal{K}'\ne\emptyset$.

(iv) There is a function $f:Z\to X$ such that $f^{-1}[E]\triangle E^*$ is negligible for every $E\in\Sigma$.

(v) Whenever (Y,T,ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous Boolean homomorphism, then there is a $g:Y\to X$ representing π.

(vi) Whenever (Y,T,ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous measure-preserving Boolean homomorphism, then there is a $g:Y\to X$ representing π.

proof (a)(i)⇒(ii) Because μ is semi-finite, there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\bar{\mu}a_i<\infty$ for each i. For each $i\in I$, let $E_i\in\Sigma$ be such that $E_i^\bullet=a_i$. Then the subspace measure μ_{E_i} on E_i is compact; let $\mathcal{K}_i\subseteq\mathcal{P}E_i$ be a compact class such that μ_{E_i} is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{K}=\bigcup_{i\in I}\mathcal{K}_i$. If $\mathcal{K}'\subseteq\mathcal{K}$ and $\mu(\bigcap\mathcal{K}_0)>0$ for every non-empty finite $\mathcal{K}_0\subseteq\mathcal{K}$, then $\mathcal{K}'\subseteq\mathcal{K}_i$ for some i, and surely has the finite intersection property, so $\bigcap\mathcal{K}'\ne\emptyset$; thus $\mathcal{K}'$ satisfies (β) of condition (ii). And if $E\in\Sigma$, $\mu E>0$ then there must be some $i\in I$ such that $E_i^\bullet\cap a_i\ne0$, that is, $\mu(E\cap E_i)>0$, in which case there is a $K\in\mathcal{K}_i\subseteq\mathcal{K}$ such that $K\subseteq E\cap E_i$ and $\mu K>0$; so that $\mathcal{K}$ satisfies condition (α).

(b)(ii)⇒(iii) Suppose that $\mathcal{K}\subseteq\Sigma$ witnesses that (ii) is true. If $\mu X=0$ then $\mathcal{K}$ already witnesses that (iii) is true, so we need consider only the case $\mu X>0$. Set $\mathcal{L}=\{K_0\cup\ldots\cup K_n:K_0,\ldots,K_n\in\mathcal{K}\}$. Then $\mathcal{L}$ witnesses that (iii) is true. **P** By 342Ba, μ is inner regular with respect to $\mathcal{L}$. Let $\mathcal{L}'\subseteq\mathcal{L}$ be such that $\mu(\bigcap\mathcal{L}_0)>0$ for every non-empty finite $\mathcal{L}_0\subseteq\mathcal{L}'$. Then

$$\mathcal{F}_0=\{A:A\subseteq X,\text{ there is a finite }\mathcal{L}_0\subseteq\mathcal{L}'\text{ such that }X\cap\bigcap\mathcal{L}_0\setminus A\text{ is negligible}\}$$

is a filter on X, so there is an ultrafilter $\mathcal{F}$ on X including $\mathcal{F}_0$. Note that every conegligible set belongs to $\mathcal{F}_0$, so no negligible set can belong to $\mathcal{F}$. Set $\mathcal{K}'=\mathcal{K}\cap\mathcal{F}$; then $\bigcap\mathcal{K}_0$ belongs to $\mathcal{F}$, so is not negligible, for every non-empty finite $\mathcal{K}_0\subseteq\mathcal{K}'$. Accordingly there is some $x\in\bigcap\mathcal{K}'$. But any member of $\mathcal{L}'$ is of the form $L=K_0\cup\ldots\cup K_n$ where each $K_i\in\mathcal{K}$; because $\mathcal{F}$ is an ultrafilter and $L\in\mathcal{F}$, there must be some $i\le n$ such that $K_i\in\mathcal{F}$, in which case $x\in K_i\subseteq L$. Thus $x\in\bigcap\mathcal{L}'$. As $\mathcal{L}'$ is arbitrary, $\mathcal{L}$ satisfies the condition (β). **Q**

(c)(iii)⇒(iv) Let $\mathcal{K} \subseteq \Sigma$ witness that (iii) is true. For any $z \in Z$, set $\mathcal{K}_z = \{K : K \in \mathcal{K}, z \in K^*\}$. If $K_0, \dots, K_n \in \mathcal{K}_z$, then $z \in \bigcap_{i \le n} K_i^* = (\bigcap_{i \le n} K_i)^*$, so $(\bigcap_{i \le n} K_i)^* \ne \emptyset$ and $\mu(\bigcap_{i \le n} K_i) > 0$. By (β) of condition (iii), $\bigcap \mathcal{K}_z \ne \emptyset$; and even if $\mathcal{K}_z = \emptyset$, $X \cap \bigcap \mathcal{K}_z \ne \emptyset$ because X is non-empty. So we may choose $f(z) \in X \cap \bigcap \mathcal{K}_z$. This defines a function $f : Z \to X$. Observe that, for $K \in \mathcal{K}$ and $z \in Z$,

$$z \in K^* \Longrightarrow K \in \mathcal{K}_z \Longrightarrow f(z) \in K \Longrightarrow z \in f^{-1}[K],$$

so that $K^* \subseteq f^{-1}[K]$.

Now take any $E \in \Sigma$. Consider

$$U_1 = \bigcup\{K^* : K \in \mathcal{K}, K \subseteq E\} \subseteq \bigcup\{E^* \cap f^{-1}[K] : K \in \mathcal{K}, K \subseteq E\} \subseteq E^* \cap f^{-1}[E],$$

$$U_2 = \bigcup\{K^* : K \in \mathcal{K}, K \subseteq X \setminus E\} \subseteq (X \setminus E)^* \cap f^{-1}[X \setminus E] = Z \setminus (f^{-1}[E] \cup E^*),$$

so that $f^{-1}[E] \triangle E^* \subseteq Z \setminus (U_1 \cup U_2)$. Now U_1 and U_2 are open subsets of Z, so $M = Z \setminus (U_1 \cup U_2)$ is closed, and in fact M is nowhere dense. **P?** Otherwise, there is a non-zero $a \in \mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}$ is included in M, and an $F \in \Sigma$ of non-zero measure such that $a = F^\bullet$. At least one of $F \cap E$, $F \setminus E$ is non-negligible and therefore includes a non-negligible member K of $\mathcal{K}$. But in this case K^* is a non-empty open subset of M which is included in either U_1 or U_2, which is impossible. **XQ**

By the definition of λ (321J-321K), M is λ-negligible, so $f^{-1}[E] \triangle E^* \subseteq M$ is negligible, as required.

(d)(iv)⇒(v) Now assume that $f : Z \to X$ witnesses (iv), and let (Y, T, ν) be a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $\nu Y = 0$ then any function from Y to X will represent π, so we may suppose that $\nu Y > 0$. Write W for the Stone space of $\mathfrak{B}$. Then we have a continuous function $\phi : W \to Z$ such that $\phi^{-1}[\widehat{a}] = \widehat{\pi a}$ for every $a \in \mathfrak{A}$ (312P), and $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense $M \subseteq Z$ (313R). It follows that $\phi^{-1}[M]$ is meager for every meager $M \subseteq Z$, that is, $\phi^{-1}[M]$ is negligible in W for every negligible $M \subseteq Z$. By 341Q, there is an inverse-measure-preserving function $h : Y \to W$ such that $h^{-1}[\widehat{b}]^\bullet = b$ for every $b \in \mathfrak{B}$. Consider $g = f\phi h : Y \to X$.

If $E \in \Sigma$, set $a = E^\bullet \in \mathfrak{A}$, so that $E^* = \widehat{a} \subseteq Z$, and $M = f^{-1}[E] \triangle E^*$ is λ-negligible; consequently $\phi^{-1}[M]$ is negligible in W. Because h is inverse-measure-preserving,

$$g^{-1}[E] \triangle h^{-1}[\phi^{-1}[E^*]] = h^{-1}[\phi^{-1}[f^{-1}[E]]] \triangle h^{-1}[\phi^{-1}[E^*]] = h^{-1}[\phi^{-1}[M]]$$

is negligible. But $\phi^{-1}[E^*] = \widehat{\pi a}$, so

$$g^{-1}[E]^\bullet = h^{-1}[\phi^{-1}[E^*]]^\bullet = \pi a.$$

As E is arbitrary, g induces the homomorphism π.

(e)(v)⇒(vi) is trivial.

(f)(vi)⇒(iv) Assume (vi). Let ν be the c.l.d. version of λ, T its domain, and $\mathfrak{B}$ its measure algebra; then ν is strictly localizable (322Qb). The embedding $\Lambda \subseteq \mathrm{T}$ corresponds to an order-continuous measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (322Db). By (vi), there is a function $f : Z \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = (E^*)^\bullet$ in $\mathfrak{B}$ for every $E \in \Sigma$. But as ν and λ have the same negligible sets (322Qb), $f^{-1}[E] \triangle E^*$ is λ-negligible for every $E \in \Sigma$, as required by (iv).

(g)(iv)⇒(i)($\boldsymbol{\alpha}$) To begin with (down to the end of (γ) below) I suppose that μ is totally finite. In this case we have a function $g : X \to Z$ such that $E \triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$ (341Q again). We are supposing also that there is a function $f : Z \to X$ such that $f^{-1}[E] \triangle E^*$ is negligible for every $E \in \Sigma$. Write $\mathcal{K}$ for the family of sets $K \subseteq E$ such that $K \in \Sigma$ and there is a compact set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$.

($\boldsymbol{\beta}$) μ is inner regular with respect to $\mathcal{K}$. **P** Take $F \in \Sigma$ and $\gamma < \mu F$. Choose $\langle V_n \rangle_{n \in \mathbb{N}}$, $\langle F_n \rangle_{n \in \mathbb{N}}$ as follows. $F_0 = F$. Given that $\mu F_n > \gamma$, then

$$\lambda(f^{-1}[F_n] \cap F_n^*) = \lambda F_n^* = \mu F_n > \gamma,$$

so there is an open-and-closed set $V_n \subseteq f^{-1}[F_n] \cap F_n^*$ with $\lambda V_n > \gamma$. Express V_n as F_{n+1}^* where $F_{n+1} \in \Sigma$; since $F_n \triangle g^{-1}[F_n^*]$ is negligible, and $V_n \subseteq F_n^*$, we may take it that $F_{n+1} \subseteq g^{-1}[F_n^*]$. Continue.

At the end of the induction, set $K = \bigcap_{n \in \mathbb{N}} F_n \in \Sigma$ and $L = \bigcap_{n \in \mathbb{N}} F_n^*$. Because $F_{n+1} \setminus F_n \subseteq g^{-1}[F_n^*] \setminus F_n$ is negligible for each n, $\mu K = \lim_{n \to \infty} \mu F_n \ge \gamma$, while $K \subseteq F$ and L is surely compact. We have

$$L \subseteq \bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}[F_n] = f^{-1}[K],$$

so $f[L] \subseteq K$. Also

$$K \subseteq \bigcap_{n\in\mathbb{N}} F_{n+1} \subseteq \bigcap_{n\in\mathbb{N}} g^{-1}[F_n^*] = g^{-1}[L].$$

So $K \in \mathcal{K}$. As F and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

(γ) Next, $\mathcal{K}$ is a compact class. **P** Suppose that $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property. If $\mathcal{K}' = \emptyset$, of course $\bigcap \mathcal{K}' \neq \emptyset$; suppose that $\mathcal{K}'$ is non-empty. Let $\mathcal{L}$ be the family of closed sets $L \subseteq Z$ such that $g^{-1}[L]$ includes some member of $\mathcal{K}'$. Then $\mathcal{L}$ has the finite intersection property, and Z is compact, so there is some $z \in \bigcap \mathcal{L}$; also $Z \in \mathcal{L}$, so $z \in Z$. For any $K \in \mathcal{K}'$, there is some closed set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$, so that $L \in \mathcal{L}$ and $z \in L$ and $f(z) \in K$. Thus $f(z) \in \bigcap \mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

So $\mathcal{K}$ witnesses that μ is a compact measure.

(δ) Now consider the general case. Take any $E \in \Sigma$ of finite measure. If $E = \emptyset$ then surely the subspace measure μ_E is compact. Otherwise, we can identify the measure algebra of μ_E with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$ generated by $E^\bullet$ (322Ja), and $E^* \subseteq Z$ with the Stone space of $\mathfrak{A}_{E^\bullet}$ (312S). Take any $x_0 \in E$ and define $\tilde{f} : E^* \to E$ by setting $\tilde{f}(z) = f(z)$ if $z \in E^* \cap f^{-1}[E]$, x_0 if $z \in E^* \setminus f^{-1}[E]$. Then f and $\tilde{f}$ agree almost everywhere on E^*, so $\tilde{f}^{-1}[F] \triangle F^*$ is negligible for every $F \in \Sigma_E$, that is, $\tilde{f}$ represents the canonical isomorphism between the measure algebras of μ_E and the subspace measure λ_{E^*} on E^*. But this means that condition (iv) is true of μ_E, so μ_E is compact, by (α)-(γ) above. As E is arbitrary, μ is locally compact.

This completes the proof.

343C Examples (a) Let κ be an infinite cardinal. We know that the usual measure ν_κ on $\{0,1\}^\kappa$ is compact (342Jd). It follows that if (X, Σ, μ) is any complete probability space such that the measure algebra $\mathfrak{B}_\kappa$ of ν_κ can be embedded as a subalgebra of the measure algebra $\mathfrak{A}$ of μ, there is an inverse-measure-preserving function from X to $\{0,1\}^\kappa$. By 332P, this is so iff every non-zero principal ideal of $\mathfrak{A}$ has Maharam type at least κ. Of course this does not depend in any way on the results of the present chapter. If $\mathfrak{B}_\kappa$ can be embedded in $\mathfrak{A}$, there must be a stochastically independent family $\langle E_\xi \rangle_{\xi<\kappa}$ of sets of measure $\frac{1}{2}$; now we get a map $h : X \to \{0,1\}^\kappa$ by saying that $h(x)(\xi) = 1$ iff $x \in E_\xi$, which by 254G is inverse-measure-preserving.

(b) In particular, if μ is atomless, there is an inverse-measure-preserving function from X to $\{0,1\}^{\mathbb{N}}$; since this is isomorphic, as measure space, to $[0,1]$ with Lebesgue measure (254K), there is an inverse-measure-preserving function from X to $[0,1]$.

(c) More generally, if (X, Σ, μ) is any complete atomless totally finite measure space, there is an inverse-measure-preserving function from X to the interval $[0, \mu X]$ endowed with Lebesgue measure. (If $\mu X > 0$, apply (b) to the normalized measure $(\mu X)^{-1}\mu$; or argue directly from 343B, using the fact that Lebesgue measure on $[0, \mu X]$ is compact; or use the idea suggested in 343Xd.)

(d) Throughout the work above – in §254 as well as in 343B – I have taken the measures involved to be complete. It does occasionally happen, in this context, that this restriction is inconvenient. Typical results not depending on completeness in the domain space X are in 343Xc-343Xd. Of course these depend not only on the very special nature of the codomain spaces $\{0,1\}^I$ or $[0,1]$, but also on the measures on these spaces being taken to be incomplete.

343D Uniqueness of realizations The results of 342E-342J, together with 343B, give a respectable number of contexts in which homomorphisms between measure algebras can be represented by functions between measure spaces. They say nothing about whether such functions are unique, or whether we can distinguish, among the possible representations of a homomorphism, any canonical one. In fact the proof of 343B, using the Lifting Theorem as it does, strongly suggests that this is like looking for a canonical lifting, and I am sure that (outside a handful of very special cases) any such search is vain. Nevertheless, we do have a weak kind of uniqueness theorem, valid in a useful number of spaces, as follows.

Definition A measure space (X, Σ, μ) is **countably separated** if there is a countable set $\mathcal{A} \subseteq \Sigma$ separating the points of X in the sense that for any distinct x, $y \in X$ there is an $E \in \mathcal{A}$ containing one but not the other. (Of course this is a property of the structure (X, Σ) rather than of (X, Σ, μ).)

343E Lemma A measure space (X, Σ, μ) is countably separated iff there is an injective measurable function from X to $\mathbb{R}$.

proof If (X, Σ, μ) is countably separated, let $\mathcal{A} \subseteq \Sigma$ be a countable set separating the points of X. Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathcal{A} \cup \{\emptyset\}$. Set

$$f = \sum_{n=0}^{\infty} 3^{-n} \chi E_n : X \to \mathbb{R}.$$

Then f is measurable (because every E_n is measurable) and injective (because if $x \neq y$ in X and $n = \min\{i : \#(E_i \cap \{x, y\}) = 1\}$ and $x \in E_n$, then

$$f(x) \geq 3^{-n} + \sum_{i<n} 3^{-i} \chi E_i(x) > \sum_{i>n} 3^{-i} + \sum_{i<n} 3^{-i} \chi E_i(y) \geq f(y).)$$

On the other hand, if $f : X \to \mathbb{R}$ is measurable and injective, then $\mathcal{A} = \{f^{-1}[\,]-\infty, q]\,] : q \in \mathbb{Q}\}$ is a countable subset of Σ separating the points of X, so (X, Σ, μ) is countably separated.

Remark The construction of the function f from the sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in the proof above is a standard trick; such f are sometimes called **Marczewski functionals**.

343F Proposition Let (X, Σ, μ) be a countably separated measure space and (Y, T, ν) any measure space. Let $f, g : Y \to X$ be two functions such that $f^{-1}[E]$ and $g^{-1}[E]$ both belong to T, and $f^{-1}[E] \triangle g^{-1}[E]$ is ν-negligible, for every $E \in \Sigma$. Then $f = g$ ν-almost everywhere, and $\{y : y \in Y,\ f(y) \neq g(y)\}$ is measurable as well as negligible.

proof Let $\mathcal{A} \subseteq \Sigma$ be a countable set separating the points of X. Then

$$\{y : f(y) \neq g(y)\} = \bigcup_{E \in \mathcal{A}} f^{-1}[E] \triangle g^{-1}[E]$$

is measurable and negligible.

343G Corollary If, in 343B, (X, Σ, μ) is countably separated, then the functions $f : Y \to X$ of 343B(v)-(vi) are almost uniquely defined in the sense that if f, g both represent the same homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $f = g$ a.e.

343H Examples Leading examples of countably separated spaces are
(i) $\mathbb{R}$ (take $\mathcal{A} = \{]-\infty, q] : q \in \mathbb{Q}\}$);
(ii) $\{0,1\}^{\mathbb{N}}$ (take $\mathcal{A} = \{E_n : n \in \mathbb{N}\}$, where $E_n = \{x : x(n) = 1\}$);
(iii) subspaces (measurable or not) of countably separated spaces;
(iv) finite products of countably separated spaces;
(v) countable products of countably separated probability spaces;
(vi) completions and c.l.d. versions of countably separated spaces.
As soon as we move away from these elementary ideas, however, some interesting difficulties arise.

343I Example Let μ be the usual measure on $X = \{0,1\}^{\mathfrak{c}}$, where $\mathfrak{c} = \#(\mathbb{R})$, and Σ its domain. Then there is a function $f : X \to X$ such that $f(x) \neq x$ for every $x \in X$, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$. **P** The set $\mathfrak{c} \setminus \omega$ is still of cardinal $\mathfrak{c}$, so there is an injection $h : \{0,1\}^{\omega} \to \mathfrak{c} \setminus \omega$. (As usual, I am identifying the cardinal number $\mathfrak{c}$ with the corresponding initial ordinal. But if you prefer to argue without the full axiom of choice, you can express all the same ideas with $\mathbb{R}$ in the place of $\mathfrak{c}$ and $\mathbb{N}$ in the place of ω.) For $x \in X$, set

$$\begin{aligned} f(x)(\xi) &= 1 - x(\xi) \text{ if } \xi = h(x \restriction \omega), \\ &= x(\xi) \text{ otherwise }. \end{aligned}$$

Evidently $f(x) \neq x$ for every x. If $E \subseteq X$ is measurable, then we can find a countable set $J \subseteq \mathfrak{c}$ and sets E', E'', both determined by coordinates in J, such that $E' \subseteq E \subseteq E''$ and $E'' \setminus E'$ is negligible (254Oc). Now for any particular $\xi \in \mathfrak{c} \setminus \omega$, $\{x : h(x\restriction\omega) = \xi\}$ is negligible, being either empty or of the form $\{x : x(n) = z(n)$ for every $n < \omega\}$ for some $z \in \{0,1\}^\omega$. So $H = \{x : h(x\restriction\omega) \in J\}$ is negligible. Now we see that for $x \in X \setminus H$, $f(x)\restriction J = x\restriction J$, so for $x \in X \setminus (H \cup (E'' \setminus E'))$,

$$x \in E \Longrightarrow x \in E' \Longrightarrow f(x) \in E' \Longrightarrow f(x) \in E,$$

$$x \notin E \Longrightarrow x \notin E'' \Longrightarrow f(x) \notin E'' \Longrightarrow f(x) \notin E.$$

Thus $E \triangle f^{-1}[E] \subseteq H \cup (E'' \setminus E')$ is negligible. **Q**

343J The split interval I introduce a construction which here will seem essentially elementary, but in other contexts is of great interest, as will appear in Volume 4.

(a) Take $I^\|$ to consist of two copies of each point of the unit interval, so that $I^\| = \{t^+ : t \in [0,1]\} \cup \{t^- : t \in [0,1]\}$. For $A \subseteq I^\|$ write $A_l = \{t : t^- \in A\}$, $A_r = \{t : t^+ \in A\}$. Let Σ be the set

$$\{E : E \subseteq I^\|,\ E_l \text{ and } E_r \text{ are Lebesgue measurable and } E_l \triangle E_r \text{ is Lebesgue negligible}\}.$$

For $E \in \Sigma$, set

$$\mu E = \mu_L E_l = \mu_L E_r$$

where μ_L is Lebesgue measure on $[0,1]$. It is easy to check that $(I^\|, \Sigma, \mu)$ is a complete probability space. Also it is compact. **P** Take $\mathcal{K}$ to be the family of sets $K \subseteq I^\|$ such that $K_l = K_r$ is a compact subset of $[0,1]$, and check that $\mathcal{K}$ is a compact class and that μ is inner regular with respect to $\mathcal{K}$; or use 343Xa below. **Q** The sets $\{t^- : t \in [0,1]\}$ and $\{t^+ : t \in [0,1]\}$ are non-measurable subsets of $I^\|$; on both of them the subspace measures correspond exactly to μ_L. We have a canonical inverse-measure-preserving function $h : I^\| \to [0,1]$ given by setting $h(t^+) = h(t^-) = t$ for every $t \in [0,1]$; h induces an isomorphism between the measure algebras of μ and μ_L.

$I^\|$ is called the **split interval** or (especially when given its standard topology, as in 343Yc below) the **double arrow space** or **two arrows space**.

Now the relevance to the present discussion is this: we have a map $f : I^\| \to I^\|$ given by setting

$$f(t^+) = t^-,\ f(t^-) = t^+ \text{ for every } t \in [0,1]$$

such that $f(x) \neq x$ for every x, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$, so that f represents the identity homomorphism on the measure algebra of μ. The function $h : I^\| \to [0,1]$ is canonical enough, but is two-to-one, and the canonical map from the measure algebra of μ to the measure algebra of μ_L is represented equally by the functions $t \mapsto t^-$ and $t \mapsto t^+$, which are nowhere equal.

(b) Consider the direct sum (Y, ν) of $(I^\|, \mu)$ and $([0,1], \mu_L)$; for definiteness, take Y to be $(I^\| \times \{0\}) \cup ([0,1] \times \{1\})$. Setting

$$h_1(t^+, 0) = h_1(t^-, 0) = (t, 1), \quad h_1(t, 1) = (t^+, 0),$$

we see that $h_1 : Y \to Y$ induces a measure-preserving involution of the measure algebra $\mathfrak{B}$ of ν, corresponding to its expression as a simple product of the isomorphic measure algebras of μ and μ_L. But h_1 is not invertible, and indeed there is no invertible function from Y to itself which induces this involution of $\mathfrak{B}$. **P?** Suppose, if possible, that $g : Y \to Y$ were such a function. Looking at the sets

$$E_q = [0, q] \times \{1\}, \quad F_q = \{(t^+, 0) : t \in [0, q]\} \cup \{(t^-, 0) : t \in [0, q]\}$$

for $q \in \mathbb{Q}$, we must have $g^{-1}[E_q] \triangle F_q$ negligible for every q, so that we must have $g(t^+, 0) = g(t^-, 0) = (t, 1)$ for almost every $t \in [0,1]$, and g cannot be injective. **XQ**

(c) Thus even with a compact probability space, and an automorphism ϕ of its measure algebra, we cannot be sure of representing ϕ and ϕ^{-1} by functions which will be inverses of each other.

343K 342L has a partial converse.

Proposition If (X, Σ, μ) is a semi-finite countably separated measure space, it is compact iff it is locally compact iff it is perfect.

proof We already know that compact measure spaces are locally compact and locally compact semi-finite measure spaces are perfect (342Ha, 342L). So suppose that (X,Σ,μ) is a perfect semi-finite countably separated measure space. Let $f:X\to\mathbb{R}$ be an injective measurable function (343E). Consider

$$\mathcal{K}=\{f^{-1}[L]:L\subseteq f[X],\,L\text{ is compact in }\mathbb{R}\}.$$

The definition of 'perfect' measure space states exactly that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. And $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}'\subseteq\mathcal{K}$ has the finite intersection property, $\mathcal{L}=\{f[K]:K\in\mathcal{K}'\}$ is a family of compact sets in $\mathbb{R}$ with the finite intersection property, and has non-empty intersection; so that $\bigcap\mathcal{K}'$ is also non-empty, because f is injective. **Q** By 342E, (X,Σ,μ) is compact.

343L The time has come to give examples of spaces which are *not* locally compact, so that we can expect to have measure-preserving homomorphisms not representable by inverse-measure-preserving functions. The most commonly arising ones are covered by the following result.

Proposition Let (X,Σ,μ) be a complete locally determined countably separated measure space, and $A\subseteq X$ a set such that the subspace measure μ_A is perfect. Then A is measurable.

proof ? Otherwise, there is a set $E\in\Sigma$ such that $\mu E<\infty$ and $B=A\cap E\notin\Sigma$. Let $f:X\to\mathbb{R}$ be an injective measurable function (343E again). Then $f\restriction B$ is Σ_B-measurable, where Σ_B is the domain of the subspace measure μ_B on B. Set

$$\mathcal{K}=\{f^{-1}[L]:L\subseteq f[B],\,L\text{ is compact in }\mathbb{R}\}.$$

Just as in the proof of 343K, $\mathcal{K}$ is a compact class and μ_B is inner regular with respect to $\mathcal{K}$. By 342Bb, there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\mu_B(B\setminus\bigcup_{n\in\mathbb{N}}K_n)=0$. But of course $\mathcal{K}\subseteq\Sigma$, because f is Σ-measurable, so $\bigcup_{n\in\mathbb{N}}K_n\in\Sigma$. Because μ is complete, $B\setminus\bigcup_{n\in\mathbb{N}}K_n\in\Sigma$ and $B\in\Sigma$. **X**

343M Example 343L tells us that any non-measurable set X of $\mathbb{R}^r$, or of $\{0,1\}^{\mathbb{N}}$, with their usual measures, is not perfect, therefore not (locally) compact, when given its subspace measure.

To find a non-representable homomorphism, we do not need to go through the whole apparatus of 343B. Take Y to be a measurable envelope of X (132Ed). Then the identity function from X to Y induces an isomorphism of their measure algebras. But there is no function from Y to X inducing the same isomorphism. **P?** Writing Z for $\mathbb{R}^r$ or $\{0,1\}^{\mathbb{N}}$ and μ for its measure, Z is countably separated; suppose $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable sets in Z separating its points. For each n, $(Y\cap E_n)^\bullet$ in the measure algebra of μ_Y corresponds to $(X\cap E_n)^\bullet$ in the measure algebra of μ_X. So if $f:Y\to X$ were a function representing the isomorphism of the measure algebras, $(Y\cap E_n)\triangle f^{-1}[E_n]$ would have to be negligible for each n, and $A=\bigcup_{n\in\mathbb{N}}(Y\cap E_n)\triangle f^{-1}[E_n]$ would be negligible. But for $y\in Y\setminus A$, $f(y)$ belongs to just the same E_n as y does, so must be equal to y. Accordingly $X\supseteq Y\setminus A$ and X is measurable. **XQ**

343X Basic exercises (a) Let (X,Σ,μ) be a semi-finite measure space. (i) Suppose that there is a set $A\subseteq X$, of full outer measure, such that the subspace measure on A is compact. Show that μ is locally compact. (*Hint*: show that μ satisfies (ii) or (v) of 343B.) (ii) Suppose that for every non-negligible $E\in\Sigma$ there is a non-negligible set $A\subseteq E$ such that the subspace measure on A is compact. Show that μ is locally compact.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty sets, with product X; write $\pi_i:X\to X_i$ for the coordinate map. Suppose we are given a σ-algebra Σ_i of subsets of X_i for each i; let $\Sigma=\widehat{\bigotimes}_{i\in I}\Sigma_i$ be the corresponding σ-algebra of subsets of X generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$. Let μ be a totally finite measure with domain Σ, and for $i\in I$ let μ_i be the image measure $\mu\pi_i^{-1}$. Check that the domain of μ_i is Σ_i. Show that if every (X_i,Σ_i,μ_i) is compact, then so is (X,Σ,μ). (*Hint*: *either* show that μ satisfies (v) of 343B *or* adapt the method of 342Gf.)

(c) Let I be any set. Let T be the σ-algebra of subsets of $\{0,1\}^I$ generated by the sets $F_i=\{z:z(i)=1\}$ for $i\in I$, and ν any probability measure with domain T; let $\mathfrak{B}$ be the measure algebra of ν. Let (X,Σ,μ) be a measure space with measure algebra $\mathfrak{A}$, and $\phi:\mathfrak{B}\to\mathfrak{A}$ an order-continuous Boolean homomorphism. Show that there is an inverse-measure-preserving function $f:X\to\{0,1\}^I$ representing ϕ. (*Hint*: for each $i\in I$, take $E_i\in\Sigma$ such that $E_i^\bullet=\phi F_i^\bullet$; set $f(x)(i)=1$ if $x\in E_i$, and use 343Ab.)

(d) Let (X,Σ,μ) be an atomless probability space. Let $\mu_{\mathcal{B}}$ be the restriction of Lebesgue measure to the σ-algebra of Borel subsets of $[0,1]$. Show that there is a function $g : X \to [0,1]$ which is inverse-measure-preserving for μ and $\mu_{\mathcal{B}}$. (*Hint*: find an $f : X \to \{0,1\}^{\mathbb{N}}$ as in 343Xc, and set $g = hf$ where $h(z) = \sum_{n=0}^{\infty} 2^{-n-1}g(n)$, as in 254K; or choose $E_q \in \Sigma$ such that $\mu E_q = q$, $E_q \subseteq E_{q'}$ whenever $q \le q'$ in $[0,1]\cap\mathbb{Q}$, and set $f(x) = \inf\{q : x \in E_q\}$ for $x \in E_1$.)

(e) Let (X,Σ,μ) be a countably separated measure space, with measure algebra $\mathfrak{A}$. (i) Show that $\{x\} \in \Sigma$ for every $x \in X$. (ii) Show that every atom of $\mathfrak{A}$ is of the form $\{x\}^\bullet$ for some $x \in X$.

(f) Let $I^{\|}$ be the split interval, with its usual measure μ described in 343J, and $h : I^{\|} \to [0,1]$ the canonical surjection. Show that the canonical isomorphism between the measure algebras of μ and Lebesgue measure on $[0,1]$ is given by the formula '$E^\bullet \mapsto h[E]^\bullet$ for every measurable $E \subseteq I^{\|}$'.

(g) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Suppose that $X\cap Y = \emptyset$ and that we have a measure-preserving isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. Set

$$\Lambda = \{W : W \subseteq X\cup Y,\ W\cap X \in \Sigma,\ W\cap Y \in \mathrm{T},\ \pi(W\cap X)^\bullet = (W\cap Y)^\bullet\},$$

and for $W \in \Lambda$ set $\lambda W = \mu(W\cap X) = \nu(W\cap Y)$. Show that $(X\cup Y,\Lambda,\lambda)$ is a measure space which is locally compact, or perfect, if (X,Σ,μ) is.

(h) Let (X,Σ,μ) be a complete perfect totally finite measure space, (Y,T,ν) a complete countably separated measure space, and $f : X \to Y$ an inverse-measure-preserving function. Show that $\mathrm{T} = \{F : F \subseteq Y,\ f^{-1}[F] \in \Sigma\}$, so that a function $h : Y \to \mathbb{R}$ is ν-integrable iff hf is μ-integrable. (*Hint*: if $A \subseteq Y$ and $E = f^{-1}[A] \in \Sigma$, $f{\restriction}E$ is inverse-measure-preserving for the subspace measures μ_E, ν_A; by 342Xk, ν_A is perfect, so by 343L $A \in \mathrm{T}$. Now use 235L.)

343Y Further exercises (a) Let (X,Σ,μ) be a semi-finite measure space, and suppose that there is a compact class $\mathcal{K} \subseteq \mathcal{P}X$ such that (α) whenever $E \in \Sigma$ and $\mu E > 0$ there is a non-negligible $K \in \mathcal{K}$ such that $K \subseteq E$ (β) whenever $K_0,\dots,K_n \in \mathcal{K}$ and $\bigcap_{i\le n} K_i = \emptyset$ then there are measurable sets $E_0,\dots,E_n$ such that $E_i \supseteq K_i$ for every i and $\bigcap_{i\le n} E_i$ is negligible. Show that μ is locally compact.

(b) (i) Show that a countably separated semi-finite measure space has magnitude at most $\mathfrak{c}$ and Maharam type at most $2^{\mathfrak{c}}$. (ii) Show that the direct sum of $\mathfrak{c}$ or fewer countably separated measure spaces is countably separated.

(c) Let $I^{\|} = \{t^+ : t \in [0,1]\} \cup \{t^- : t \in [0,1]\}$ be the split interval (343J). (i) Show that the rules

$$s^- \le t^- \iff s^+ \le t^+ \iff s \le t, \quad s^+ \le t^- \iff s < t,$$

$$t^- \le t^+ \text{ for all } t \in [0,1]$$

define a Dedekind complete total order on $I^{\|}$ with greatest and least elements. (ii) Show that the intervals $[0^-,t^-]$, $[t^+,1^+]$, interpreted for this ordering, generate a compact Hausdorff topology on $I^{\|}$ for which the map $h : I^{\|} \to [0,1]$ of 343J is continuous. (iii) Show that a subset E of $I^{\|}$ is Borel for this topology iff the sets E_r, $E_l \subseteq [0,1]$, as described in 343J, are Borel and $E_r \triangle E_l$ is countable. (iv) Show that if $f : [0,1] \to \mathbb{R}$ is of bounded variation then there is a continuous $g : I^{\|} \to \mathbb{R}$ such that $g = fh$ except perhaps at countably many points. (v) Show that the measure μ of 343J is inner regular with respect to the compact subsets of $I^{\|}$. (vi) Show that we have a lower density ϕ for μ defined by setting

$$\phi E = \{t^- : 0 < t \le 1,\ \lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\cap[(t-\delta)^+,t^-]) = 1\}$$

$$\cup\,\{t^+ : 0 \le t < 1,\ \lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\cap[t^+,(t+\delta)^-]) = 1\}$$

for measurable sets $E \subseteq I^{\|}$.

(d) Set $X=\{0,1\}^{\mathfrak c}$, with its usual measure μ. Show that there is an inverse-measure-preserving function $f:X\to X$ such that $f[X]$ is non-measurable but f induces the identity automorphism of the measure algebra of μ. (*Hint*: use the idea of 343I.) Show that under these conditions $f[X]$, with its subspace measure, must be compact. (*Hint*: use 343B(iv).)

(e) Let μ_{Hr} be r-dimensional Hausdorff measure on $\mathbb{R}^s$, where $s\ge 1$ is an integer and $r\ge 0$ (§264). (i) Show that μ_{Hr} is countably separated. (ii) Show that the c.l.d. version of μ_{Hr} is compact. (*Hint*: 264Yi.)

(f) Give an example of a countably separated probability space (X,Σ,μ) and a function f from X to a set Y such that the image measure μf^{-1} is not countably separated. (*Hint*: use 223B to show that if $E\subseteq\mathbb{R}$ is Lebesgue measurable and not negligible, then $E+\mathbb{Q}$ is conegligible; or use the zero-one law to show that if $E\subseteq\mathcal{P}\mathbb{N}$ is measurable and not negligible for the usual measure on $\mathcal{P}\mathbb{N}$, then $\{a\triangle b: a\in E,\, b\in[\mathbb{N}]^{<\omega}\}$ is conegligible.)

343 Notes and comments The points at which the Lifting Theorem impinges on the work of this section are in the proofs of (iv)⇒(i) and (iv)⇒(v) in Theorem 343B. In fact the ideas can be rearranged to give a proof of 343B which does not rely on the Lifting Theorem; I give a hint in Volume 4 (413Yc).

I suppose the significant new ideas of this section are in 343B and 343K. The rest is mostly a matter of being thorough and careful. But I take this material at a slow pace because there are some potentially confusing features, and the underlying question is of the greatest importance: when, given a Boolean homomorphism from one measure algebra to another, can we be sure of representing it by a measurable function between measure spaces? The concept of 'compact' space puts the burden firmly on the measure space corresponding to the *domain* of the Boolean homomorphism, which will be the *codomain* of the measurable function. So the first step is to try to understand properly which spaces are compact, and what other properties they can be expected to have; which accounts for much of the length of §342. But having understood that many of our favourite spaces are compact, we have to come to terms with the fact that we still cannot count on a measure algebra isomorphism corresponding to a measure space isomorphism. I introduce the split interval (343J, 343Xf, 343Yc) as a close approximation to Lebesgue measure on $[0,1]$ which is not isomorphic to it. Of course we have already seen a more dramatic example: the Stone space of the Lebesgue measure algebra also has the same measure algebra as Lebesgue measure, while being in almost every other way very much more complex, as will appear in Volumes 4 and 5.

As 343C suggests, elementary cases in which 343B can be applied are often amenable to more primitive methods, avoiding not only the concept of 'compact' measure, but also Stone spaces and the Lifting Theorem. For substantial examples in which we can prove that a measure space (X,μ) is compact, without simultaneously finding direct constructions for inverse-measure-preserving functions into X (as in 343Xc-343Xd), I think we shall have to wait until Volume 4.

The concept of 'countably separated' measure space does not involve the measure at all, nor even the ideal of negligible sets; it belongs to the theory of σ-algebras of sets. Some simple permanence properties are in 343H and 343Yb(ii). Let us note in passing that 343Xh describes some more situations in which the 'image measure catastrophe', described in 235J, cannot arise.

I include the variants 343B(ii), 343B(iii) and 343Ya of the notion of 'local compactness' because they are not obvious and may illuminate it.

344 Realization of automorphisms

In 343Jb, I gave an example of a 'good' (compact, complete) probability space X with an automorphism ϕ of its measure algebra such that both ϕ and ϕ^{-1} are representable by functions from X to itself, but there is no such representation in which the two functions are inverses of each other. The present section is an attempt to describe the further refinements necessary to ensure that automorphisms of measure algebras can be represented by automorphisms of the measure spaces. It turns out that in the most important contexts in which this can be done, a little extra work yields a significant generalization: the simultaneous realization of countably many homomorphisms by a consistent family of functions.

I will describe three cases in which such simultaneous realizations can be achieved: Stone spaces (344A), perfect complete countably separated spaces (344C) and suitable measures on $\{0,1\}^I$ (344E-344G). The arguments for 344C, suitably refined, give a complete description of perfect complete countably separated strictly localizable spaces which are not purely atomic (344I, 344Xc). At the same time we find that Lebesgue measure, and the usual measure on $\{0,1\}^I$, are 'homogeneous' in the strong sense that two measurable subspaces (of non-zero measure) are isomorphic iff they have the same measure (344J, 344L).

344A Stone spaces The first case is immediate from the work of §§312, 313 and 321, as collected in 324E. If (Z,Σ,μ) is actually the Stone space of a measure algebra $(\mathfrak{A},\bar{\mu})$, then every order-continuous Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ corresponds to a unique continuous function $f_\phi:Z\to Z$ (312P) which represents ϕ (324E). The uniqueness of f_ϕ means that we can be sure that $f_{\phi\psi}=f_\psi f_\phi$ for all order-continuous homomorphisms ϕ and ψ; and of course f_ι is the identity map on Z, so that $f_{\phi^{-1}}$ will have to be f_ϕ^{-1} whenever ϕ is invertible. Thus in this special case we can consistently, and canonically, represent all order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself.

Now for two cases where we have to work for the results.

344B Theorem Let (X,Σ,μ) be a countably separated measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Then a family $\langle f_\phi\rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

proof (a) Because $G\cup\{\iota\}$ satisfies the same conditions as G, we may suppose from the beginning that ι belongs to G itself. Let $\mathcal{A}\subseteq\Sigma$ be a countable set separating the points of X. For each $\phi\in G$ take some representing function $g_\phi:X\to X$; take g_ι to be the identity function. If ϕ, $\psi\in G$, then of course

$$\begin{aligned}((g_\phi g_\psi)^{-1}[E])^\bullet&=(g_\psi^{-1}[g_\phi^{-1}[E]])^\bullet=\psi(g_\phi^{-1}[E])^\bullet\\&=\psi\phi E^\bullet=(g_{\psi\phi}^{-1}[E])^\bullet\end{aligned}$$

for every $E\in\Sigma$. By 343F, the set

$$H_{\phi\psi}=\{x:g_{\psi\phi}(x)\ne g_\phi g_\psi(x)\}$$

is negligible and belongs to Σ.

(b) Set

$$H=\bigcup_{\phi,\psi\in G}H_{\phi\psi};$$

because G is countable, H is also measurable and negligible. Try defining $f_\phi:X\to X$ by setting $f_\phi(x)=g_\phi(x)$ if $x\in X\setminus H$, $f_\phi(x)=x$ if $x\in H$. Because H is measurable, $f_\phi^{-1}[E]\in\Sigma$ for every $E\in\Sigma$; because H is negligible,

$$(f_\phi^{-1}[E])^\bullet=(g_\phi^{-1}[E])^\bullet=\phi E^\bullet$$

for every $E\in\Sigma$, and f_ϕ represents ϕ, for every $\phi\in G$. Of course $f_\iota=g_\iota$ is the identity function on X.

(c) If $\theta\in G$ then $f_\theta^{-1}[H]=H$. **P** (i) If $x\in H$ then $f_\theta(x)=x\in H$. (ii) If $f_\theta(x)\in H$ and $f_\theta(x)=x$ then of course $x\in H$. (iii) If $f_\theta(x)=g_\theta(x)\in H$ then there are ϕ, $\psi\in G$ such that $g_\phi g_\psi g_\theta(x)\ne g_{\psi\phi}g_\theta(x)$. So either

$$g_\psi g_\theta(x)\ne g_{\theta\psi}(x),$$

or

$$g_\phi g_{\theta\psi}(x)\ne g_{\theta\psi\phi}(x)$$

or

$$g_{\theta\psi\phi}(x)\ne g_{\psi\phi}g_\theta(x),$$

and in any case $x\in H$. **Q**

(d) It follows that $f_\phi f_\psi = f_{\psi\phi}$ for every ϕ, $\psi \in G$. **P** (i) If $x \in H$ then

$$f_\phi f_\psi(x) = x = f_{\psi\phi}(x).$$

(ii) If $x \in X \setminus H$ then $f_\psi(x) \notin H$, by (c), so

$$f_\phi f_\psi(x) = g_\phi g_\psi(x) = g_{\psi\phi}(x) = f_{\psi\phi}(x). \quad \mathbf{Q}$$

344C Corollary Let (X,Σ,μ) be a countably separated perfect complete strictly localizable measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible, and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ and ϕ^{-1} are measure-preserving, f_ϕ will be an automorphism of the measure space (X,Σ,μ).

proof By 343K, (X,Σ,μ) is compact. So 343B(v) tells us that every member of G is representable, and we can apply 344B.

Reminder: Spaces satisfying the conditions of this corollary include Lebesgue measure on $\mathbb{R}^r$, the usual measure on $\{0,1\}^{\mathbb{N}}$, and their measurable subspaces; see also 342J, 342Xe, 343H and 343Ye.

344D The third case I wish to present requires a more elaborate argument. I start with a kind of Schröder-Bernstein theorem for measurable spaces.

Lemma Let X and Y be sets, and $\Sigma \subseteq \mathcal{P}X$, $\mathrm{T} \subseteq \mathcal{P}Y$ σ-algebras. Suppose that there are $f : X \to Y$, $g : Y \to X$ such that $F = f[X] \in \mathrm{T}$, $E = g[Y] \in \Sigma$, f is an isomorphism between (X,Σ) and (F,T_F) and g is an isomorphism between (Y,T) and (E,Σ_E), writing Σ_E, T_F for the subspace σ-algebras (see 121A). Then (X,Σ) and (Y,T) are isomorphic, and there is an isomorphism $h : X \to Y$ which is covered by f and g in the sense that

$$\{(x,h(x)) : x \in X\} \subseteq \{(x,f(x)) : x \in X\} \cup \{(g(y),y) : y \in Y\}.$$

proof Set $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g[Y_n]$ and $Y_{n+1} = f[X_n]$ for each $n \in \mathbb{N}$; then $\langle X_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ and $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T. Set $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$, $Y_\infty = \bigcap_{n\in\mathbb{N}} Y_n$. Then $f\restriction X_{2k} \setminus X_{2k+1}$ is an isomorphism between $X_{2k} \setminus X_{2k+1}$ and $Y_{2k+1} \setminus Y_{2k+2}$, while $g\restriction Y_{2k} \setminus Y_{2k+1}$ is an isomorphism between $Y_{2k} \setminus Y_{2k+1}$ and $X_{2k+1} \setminus X_{2k+2}$; and $g\restriction Y_\infty$ is an isomorphism between Y_∞ and X_∞. So the formula

$$\begin{aligned} h(x) &= f(x) \text{ if } x \in \bigcup_{k\in\mathbb{N}} X_{2k} \setminus X_{2k+1},\\ &= g^{-1}(x) \text{ for other } x \in X \end{aligned}$$

gives the required isomorphism between X and Y.

Remark You will recognise the ordinary Schröder-Bernstein theorem (2A1G) as the case $\Sigma = \mathcal{P}X$, $\mathrm{T} = \mathcal{P}Y$.

344E Theorem Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$ with domain the σ-algebra $\mathcal{B}$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I; write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space $(X,\mathcal{B},\mu)$.

proof (a) As in 344C, we may as well suppose from the beginning that $\iota \in G$. The case of finite I is trivial, so I will suppose that I is infinite. For $i \in I$, set $E_i = \{x : x(i) = 1\}$; for $J \subseteq I$, let $\mathcal{B}_J$ be the σ-subalgebra of $\mathcal{B}$ generated by $\{E_i : I \in J\}$. For $i \in I$, $\phi \in G$ choose $F_{\phi i} \in \mathcal{B}$ such that $F_{\phi i}^\bullet = \phi E_i^\bullet$. Let $\mathcal{J}$ be the family of those subsets J of I such that $F_{\phi i} \in \mathcal{B}_J$ for every $i \in J$, $\phi \in G$.

(b) For the purposes of this proof, I will say that a pair $(J,\langle g_\phi\rangle_{\phi\in G})$ is **consistent** if $J\in\mathcal{J}$ and, for each $\phi\in G$, g_ϕ is a function from X to itself such that

$g_\phi^{-1}[E_i]\in\mathcal{B}_J$ and $(g_\phi^{-1}[E_i])^\bullet=\phi E_i^\bullet$ whenever $i\in J$, $\phi\in G$,

$g_\phi^{-1}[E_i]=E_i$ whenever $i\in I\setminus J$, $\phi\in G$,

$g_\phi g_\psi=g_{\psi\phi}$ whenever ϕ, $\psi\in G$,

$g_\iota(x)=x$ for every $x\in X$.

Now the key to the proof is the following fact: if $(J,\langle g_\phi\rangle_{\phi\in G})$ is consistent, and $\tilde J$ is a member of $\mathcal{J}$ such that $\tilde J\setminus J$ is countably infinite, then there is a family $\langle\tilde g_\phi\rangle_{\phi\in G}$ such that $(\tilde J,\langle\tilde g_\phi\rangle_{\phi\in G})$ is consistent and $\tilde g_\phi^{-1}[E_i]=g_\phi^{-1}[E_i]$ whenever $i\in J$, $\phi\in G$. The construction is as follows.

(i) Start by fixing on any infinite set $K\subseteq\tilde J\setminus J$ such that $(\tilde J\setminus J)\setminus K$ is also infinite. For $z\in\{0,1\}^K$, set $V_z=\{x:x\in X,\ x{\restriction}K=z\}$; then $V_z\in\mathcal{B}_{\tilde J}$. All the sets V_z, as z runs over the uncountable set $\{0,1\}^K$, are disjoint, so they cannot all have non-zero measure (because μ is σ-finite), and we can choose z such that V_z is μ-negligible.

(ii) Define $h_\phi:X\to X$, for $\phi\in G$, by setting

$$\begin{aligned}h_\phi(x)(i)&=g_\phi(x)(i)\text{ if }i\in J,\\&=x(i)\text{ if }i\in I\setminus\tilde J,\\&=x(i)\text{ if }i\in\tilde J\setminus J\text{ and }x\in V_z,\\&=1\text{ if }i\in\tilde J\setminus J\text{ and }x\in F_{\phi i}\setminus V_z,\\&=0\text{ if }i\in\tilde J\setminus J\text{ and }x\notin F_{\phi i}\cup V_z.\end{aligned}$$

Because $V_z\in\mathcal{B}_{\tilde J}$ and $\mu V_z=0$, we see that

(α) $h_\phi^{-1}[E_i]=g_\phi^{-1}[E_i]\in\mathcal{B}_J$ if $i\in J$,

(β) $h_\phi^{-1}[E_i]\in\mathcal{B}_{\tilde J}$ and $h_\phi^{-1}[E_i]\triangle F_{\phi i}$ is negligible if $i\in\tilde J\setminus J$,

and consequently

(γ) $(h_\phi^{-1}[E_i])^\bullet=\phi E_i^\bullet$ for every $i\in\tilde J$,

(δ) $(h_\phi^{-1}[E])^\bullet=\phi E^\bullet$ for every $E\in\mathcal{B}_{\tilde J}$

(by 343Ab); moreover,

(ϵ) $h_\phi^{-1}[E]=g_\phi^{-1}[E]$ for every $E\in\mathcal{B}_J$,

(ζ) $h_\phi^{-1}[E]\in\mathcal{B}_{\tilde J}$ for every $E\in\mathcal{B}_{\tilde J}$,

(η) $h_\phi^{-1}[E_i]=E_i$ if $i\in I\setminus\tilde J$,

so that

(θ) $h_\phi^{-1}[E]\in\mathcal{B}$ for every $E\in\mathcal{B}$;

finally

(ι) $h_\iota(x)=x$ for every $x\in X$.

(iii) The next step is to note that if ϕ, $\psi\in G$ then

$$H_{\phi,\psi}=\{x:x\in X,\ h_\phi h_\psi(x)\ne h_{\psi\phi}(x)\}$$

belongs to $\mathcal{B}_{\tilde J}$ and is negligible. **P**

$$H_{\phi,\psi}=\bigcup_{i\in I}h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i].$$

Now if $i\in J$, then $h_\phi^{-1}[E_i]=g_\phi^{-1}[E_i]\in\mathcal{B}_J$, so

$$h_\psi^{-1}[h_\phi^{-1}[E_i]]=h_\psi^{-1}[g_\phi^{-1}[E_i]]=g_\psi^{-1}[g_\phi^{-1}[E_i]]=g_{\psi\phi}^{-1}[E_i]=h_{\psi\phi}^{-1}[E_i].$$

Next, for $i\in I\setminus\tilde J$,

$$h_\psi^{-1}[h_\phi^{-1}[E_i]]=h_\psi^{-1}[E_i]=E_i=h_{\psi\phi}^{-1}[E_i].$$

So

$$H_{\phi,\psi}=\bigcup_{i\in\tilde J\setminus J}h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i].$$

But for any particular $i\in\tilde J\setminus J$, E_i and $h_\phi^{-1}[E_i]$ belong to $\mathcal{B}_{\tilde J}$, so

$$(h_\psi^{-1}[h_\phi^{-1}[E_i]])^\bullet=\psi(h_\phi^{-1}[E_i])^\bullet=\psi\phi E_i^\bullet=(h_{\psi\phi}^{-1}[E_i])^\bullet,$$

and $h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i]$ is a negligible set, which by (ii-ζ) belongs to $\mathcal{B}_{\tilde J}$. So $H_{\phi,\psi}$ is a countable union of sets of measure 0 in $\mathcal{B}_{\tilde J}$ and is itself a negligible member of $\mathcal{B}_{\tilde J}$, as claimed. **Q**

(iv) Set

$$H=\bigcup_{\phi,\psi\in G}H_{\phi,\psi}\cup\bigcup_{\phi\in G}h_\phi^{-1}[V_z].$$

Then $H\in\mathcal{B}_{\tilde J}$ and $\mu H=0$. **P** We know that every $H_{\phi,\psi}$ is negligible and belongs to $\mathcal{B}_{\tilde J}$ ((iii) above), that every $h_\phi^{-1}[V_z]$ belongs to $\mathcal{B}_{\tilde J}$ (by (ii-ζ), and that $(h_\phi^{-1}[V_z])^\bullet=\phi V_z^\bullet=0$, so that $h_\phi^{-1}[V_z]$ is negligible, for every $\phi\in G$ (by (ii-δ)). Consequently H is negligible and belongs to $\mathcal{B}_{\tilde J}$. **Q** Also, of course, $V_z=h_\iota^{-1}[V_z]\subseteq H$.

Next, $h_\phi(x)\notin H$ whenever $x\in X\setminus H$ and $\phi\in G$. **P** If ψ, $\theta\in G$ then

$$h_{\theta\psi}h_\phi(x)=h_{\phi\theta\psi}(x)=h_\psi h_{\phi\theta}(x)=h_\psi h_\theta h_\phi(x),$$

$$h_\psi h_\phi(x)=h_{\phi\psi}(x)\notin V_z$$

because

$$x\notin H_{\theta\psi,\phi}\cup H_{\phi\theta,\psi}\cup H_{\theta,\phi}\cup H_{\psi,\phi}\cup h_{\phi\psi}^{-1}[V_z];$$

thus $h_\phi(x)\notin H_{\psi,\theta}\cup h_\psi^{-1}[V_z]$; as ψ and θ are arbitrary, $h_\phi(x)\notin H$. **Q**

(v) The next fact we need is that there is a bijection $q:X\to H$ such that (α) for $E\subseteq H$, $E\in\mathcal{B}_{\tilde J}$ iff $q^{-1}[E]\in\mathcal{B}_{\tilde J}$ (β) $q(x)(i)=x(i)$ for every $i\in I\setminus(\tilde J\setminus J)$, $x\in X$. **P** Fix any bijection $r:\tilde J\setminus J\to\tilde J\setminus(J\cup K)$. Consider the maps $p_1:X\to H$, $p_2:H\to X$ given by

$$\begin{aligned}p_1(x)(i)&=x(r^{-1}(i))\text{ if }i\in\tilde J\setminus(J\cup K),\\&=z(i)\text{ if }i\in K,\\&=x(i)\text{ if }i\in X\setminus(\tilde J\setminus J),\\p_2(y)&=y\end{aligned}$$

for $x\in X$, $y\in H$. Then p_1 is actually an isomorphism between $(X,\mathcal{B}_{\tilde J})$ and $(V_z,\mathcal{B}_{\tilde J}\cap\mathcal{P}V_z)$. So p_1, p_2 are isomorphisms between $(X,\mathcal{B}_{\tilde J})$, $(H,\mathcal{B}_{\tilde J}\cap\mathcal{P}H)$ and measurable subspaces of H, X respectively. By 344D, there is an isomorphism q between X and H such that, for every $x\in X$, either $q(x)=p_1(x)$ or $p_2(q(x))=x$. Since $p_1(x)\restriction I\setminus(\tilde J\setminus J)=x\restriction I\setminus(\tilde J\setminus J)$ for every $x\in X$, and $p_2(y)\restriction I\setminus(\tilde J\setminus J)=y\restriction I\setminus(\tilde J\setminus J)$ for every $y\in H$, $q(x)\restriction I\setminus(\tilde J\setminus J)=x\restriction I\setminus(\tilde J\setminus J)$ for every $x\in X$. **Q**

(vi) An incidental fact which will be used below is the following: if $i\in\tilde J$, $\phi\in G$ then $g_\phi^{-1}[E_i]$ belongs to $\mathcal{B}_{\tilde J}$, because it belongs to $\mathcal{B}_J$ if $i\in J$, and otherwise is equal to E_i. Consequently $g_\phi^{-1}[E]\in\mathcal{B}_{\tilde J}$ for every $E\in\mathcal{B}_{\tilde J}$.

(vii) I am at last ready to give a formula for $\tilde g_\phi$. For $\phi\in G$ set

$$\begin{aligned}\tilde g_\phi(x)&=h_\phi(x)\text{ if }x\in X\setminus H,\\&=qg_\phi q^{-1}(x)\text{ if }x\in H.\end{aligned}$$

Now $(\tilde J,\langle\tilde g_\phi\rangle_{\phi\in G})$ is consistent. **P**

(α) If $i\in\tilde J$, $\phi\in G$,

$$\tilde g_\phi^{-1}[E_i]=(h_\phi^{-1}[E_i]\setminus H)\cup q[g_\phi^{-1}[q^{-1}[E_i\cap H]]]\in\tilde{\mathcal{B}}_J$$

because $H\in\mathcal{B}_{\tilde J}$ and $h_\phi^{-1}[E]$, $q^{-1}[H\cap E]$, $g_\phi^{-1}[E]$ and $q[E]$ all belong to $\mathcal{B}_{\tilde J}$ for every $E\in\mathcal{B}_{\tilde J}$. At the same time, because $\tilde g_\phi$ agrees with h_ϕ on the conegligible set $X\setminus H$,

$$(\tilde g_\phi^{-1}[E_i])^\bullet=(h_\phi^{-1}[E_i])^\bullet=\phi E_i^\bullet.$$

(β) If $i \in I \setminus \tilde{J}$, $\phi \in G$, $x \in X$ then

$$g_\phi(x)(i) = h_\phi(x)(i) = q(x)(i) = x(i),$$

and if $x \in H$ then $q^{-1}(x)(i)$ is also equal to $x(i)$; so $\tilde{g}_\phi(x)(i) = x(i)$. But this means that $\tilde{g}_\phi^{-1}[E_i] = E_i$.

(γ) If ϕ, $\psi \in G$ and $x \in X \setminus H$, then

$$\tilde{g}_\psi(x) = h_\psi(x) \in X \setminus H$$

by (iv) above. So

$$\tilde{g}_\phi \tilde{g}_\psi(x) = h_\phi h_\psi(x) = h_{\psi\phi}(x) = g_{\psi\phi}(x)$$

because $x \notin H_{\phi,\psi}$. While if $x \in H$, then

$$\tilde{g}_\psi(x) = q g_\psi q^{-1}(x) \in H,$$

so

$$\tilde{g}_\phi \tilde{g}_\psi(x) = q g_\phi q^{-1} q g_\psi q^{-1}(x) = q g_\phi g_\psi q^{-1}(x) = q g_{\psi\phi} q^{-1}(x) = \tilde{g}_{\psi\phi}(x).$$

Thus $\tilde{g}_\phi \tilde{g}_\psi = \tilde{g}_{\psi\phi}$.

(δ) Because $g_\iota(x) = h_\iota(x) = x$ for every x, $\tilde{g}_\iota(x) = x$ for every x. **Q**

(viii) Finally, if $i \in J$ and $\phi \in G$, $q^{-1}[E_i] = E_i$, so that $q[E_i \cap H] = E_i$. Accordingly $q(x){\restriction}J = x{\restriction}J$ for every $x \in X$, while $q^{-1}(x){\restriction}J = x{\restriction}J$ for $x \in H$. So $g_\phi q^{-1}(x){\restriction}J = g_\phi(x){\restriction}J$ for $x \in H$, and

$$\begin{aligned}\tilde{g}_\phi(x)(i) &= h_\phi(x)(i) = g_\phi(x)(i) \text{ if } x \in X \setminus H,\\ &= q g_\phi q^{-1}(x)(i) = g_\phi q^{-1}(x)(i) = g_\phi(x)(i) \text{ if } x \in X \setminus H.\end{aligned}$$

Thus $(\tilde{J}, \langle \tilde{g}_\phi \rangle_{\phi \in G})$ satisfies all the required conditions.

(c) The remaining idea we need is the following: there is a non-decreasing family $\langle J_\xi \rangle_{\xi \le \kappa}$ in $\mathcal{J}$, for some cardinal κ, such that $J_{\xi+1} \setminus J_\xi$ is countably infinite for every $\xi < \kappa$, $J_\xi = \bigcup_{\eta<\xi} J_\eta$ for every limit ordinal $\eta < \kappa$, and $J_\kappa = I$. **P** Recall that I am already supposing that I is infinite. If I is countable, set $\kappa = 1$, $J_0 = \emptyset$, $J_1 = I$. Otherwise, set $\kappa = \#(I)$ and let $\langle i_\xi \rangle_{\xi<\kappa}$ be an enumeration of I. For $i \in I$, $\phi \in G$ let $K_{\phi i} \subseteq I$ be a countable set such that $F_{\phi i} \in \mathcal{B}_{K_{\phi i}}$. Choose the J_ξ inductively, as follows. The inductive hypothesis must include the requirement that $\#(J_\xi) \le \max(\omega, \#(\xi))$ for every ξ. Start by setting $J_0 = \emptyset$. Given $\xi < \kappa$ and $J_\xi \in \mathcal{J}$ with $\#(J_\xi) \le \max(\omega, \#(\xi)) < \kappa$, take an infinite set $L \subseteq \kappa \setminus J_\xi$ and set $J_{\xi+1} = J_\xi \cup \bigcup_{n\in\mathbb{N}} L_n$, where

$$L_0 = L \cup \{i_\xi\},$$

$$L_{n+1} = \bigcup_{i \in L_n, \phi \in G} K_{\phi i}$$

for $n \in \mathbb{N}$, so that every L_n is countable,

$$F_{\phi i} \in \mathcal{B}_{L_{n+1}} \text{ whenever } i \in L_n,\ \phi \in G$$

and $J_{\xi+1} \in \mathcal{J}$; since $L \subseteq J_{\xi+1} \setminus J_\xi \subseteq \bigcup_{n\in\mathbb{N}} L_n$, $J_{\xi+1} \setminus J_\xi$ is countably infinite, and

$$\#(J_{\xi+1}) = \max(\omega, \#(J_\xi)) \le \max(\omega, \#(\xi)) = \max(\omega, \#(\xi+1)).$$

For non-zero limit ordinals $\xi < \kappa$, set $J_\xi = \bigcup_{\eta<\xi} J_\eta$; then

$$\#(J_\xi) \le \max(\omega, \#(\xi), \sup_{\eta<\xi} \#(J_\eta)) \le \max(\omega, \#(\xi)).$$

Thus the induction proceeds. Observing that the construction puts i_ξ into $J_{\xi+1}$ for every ξ, we see that J_κ will be the whole of I, as required. **Q**

(d) Now put (b) and (c) together, as follows. Take $\langle J_\xi \rangle_{\xi \le \kappa}$ from (c). Set $f_{\phi 0}(x) = x$ for every $\phi \in G$, $x \in X$; then, because $J_0 = \emptyset$, $(J_0, \langle f_{\phi 0} \rangle_{\phi \in G})$ is consistent in the sense of (b). Given that $(J_\xi, \langle f_{\phi\xi} \rangle_{\phi \in G})$ is consistent, where $\xi < \kappa$, use the construction of (b) to find a family $\langle f_{\phi,\xi+1} \rangle_{\phi \in G}$ such that $(J_{\xi+1}, \langle f_{\phi,\xi+1} \rangle_{\phi \in G})$ is consistent and $f_{\phi,\xi+1}(x)(i) = f_{\phi\xi}(x)(i)$ for every $i \in J_\xi$. At a non-zero limit ordinal $\xi \le \kappa$, set

$$\begin{aligned}f_{\phi\xi}(x)(i) &= f_{\phi\eta}(x)(i) \text{ if } x \in X,\ \eta < \xi,\ i \in J_\eta,\\ &= x(i) \text{ if } i \in I \setminus J_\xi.\end{aligned}$$

(The inductive hypothesis includes the requirement that $f_{\phi\eta}(x)\restriction J_\zeta = f_{\phi\zeta}(x)\restriction J_\zeta$ whenever $\phi \in G$, $x \in X$ and $\zeta \le \eta < \xi$.) To see that $(J_\xi, \langle f_{\phi\xi}\rangle_{\phi\in G})$ is consistent, the only non-trivial point to check is that

$$f_{\phi,\xi}f_{\psi,\xi} = f_{\psi\phi,\xi}$$

for all ϕ, $\psi \in G$. But if $i \in J_\xi$ there is some $\eta < \xi$ such that $i \in J_\eta$, and in this case

$$f_{\psi,\xi}^{-1}[E_i] = f_{\psi,\eta}^{-1}[E_i] \in \mathcal{B}_{J_\eta}$$

is determined by coordinates in J_η, so that (because $f_{\phi,\xi}(x)\restriction J_\eta = f_{\phi,\eta}(x)\restriction J_\eta$ for every x)

$$f_{\phi,\xi}^{-1}[f_{\psi,\xi}^{-1}[E_i]] = f_{\phi,\eta}^{-1}[f_{\psi,\eta}^{-1}[E_i]] = f_{\psi\phi,\eta}^{-1}[E_i] = f_{\psi\phi,\xi}^{-1}[E_i];$$

while if $i \in I \setminus J_\xi$ then

$$f_{\psi\phi,\xi}^{-1}[E_i] = E_i = f_{\phi,\xi}^{-1}[E_i] = f_{\psi,\xi}^{-1}[E_i] = f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]].$$

Thus

$$f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]] = f_{\psi\phi,\xi}^{-1}[E_i]$$

for every i, and $f_{\phi,\xi}f_{\psi,\xi} = f_{\phi\psi,\xi}$.

On completing the induction, set $f_\phi = f_{\phi\kappa}$ for every $\phi \in G$; it is easy to see that $\langle f_\phi\rangle_{\phi\in G}$ satisfies the conditions of the theorem.

344F Corollary Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$. Suppose that μ is the completion of its restriction to the σ-algebra $\mathcal{B}$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space (X, Σ, μ).

proof Apply 344E to $\mu\restriction\mathcal{B}$; of course $\mathfrak{A}$ is canonically isomorphic to the measure algebra of $\mu\restriction\mathcal{B}$ (322Da). The functions f_ϕ provided by 344E still represent the homomorphisms ϕ when re-interpreted as functions on the completed measure space $(\{0,1\}^I, \mu)$, by 343Ac.

344G Corollary Let I be any set, ν the usual measure on $\{0,1\}^I$, and $\mathfrak{A}$ its measure algebra. Then any measure-preserving automorphism of $\mathfrak{A}$ is representable by a measure space automorphism of $(\{0,1\}^I, \nu)$.

344H Lemma Let (X, Σ, μ) be a locally compact semi-finite measure space which is not purely atomic. Then there is a negligible subset of X of cardinal $\mathfrak{c}$.

proof Let E be a set of non-zero finite measure not including any atom. Let $\mathcal{K} \subseteq \mathcal{P}E$ be a compact class such that the subspace measure μ_E is inner regular with respect to $\mathcal{K}$. Set $S = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, and choose $\langle K_z\rangle_{z\in S}$ inductively , as follows. $K_\emptyset$ is to be any non-negligible member of $\mathcal{K}\cap\Sigma$ included in E. Given that $K_z \subseteq E$ and $\mu K_z > 0$, where $z \in \{0,1\}^n$, take F_z, $F'_z \subseteq K_z$ to be disjoint non-negligible measurable sets both of measure at most 3^{-n}; such exist because μ is semi-finite and E does not include any atom. Choose $K_{z^\frown 0} \subseteq F_z$, $K_{z^\frown 1} \subseteq F'_z$ to be non-negligible members of $\mathcal{K}\cap\Sigma$.

For each $w \in \{0,1\}^{\mathbb{N}}$, $\langle K_{w\restriction n}\rangle_{n\in\mathbb{N}}$ is a decreasing sequence of members of $\mathcal{K}$ all of non-zero measure, so has non-empty intersection; choose a point $x_w \in \bigcap_{n\in\mathbb{N}} K_{w\restriction n}$. Since $K_{z^\frown 0}\cap K_{z^\frown 1} = \emptyset$ for every $z \in S$, all the x_w are distinct, and $A = \{x_w : w \in \{0,1\}^{\mathbb{N}}\}$ has cardinal $\mathfrak{c}$. Also

$$A \subseteq \bigcup_{z\in\{0,1\}^n} K_z$$

which has measure at most $2^n 3^{-(n-1)}$ for every $n \ge 1$, so $\mu^*A = 0$ and A is negligible.

Remark I see that in this proof I have slipped into a notation which is a touch more sophisticated than what I have used so far. See 3A1H for a note on the interpretations of '$\{0,1\}^n$', '$\{0,1\}^{\mathbb{N}}$' which make sense of the formulae here.

I ought to note also that the lemma is valid for all perfect spaces; see 344Yf.

344I Theorem Let (X,Σ,μ) and (Y,T,ν) be atomless, perfect, complete, strictly localizable, countably separated measure spaces of the same non-zero magnitude. Then they are isomorphic.

proof (a) The point is that the measure algebra $(\mathfrak{A},\bar{\mu})$ of μ has Maharam type ω. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X. Let Σ_0 be the σ-subalgebra of Σ generated by $\{E_n : n\in\mathbb{N}\}$, and $\mathfrak{A}_0$ the order-closed subalgebra of $\mathfrak{A}$ generated by $\{E_n^\bullet : n\in\mathbb{N}\}$; then $E^\bullet\in\mathfrak{A}_0$ for every $E\in\Sigma_0$, and $(X,\Sigma_0,\mu\restriction\Sigma_0)$ is countably separated. Let $f:X\to\mathbb{R}$ be Σ_0-measurable and injective (343E). Of course f is also Σ-measurable. If $a\in\mathfrak{A}\setminus\{0\}$, express a as $E^\bullet$ where $E\in\Sigma$. Because (X,Σ,μ) is perfect, there is a compact $K\subseteq\mathbb{R}$ such that $K\subseteq f[E]$ and $\mu f^{-1}[K]>0$. K is surely a Borel set, so $f^{-1}[K]\in\Sigma_0$ and

$$b=f^{-1}[K]^\bullet\in\mathfrak{A}_0\setminus\{0\}.$$

But because f is injective, we also have $f^{-1}[K]\subseteq E$ and $b\subseteq a$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$; but $\mathfrak{A}_0$ is order-closed, so must be the whole of $\mathfrak{A}$. Thus $\mathfrak{A}$ is τ-generated by the countable set $\{E_n^\bullet : n\in\mathbb{N}\}$, and $\tau(\mathfrak{A})\le\omega$. **Q**

On the other hand, because $\mathfrak{A}$ is atomless, and not $\{0\}$, none of its principal ideals can have finite Maharam type, and it is Maharam-type-homogeneous, with type ω.

(b) Writing $(\mathfrak{B},\bar{\nu})$ for the measure algebra of ν, we see that the argument of (a) applies equally to $(\mathfrak{B},\bar{\nu})$, so that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are atomless localizable measure algebras, of Maharam type ω and the same magnitude. Consequently they are isomorphic as measure algebras, by 332J. Let $\phi:\mathfrak{A}\to\mathfrak{B}$ be a measure-preserving isomorphism.

By 343B, there are functions $g:Y\to X$ and $f:X\to Y$ representing ϕ and ϕ^{-1}, because μ and ν are compact and complete and strictly localizable. Now $fg:Y\to Y$ and $gf:X\to X$ represent the identity automorphisms on $\mathfrak{B}$, $\mathfrak{A}$, so by 343F are equal almost everywhere to the identity functions on Y, X respectively. Set

$$E=\{x : x\in X,\ gf(x)=x\},\quad F=\{y : y\in Y,\ fg(y)=y\};$$

then both E and F are conegligible. Of course $f[E]\subseteq F$ (since $fgf(x)=f(x)$ for every $x\in E$), and similarly $g[F]\subseteq E$; consequently $f\restriction E$, $g\restriction F$ are the two halves of a one-to-one correspondence between E and F. Because ϕ is measure-preserving, $\mu f^{-1}[H]=\nu H$, $\nu g^{-1}[G]=\mu G$ for every $G\in\Sigma$, $H\in\mathrm{T}$; accordingly $f\restriction E$ is an isomorphism between the subspace measures on E and F.

(c) By 344H, applied to the subspace measure on E, there is a negligible set $A\subseteq E$ of cardinal $\mathfrak{c}$. Now X and Y, being countably separated, both have cardinal at most $\mathfrak{c}$. (There are injective functions from X and Y to $\mathbb{R}$.) Set

$$B=A\cup(X\setminus E),\quad C=f[A]\cup(Y\setminus F).$$

Then B and C are negligible subsets of X, Y respectively, and both have cardinal $\mathfrak{c}$ precisely, so there is a bijection $h:B\to C$. Set

$$\begin{aligned} f_1(x)&=f(x)\text{ if }x\in X\setminus B=E\setminus A,\\ &=h(x)\text{ if }x\in B.\end{aligned}$$

Then, because μ and ν are complete, f_1 is an isomorphism between the measure spaces (X,Σ,μ) and (Y,T,ν), as required.

344J Corollary Suppose that E, F are two Lebesgue measurable subsets of $\mathbb{R}^r$ of the same non-zero measure. Then the subspace measures on E and F are isomorphic.

344K Corollary (a) A measure space is isomorphic to Lebesgue measure on $[0,1]$ iff it is an atomless countably separated compact (or perfect) complete probability space; in this case it is also isomorphic to the usual measure on $\{0,1\}^{\mathbb{N}}$.

(b) A measure space is isomorphic to Lebesgue measure on $\mathbb{R}$ iff it is an atomless countably separated compact (or perfect) σ-finite measure space which is not totally finite; in this case it is also isomorphic to Lebesgue measure on any Euclidean space $\mathbb{R}^r$.

(c) Let μ be Lebesgue measure on $\mathbb{R}$. If $0<\mu E<\infty$ and we set $\nu F=\dfrac{1}{\mu E}\mu F$ for every measurable $F\subseteq E$, then (E,ν) is isomorphic to Lebesgue measure on $[0,1]$.

344L The homogeneity property of Lebesgue measure described in 344J is repeated in $\{0,1\}^I$ for any I.

Theorem Let I be any set, and μ the usual measure on $\{0,1\}^I$. If E, $F \subseteq \{0,1\}^I$ are two measurable sets of the same non-zero finite measure, the subspace measures on E and F are isomorphic.

proof Write $X = \{0,1\}^I$.

(a) If I is finite, then X, E and F are all finite, and $\#(E) = \#(F)$, so the result is trivial. If I is countably infinite, then the subspace measures are perfect and complete and countably separated, so the result follows from 344I. So let us suppose that I is uncountable.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ. Then $\mathfrak{A}$ is homogeneous, and $\bar{\mu}E^\bullet = \bar{\mu}F^\bullet$, so there is an automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\phi E^\bullet = F^\bullet$. (Apply 333D to the finite subalgebra $\mathfrak{C}$ generated by $\{E^\bullet, F^\bullet\}$; or argue directly from 331I.) By 344F, there is a measure space isomorphism $f : X \to X$ representing ϕ, so that $f^{-1}[E] \triangle F$ and $f[F] \triangle E$ are negligible.

(c) We can find a countably infinite set $J \subseteq I$ and a measurable set E' such that E' is determined by coordinates in J, $E' \subseteq E \cap f[F]$ and $E \setminus E'$ is negligible; so that E' is non-empty. Take any $x_0 \in E'$ and set $V = \{x : x \in X, x{\restriction}J = x_0{\restriction}J\}$; then V is a negligible subset of E and $\#(V) = \#(X)$ (because $\#(I \setminus J) = \#(I)$ and $x \mapsto x{\restriction}I \setminus J : V \to \{0,1\}^{I \setminus J}$ is a bijection). Setting $E_1 = E \cap f[F]$, $F_1 = F \cap f^{-1}[E]$, $f{\restriction}E_1$ is a bijection between E_1 and F_1. Now

$$\#(V \cup (E \setminus E_1)) = \#(X) = \#(f^{-1}[V] \cup (F \setminus F_1)),$$

so there is a bijection $h : f^{-1}[V] \cup (F \setminus F_1) \to V \cup (E \setminus E_1)$. Define $g : E \to F$ by writing

$$\begin{aligned} g(x) &= f(x) \text{ if } x \in F_1 \setminus f^{-1}[V], \\ &= h(x) \text{ if } x \in f^{-1}[V] \cup (F \setminus F_1). \end{aligned}$$

Then g is a bijection, and because $g = f$ almost everywhere on F, $g^{-1} = f^{-1}$ almost everywhere on E, g is an isomorphism for the subspace measures.

344X Basic exercises (a) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and suppose that there are $E \in \Sigma$, $F \in \mathrm{T}$ such that (X, Σ, μ) is isomorphic to the subspace (F, T_F, ν_F), while (Y, T, ν) is isomorphic to (E, Σ_E, μ_E). Show that (X, Σ, μ) and (Y, T, ν) are isomorphic.

(b) Let (X, Σ, μ) and (Y, T, ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Show that there are conegligible subsets $X' \subseteq X$, $Y' \subseteq Y$ such that X' and Y', with the subspace measures, are isomorphic.

(c) Let (X, Σ, μ) and (Y, T, ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Suppose that they are not purely atomic. Show that they are isomorphic.

(d) Give an example of two perfect countably separated complete probability spaces, with isomorphic measure algebras, which are not isomorphic.

(e) Let (Z, Σ, μ) be the Stone space of a Maharam-type-homogeneous measure algebra. Show that if E, $F \in \Sigma$ have the same non-zero finite measure, then the subspace measures on E and F are isomorphic.

(f) Let $(I^{\|}, \Sigma, \mu)$ be the split interval with its usual measure (343J), and $\mathfrak{A}$ its measure algebra. (i) Show that every automorphism of $\mathfrak{A}$ is represented by a measure space automorphism of $I^{\|}$. (ii) Show that if E, $F \in \Sigma$ and $\mu E = \mu F > 0$ then the subspace measures on E and F are isomorphic.

(g) Let I be an infinite set, and μ the usual measure on $\{0,1\}^I$. Show that if $E \subseteq \{0,1\}^I$ is any set of non-zero measure, then the subspace measure on E is isomorphic to a multiple of μ.

344Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ, and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that there is a countable set $\mathcal{A} \subseteq \Sigma$ separating the points of X. Let G be a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Show that a family $\langle f_\phi \rangle_{\phi \in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras. Suppose that each is isomorphic to a principal ideal of the other. Show that they are isomorphic.

(c) Let I be an infinite set, and write $\mathcal{B}$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let μ and ν be σ-finite measures on X, both with domain $\mathcal{B}$, and with measure algebras $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$. Show that any Boolean isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is represented by a bijection $f : X \to X$ such that f^{-1} represents $\phi^{-1} : \mathfrak{B} \to \mathfrak{A}$, and hence that $(\mathfrak{A}, \bar{\mu})$ is isomorphic to $(\mathfrak{B}, \bar{\nu})$ iff $(X, \mathcal{B}, \mu)$ is isomorphic to $(X, \mathcal{B}, \nu)$.

(d) Let I be any set, and write $\mathcal{B}$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let $\mathcal{I}$ be an ω_1-saturated ideal of $\mathcal{B}$, and write $\mathfrak{A}$ for the quotient Boolean algebra $\mathcal{B}/\mathcal{I}$. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Show that we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, f_ϕ will be an automorphism of the structure $(X, \mathcal{B}, \mathcal{I})$.

(e) Let I be any set, and write $\mathcal{B}$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let $\mathcal{I}$, $\mathcal{J}$ be ω_1-saturated ideals of $\mathcal{B}$. Show that if the Boolean algebras $\mathcal{B}/\mathcal{I}$ and $\mathcal{B}/\mathcal{J}$ are isomorphic, so are the structures $(X, \mathcal{B}, \mathcal{I})$ and $(X, \mathcal{B}, \mathcal{J})$.

(f) Show that if (X, Σ, μ) is a perfect semi-finite measure space which is not purely atomic, there is a negligible set of cardinal $\mathfrak{c}$. (*Hint*: reduce to the case in which μ is atomless and totally finite; in this case, construct a measurable function $f : X \to \mathbb{R}$ such that the image measure $\nu = \mu f^{-1}$ is atomless, and apply 344H to ν.)

344 Notes and comments In this section and the last, I have allowed myself to drift some distance from the avowed subject of this chapter; but it seemed a suitable place for this material, which is fundamental to abstract measure theory. We find that the concepts of §§342-343 are just what is needed to characterise Lebesgue measure (344K), and the characterization shows that among non-negligible measurable subspaces of $\mathbb{R}^r$ the isomorphism classes are determined by a single parameter, the measure of the subspace. Of course a very large number of other spaces – indeed, most of those appearing in ordinary applications of measure theory to other topics – are perfect and countably separated (for example, those of 342Xe and 343Ye), and therefore covered by this classification. I note that it includes, as a special case, the isomorphism between Lebesgue measure on $[0,1]$ and the usual measure on $\{0,1\}^{\mathbb{N}}$ already described in 254K.

In 344I, the first part of the proof is devoted to showing that a perfect countably separated measure space has countable Maharam type; I ought perhaps to note here that we must resist the temptation to suppose that all countably separated measure spaces have countable Maharam type. In fact there are countably separated probability spaces with Maharam type as high as $2^{\mathfrak{c}}$. The arguments are elementary but seem to fit better into Volume 5 than here.

I have offered three contexts in which automorphisms of measure algebras are represented by automorphisms of measure spaces (344A, 344C, 344E). In the first case, every automorphism can be represented simultaneously in a consistent way. In the other two cases, there is, I am sure, no such consistent family of representations which can be constructed within ZFC; but the theorems I give offer consistent simultaneous representations of countably many homomorphisms. The question arises, whether 'countably many' is the true natural limit of the arguments. In fact it is possible to extend both results to families of at most ω_1 automorphisms. I hope to return to this in Volume 5.

Having successfully characterized Lebesgue measure – or, what is very nearly the same thing, the usual measure on $\{0,1\}^{\mathbb{N}}$ – it is natural to seek similar characterizations of the usual measures on $\{0,1\}^\kappa$ for uncountable cardinals κ. This seems to be hard. A variety of examples (which I hope to describe in Volume 5) show that none of the most natural conjectures can be provable in ZFC.

In fact the principal new ideas of this section do not belong specifically to measure theory; rather, they belong to the general theory of σ-algebras and σ-ideals of sets. In the case of the Schröder-Bernstein-type theorem 344D, this is obvious from the formulation I give. (See also 344Yb.) In the case of 344B and 344E, I offer generalizations in 344Ya-344Ye. Of course the applications of 344B here, in 344C and its corollaries, depend on Maharam's theorem and the concept of 'compact' measure space. The former has no generalization to the wider context, and the value of the latter is based on the equivalences in Theorem 343B, which also do not have simple generalizations.

The property described in 344J and 344L – a measure space (X,Σ,μ) in which any two measurable subsets of the same non-zero measure are isomorphic – seems to be a natural concept of 'homogeneity' for measure spaces; it seems unreasonable to ask for all sets of zero measure to be isomorphic, since finite sets of different cardinalities can be expected to be of zero measure. An extra property, shared by Lebesgue measure and the usual measure on $\{0,1\}^I$ (and by the measure on the split interval, 344Xf) but not by counting measure, would be the requirement that measurable sets of different non-zero finite measures should be isomorphic up to a scalar multiple of the measure. All these examples have the further property, that all automorphisms of their measure algebras correspond to automorphisms of the measure spaces.

345 Translation-invariant liftings

In this section and the next I complement the work of §341 by describing some important special properties which can, in appropriate circumstances, be engineered into our liftings. I begin with some remarks on translation-invariance. I restrict my attention to measure spaces which we have already seen, delaying a general discussion of translation-invariant measures on groups until Volume 4, and to results which can be proved without special axioms, delaying the use of the continuum hypothesis, in particular, until Volume 5.

345A Translation-invariant liftings In this section I shall consider two forms of translation-invariance, as follows.

(a) Let μ be Lebesgue measure on $\mathbb{R}^r$, and Σ its domain. A lifting $\phi:\Sigma\to\Sigma$ is **translation-invariant** if $\phi(E+x)=\phi E+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$. (Recall from 134A that $E+x=\{y+x:y\in E\}$ belongs to Σ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.)

Similarly, writing $\mathfrak{A}$ for the measure algebra of μ, a lifting $\theta:\mathfrak{A}\to\Sigma$ is **translation-invariant** if $\theta(E+x)^\bullet=\theta E^\bullet+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.

It is easy to see that if θ and ϕ correspond to each other in the manner of 341B, then one is translation-invariant if and only it the other is.

(b) Now let I be any set, and let μ be the usual measure on $X=\{0,1\}^I$, with Σ its domain and $\mathfrak{A}$ its measure algebra. For x, $y\in X$, define $x+y\in X$ by setting $(x+y)(i)=x(i)+_2 y(i)$ for every $i\in I$; that is, give X the group structure of the product group $\mathbb{Z}_2^I$. This makes X an abelian group (isomorphic to the additive group $(\mathcal{P}I,\triangle)$ of the Boolean algebra $\mathcal{P}I$, if we match $x\in X$ with $\{i:x(i)=1\}\subseteq I$).

Recall that the measure μ is a product measure (254J), being the product of copies of the fair-coin probability on the two-element set $\{0,1\}$. If $x\in X$, then for each $i\in I$ the map $\epsilon\mapsto\epsilon+_2x(i):\{0,1\}\to\{0,1\}$ is a measure space automorphism of $\{0,1\}$, since the two singleton sets $\{0\}$ and $\{1\}$ have the same measure $\frac{1}{2}$. It follows at once that the map $y\mapsto y+x:X\to X$ is a measure space automorphism.

Accordingly we can again say that a lifting $\theta:\mathfrak{A}\to\Sigma$, or $\phi:\Sigma\to\Sigma$, is **translation-invariant** if

$$\theta(E+x)^\bullet=\theta E^\bullet+x,\quad \phi(E+x)=\phi E+x$$

for every $E\in\Sigma$, $x\in X$.

345B Theorem For any $r \geq 1$, there is a translation-invariant lifting of Lebesgue measure on $\mathbb{R}^r$.

proof (a) Write μ for Lebesgue measure on $\mathbb{R}^r$, Σ for its domain. Let $\underline{\phi} : \Sigma \to \Sigma$ be lower Lebesgue density (341E). Then $\underline{\phi}$ is translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for every $E \in \Sigma$, $x \in \mathbb{R}^r$. **P**

$$\underline{\phi}(E + x) = \{y : y \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \tfrac{\mu(E+x)\cap B(y,\delta)}{\mu(B(y,\delta))} = 1\}$$

$$= \{y : y \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \tfrac{\mu(E\cap B(y-x,\delta))}{\mu(B(y-x,\delta))} = 1\}$$

(because μ is translation-invariant)

$$= \{y + x : y \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \tfrac{\mu(E\cap B(y,\delta))}{\mu(B(y,\delta))} = 1\}$$

$$= \underline{\phi}E + x. \text{ **Q**}$$

(b) Let ϕ_0 be any lifting of μ such that $\phi_0 E \supseteq \underline{\phi}E$ for every $E \in \Sigma$ (341J). Consider

$$\phi E = \{y : \mathbf{0} \in \phi_0(E - y)\}$$

for $E \in \Sigma$. It is easy to check that $\phi : \Sigma \to \Sigma$ is a Boolean homomorphism because ϕ_0 is, so that, for instance,

$$\begin{aligned} y \in \phi E \triangle \phi F &\iff \mathbf{0} \in \phi_0(E - y) \triangle \phi_0(F - y) \\ &\iff \mathbf{0} \in \phi_0((E - y) \triangle (F - y)) = \phi_0((E \triangle F) - y)) \\ &\iff y \in \phi(E \triangle F). \end{aligned}$$

(c) If $\mu E = 0$, then $E - y$ is negligible for every $y \in \mathbb{R}^r$, so $\phi_0(E - y)$ is always empty and $\phi E = \emptyset$.

(d) Next, $\underline{\phi}E \subseteq \phi E$ for every $E \in \Sigma$. **P** If $y \in \underline{\phi}E$, then

$$\mathbf{0} = y - y \in \underline{\phi}E - y = \underline{\phi}(E - y) \subseteq \phi_0(E - y),$$

so $y \in \phi E$. **Q** By 341Ib, ϕ is a lifting for μ.

(e) Finally, ϕ is translation-invariant, because if $E \in \Sigma$ and x, $y \in \mathbb{R}^r$ then

$$\begin{aligned} y \in \phi(E + x) &\iff \mathbf{0} \in \phi_0(E + x - y) = \phi_0(E - (y - x)) \\ &\iff y - x \in \phi E \\ &\iff y \in \phi E + x. \end{aligned}$$

345C Theorem For any set I, there is a translation-invariant lifting of the usual measure on $\{0, 1\}^I$.

proof I base the argument on the same programme as in 345B. This time we have to work rather harder, as we have no simple formula for a translation-invariant lower density. However, the ideas already used in 341F-341H are in fact adequate, if we take care, to produce one.

(a) Since there is certainly a bijection between I and its cardinal $\kappa = \#(I)$, it is enough to consider the case $I = \kappa$. Write μ for the usual measure on $X = \{0, 1\}^I = \{0, 1\}^\kappa$ and Σ for its domain. For each $\xi < \kappa$ set $E_\xi = \{x : x \in X, x(\xi) = 1\}$, and let Σ_ξ be the σ-algebra generated by $\{E_\eta : \eta < \xi\}$. Because $x + E_\eta$ is either E_η or $X \setminus E_\eta$, and in either case belongs to Σ_ξ, for every $\eta < \xi$ and $x \in X$, Σ_ξ is translation-invariant. (Consider the algebra

$$\Sigma'_\xi = \{E : E + x \in \Sigma_\xi \text{ for every } x \in X\};$$

this must be Σ_ξ.) Let Φ_ξ be the set of partial lower densities $\underline{\phi} : \Sigma_\xi \to \Sigma$ which are translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for any $E \in \Sigma_\xi$, $x \in X$.

(b)(i) For $\xi < \kappa$, $\Sigma_{\xi+1}$ is just the algebra of subsets of X generated by $\Sigma_\xi \cup \{E_\xi\}$, that is, sets of the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ where F, $G \in \Sigma_\xi$ (312M). Moreover, the expression is unique. **P** Define $x_\xi \in X$ by setting $x_\xi(\xi)=1$, $x_\xi(\eta)=0$ if $\eta\ne\xi$. Then $x_\xi+E_\eta=E_\eta$ for every $\eta<\xi$, so $x_\xi+F=F$ for every $F\in\Sigma_\xi$. If $H=(F\cap E_\xi)\cup(G\setminus E_\xi)$ where F, $G\in\Sigma_\xi$, then

$$x_\xi+H=((x_\xi+F)\cap(x_\xi+E_\xi))\cup((x_\xi+G)\setminus(x_\xi+E_\xi))=(F\setminus E_\xi)\cup(G\cap E_\xi),$$

so

$$F=(H\cap E_\xi)\cup((x_\xi+H)\setminus E_\xi)=F_H,$$

$$G=(H\setminus E_\xi)\cup((x_\xi+H)\cap E_\xi)=G_H$$

are determined by H. **Q**

(ii) The functions $H\mapsto F_H$, $H\mapsto G_H:\Sigma_{\xi+1}\to\Sigma_\xi$ defined above are clearly Boolean homomorphisms; moreover, if H, $H'\in\Sigma_{\xi+1}$ and $H\triangle H'$ is negligible, then

$$(F_H\triangle F_{H'})\cup(G_H\triangle G_{H'})\subseteq(H\triangle H')\cup(x_\xi+(H\triangle H'))$$

is negligible. It follows at once that if $\xi<\kappa$ and $\underline{\phi}\in\Phi_\xi$, we can define $\underline{\phi}_1:\Sigma_{\xi+1}\to\Sigma$ by setting

$$\underline{\phi}_1H=(\underline{\phi}F_H\cap E_\xi)\cup(\underline{\phi}G_H\setminus E_\xi),$$

and $\underline{\phi}_1$ will be a lower density. If $H\in\Sigma_\xi$ then $F_H=G_H=H$, so $\underline{\phi}_1H=\underline{\phi}H$.

(iii) To see that $\underline{\phi}_1$ is translation-invariant, observe that if $x\in X$ and $x(\xi)=0$ then $x+E_\xi=E_\xi$, so, for any F, $G\in\Sigma_\xi$,

$$\begin{aligned}\underline{\phi}_1(x+((F\cap E_\xi)\cup(G\setminus E_\xi)))&=\underline{\phi}_1(((F+x)\cap E_\xi)\cup((G+x)\setminus E_\xi))\\&=(\underline{\phi}(F+x)\cap E_\xi)\cup(\underline{\phi}(G+x)\setminus E_\xi)\\&=((\underline{\phi}F+x)\cap E_\xi)\cup((\underline{\phi}G+x)\setminus E_\xi)\\&=x+(\underline{\phi}F\cap E_\xi)\cup(\underline{\phi}G\setminus E_\xi)\\&=x+\underline{\phi}_1((F\cap E_\xi)\cup(G\setminus E_\xi)).\end{aligned}$$

While if $x(\xi)=1$ then $x+E_\xi=X\setminus E_\xi$, so

$$\begin{aligned}\underline{\phi}_1(x+((F\cap E_\xi)\cup(G\setminus E_\xi)))&=\underline{\phi}_1(((F+x)\setminus E_\xi)\cup((G+x)\cap E_\xi))\\&=(\underline{\phi}(F+x)\setminus E_\xi)\cup(\underline{\phi}(G+x)\cap E_\xi)\\&=((\underline{\phi}F+x)\setminus E_\xi)\cup((\underline{\phi}G+x)\cap E_\xi)\\&=x+(\underline{\phi}F\cap E_\xi)\cup(\underline{\phi}G\setminus E_\xi)\\&=x+\underline{\phi}_1((F\cap E_\xi)\cup(G\setminus E_\xi)).\end{aligned}$$

So $\underline{\phi}_1\in\Phi_{\xi+1}$.

(iv) Thus every member of Φ_ξ has an extension to a member of $\Phi_{\xi+1}$.

(c) Now suppose that $\langle\zeta(n)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in κ with supremum $\xi<\kappa$. Then Σ_ξ is just the σ-algebra generated by $\bigcup_{n\in\mathbb{N}}\Sigma_{\zeta(n)}$. If we have a sequence $\langle\underline{\phi}_n\rangle_{n\in\mathbb{N}}$ such that $\underline{\phi}_n\in\Phi_{\zeta(n)}$ and $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n, then there is a $\underline{\phi}\in\Phi_\xi$ extending every $\underline{\phi}_n$. **P** I repeat the ideas of 341G.

(i) For $E\in\Sigma_\xi$, $n\in\mathbb{N}$ choose g_{En} such that g_{En} is a conditional expectation of χE on $\Sigma_{\zeta(n)}$; that is,

$$\int_F g_{En}=\int_F\chi E=\mu(F\cap E)$$

for every $E\in\Sigma_{\zeta(n)}$. Moreover, make these choices in such a way that (α) every g_{En} is $\Sigma_{\zeta(n)}$-measurable and defined everywhere on X (β) $g_{En}=g_{E'n}$ for every n if $E\triangle E'$ is negligible. Now $\lim_{n\to\infty}g_{En}$ exists and is equal to χE almost everywhere, by Lévy's martingale theorem (275I).

(ii) For $E\in\Sigma_\xi$, $k\ge1$, $n\in\mathbb{N}$ set

$$H_{kn}(E)=\{x:x\in X,\,g_{En}(x)\ge1-2^{-k}\}\in\Sigma_{\zeta(n)},\quad\tilde{H}_{kn}(E)=\underline{\phi}_n(H_{kn}(E)),$$

$$\underline{\phi}E=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde H_{km}(E).$$

(iii) Every $g_{\emptyset n}$ is zero almost everywhere, every $H_{kn}(\emptyset)$ is negligible and every $\tilde H_{kn}(\emptyset)$ is empty; so $\underline{\phi}\emptyset=\emptyset$. If E, $E'\in\Sigma_\xi$ and $E\triangle E'$ is negligible, $g_{En}=g_{E'n}$ for every n, $H_{nk}(E)=H_{nk}(E')$ and $\tilde H_{nk}(E)=\tilde H_{nk}(E')$ for all n, k, and $\underline{\phi}E=\underline{\phi}E'$.

(iv) If $E\subseteq F$ in Σ_ξ, then $g_{En}\le g_{Fn}$ almost everywhere for every n, every $H_{kn}(E)\setminus H_{kn}(F)$ is negligible, $\tilde H_{kn}(E)\subseteq\tilde H_{kn}(F)$ for every n, k, and $\underline{\phi}E\subseteq\underline{\phi}F$.

(v) If E, $F\in\Sigma_\xi$ then $\chi(E\cap F)\ge\chi E+\chi F-1$ a.e. so $g_{E\cap F,n}\ge g_{En}+g_{Fn}-1$ a.e. for every n. Accordingly

$$H_{k+1,n}(E)\cap H_{k+1,n}(F)\setminus H_{kn}(E\cap F)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde H_{kn}(E\cap F)\supseteq\underline{\phi}_n(H_{k+1,n}(E)\cap H_{k+1,n}(F))=\tilde H_{k+1,n}(E)\cap\tilde H_{k+1,n}(F)$$

for all $k\ge 1$, $n\in\mathbb{N}$. Now, if $x\in\underline{\phi}E\cap\underline{\phi}F$, then, for any $k\ge 1$, there are n_1, $n_2\in\mathbb{N}$ such that

$$x\in\bigcap_{m\ge n_1}\tilde H_{k+1,m}(E),\quad x\in\bigcap_{m\ge n_2}\tilde H_{k+1,m}(F).$$

But this means that

$$x\in\bigcap_{m\ge\max(n_1,n_2)}\tilde H_{km}(E\cap F).$$

As k is arbitrary, $x\in\underline{\phi}(E\cap F)$; as x is arbitrary, $\underline{\phi}E\cap\underline{\phi}F\subseteq\underline{\phi}(E\cap F)$. We know already from (iv) that $\underline{\phi}(E\cap F)\subseteq\underline{\phi}E\cap\underline{\phi}F$, so $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$.

(vi) If $E\in\Sigma_\xi$, then $g_{En}\to\chi E$ a.e., so setting

$$V=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}H_{km}(E)=\{x:\limsup_{n\to\infty}g_{En}(x)\ge 1\},$$

$V_E\triangle E$ is negligible; but

$$\underline{\phi}E\triangle V\subseteq\bigcup_{k\ge 1,n\in\mathbb{N}}H_{kn}(E)\triangle\tilde H_{kn}(E)$$

is also negligible, so $\underline{\phi}E\triangle E$ is negligible. Thus $\underline{\phi}$ is a partial lower density with domain Σ_ξ.

(vii) If $E\in\Sigma_{\zeta(n)}$, then $E\in\Sigma_{\zeta(m)}$ for every $m\ge n$, so $g_{Em}=\chi E$ a.e. for every $m\ge n$; $H_{km}(E)\triangle E$ is negligible for $k\ge 1$, $m\ge n$;

$$\tilde H_{km}(E)=\underline{\phi}_mE=\underline{\phi}_nE$$

for $k\ge 1$, $m\ge n$; and $\underline{\phi}E=\underline{\phi}_nE$. Thus $\underline{\phi}$ extends every $\underline{\phi}_n$.

(viii) I have still to check the translation-invariance of $\underline{\phi}$. If $E\in\Sigma_\xi$ and $x\in X$, consider g'_n, defined by setting

$$g'_n(y)=g_{En}(y-x)$$

for every $y\in X$, $n\in\mathbb{N}$; that is, g'_n is the composition $g_{En}\psi$, where $\psi(y)=y-x$ for $y\in X$. (I am not sure whether it is more, or less, confusing to distinguish between the operations of addition and subtraction in X. Of course $y-x=y+(-x)=y+x$ for every y.) Because ψ is a measure space automorphism, and in particular is inverse-measure-preserving, we have

$$\int_{F+x}g'_n=\int_{\psi^{-1}[F]}g'_n=\int_F g_{En}=\mu(E\cap F)$$

whenever $F\in\Sigma_{\zeta(n)}$ (235Ic). But because $\Sigma_{\zeta(n)}$ is itself translation-invariant, we can apply this to $F-x$ to get

$$\int_F g'_n=\mu(E\cap(F-x))=\mu((E+x)\cap F)$$

for every $F\in\Sigma_{\zeta(n)}$. Moreover, for any $\alpha\in\mathbb{R}$,

$$\{y:g'_n(y)\le\alpha\}=\{y:g_{En}(y)\ge\alpha\}+x\in\Sigma_{\zeta(n)}$$

for every α, and g'_n is $\Sigma_{\zeta(n)}$-measurable. So g'_n is a conditional expectation of $\chi(E+x)$ on $\Sigma_{\zeta(n)}$, and must be equal almost everywhere to $g_{E+x,n}$.

This means that if we set

$$H'_{kn} = \{y : g'_n(y) \ge 1 - 2^{-k}\} = H_{kn}(E) + x$$

for k, $n \in \mathbb{N}$, we shall have $H'_{kn} \in \Sigma_{\zeta(n)}$ and $H'_{kn} \triangle H_{kn}(E+x)$ will be negligible, so

$$\begin{aligned}\tilde{H}_{kn}(E+x) &= \underline{\phi}_n(H_{kn}(E+x)) = \underline{\phi}_n(H'_{kn})\\ &= \underline{\phi}_n(H_{kn}(E)+x) = \underline{\phi}_n(H_{kn}(E)) + x = \tilde{H}_{kn}(E) + x.\end{aligned}$$

Consequently

$$\begin{aligned}\underline{\phi}(E+x) &= \bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{kn}(E+x)\\ &= \bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{kn}(E) + x = \underline{\phi}E + x.\end{aligned}$$

As E and x are arbitrary, $\underline{\phi}$ is translation-invariant and belongs to Φ_ξ. **Q**

(d) We are now ready for the proof that there is a translation-invariant lower density on X. **P** Build inductively a family $\langle\underline{\phi}_\xi\rangle_{\xi\le\kappa}$ such that (α) $\underline{\phi}_\xi \in \Phi_\xi$ for each ξ (β) $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta \le \xi \le \kappa$. The induction starts with $\Sigma_0 = \{\emptyset, X\}$, $\underline{\phi}_0\emptyset = \emptyset$, $\underline{\phi}_0 X = X$. The inductive step to a successor ordinal is dealt with in (b), and the inductive step to a non-zero ordinal of countable cofinality is dealt with in (c). If $\xi \le \kappa$ has uncountable cofinality, then $\Sigma_\xi = \bigcup_{\eta<\xi}\Sigma_\eta$, so we can (and must) take $\underline{\phi}_\xi$ to be the unique common extension of all the previous $\underline{\phi}_\eta$.

The induction ends with $\underline{\phi}_\kappa : \Sigma_\kappa \to \Sigma$. Note that Σ_κ is not in general the whole of Σ. But for every $E \in \Sigma$ there is an $F \in \Sigma_\kappa$ such that $E\triangle F$ is negligible (254Ff). So we can extend $\underline{\phi}_\kappa$ to a function $\underline{\phi}$ defined on the whole of Σ by setting

$$\underline{\phi}E = \underline{\phi}_\kappa F \text{ whenever } E \in \Sigma,\ F \in \Sigma_\kappa \text{ and } \mu(E\triangle F) = 0$$

(the point being that $\underline{\phi}_\kappa F = \underline{\phi}_\kappa F'$ if F, $F' \in \Sigma_\kappa$ and $\mu(E\triangle F) = \mu(E\triangle F') = 0$). It is easy to check that $\underline{\phi}$ is a lower density, and it is translation-invariant because if $E \in \Sigma$, $x \in X$, $F \in \Sigma_\kappa$ and $E\triangle F$ is negligible, then $(E+x)\triangle(F+x) = (E\triangle F) + x$ is negligible, so

$$\underline{\phi}(E+x) = \underline{\phi}_\kappa(F+x) = \underline{\phi}_\kappa F + x = \underline{\phi}E + x. \text{ } \mathbf{Q}$$

(e) The rest of the argument is exactly that of parts (b)-(e) of the proof of 345B; you have to change $\mathbb{R}^r$ into X wherever it appears, but otherwise you can use it word for word, interpreting '$\mathbf{0}$' as the identity of the group X, that is, the constant function with value 0.

345D Translation-invariant liftings are of great importance, and I will return to them in §447 with a theorem dramatically generalizing the results above. Here I shall content myself with giving one of their basic properties, set out for the two kinds of translation-invariant lifting we have seen.

Proposition Let (X, Σ, μ) be *either* Lebesgue measure on $\mathbb{R}^r$ *or* the usual measure on $\{0,1\}^I$ for some set I, and let $\phi : \Sigma \to \Sigma$ be a translation-invariant lifting. Then for any open set $G \subseteq X$ we must have $G \subseteq \phi G \subseteq \overline{G}$, and for any closed set F we must have $\operatorname{int} F \subseteq \phi F \subseteq F$.

proof (a) Suppose that $G \subseteq X$ is open and that $x \in G$. Then there is an open set U such that $\mathbf{0} \in U$ and $x+U-U = \{x+y-z : y,\ z \in U\} \subseteq G$. **P** (α) If $X = \mathbb{R}^r$, take $\delta > 0$ such that $\{y : \|y-x\| \le \delta\} \subseteq G$, and set $U = \{y : \|y-x\| < \frac{1}{2}\delta\}$. (β) If $X = \{0,1\}^I$, then there is a finite set $K \subseteq I$ such that $\{y : y{\restriction}K = x{\restriction}K\} \subseteq G$ (3A3K); set $U = \{y : y(i) = 0 \text{ for every } i \in K\}$. **Q**

It follows that $x \in \phi G$. **P** Consider $H = x + U$. Then $\mu H = \mu U > 0$ so $H \cap \phi H \ne \emptyset$. Let $y \in U$ be such that $x + y \in \phi H$. Then

$$x = (x+y) - y \in \phi(H-y) \subseteq \phi G$$

because

$$H - y \subseteq x + U - U \subseteq G. \quad \mathbf{Q}$$

(b) Thus $G \subseteq \phi G$ for every open set $G \subseteq X$. But it follows at once that if G is open and F is closed,

$$\operatorname{int} F \subseteq \phi(\operatorname{int} F) \subseteq \phi F,$$

$$\overline{G} = X \setminus \operatorname{int}(X \setminus G) \supseteq X \setminus \phi(X \setminus G) = \phi G,$$

$$F = X \setminus (X \setminus F) \supseteq X \setminus \phi(X \setminus F) = \phi F.$$

345E I remarked in 341Lg that it is undecidable in ordinary set theory whether there is a lifting for Borel measure on $\mathbb{R}$. It is however known that there can be no translation-invariant Borel lifting. The argument depends on the following fact about measurable sets in $\{0,1\}^{\mathbb{N}}$.

Lemma Let μ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, and E any non-negligible measurable set. Then there are x, $x' \in E$ which differ at exactly one coordinate.

proof By 254Fe, there is a set F, determined by coordinates in a finite set, such that $\mu(E \triangle F) \le \frac{1}{4}\mu E$; we have $\mu F \ge \frac{3}{4}\mu E$, so $\mu(E \triangle F) \le \frac{1}{3}\mu F$. Suppose that F is determined by coordinates in $\{0, \dots, n-1\}$. Then the map $\psi : X \to X$, defined by setting $(\psi x)(n) = 1 - x(n)$, $(\psi x)(i) = x(i)$ for $i \ne n$, is a measure space automorphism, and

$$\mu(\psi^{-1}[E \triangle F] \cup (E \triangle F)) \le 2\mu(E \triangle F) < \mu F.$$

Take any $x \in F \setminus ((E \triangle F) \cup \psi^{-1}[E \triangle F])$. Then $x' = \psi x$ differs from x at exactly one coordinate; but also $x' \in F$, by the choice of n, so both x and x' belong to E.

345F Proposition Let μ be Borel measure on $\mathbb{R}$, that is, the restriction of Lebesgue measure to the algebra $\mathcal{B}$ of Borel sets. Then μ is translation-invariant, but has no translation-invariant lifting.

proof (a) To see that μ is translation-invariant all we have to know is that $\mathcal{B}$ is translation-invariant and that Lebesgue measure is translation-invariant. I have already cited 134A for the proof that Lebesgue measure is invariant, and $\mathcal{B}$ is invariant because $G + x$ is open for every open set G and every $x \in \mathbb{R}$.

(b) The argument below is most easily expressed in terms of the geometry of the Cantor set C. Recall that C is defined as the intersection $\bigcap_{n\in\mathbb{N}} C_n$ of a sequence of closed subsets of $[0,1]$; each C_n consists of 2^n closed intervals of length 3^{-n}; C_{n+1} is obtained from C_n by deleting the middle third of each interval of C_n. Any point of C is uniquely expressible as $f(e) = \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n} e(n)$ for some $e \in \{0,1\}^{\mathbb{N}}$. (See 134G.) Let ν be the usual measure of $\{0,1\}^{\mathbb{N}}$. Because the map $e \mapsto e(n) : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is measurable for each n, $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is measurable.

We can label the closed intervals constituting C_n as $\langle J_z \rangle_{z \in \{0,1\}^n}$, taking $J_\emptyset$ to be the unit interval $[0,1]$ and, for $z \in \{0,1\}^n$, taking $J_{z^\frown 0}$ to be the left-hand third of J_z and $J_{z^\frown 1}$ to be the right-hand third of J_z. (If the notation here seems odd to you, there is an explanation in 3A1H.)

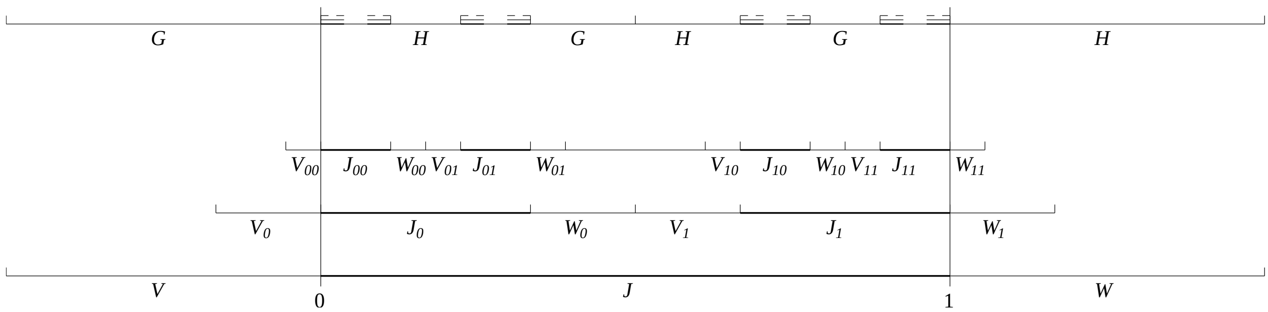
For $n \in \mathbb{N}$, $z \in \{0,1\}^n$, let J'_z be the open interval with the same centre as J_z and twice the length. Then $J'_z \setminus J_z$ consists of two open intervals of length $3^{-n}/2$ on either side of J_z; call the left-hand one V_z and the right-hand one W_z. Thus $V_{z^\frown 1}$ is the right-hand half of the middle third of J_z, and $W_{z^\frown 0}$ is the left-hand half of the middle third of J_z.

Construct sets G, $H \subseteq \mathbb{R}$ as follows.

> G is to be the union of the intervals V_z where z takes the value 1 an even number of times, together with the intervals W_z where z takes the value 0 an odd number of times;
>
> H is to be the union of the intervals V_z where z takes the value 1 an odd number of times, together with the intervals W_z where z takes the value 0 an even number of times.

G and H are open sets. The intervals V_z, W_z between them cover the whole of the interval $\left]-\frac{1}{2}, \frac{3}{2}\right[$ with the exception of the set C and the countable set of midpoints of the intervals J_z; so that $\left]-\frac{1}{2}, \frac{3}{2}\right[\setminus (G \cup H)$ is negligible. We have to observe that $G \cap H = \emptyset$. $\mathbf{P}$ For each z, $J'_{z^\frown 0}$ and $J'_{z^\frown 1}$ are disjoint subsets of J'_z. Consequently $J'_z \cap J'_w$ is non-empty just when one of z, w extends the other, and we need consider only the intersections of the four sets V_z, W_z, V_w, W_w when w is a proper extension of z; say $w \in \{0,1\}^n$ and $z = w{\restriction}m$, where $m < n$. (α) If in the extension $(w(m), \dots, w(n-1))$ both values 0 and 1 appear, J'_w will be a subset of J_z, and certainly the four sets will all be disjoint. (β) If $w(i) = 0$ for $m \le i < n$, then $W_w \subseteq J_z$ is disjoint from the rest, while $V_w \subseteq V_z$; but z and w take the value 1 the same number of times, so V_w is assigned to G iff V_z is, and otherwise both are assigned to H. (γ) Similarly, if $w(i) = 1$ for $m \le i < n$, $V_w \subseteq J_z$, $W_w \subseteq W_z$ and z, w take the value 0 the same number of times, so W_z and W_w are assigned to the same set. **Q**

The following diagram may help you to see what is supposed to be happening:



The assignment rule can be restated as follows:

$V = V_\emptyset$ is assigned to G, $W = W_\emptyset$ is assigned to H;

$V_{z^\frown 0}$ is assigned to the same set as V_z, and $V_{z^\frown 1}$ to the other;

$W_{z^\frown 1}$ is assigned to the same set as W_z, and $W_{z^\frown 0}$ to the other.

(c) Now take any $n \in \mathbb{N}$ and $z \in \{0,1\}^n$. Consider the two open intervals $I_0 = J'_{z^\frown 0}$, $I_1 = J'_{z^\frown 1}$. These are both of length $\gamma = 2 \cdot 3^{-n-1}$ and abut at the centre of J_z, so I_1 is just the translate $I_0 + \gamma$. I claim that $I_1 \cap H = (I_0 \cap G) + \gamma$. **P** Let A be the set

$$\bigcup_{m>n} \{w : w \in \{0,1\}^m,\ w \text{ extends } z^\frown 0\},$$

and for $w \in A$ let w' be the finite sequence obtained from w by changing $w(n) = 0$ into $w'(n) = 1$ but leaving the other values of w unaltered. Then $V_{w'} = V_w + \gamma$ and $W_{w'} = W_w + \gamma$ for every $w \in A$. Now

$$I_0 \cap G = \bigcup\{V_w : w \in A,\ w \text{ takes the value 1 an even number of times}\}$$
$$\cup \bigcup\{W_w : w \in A,\ w \text{ takes the value 0 an odd number of times}\},$$

so

$$\begin{aligned}(I_0 \cap G) + \gamma &= \bigcup\{V_{w'} : w \in A,\ w \text{ takes the value 1 an even number of times}\}\\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w \text{ takes the value 0 an odd number of times}\}\\ &= \bigcup\{V_{w'} : w \in A,\ w' \text{ takes the value 1 an odd number of times}\}\\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w' \text{ takes the value 0 an even number of times}\}\\ &= I_1 \cap H. \ \mathbf{Q}\end{aligned}$$

(d) ? Now suppose, if possible, that $\phi : \mathcal{B} \to \mathcal{B}$ is a translation-invariant lifting. Note first that $U \subseteq \phi U$ for every open $U \subseteq \mathbb{R}$. **P** The argument is exactly that of 345D as applied to $\mathbb{R} = \mathbb{R}^1$. **Q** Consequently

$$J'_\emptyset = \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi J'_\emptyset.$$

But as $J'_\emptyset \setminus (G \cup H)$ is negligible,

$$C \subseteq \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi G \cup \phi H.$$

Consider the sets $E = f^{-1}[\phi G]$, $F = \{0,1\}^{\mathbb{N}} \setminus E = f^{-1}[\phi H]$. Because f is measurable and ϕG, ϕH are Borel sets, E and F are measurable subsets of $\{0,1\}^{\mathbb{N}}$, and at least one of them has positive measure. There must therefore be e, $e' \in \{0,1\}^{\mathbb{N}}$, differing at exactly one coordinate, such that either both belong to E or both belong to F (345E). Let us suppose that n is such that $e(n) = 0$, $e'(n) = 1$ and $e(i) = e'(i)$ for $i \neq n$. Set $z = e\restriction n = e'\restriction n$. Then $f(e)$ belongs to the open interval $I_0 = J'_{z^\frown 0}$, so $f(e) \in \phi I_0$ and $f(e) \in \phi G$ iff $f(e) \in \phi(I_0 \cap G)$. But now

$$f(e') = f(e) + 2 \cdot 3^{-n-1} \in I_1 = J'_{z^\frown 1},$$

so

$$\begin{aligned} e \in E &\iff f(e) \in \phi G \iff f(e) \in \phi(I_0 \cap G) \\ &\iff f(e') \in \phi((I_0 \cap G) + 2 \cdot 3^{-n-1}) \end{aligned}$$

(because ϕ is translation-invariant)

$$\iff f(e') \in \phi(I_1 \cap H)$$

(by (c) above)

$$\iff f(e') \in \phi H$$

(because $f(e') \in I_1 \subseteq \phi I_1$)

$$\iff e' \in F.$$

But this contradicts the choice of e. **X**

Thus there is no translation-invariant lifting of μ.

Remark This result is due to Johnson 80; the proof here follows Talagrand 82b. For references to various generalizations see Burke 93, §3.

345X Basic exercises (a) In 345Ab I wrote 'It follows at once that the map $y \mapsto y + x : X \to X$ is a measure space automorphism'. Write the details out in full, using 254G or otherwise.

(b) Let S^1 be the unit circle in $\mathbb{R}^2$, and let μ be one-dimensional Hausdorff measure on S^1 (§§264-265). Show that μ is translation-invariant, if S^1 is given its usual group operation corresponding to complex multiplication (255M), and that it has a translation-invariant lifting ϕ. (*Hint*: Identifying S^1 with $]-\pi,\pi]$ with the group operation $+_{2\pi}$, show that we can set $\phi E = \,]-\pi,\pi] \cap \phi'(\bigcup_{n\in\mathbb{Z}} E + 2\pi n)$, where ϕ' is any translation-invariant lifting for Lebesgue measure.)

>(c) Show that there is no lifting ϕ of Lebesgue measure on $\mathbb{R}$ which is 'symmetric' in the sense that $\phi(-E) = -\phi E$ for every measurable set E, writing $-E = \{-x : x \in E\}$. (*Hint*: can 0 belong to $\phi([0,\infty[)$?)

>(d) Let μ be Lebesgue measure on $X = \mathbb{R}\setminus\{0\}$. Show that there is a lifting ϕ of μ such that $\phi(xE) = x\phi E$ for every $x \in X$ and every measurable $E \subseteq X$, writing $xE = \{xy : y \in E\}$.

(e) Let μ be the usual measure on $X = \{0,1\}^I$, for some set I, Σ its domain, and $(\mathfrak{A}, \bar\mu)$ its measure algebra. (i) Show that we can define $\pi_x(a) = a + x$, for $a \in \mathfrak{A}$ and $x \in X$, by the formula $E^\bullet + x = (E + x)^\bullet$; and that $x \mapsto \pi_x$ is a group homomorphism from X to the group of measure-preserving automorphisms of $\mathfrak{A}$. (ii) Define Σ_ξ as in the proof of 345C, and set $\mathfrak{A}_\xi = \{E^\bullet : E \in \Sigma_\xi\}$. Say that a partial lifting $\underline{\theta} : \mathfrak{A}_\xi \to \Sigma$ is translation-invariant if $\underline{\theta}(a + x) = \underline{\theta}a + x$ for every $a \in \mathfrak{A}_\xi$ and $x \in X$. Show that any such partial lifting can be extended to a translation-invariant partial lifting on $\mathfrak{A}_{\xi+1}$. (iii) Write out a proof of 345C in the language of 341F-341H.

>(f) Let $\underline{\phi}$ be a lower density for Lebesgue measure on $\mathbb{R}^r$ which is translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for every $x \in \mathbb{R}^r$ and every measurable set E. Show that $\underline{\phi}G \subseteq G$ for every open set $G \subseteq \mathbb{R}^r$.

(g) Let μ be 1-dimensional Hausdorff measure on S^1, as in 345Xb. Show that there is no translation-invariant lifting ϕ of μ such that ϕE is a Borel set for every $E \in \operatorname{dom}\mu$.

345Y Further exercises (a) Let (X, Σ, μ) be a complete measure space, and suppose that X has a group operation $(x, y) \mapsto xy$ (not necessarily abelian!) such that μ is left-translation-invariant, in the sense that $xE = \{xy : y \in E\} \in \Sigma$ and $\mu(xE) = \mu E$ whenever $E \in \Sigma$ and $x \in X$. Suppose that $\underline{\phi} : \Sigma \to \Sigma$ is a lower density which is left-translation-invariant in the sense that $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $E \in \Sigma$, $x \in X$. Show that there is a left-translation-invariant lifting $\phi : \Sigma \to \Sigma$ such that $\underline{\phi}E \subseteq \phi E$ for every $E \in \Sigma$.

(b) Write Σ for the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{L}^0(\Sigma)$ for the linear space of Σ-measurable functions from $\mathbb{R}$ to itself. Show that there is a linear operator $T : L^0(\mu) \to \mathcal{L}^0(\Sigma)$ such that (α) $(Tu)^\bullet = u$ for every $u \in L^0(\mu)$ (β) $\sup_{x\in\mathbb{R}} |(Tu)(x)| = \|u\|_\infty$ for every $u \in L^\infty(\mu)$ (γ) $Tu \ge 0$ whenever $u \in L^\infty(\mu)$ and $u \ge 0$ (δ) T is translation-invariant in the sense that $T(S_x f)^\bullet = S_x T f^\bullet$ for every $x \in \mathbb{R}$ and $f \in \mathcal{L}^0(\Sigma)$, where $(S_x f)(y) = f(x + y)$ for $f \in \mathcal{L}^0(\Sigma)$ and x, $y \in \mathbb{R}$ (ϵ) T is reflection-invariant in the sense that $T(Rf)^\bullet = RTf^\bullet$ for every $f \in \mathcal{L}^0(\Sigma)$, where $(Rf)(x) = f(-x)$ for $f \in \mathcal{L}^0(\Sigma)$ and $x \in \mathbb{R}$. (*Hint*: for $f \in \mathcal{L}^0(\Sigma)$, set

$$p(f^\bullet) = \inf\{\alpha : \alpha \in [0, \infty], \lim_{\delta\downarrow 0} \frac{1}{2\delta}\mu\{x : |x| \le \delta, |f(x)| > \alpha\} = 0\}.$$

Set $V = \{u : u \in L^0(\mu), p(u) < \infty\}$ and show that V is a linear subspace of $L^0(\mu)$ and that $p\restriction V$ is a seminorm. Let $h_0 : V \to \mathbb{R}$ be a linear functional such that $h_0(\chi\mathbb{R})^\bullet = 1$ and $h_0(u) \le p(u)$ for every $u \in V$. Extend h_0 arbitrarily to a linear functional $h_1 : L^0(\mu) \to \mathbb{R}$; set $h(f^\bullet) = \frac{1}{2}(h_1(f^\bullet) + h_1(Rf)^\bullet)$. Set $(Tf^\bullet)(x) = h(S_{-x}f)^\bullet$. You will need 231C.) Show that there must be a $u \in L^1(\mu)$ such that $u \ge 0$ but $Tu \not\ge 0$.

(c) Show that there is no translation-invariant lifting ϕ of the usual measure on $\{0, 1\}^{\mathbb{N}}$ such that ϕE is a Borel set for every measurable set E.

345 Notes and comments I have taken a great deal of care over the concept of 'translation-invariance'. I hope that you are already a little impatient with some of the details as I have written them out; but while it is very easy to guess at the structure of such arguments as part (e) of the proof of 345B, or (b-iii) and (c-viii) in the proof of 345C, I am not sure that one can always be certain of guessing correctly. A fair test of your intuition will be how quickly you can generate the formulae appropriate to a non-abelian group operation, as in 345Ya.

Part (b) of the proof of 345C is based on the same idea as the proof of 341F. There is a useful simplification because the set E_ξ in 345C, corresponding to the set E of the proof of 341F, is independent of the algebra Σ_ξ in a very strong sense, so that the expression of an element of $\Sigma_{\xi+1}$ in the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ is unique. Interpreted in the terms of 341F, we have $w = v = 1$, so that the formula

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr)$$

used there becomes

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = (\underline{\theta}a \cap E) \cup (\underline{\theta}b \setminus E),$$

matching the formula for $\underline{\phi}_1$ in the proof of 345C.

The results of this section are satisfying and natural; they have obvious generalizations, many of which are true. The most important measure spaces come equipped with a variety of automorphisms, and we can always ask which of these can be preserved by a lifting. The answers are not always obvious; I offer 345Xc and 346Xb as warnings, and 345Xd as an encouragement. 345Yb is striking (I have made it as striking as I can), but slightly off the most natural target; the sting is in the last sentence (see 341Ye).

346 Consistent liftings

I turn now to a different type of condition which we should naturally prefer our liftings to satisfy. If we have a product measure μ on a product $X = \prod_{i\in I} X_i$ of probability spaces, then we can look for liftings ϕ which 'respect coordinates', that is, are compatible with the product structure in the sense that they factor through subproducts (346A). There seem to be obstacles in the way of the natural conjecture (346Za), and I give the partial results which are known. For Maharam-type-homogeneous spaces X_i, there is always a lifting which respects coordinates (346E), and indeed the translation-invariant liftings of §345 on $\{0,1\}^I$ already have this property (346C). There is always a lower density on the product which respects coordinates, and we can ask for a little more (346G); using the full strength of 346G, we can enlarge this lower density to a lifting which respects single coordinates and initial segments of a well-ordered product (346H). In the case in which all the factors are copies of each other, we can arrange for the induced liftings on the factors to be copies also (346I, 346J, 346Yd). I end the section with an important fact about Stone spaces which is relevant here (346K-346L).

346A Definition Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Σ, μ). I will say that a lifting $\phi : \Sigma \to \Sigma$ **respects coordinates** if ϕE is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in $J \subseteq I$.

Remark Recall that a set $E \subseteq X$ is 'determined by coordinates in J' if $x' \in E$ whenever $x \in E$, $x' \in X$ and $x'\restriction J = x\restriction J$; that is, if E is expressible as $\pi_J^{-1}[F]$ for some $F \subseteq \prod_{i\in J} X_i$, where $\pi_J(x) = x\restriction J$ for every $x \in X$; that is, if $E = \pi_J^{-1}[\pi_J[E]]$. See 254M. Recall also that in this case, if E is measurable for the product measure on X, then $\pi_J[E]$ is measurable for the product measure on $\prod_{i\in J} X_i$ (254Ob).

346B Proposition Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (Z, Λ, λ). For $J \subseteq I$ let $(Z_J, \Lambda_J, \lambda_J)$ be the product of $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in J}$, and $\pi_J : Z \to Z_J$ the canonical map. Let $\phi : \Lambda \to \Lambda$ be a lifting. If $J \subseteq I$ is such that ϕW is determined by coordinates in J whenever $W \in \Lambda$ is determined by coordinates in J, then ϕ induces a lifting $\phi_J : \Lambda_J \to \Lambda_J$ defined by the formula

$$\pi_J^{-1}[\phi_J E] = \phi(\pi_J^{-1}[E]) \text{ for every } E \in \Lambda_J.$$

proof If $E \in \Lambda_J$, then $\pi^{-1}[E]$ and $\phi(\pi_J^{-1}[E])$ are determined by coordinates in J, so $\phi(\pi_J^{-1}[E])$ is of the form $\pi_J^{-1}[F]$ for some $F \in \Lambda_J$; this defines $\phi_J : \Lambda_J \to \Lambda_J$. It is now easy to see that ϕ_J is a lifting.

Remark Of course we frequently wish to use this result with a singleton set $J = \{j\}$. In this case we must remember that $(Z_J, \Sigma_J, \lambda_J)$ corresponds to the *completion* of the probability space (X_j, Σ_j, μ_j).

346C Theorem Let I be any set, and μ the usual measure on $X = \{0,1\}^I$. Then any translation-invariant lifting of μ respects coordinates.

proof Suppose that $E \subseteq X$ is a measurable set determined by coordinates in $J \subseteq I$; take $x \in \phi E$ and $x' \in X$ such that $x'\restriction J = x\restriction J$. Set $y = x' - x$; then $y(i) = 0$ for $i \in J$, so that $E + y = y$. Now

$$x' = x + y \in \phi E + y = \phi(E + y) = \phi E$$

because ϕ is translation-invariant. As x, x' are arbitrary, ϕE is determined by coordinates in J. As E and J are arbitrary, ϕ respects coordinates.

346D I describe a standard method of constructing liftings from other liftings.

Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$, $\mathfrak{B}$; suppose that $f : X \to Y$ is a (Σ, T)-measurable function inducing an isomorphism $F^\bullet \mapsto f^{-1}[F]^\bullet : \mathfrak{B} \to \mathfrak{A}$. Then if $\phi : \mathrm{T} \to \mathrm{T}$ is a lifting for ν, there is a corresponding lifting $\phi' : \Sigma \to \Sigma$ given by the formula

$$\phi' E = f^{-1}[\phi F] \text{ whenever } \mu(E \triangle f^{-1}[F]) = 0.$$

proof If we say that $\pi : \mathfrak{B} \to \mathfrak{A}$ is the isomorphism induced by f, then

$$\phi' E = f^{-1}[\theta(\pi^{-1}E^\bullet)],$$

where $\theta : \mathfrak{B} \to \mathrm{T}$ is the lifting corresponding to $\phi : \mathrm{T} \to \mathrm{T}$. Since θ, π^{-1} and $F \mapsto f^{-1}[F]$ are all Boolean homomorphisms, so is ϕ', and it is easy to check that $(\phi' E)^\bullet = E^\bullet$ for every $E \in \Sigma$ and that $\phi' E = \emptyset$ if $\mu E = 0$.

Remark Compare the construction in 341P.

346E Theorem Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Maharam-type-homogeneous probability spaces, with product (X, Σ, μ). Then there is a lifting of μ which respects coordinates.

proof (a) Replacing each μ_i by its completion does not change μ (254I), so we may suppose that all the μ_i are complete. In this case there is for each i an isomorphism between the measure algebra $(\mathfrak{A}_i, \bar\mu_i)$ of μ_i and the measure algebra $(\mathfrak{B}_i, \bar\nu_i)$ of some $\{0,1\}^{J_i}$ with its usual measure ν_i (331L). We may suppose that the sets J_i are disjoint. Each ν_i is compact (342Jd), so the isomorphisms are represented by inverse-measure-preserving functions $f_i : X_i \to \{0,1\}^{J_i}$ (343Ca).

Set $K = \bigcup_{i\in I} J_i$, and let ν be the usual measure on $Y = \{0,1\}^K$, T its domain. We have a natural bijection between $\prod_{i\in I}\{0,1\}^{J_i}$ and Y, so we obtain a function $f : X \to Y$; literally speaking,

$$f(x)(j) = f_i(x(i))(j)$$

for $i \in I$, $j \in J_i$, $x \in X$.

(b) Now f is inverse-measure-preserving and induces an isomorphism between the measure algebras $\mathfrak{A}$, $\mathfrak{B}$ of μ, ν.

P(i) If $L \subseteq K$ is finite and $z \in \{0,1\}^L$, then, setting $L_i = LcapJ_i$ for $i \in I$,

$$\begin{aligned}\mu\{x : x \in X,\, f(x){\restriction}L = z\} &= \mu\Bigl(\prod_{i\in I}\{w : w \in X_i,\, f_i(w){\restriction}L_i = z{\restriction}L_i\}\Bigr)\\ &= \prod_{i\in I}\mu_i\{w : w \in X_i,\, f_i(w){\restriction}L_i = z{\restriction}L_i\}\\ &= \prod_{i\in I}\nu_i\{v : v \in \{0,1\}^{J_i},\, v{\restriction}L_i = z{\restriction}L_i\}\end{aligned}$$

(because every f_i is inverse-measure-preserving)

$$= \prod_{i\in I} 2^{-\#(L_i)} = 2^{-\#(L)} = \nu\{y : y \in Y,\, y{\restriction}L = z\}.$$

So $\mu f^{-1}[C] = \nu C$ for every basic cylinder set $C \subseteq Y$. By 254G, f is inverse-measure-preserving.

(ii) Accordingly f induces a measure-preserving homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$. To see that π is surjective, consider

$$\Lambda' = \{E : E \text{ is } \Sigma\text{-measurable, } E^\bullet \in \pi[\mathfrak{B}]\}.$$

Because $\pi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{A}$ (324Kb), Λ' is a σ-subalgebra of the domain Λ of μ, and of course it contains all μ-negligible sets. If $i \in J$ and $G \in \Sigma_i$, then there is an $H \subseteq \{0,1\}^{J_i}$ such that $G \triangle f_i^{-1}[H]$ is μ_i-negligible. Now if $E = \{x : x \in X,\, x(i) \in G\}$ and $F = \{y : y \in Y,\, y{\restriction}J_i \in H\}$,

$$E \triangle f^{-1}[F] = \{x : x(i) \in G \triangle f_i^{-1}[H]\}$$

is μ-negligible, and $E \in \Lambda'$. But this means that $\Lambda' \supseteq \widehat{\bigotimes}_{i\in I}\Sigma_i$, and must therefore be the whole of Λ (254Ff). **Q**

(c) By 345C, there is a translation-invariant lifting ϕ for ν; by 346C, this respects coordinates. By 346D, we have a corresponding lifting ϕ' for μ such that

$$\phi' f^{-1}[F] = f^{-1}[\phi F]$$

for every $F \in \mathrm{T}$. Now suppose that $E \in \Lambda$ is determined by coordinates in $L \subseteq I$. Then there is an E' belonging to the σ-algebra Λ'_L generated by

$$\{\{x : x(i) \in G\} : i \in L,\, G \in \Sigma_i\}$$

such that $\mu(E\triangle E') = 0$ (254Ob). Write T_L for the family of sets in T determined by coordinates in $\bigcup_{i\in L} J_i$. Then, just as in (b-ii), every member of Λ'_L differs by a negligible set from some set of the form $f^{-1}[F]$ with $F \in \mathrm{T}_L$. So there is an $F \in \mathrm{T}_L$ such that $E\triangle f^{-1}[F]$ is μ-negligible. Consequently

$$\phi' E = \phi' f^{-1}[F] = f^{-1}[\phi F].$$

But ϕ respects coordinates, so ϕF is determined by coordinates in $\bigcup_{i\in L} J_i$. It follows at once that $f^{-1}[\phi F]$ is determined by coordinates in L; that is, that $\phi' E$ is determined by coordinates in L. As E and L are arbitrary, ϕ' respects coordinates, and witnesses the truth of the theorem.

346F It seems to be unknown whether 346E is true of arbitrary probability spaces (346Za); I give some partial results in this direction. The following general method of constructing lower densities will be useful.

Lemma Let (X,Σ,μ) and (Y,T,ν) be complete probability spaces, with product $(X\times Y,\Lambda,\lambda)$. If $\underline{\phi} : \Lambda \to \Lambda$ is a lower density, then we have a lower density $\underline{\phi}_1 : \Sigma \to \Sigma$ defined by saying that

$$\underline{\phi}_1 E = \{x : x \in X,\ \{y : (x,y) \in \underline{\phi}(E\times Y)\} \text{ is conegligible in } Y\}$$

for every $E \in \Sigma$.

proof For $E \in \Sigma$, $(E\times Y)\triangle\underline{\phi}(E\times Y)$ is negligible, so that

$$H_x = \{y : (x,y) \in (E\times Y)\triangle\underline{\phi}(E\times Y)\}$$

is ν-negligible for almost every $x \in X$ (252D). Now $E\triangle\underline{\phi}_1 E = \{x : H_x \text{ is not negligible}\}$ is negligible, so $\underline{\phi}_1 E \in \Sigma$. If E, $F \in \Sigma$, then

$$\underline{\phi}((E\cap F)\times Y) = \underline{\phi}((E\times Y)\cap(F\times Y)) = \underline{\phi}(E\times Y)\cap\underline{\phi}(F\times Y),$$

so that

$$\{y : (x,y) \in \underline{\phi}((E\cap F)\times Y)\} = \{y : (x,y) \in \underline{\phi}(E\times Y)\} \cap \{y : (x,y) \in \underline{\phi}(F\times Y)\}$$

is conegligible iff both $\{y : (x,y) \in \underline{\phi}(E\times Y)\}$ and $\{y : (x,y) \in \underline{\phi}(F\times Y)\}$ are conegligible, and $\underline{\phi}_1(E\cap F) = \underline{\phi}_1 E \cap \underline{\phi}_1 F$.

The rest is easy. Of course $\underline{\phi}(\emptyset\times Y) = \emptyset$ so $\underline{\phi}_1\emptyset = \emptyset$. If E, $F \in \Sigma$ and $E\triangle F$ is negligible, then $(E\times Y)\triangle(F\times Y)$ is negligible, $\underline{\phi}(E\times Y) = \underline{\phi}(F\times Y)$ and $\underline{\phi}_1 E = \underline{\phi}_1 F$. So $\underline{\phi}_1$ is a lower density, as claimed.

346G Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with product (X,Σ,μ). For $J \subseteq I$ let Σ_J be the set of members of Σ which are determined by coordinates in J. Then there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ such that

(i) whenever $J \subseteq I$ and $E \in \Sigma_J$ then $\underline{\phi}E \in \Sigma_J$,

(ii) whenever J, $K \subseteq I$ are disjoint, $E \in \Sigma_J$ and $F \in \Sigma_K$ then $\underline{\phi}(E\cup F) = \underline{\phi}E \cup \underline{\phi}F$.

proof For each $i \in I$, set $Y_i = X_i^{\mathbb{N}}$, with the product measure ν_i; set $Y = \prod_{i\in I} Y_i$, with its product measure ν; set $Z_i = X_i \times Y_i$, with its product measure λ_i, and $Z = \prod_{i\in I} Z_i$, with its product measure λ. Then the natural identification of $Z = \prod_{i\in I} X_i \times Y_i$ with $\prod_{i\in I} X_i \times \prod_{i\in I} Y_i = X \times Y$ makes λ correspond to the product of μ and ν (254N).

Each (Z_i,λ_i) can be identified with an infinite power of (X_i,μ_i), and is therefore Maharam-type-homogeneous (334E). Consequently there is a lifting $\phi : \Lambda \to \Lambda$ which respects coordinates (346E). Regarding (Z,λ) as the product of (X,μ) and (Y,ν), we see that ϕ induces a lower density $\underline{\phi} : \Sigma \to \Sigma$ by the formula of 346F.

If $J \subseteq I$ and $E \in \Sigma$ is determined by coordinates in J, then $E \times Y$ (regarded as a subset of $\prod_{i\in I} Z_i$) is determined by coordinates in J, so $\phi(E\times Y)$ also is. Now suppose that $x \in \underline{\phi}E$, $x' \in X$ and $x\restriction J = x'\restriction J$. Then for any $y \in Y$, $(x\restriction J, y\restriction J) = (x'\restriction J, y\restriction J)$, so $(x,y) \in \phi(E\times Y)$ iff $(x',y) \in \phi(E\times Y)$. Thus

$$\{y : (x',y) \in \phi(E\times Y)\} = \{y : (x,y) \in \phi(E\times Y)\}$$

is conegligible in Y, and $x' \in \underline{\phi}E$. This shows that $\underline{\phi}E$ is determined by coordinates in J.

Now suppose that J and K are disjoint subsets of I, that E, $F \in \Sigma$ are determined by coordinates in J, K respectively, and that $x \notin \underline{\phi}E \cup \underline{\phi}F$. Then $A = \{y : (x,y) \notin \phi(E \times Y)\}$ and $B = \{y : (x,y) \notin \phi(F \times Y)\}$ are non-negligible. As noted just above, $\phi(E \times Y)$ is determined by coordinates in J, so A is determined by coordinates in J, and can be expressed as $\{y : y\restriction J \in A'\}$, where $A' \subseteq Y_J = \prod_{i\in J} Y_i$. Because $y \mapsto y\restriction J : Y \to Y_J$ is inverse-measure-preserving, A' cannot be negligible in Y_J. Similarly, B can be expressed as $\{y : y\restriction K \in B'\}$ for some non-negligible $B' \subseteq Y_K$.

By 251R/251Wm, $A' \times B' \times Y_{I\setminus(J\cup K)}$, regarded as a subset of Y, is non-negligible, that is,

$$C = \{y : y \in Y,\, y\restriction J \in A',\, y\restriction K \in B'\}$$

is non-negligible. But

$$C = A \cap B = \{y : (x,y) \notin \phi(E \times Y) \cup \phi(F \times Y)\} = \{y : (x,y) \notin \phi((E \cup F) \times Y\}.$$

So $x \notin \underline{\phi}(E \cup F)$. As x is arbitrary, $\underline{\phi}(E \cup F) \subseteq \underline{\phi}E \cup \underline{\phi}F$; but of course $\underline{\phi}E \cup \underline{\phi}F \subseteq \underline{\phi}(E \cup F)$, because $\underline{\phi}$ is a lower density, so that $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$, as required.

Remark See Macheras Musiał & Strauss p99 for an alternative proof.

346H Theorem Let ζ be an ordinal, and $\langle(X_\xi, \Sigma_\xi, \mu_\xi)\rangle_{\xi<\zeta}$ any family of probability spaces, with product (Z, Λ, λ). For $J \subseteq \zeta$ let Λ_J be the set of those $W \in \Lambda$ which are determined by coordinates in J. Then there is a lifting $\phi : \Lambda \to \Lambda$ such that $\phi W \in \Lambda_J$ whenever $W \in \Lambda_J$ and J is *either* a singleton subset of ζ *or* an initial segment of ζ.

proof (a) Let P be the set of all lower densities $\underline{\phi} : \Lambda \to \Lambda$ such that, for every $\xi < \zeta$, (i) whenever $E \in \Lambda_\xi$ then $\underline{\phi}E \in \Lambda_\xi$ (ii) whenever $E \in \Lambda_{\{\xi\}}$ then $\underline{\phi}E \in \Lambda_{\{\xi\}}$ (iii) whenever $E \in \Lambda_\xi$ and $F \in \Lambda_{\zeta\setminus\xi}$ then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$. By 346G, P is not empty. Order P by saying that $\underline{\phi} \le \underline{\phi}'$ if $\underline{\phi}E \subseteq \underline{\phi}'E$ for every $E \in \Lambda$; then P is a partially ordered set. Note that if $\underline{\phi} \in P$ then $\underline{\phi}Z = Z$ (because $\Lambda_0 = \{\emptyset, Z\}$).

(b) Any non-empty totally ordered subset Q of P has an upper bound in P. **P** Define $\underline{\phi}^* : \Lambda \to \mathcal{P}X$ by setting $\underline{\phi}^*E = \bigcup_{\underline{\phi}\in Q} \underline{\phi}E$ for every $E \in \Lambda$. (i)

$$\underline{\phi}^*\emptyset = \bigcup_{\underline{\phi}\in Q} \emptyset = \emptyset.$$

(ii) If E, $F \in \Lambda$ and $\lambda(E \triangle F) = 0$ then $\underline{\phi}E = \underline{\phi}F$ for every $\underline{\phi} \in Q$ so $\underline{\phi}^*E = \underline{\phi}^*F$. (iii) If E, $F \in \Lambda$ and $E \subseteq F$ then $\underline{\phi}E \subseteq \underline{\phi}F$ for every $\underline{\phi} \in Q$ so $\underline{\phi}^*E \subseteq \underline{\phi}^*F$. (iv) If E, $F \in \Lambda$ and $x \in \underline{\phi}^*E \cap \underline{\phi}^*F$, then there are $\underline{\phi}_1$, $\underline{\phi}_2 \in Q$ such that $x \in \underline{\phi}_1 E \cap \underline{\phi}_2 F$; now either $\underline{\phi}_1 \le \underline{\phi}_2$ or $\underline{\phi}_2 \le \underline{\phi}_1$, so that

$$x \in (\underline{\phi}_1 E \cap \underline{\phi}_1 F) \cup (\underline{\phi}_2 E \cap \underline{\phi}_2 F) = \underline{\phi}_1(E \cap F) \cup \underline{\phi}_2(E \cap F) \subseteq \underline{\phi}^*(E \cap F).$$

Accordingly $\underline{\phi}^*E \cap \underline{\phi}^*F \subseteq \underline{\phi}^*(E \cap F)$ and $\underline{\phi}^*E \cap \underline{\phi}^*F = \underline{\phi}^*(E \cap F)$. (v) Taking any $\underline{\phi}_0 \in Q$, we have $\underline{\phi}_0 E \subseteq \underline{\phi}^*E$ for every $E \in \Lambda$, so (because λ is complete) $\underline{\phi}^*$ is a lower density, by 341Ib. (vi) Now suppose that $J \subseteq I$ is either a singleton $\{\xi\}$ or an initial segment ξ, and that $E \in \Lambda_J$. Then $\underline{\phi}E$ is determined by coordinates in J for every $\underline{\phi} \in Q$, so $\underline{\phi}^*E$ is determined by coordinates in J. (vii) Finally, suppose that $\xi < \zeta$ and that $E \in \Lambda_\xi$, $F \in \Lambda_{\zeta\setminus\xi}$. If $x \in \underline{\phi}^*(E \cup F)$ then there is a $\underline{\phi} \in Q$ such that

$$x \in \underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F \subseteq \underline{\phi}^*E \cup \underline{\phi}^*F.$$

So $\underline{\phi}^*(E \cup F) \subseteq \underline{\phi}^*E \cup \underline{\phi}^*F$ and (using (iii) again) $\underline{\phi}^*(E \cup F) = \underline{\phi}^*E \cup \underline{\phi}^*F$. Thus $\underline{\phi}^*$ belongs to P and is an upper bound for Q in P. **Q**

By Zorn's Lemma, P has a maximal element $\tilde{\underline{\phi}}$.

(c) For any $H \in \Lambda$ we may define a function $\underline{\phi}_H$ as follows. Set $A_H = Z \setminus (\tilde{\underline{\phi}}H \cup \tilde{\underline{\phi}}(Z \setminus H))$,

$$\underline{\phi}_H E = \tilde{\underline{\phi}}E \cup (A_H \cap \tilde{\underline{\phi}}(H \cup E))$$

for $E \in \Lambda$. Then $\underline{\phi}_H$ is a lower density. **P** (i) Because $H \triangle \tilde{\underline{\phi}}H$ and $(Z\setminus H) \triangle \tilde{\underline{\phi}}(Z\setminus H)$ are both negligible, A_H is negligible and $\underline{\phi}_H E$ is measurable and $(\underline{\phi}_H E)^\bullet = (\tilde{\underline{\phi}}E)^\bullet = E^\bullet$ for every $E \in \Lambda$. (ii) Because $A_H \cap \tilde{\underline{\phi}}H = \emptyset$, $\underline{\phi}_H\emptyset = \emptyset$. (iii) If E, $F \in \Lambda$ and $\lambda(E \triangle F) = 0$ then $\tilde{\underline{\phi}}E = \tilde{\underline{\phi}}F$ and $\tilde{\underline{\phi}}(E \cup H) = \tilde{\underline{\phi}}(F \cup H)$, so $\underline{\phi}_H E = \underline{\phi}_H F$. (iv) If E, $F \in \Lambda$ and $E \subseteq F$ then $\tilde{\underline{\phi}}E \subseteq \tilde{\underline{\phi}}F$ and $\tilde{\underline{\phi}}(E \cup H) \subseteq \tilde{\underline{\phi}}(F \cup H)$, so $\underline{\phi}_H E \subseteq \underline{\phi}_H F$. (v) If E, $F \in \Lambda$ and $x \in \underline{\phi}_H E \cap \underline{\phi}_H F$, then

(α) if $x \notin A_H$,

$$x \in \underline{\tilde{\phi}}E \cap \underline{\tilde{\phi}}F = \underline{\tilde{\phi}}(E \cap F) \subseteq \underline{\phi}_H(E \cap F),$$

(β) if $x \in A_H$,

$$x \in \underline{\tilde{\phi}}(E \cup H) \cap \underline{\tilde{\phi}}(F \cup H) = \underline{\tilde{\phi}}((E \cap F) \cup H) \subseteq \underline{\phi}_H(E \cap F).$$

Thus $\underline{\phi}_H E \cap \underline{\phi}_H F \subseteq \underline{\phi}_H(E \cap F)$ and $\underline{\phi}_H E \cap \underline{\phi}_H F = \underline{\phi}_H(E \cap F)$. **Q**

(d) It is worth noting the following.

(i) If E, $H \in \Lambda$ and $\underline{\tilde{\phi}}(E \cup H) = \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}H$ then $\underline{\phi}_H E = \underline{\tilde{\phi}}E$. **P** We have

$$\underline{\phi}_H E = \underline{\tilde{\phi}}E \cup (A_H \cap \underline{\tilde{\phi}}(E \cup H)) = \underline{\tilde{\phi}}E \cup (A_H \cap \underline{\tilde{\phi}}E) \cup (A_H \cap \underline{\tilde{\phi}}H) = \underline{\tilde{\phi}}E$$

because $A_H \cap \underline{\tilde{\phi}}H = \emptyset$. **Q**

(ii) If $H \in \Lambda$ and $\underline{\phi}_H \in P$ then $\underline{\tilde{\phi}}H \cup \underline{\tilde{\phi}}(Z \setminus H) = Z$. **P** By the maximality of $\underline{\tilde{\phi}}$, we must have $\underline{\phi}_H = \underline{\tilde{\phi}}$. But

$$A_H = \underline{\phi}_H(Z \setminus H) \setminus \underline{\tilde{\phi}}(Z \setminus H),$$

so $A_H = \emptyset$, that is, $\underline{\tilde{\phi}}H \cup \underline{\tilde{\phi}}(Z \setminus H) = Z$. **Q**

(iii) If E, $F \in \Lambda$ and $\underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}(Z \setminus E) = Z$, then $\underline{\tilde{\phi}}(E \cup F) = \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}F$. **P**

$$\underline{\tilde{\phi}}(E \cup F) \setminus \underline{\tilde{\phi}}E = \underline{\tilde{\phi}}(E \cup F) \cap \underline{\tilde{\phi}}(Z \setminus E) = \underline{\tilde{\phi}}((E \cup F) \cap (Z \setminus E)) = \underline{\tilde{\phi}}(F \setminus E) \subseteq \underline{\tilde{\phi}}F,$$

so $\underline{\tilde{\phi}}(E \cup F) \subseteq \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}F$; as the reverse inclusion is true for all E and F, we have the result. **Q**

(e) If $\xi < \zeta$ and $H \in \Lambda_{\{\xi\}}$, then $\underline{\phi}_H \in P$.

P(i) If $J \subseteq I$ is either a singleton or an inital segment, and $E \in \Lambda_J$, then

(α) if $\xi \in J$, $E \cup H$ and $\underline{\tilde{\phi}}E$ and $\underline{\tilde{\phi}}(E \cup H)$ and A_H all belong to Λ_J, so $\underline{\phi}_H E \in \Lambda_J$.

(β) If $\xi \notin J$, $\underline{\tilde{\phi}}(E \cup H) = \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}H$, because there is some η such that $J \subseteq \eta$ and $\{\xi\} \subseteq \zeta \setminus \eta$); so $\underline{\phi}_H E = \underline{\tilde{\phi}}E \in \Lambda_J$ by (d-i).

(ii) If $\eta < \zeta$ and $E \in \Lambda_\eta$, $F \in \Lambda_{\zeta \setminus \eta}$, then

if $\xi < \eta$, $E \cup H \in \Lambda_\eta$ so $\underline{\tilde{\phi}}(E \cup F \cup H) = \underline{\tilde{\phi}}(E \cup H) \cup \underline{\tilde{\phi}}F$, and

$$\begin{aligned}\underline{\phi}_H(E \cup F) &= \underline{\tilde{\phi}}(E \cup F) \cup (A_H \cap \underline{\tilde{\phi}}(E \cup F \cup H)) \\ &= \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}F \cup (A_H \cap \underline{\tilde{\phi}}(E \cup H)) \cup (A_H \cap \underline{\tilde{\phi}}F) \subseteq \underline{\phi}_H E \cup \underline{\phi}_H F;\end{aligned}$$

if $\eta \le \xi$, $F \cup H \in \Lambda_{\zeta \setminus \eta}$ so $\underline{\tilde{\phi}}(E \cup F \cup H) = \underline{\tilde{\phi}}(E) \cup \underline{\tilde{\phi}}(F \cup H)$, and

$$\begin{aligned}\underline{\phi}_H(E \cup F) &= \underline{\tilde{\phi}}(E \cup F) \cup (A_H \cap \underline{\tilde{\phi}}(E \cup F \cup H)) \\ &= \underline{\tilde{\phi}}E \cup \underline{\tilde{\phi}}F \cup (A_H \cap \underline{\tilde{\phi}}E)) \cup (A_H \cap \underline{\tilde{\phi}}(F \cup H)) \subseteq \underline{\phi}_H E \cup \underline{\phi}_H F;\end{aligned}$$

accordingly $\underline{\phi}_H(E \cup F) = \underline{\phi}_H E \cup \underline{\phi}_H F$. **Q**

By (d-ii) we have

$$\underline{\tilde{\phi}}H \cup \underline{\tilde{\phi}}(Z \setminus H) = Z$$

whenever $\xi < \zeta$ and $H \in \Lambda_{\{\xi\}}$.

(f) If $\xi \le \zeta$ and $H \in \Lambda_\xi$, then $\underline{\phi}_H \in P$. **P** Induce on ξ. For $\xi = 0$, $H \in \Lambda_0 = \{\emptyset, Z\}$ so $\underline{\tilde{\phi}}H$ is either $\emptyset$ or Z and the result is trivial. For the inductive step to $\xi \le \zeta$, we have the following.

(i) If $\eta < \zeta$ and $E \in \Lambda_\eta$, then

(α) if $\xi \le \eta$, $E \cup H$ and $\underline{\tilde{\phi}}E$ and $\underline{\tilde{\phi}}(E \cup H)$ and A_H all belong to Λ_η, so $\underline{\phi}_H E \in \Lambda_\eta$.

(β) if $\eta<\xi$, then, by the inductive hypothesis, $\underline{\phi}_E\in P$, $\tilde{\underline{\phi}}E=Z\setminus\tilde{\underline{\phi}}(Z\setminus E)$ and $\tilde{\underline{\phi}}(E\cup H)=\tilde{\underline{\phi}}E\cup\tilde{\underline{\phi}}H$, by (d-ii) and (d-iii) above; so $\underline{\phi}_H E=\tilde{\underline{\phi}}E\in\Lambda_\eta$ by (d-i).

(ii) If $\eta<\zeta$ and $E\in\Lambda_{\{\eta\}}$, then, by (e), $\tilde{\underline{\phi}}E\cup\tilde{\underline{\phi}}(Z\setminus E)=Z$, so that $\tilde{\underline{\phi}}(E\cup H)=\tilde{\underline{\phi}}E\cup\tilde{\underline{\phi}}H$, by (d-iii), and $\underline{\phi}_H E=\tilde{\underline{\phi}}E\in\Lambda_{\{\eta\}}$, by (d-i).

(iii) If $\eta<\zeta$ and $E\in\Lambda_\eta$, $F\in\Lambda_{\zeta\setminus\eta}$, then

(α) if $\xi\le\eta$, then $E\cup H\in\Lambda_\eta$ and $F\in\Lambda_{\zeta\setminus\eta}$, so that $\tilde{\underline{\phi}}(E\cup F\cup H)=\tilde{\underline{\phi}}(E\cup H)\cup\tilde{\underline{\phi}}F$, and

$$\begin{aligned}\underline{\phi}_H(E\cup F)&=\tilde{\underline{\phi}}(E\cup F)\cup(A_H\cap\tilde{\underline{\phi}}(E\cup F\cup H))\\&=\tilde{\underline{\phi}}E\cup\tilde{\underline{\phi}}F\cup(A_H\cap\tilde{\underline{\phi}}(E\cup H))\cup(A_H\cap\tilde{\underline{\phi}}F)\subseteq\underline{\phi}_H E\cup\underline{\phi}_H F,\end{aligned}$$

as in (e-ii) above, and accordingly $\underline{\phi}_H(E\cup F)=\underline{\phi}_H E\cup\underline{\phi}_H F$.

(β) If $\eta<\xi$ then, as in (ii), using the inductive hypothesis, we have $\tilde{\underline{\phi}}(E\cup F\cup H)=\tilde{\underline{\phi}}E\cup\tilde{\underline{\phi}}(F\cup H)$, and (just as in ($\alpha$)) we get $\underline{\phi}_H(E\cup F)=\underline{\phi}_H E\cup\underline{\phi}_H F$.

Thus $\underline{\phi}_H\in P$ and the induction continues. **Q**

(g) But the case $\xi=\zeta$ of (f) just tells us that

$$\tilde{\underline{\phi}}H\cup\tilde{\underline{\phi}}(Z\setminus H)=Z$$

for every $H\in\Lambda$. This means that $\tilde{\underline{\phi}}$ is actually a lifting (since it preserves intersections and complements). And the definition of P is just what is needed to ensure that it is a lifting of the right type.

Remark This result is due to Macheras & Strauss 96b.

346I Theorem Let (X,Σ,μ) be a complete probability space. For any set I, write λ_I for the product measure on X^I, Λ_I for its domain and $\pi_{Ii}(x)=x(i)$ for $x\in X^I$, $i\in I$. Then there is a lifting $\psi:\Sigma\to\Sigma$ such that for every set I there is a lifting $\phi:\Lambda_I\to\Lambda_I$ such that $\phi(\pi_{Ii}^{-1}[E])=\pi_{Ii}^{-1}[\psi E]$ for every $E\in\Sigma$, $i\in I$.

proof ? Suppose, if possible, otherwise.

Let Ψ be the set of all liftings for μ. We are supposing that for every $\psi\in\Psi$ there is a set I_ψ for which there is no lifting for λ_{I_ψ} consistent with ψ in the sense above. Let κ be a cardinal greater than $\max(\omega,\#(\Psi),\sup_{\psi\in\Psi}\#(I_\psi))$. Let $\phi_0:\Lambda_\kappa\to\Lambda_\kappa$ be a lifting satisfying the conditions of 346H. 346B tells us that for every $\xi<\kappa$ we have a lifting ψ for μ defined by the formula $\pi_{\kappa\xi}^{-1}[\psi E]=\phi_0(\pi_{\kappa\xi}^{-1}[E])$. For $\psi\in\Psi$ set

$$K_\psi=\{\xi:\xi<\kappa,\ \phi_0(\pi_{\kappa\xi}^{-1}[E])=\pi_{\kappa\xi}^{-1}[\psi E]\text{ for every }E\in\Sigma\}.$$

Then $\bigcup_{\psi\in\Psi}K_\psi=\kappa$, so $\kappa\le\max(\omega,\#(\Psi),\sup_{\psi\in\Psi}\#(K_\psi))$ and there is some $\psi\in\Psi$ such that $\#(K_\psi)>\#(I_\psi)$. Take $I\subseteq K_\psi$ such that $\#(I)=\#(I_\psi)$.

We may regard X^κ as $X^I\times X^{\kappa\setminus I}$, and in this form we can use the method of 346F to obtain a lower density $\underline{\phi}:\Lambda_I\to\Lambda_I$ from $\phi_0:\Lambda_\kappa\to\Lambda_\kappa$. Now

$$\underline{\phi}(\pi_{I\xi}^{-1}[E])=\pi_{I\xi}^{-1}[\psi E]\text{ for every }E\in\Sigma,\ \xi\in I.$$

P The point is that $\pi_{I\xi}^{-1}[E]\times X^{\kappa\setminus I}$ corresponds to $\pi_{\kappa\xi}^{-1}[E]\subseteq X^\kappa$, while $\phi_0(\pi_{\kappa\xi}^{-1}[E])=\pi_{\kappa\xi}^{-1}[\psi E]$ can be identified with $\pi_{I\xi}^{-1}[\psi E]\times X^{\kappa\setminus I}$. Now the construction of 346F obviously makes $\underline{\phi}(\pi_{I\xi}^{-1}[E])$ equal to $\pi_{I\xi}^{-1}[\psi E]$. **Q**

By 341Jb, there is a lifting $\phi:\Lambda_I\to\Lambda_I$ such that $\phi W\supseteq\underline{\phi}W$ for every $W\in\Lambda_I$. But now we must have

$$\begin{aligned}\pi_{I\xi}^{-1}[\psi E]&=\underline{\phi}(\pi_{I\xi}^{-1}[E])\subseteq\phi(\pi_{I\xi}^{-1}[E])\\&=X^I\setminus\phi(\pi_{I\xi}^{-1}[X\setminus E])\subseteq X^I\setminus\underline{\phi}(\pi_{I\xi}^{-1}[X\setminus E])\\&=X^I\setminus\pi_{I\xi}^{-1}[\psi(X\setminus E)]=X^I\setminus\pi_{I\xi}^{-1}[X\setminus\psi E]=\pi_{I\xi}^{-1}[\psi E]\end{aligned}$$

and $\phi(\pi_{I\xi}^{-1}[E])=\pi_{I\xi}^{-1}[\psi E]$ for every $E\in\Sigma$, $\xi\in I$. But since $\#(I)=\#(I_\psi)$, this must be impossible, by the choice of I_ψ. **X**

This contradiction proves the theorem.

346J Consistent liftings Let (X,Σ,μ) be a measure space. A lifting $\psi:\Sigma\to\Sigma$ is **consistent** if for every $n\ge 1$ there is a lifting ϕ_n of the product measure on X^n such that $\phi_n(E_1\times\ldots\times E_n)=\psi E_1\times\ldots\times\psi E_n$ for all $E_1,\ldots,E_n\in\Sigma$. Thus 346I tells us, in part, that every complete probability space has a consistent lifting; it follows that every non-trivial complete totally finite measure space has a consistent lifting.

I do not suppose you will be surprised to be told that not all liftings on probability spaces are consistent. What may be surprising is the fact that one of the standard liftings already introduced is not consistent. This depends on a general fact about Stone spaces of measure algebras which has further important applications, so I present it as a lemma.

346K Lemma Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and let λ be the product measure on $Z\times Z$, with Λ its domain. Then there is a set $W\in\Lambda$, with $\lambda W<1$, such that $\lambda^*\tilde W=1$, where

$$\tilde W=\bigcup\{G\times H: G,\ H\subseteq Z \text{ are open-and-closed, } (G\times H)\setminus W \text{ is negligible}\}.$$

Remark For the sake of anybody who has already become acquainted with the alternative measures which can be put on the product of topological measure spaces, I ought to insist here that the 'product measure' λ is, as always in this volume, the ordinary completed product measure as defined in Chapter 25.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $[0,1]$, stochastically independent for Lebesgue measure μ on $[0,1]$, such that $\mu E_n=\frac{1}{n+2}$ for each n. Set $a_n=E_n^\bullet$ in the measure of algebra of μ, and $E_n^*=\widehat{a}_n$ the corresponding compact open subset of Z. Set $W=\bigcup_{n\in\mathbb{N}}E_n^*\times E_n^*$. Then

$$\lambda W\le\sum_{n=0}^\infty(\nu E_n)^2=\sum_{n=2}^\infty\frac{1}{n^2}<1.$$

? Suppose, if possible, that $\lambda^*\tilde W<1$. Then there are sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ in T such that $\tilde W\subseteq\bigcup_{n\in\mathbb{N}}G_n\times H_n$ and $\lambda(\bigcup_{n\in\mathbb{N}}G_n\times H_n)<1$. Recall from 322Qc that

$$\nu F=\inf\{\nu G: G \text{ is compact and open, } F\subseteq G\}$$

for every $F\in\mathrm{T}$. Accordingly we can find compact open sets $\tilde G_n$, $\tilde H_n$ such that $G_n\subseteq\tilde G_n$, $H_n\subseteq\tilde H_n$ for every $n\in\mathbb{N}$ and

$$\sum_{n=0}^\infty\nu(\tilde G_n\setminus G_n)+\sum_{n=0}^\infty\nu(\tilde H_n\setminus H_n)<1-\lambda(\bigcup_{n\in\mathbb{N}}G_n\times H_n),$$

so that $\lambda(\bigcup_{n\in\mathbb{N}}\tilde G_n\times\tilde H_n)<1$.

Let $\mathcal{U}_0$ be the family

$$\{Z\}\cup\{E_n^*:n\in\mathbb{N}\}\cup\{Z\setminus\tilde G_n:n\in\mathbb{N}\}\cup\{Z\setminus\tilde H_n:n\in\mathbb{N}\},$$

so that $\mathcal{U}_0$ is a countable subset of T. Let $\mathcal{U}$ be the set of finite intersections $U_0\cap U_1\cap\ldots\cap U_n$ where $U_0,\ldots,U_n\in\mathcal{U}_0$, so that $\mathcal{U}$ is also a countable subset of T, and $\mathcal{U}$ is closed under $\cap$.

(b) For $U\in\mathcal{U}$, define $Q(U)$ as follows. If $\nu U=0$, then $Q(U)=U$. Otherwise,

$$Q(U)=Z\setminus\bigcup\{E_n^*:n\in\mathbb{N},\ \nu(E_n^*\cap U)>0\}.$$

Then $\nu Q(U)$ is always 0. **P** Of course this is true if $\nu U=0$, so suppose that $\nu U>0$. Set $I=\{n:\nu(E_n^*\cap U)=0\}$. Then we have $\nu U'>0$, where $U'=U\setminus\bigcup_{n\in I}E_n^*$, and $Z\setminus E_n^*\supseteq U'$ for every $n\in I$. Because $\langle E_n\rangle_{n\in\mathbb{N}}$ is stochastically independent for μ, $\langle E_n^*\rangle_{n\in\mathbb{N}}$ is stochastically independent for ν, while

$$\nu(\bigcup_{n\in I}E_n^*)\le 1-\nu U'<1.$$

By the Borel-Cantelli lemma (273K), $\sum_{n\in I}\nu E_n^*<\infty$. Consequently $\sum_{n\in\mathbb{N}\setminus I}\nu E_n^*=\infty$, because $\sum_{n=0}^\infty\frac{1}{n+2}$ is infinite, so

$$\nu(Z\setminus Q(U))=\nu(\bigcup_{n\in\mathbb{N}\setminus I}E_n^*)=1,$$

and $\nu Q(U)=0$. **Q**

(c) Set $Q_0=\bigcup_{U\in\mathcal{U}}Q(U)$; because $\mathcal{U}$ is countable, Q_0 is negligible. Accordingly $(Z\setminus Q_0)^2$ has measure 1 and cannot be included in $\bigcup_{n\in\mathbb{N}}\tilde G_n\times\tilde H_n$; take $(w,z)\in(Z\setminus Q_0)^2\setminus\bigcup_{n\in\mathbb{N}}\tilde G_n\times\tilde H_n$.

(d) We can find sequences $\langle C_n\rangle_{n\in\mathbb{N}}$, $\langle D_n\rangle_{n\in\mathbb{N}}$, $\langle U_n\rangle_{n\in\mathbb{N}}$ and $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that

$w\in U_{n+1}\subseteq U_n$, $z\in V_{n+1}\subseteq V_n$, $(U_{n+1}\times V_{n+1})\cap(\tilde G_n\times\tilde H_n)=\emptyset$,

$\nu C_n>0$, $\nu D_n>0$,

$C_n\subseteq U_n$, $D_n\subseteq V_{n+1}$,

$C_n\times V_{n+1}\subseteq W$, $U_{n+1}\times D_n\subseteq W$

for every $n\in\mathbb{N}$. **P** Build the sequences inductively, as follows. Start with $U_0=V_0=Z$. Given that $w\in U_n\in\mathcal{U}$, $z\in V_n\in\mathcal{U}$, then we know that $(w,z)\notin\tilde G_n\times\tilde H_n$. If $w\notin\tilde G_n$, set $U'_n=U_n\setminus\tilde G_n$, $V'_n=V_n$; otherwise set $U'_n=U_n$, $V'_n=V_n\setminus\tilde H_n$. In either case, we have $w\in U'_n\in\mathcal{U}$, $z\in V'_n\in\mathcal{U}$ and $(U'_n\times V'_n)\cap(\tilde G_n\times\tilde H_n)=\emptyset$.

Because $U'_n\in\mathcal{U}$, $w\notin Q(U'_n)$. But $w\in U'_n$, so this must be because $\nu U'_n>0$. Now $z\notin Q(U'_n)$, so $z\in\bigcup\{E^*_k:k\in\mathbb{N},\,\nu(E^*_k\cap U'_n)>0\}$. Take some $k\in\mathbb{N}$ such that $z\in E^*_k$ and $\nu(E^*_k\cap U'_n)>0$, and set

$$V_{n+1}=V'_n\cap E^*_k,\quad C_n=E^*_k\cap U'_n,$$

so that

$$z\in V_{n+1}\in\mathcal{U},\quad C_n\subseteq U_n,\quad C_n\times V_{n+1}\subseteq E^*_k\times E^*_k\subseteq W,\quad \nu C_n>0.$$

Next, $z\notin Q(V_{n+1})$ and $\nu V_{n+1}>0$; also $w\notin Q(V_{n+1})$, so there is an l such that $w\in E^*_l$ and $\nu(E^*_l\cap V_{n+1})>0$. Set

$$U_{n+1}=U'_n\cap E^*_l,\quad D_n=E^*_l\cap V_{n+1},$$

so that

$$w\in U_{n+1}\in\mathcal{U},\quad D_n\subseteq V_{n+1},\quad U_{n+1}\times D_n\subseteq E^*_l\times E^*_l\subseteq W,\quad \nu D_n>0,$$

$$(U_{n+1}\times V_{n+1})\cap(\tilde G_n\times\tilde H_n)\subseteq(U'_n\times V'_n)\cap(\tilde G_n\times\tilde H_n)=\emptyset,$$

and continue the process. **Q**

(e) Setting $C=\bigcup_{n\in\mathbb{N}}C_n$, $D=\bigcup_{n\in\mathbb{N}}D_n$ we see that $C\times D\subseteq W$. **P** If $m\le n$, $D_n\subseteq V_{n+1}\subseteq V_{m+1}$, so $C_m\times D_n\subseteq W$. If $m>n$, $C_m\subseteq U_m\subseteq U_{n+1}$, so $C_m\times D_n\subseteq W$. **Q**

Recall from 322Qa that the measurable sets of Z are precisely those of the form $G\triangle M$ where M is nowhere dense and negligible and G is compact and open. There must therefore be compact open sets G, $H\subseteq Z$ such that $G\triangle C$ and $H\triangle D$ are negligible. Consequently

$$(G\times H)\setminus W\subseteq((G\setminus C)\times Z)\cup(Z\times(H\setminus D))$$

is negligible, and

$$G\times H\subseteq\tilde W\subseteq\textstyle\bigcup_{n\in\mathbb{N}}\tilde G_n\times\tilde H_n.$$

But because $G\times H$ is compact (3A3J), and all the $\tilde G_n\times\tilde H_n$ are open, there must be some n such that $G\times H\subseteq\bigcup_{k\le n}\tilde G_k\times\tilde H_k=S$ say. Now $(U_{k+1}\times V_{k+1})\cap(\tilde G_k\times\tilde H_k)=\emptyset$ for every k, so

$$(C_{n+2}\times D_{n+2})\cap(G\times H)\subseteq(U_{n+1}\times V_{n+1})\cap S=\emptyset,$$

and either $C_{n+2}\cap G=\emptyset$ or $D_{n+2}\cap H=\emptyset$. Since

$$C_{n+2}\setminus G\subseteq C\setminus G,\quad D_{n+2}\setminus H\subseteq D\setminus H$$

are both negligible, one of C_{n+2}, D_{n+2} is negligible. But the construction took care to ensure that all the C_k, D_k were non-negligible. **X**

(f) Thus $\lambda^*\tilde W=1$, as required.

346L Proposition Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$. Let $\psi:\mathrm{T}\to\mathrm{T}$ be the canonical lifting, defined by setting $\psi E=G$ whenever $E\in\mathrm{T}$, G is open-and-closed and $E\triangle G$ is negligible (341O). Then ψ is not consistent.

proof ? Suppose, if possible, that ϕ is a lifting on $Z\times Z$ such that $\phi(E\times F)=\psi E\times\psi F$ for every E, $F\in\mathrm{T}$. Let $W\subseteq Z\times Z$ be a set as in 346K, and consider ϕW. If G, $H\subseteq Z$ are open-and-closed and $(G\times H)\setminus W$ is negligible, then

$$G \times H = \psi G \times \psi H = \phi(G \times H) \subseteq \phi W;$$

that is, in the language of 346K, we must have $\tilde{W} \subseteq \phi W$. But this means that

$$\lambda(\phi W) \geq \lambda^* \tilde{W} = 1 > \lambda W,$$

which is impossible. **X**

Thus ψ fails the first test and cannot be consistent.

346X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\underline{\phi}$ a lower density for μ. Take $H \in \Sigma$ and set $A = X \setminus (\underline{\phi}H \cup \underline{\phi}(Z \setminus H))$, $\underline{\phi}'E = \underline{\phi}E \cup (A \cap \underline{\phi}(H \cup E))$ for $E \in \Sigma$. Show that $\underline{\phi}'$ is a lower density.

>(b) Show that there is no lifting ϕ of Lebesgue measure on $[0,1]^2$ which is 'symmetric' in the sense that $\phi(E^{-1}) = (\phi E)^{-1}$ for every measurable set E, writing $E^{-1} = \{(y,x) : (x,y) \in E\}$.

(c) Let (X, Σ, μ) be a measure space and $\langle \underline{\phi}_n \rangle_{n \in \mathbb{N}}$ a sequence of lower densities for μ. (i) Show that $E \mapsto \bigcap_{n \in \mathbb{N}} \underline{\phi}_n E$ and $E \mapsto \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \underline{\phi}_m E$ are also lower densities for μ. (ii) Show that if μ is complete and $\mathcal{F}$ is any filter on $\mathbb{N}$, then $E \mapsto \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} \underline{\phi}_n E$ is a lower density for μ.

(d) Let (X, Σ, μ) be a strictly localizable measure space, and G a countable group of measure space automorphisms from X to itself. Show that there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ which is G-invariant in the sense that $\phi(g^{-1}[E]) = g^{-1}[\phi E]$ for every $E \in \Sigma$, $g \in G$. (*Hint*: set $\underline{\phi}E = \bigcap_{g \in G} g[\underline{\phi}_0(g^{-1}[E])]$.)

>(e) Let $\underline{\phi}$ be lower Lebesgue density on $\mathbb{R}$, and ϕ any lifting of Lebesgue measure on $\mathbb{R}$ such that $\phi E \supseteq \underline{\phi}E$ for every measurable set E. Show that ϕ is consistent. (*Hint*: given $n \geq 1$, let $\underline{\phi}_n$ be lower Lebesgue density on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ let $\mathcal{I}_x$ be the ideal generated by

$$\{W : x \in \underline{\phi}_n(\mathbb{R}^n \setminus W)\} \cup \bigcup_{i<n} \{\pi_i^{-1}[E] : x(i) \in \phi(\mathbb{R} \setminus E)\};$$

show that $\mathbb{R}^n \notin \mathcal{I}_x$, so that we can use the method of 341J to construct a lifting of Lebesgue measure on $\mathbb{R}^n$.)

(f) Show that Lemma 346K is valid for any (Z, T, ν) which is the Stone space of an atomless probability space.

>(g) Suppose, in 341H, that (X, Σ, μ) is a product of probability spaces, and that in the proof, instead of taking $\langle a_\xi \rangle_{\xi < \kappa}$ to run over the whole measure algebra $\mathfrak{A}$, we take it to run over the elements of $\mathfrak{A}$ expressible as $E^\bullet$ where $E \in \Sigma$ is determined by a single coordinate. Show that the resulting lower density $\underline{\theta}$ respects coordinates in the sense that $\underline{\theta}E^\bullet$ is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in J. (Compare Macheras & Strauss 95, Theorem 2.)

346Y Further exercises (a) Suppose that (X, Σ, μ) and (Y, T, ν) are complete probability spaces with product $(X \times Y, \Lambda, \lambda)$. Show that for any lifting $\psi_1 : \Sigma \to \Sigma$ there are liftings $\psi_2 : \mathrm{T} \to \mathrm{T}$ and $\phi : \Lambda \to \Lambda$ such that $\phi(E \times F) = \psi_1 E \times \psi_2 F$ for all $E \in \Sigma$, $F \in \mathrm{T}$. (*Hint*: use the methods of §341. In the inductive construction of 341H, start with $\underline{\phi}_0(E \times Y) = \psi_1 E \times Y$ for every $E \in \Sigma$. Extend each lower density $\underline{\phi}_\xi$ to the algebra generated by $\mathrm{dom}(\underline{\phi}_\xi) \cup \{X \times F_\xi\}$ for some $F_\xi \in \mathrm{T}$. Make sure that $\underline{\phi}_\xi(X \times F)$ is always of the form $X \times F'$, and that $\underline{\phi}_\xi((E \times Y) \cup (X \times F)) = \underline{\phi}_\xi(E \times Y) \cup \underline{\phi}_\xi(X \times F)$; adapt the construction of 341G to maintain this. Use the method of 346H to generate a lifting from the final lower density $\underline{\phi}$. See Macheras & Strauss 96a, Theorem 4.)

(b) Use 346Ya and induction on ζ to prove 346H. (Macheras & Strauss 96b.)

(c) Let $(X_1, \Sigma_1, \mu_1), \dots, (X_n, \Sigma_n, \mu_n)$ be probability spaces with product (X, Σ, μ). Show that there is a lifting for μ which respects coordinates. (Burke n95.)

(d) Let (X, Σ, μ) be a complete probability space. Show that there is a lifting $\psi : \Sigma \to \Sigma$ such that whenever $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of probability spaces, with product measure λ, there is a lifting ϕ for λ such that $\phi(\pi_i^{-1}[E]) = \pi_i^{-1}[\psi E]$ whenever $E \in \Sigma$ and $i \in I$ is such that $(X_i, \Sigma_i, \mu_i) = (X, \Sigma, \mu)$, writing $\pi_i(x) = x(i)$ for $x \in \prod_{i \in I} X_i$.

(e) In 346Ya, find a modification of the countable-cofinality inductive step of 341G which will ensure that the lower density obtained in the product satisfies both conditions of 346G.

(f) Let (X, Σ, μ) be a probability space, I any set, and λ the product measure on X^I. Show that there is a lower density for λ which is invariant under transpositions of pairs of coordinates.

346Z Problems (a) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product (X, Σ, μ). Is there always a lifting for μ which respects coordinates in the sense of 346A?

(b) Is there a lower density $\underline{\phi}$ for the usual measure on $\{0,1\}^{\mathbb{N}}$ which is invariant under all permutations of coordinates?

346 Notes and comments I ought to say at once that in writing this section I have been greatly assisted by M.R.Burke.

The theorem that every complete probability space has a consistent lifting (346J) is due to Talagrand 82a; it is the inspiration for the whole of the section. 'Consistent' liftings were devised in response to some very interesting questions (see Talagrand 84, §6) which I do not discuss here; one will be mentioned in §465 in Volume 4. My aim here is rather to suggest further ways in which a lifting on a product space can be consistent with the product structure. The labour is substantial and the results achieved are curiously partial. I offer 346Za as the easiest natural question which does not appear amenable to the methods I describe.

The arguments I use are based on the fact that the translation-invariant measures of 345C already respect coordinates. Maharam's theorem now makes it easy to show that any product of Maharam-type-homogeneous probability spaces has a lifting which respects coordinates. A kind of projection argument (346F) makes it possible to obtain a lower density which respects coordinates on any product of probability spaces (346G). In fact the methods of §341, very slightly refined, automatically produce such lower densities (346Xg). But the extra power of 346G lies in the condition (ii): if E and F are 'fully independent' in the sense of being determined by coordinates in disjoint sets, then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$, that is, $\underline{\phi}$ is making a tentative step towards being a lifting. (Remember that the difference between a lifting and a lower density is mostly that a lifting preserves finite unions as well as finite intersections; see 341Xa.) This can also be achieved by a modification of the previous method, but we have to work harder at one point in the proof (346Ye).

The next step is to move to liftings which continue, as far as possible, to respect coordinates. Here there seem to be quite new obstacles, and 346H is the best result I know; the lifting respects *individual* coordinates, and also, for a given well-ordering of the index set, initial segments of the coordinates. The treatment of initial segments makes essential use of the well-ordering, which is what leaves 346Za open.

Finally, if all the factors are identical, we can seek lower densities and liftings which are invariant under permutation of coordinates. I give 345Xc and 346Xb as examples to show that we must not just assume that a symmetry in the underlying measure space can be reflected in a symmetry of a lifting. The problems there concern liftings themselves, not lower densities, since we can frequently find lower densities which share symmetries (346Xd, 346Yf). (Even for lower densities there seem to be difficulties if we are more ambitious (346Zb).) However a very simple argument (346I) shows that at least we can make each individual coordinate look more or less the same, as long as we do not investigate its relations with others.

Still on the question of whether, and when, liftings can be 'good', note 346L/346Xf and 346Xe. The most natural liftings of Lebesgue measure are necessarily consistent; but the only example we have of a truly canonical lifting is not consistent in any non-trivial context.

I have deliberately used a variety of techniques here, even though 346H (for instance) has an alternative proof based on the ideas of §341 (346Ya-346Yb). In particular, I give some of the standard methods of constructing liftings and lower densities (346D, 346F, 346B, 346Xa, 346Xc). In fact 346D was one of the elements of Maharam's original proof of the lifting theorem (Maharam 58).

Chapter 35

Riesz spaces

The next three chapters are devoted to an abstract description of the 'function spaces' described in Chapter 24, this time concentrating on their internal structure and their relationships with their associated measure algebras. I find that any convincing account of these must involve a substantial amount of general theory concerning partially ordered linear spaces, and in particular various types of Riesz space or vector lattice. I therefore provide an introduction to this theory, a kind of appendix built into the middle of the volume. The relation of this chapter to the next two is very like the relation of Chapter 31 to Chapter 32. As with Chapter 31, it is not really meant to be read for its own sake; those with a particular interest in Riesz spaces might be better served by Luxemburg & Zaanen 71, Schaefer 74, Zaanen 83 or my own book Fremlin 74a.

I begin with three sections in an easy gradation towards the particular class of spaces which we need to understand: partially ordered linear spaces (§351), general Riesz spaces (§352) and Archimedean Riesz spaces (§353); the last includes notes on Dedekind (σ)-complete spaces. These sections cover the fragments of the algebraic theory of Riesz spaces which I will use. In the second half of the chapter, I deal with normed Riesz spaces (in particular, L- and M-spaces)(§354), spaces of linear operators (§355) and dual Riesz spaces (§356).

351 Partially ordered linear spaces

I begin with an account of the most basic structures which involve an order relation on a linear space, partially ordered linear spaces. As often in this volume, I find myself impelled to do some of the work in very much greater generality than is strictly required, in order to show more clearly the nature of the arguments being used. I give the definition (351A) and most elementary properties (351B-351L) of partially ordered linear spaces; then I describe a general representation theorem for arbitrary partially ordered linear spaces as subspaces of reduced powers of $\mathbb{R}$ (351M-351Q). I end with a brief note on Archimedean partially ordered linear spaces (351R).

351A Definition I repeat a definition mentioned in 241E. A **partially ordered linear space** is a linear space $(U,+,\cdot)$ over $\mathbb{R}$ together with a partial order $\le$ on U such that

$$u \le v \Longrightarrow u+w \le v+w,$$

$$u \ge 0,\ \alpha \ge 0 \Longrightarrow \alpha u \ge 0$$

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

351B Elementary facts Let U be a partially ordered linear space. We have the following elementary consequences of the definition above, corresponding to the familiar rules for manipulating inequalities among real numbers.

(a) For u, $v \in U$,

$$u \le v \Longrightarrow 0 = u+(-u) \le v+(-u) = v-u \Longrightarrow u = 0+u \le v-u+u = v,$$

$$u \le v \Longrightarrow -v = u+(-v-u) \le v+(-v-u) = -u.$$

(b) Suppose that u, $v \in U$ and $u \le v$. Then $\alpha u \le \alpha v$ for every $\alpha \ge 0$ and $\alpha v \le \alpha u$ for every $\alpha \le 0$. **P** (i) If $\alpha \ge 0$, then $\alpha(v-u) \ge 0$ so $\alpha v \ge \alpha u$. (ii) If $\alpha \le 0$ then $(-\alpha)u \le (-\alpha)v$ so

$$\alpha v = -(-\alpha)v \le -(-\alpha u) = u. \text{ **Q**}$$

(c) If $u \ge 0$ and $\alpha \le \beta$ in $\mathbb{R}$, then $(\beta-\alpha)u \ge 0$, so $\alpha u \le \beta u$. If $0 \le u \le v$ in U and $0 \le \alpha \le \beta$ in $\mathbb{R}$, then $\alpha u \le \beta u \le \beta v$.

351C Positive cones Let U be a partially ordered linear space.

(a) I will write U^+ for the **positive cone** of U, the set $\{u : u \in U,\ u \ge 0\}$.

(b) By 351Ba, the ordering is determined by the positive cone U^+, in the sense that $u \le v \iff v - u \in U^+$.

(c) It is easy to characterize positive cones. If U is a real linear space, a set $C \subseteq U$ is the positive cone for some ordering rendering U a partially ordered linear space iff

$$u + v \in C, \quad \alpha u \in C \text{ whenever } u,\ v \in C \text{ and } \alpha \ge 0,$$

$$0 \in C, \quad u \in C \mathbin{\&} -u \in C \Longrightarrow u = 0.$$

P (i) If $C = U^+$ for some partially ordered linear space ordering $\le$ on U, then

$$u,\ v \in C \Longrightarrow 0 \le u \le u + v \Longrightarrow u + v \in C,$$

$$u \in C,\ \alpha \ge 0 \Longrightarrow \alpha u \ge 0, \text{ i.e., } \alpha u \in C,$$

$$0 \le 0 \text{ so } 0 \in C,$$

$$u,\ -u \in C \Longrightarrow u = 0 + u \le (-u) + u = 0 \le u \Longrightarrow u = 0.$$

(ii) On the other hand, if C satisfies the conditions, define the relation $\le$ by writing $u \le v \iff v - u \in C$; then

$$u - u = 0 \in C \text{ so } u \le u \text{ for every } u \in U,$$

$$\text{if } u \le v \text{ and } v \le w \text{ then } w - u = (w - v) + (v - u) \in C \text{ so } u \le w,$$

$$\text{if } u \le v \text{ and } v \le u \text{ then } u - v,\ v - u \in C \text{ so } u - v = 0 \text{ and } u = v$$

and $\le$ is a partial order; moreover,

$$\text{if } u \le v \text{ and } w \in U \text{ then } (v + w) - (u + w) = v - u \in C \text{ and } u + w \le v + w,$$

$$\text{if } u,\ \alpha \ge 0 \text{ then } \alpha u \in C \text{ and } \alpha u \ge 0,$$

$$u \ge 0 \iff u \in C.$$

So $\le$ makes U a partially ordered linear space in which C is the positive cone. **Q**

(d) An incidental useful fact. Let U be a partially ordered linear space, and $u \in U$. Then $u \ge 0$ iff $u \ge -u$. **P** If $u \ge 0$ then $0 \ge -u$ so $u \ge -u$. If $u \ge -u$ then $2u \ge 0$ so $u = \frac{1}{2} \cdot 2u \ge 0$. **Q**

(e) I have called U^+ a 'positive cone' without defining the term 'cone'. I think this is something we can pass by for the moment; but it will be useful to recognise that U^+ is always convex, for if $u,\ v \in U^+$ and $\alpha \in [0,1]$ then $\alpha u,\ (1 - \alpha)v \ge 0$ and $\alpha u + (1 - \alpha)v \in U^+$.

351D Suprema and infima Let U be a partially ordered linear space.

(a) The definition of 'partially ordered linear space' implies that $u \mapsto u + w$ is always an order-isomorphism; on the other hand, $u \mapsto -u$ is order-reversing, by 351Ba.

(b) It follows that if $A \subseteq U$, $v \in U$ then

$$\sup_{u \in A} v + u = v + \sup A \text{ if either side is defined,}$$

$$\inf_{u \in A} v + u = v + \inf A \text{ if either side is defined,}$$

$$\sup_{u \in A} v - u = v - \inf A \text{ if either side is defined,}$$

$$\inf_{u \in A} v - u = v - \sup A \text{ if either side is defined.}$$

(c) Moreover, we find that if A, $B \subseteq U$ and $\sup A$ and $\sup B$ are defined, then $\sup(A+B)$ is defined and equal to $\sup A + \sup B$, writing $A + B = \{u + v : u \in A,\ v \in B\}$ as usual. **P** Set $u_0 = \sup A$, $v_0 = \sup B$. Using (b), we have

$$\begin{aligned} u_0 + v_0 &= \sup_{u \in A}(u + v_0) \\ &= \sup_{u \in A}(\sup_{v \in B}(u + v)) = \sup(A + B). \ \mathbf{Q} \end{aligned}$$

Similarly, if A, $B \subseteq U$ and $\inf A$, $\inf B$ are defined then $\inf(A + B) = \inf A + \inf B$.

(d) If $\alpha > 0$ then $u \mapsto \alpha u$ is an order-isomorphism, so we have $\sup(\alpha A) = \alpha \sup A$ if either side is defined; similarly, $\inf(\alpha A) = \alpha \inf A$.

351E Linear subspaces If U is a partially ordered linear space, and V is any linear subspace of U, then V, with the induced linear and order structures, is a partially ordered linear space; this is obvious from the definition.

351F Positive linear operators Let U and V be partially ordered linear spaces, and write $\mathsf{L}(U;V)$ for the linear space of all linear operators from U to V. For S, $T \in \mathsf{L}(U;V)$ say that $S \le T$ iff $Su \le Tu$ for every $u \in U^+$. Under this ordering, $\mathsf{L}(U;V)$ is a partially ordered linear space; its positive cone is $\{T : Tu \ge 0$ for every $u \in U^+\}$. **P** This is an elementary verification. **Q** Note that, for $T \in \mathsf{L}(U;V)$,

$$\begin{aligned} T \ge 0 &\Longrightarrow Tu \le Tu + T(v - u) = Tv \text{ whenever } u \le v \text{ in } U \\ &\Longrightarrow 0 = T0 \le Tu \text{ for every } u \in U^+ \\ &\Longrightarrow T \ge 0, \end{aligned}$$

so that $T \ge 0$ iff T is order-preserving. In this case we say that T is a **positive** linear operator.

Clearly ST is a positive linear operator whenever U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are positive linear operators (cf. 313Ia).

351G Order-continuous positive linear operators: Proposition Let U and V be partially ordered linear spaces and $T : U \to V$ a positive linear operator.

(a) The following are equiveridical:

(i) T is order-continuous;

(ii) $\inf T[A] = 0$ in V whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U;

(iii) $\sup T[A] = Tw$ in V whenever $A \subseteq U^+$ is a non-empty upwards-directed set with supremum w in U.

(b) The following are equiveridical:

(i) T is sequentially order-continuous;

(ii) $\inf_{n\in\mathbb{N}} Tu_n = 0$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum 0 in U;

(iii) $\sup_{n\in\mathbb{N}} Tu_n = Tw$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum w in U.

proof (a)(i)⇒(iii) is trivial.

(iii)⇒(ii) Assuming (iii), and given that A is non-empty, downwards-directed and has infimum 0, take any $u_0 \in A$ and consider $A' = \{u : u \in A,\ u \le u_0\}$, $B = u_0 - A'$. Then A' is non-empty, downwards-directed and has infimum 0, so B is non-empty, upwards-directed and has supremum u_0 (using 351Db); by (iii), $\sup T[B] = Tu_0$ and (inverting again)

$$\inf T[A'] = \inf T[u_0 - B] = \inf Tu_0 - T[B] = Tu_0 - \sup T[B] = 0.$$

But (because T is positive) 0 is surely a lower bound for $T[A]$, so it is also the infimum of $T[A]$. As A is arbitrary, (ii) is true.

(ii)⇒(i) Suppose now that (ii) is true. **(α)** If $A \subseteq U$ is non-empty, downwards-directed and has infimum w, then $A - w$ is non-empty, downwards-directed and has infimum 0, so

$$\inf T[A-w]=0, \quad \inf T[A]=\inf(T[A-w]+Tw)=Tw+\inf T[A-w]=Tw.$$

(β) If $A\subseteq U$ is non-empty, upwards-directed and has supremum w, then $-A$ is non-empty, downwards-directed and has infimum $-w$, so

$$\sup T[A]=-\inf(-T[A])=-\inf T[-A]=-T(-w)=Tw.$$

Putting these together, T is order-continuous.

(b) The arguments are identical, replacing each directed set by an appropriate sequence.

351H Riesz homomorphisms (a) For the sake of a representation theorem below (351Q), I introduce the following definition. Let U, V be partially ordered linear spaces. A **Riesz homomorphism** from U to V is a linear operator $T:U\to V$ such that whenever $A\subseteq U$ is a finite non-empty set and $\inf A=0$ in U, then $\inf T[A]=0$ in V. The following facts are now nearly obvious.

(b) Any Riesz homomorphism is a positive linear operator. (For if T is a Riesz homomorphism and $u\geq 0$, then $\inf\{0,u\}=0$ so $\inf\{0,Tu\}=0$ and $Tu\geq 0$.)

(c) Let U and V be partially ordered linear spaces and $T:U\to V$ a Riesz homomorphism. Then

$$\inf T[A] \text{ exists} = T(\inf A), \quad \sup T[A] \text{ exists} = T(\sup A)$$

whenever $A\subseteq U$ is a finite non-empty set and $\inf A$, $\sup A$ exist. (Apply the definition in (a) to

$$A'=\{u-\inf A:u\in A\}, \quad A''=\{\sup A-u:u\in A\}.)$$

(d) If U, V and W are partially ordered linear spaces and $T:U\to V$, $S:V\to W$ are Riesz homomorphisms then $ST:U\to W$ is a Riesz homomorphism.

351I Solid sets Let U be a partially ordered linear space. I will say that a subset A of U is **solid** if

$$A=\{v:v\in U,\ -u\leq v\leq u \text{ for some } u\in A\}=\bigcup_{u\in U}[-u,u]$$

in the notation of 2A1Ab. (I should perhaps remark that while this definition is well established in the case of Riesz spaces (§352), the extension to general partially ordered linear spaces is not standard. See 351Yb for a warning.)

351J Proposition Let U be a partially ordered linear space and V a solid linear subspace of U. Then the quotient linear space U/V has a partially ordered linear space structure defined by either of the rules

$u^\bullet\leq w^\bullet$ iff there is some $v\in V$ such that $u\leq v+w$,

$(U/V)^+=\{u^\bullet:u\in U^+\}$,

and for this partial ordering on U/V the map $u\mapsto u^\bullet:U\to U/V$ is a Riesz homomorphism.

proof (a) I had better start by giving priority to one of the descriptions of the relation $\leq$ on U/V; I choose the first. To see that this makes U/V a partially ordered linear space, we have to check the following.

(i) $0\in V$ and $u\leq u+0$, so $u^\bullet\leq u^\bullet$ for every $u\in U$.

(ii) If u_1, u_2, $u_3\in U$ and $u_1^\bullet\leq u_2^\bullet$, $u_2^\bullet\leq u_3^\bullet$ then there are v_1, $v_2\in V$ such that $u_1\leq u_2+v_1$, $u_2\leq u_3+v_2$; in which case $v_1+v_2\in V$ and $u_1\leq u_3+v_1+v_2$, so $u_1^\bullet\leq u_3^\bullet$.

(iii) If u, $w\in U$ and $u^\bullet\leq w^\bullet$, $w^\bullet\leq u^\bullet$ then there are v, $v'\in V$ such that $u\leq w+v$, $w\leq u+v'$. Now there are v_0, $v_0'\in V$ such that $-v_0\leq v\leq v_0$, $-v_0'\leq v'\leq v_0'$, and in this case v_0, $v_0'\geq 0$ (351Cd), so

$$-v_0-v_0'\leq -v'\leq u-w\leq v\leq v_0+v_0'\in V,$$

.

Accordingly $u-w\in V$ and $u^\bullet=w^\bullet$. Thus U/V is a partially ordered set.

(iv) If u_1, u_2, $w\in U$ and $u_1^\bullet\leq u_2^\bullet$, then there is a $v\in V$ such that $u_1\leq u_2+v$, in which case $u_1+w\leq u_2+w+v$ and $u_1^\bullet+w^\bullet\leq u_2^\bullet+w^\bullet$.

(v) If $u\in U$, $\alpha\in\mathbb{R}$ and $u^\bullet\geq 0$, $\alpha\geq 0$ then there is a $v\in V$ such that $u+v\geq 0$; now $\alpha v\in V$ and $\alpha u+\alpha v\geq 0$, so $\alpha u^\bullet=(\alpha u)^\bullet\geq 0$.

Thus U/V is a partially ordered linear space.

(b) Now $(U/V)^+ = \{u^\bullet : u \ge 0\}$. **P** If $u \ge 0$ then of course $u^\bullet \ge 0$ because $0 \in V$ and $u + 0 \ge 0$. On the other hand, if we have any element p of $(U/V)^+$, there are $u \in U$, $v \in V$ such that $u^\bullet = p$ and $u + v \ge 0$; but now $p = (u+v)^\bullet$ is of the required form. **Q**

(c) Finally, $u \mapsto u^\bullet$ is a Riesz homomorphism. **P** Suppose that $A \subseteq U$ is a non-empty finite set and that $\inf A = 0$ in U. Then $u^\bullet \ge 0$ for every $u \in A$, that is, 0 is a lower bound for $\{u^\bullet : u \in A\}$. Let $p \in U/V$ be any other lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$ where $w \in U$. For each $u \in A$, $w^\bullet \le u^\bullet$ so there is a $v_u \in V$ such that $w \le u + v_u$. Next, there is a $v'_u \in V$ such that $-v'_u \le v_u \le v'_u$. Set $v^* = \sum_{u \in A} v'_u \in V$. Then $v_u \le v'_u \le v^*$ so $w \le u + v^*$ for every $u \in A$, and $w - v^*$ is a lower bound for A in U. Accordingly $w - v^* \le 0$, $w \le 0 + v^*$ and $p = w^\bullet \le 0$. As p is arbitrary, $\inf\{u^\bullet : u \in A\} = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is a Riesz homomorphism. **Q**

351K Lemma Suppose that U is a partially ordered linear space, and that W, V are solid linear subspaces of U such that $W \subseteq V$. Then $V_1 = \{v^\bullet : v \in V\}$ is a solid linear subspace of U/W.

proof (i) Because the map $u \mapsto u^\bullet$ is linear, V_1 is a linear subspace of U/W. (ii) If $p \in V_1$, there is a $v \in V$ such that $p = v^\bullet$; because V is solid in U, there is a $v_0 \in V$ such that $-v_0 \le v \le v_0$; now $v_0^\bullet \in V_1$ and $-v_0^\bullet \le p \le v_0^\bullet$. (iii) If $p \in V_1$, $q \in U/W$ and $-p \le q \le p$, take $v_0 \in V$, $u \in U$ such that $v_0^\bullet = p$, $u^\bullet = q$. Because $-v_0^\bullet \le u^\bullet \le v_0^\bullet$, there are w, $w' \in W$ such that $-v_0 - w \le u \le v_0 + w'$. Now $-v_0 - w$, $v_0 + w'$ both belong to V, which is solid, so both are mapped to 0 by the canonical Riesz homomorphism from U to U/V, and u must also be, that is, $u \in V$ and $q = u^\bullet \in V_1$. (iv) Putting (ii) and (iii) together, V_1 is solid.

351L Products If $\langle U_i \rangle_{i \in I}$ is any family of partially ordered linear spaces, we have a product linear space $U = \prod_{i \in I} U_i$; if we set $u \le v$ in U iff $u(i) \le v(i)$ for every $i \in I$, U becomes a partially ordered linear space, with positive cone $\{u : u(i) \ge 0$ for every $i \in I\}$. For each $i \in I$ the map $u \mapsto u(i) : U \to U_i$ is an order-continuous Riesz homomorphism (in fact, it preserves arbitrary suprema and infima).

351M Reduced powers of $\mathbb{R}$ (a) Let X be any set. Then $\mathbb{R}^X$ is a partially ordered linear space if we say that $f \le g$ means that $f(x) \le g(x)$ for every $x \in X$, as in 351L. If now $\mathcal{F}$ is a filter on X, we have a corresponding set

$$V = \{f : f \in \mathbb{R}^X,\ \{x : f(x) = 0\} \in \mathcal{F}\};$$

it is easy to see that V is a linear subspace of $\mathbb{R}^X$, and is solid because $f \in V$ iff $|f| \in V$. By the **reduced power** $\mathbb{R}^X|\mathcal{F}$ I shall mean the quotient partially ordered linear space $\mathbb{R}^X/V$.

(b) Note that for $f \in \mathbb{R}^X$,

$$f^\bullet \ge 0 \text{ in } \mathbb{R}^X|\mathcal{F} \iff \{x : f(x) \ge 0\} \in \mathcal{F}.$$

P (i) If $f^\bullet \ge 0$, there is a $g \in V$ such that $f + g \ge 0$; now

$$\{x : f(x) \ge 0\} \supseteq \{x : g(x) = 0\} \in \mathcal{F}.$$

(ii) If $\{x : f(x) \ge 0\} \in \mathcal{F}$, then $\{x : (|f| - f)(x) = 0\} \in \mathcal{F}$, so $f^\bullet = |f|^\bullet \ge 0$. **Q**

351N On the way to the next theorem, the main result (in terms of mathematical depth) of this section, we need a string of lemmas.

Lemma Let U be a partially ordered linear space. If $u, v_0, \dots, v_n \in U$ are such that $u \ne 0$ and $v_0, \dots, v_n \ge 0$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \ne 0$ and $f(v_i) \ge 0$ for every i.

proof The point is that at most one of u, $-u$ can belong to the convex cone C generated by $\{v_0, \dots, v_n\}$, because this is included in the convex cone set U^+, and since $u \ne 0$ at most one of u, $-u$ can belong to U^+.

Now however the Hahn-Banach theorem, in the form 3A5D, tells us that if $u \notin C$ there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) < 0$, $f(v_i) \ge 0$ for every i; while if $-u \notin C$ we can get $f(-u) < 0$ and $f(v_i) \ge 0$ for every i. Thus in either case we have a functional of the required type.

351O Lemma Let U be a partially ordered linear space, and u_0 a non-zero member of U. Then there is a solid linear subspace V of U such that $u_0 \notin V$ and whenever $A \subseteq U$ is finite, non-empty and has infimum 0 then $A \cap V \neq \emptyset$.

proof (a) Let $\mathcal{W}$ be the family of all solid linear subspaces of U not containing u_0. Then any non-empty totally ordered $\mathcal{V} \subseteq \mathcal{W}$ has an upper bound $\bigcup \mathcal{V}$ in $\mathcal{W}$. By Zorn's Lemma, $\mathcal{W}$ has a maximal element V say. This is surely a solid linear subspace of U not containing u_0.

(b) Now for any $w \in U^+ \setminus V$ there are $\alpha \geq 0$, $v \in V^+$ such that $-\alpha w - v \leq u_0 \leq \alpha w + v$. **P** Let V_1 be

$$\{u : u \in U, \text{ there are } \alpha \geq 0,\ v \in V^+ \text{ such that } -\alpha w - v \leq u \leq \alpha w + v\}.$$

Then it is easy to check that V_1 is a solid linear subspace of U, including V, and containing w; because $w \notin V$, $V_1 \neq V$, so $V_1 \notin \mathcal{W}$ and $u \in V_1$, as claimed. **Q**

(c) It follows that if $A \subseteq U$ is finite and non-empty and $\inf A = 0$ in U then $A \cap V \neq \emptyset$. **P?** Otherwise, for every $w \in A$ there must be $\alpha_w \geq 0$, $v_w \in V^+$ such that $-\alpha_w w - v_w \leq u_0 \leq \alpha_w w + v_w$. Set $\alpha = 1 + \sum_{w \in A} \alpha_w$, $v = \sum_{w \in A} v_w \in V$; then $-\alpha w - v \leq u_0 \leq \alpha w + v$ for every $w \in A$. Accordingly $\frac{1}{\alpha}(u_0 - v) \leq w$ for every $w \in A$ and $\frac{1}{\alpha}(u_0 - v) \leq 0$, so $u_0 \leq v$. Similarly, $-\frac{1}{\alpha}(v + u_0) \leq w$ for every $w \in A$ and $-v \leq u_0$. But (because V is solid) this means that $u_0 \in V$, which is not so. **XQ**

Accordingly V has the required properties.

351P Lemma Let U be a partially ordered linear space and u a non-zero element of U, and suppose that $A_0, \dots, A_n$ are finite non-empty subsets of U such that $\inf A_j = 0$ for every $j \leq n$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \neq 0$ and $\min f[A_j] = 0$ for every $j \leq n$.

proof By 351O, there is a solid linear subspace V of U such that $u \notin V$ and $A_j \cap V \neq 0$ for every $j \leq n$. Give the quotient space U/V its standard partial ordering (351J), and in U/V set $C = \{v^\bullet : v \in \bigcup_{j \leq n} A_j\}$. Then C is a finite subset of $(U/V)^+$, while $u^\bullet \neq 0$, so by 351N there is a linear functional $g : U/V \to \mathbb{R}$ such that $g(u^\bullet) \neq 0$ but $g(p) \geq 0$ for every $p \in C$. Set $f(v) = g(v^\bullet)$ for $v \in U$; then $f : U \to \mathbb{R}$ is linear, $f(u) \neq 0$ and $f(v) \geq 0$ for every $v \in \bigcup_{j \leq n} A_j$. But also, for each $j \leq n$, there is a $v_j \in A_j \cap V$, so that $f(v_j) = 0$; and this means that $\min f[A_j]$ must be 0, as required.

351Q Now we are ready for the theorem.

Theorem Let U be any partially ordered linear space. Then we can find a set X, a filter $\mathcal{F}$ on X and an injective Riesz homomorphism from U to the reduced power $\mathbb{R}^X|\mathcal{F}$ described in 351M.

proof Let X be the set of all linear functionals $f : U \to \mathbb{R}$; define $\phi : U \to \mathbb{R}^X$ by setting $\phi(u)(f) = f(u)$ for every $f \in X$, $u \in U$, so that ϕ is linear. Let $\mathcal{A}$ be the family of non-empty finite sets $A \subseteq U$ such that $\inf A = 0$. For $A \in \mathcal{A}$ let F_A be the set of those $f \in X$ such that $\min f[A] = 0$. Since $0 \in F_A$ for every $A \in \mathcal{A}$, the set

$$\mathcal{F} = \{F : F \subseteq X, \text{ there are } A_0, \dots, A_n \in \mathcal{A} \text{ such that } F \supseteq \textstyle\bigcap_{j \leq n} F_{A_j}\}$$

is a filter on X. Set $\psi(u) = \phi(u)^\bullet \in \mathbb{R}^X|\mathcal{F}$ for $u \in U$. The $\psi : U \to \mathbb{R}^X|\mathcal{F}$ is an injective Riesz homomorphism.

P (i) ψ is linear because ϕ and $h \mapsto h^\bullet : \mathbb{R}^X \to \mathbb{R}^x|\mathcal{F}$ are. (ii) If $A \in \mathcal{A}$, then $F_A \in \mathcal{F}$. So, first, if $v \in A$, then $\{f : \phi(v)(f) \geq 0\} \in \mathcal{F}$, so that $\psi(v) = \phi(v)^\bullet \geq 0$ in $\mathbb{R}^X|\mathcal{F}$ (351Mb). Next, if $w \in \mathbb{R}^X|\mathcal{F}$ and $w \leq \psi(v)$ for every $v \in A$, we can express w as $h^\bullet$ where $h^\bullet \leq \phi(v)^\bullet$ for every $v \in A$, that is, $H_v = \{f : h(f) \leq \phi(v)(f)\} \in \mathcal{F}$ for every $v \in A$. But now $H = F_A \cap \bigcap_{v \in A} H_v \in \mathcal{F}$, and for $f \in H$ we have $h(f) \leq \min_{v \in A} f(v) = 0$. This means that $w = h^\bullet \leq 0$. As w is arbitrary, $\inf \psi[A] = 0$. As A is arbitrary, ψ is a Riesz homomorphism. (iii) Finally, **?** suppose, if possible, that there is a non-zero $u \in U$ such that $\psi(u) = 0$. Then $F = \{f : f(u) = 0\} \in \mathcal{F}$, and there are $A_0, \dots, A_n \in \mathcal{A}$ such that $F \supseteq \bigcap_{j \leq n} F_{A_j}$. By 351P, there is an $f \in \bigcap_{j \leq n} F_{A_j}$ such that $f(u) \neq 0$; which is impossible. **X** Accordingly ψ is injective, as claimed. **Q**

351R Archimedean spaces (a) For a partially ordered linear space U, the following are equiveridical: (i) if u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$ then $u \le 0$ (ii) if $u \ge 0$ in U then $\inf_{\delta>0} \delta u = 0$. **P** (i)$\Rightarrow$(ii) If (i) is true and $u \ge 0$, then of course $\delta u \ge 0$ for every $\delta > 0$; on the other hand, if $v \le \delta u$ for every $\delta > 0$, then $nv \le n \cdot \frac{1}{n} u = u$ for every $n \ge 1$, while of course $0v = 0 \le u$, so $v \le 0$. Thus 0 is the greatest lower bound of $\{\delta u : \delta > 0\}$. (ii)$\Rightarrow$(i) If (ii) is true and $nu \le v$ for every $n \in \mathbb{N}$, then $0 \le v$ and $u \le \frac{1}{n} v$ for every $n \ge 1$. If now $\delta > 0$, then there is an $n \ge 1$ such that $\frac{1}{n} \le \delta$, so that $u \le \frac{1}{n} v \le \delta v$ (351Bc). Accordingly u is a lower bound for $\{\delta v : \delta > 0\}$ and $u \le 0$. **Q**

(b) I will say that partially ordered linear spaces satisfying the equiveridical conditions of (a) above are **Archimedean**.

(c) Any linear subspace of an Archimedean partially ordered linear space, with the induced partially ordered linear space structure, is Archimedean.

(d) Any product of Archimedean partially ordered linear spaces is Archimedean. **P** If $U = \prod_{i \in I} U_i$ is a product of Archimedean spaces, and $nu \le v$ in U for every $n \in \mathbb{N}$, then for each $i \in I$ we must have $nu(i) \le v(i)$ for every n, so that $u(i) \le 0$; accordingly $u \le 0$. **Q** In particular, $\mathbb{R}^X$ is Archimedean for any set X.

351X Basic exercises >(a) Let ζ be any ordinal. The **lexicographic ordering** on $\mathbb{R}^\zeta$ is defined by saying that $f \le g$ iff either $f = g$ or there is a $\xi < \zeta$ such that $f(\eta) = g(\eta)$ for $\eta < \xi$ and $f(\xi) < g(\xi)$. Show that this is a total order on $\mathbb{R}^\zeta$ which renders $\mathbb{R}^\zeta$ a partially ordered linear space.

(b) Let U be a partially ordered linear space and V a linear subspace of U. Show that the formulae of 351J define a partially ordered linear space structure on the quotient U/V iff V is **order-convex**, that is, $u \in V$ whenever v_1, $v_2 \in V$ and $v_1 \le u \le v_2$.

(c) Let $\langle U_i \rangle_{i \in I}$ be a family of partially ordered linear spaces with product U. Define $T_i : U_i \to U$ by setting $T_i x = u$ where $u(i) = x$, $u(j) = 0$ for $j \ne i$. Show that T_i is an injective order-continuous Riesz homomorphism.

>(d) Let U be a partially ordered linear space and $\langle V_i \rangle_{i \in I}$ a family of partially ordered linear spaces with product V. Show that $\mathsf{L}(U;V)$ can be identified, as partially ordered linear space, with $\prod_{i \in I} \mathsf{L}(U;V_i)$.

>(e) Show that if U, V are partially ordered linear spaces and V is Archimedean, then $\mathsf{L}(U;V)$ is Archimedean.

351Y Further exercises (a) Give an example of two partially ordered linear spaces U and V and a bijective Riesz homomorphism $T : U \to V$ such that $T^{-1} : V \to U$ is not a Riesz homomorphism.

(b) (i) Let U be a partially ordered linear space. Show that U is a solid subset of itself (on the definition 351I) iff $U = U^+ - U^+$. (ii) Give an example of a partially ordered linear space U satisfying this condition with an element $u \in U$ such that the intersection of the solid sets containing u is not solid.

(c) Let U be a partially ordered linear space, and suppose that A, $B \subseteq U$ are two non-empty finite sets such that (α) $u \vee v = \sup\{u, v\}$ is defined for every $u \in A$, $v \in B$ (β $\inf A$ and $\inf B$ and $(\inf A) \vee (\inf B)$ are defined. Show that $\inf\{u \vee v : u \in A, v \in B\} = (\inf A) \vee (\inf B)$. (*Hint*: show that this is true if $U = \mathbb{R}$, if $U = \mathbb{R}^X$ and if $U = \mathbb{R}^X|\mathcal{F}$, and use 351Q.)

(d) Show that a reduced power $\mathbb{R}^X|\mathcal{F}$, as described in 351M, is totally ordered iff $\mathcal{F}$ is an ultrafilter, and in this case has a natural structure as a totally ordered field.

351 Notes and comments The idea of 'partially ordered linear space' is a very natural abstraction from the elementary examples of $\mathbb{R}^X$ and its subspaces, and the only possible difficulty lies in guessing the exact boundary at which one's standard manipulations with such familiar spaces cease to be valid in the general case. (For instance, most people's favourite examples are Archimedean, in the sense of 351R, so it is prudent to check your intuitions against a non-Archimedean space like that of 351Xa.) There is really no room for any deep idea to appear in 351B-351F. When I come to what I call 'Riesz homomorphisms', however (351H), there are some more interesting possibilities in the background.

I shall not discuss the applications of Theorem 351Q to general partially ordered linear spaces; it is here for the sake of its application to Riesz spaces in the next section. But I think it is a very striking fact that not only does any partially ordered linear space U appear as a linear subspace of some reduced power of $\mathbb{R}$, but the embedding can be taken to preserve any suprema and infima of finite sets which exist in U. This is in a sense a result of the same kind as the Stone representation theorem for Boolean algebras; it gives us a chance to confirm that an intuition valid for $\mathbb{R}$ or $\mathbb{R}^X$ may in fact apply to arbitrary partially ordered linear spaces. If you like, this provides a metamathematical foundation for such results as those in 351B. I have to say that for partially ordered linear spaces it is generally quicker to find a proof directly from the definition than to trace through an argument relying on 351Q; but this is not always the case for Riesz spaces. I offer 351Yc as an example of a result where a direct proof does at least call for a moment's thought, while the argument through 351Q is straightforward.

'Reduced powers' are of course of great importance for other reasons; I mention 351Yd as a hint of what can be done.

352 Riesz spaces

In this section I sketch those fragments of the theory we need which can be expressed as theorems about general Riesz spaces or vector lattices. I begin with the definition (352A) and most elementary properties (352C-352F). In 352G-352J I discuss Riesz homomorphisms and the associated subspaces (Riesz subspaces, solid linear subspaces); I mention product spaces (352K, 352T) and quotient spaces (352Jb, 352U) and the form the representation theorem 351Q takes in the present context (352L-352M). Most of the second half of the section concerns the theory of 'bands' in Riesz spaces, with the algebras of complemented bands (352Q) and projection bands (352S) and a description of bands generated by upwards-directed sets (352V). I conclude with a description of 'f-algebras' (352W).

352A I repeat a definition from 241E.

Definition A **Riesz space** or **vector lattice** is a partially ordered linear space which is a lattice.

352B Lemma If U is a partially ordered linear space, then it is a Riesz space iff $\sup\{0,u\}$ is defined for every $u \in U$.

proof If U is a lattice, then of course $\sup\{u,0\}$ is defined for every u. If $\sup\{u,0\}$ is defined for every u, and v_1, v_2 are any two members of U, consider $w = v_1 + \sup\{0, v_2 - v_1\}$; by 351Db, $w = \sup\{v_1, v_2\}$. Next,

$$\inf\{v_1, v_2\} = -\sup\{-v_1, -v_2\}$$

must also be defined in U; as v_1 and v_2 are arbitrary, U is a lattice.

352C Notation In any Riesz space U I will write

$$u^+ = u \vee 0, \quad u^- = (-u) \vee 0 = (-u)^+, \quad |u| = u \vee (-u)$$

where (as in any lattice) $u \vee v = \sup\{u, v\}$ (and $u \wedge v = \inf\{u, v\}$).

I mention immediately a term which will be useful: a family $\langle u_i \rangle_{i \in I}$ in U is **disjoint** if $|u_i| \wedge |u_j| = 0$ for all distinct i, $j \in I$. Similarly, a set $C \subseteq U$ is **disjoint** if $|u| \wedge |v| = 0$ for all distinct u, $v \in C$.

352D Elementary identities Let U be a Riesz space. The translation-invariance of the order, and its invariance under positive scalar multiplication, reversal under negative multiplication, lead directly to the following, which are in effect special cases of 351D:

$$u + (v \vee w) = (u + v) \vee (u + w), \quad u + (v \wedge w) = (u + v) \wedge (u + w),$$

$$\alpha(u \vee v) = \alpha u \vee \alpha v, \quad \alpha(u \wedge v) = \alpha u \wedge \alpha v \text{ if } \alpha \geq 0,$$

$$-(u \vee v) = (-u) \wedge (-v).$$

Combining and elaborating on these facts, we get

$$u^+ - u^- = (u \vee 0) - ((-u) \vee 0) = u + (0 \vee (-u)) - ((-u) \vee 0) = u,$$

$$u^+ + u^- = 2u^+ - u = (2u \vee 0) - u = u \vee (-u) = |u|,$$

$$u \geq 0 \iff -u \leq 0 \iff u^- = 0 \iff u = u^+ \iff u = |u|,$$

$$|-u| = |u|, \quad |\,|u|\,| = |u|, \quad |\alpha u| = |\alpha||u|$$

(looking at the cases $\alpha \geq 0$, $\alpha \leq 0$ separately),

$$\begin{aligned} u \vee v + u \wedge v &= u + (0 \vee (v-u)) + v + ((u-v) \wedge 0) \\ &= u + (0 \vee (v-u)) + v - ((v-u) \vee 0) = u + v, \end{aligned}$$

$$u \vee v = u + (0 \vee (v-u)) = u + (v-u)^+,$$

$$u \wedge v = u + (0 \wedge (v-u)) = u - (-0 \vee (u-v)) = u - (u-v)^+,$$

$$u \vee v = \frac{1}{2}(2u \vee 2v) = \frac{1}{2}(u + v + (u-v) \vee (v-u)) = \frac{1}{2}(u + v + |u-v|),$$

$$u \wedge v = u + v - u \vee v = \frac{1}{2}(u + v - |u-v|),$$

$$u^+ \vee u^- = u \vee (-u) \vee 0 = |u|, \qquad u^+ \wedge u^- = u^+ + u^- - (u^+ \vee u^-) = 0,$$

$$|u+v| = (u+v) \wedge ((-u) + (-v)) \leq (|u| + |v|) \wedge (|u| + |v|) = |u| + |v|,$$

$$||u| - |v|| = (|u| - |v|) \wedge (|v| - |u|) \leq |u-v| + |v-u| = |u-v|,$$

$$|u \vee v| \leq |u| + |v|$$

(because $-|u| \leq u \vee v \leq |u| \vee |v| \leq |u| + |v|$)

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

352E Distributive laws Let U be a Riesz space.

(a) If A, $B \subseteq U$ have suprema a, b in U, then $C = \{u \wedge v : u \in A, v \in B\}$ has supremum $a \wedge b$. **P** Of course $u \wedge v \leq a \wedge b$ for all $u \in A$, $v \in B$, so $a \wedge b$ is an upper bound for C. Now suppose that c is any upper bound for C. If $u \in A$, $v \in B$ then

$$u - (u-v)^+ = u \wedge v \leq c, \qquad u \leq c + (u-v)^+ \leq c + (a-v)^+$$

(because $(a-v)^+ = \sup\{a-v, 0\} \geq \sup\{u-v, 0\} = (u-v)^+$). As u is arbitrary, $a \leq c + (a-v)^+$ and $a \wedge v \leq c$. Now turn the argument round:

$$v = (a \wedge v) + (v-a)^+ \leq c + (v-a)^+ \leq c + (b-a)^+,$$

and this is true for every $v \in B$, so $b \leq c + (b-a)^+$ and $a \wedge b \leq c$. As c is arbitrary, $a \wedge b = \sup C$, as claimed. **Q**

(b) Similarly, or applying (a) to $-A$ and $-B$, $\inf\{u \vee v : u \in A, v \in B\} = \inf A \vee \inf B$ whenever A, $B \subseteq U$ and the right-hand-side is defined.

(c) In particular, U is a distributive lattice (definition: 3A1Ic).

352F Further identities and inequalities At a slightly deeper level we have the following facts.

Proposition Let U be a Riesz space.

(a) If u, v, $w \ge 0$ in U then $u \wedge (v + w) \le (u \wedge v) + (u \wedge w)$.

(b) If $u_0, \dots , u_n \in U$ and $|u_i| \wedge |u_j| = 0$ for $i \ne j$, then $|\sum_{i=0}^n \alpha_i u_i| = \sum_{i=0}^n |\alpha_i||u_i|$ for any $\alpha_0, \dots , \alpha_n$.

(c) If u, $v \in U$ then

$$u^+ \wedge v^+ \le (u + v)^+ \le u^+ + v^+.$$

(d) If $u_0, \dots , u_m, v_0, \dots , v_n \in U^+$ and $\sum_{i=0}^m u_i = \sum_{j=0}^n v_j$, then there is a family $\langle w_{ij}\rangle_{i\le m,j\le n}$ in U^+ such that $\sum_{i=0}^m w_{ij} = v_j$ for every $j \le n$ and $\sum_{j=0}^n w_{ij} = u_i$ for every $i \le m$.

proof (a)

$$\begin{aligned} u \wedge (v + w) &\le [(u + w) \wedge (v + w)] \wedge u \\ &\le [(u \wedge v) + w] \wedge [(u \wedge v) + u] = (u \wedge v) + (u \wedge w). \end{aligned}$$

(b)(i)($\boldsymbol{\alpha}$) A simple induction, using (a) for the inductive step, shows that if $v_0, \dots , v_m$, $w_0, \dots , w_n$ are non-negative then $\sum_{i=0}^m v_i \wedge \sum_{j=0}^n w_j \le \sum_{i=0}^m \sum_{j=0}^n v_i \wedge w_j$. **($\boldsymbol{\beta}$)** Next, if $u \wedge v = 0$ then

$$(u - v)^+ = u - (u \wedge v) = u, \quad (u - v)^- = (v - u)^+ = v - (v \wedge u) = v,$$

$$|u - v| = (u - v)^+ + (u - v)^- = u + v = |u + v|,$$

so if $|u| \wedge |v| = 0$ then

$$\begin{aligned} (u^+ + v^+) \wedge (u^- + v^-) &\le (u^+ \wedge u^-) + (u^+ \wedge v^-) + (v^+ \wedge u^-) + (v^+ \wedge v^-) \\ &\le 0 + (|u| \wedge |v|) + (|v| \wedge |u|) + 0 = 0 \end{aligned}$$

and

$$|u + v| = |(u^+ + v^+) - (u^- + v^-)| = u^+ + v^+ + u^- + v^- = |u| + |v|.$$

($\boldsymbol{\gamma}$) Finally, if $|u| \wedge |v| = 0$ and α, $\beta \in \mathbb{R}$,

$$|\alpha u| \wedge |\beta v| = |\alpha||u| \wedge |\beta||v| \le (|\alpha| + |\beta|)|u| \wedge (|\alpha| + |\beta|)|v| = (|\alpha| + |\beta|)(|u| \wedge |v|) = 0.$$

(ii) We may therefore proceed by induction. The case $n = 0$ is trivial. For the inductive step to $n + 1$, setting $u'_i = \alpha_i u_i$ we have $|u'_i| \wedge |u'_j| = 0$ for all $i \ne j$, by (i-γ). By (i-α),

$$|u'_{n+1}| \wedge |\sum_{i=0}^n u'_i| \le |u'_{n+1}| \wedge \sum_{i=0}^n |u'_i| \le \sum_{i=0}^n |u'_{n+1}| \wedge |u'_i| = 0,$$

so by (i-β) and the inductive hypothesis

$$|\sum_{i=0}^{n+1} u'_i| = |u'_{n+1}| + |\sum_{i=0}^n u'_i| = \sum_{i=0}^{n+1} |u'_i|$$

as required.

(c) By 352E,

$$u^+ \wedge v^+ = (u \vee 0) \wedge (v \vee 0) = (u \wedge v) \vee 0.$$

Now

$$u \wedge v = \frac{1}{2}(u + v - |u - v|) \le \frac{1}{2}(u + v + |u + v|) = (u + v)^+,$$

and of course $0 \le (u + v)^+$, so $u^+ \wedge v^+ \le (u + v)^+$.

For the other inequality we need only note that $u + v \le u^+ + v^+$ (because $u \le u^+$, $v \le v^+$) and $0 \le u^+ + v^+$.

(d) Write w for the common value of $\sum_{i=0}^m u_i$ and $\sum_{j=0}^n v_j$.

Induce on $k = \#(\{(i, j) : i \le m, j \le n, u_i \wedge v_j > 0\})$. If $k = 0$, that is, $u_i \wedge v_j = 0$ for all i, j, then (by (a), used repeatedly) we must have $w \wedge w = 0$, that is, $w = 0$, and we can take $w_{ij} = 0$ for all i, j. For the inductive step to $k \ge 1$, take i^*, j^* such that $\tilde{w} = u_{i^*} \wedge v_{j^*} > 0$. Set

$$\tilde{u}_{i^*} = u_{i^*} - \tilde{w}, \quad \tilde{u}_i = u_i \text{ for } i \neq i^*,$$

$$\tilde{v}_{j^*} = v_{j^*} - \tilde{w}, \quad \tilde{v}_j = v_j \text{ for } j \neq j^*.$$

Then $\sum_{i=0}^{m} \tilde{u}_i = \sum_{j=0}^{n} \tilde{v}_j = w - \tilde{w}$ and $\tilde{u}_i \wedge \tilde{v}_j \leq u_i \wedge v_j$ for all i, j, while $\tilde{u}_{i^*} \wedge \tilde{v}_{j^*} = 0$; so that

$$\#(\{(i,j) : \tilde{u}_i \wedge v_j > 0\}) < k.$$

By the inductive hypothesis, there are $\tilde{w}_{ij} \geq 0$, for $i \leq m$ and $j \leq n$, such that $\tilde{u}_i = \sum_{j=0}^{n} \tilde{w}_{ij}$ for each i, $\tilde{v}_j = \sum_{i=0}^{m} \tilde{w}_{ij}$ for each j. Set $w_{i^*j^*} = \tilde{w}_{i^*j^*} + \tilde{w}$, $w_{ij} = \tilde{w}_{ij}$ for $(i,j) \neq (i^*,j^*)$; then $u_i = \sum_{j=0}^{n} w_{ij}$, $v_j = \sum_{i=0}^{m} w_{ij}$ so the induction proceeds.

352G Riesz homomorphisms: Proposition Let U be a Riesz space, V a partially ordered linear space and $T : U \to V$ a linear operator. Then the following are equiveridical:

(i) T is a Riesz homomorphism in the sense of 351H;

(ii) $(Tu)^+ = \sup\{Tu, 0\}$ is defined and equal to $T(u^+)$ for every $u \in U$;

(iii) $\sup\{Tu, -Tu\}$ is defined and equal to $T|u|$ for every $u \in U$;

(iv) $\inf\{Tu, Tv\} = 0$ in V whenever $u \wedge v = 0$ in U.

proof (i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv) are special cases of 351Hc. For (iii)$\Rightarrow$(ii) we have

$$\sup\{Tu, 0\} = \frac{1}{2}Tu + \sup\{\frac{1}{2}Tu, -\frac{1}{2}Tu\} = \frac{1}{2}Tu + \frac{1}{2}T|u| = T(u^+).$$

For (ii)$\Rightarrow$(i), argue as follows. If (ii) is true and u, $v \in U$, then

$$Tu \wedge Tv = \inf\{Tu, Tv\} = Tu + \inf\{0, Tv - Tu\} = Tu - \sup\{0, T(u - v)\}$$

is defined and equal to

$$Tu - T((u - v)^+) = T(u - (u - v)^+) = T(u \wedge v).$$

Inducing on n,

$$\inf_{i \leq n} Tu_i = T(\inf_{i \leq n} u_i)$$

for all $u_0, \dots, u_n \in U$; in particular, if $\inf_{i \leq n} u_i = 0$ then $\inf_{i \leq n} Tu_i = 0$; which is the definition I gave of Riesz homomorphism.

Finally, for (iv)$\Rightarrow$(ii), we know from (iv) that $0 = \inf\{T(u^+), T(u^-)\}$, so $-T(u^+) = \inf\{0, -Tu\}$ and $T(u^+) = \sup\{0, Tu\}$.

352H Proposition If U and V are Riesz spaces and $T : U \to V$ is a bijective Riesz homomorphism, then T is a partially-ordered-linear-space isomorphism, and $T^{-1} : V \to U$ is a Riesz homomorphism.

proof Use 352G(ii). If $v \in V$, set $u = T^{-1}v$; then $T(u^+) = v^+$ so $T^{-1}(v^+) = u^+ = (T^{-1}v)^+$. Thus T^{-1} is a Riesz homomorphism; in particular, it is order-preserving, so T is an isomorphism for the order structures as well as for the linear structures.

352I Riesz subspaces (a) If U is a partially ordered linear space, a **Riesz subspace** of U is a linear subspace V such that $\sup\{u, v\}$ and $\inf\{u, v\}$ are defined in U and belong to V for every u, $v \in V$. In this case they are the supremum and infimum of $\{u, v\}$ in V, so V, with the induced order and linear structure, is a Riesz space in its own right, and the embedding map $u \mapsto u : V \to U$ is a Riesz homomorphism.

(b) Generally, if U is a Riesz space, V is a partially ordered linear space and $T : U \to V$ is a Riesz homomorphism, then $T[U]$ is a Riesz subspace of V (because, by 351Hc, $Tu \vee Tu' = T(u \vee u')$, $T(u \wedge u') = Tu \wedge Tu'$ belong to $T[U]$ for all u, $u' \in U$).

(c) If U is a Riesz space and V is a linear subspace of U, then V is a Riesz subspace of U iff $|u| \in V$ for every $u \in V$. **P** In this case,

$$u \vee v = \frac{1}{2}(u + v + |u - v|), \quad u \wedge v = \frac{1}{2}(u + v - |u - v|$$

belong to V for all u, $v \in V$. **Q**

352J Solid subsets (a) If U is a Riesz space, a subset A of U is solid (in the sense of 351I) iff $v \in A$ whenever $u \in A$ and $|v| \le |u|$. **P** (α) If A is solid, $u \in V$ and $|v| \le |u|$, then there is some $w \in A$ such that $-w \le u \le w$; in this case $|v| \le |u| \le w$ and $-w \le v \le w$ and $v \in A$. (β) Suppose that A satisfies the condition. If $u \in A$, then $|u| \in A$ and $-|u| \le u \le |u|$. If $w \in A$ and $-w \le u \le w$ then $-u \le w$, $|u| \le w = |w|$ and $u \in A$. Thus A is solid. **Q** In particular, if A is solid, then $v \in A$ iff $|v| \in A$.

For any set $A \subseteq U$, the set

$$\{u : \text{there is some } v \in A \text{ such that } |u| \le |v|\}$$

is a solid subset of U, the smallest solid set including A; we call it the **solid hull** of A in U.

Any solid linear subspace of U is a Riesz subspace (use 352Fc). If $V \subseteq U$ is a Riesz subspace, then the solid hull of V in U is

$$\{u : \text{there is some } v \in V \text{ such that } |u| \le v\}$$

and is a solid linear subspace of U.

(b) If T is a Riesz homomorphism from a Riesz space U to a partially ordered linear space V, then its kernel W is a solid linear subspace of U. **P** If $u \in W$ and $|v| \le |u|$, then $T|u| = \sup\{Tu, T(-u)\} = 0$, while $-|u| \le v \le |u|$, so that $-0 \le Tv \le 0$ and $v \in W$. **Q** Now the quotient space U/W, as defined in 351J, is a Riesz space, and is isomorphic, as partially ordered linear space, to the Riesz space $T[U]$. **P** Because U/W is the linear space quotient of V by the kernel of the linear operator T, we have an induced linear space isomorphism $S : U/W \to T[U]$ given by setting $Su^\bullet = Tu$ for every $u \in U$. If $p \ge 0$ in U/W there is a $u \in U^+$ such that $u^\bullet = p$ (351J), so that $Sp = Tu \ge 0$. On the other hand, if $p \in U/W$ and $Sp \ge 0$, take $u \in V$ such that $u^\bullet = p$. We have

$$T(u^+) = (Tu)^+ = (Sp)^+ = Sp = Tu,$$

so that $T(u^-) = Tu^+ - Tu = 0$ and $u^- \in W$, $p = (u^+)^\bullet \ge 0$. Thus $Sp \ge 0$ iff $p \ge 0$, and S is a partially-ordered-linear-space isomorphism. **Q**

(c) Because a subset of a Riesz space is a solid linear subspace iff it is the kernel of a Riesz homomorphism, such subspaces are sometimes called **ideals**.

352K Products If $\langle U_i\rangle_{i\in I}$ is any family of Riesz spaces, then the product partially ordered linear space $U = \prod_{i\in I} U_i$ (351L) is a Riesz space, with

$$u \vee v = \langle u(i) \vee v(i)\rangle_{i\in I}, \quad u \wedge v = \langle u(i) \wedge v(i)\rangle_{i\in I}, \quad |u| = \langle |u(i)|\rangle_{i\in I}$$

for all u, $v \in U$.

352L Theorem Let U be any Riesz space. Then there are a set X, a filter $\mathcal{F}$ on X and a Riesz subspace of the Riesz space $\mathbb{R}^X|\mathcal{F}$ (351M) which is isomorphic, as Riesz space, to U.

proof By 351Q, we can find such X and $\mathcal{F}$ and an injective Riesz homomorphism $T : U \to \mathbb{R}^X|\mathcal{F}$. By 352K, or otherwise, $\mathbb{R}^X$ is a Riesz space; by 352Ib, $\mathbb{R}^X|\mathcal{F}$ is a Riesz space (recall that it is a quotient of $\mathbb{R}^X$ by a solid linear subspace, as explained in 351M); by 352I, $T[U]$ is a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and by 352H it is isomorphic to U.

352M Corollary Any identity involving the operations $+$, $-$, $\vee$, $\wedge$, $^+$, $^-$, $|\ |$ and scalar multiplication, and the relation $\le$, which is valid in $\mathbb{R}$, is valid in all Riesz spaces.

Remark I suppose some would say that a strict proof of this must begin with a formal description of what the phrase 'any identity involving the operations...' means. However I think it is clear in practice what is involved. Given a proposed identity like

$$0 \le \sum_{i=0}^n |\alpha_i||u_i| - |\sum_{i=0}^n \alpha_i u_i| \le \sum_{i\ne j}(|\alpha_i| + |\alpha_j|)(|u_i| \wedge |u_j|),$$

(compare 352Fb), then to check that it is valid in all Riesz spaces you need only check (i) that it is true in $\mathbb{R}$ (ii) that it is true in $\mathbb{R}^X$ (iii) that it is true in any $\mathbb{R}^X|\mathcal{F}$ (iv) that it is true in any Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and you can hope that the arguments for (ii)-(iv) will be nearly trivial, since (ii) is generally nothing but a coordinate-by-coordinate repetition of (i), and (iii) and (iv) involve only transformations of the formula by Riesz homomorphisms which preserve its structure.

352N Order-density and order-continuity Let U be a Riesz space.

(a) Definition A Riesz subspace V of U is **quasi-order-dense** if for every $u>0$ in U there is a $v \in V$ such that $0<v \le u$; it is **order-dense** if $u=\sup\{v : v \in V,\, 0 \le v \le u\}$ for every $u \in U^+$.

(b) If U is a Riesz space and V is a quasi-order-dense Riesz subspace of U, then the embedding $V \subseteq U$ is order-continuous. **P** Let $A \subseteq V$ be a non-empty set such that $\inf A=0$ in V. **?** If 0 is not the infimum of A in U, then there is a $u>0$ such that u is a lower bound for A in U; now there is a $v \in V$ such that $0<v \le u$, and v is a lower bound for A in V which is strictly greater than 0. **X** Thus $0=\inf A$ in U. As A is arbitrary, the embedding is order-continuous. **Q**

(c) (i) If $V \subseteq U$ is an order-dense Riesz subspace, it is quasi-order-dense. (ii) If V is a quasi-order-dense Riesz subspace of U and W is a quasi-order-dense Riesz subspace of V, then W is a quasi-order-dense Riesz subspace of U. (iii) If V is an order-dense Riesz subspace of U and W is an order-dense Riesz subspace of V, then W is an order-dense Riesz subspace of U. (Use (b).) (iv) If V is a quasi-order-dense solid linear subspace of U and W is a quasi-order-dense Riesz subspace of U then $V \cap W$ is quasi-order-dense in V, therefore in U.

(d) I ought somewhere to remark that a Riesz homomorphism, being a lattice homomorphism, is order-continuous iff it preserves arbitrary suprema and infima; compare 313L(b-iv) and (b-v).

(e) If V is a Riesz subspace of U, we say that it is **regularly embedded** in U if the identity map from V to U is order-continuous, that is, whenever $A \subseteq V$ is non-empty and has infimum 0 in V, then 0 is still its greatest lower bound in U. Thus quasi-order-dense Riesz subspaces and solid linear subspaces are regularly embedded.

352O Bands Let U be a Riesz space.

(a) Definition A **band** or **normal subspace** of U is an order-closed solid linear subspace.

(b) If $V \subseteq U$ is a solid linear subspace then it is a band iff $\sup A \in V$ whenever $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U. **P** Of course the condition is necessary; I have to show that it is sufficient. (i) Let $A \subseteq V$ be any non-empty upwards-directed set with a supremum in V. Take any $u_0 \in A$ and set $A_1=\{u-u_0 : u \in A,\, u \ge u_0\}$. Then A_1 is a non-empty upwards-directed subset of V^+, and $u_0+A_1=\{u : u \in A,\, u \ge u_0\}$ has the same upper bounds as A, so $\sup A_1=\sup A-u_0$ is defined in U and belongs to V. Now $\sup A=u_0+\sup A_1$ also belongs to V. (ii) If $A \subseteq V$ is non-empty, downwards-directed and has an infimum in U, then $-A \subseteq V$ is upwards-directed, so $\inf A=\sup(-A)$ belongs to V. Thus V is order-closed. **Q**

(c) For any set $A \subseteq U$ set $A^\perp=\{v : v \in U,\, |u| \wedge |v|=0$ for every $u \in A\}$. Then $A^\perp$ is a band. **P** (i) Of course $0 \in A^\perp$. (ii) If v, $w \in A^\perp$ and $u \in A$, then

$$|u| \wedge |v+w| \le (|u| \wedge |v|)+(|u| \wedge |w|)=0,$$

so $v+w \in A^\perp$. (iii) If $v \in A^\perp$ and $|w| \le |v|$ then

$$0 \le |u| \wedge |w| \le |u| \wedge |v|=0$$

for every $u \in A$, so $w \in A^\perp$. (iv) If $v \in A^\perp$ then $nv \in A^\perp$ for every n, by (ii). So if $\alpha \in \mathbb{R}$, take $n \in \mathbb{N}$ such that $|\alpha| \le n$; then

$$|\alpha v|=|\alpha||v| \le n|v| \in A^\perp$$

and $\alpha v \in A^\perp$. Thus $A^\perp$ is a solid linear subspace of U. (v) If $B \subseteq (A^\perp)^+$ is non-empty and upwards-directed and has a supremum w in U, then

$$|u| \wedge |w|=|u| \wedge w=\sup_{v \in B} |u| \wedge v=0$$

by 352Ea, so $w \in A^\perp$. Thus $A^\perp$ is a band. **Q**

(d) For any $A \subseteq U$, $A \subseteq (A^\perp)^\perp$. Also $B^\perp \subseteq A^\perp$ whenever $A \subseteq B$. So

$$A^{\perp\perp\perp} \subseteq A^\perp \subseteq A^{\perp\perp\perp}$$

and $A^\perp = A^{\perp\perp\perp}$.

(e) If W is another Riesz space and $T : U \to W$ is an order-continuous Riesz homomorphism then its kernel is a band. (For $\{0\}$ is order-closed in W and the inverse image of an order-closed set by an order-continuous order-preserving function is order-closed.)

352P Complemented bands Let U be a Riesz space. A band $V \subseteq U$ is **complemented** if $V^{\perp\perp} = V$, that is, if V is of the form $A^\perp$ for some $A \subseteq U$ (352Od). In this case its **complement** is the complemented band $V^\perp$.

352Q Theorem In any Riesz space U, the set $\mathfrak{C}$ of complemented bands forms a Dedekind complete Boolean algebra, with

$$V \cap_{\mathfrak{C}} W = V \cap W, \quad V \cup_{\mathfrak{C}} W = (V+W)^{\perp\perp},$$

$$1_{\mathfrak{C}} = U, \quad 0_{\mathfrak{C}} = \{0\}, \quad 1_{\mathfrak{C}} \setminus_{\mathfrak{C}} V = V^\perp,$$

$$V \subseteq_{\mathfrak{C}} W \iff V \subseteq W$$

for V, $W \in \mathfrak{C}$.

proof To show that $\mathfrak{C}$ is a Boolean algebra, I use the identification of Boolean algebras with complemented distributive lattices (311L).

(a) Of course $\mathfrak{C}$ is partially ordered by $\subseteq$. If V, $W \in \mathfrak{C}$ then

$$V \cap W = V^{\perp\perp} \cap W^{\perp\perp} = (V^\perp \cup W^\perp)^\perp \in \mathfrak{C},$$

and $V \cap W$ must be $\inf\{V, W\}$ in C. The map $V \mapsto V^\perp : \mathfrak{C} \to \mathfrak{C}$ is an order-reversing bijection, so that $V \subseteq W$ iff $W^\perp \subseteq V^\perp$ and $V \vee W = \sup\{V, W\}$ will be $(V^\perp \cap W^\perp)^\perp$; thus $\mathfrak{C}$ is a lattice. Note also that $V \vee W$ must be the smallest complemented band including $V + W$, that is, it is $(V+W)^{\perp\perp}$.

(b) If V_1, V_2, $W \in \mathfrak{C}$ then $(V_1 \vee V_2) \wedge W = (V_1 \wedge W) \vee (V_2 \wedge W)$. **P** Of course $(V_1 \vee V_2) \wedge W \supseteq (V_1 \wedge W) \vee (V_2 \wedge W)$. **?** Suppose, if possible, that there is a $u \in (V_1 \vee V_2) \cap W \setminus ((V_1 \cap W) \vee (V_2 \cap W))$. Then $u \notin ((V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp)^\perp$, so there is a $v \in (V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp$ such that $u_1 = |u| \wedge |v| > 0$. Now $u_1 \in V_1 \vee V_2 = (V_1^\perp \cap V_2^\perp)^\perp$ so $u_1 \notin V_1^\perp \cap V_2^\perp$; say $u_1 \notin V_j^\perp$, and there is a $v_j \in V_j$ such that $u_2 = u_1 \wedge |v_j| > 0$. In this case we still have $u_2 \in (V_j \cap W)^\perp$, because $u_2 \le |v|$, but also $u_2 \in V_j$ and $u_2 \in W$ because $u_2 \le |u|$; but this means that $u_2 = u_2 \wedge u_2 = 0$, which is absurd. **X** Thus $(V_1 \vee V_2) \wedge W \subseteq (V_1 \wedge W) \vee (V_2 \wedge W)$ and the two are equal. **Q**

(c) Now if $V \in \mathfrak{C}$,

$$V \wedge V^\perp = \{0\}$$

is the least member of $\mathfrak{C}$, because if $v \in V \cap V^\perp$ then $|v| = |v| \wedge |v| = 0$. By 311L, $\mathfrak{C}$ has a Boolean algebra structure, with the Boolean relations described; by 312L, this structure is uniquely defined.

(d) Finally, if $\mathcal{V} \subseteq \mathfrak{C}$ is non-empty, then

$$\bigcap \mathcal{V} = (\bigcup_{V \in \mathcal{V}} V^\perp)^\perp \in \mathfrak{C}$$

and is $\inf \mathcal{V}$ in $\mathfrak{C}$. So $\mathfrak{C}$ is Dedekind complete.

352R Projection bands Let U be a Riesz space.

(a) A **projection band** in U is a set $V \subseteq U$ such that $V + V^\perp = U$. In this case V is a complemented band. **P** If $v \in V^{\perp\perp}$ then v is expressible as $v_1 + v_2$ where $v_1 \in V$, $v_2 \in V^\perp$. Now $|v| = |v_1| + |v_2| \ge |v_2|$ (352Fb), so

$$|v_2| = |v_2| \wedge |v_2| \leq |v_2| \wedge |v| = 0$$

and $v = v_1 \in V$. Thus $V = V^{\perp\perp}$ is a complemented band. **Q** Observe that $U = V^{\perp} + V^{\perp\perp}$ so $V^{\perp}$ is also a projection band.

(b) Because $V \cap V^{\perp}$ is always $\{0\}$, we must have $U = V \oplus V^{\perp}$ for any projection band $V \subseteq U$; accordingly there is a corresponding **band projection** $P_V : U \to U$ defined by setting $P(v + w) = v$ whenever $v \in V$, $w \in V^{\perp}$. In this context I will say that v is the **component** of $v + w$ in V. The kernel of P is $V^{\perp}$, the set of values is V, and $P^2 = P$. Moreover, P is an order-continuous Riesz homomorphism. **P** (i) P is a linear operator because V and $V^{\perp}$ are linear subspaces. (ii) If $v \in V$, $w \in V^{\perp}$ then $|v+w| = |v|+|w|$, by 352Fb, so $P|v + w| = |v| = |P(v + w)|$; consequently P is a Riesz homomorphism (352G). (iii) If $A \subseteq U$ is downwards-directed and has infimum 0, then $Pu \leq u$ for every $u \in A$, so $\inf P[A] = 0$; thus P is order-continuous. **Q**

(c) Note that for any band projection P, and any $u \in U$, we have $|Pu| \wedge |u - Pu| = 0$, so that $|u| = |Pu| + |u - Pu|$ and (in particular) $|Pu| \leq |u|$; consequently $P[W] \subseteq W$ for any solid linear subspace W of U.

(d) A linear operator $P : U \to U$ is a band projection iff $Pu \wedge (u - Pu) = 0$ for every $u \in U^+$. **P** I remarked in (c) that the condition is satisfied for any band projection. Now suppose that P has the property. (i) For any $u \in U^+$, $Pu \geq 0$ and $u - Pu \geq 0$; in particular, P is a positive linear operator. (ii) If u, $v \in U^+$ then $u - Pu \leq (u + v) - P(u + v)$, so

$$Pv \wedge (u - Pu) \leq P(u + v) \wedge ((u + v) - P(u + v)) = 0$$

and $Pv \wedge (u - Pu) = 0$. (iii) If u, $v \in U$ then $|Pv| \leq P|v|$, $|u - Pu| \leq |u| - P|u|$ (because $w \mapsto w - Pw$ is a positive linear operator), so

$$|Pv| \wedge |u - Pu| \leq P|v| \wedge (|u| - P|u|) = 0.$$

(iv) Setting $V = P[U]$, we see that $u - Pu \in V^{\perp}$ for every $u \in U$, so that

$$u = u + (u - Pu) \in V + V^{\perp}$$

for every u, and $U = V + V^{\perp}$; thus V is a projection band. (v) Since $Pu \in V$ and $u - Pu \in V^{\perp}$ for every $u \in U$, P is the band projection onto V. **Q**

352S Proposition Let U be any Riesz space.

(a) The family $\mathfrak{B}$ of projection bands in U is a subalgebra of the Boolean algebra $\mathfrak{C}$ of complemented bands in U.

(b) For $V \in \mathfrak{B}$ let $P_V : U \to V$ be the corresponding projection. Then for any $e \in U^+$,

$$P_{V\cap W}e = P_V e \wedge P_W e = P_V P_W e, \qquad P_{V\vee W}e = P_V e \vee P_W e$$

for all V, $W \in \mathfrak{B}$. In particular, band projections commute.

(c) If $V \in \mathfrak{B}$ then the algebra of projection bands of V is just the principal ideal of $\mathfrak{B}$ generated by V.

proof (a) Of course $0_{\mathfrak{C}} = \{0\} \in \mathfrak{B}$. If $V \in \mathfrak{B}$ then $V^{\perp} = 1_{\mathfrak{C}} \setminus V$ belongs to $\mathfrak{B}$. If now W is another member of $\mathfrak{B}$, then

$$(V \cap W) + (V \cap W)^{\perp} \supseteq (V \cap W) + V^{\perp} + W^{\perp}.$$

But if $u \in U$ then we can express u as $v + v'$, where $v \in V$ and $v' \in V^{\perp}$, and v as $w + w'$, where $w \in W$ and $w' \in W^{\perp}$; and as $|w| \leq |v|$, we also have $w \in V$, so that

$$u = w + v' + w' \in (V \cap W) + V^{\perp} + W^{\perp}.$$

This shows that $V \cap W \in \mathfrak{B}$. Thus $\mathfrak{B}$ is closed under intersection and complements and is a subalgebra of $\mathfrak{C}$.

(b) If V, $W \in \mathfrak{B}$ and $e \in U^+$, we have $e = e_1 + e_2 + e_3 + e_4$ where

$$e_1 = P_W P_V e \in V \cap W, \quad e_2 = P_{W^{\perp}} P_V e \in V \cap W^{\perp},$$

$$e_3 = P_W P_{V^\perp} e \in V^\perp \cap W, \quad e_4 = P_{W^\perp} P_{V^\perp} e \in V^\perp \cap W^\perp,$$

$$e_1 + e_2 = P_V e, \quad e_1 + e_3 = P_W e.$$

Now $e_2 + e_3 + e_4$ belongs to $(V \cap W)^\perp$, so e_1 must be the component of e in $V \cap W$; similarly e_4 is the component of e in $V^\perp \cap W^\perp$, and $e_1 + e_2 + e_3$ is the component of e in $V \vee W$. But as $e_2 \wedge e_3 = 0$, we have

$$P_{V \cap W} e = e_1 = (e_1 + e_2) \wedge (e_1 + e_3) = P_V e \wedge P_W e,$$

$$P_{V \vee W} e = e_1 + e_2 + e_3 = (e_1 + e_2) \vee (e_1 + e_3) = P_V e \vee P_W e,$$

as required.

It follows that

$$P_V P_W = P_{V \cap W} = P_{W \cap V} = P_W P_V.$$

(c) If V, $W \in \mathfrak{B}$ and $W \subseteq V$, then of course W is a band in the Riesz space V (because V is order-closed in U, so that for any set $A \subseteq W$ its supremum in U will be its supremum in V). For any $v \in V$, we have an expression of it as $w + w'$, where $w \in W$ and $w' \in W^\perp$, taken in U; but as $|w| + |w'| = |w + w'| = |v| \in V$, w' belongs to V, and is in $W^\perp_V$, the band in V orthogonal to W. Thus $W + W^\perp_V = V$ and W is a projection band in V. Conversely, if W is a projection band in V, then $W^\perp$ (taken in U) includes $W^\perp_V + V^\perp$, so that

$$W + W^\perp \supseteq W + W^\perp_V + V^\perp = V + V^\perp = U$$

and $W \in \mathfrak{B}$.

Thus the algebra of projection bands in V is, as a set, equal to the principal ideal $\mathfrak{B}_V$; because their orderings agree, or otherwise, their Boolean algebra structures coincide.

352T Products again (a) If $U = \prod_{i \in I} U_i$ is a product of Riesz spaces, then for any $J \subseteq I$ we have a subspace

$$V_J = \{u : u \in U,\, u(i) = 0 \text{ for all } i \in I \setminus J\}$$

of U, canonically isomorphic to $\prod_{i \in J} U_i$. Each V_J is a projection band, its complement being $V_{I \setminus J}$; the map $J \mapsto V_J$ is a Boolean homomorphism from $\mathcal{P}I$ to the algebra $\mathfrak{B}$ of projection bands in U, and $\langle V_{\{i\}} \rangle_{i \in I}$ is a partition of unity in $\mathfrak{B}$.

(b) Conversely, if U is a Riesz space and $(V_0, \dots, V_n)$ is a *finite* partition of unity in the algebra $\mathfrak{B}$ of projection bands in U, then every element of U is uniquely expressible as $\sum_{i=0}^n u_i$ where $u_i \in V_i$ for each i. (Induce on n.) This decomposition corresponds to a Riesz space isomorphism between U and $\prod_{i \le n} V_i$.

352U Quotient spaces (a) If U is a Riesz space and V is a solid linear subspace, then the quotient partially ordered linear space U/V (351J) is a Riesz space; if U and W are Riesz spaces and $T : U \to W$ a Riesz homomorphism, then the kernel V of T is a solid linear subspace of U and the Riesz subspace $T[U]$ of W is isomorphic to U/V (352Jb).

(b) Suppose that U is a Riesz space and V a solid linear space. Then the canonical map from U to U/V is order-continuous iff V is a band. **P** (i) If $u \mapsto u^\bullet$ is order-continuous, its kernel V is a band, by 352Oe. (ii) If V is a band, and $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, let $p \in U/V$ be any lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$. Then $((u - w)^-)^\bullet = (u^\bullet - w^\bullet)^- = 0$, that is, $(w - u)^+ = (u - w)^- \in V$ for every $u \in A$. But this means that

$$w^+ = \sup_{u \in A} (w - u)^+ \in V, \quad p^+ = (w^+)^\bullet = 0,$$

that is, $p \le 0$. As p is arbitrary, $\inf_{u \in A} u^\bullet = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is order-continuous. **Q**

352V Principal bands Let U be a Riesz space. Evidently the intersection of any family of Riesz subspaces of U is a Riesz subspace, the intersection of any family of solid linear subspaces is a solid linear subspace, the intersection of any family of bands is a band; we may therefore speak of the band generated by a subset A of U, the intersection of all the bands including A. Now we have the following description of the band generated by a single element.

Lemma Let U be a Riesz space.

(a) If $A \subseteq U^+$ is upwards-directed and $2w \in A$ for every $w \in A$, then an element u of U belongs to the band generated by A iff $|u| = \sup_{w\in A} |u| \wedge w$.

(b) If $u \in U$ and $w \in U^+$, then u belongs to the band of U generated by w iff $|u| = \sup_{n\in\mathbb{N}} |u| \wedge nw$.

proof (a) Let W be the band generated by A and W' the set of elements of U satisfying the condition.

(i) If $u \in W'$ then $|u| \wedge w \in W$ for every $w \in A$, because W is a solid linear subspace; because W is also order-closed, $|u|$ and u belong to W. Thus $W' \subseteq W$.

(ii) Now W' is a band.

$\mathbf{P}$**(α)** If $u \in W'$ and $|v| \le |u|$ then

$$\sup_{w\in A} |v| \wedge w = \sup_{w\in A} |v| \wedge |u| \wedge w = |v| \wedge \sup_{w\in A} |u| \wedge w = |v| \wedge |u| = |v|$$

by 352Ea, so $v \in W'$.

(β) If u, $v \in W'$ then, for any w_1, $w_2 \in A$ there is a $w \in A$ such that $w \ge w_1 \vee w_2$. Now $w_1 + w_2 \le 2w \in A$, and

$$(|u| + |v|) \wedge 2w \ge (|u| \wedge w_1) + (|v| \wedge w_2).$$

So any upper bound for $\{(|u|+|v|)\wedge w : w \in A\}$ must also be an upper bound for $\{|u|\wedge w : w \in A\}+\{|v|\wedge w : w \in A\}$ and therefore greater than or equal to

$$\sup(\{|u| \wedge w : w \in A\} + \{|v| \wedge w : w \in A\}) = \sup_{w\in A} |u| \wedge w + \sup_{w\in A} |v| \wedge w = |u| + |v|$$

(351Dc). But this means that $\sup_{w\in A}(|u| + |v|) \wedge w$ must be $|u| + |v|$, and $|u| + |v|$ belongs to W'; it follows from (i) that $u + v$ belongs to W'.

(γ) Just as in 352Oc, we now have

$$nu \in W' \text{ for every } n \in \mathbb{N},\ u \in W',$$

and therefore $\alpha u \in W'$ for every $\alpha \in \mathbb{R}$, $u \in W'$, since $|\alpha u| \le |nu|$ if $|\alpha| \le n$. Thus W' is a solid linear subspace of U.

(δ) Now suppose that $C \subseteq (W')^+$ has a supremum v in U. Then any upper bound of $\{v\wedge w : w \in A\}$ must also be an upper bound of $\{u \wedge w : u \in C,\ w \in A\}$ and greater than or equal to $u = \sup_{w\in A} u \wedge w$ for every $u \in C$, therefore greater than or equal to $v = \sup C$. Thus $v = \sup_{w\in A} v \wedge w$ and $v \in W'$. As C is arbitrary, W' is a band (352Ob). $\mathbf{Q}$

(iii) Since A is obviously included in W', W' must include W; putting this together with (i), $W = W'$, as claimed.

(b) Apply (a) with $A = \{nw : n \in \mathbb{N}\}$.

352W f-algebras Some of the most important Riesz spaces have multiplicative structures as well as their order and linear structures. A particular class of these structures appears sufficiently often for it to be useful to develop a little of its theory. The following definition is a common approach.

(a) Definition An **f-algebra** is a Riesz space U with a multiplication $\times : U \times U \to U$ such that

$$u \times (v \times w) = (u \times v) \times w,$$

$$(u + v) \times w = (u \times w) + (v \times w), \quad u \times (v + w) = (u \times v) + (u \times w),$$

$$\alpha(u \times v) = (\alpha u) \times v = u \times (\alpha v)$$

for all u, v, $w \in U$ and $\alpha \in \mathbb{R}$, and

$$u \times v \ge 0 \text{ whenever } u,\ v \ge 0,$$

$$\text{if } u \wedge v = 0 \text{ then } (u \times w) \wedge v = (w \times u) \wedge v = 0 \text{ for every } w \ge 0.$$

An f-algebra is **commutative** if $u \times v = v \times u$ for all u, v.

(b) Let U be an f-algebra.

(i) If $u \wedge v = 0$ in U, then $u \times v = 0$. **P** $u \wedge (u \times v) = 0$ so $(u \times v) \wedge (u \times v) = 0$. **Q**

(ii) $u \times u \geq 0$ for every $u \in U$. **P**

$$\begin{aligned}(u^+ - u^-) \times (u^+ - u^-) &= u^+ \times u^+ - u^+ \times u^- - u^- \times u^+ + u^- \times u^- \\ &= u^+ \times u^+ + u^- \times u^- \geq 0. \textbf{ Q}\end{aligned}$$

(iii) If u, $v \in U$ then $|u \times v| = |u| \times |v|$. **P** $u^+ \times v^+$, $u^+ \times v^-$, $u^- \times v^+$ and $u^+ \times u^-$ are disjoint, so

$$\begin{aligned}|u \times v| &= |u^+ \times v^+ - u^+ \times v^- - u^- \times v^+ + u^- \times v^-| \\ &= u^+ \times v^+ + u^+ \times v^- + u^- \times v^+ + u^- \times v^- \\ &= |u| \times |v|\end{aligned}$$

by 352Fb. **Q**

(iv) If $v \in U^+$ the maps $u \mapsto u \times v$, $u \mapsto v \times u : U \to U$ are Riesz homomorphisms. **P** The first four clauses of the definition in (a) ensure that they are linear operators. If $u \in U$, then

$$|u| \times v = |u \times v|, \quad v \times |u| = |v \times u|$$

by (iii), so we have Riesz homomorphisms, by 352G(iii). **Q**

(c) Let $\langle U_i \rangle_{i \in I}$ be a family of f-algebras, with Riesz space product U (352K). If we set $u \times v = \langle u(i) \times v(i) \rangle_{i \in I}$ for all u, $v \in U$, then U becomes an f-algebra.

352X Basic exercises >(a) Let U be any Riesz space. Show that $|u^+ - v^+| \leq |u - v|$ for all u, $v \in U$.

>(b) Let U, V be a Riesz spaces and $T : U \to V$ a linear operator. Show that the following are equiveridical: (i) T is a Riesz homomorphism; (ii) $T(u \vee v) = Tu \vee Tv$ for all u, $v \in U$; (iii) $T(u \wedge v) = Tu \wedge Tv$ for all u, $v \in U$; (iv) $|Tu| = T|u|$ for every $u \in U$.

(c) Let U be a Riesz space and V a solid linear subspace; for $u \in U$ write $u^\bullet$ for the corresponding element of U/V. Show that if $A \subseteq U$ is solid then $\{u^\bullet : u \in A\}$ is solid in U/W.

(d) Let U and V be Riesz spaces and $T : U \to V$ a Riesz homomorphism with kernel W. Show that if W is a band in U and $T[U]$ is regularly embedded in V then T is order-continuous.

(e) Give $U = \mathbb{R}^2$ its lexicographic ordering (351Xa). Show that it has a band V which is not complemented.

(f) Let U be a Riesz space, $\mathfrak{C}$ the algebra of complemented bands in U. Show that for any $V \in \mathfrak{C}$ the algebra of complemented bands of V is just the principal ideal of $\mathfrak{C}$ generated by V.

>(g) Let $U = C([0,1])$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$, with its usual linear and order structures, so that it is a Riesz subspace of $\mathbb{R}^{[0,1]}$. Set $V = \{u : u \in U, u(t) = 0 \text{ if } t \leq \frac{1}{2}\}$. Show that V is a band in U and that $V^\perp = \{u : u(t) = 0 \text{ if } t \geq \frac{1}{2}\}$, so that V is complemented but is not a projection band.

(h) Show that the Boolean homomorphism $J \mapsto V_J : \mathcal{P}I \to \mathfrak{B}$ of 352Ta is order-continuous.

(i) Let U be a Riesz space and $A \subseteq U^+$ an upwards-directed set. Show that the band generated by A is $\{u : |u| = \sup_{n \in \mathbb{N}, w \in A} |u| \wedge nw\}$.

>(j) (i) Let X be any set. Setting $(u \times v)(x) = u(x)v(x)$ for u, $v \in \mathbb{R}^X$, $x \in X$, show that $\mathbb{R}^X$ is a commutative f-algebra. (ii) With the same definition of $\times$, show that $\ell^\infty(X)$ is an f-algebra. (iii) If X is a topological space, show that $C(X)$, $C_b(X)$ are f-algebras. (iv) If (X, Σ, μ) is a measure space, show that $L^0(\mu)$, $L^\infty(\mu)$ (§241, §243) are f-algebras.

(k) Let $U \subseteq \mathbb{R}^{\mathbb{Z}}$ be the set of sequences u such that $\{n : u(n) \neq 0\}$ is bounded above in $\mathbb{Z}$. For u, $v \in U$ (i) say that $u \leq v$ if either $u = v$ or there is an $n \in \mathbb{Z}$ such that $u(n) < v(n)$, $u(i) = v(i)$ for every $i > n$ (ii) say that $(u * v)(n) = \sum_{i=-\infty}^{\infty} u(i)v(n-i)$ for every $n \in \mathbb{Z}$. Show that U is an f-algebra under this ordering and multiplication.

(l) Let U be an f-algebra. (i) Show that any complemented band of U is an ideal in the ring $(U, +, \times)$. (ii) Show that if $P : U \to U$ is a band projection, then $P(u \times v) = Pu \times Pv$ for every u, $v \in U$.

(m) Let U be an f-algebra with multiplicative identity e. Show that $u - \gamma e \leq \dfrac{1}{\gamma}u^2$ for every $u \in U$, $\gamma > 0$. (*Hint*: $(u^+ - \gamma e)^2 \geq 0$.)

352 Notes and comments In this section we begin to see a striking characteristic of the theory of Riesz spaces: repeated reflections of results in Boolean algebra. Without spelling out a complete list, I mention the distributive laws (313Bc, 352Ea) and the behaviour of order-continuous homomorphisms (313Pa, 352N, 352Oe, 352Ub, 352Xd). Riesz subspaces correspond to subalgebras, solid linear subspaces to ideals and Riesz homomorphisms to Boolean homomorphisms. We even have a correspondence, though a weaker one, between the representation theorems available; every Boolean algebra is isomorphic to a subalgebra of a power of $\mathbb{Z}_2$ (311D-311E), while every Riesz space is isomorphic to a Riesz subspace of a quotient of a power of $\mathbb{R}$ (352L). It would be a closer parallel if every Riesz space were embeddable in some $\mathbb{R}^X$; I must emphasize that the differences are as important as the agreements. Subspaces of $\mathbb{R}^X$ are of great importance, but are by no means adequate for our needs. And of course the details – for instance, the identities in 352D-352F, or 352V – frequently involve new techniques in the case of Riesz spaces. Elsewhere, as in 352G, I find myself arguing rather from the opposite side, when applying results from the theory of general partially ordered linear spaces, which has little to do with Boolean algebra.

In the theory of bands in Riesz spaces – corresponding to order-closed ideals in Boolean algebras – we have a new complication in the form of bands which are not complemented, which does not arise in the Boolean algebra context; but it disappears again when we come to specialize to Archimedean Riesz spaces (353B). (Similarly, order-density and quasi-order-density coincide in both Boolean algebras (313K) and Archimedean Riesz spaces (353A).) Otherwise the algebra of complemented bands in a Riesz space looks very like the algebra of order-closed ideals in a Boolean algebra (314Yh, 352Q). The algebra of projection bands in a Riesz space (352S) would correspond, in a Boolean algebra, to the algebra itself.

I draw your attention to 352H. The result is nearly trivial, but it amounts to saying that the theory of Riesz spaces will be 'algebraic', like the theories of groups or linear spaces, rather than 'analytic', like the theories of partially ordered linear spaces or topological spaces, in which we can have bijective morphisms which are not isomorphisms.

353 Archimedean and Dedekind complete Riesz spaces

I take a few pages over elementary properties of Archimedean and Dedekind (σ)-complete Riesz spaces.

353A Proposition Let U be an Archimedean Riesz space. Then every quasi-order-dense Riesz subspace of U is order-dense.

proof Let $V \subseteq U$ be a quasi-order-dense Riesz subspace, and $u \geq 0$ in U. Set $A = \{v : v \in V, v \leq u\}$. **?** Suppose, if possible, that u is not the least upper bound of A. Then there is a $u_1 < u$ such that $v \leq u_1$ for every $v \in A$. Because $0 \in A$, $u_1 \geq 0$. Because V is quasi-order-dense, there is a $v > 0$ in V such that $v \leq u - u_1$. Now $nv \leq u_1$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given $nv \leq u_1$, then $(n+1)v \leq u_1 + v \leq u$, so $(n+1)v \in A$ and $(n+1)v \leq u_1$. Thus the induction proceeds. **Q** But this is impossible, because $v > 0$ and U is supposed to be Archimedean. **X**

So $u = \sup A$. As u is arbitrary, V is order-dense.

353B Proposition Let U be an Archimedean Riesz space. Then
(a) for every $A \subseteq U$, the band generated by A is $A^{\perp\perp}$,
(b) every band in U is complemented.

proof (a) Let V be the band generated by A. Then V is surely included in $A^{\perp\perp}$, because this is a band including A (352O). **?** Suppose, if possible, that $V \neq A^{\perp\perp}$. Then there is a $w \in A^{\perp\perp} \setminus V$, so that $|w| \notin V$. Set $B = \{v : v \in V, v \leq |w|\}$; then B is upwards-directed and non-empty. Because V is order-closed, $|w|$ cannot be the supremum of A, and there is a $u_0 > 0$ such that $|w| - u_0 \geq v$ for every $v \in B$. Now $u_0 \wedge |w| \neq 0$, so $u_0 \notin A^{\perp}$, and there is a $u_1 \in A$ such that $v = u_0 \wedge |u_1| > 0$. In this case $nv \in B$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given that $nv \in B$, then $nv \leq |w| - u_0$ so $(n+1)v \leq nv + u_0 \leq |w|$; but also $(n+1)v \leq nv + |u_1| \in V$, so $(n+1)v \in B$. **Q** But this means that $nv \leq |w|$ for every n, which is impossible, because U is Archimedean. **X**

(b) Now if $V \subseteq U$ is any band, it is surely the band generated by itself, so is equal to $V^{\perp\perp}$, and is complemented.

Remark We may therefore speak of the **band algebra** of an Archimedean Riesz space, rather than the 'complemented band algebra' (352Q).

353C Corollary Let U be an Archimedean Riesz space and $v \in U$. Let V be the band in U generated by v. If $u \in U$, then $u \in V$ iff there is no w such that $0 < w \leq |u|$ and $w \wedge |v| = 0$.

proof By 353B, $V = \{v\}^{\perp\perp}$. Now, for $u \in U$,

$$u \notin V \iff \exists\, w \in \{v\}^{\perp},\ |u| \wedge |w| > 0 \iff \exists\, w \in \{v\}^{\perp},\ 0 < w \leq |u|.$$

Turning this round, we have the condition announced.

353D Proposition Let U be an Archimedean Riesz space and V an order-dense Riesz subspace of U. Then the map $W \mapsto W \cap V$ is an isomorphism between the band algebras of U and V.

proof If $W \subseteq U$ is a band, then $W \cap V$ is surely a band in V (it is order-closed in V because it is the inverse image of the order-closed set W under the embedding $V \subseteq U$, which is order-continuous by 352Nc and 352Nb). If W, W' are distinct bands in U, say $W' \not\subseteq W$, then $W' \not\subseteq W^{\perp\perp}$, by 353B, so $W' \cap W^{\perp} \neq \{0\}$; because V is order-dense, $V \cap W' \cap W^{\perp} \neq \{0\}$, and $V \cap W' \neq V \cap W$. Thus $W \mapsto W \cap V$ is injective.

If $Q \subseteq V$ is a band in V, then its complementary band in V is just $Q^{\perp} \cap V$, where $Q^{\perp}$ is taken in U. So (because V, like U, is Archimedean, by 351Rc) $Q = (Q^{\perp} \cap V)^{\perp} \cap V = W \cap V$, where $W = (Q^{\perp} \cap V)^{\perp}$ is a band in U. Thus the map $W \mapsto W \cap V$ is an order-preserving bijection between the two band algebras. By 312L, it is a Boolean isomorphism, as claimed.

353E Lemma Let U be an Archimedean Riesz space and $V \subseteq U$ a band such that $\sup\{v : v \in V, 0 \leq v \leq u\}$ is defined for every $u \in U^+$. Then V is a projection band.

proof Take any $u \in U^+$ and set $v = \sup\{v' : v' \in V^+, v' \leq u\}$, $w = u - v$. $v \in V$ because V is a band. Also $w \in V^{\perp}$. **P?** If not, there is some $v_0 \in V^+$ such that $w \wedge v_0 > 0$. Now for any $n \in \mathbb{N}$ we see that

$$nv_0 \leq u \Longrightarrow nv_0 \leq v \Longrightarrow (n+1)v_0 \leq v + w = u,$$

so an induction on n shows that $nv_0 \leq u$ for every n; which is impossible, because U is supposed to be Archimedean. **XQ** Accordingly $u = v + w \in V + V^{\perp}$. As u is arbitrary, $U^+ \subseteq V + V^{\perp}$, and V is a projection band (352R).

353F Lemma Let U be an Archimedean Riesz space. If $A \subseteq U$ is non-empty and bounded above and B is the set of its upper bounds, then $\inf(B - A) = 0$.

proof ? If not, let $w > 0$ be a lower bound for $B - A$. If $u \in A$ and $v \in B$, then $v - u \geq w$, that is, $u \leq v - w$; as u is arbitrary, $v - w \in B$. Take any $u_0 \in A$, $v_0 \in B$. Inducing on n, we see that $v_0 - nw \in B$ for every $n \in \mathbb{N}$, so that $v_0 - nw \geq u_0$, $nw \leq v_0 - u_0$ for every n; but this is impossible, because U is supposed to be Archimedean. **X**

353G Dedekind completeness Recall from 314A that a partially ordered set P is Dedekind (σ)-complete if (countable) non-empty sets with upper and lower bounds have suprema and infima in P. For a Riesz space U, U is Dedekind complete iff every non-empty upwards-directed subset of U^+ with an upper bound has a least upper bound, and is Dedekind σ-complete iff every non-decreasing sequence in U^+ with an upper bound has a least upper bound. **P** (Compare 314Bc.) (i) Suppose that any non-empty upwards-directed order-bounded subset of U^+ has an upper bound, and that $A \subseteq U$ is any non-empty set with an upper bound. Take $u_0 \in A$ and set

$$B = \{u_0 \vee u_1 \vee \ldots \vee u_n - u_0 : u_1, \ldots , u_n \in A\}.$$

Then B is an upwards-directed subset of U^+, and if w is an upper bound of A then $w - u_0$ is an upper bound of B. So $\sup B$ is defined in U, and in this case $u_0 + \sup B = \sup A$. As A is arbitrary, U is Dedekind complete. (ii) Suppose that order-bounded non-decreasing sequences in U^+ have suprema, and that $A \subseteq U$ is any countable non-empty set with an upper bound. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $v_n = \sup_{i\le n} u_i - u_0$ for each n. Then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U^+, and $u_0 + \sup_{n\in\mathbb{N}} v_n = \sup A$. (iii) Finally, still supposing that order-bounded non-decreasing sequences in U^+ have suprema, if $A \subseteq U$ is non-empty, countable and bounded below, $\inf A$ will be defined and equal to $-\sup(-A)$. **Q**

353H Proposition Let U be a Dedekind σ-complete Riesz space.

(a) U is Archimedean.

(b) For any $v \in U$ the band generated by v is a projection band.

(c) If u, $v \in U$, then u is uniquely expressible as $u_1 + u_2$, where u_1 belongs to the band generated by v and $|u_2| \wedge |v| = 0$.

proof (a) Suppose that u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$. Then $nu^+ \le v^+$ for every n, and $A = \{nu^+ : n \in \mathbb{N}\}$ is a countable non-empty upwards-directed set with an upper bound; say $w = \sup A$. Since $A + u^+ \subseteq A$, $w + u^+ = \sup(A + u^+) \le w$, and $u \le u^+ \le 0$. As u, v are arbitrary, U is Archimedean.

(b) Let V be the band generated by v. Take any $u \in U^+$ and set $A = \{v' : v' \in V, 0 \le v' \le u\}$. Then $\{u \wedge n|v| : n \in \mathbb{N}\}$ is a countable set with an upper bound, so has a supremum u_1 say in U. Now u_1 is an upper bound for A. **P** If $v' \in A$, then

$$v' = \sup_{n\in\mathbb{N}} v' \wedge n|v| \le u_1$$

by 352V. **Q** Since $u \wedge n|v| \in A \subseteq V$ for every n, $u_1 \in V$ and $u_1 = \sup A$.

As u is arbitrary, 353E tells us that V is a projection band.

(c) Again let V be the band generated by v. Then $\{v\}^{\perp\perp}$ is a band containing v, so

$$\{v\} \subseteq V \subseteq \{v\}^{\perp\perp}, \quad \{v\}^\perp \supseteq V^\perp \supseteq \{v\}^{\perp\perp\perp} = \{v\}^\perp$$

(352Od), and $V^\perp = \{v\}^\perp$.

Now, if $u \in U$, u is uniquely expressible in the form $u_1 + u_2$ where $u_1 \in V$ and $u_2 \in V^\perp$, by (b). But

$$u_2 \in V^\perp \iff u_2 \in \{v\}^\perp \iff |u_2| \wedge |v| = 0.$$

So we have the result.

353I Proposition In a Dedekind complete Riesz space, all bands are projection bands.

proof Use 353E, noting that the sets $\{v : v \in V, 0 \le v \le u\}$ there are always non-empty, upwards-directed and bounded above, so always have suprema.

353J Proposition (a) Let U be a Dedekind σ-complete Riesz space.

(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind σ-complete.

(ii) If W is a sequentially order-closed Riesz subspace of U then W is Dedekind σ-complete.

(iii) If V is a sequentially order-closed solid linear subspace of U, the canonical map from U to V is sequentially order-continuous, and the quotient Riesz space U/V is also Dedekind σ-complete.

(b) Let U be a Dedekind complete Riesz space.

(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind complete.

(ii) If $W \subseteq U$ is an order-closed Riesz subspace then W is Dedekind complete.

proof (a)(i) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V^+ with an upper bound $v \in V$, then $w = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but as $0 \le w \le v$, $w \in V$ and $w = \sup_{n\in\mathbb{N}} u_n$ in V. Thus V is Dedekind σ-complete.

(ii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in W, then $u = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but because W is sequentially order-closed, $u \in W$ and $u = \sup_{n\in\mathbb{N}} u_n$ in W.

(iii) Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in U with supremum u. Then of course $u^\bullet$ is an upper bound for $A = \{u_n^\bullet : n \in \mathbb{N}\}$ in U/V. Now let p be any other upper bound for A. Express p as $v^\bullet$. Then for each $n \in \mathbb{N}$ we have $u_n^\bullet \le p$, so that $(u_n - v)^+ \in V$. Because V is sequentially order-closed, $(u-v)^+ = \sup_{n\in\mathbb{N}}(u_n - v)^+ \in V$ and $u^\bullet \le p$. Thus $u^\bullet$ is the least upper bound of A.

Similarly, if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum u, then $u^\bullet = \inf_{n\in\mathbb{N}} u_n^\bullet$ in U/V. Thus $u \mapsto u^\bullet$ is sequentially order-continuous.

Now suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $(U/V)^+$ with an upper bound $p \in (U/V)^+$. Let $u \in U^+$ be such that $u^\bullet = p$, and for each $n \in \mathbb{N}$ let $u_n \in U^+$ be such that $u_n^\bullet = p_n$. Set $v_n = u \wedge \sup_{i\le n} u_i$ for each n; then $v_n^\bullet = p_n$ for each n, and $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U. Set $v = \sup_{n\in\mathbb{N}} v_n$; by the last paragraph, $v^\bullet = \sup_{n\in\mathbb{N}} p_n$ in U/V. As $\langle p_n\rangle_{n\in\mathbb{N}}$ is arbitrary, U/V is Dedekind σ-complete, as claimed.

(b) The argument is the same as parts (i) and (ii) of the proof of (a).

353K Proposition Let U be a Riesz space and V a quasi-order-dense Riesz subspace of U which is (in itself) Dedekind complete. Then V is a solid linear subspace of U.

proof Suppose that $v \in V$, $u \in U$ and that $|u| \le |v|$. Consider $A = \{w : w \in V,\ 0 \le w \le u^+\}$. Then A is a non-empty subset of V with an upper bound in V (viz., $|v|$). So A has a supremum v_0 in V. Because the embedding $V \subseteq U$ is order-continuous (352N), v_0 is the supremum of A in U. But as V is order-dense (353A), $v_0 = u^+$ and $u^+ \in V$. Similarly, $u^- \in V$ and $u \in V$. As u, v are arbitrary, V is solid.

353L Order units Let U be a Riesz space.

(a) An element e of U^+ is an **order unit** in U if U is the solid linear subspace of itself generated by e; that is, if for every $u \in U$ there is an $n \in \mathbb{N}$ such that $|u| \le ne$. (For the solid linear subspace generated by $v \in U^+$ is $\bigcup_{n\in\mathbb{N}}[-nv, nv]$.)

(b) An element e of U^+ is a **weak order unit** in U if U is the principal band generated by e; that is, if $u = \sup_{n\in\mathbb{N}} u \wedge ne$ for every $u \in U^+$ (352Vb).

Of course an order unit is a weak order unit.

(c) If U is Archimedean, then an element e of U^+ is a weak order unit iff $\{e\}^{\perp\perp} = U$ (353B), that is, iff $\{e\}^\perp = \{0\}$ (because

$$\{e\}^\perp = \{0\} \Longrightarrow \{e\}^{\perp\perp} = \{0\}^\perp = U \Longrightarrow \{e\}^\perp = \{e\}^{\perp\perp\perp} = U^\perp = \{0\},)$$

that is, iff $u \wedge e > 0$ whenever $u > 0$.

353M Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX; moreover, this embedding is essentially unique.

Remark Here $C(X)$ is the space of all continuous functions from X to $\mathbb{R}$; because X is compact, they are all bounded, so that χX is an order unit in $C(X) = C_b(X)$.

proof (a) Let X be the set of Riesz homomorphisms x from U to $\mathbb{R}$ such that $x(e) = 1$. Define $T : U \to \mathbb{R}^X$ by setting $(Tu)(x) = x(u)$ for $x \in X$, $u \in U$; then it is easy to check that T is a Riesz homomorphism, just because every member of X is a Riesz homomorphism, and of course $Te = \chi X$.

(b) The key to the proof is the fact that X separates the points of U, that is, that T is injective. I choose the following method to show this. Suppose that $w \in U$ and $w > 0$. Because U is Archimedean, there is a $\delta > 0$ such that $(w - \delta e)^+ \neq 0$. Now there is an $x \in X$ such that $x(w) \geq \delta$. **P** (i) By 351O, there is a solid linear subspace V of U such that $(w - \delta e)^+ \notin V$ and whenever $u \wedge v = 0$ in U then one of u, v belongs to V. (ii) Because $V \neq U$, $e \notin V$, so no non-zero multiple of e can belong to V. Also observe that if $u, v \in U \setminus V$, then one of $(u-v)^+$, $(v-u)^+$ must belong to V, while neither $u = u \wedge v + (u-v)^+$ nor $v = u \wedge v + (v-u)^+$ does; so $u \wedge v \notin V$. (iii) For each $u \in U$ set $A_u = \{\alpha : \alpha \in \mathbb{R}, (u - \alpha e)^+ \in V\}$. Then

$$\alpha \geq \beta \in A_u \Longrightarrow 0 \leq (u - \alpha e)^+ \leq (u - \beta e)^+ \in V \Longrightarrow \alpha \in A_u.$$

Also A_u is non-empty and bounded below, because if $\alpha \geq 0$ is such that $-\alpha e \leq u \leq \alpha e$ then $\alpha \in A_u$ and $-\alpha - 1 \notin A_u$ (since $(u - (-\alpha - 1)e)^+ \geq e \notin V$). (iv) Set $x(u) = \inf A_u$ for every $u \in U$; then $\alpha \in A_u$ for every $\alpha > x(u)$, $\alpha \notin A_u$ for every $\alpha < x(u)$. (v) If $u, v \in U$ and $\alpha > x(u)$, $\beta > x(v)$ then

$$((u + v) - (\alpha + \beta)e)^+ \leq (u - \alpha e)^+ + (v - \beta e)^+ \in V$$

(352Fc), so $\alpha + \beta \in A_{u+v}$; as α and β are arbitrary, $x(u + v) \leq x(u) + x(v)$. (vi) If $u, v \in U$ and $\alpha < x(u)$, $\beta < x(v)$ then

$$((u + v) - (\alpha + \beta)e)^+ \geq (u - \alpha e)^+ \wedge (v - \beta e)^+ \notin V,$$

using (ii) of this argument and 352Fc, so $\alpha + \beta \notin A_{u+v}$. As α and β are arbitrary, $x(u + v) \geq x(u) + x(v)$. (vii) Thus $x : U \to \mathbb{R}$ is additive. (viii) If $u \in U$, $\gamma > 0$ then

$$\alpha \in A_u \Longrightarrow (\gamma u - \alpha\gamma e)^+ = \gamma(u - \alpha e)^+ \in V \Longrightarrow \gamma\alpha \in A_{\gamma u};$$

thus $A_{\gamma u} \supseteq \gamma A_u$; similarly, $A_u \supseteq \gamma^{-1} A_{\gamma u}$ so $A_{\gamma u} = \gamma A_u$ and $x(\gamma u) = \gamma x(u)$. (ix) Consequently x is linear, since we know already from (vii) that $x(0u) = 0.x(u)$, $x(-u) = -x(u)$. (x) If $u \geq 0$ then $u + \alpha e \geq \alpha e \notin V$ for every $\alpha > 0$, that is, $-\alpha \notin A_u$ for every $\alpha > 0$, and $x(u) \geq 0$; thus x is a positive linear functional. (xi) If $u \wedge v = 0$, then one of u, v belongs to V, so $\min(x(u), x(v)) \leq 0$ and (using (x)) $\min(x(u), x(v)) = 0$; thus x is a Riesz homomorphism (352G(iv)). (xii) $A_e = [1, \infty[$ so $x(e) = 1$. Thus $x \in X$. (xiii) $\delta \notin A_w$ so $x(w) \geq \delta$. **Q**

(c) Thus $Tw \neq 0$ whenever $w > 0$; consequently $|Tw| = T|w| \neq 0$ whenever $w \neq 0$, and T is injective. I now have to define the topology of X. This is just the subspace topology on X if we regard X as a subset of $\mathbb{R}^U$ with its product topology. To see that X is compact, observe that if for each $u \in U$ we choose an α_u such that $|u| \leq \alpha_u e$, then X is a subspace of $Q = \prod_{u \in U} [-\alpha_u, \alpha_u]$. Because Q is a product of compact spaces, it is compact, by Tychonoff's theorem (3A3J). Now X is a closed subset of Q. **P** X is just the intersection of the sets

$$\{x : x(u + v) = x(u) + x(v)\}, \quad \{x : x(\alpha u) = \alpha x(u)\},$$

$$\{x : x(u^+) = \max(x(u), 0)\}, \quad \{x : x(e) = 1\}$$

as u, v run over U and α over $\mathbb{R}$; and each of these is closed, so X is an intersection of closed sets and therefore itself closed. **Q** Consequently X also is compact. Moreover, the coordinate functionals $x \mapsto x(u)$ are continuous on Q, therefore on X also, that is, $Tu : X \to \mathbb{R}$ is a continuous function for every $u \in U$.

Note also that because Q is a product of Hausdorff spaces, Q and X are Hausdorff (3A3Id).

(d) So T is a Riesz homomorphism from U to $C(X)$. Now $T[U]$ is a Riesz subspace of $C(X)$, containing χX, and such that if $x, y \in X$ are distinct there is an $f \in T[U]$ such that $f(x) \neq f(y)$ (because there is surely a $u \in U$ such that $x(u) \neq y(u)$). By the Stone-Weierstrass theorem (281A), $T[U]$ is $\| \|_\infty$-dense in $C(X)$.

Consequently it is also order-dense. **P** If $f > 0$ in $C(X)$, set $\epsilon = \frac{1}{3}\|f\|_\infty$, and let $u \in U$ be such that $\|f - Tu\|_\infty \leq \epsilon$; set $v = (u - \epsilon e)^+$. Since

$$0 < (f - 2\epsilon\chi X)^+ \leq (Tu - \epsilon\chi X)^+ \leq f^+ = f,$$

$0 < Tv \leq f$. As f is arbitrary, $T[U]$ is quasi-order-dense, therefore order-dense (353A). **Q**

(e) I have still to show that the representation is (essentially) unique. Suppose, then, that we have another representation of U as a norm-dense Riesz subspace of $C(Z)$, with e this time corresponding to χZ; to simplify the notation, let us suppose that U is actually a subspace of $C(Z)$. Then for each $z \in Z$, we have a functional $\hat{z} : U \to \mathbb{R}$ defined by setting $\hat{z}(u) = u(z)$ for every $u \in U$; of course $\hat{z}$ is a Riesz homomorphism such that $\hat{z}(e) = 1$, that is, $\hat{z} \in X$. Thus we have a function $z \mapsto \hat{z} : Z \to X$. For any $u \in U$, the function $z \mapsto \hat{z}(u) = u(z)$ is continuous, so the function $z \mapsto \hat{z}$ is continuous (3A3Ib). If z_1, z_2 are distinct members of Z, there is an $f \in C(Z)$ such that $f(z_1) \neq f(z_2)$ (3A3Bf); now there is a $u \in U$ such that $\|f - u\|_\infty \le \frac{1}{3}|f(z_1) - f(z_2)|$, so that $u(z_1) \neq u(z_2)$ and $\hat{z}_1 \neq \hat{z}_2$. Thus $z \mapsto \hat{z}$ is injective. Finally, it is also surjective. **P** Suppose that $x \in X$. Set $V = \{u : u \in U, x(u) = 0\}$; then V is a solid linear subspace of U (352Jb), not containing e. For $z \in V^+$ set $G_v = \{z : v(z) > 1\}$. Because $e \notin V$, $G_v \neq Z$. $\mathcal{G} = \{G_v : v \in V^+\}$ is an upwards-directed family of open sets in Z, not containing Z; consequently, because Z is compact, $\mathcal{G}$ cannot be an open cover of Z. Take $z \in Z \setminus \bigcup \mathcal{G}$. Then $v(z) \le 1$ for every $v \in V^+$; because $\alpha|v| \in V^+$ whenever $v \in V$, $\alpha \ge 0$, we must have $v(z) = 0$ for every $v \in V$. Now, given any $u \in U$, consider $v = u - x(u)e$. Then $x(v) = 0$ so $v \in V$ and $v(z) = 0$, that is,

$$u(z) = (v + x(u)e)(z) = v(z) + x(u)e(z) = x(u).$$

As u is arbitrary, $\hat{z} = x$; as x is arbitrary, we have the result. **Q**

Thus $z \mapsto \hat{z}$ is a continuous bijection from the compact Hausdorff space Z to the compact Hausdorff space X; it must therefore be a homeomorphism (3A3Dd).

This argument shows that if U is embedded as a norm-dense Riesz subspace of $C(Z)$, where Z is compact and Hausdorff, then Z must be homeomorphic to X. But it shows also that a homeomorphism is canonically defined by the embedding; $z \in Z$ corresponds to the Riesz homomorphism $u \mapsto u(z)$ in X.

353N Lemma Let U be a Riesz space, V an Archimedean Riesz space and S, $T : U \to V$ Riesz homomorphisms such that $Su \wedge Tu' = 0$ in V whenever $u \wedge u' = 0$ in U. Set $W = \{u : Su = Tu\}$. Then W is a solid linear subspace of U; if S and T are order-continuous, W is a band.

proof (a) It is easy to check that, because S and T are Riesz homomorphisms, W is a Riesz subspace of U.

(b) If $w \in W$ and $0 \le u \le w$ in U, then $Su \le Tu$. **P?** Otherwise, set $e = Sw = Tw$, and let V_e be the solid linear subspace of V generated by e, so that V_e is an Archimedean Riesz space with order unit, containing both Su and Tu. By 353M (or its proof), there is a Riesz homomorphism $x : V_e \to \mathbb{R}$ such that $x(e) = 1$ and $x(Su) > x(Tu)$. Take α such that $x(Su) > \alpha > x(Tu)$, and consider $u' = (u - \alpha w)^+$, $u'' = (\alpha w - u)^+$. Then

$$x(Su') = \max(0, x(Su) - \alpha x(Sw)) = \max(0, x(Su) - \alpha) > 0,$$

$$x(Tu'') = \max(0, \alpha x(Tw) - x(Tu)) = \max(0, \alpha - x(Tu)) > 0,$$

so

$$x(Su' \wedge Tu'') = \min(x(Su'), x(Tu'')) > 0$$

and $Su' \wedge Tu'' > 0$, while $u' \wedge u'' = 0$. **XQ**

Similarly, $Tu \le Su$ and $u \in W$. As u and w are arbitrary, W is a solid linear subspace.

(c) Finally, suppose that S and T are order-continuous, and that $A \subseteq W$ is a non-empty upwards-directed set with supremum u in U. Then

$$Su = \sup S[A] = \sup T[A] = Tu$$

and $u \in W$. As u and A are arbitrary, W is a band (352Ob).

353O f-algebras I give two results on f-algebras, intended to clarify the connexions between the multiplicative and lattice structures of the Riesz spaces in Chapter 36.

Proposition Let U be an Archimedean f-algebra (352W). Then

(a) the multiplication is separately order-continuous in the sense that the maps $u \mapsto u \times w$, $u \mapsto w \times u$ are order-continuous for every $w \in U^+$;

(b) the multiplication is commutative.

proof (a) Let $A \subseteq U$ be a non-empty set with infimum 0, and $v_0 \in U^+$ a lower bound for $\{u \times w : u \in A\}$. Fix $u_0 \in A$. If $u \in A$ and $\delta > 0$, then $v_0 \wedge (u_0 - \frac{1}{\delta}u)^+ \le \delta u_0 \times w$. **P** Set $v = v_0 \wedge (u_0 - \frac{1}{\delta}u)^+$. Then

$$\delta v \wedge (u - \delta u_0)^+ \le (\delta u_0 - u)^+ \wedge (u - \delta u_0)^+ = 0,$$

so $v \wedge (u - \delta u_0)^+ = 0$ and $v \wedge ((u - \delta u_0)^+ \times w) = 0$. But

$$v \le v_0 \le u \times w \le (u - \delta u_0)^+ \times w + \delta u_0 \times w,$$

so

$$v \le ((u - \delta u_0)^+ \times w) \wedge v + (\delta u_0 \times w) \wedge v \le \delta u_0 \times w,$$

by 352Fa. **Q**

Taking the infimum over u, and using the distributive laws (352E), we get

$$v_0 \wedge u_0 \le \delta u_0 \times w.$$

Taking the infimum over δ, and using the hypothesis that U is Archimedean,

$$v_0 \wedge u_0 = 0.$$

But this means that $v_0 \wedge (u_0 \times w) = 0$, while $v_0 \le u_0 \times w$, so $v_0 = 0$. As v_0 is arbitrary, $\inf_{u \in A} u \times w = 0$; as A is arbitrary, $u \mapsto u \times w$ is order-continuous. Similarly, $u \mapsto w \times u$ is order-continuous.

(b)(i) Fix $v \in U^+$, and for $u \in U$ set

$$Su = u \times v, \quad Tu = v \times u.$$

Then S and T are both order-continuous Riesz homomorphisms from U to itself (352W(b-iv) and (a) above). Also, $Su \wedge Tu' = 0$ whenever $u \wedge u' = 0$. **P**

$$0 = (u \times v) \wedge u' = (u \times v) \wedge (v \times u'). \ \mathbf{Q}$$

So $W = \{u : u \times v = v \times u\}$ is a band in U (353N). Of course $v \in W$ (because $Sv = Tv = v^2$). If $u \in W^\perp$, then $v \wedge |u| = 0$ so $Su = Tu = 0$ (352W(b-i)), and $u \in W$; but this means that $W^\perp = \{0\}$ and $W = W^{\perp\perp} = U$ (353Bb). Thus $v \times u = u \times v$ for every $u \in U$.

(ii) This is true for every $v \in U^+$. Of course it follows that $v \times u = u \times v$ for every u, $v \in U$, so that multiplication is commutative.

353P Proposition Let U be an Archimedean f-algebra with multiplicative identity e.

(a) e is a weak order unit in U.

(b) If u, $v \in U$ then $u \times v = 0$ iff $|u| \wedge |v| = 0$.

(c) If $u \in U$ has a multiplicative inverse u^{-1} then $|u|$ also has a multiplicative inverse; if $u \ge 0$ then $u^{-1} \ge 0$ and u is a weak order unit.

(d) If V is another Archimedean f-algebra with multiplicative identity e', and $T : U \to V$ is a positive linear operator such that $Te = e'$, then T is a Riesz homomorphism iff $T(u \times v) = Tu \times Tv$ for all u, $v \in U$.

proof (a) $e = e^2 \ge 0$ by 352W(b-iii). If $u \in U$ and $e \wedge |u| = 0$ then $|u| = (e \times |u|) \wedge |u| = 0$; by 353Lc, e is a weak order unit.

(b) If $|u| \wedge |v| = 0$ then $u \times v = 0$, by 352W(b-ii). If $w = |u| \wedge |v| > 0$, then $w^2 \le |u| \times |v|$. Let $n \in \mathbb{N}$ be such that $nw \not\le e$, and set $w_1 = (nw - e)^+$, $w_2 = (e - nw)^+$. Then

$$\begin{aligned} 0 \ne w_1 = w_1 \times e &= w_1 \times w_2 + w_1 \times (e \wedge nw) \\ &= w_1 \times (e \wedge nw) \le (nw)^2 \le n^2 |u| \times |v| = n^2 |u \times v|, \end{aligned}$$

so $u \times v \ne 0$.

(c) $u \times u^{-1} = e$ so $|u| \times |u^{-1}| = |e| = e$ (352W(b-iii)), and $|u^{-1}| = |u|^{-1}$. (Recall that inverses in any semigroup with identity are unique, so that we need have no inhibitions in using the formulae u^{-1}, $|u|^{-1}$.)

Now suppose that $u \ge 0$. Then $u^{-1} = |u^{-1}| \ge 0$. If $u \wedge |v| = 0$ then

$$e \wedge |v| = (u \times u^{-1}) \wedge |v| = 0,$$

so $v = 0$; accordingly u is a weak order unit.

(d)(i) If T is multiplicative, and $u \wedge v = 0$ in U, then $Tu \times Tv = T(u \times v) = 0$ and $Tu \wedge Tv = 0$, by (b). So T is a Riesz homomorphism, by 352G.

(ii) Accordingly I shall henceforth assume that T is a Riesz homomorphism and seek to show that it is multiplicative.

If u, $v \in U^+$, then $T(u \times v)$ and $Tu \times Tv$ both belong to the band generated by Tu. **P** Write W for this band. (α) For any $n \geq 1$ we have $(v - ne)^2 \geq 0$, that is, $2nv \leq v^2 + n^2e$, so

$$n(v - ne) \leq 2nv - n^2e \leq v^2.$$

Consequently

$$T(u \times v) - nTu = T(u \times v) - nT(u \times e) = T(u \times (v - ne)) \leq \frac{1}{n}T(u \times v^2)$$

because $v' \mapsto T(u \times v')$ is a positive linear operator; as V is Archimedean, $\inf_{n\in\mathbb{N}}(T(u \times v) - nTu)^+ = 0$ and $T(u \times v) = \sup_{n\in\mathbb{N}} T(u \times v) \wedge nTu$ belongs to W. (β) If $w \wedge |Tu| = 0$ then

$$w \wedge |Tu \times Tv| = w \wedge (|Tu| \times |Tv|) = 0;$$

so $Tu \times Tv \in W^{\perp\perp} = W$. **Q**

(iii) Fix $v \in U^+$. For $u \in U$, set $S_1u = Tu \times Tv$ and $S_2u = T(u \times v)$. Then S_1 and S_2 are both Riesz homomorphisms from U to V. If $u \wedge u' = 0$ in U, then $S_1u \wedge S_2u' = 0$ in V, because (by (ii) just above) S_1u belongs to the band generated by Tu, while S_2u' belongs to the band generated by Tu', and $Tu \wedge Tu' = T(u \wedge u') = 0$. By 353N, $W = \{u : S_1u = S_2u\}$ is a solid linear subspace of U. Of course it contains e, since

$$S_1e = Te \times Tv = e' \times Tv = Tv = T(e \times v) = S_2e.$$

In fact $u \in W$ for every $u \in U^+$. **P** As noted in (ii) just above, $u - ne \leq \frac{1}{n}u^2$ for every $n \geq 1$. So

$$\begin{aligned}|S_1u - S_2u| &= |S_1(u - ne)^+ + S_1(u \wedge ne) - S_2(u - ne)^+ - S_2(u \wedge ne)|\\ &\leq S_1(u - ne)^+ + S_2(u - ne)^+ \leq \frac{1}{n}(S_1u^2 + S_2u^2)\end{aligned}$$

for every $n \geq 1$, and $|S_1u - S_2u| = 0$, that is, $S_1u = S_2u$. **Q**

So $W = U$, that is, $Tu \times Tv = T(u \times v)$ for every $u \in U$. And this is true for every $v \in U^+$. It follows at once that it is true for every $v \in U$, so that T is multiplicative, as claimed.

353X Basic exercises >(a) Let U be a Riesz space in which every band is complemented. Show that U is Archimedean.

(b) A Riesz space U has the **principal projection property** iff the band generated by any single member of U is a projection band. Show that any Dedekind σ-complete Riesz space has the principal projection property, and that any Riesz space with the principal projection property is Archimedean.

>(c) Fill in the missing part (b-iii) of 353J.

(d) Let U be an Archimedean f-algebra with an order-unit which is a multiplicative identity. Show that U can be identified, as f-algebra, with a subspace of $C(X)$ for some compact Hausdorff space X.

353Y Further exercises (a) Let U be a Riesz space in which every quasi-order-dense solid linear subspace is order-dense. Show that U is Archimedean.

(b) Let X be a completely regular Hausdorff space. Show that $C(X)$ is Dedekind complete iff $C_b(X)$ is Dedekind complete iff X is extremally disconnected.

(c) Let X be a compact Hausdorff space. Show that $C(X)$ is Dedekind σ-complete iff $\overline{G}$ is open for every cozero set $G \subseteq X$. (Cf. 314Yf.) Show that in this case X is zero-dimensional.

(d) Let U be an Archimedean Riesz space such that $\{u_n : n \in \mathbb{N}\}$ has a supremum in U whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $u_m \wedge u_n = 0$ whenever $m \neq n$. Show that U has the principal projection property, but need not be Dedekind σ-complete.

(e) Let U be an Archimedean Riesz space. Show that the following are equiveridical: (i) U has the countable sup property (241Yd) (ii) for every $A \subseteq U$ there is a countable $B \subseteq A$ such that A and B have the same upper bounds; (iii) every disjoint subset of U^+ is countable.

(f) Let U be an Archimedean Riesz space with order unit e, and $\| \|_e$ the corresponding norm. Let Z be the unit ball of U^*. Show that for a linear functional $f : U \to \mathbb{R}$ the following are equiveridical: (i) f is an **extreme point** of Z, that is, $f \in Z$ and $Z \setminus \{f\}$ is convex (ii) $|f(e)| = 1$ and one of f, $-f$ is a Riesz homomorphism.

(g) Let U be an Archimedean f-algebra. Show that an element e of U is a multiplicative identity iff $e^2 = e$ and e is a weak order unit. (*Hint*: start by showing that under these conditions, $e \times u = 0 \Rightarrow u = 0$.)

(h) Let U be an Archimedean f-algebra with a multiplicative identity. Show that if $u \in U$ then u is invertible iff $|u|$ is invertible.

353 Notes and comments As in the last section, many of the results above have parallels in the theory of Boolean algebras; thus 353A corresponds to 313K, 353G corresponds in part to remarks in 314Bc and 314Xa, and 353J corresponds to 314C-314E. Riesz spaces are more complicated; for instance, principal ideals in Boolean algebras are straightforward, while in Riesz spaces we have to distinguish between the solid linear subspace generated by an element and the band generated by the same element. Thus an 'order unit' in a Boolean ring would just be an identity, while in a Riesz space we must distinguish between 'order unit' and 'weak order unit'. As this remark may suggest to you, (Archimedean) Riesz spaces are actually closer in spirit to arbitrary Boolean rings than to the Boolean algebras we have been concentrating on so far; to the point that in §361 below I will return briefly to general Boolean rings.

Note that the standard definition of 'order-dense' in Boolean algebras, as given in 313J, corresponds to the definition of 'quasi-order-dense' in Riesz spaces (352Na); the point here being that Boolean algebras behave like Archimedean Riesz spaces, in which there is no need to make a distinction.

I give the representation theorem 353M more for completeness than because we need it in any formal sense. In 351Q and 352L I have given representation theorems for general partially ordered linear spaces, and general Riesz spaces, as quotients of spaces of functions; in 368F below I give a theorem for Archimedean Riesz spaces corresponding rather more closely to the expressions of the L^p spaces as quotients of spaces of measurable functions. In 353M, by contrast, we have a theorem expressing Archimedean Riesz spaces with order units as true spaces of functions, rather than as spaces of equivalence classes of functions. All these theorems are important in forming an appropriate mental picture of ordered linear spaces, as in 352M.

I give a bare-handed proof of 353M, using only the Riesz space structure of $C(X)$; if you know a little about extreme points of dual unit balls you can approach from that direction instead, using 353Yf. The point is that (as part (d) of the proof makes clear) the space X can be regarded as a subset of the normed space dual U^* of U with its weak* topology. In this treatise generally, and in the present chapter in particular, I allow myself to be slightly prejudiced against normed-space methods; you can find them in any book on functional analysis, and I prefer here to develop techniques like those in part (b) of the proof of 353M, which will be a useful preparation for such theorems as 368E.

There is a very close analogy between 353M and the Stone representation of Boolean algebras (311E, 311I-311K). Just as the proof of 311E looked at the set of ring homomorphisms from $\mathfrak{A}$ to the elementary Boolean algebra $\mathbb{Z}_2$, so the proof of 353M looks at Riesz homomorphisms from U to the elementary M-space $\mathbb{R}$. Later on, the most important M-spaces, from the point of view of this treatise, will be the L^∞ spaces of §363, explicitly defined in terms of Stone representations (363A).

Of the two parts of 353O, it is (a) which is most important for the purposes of this book. The f-algebras we shall encounter in Chapter 36 can be seen to be commutative for different, and more elementary, reasons. The (separate) order-continuity of multiplication, however, is not always immediately obvious. Similarly, the uniferent Riesz homomorphisms we shall encounter can generally be seen to be multiplicative without relying on the arguments of 353Pd.

354 Banach lattices

The next step is a brief discussion of norms on Riesz spaces. I start with the essential definitions (354A, 354D) with the principal properties of general Riesz norms (354B-354C) and order-continuous norms (354E). I then describe two of the most important classes of Banach lattice: M-spaces (354F-354L) and L-spaces (354M-354R), with their elementary properties. For M-spaces I give the basic representation theorem (354K-354L), and for L-spaces I give a note on uniform integrability (354P-354R).

354A Definitions (a) If U is a Riesz space, a **Riesz norm** or **lattice norm** on U is a norm $\| \|$ such that $\|u\| \le \|v\|$ whenever $|u| \le |v|$; that is, a norm such that $\||u|\| = \|u\|$ for every u and $\|u\| \le \|v\|$ whenever $0 \le u \le v$.

(b) A **Banach lattice** is a Riesz space with a Riesz norm under which it is complete.

Remark We have already seen many examples of Banach lattices; I list some in 354Xa below.

354B Lemma Let U be a Riesz space with a Riesz norm $\| \|$.

(a) U is Archimedean.

(b) The maps $u \mapsto |u|$ and $u \mapsto u^+$ are uniformly continuous.

(c) For any $u \in U$, the sets $\{v : v \le u\}$ and $\{v : v \ge u\}$ are closed; in particular, the positive cone of U is closed.

(d) Any band in U is closed.

(e) If V is a norm-dense Riesz subspace of U, then $V^+ = \{v : v \in V,\ v \ge 0\}$ is norm-dense in the positive cone U^+ of U.

proof (a) If u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$, then $nu^+ \le v^+$ so $n\|u^+\| \le \|v^+\|$ for every n, and $\|u^+\| = 0$, that is, $u^+ = 0$ and $u \le 0$. As u, v are arbitrary, U is Archimedean.

(b) For any u, $v \in U$, $||u| - |v|| \le |u - v|$ (352D), so $\||u| - |v|\| \le \|u - v\|$; thus $u \mapsto |u|$ is uniformly continuous. Consequently $u \mapsto \frac{1}{2}(u + |u|) = u^+$ is uniformly continuous.

(c) Now $\{v : v \le u\} = \{v : (v - u)^+ = 0\}$ is closed because the function $v \mapsto (v - u)^+$ is continuous and $\{0\}$ is closed. Similarly $\{v : v \ge u\} = \{v : (u - v)^+ = 0\}$ is closed.

(d) If $V \subseteq U$ is a band, then $V = V^{\perp\perp}$ (353Bb), that is, $V = \{v : |v| \wedge |w| = 0$ for every $w \in V^\perp\}$. Because the function $v \mapsto |v| \wedge |w| = \frac{1}{2}(|v| + |w| - ||v| - |w||)$ is continuous, all the sets $\{v : |v| \wedge |w| = 0\}$ are closed, and so is their intersection V.

(e) Observe that $V^+ = \{v^+ : v \in V\}$ and $U^+ = \{u^+ : u \in U\}$; recall that $u \mapsto u^+$ is continuous, and apply 3A3Eb.

354C Lemma If U is a Banach lattice and $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\sum_{n=0}^\infty \|u_n\| < \infty$, then $\sup_{n\in\mathbb{N}} u_n$ is defined in U, with $\|\sup_{n\in\mathbb{N}} u_n\| \le \sum_{n=0}^\infty \|u_n\|$.

proof Set $v_n = \sup_{i\le n} u_i$ for each n. Then

$$0 \le v_{n+1} - v_n \le (u_{n+1} - u_n)^+ \le |u_{n+1} - u_n|$$

for each $n \in \mathbb{N}$, so

$$\sum_{n=0}^\infty \|v_{n+1} - v_n\| \le \sum_{n=0}^\infty \|u_{n+1} - u_n\| \le \sum_{n=0}^\infty \|u_{n+1}\| + \|u_n\|$$

is finite, and $\langle v_n \rangle_{n\in\mathbb{N}}$ is Cauchy. Let u be its limit; because $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, and the sets $\{v : v \ge v_n\}$ are all closed, $u \ge v_n$ for each $n \in \mathbb{N}$. On the other hand, if $v \ge v_n$ for every n, then

$$(u - v)^+ = \lim_{n\to\infty} (v_n - v)^+ = 0,$$

and $u \le v$. So

$$u = \sup_{n\in\mathbb{N}} v_n = \sup_{n\in\mathbb{N}} u_n$$

is the required supremum.

To estimate its norm, observe that $|v_n| \le \sum_{i=0}^{n} |u_i|$ for each n (induce on n, using the last item in 352D for the inductive step), so that

$$\|u\| = \lim_{n\to\infty} \|v_n\| \le \sum_{i=0}^{\infty} \||u_i|\| = \sum_{i=0}^{\infty} \|u_i\|.$$

354D I come now to the basic properties according to which we classify Riesz norms.

Definitions (a) A **Fatou norm** on a Riesz space U is a Riesz norm on U such that whenever $A \subseteq U^+$ is non-empty and upwards-directed and has a least upper bound in U, then $\|\sup A\| = \sup_{u\in A} \|u\|$. (Observe that, once we know that $\|\,\|$ is a Riesz norm, we can be sure that $\|u\| \le \|\sup A\|$ for every $u \in A$, so that all we shall need to check is that $\|\sup A\| \le \sup_{u\in A} \|u\|$.)

(b) A Riesz norm on a Riesz space U has the **Levi property** if every upwards-directed norm-bounded set is bounded above.

(c) A Riesz norm on a Riesz space U is **order-continuous** if $\inf_{u\in A} \|u\| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0.

354E Proposition Let U be a Riesz space with an order-continuous Riesz norm $\|\,\|$.

(a) If $A \subseteq U$ is non-empty and upwards-directed and has a supremum, then $\sup A \in \overline{A}$.

(b) $\|\,\|$ is Fatou.

(c) If $A \subseteq U$ is non-empty and upwards-directed and bounded above, then for every $\epsilon > 0$ there is a $u \in A$ such that $\|(v-u)^+\| \le \epsilon$ for every $v \in A$.

(d) Any non-decreasing order-bounded sequence in U is Cauchy.

(e) If U is a Banach lattice it is Dedekind complete.

(f) Every order-dense Riesz subspace of U is norm-dense.

proof (a) Suppose that $A \subseteq U$ is non-empty and upwards-directed and has a least upper bound u_0. Then $B = \{u_0 - u : u \in A\}$ is downwards-directed and has infimum 0. So $\inf_{u\in A} \|u_0 - u\| = 0$, and $u_0 \in \overline{A}$.

(b) If, in (a), $A \subseteq U^+$, then we must have

$$\|u_0\| \le \inf_{u\in A} \|u\| + \|u - u_0\| \le \sup_{u\in A} \|u\|.$$

As A is arbitary, $\|\,\|$ is a Fatou norm.

(c) Let B be the set of upper bounds for A. Then B is downwards-directed; because A is upwards-directed, $B - A = \{v - u : v \in B,\ u \in A\}$ is downwards-directed. By 353F, $\inf(B - A) = 0$. So there are $w \in B$, $u \in A$ such that $\|w - u\| \le \epsilon$. Now if $v \in A$,

$$(v-u)^+ = (v \vee u) - u \le w - u,$$

so $\|(v-u)^+\| \le \epsilon$.

(d) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence, and $\epsilon > 0$, then, applying (c) to $\{u_n : n \in \mathbb{N}\}$, we find that there is an $m \in \mathbb{N}$ such that $\|u_m - u_n\| \le \epsilon$ whenever $m \ge n$.

(e) Now suppose that U is a Banach lattice. Let $A \subseteq U$ be any non-empty set with an upper bound. Set $A' = \{u_0 \vee \ldots \vee u_n : u_0, \ldots, u_n \in A\}$, so that A' is upwards-directed and has the same upper bounds as A. For each n, choose $u_n \in A'$ such that $\|(u - u_n)^+\| \le 2^{-n}$ for every $u \in A'$. Set $v_n = \sup_{i\le n} u_i$ for each n; then $v_n \in A'$ and $\|v_m - v_n\| \le \|(v_m - u_n)^+\| \le 2^{-n}$ for all $m \ge n$. So $\langle v_n\rangle_{n\in\mathbb{N}}$ is Cauchy and has a limit v say. If $u \in A$, then $\|(u - v)^+\| = \lim_{n\to\infty} \|(u - v_n)^+\| = 0$, so $u \le v$; while if w is any upper bound for A, then $\|(v - w)^+\| = \lim_{n\to\infty} \|(v_n - w)^+\| = 0$ and $v \le w$. Thus $v = \sup A$ and A has a supremum.

(f) If V is an order-dense Riesz subspace of U and $u \in U^+$, set $A = \{v : v \in V,\ v \le u\}$. Then A is upwards-directed and has supremum u, so $u \in \overline{A} \subseteq \overline{V}$, by (a). Thus $U^+ \subseteq \overline{V}$; it follows at once that $U = U^+ - U^+ \subseteq \overline{V}$.

354F Lemma If U is an Archimedean Riesz space with an order unit e (definition: 353L), there is a Riesz norm $\| \|_e$ defined on U by the formula

$$\|u\|_e = \min\{\alpha : |u| \le \alpha e\}$$

for every $u \in U$.

proof This is a routine verification. Because e is an order-unit, $\{\alpha : |u| \le \alpha e\}$ is always non-empty, so always has an infimum α_0 say; now $|u| - \alpha_0 e \le \delta e$ for every $\delta > 0$, so (because U is Archimedean) $|u| - \alpha_0 e \le 0$ and $|u| \le \alpha_0 e$, so that the minimum is attained. In particular, $\|u\|_e = 0$ iff $u = 0$. The subadditivity and homogeneity of $\| \|_e$ are immediate from the facts that $|u + v| \le |u| + |v|$, $|\alpha u| = |\alpha||u|$.

354G Definitions (a) If U is an Archimedean Riesz space and e an order unit in U, the norm $\| \|_e$ as defined in 354F is the **order-unit norm** on U associated with e.

(b) An **M-space** is a Banach lattice in which the norm is an order-unit norm.

(c) If U is an M-space, its **standard order unit** is the order unit e such that $\| \|_e$ is the norm of U. (To see that e is uniquely defined, observe that it is $\sup\{u : u \in U, \|u\| \le 1\}$.)

354H Examples (a) For any set X, $\ell^\infty(X)$ is an M-space with standard order unit χX. (As remarked in 243Xl, the completeness of $\ell^\infty(X)$ can be regarded as the special case of 243E in which X is given counting measure.)

(b) For any topological space X, the space $C_b(X)$ of bounded continuous real-valued functions on X is an M-space with standard order unit χX. (It is a Riesz subspace of $\ell^\infty(X)$ containing the order unit of $\ell^\infty(X)$, therefore in its own right an Archimedean Riesz space with order unit. To see that it is complete, it is enough to observe that it is closed in $\ell^\infty(X)$ because a uniform limit of continuous functions is continuous (3A3Nb).)

(c) For any measure space (X, Σ, μ), the space $L^\infty(\mu)$ is an M-space with standard order unit $\chi X^\bullet$.

354I Lemma Let U be an Archimedean Riesz space with order unit e, and V a subset of U which is dense for the order-unit norm $\| \|_e$. Then for any $u \in U$ there are sequences $\langle v_n \rangle_{n \in \mathbb{N}}$, $\langle w_n \rangle_{n \in \mathbb{N}}$ in V such that $v_n \le v_{n+1} \le u \le w_{n+1} \le w_n$ and $\|w_n - v_n\|_e \le 2^{-n}$ for every n; so that $u = \sup_{n \in \mathbb{N}} v_n = \inf_{n \in \mathbb{N}} w_n$ in U.

If V is a Riesz subspace of U, and $u \ge 0$, we may suppose that $v_n \ge 0$ for every n. Consequently V is order-dense in U.

proof For each $n \in \mathbb{N}$, take v_n, $w_n \in V$ such that

$$\|u - \frac{3}{2^{n+3}}e - v_n\|_e \le \frac{1}{2^{n+3}}, \quad \|u + \frac{3}{2^{n+3}}e - w_n\|_e \le \frac{1}{2^{n+3}}.$$

Then

$$u - \frac{1}{2^{n+1}}e \le v_n \le u - \frac{1}{2^{n+2}}e \le u \le u + \frac{1}{2^{n+2}}e \le w_n \le u + \frac{1}{2^{n+1}}e.$$

Accordingly $\langle v_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, $\langle w_n \rangle_{n \in \mathbb{N}}$ is non-increasing and $\|w_n - v_n\|_e \le 2^{-n}$ for every n. Because U is Archimedean, $\sup_{n \in \mathbb{N}} v_n = \inf_{n \in \mathbb{N}} w_n = u$.

If V is a Riesz subspace of U, then replacing v_n by v_n^+ if necessary we may suppose that every v_n is non-negative; and V is order-dense by the definition in 352N.

354J Proposition Let U be an Archimedean Riesz space with an order unit e. Then $\| \|_e$ is Fatou and has the Levi property.

proof This is elementary. If $A \subseteq U^+$ is non-empty, upwards-directed and norm-bounded, then it is bounded above by αe, where $\alpha = \sup_{u \in A} \|u\|_e$. This is all that is called for in the Levi property. If moreover $\sup A$ is defined, then $\sup A \le \alpha e$ so $\|\sup A\| \le \alpha$, as required in the Fatou property.

354K Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX and $\| \|_e$ corresponds to $\| \|_\infty$; moreover, this embedding is essentially unique.

proof This is nearly word-for-word a repetition of 353M. The only addition is the mention of the norms. But let X and $T : U \to C(X)$ be as in 353M. Then, for any $u \in U$, $|u| \le \|u\|_e e$, so that

$$|Tu| = T|u| \le \|u\|_e Te = \|u\|_e \chi X,$$

and $\|Tu\|_\infty \le \|u\|_e$. On the other hand, if $0 < \delta < \|u\|_e$ then $u_1 = (|u| - \delta e)^+ > 0$, so that $Tu_1 = (|Tu| - \delta\chi X)^+ > 0$ and $\|Tu\|_\infty \ge \delta$; as δ is arbitrary, $\|Tu\|_\infty \ge \|u\|_e$.

354L Corollary Any M-space U is isomorphic, as Banach lattice, to $C(X)$ for some compact Hausdorff X, and the isomorphism is essentially unique. X can be identified with the set of Riesz homomorphisms $x : U \to \mathbb{R}$ such that $x(e) = 1$, where e is the standard order unit of U, with the topology induced by the product topology on $\mathbb{R}^U$.

proof By 354K, there are a compact Hausdorff space X and an embedding of U as a norm-dense Riesz subspace of $C(X)$ matching $\| \|_e$ to $\| \|_\infty$. Since U is complete under $\| \|_e$, its image is closed in $C(X)$ (3A4Ff), and must be the whole of $C(X)$. The expression is unique just in so far as the expression of 353M/354K is unique. In particular, we may, if we wish, take X to be the set of normalized Riesz homomorphisms from U to $\mathbb{R}$, as in the proof of 353M.

Remark If U is an M-space, then the construction of 353M represents U as $C(X)$, where X is the set of uniferent Riesz homomorphisms from U to $\mathbb{R}$; this is sometimes called the **spectrum** of U.

354M I come now to a second fundamental class of Banach lattices, in a strong sense 'dual' to the class of M-spaces, as will appear in §356.

Definition An **L-space** is a Banach lattice U such that $\|u + v\| = \|u\| + \|v\|$ whenever $u, v \in U^+$.

Example If (X, Σ, μ) is any measure space, then $L^1(\mu)$, with its norm $\| \|_1$, is an L-space (242D, 242F). In particular, taking μ to be counting measure on $\mathbb{N}$, ℓ^1 is an L-space (242Xa).

354N Theorem If U is an L-space, then its norm is order-continuous and has the Levi property.

proof (a) Both of these are consequences of the following fact: if $A \subseteq U$ is norm-bounded and non-empty and upwards-directed, then $\sup A$ is defined in U and belongs to the norm-closure of A in U. **P** For $u \in A$, set $\gamma(u) = \sup\{\|v - u\| : v \in A, v \ge u\}$. We surely have $\gamma(u) \le \|u\| + \sup_{v\in A} \|v\| < \infty$. Choose a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A such that $u_{n+1} \ge u_n$ and $\|u_{n+1} - u_n\| \ge \frac{1}{2}\gamma(u_n)$ for each n. Then

$$\|u_{n+1} - u_0\| = \sum_{i=0}^{n} \|u_{i+1} - u_i\| \ge \frac{1}{2} \sum_{i=0}^{n} \gamma(u_i)$$

for every n, using the definition of 'L-space'. Because A is bounded, $\sum_{i=0}^{\infty} \gamma(u_i) < \infty$ and $\lim_{n\to\infty} \gamma(u_n) = 0$. But $\|u_m - u_n\| \le \gamma(u_n)$ whenever $m \ge n$, so $\langle u_n\rangle_{n\in\mathbb{N}}$ is Cauchy and has a limit u^* in U.

For each $n \in \mathbb{N}$, $u^* \ge u_n$ because $u_m \ge u_n$ for every $m \ge n$ (see 354Bc). If $u \in A$, $n \in \mathbb{N}$ then there is a $u' \in A$ such that $u' \ge u \vee u_n$; now $(u - u^*)^+ \le u' - u_n$ so $\|(u - u^*)^+\| \le \|u' - u_n\| \le \gamma(u_n)$; as n is arbitrary, $\|(u - u^*)^+\| = 0$ and $u \le u^*$. Thus u^* is an upper bound for A. But if v is any upper bound for A, then $u_n \le v$ for every n so $u^* \le v$. Thus u^* is the least upper bound of A; and $u^* \in \overline{A}$ because it is the norm limit of $\langle u_n\rangle_{n\in\mathbb{N}}$. **Q**

(b) This shows immediately that the norm has the Levi property. But also it must be order-continuous. **P** If $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, take any $u_0 \in A$ and consider $B = \{u_0 - u : u \in A, u \le u_0\}$. Then B is upwards-directed and has supremum u_0, so $u_0 \in \overline{B}$ and

$$\inf_{u\in A} \|u\| \le \inf_{v\in B} \|u_0 - v\| = 0. \text{ **Q**}$$

354O Proposition If U is an L-space and V is a norm-closed Riesz subspace of U, then V is an L-space in its own right. In particular, any band of U is an L-space.

proof For any Riesz subspace V of U, we surely have $\|u+v\| = \|u| + \|v\|$ whenever u, $v \in V^+$; so if V is norm-closed, therefore a Banach lattice, it must be an L-space. But in any Banach lattice, a band is norm-closed (354Bd), so a band in an L-space is again an L-space.

354P Uniform integrability in L-spaces Some of the ideas of §246 can be readily expressed in this abstract context.

Definition Let U be an L-space. A set $A \subseteq U$ is **uniformly integrable** if for every $\epsilon > 0$ there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$.

354Q Since I have already used the phrase 'uniformly integrable' based on a different formula, I had better check instantly that the two definitions are consistent.

Proposition If (X, Σ, μ) is any measure space, then a subset of $L^1 = L^1(\mu)$ is uniformly integrable in the sense of 354P iff it is uniformly integrable in the sense of 246A.

proof (a) If $A \subseteq L^1$ is uniformly integrable in the sense of 246A, then for any $\epsilon > 0$ there are $M \ge 0$, $E \in \Sigma$ such that $\mu E < \infty$ and $\int(|u| - M\chi E^\bullet)^+ \le \epsilon$ for every $u \in A$; now $w = M\chi E^\bullet$ belongs to $(L^1)^+$ and $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

(b) Now suppose that A is uniformly integrable in the sense of 354P. Let $\epsilon > 0$. Then there is a $w \in (L^1)^+$ such that $\|(|u|-w)^+\| \le \frac{1}{2}\epsilon$ for every $u \in A$. There is a simple function $h : X \to \mathbb{R}$ such that $\|w - h^\bullet\| \le \frac{1}{2}\epsilon$ (242M); now take $E = \{x : h(x) \ne 0\}$, $M = \sup_{x\in X} |h(x)|$ (I pass over the trivial case $X = \emptyset$), so that $h \le M\chi E$ and

$$(|u| - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - h^\bullet)^+,$$

$$\int(|u| - M\chi E^\bullet)^+ \le \|(|u| - w)^+\| + \|w - h^\bullet\| \le \epsilon$$

for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

354R I give abstract versions of the easiest results from §246.

Theorem Let U be an L-space.

(a) If $A \subseteq U$ is uniformly integrable, then

(i) A is norm-bounded;

(ii) every subset of A is uniformly integrable;

(iii) for any $\alpha \in \mathbb{R}$, αA is uniformly integrable;

(iv) there is a uniformly integrable, solid, convex, norm-closed set $C \supseteq A$;

(v) for any other uniformly integrable set $B \subseteq U$, $A \cup B$ and $A + B$ are uniformly integrable.

(b) For any set $A \subseteq U$, the following are equiveridical:

(i) A is uniformly integrable;

(ii) $\lim_{n\to\infty}(|u_n| - \sup_{i<n} |u_i|)^+ = 0$ for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A;

(iii) *either* A is empty *or* for every $\epsilon > 0$ there are $u_0, \dots, u_n \in A$ such that $\|(|u| - \sup_{i\le n} |u_i|)^+\| \le \epsilon$ for every $u \in A$;

(iv) A is norm-bounded and any disjoint sequence in the solid hull of A is norm-convergent to 0.

(c) If $V \subseteq U$ is a closed Riesz subspace then a subset of V is uniformly integrable when regarded as a subset of V iff it is uniformly integrable when regarded as a subset of U.

proof (a)(i) There must be a $w \in U^+$ such that $\int(|u| - w)^+ \le 1$ for every $u \in A$; now

$$|u| \le |u| - w + |w| \le (|u| - w)^+ + |w|, \quad \|u\| \le \|(|u| - w)^+\| + \|w\| \le 1 + \|w\|$$

for every $u \in A$, so A is norm-bounded.

(ii) This is immediate from the definition.

(iii) Given $\epsilon > 0$, we can find $w \in U^+$ such that $|\alpha|\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$; now $\|(|v| - |\alpha|w)^+\| \le \epsilon$ for every $v \in \alpha A$.

(iv) If A is empty, take $C = A$. Otherwise, try

$$C = \{v : v \in U,\ \|(|v| - w)^+\| \le \sup_{u\in A} \|(|u| - w)^+\| \text{ for every } w \in U^+\}.$$

Evidently $A \subseteq C$, and C satisfies the definition 354M because A does. The functionals

$$v \mapsto \|(|v| - w)^+\| : U \to \mathbb{R}$$

are all continuous for $\| \|$ (because the operators $v \mapsto |v|$, $v \mapsto v - w$, $v \mapsto v^+$, $v \mapsto \|v\|$ are continuous), so C is closed. If $|v'| \le |v|$ and $v \in C$, then

$$\|(|v'| - w)^+\| \le \|(|v| - w)^+\| \le \sup_{u\in A} \|(|u| - w)^+\|$$

for every w, and $v' \in C$. If $v = \alpha v_1 + \beta v_2$ where v_1, $v_2 \in C$, $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$, then $|v| \le \alpha|v_1| + \beta|v_2|$, so

$$|v| - w \le (\alpha|v_1| - \alpha w) + (\beta|v_2| - \beta w) \le (\alpha|v_1| - \alpha w)^+ + (\beta|v_2| - \beta w)^+$$

and

$$(|v| - w)^+ \le \alpha(|v_1| - w)^+ + \beta(|v_2| - w)^+$$

for every w; accordingly

$$\begin{aligned}\|(|v| - w)^+\| &\le \alpha\|(|v_1| - w)^+\| + \beta\|(|v_2| - w)^+\| \\ &\le (\alpha + \beta) \sup_{u\in A} \|(|u| - w)^+\| = \sup_{u\in A} \|(|u| - w)^+\|\end{aligned}$$

for every w, and $v \in C$.

Thus C has all the required properties.

(v) I show first that $A \cup B$ is uniformly integrable. **P** Given $\epsilon > 0$, let w_1, $w_2 \in U^+$ be such that

$$\|(|u| - w_1)^+\| \le \epsilon \text{ for every } u \in A, \quad \|(|u| - w_2)^+\| \le \epsilon \text{ for every } u \in B.$$

Set $w = w_1 \vee w_2$; then $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A \cup B$. As ϵ is arbitrary, $A \cup B$ is uniformly integrable. **Q**

Now (iv) tells us that there is a convex uniformly integrable set C including $A \cup B$, and in this case $A + B \subseteq 2C$, so $A + B$ is also uniformly integrable, using (ii) and (iii).

(b)(i)⇒(ii)&(iv) Suppose that A is uniformly integrable and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is any sequence in the solid hull of A. Set $v_n = \sup_{i\le n} |u_i|$ for $n \in \mathbb{N}$ and

$$v'_0 = v_0 = |u_0|, \quad v'_n = v_n - v_{n-1} = (|u_n| - \sup_{i<n} |u_i|)^+$$

for $n \ge 1$. Given $\epsilon > 0$, there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, and therefore for every u in the solid hull of A. Of course $\sup_{n\in\mathbb{N}} \|v_n \wedge w\| \le \|w\|$ is finite, so there is an $n \in \mathbb{N}$ such that $\|v_i \wedge w\| \le \epsilon + \|v_n \wedge w\|$ for every $i \in \mathbb{N}$. But now, for $m > n$,

$$\begin{aligned}v'_m \le (|u_m| - v_n)^+ &\le (|u_m| - |u_m| \wedge w)^+ + ((|u_m| \wedge w) - v_n)^+ \\ &\le (|u_m| - w)^+ + (v_m \wedge w) - (v_n \wedge w),\end{aligned}$$

so that

$$\begin{aligned}\|v'_m\| &\le \|(|u_m| - w)^+\| + \|(v_m \wedge w) - (v_n \wedge w)\| \\ &= \|(|u_m| - w)^+\| + \|v_m \wedge w\| - \|v_n \wedge w\| \le 2\epsilon,\end{aligned}$$

using the L-space property of the norm for the equality in the middle. As ϵ is arbitrary, $\lim_{n\to\infty} v'_n = 0$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, condition (ii) is satisfied; but so is condition (iv), because we know from (a-i) that A is norm-bounded, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is disjoint then $v'_n = |u_n|$ for every n, so that in this case $\lim_{n\to\infty} u_n = 0$.

(ii)⇒(iii)⇒(i) are elementary.

not-(i)⇒not-(iv) Now suppose that A is not uniformly integrable. If it is not norm-bounded, we can stop. Otherwise, there is some $\epsilon > 0$ such that $\sup_{u\in A} \|(|u| - w)^+\| > \epsilon$ for every $w \in U^+$. Consequently we shall be able to choose inductively a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A such that $\|(|u_n| - 2^n \sup_{i<n} |u_i|)^+\| > \epsilon$ for every $n \ge 1$. Because A is norm-bounded, $\sum_{i=0}^{\infty} 2^{-i}\|u_i\|$ is finite, and we can set

$$v_n = (|u_n| - 2^n \sup_{i<n} |u_i| - \sum_{i=n+1}^{\infty} 2^{-i}|u_i|)^+$$

for each n. (The sum $\sum_{i=n+1}^{\infty} 2^{-i}|u_i|$ is defined because $\langle \sum_{i=n+1}^{m} 2^{-i}|u_i| \rangle_{m \ge n+1}$ is a Cauchy sequence. We have $v_m \le |u_m|$,

$$\begin{aligned} v_m \wedge v_n &\le (|u_m| - 2^{-n}|u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ \\ &\le (2^n|u_m| - |u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ = 0 \end{aligned}$$

whenever $m < n$, so $\langle v_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in the solid hull of A; while

$$\|v_n\| \ge \|(|u_n| - 2^n \sup_{i<n} |u_i|)^+\| - \sum_{i=n+1}^{\infty} 2^{-i}\|u_i\| \ge \epsilon - 2^{-n} \sup_{u\in A} \|u\| \to \epsilon$$

as $n \to \infty$, so condition (iv) is not satisfied.

(c) Now this follows at once, because conditions (b-ii) and (b-iv) are satisfied in V iff they are satsified in U.

354X Basic exercises >(a) Work through the proofs that the following are all Banach lattices: (i) $\mathbb{R}^r$ with (α) $\|x\|_1 = \sum_{i=1}^{r} |\xi_i|$ (β) $\|x\|_2 = \sqrt{\sum_{i=1}^{r} |\xi_i|^2}$ (γ) $\|x\|_\infty = \max_{i\le r} |\xi_i|$, where $x = (\xi_1, \dots, \xi_r)$. (ii) $\ell^p(X)$, for any set X and any $p \in [1, \infty]$ (242Xa, 243Xl, 244Xn). (iii) $L^p(\mu)$, for any measure space (X, Σ, μ) and any $p \in [1, \infty]$ (242F, 243E, 244G). (iv) $\boldsymbol{c}_0$, the space of sequences convergent to 0, with the norm $\| \|_\infty$ inherited from ℓ^∞.

(b) Let $\langle U_i \rangle_{i\in I}$ be any family of Banach lattices. Write U for their Riesz space product (352K), and in U set

$$\|u\|_1 = \sum_{i\in I} \|u(i)\|, \quad V_1 = \{u : \|u\|_1 < \infty\},$$

$$\|u\|_\infty = \sup_{i\in I} \|u(i)\|, \quad V_\infty = \{u : \|u\|_\infty < \infty\}.$$

Show that V_1, V_∞ are solid linear subspaces of U and are Banach lattices under their norms $\| \|_1$, $\| \|_\infty$.

(c) Let U be a Riesz space with a Riesz norm. Show that the maps $(u, v) \mapsto u \wedge v$, $(u, v) \mapsto u \vee v : U \times U \to U$ are uniformly continuous.

>(d) Let U be a Riesz space with a Riesz norm. (i) Show that any order-bounded set in U is norm-bounded. (ii) Show that in $\mathbb{R}^r$, with any of the standard Riesz norms (354Xa(i)), norm-bounded sets are order-bounded. (iii) Show that in $\ell^1(\mathbb{N})$ there is a sequence converging to 0 (for the norm) which is not order-bounded. (iv) Show that in $\boldsymbol{c}_0$ any sequence converging to 0 is order-bounded, but there is a norm-bounded set which is not order-bounded.

(e) Let U be a Riesz space with a Riesz norm. Show that it is a Banach lattice iff non-decreasing Cauchy sequences are convergent. (*Hint*: if $\|u_{n+1} - u_n\| \le 2^{-n}$ for every n, show that $\langle \sup_{i\le n} u_i \rangle_{n\in\mathbb{N}}$ is Cauchy, and that $\langle u_n \rangle_{n\in\mathbb{N}}$ converges to $\inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m$.)

(f) Let U be a Riesz space with a Riesz norm. Show that U is a Banach lattice iff every non-decreasing Cauchy sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in U^+ has a least upper bound u with $\|u\| = \lim_{n\to\infty} \|u_n\|$.

(g) Let U be a Banach lattice. Suppose that $B \subseteq U$ is solid and $\sup_{n\in\mathbb{N}} u_n \in B$ whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in B with a supremum in U. Show that B is closed. (*Hint*: show first that $u \in B$ whenever there is a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $B \cap U^+$ such that $\|u - u_n\| \le 2^{-n}$ for every n; do this by considering $v_m = \inf_{n\ge m} u_n$.)

(h) Let U be any Riesz space with a Riesz norm. Show that the Banach space completion of U (3A5Ib) has a unique partial ordering under which it is a Banach lattice.

>(i) Show that the space $\boldsymbol{c}_0$ of sequences convergent to 0, with $\| \|_\infty$, is a Banach lattice with an order-continuous norm which does not have the Levi property.

>(j) Show that ℓ^∞, with $\| \|_\infty$, is a Banach lattice with a Fatou norm which has the Levi property but is not order-continuous.

(k) Let U be a Riesz space with a Fatou norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace (definition: 352Ne) then the induced norm on V is a Fatou norm.

(l) Let U be a Riesz space and $\| \|$ a Riesz norm on U which is order-continuous in the sense of 354Dc. Show that it is order-continuous in the sense of 313H when regarded as a function from U^+ to $[0, \infty[$.

(m) Let U be a Riesz space with an order-continuous norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace then the induced norm on V is order-continuous.

(n) Let U be a Dedekind σ-complete Riesz space with a Fatou norm which has the Levi property. Show that it is a Banach lattice. (*Hint*: 354Xf.)

(o) Let $\langle U_i \rangle_{i \in I}$ be any family of Banach lattices and let V_1, V_∞ be the subspaces of $U = \prod_{i \in I} U_i$ as described in 354Xb. Show that V_1, V_∞ have norms which are Fatou, or have the Levi property, iff every U_i has. Show that the norm of V_1 is order-continuous iff the norm of every U_i is.

(p) Let U be a Banach lattice with an order-continuous norm. Show that a Riesz subspace of U (indeed, any sublattice of U) is norm-closed iff it is order-closed in the sense of 313D, and in this case is itself a Banach lattice with an order-continuous norm.

>(q) Let U be an M-space and V a norm-closed Riesz subspace of U containing the standard order unit of U. (i) Show that V, with the induced norm, is an M-space. (ii) Deduce that the space $\boldsymbol{c}$ of convergent sequences is an M-space if given the norm $\| \|_\infty$ inherited from ℓ^∞.

(r) Show that a Banach lattice U is an M-space iff (i) its norm is a Fatou norm with the Levi property (ii) $\|u \vee v\| = \max(\|u\|, \|v\|)$ for all u, $v \in U^+$.

>(s) Describe a topological space X such that the space $\boldsymbol{c}$ of convergent sequences (354Xq) can be identified with $C(X)$.

(t) Let $D \subseteq \mathbb{R}$ be any non-empty set, and V the space of functions $f : D \to \mathbb{R}$ of bounded variation (§224). For $f \in V$ set $\|f\| = \sup\{|f(t_0)| + \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : t_0 \le t_1 \le \ldots \le t_n \text{ in } D\}$ (224Yb). Let V^+ be the set of non-negative, non-decreasing functions in V. Show that V^+ is the positive cone of V for a Riesz space ordering under which V is an L-space.

354Y Further exercises (a) Let U be a Riesz space with a Riesz norm, and V a norm-dense Riesz subspace of U. Suppose that the induced norm on V is Fatou, when regarded as a norm on the Riesz space V. Show (i) that V is order-dense in U (ii) that the norm of U is Fatou. (*Hint*: for (i), show that if $u \in U^+$, $v_n \in V^+$ and $\|u - v_n\| \le 2^{-n-2}\|u\|$ for every n, then $\|v_0 - \inf_{i \le n} v_i\| \ge \frac{1}{4}\|u\|$ for every n, so that 0 cannot be $\inf_{n \in \mathbb{N}} v_n$ in V.)

(b) Let U be a Riesz space with a Riesz norm. Show that the following are equiveridical: (i) $\lim_{n \to \infty} u_n = 0$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is a disjoint order-bounded sequence in U^+ (ii) $\lim_{n \to \infty} u_{n+1} - u_n = 0$ for every order-bounded non-decreasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U (iii) whenever $A \subseteq U^+$ is a non-empty downwards-directed set in U^+ with infimum 0, $\inf_{u \in A} \sup_{v \in A, v \le u} \|u - v\| = 0$. (*Hint*: for (i)⇒(ii), show by induction that $\lim_{n \to \infty} u_n = 0$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is an order-bounded sequence such that, for some fixed k, $\inf_{i \in K} u_i = 0$ for every $K \subseteq \mathbb{N}$ of size k; now show that if $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and $0 \le u_n \le u$ for every n, then $\inf_{i \in K}(u_{i+1} - u_i - \frac{1}{k}u)^+ = 0$ whenever $K \subseteq \mathbb{N}$, $\#(K) = k \ge 1$. For (iii)⇒(i), set $A = \{u : \exists\, n, u \ge u_i \;\forall\, i \ge n\}$. See Fremlin 74a, 24H.)

(c) Show that any Riesz space with an order-continuous norm has the countable sup property (definition: 241Yd).

(d) Let U be a Banach lattice. Show that the following are equiveridical: (i) the norm on U is order-continuous; (ii) U satisfies the conditions of 354Yb; (iii) every order-bounded monotonic sequence in U is Cauchy.

(e) Let U be a Riesz space with a Fatou norm. Show that the norm on U is order-continuous iff it satisfies the conditions of 354Yb.

(f) For $f \in C([0,1])$, set $\|f\|_1 = \int |f(x)|dx$. Show that $\| \|_1$ is a Riesz norm on $C([0,1])$ satisfying the conditions of 354Yb, but is not order-continuous.

(g) Let U be a Riesz space with a Riesz norm $\| \|$. Show that $(U, \| \|)$ satisfies the conditions of 354Yb iff the norm of its completion is order-continuous.

(h) Let U be a Riesz space with a Riesz norm, and $V \subseteq U$ a norm-dense Riesz subspace such that the induced norm on V is order-continuous. Show that the norm of U is order-continuous. (*Hint*: use 354Ya.)

(i) Let U be an Archimedean Riesz space. For any $e \in U^+$, let U_e be the solid linear subspace of U generated by e, so that e is an order unit in U_e, and let $\| \|_e$ be the corresponding order-unit norm on U_e. We say that U is **uniformly complete** if U_e is complete under $\| \|_e$ for every $e \in U^+$. (i) Show that any Banach lattice is uniformly complete. (ii) Show that any Dedekind σ-complete Riesz space is uniformly complete (cf. 354Xn). (iii) Show that if U is a uniformly complete Riesz space with a Riesz norm which has the Levi property, then U is a Banach lattice. (iv) Show that if U is a Banach lattice then a set $A \subseteq U$ is closed, for the norm topology, iff $A \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (v) Let V be a solid linear subspace of U. Show that the quotient Riesz space U/V (352U) is Archimedean iff $V \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (vi) Show that if U is uniformly complete and $V \subseteq U$ is a solid linear subspace such that U/V is Archimedean, then U/V is uniformly complete. (vii) Show that U is Dedekind σ-complete iff it is uniformly complete and has the principal projection property (353Xb). (*Hint*: for (vii), use 353Yc.)

(j) Let U be a Banach lattice such that $\|u+v\| = \|u\| + \|v\|$ whenever $u \wedge v = 0$. Show that U is an L-space. (*Hint*: by 354Yd, the norm is order-continuous, so U is Dedekind complete. If u, $v \geq 0$, set $e = u+v$, and represent U_e as $C(X)$ where X is extremally disconnected (353Yb); now approximate u and v by functions taking only finitely many values to show that $\|u+v\| = \|u\| + \|v\|$.)

(k) Let U be a uniformly complete Archimedean Riesz space (354Yi). Set $U_{\mathbb{C}} = U \times U$ with the complex linear structure defined by identifying $(u,v) \in U \times U$ as $u+iv \in U_{\mathbb{C}}$, so that $u = \mathcal{R}\mathrm{e}(u+iv)$, $v = \mathcal{I}\mathrm{m}(u+iv)$ and $(\alpha+i\beta)(u+iv) = (\alpha u - \beta v) + i(\alpha v + \beta u)$. (i) Show that for $w \in U_{\mathbb{C}}$ we can define $|w| \in U$ by setting $|w| = \sup_{|\zeta|=1} \mathcal{R}\mathrm{e}(\zeta w)$. (ii) Show that if U is a uniformly complete Riesz subspace of $\mathbb{R}^X$ for some set X, then we can identify $U_{\mathbb{C}}$ with the linear subspace of $\mathbb{C}^X$ generated by U. (iii) Show that $|w+w'| \leq |w|+|w'|$, $|\gamma w| = |\gamma||w|$ for all $w \in U_{\mathbb{C}}$, $\gamma \in \mathbb{C}$. (iv) Show that if $w \in U_{\mathbb{C}}$ and $|w| \leq u_1 + u_2$, where u_1, $u_2 \in U^+$, then w is expressible as $w_1 + w_2$ where $|w_j| \leq u_j$ for both j. (*Hint*: set $e = u_1 + u_2$ and represent U_e as $C(X)$.) (v) Show that if U_0 is a solid linear subspace of U, then, for $w \in U_{\mathbb{C}}$, $|w| \in U_0$ iff $\mathcal{R}\mathrm{e}\, w$, $\mathcal{I}\mathrm{m}\, w$ both belong to U_0. (vi) Show that if U has a Riesz norm then we have a norm on $U_{\mathbb{C}}$ defined by setting $\|w\| = \||w|\|$, and that if U is a Banach lattice then $U_{\mathbb{C}}$ is a (complex) Banach space. (vii) Show that if $U = L^p(\mu)$, where (X, Σ, μ) is a measure space and $p \in [1,\infty]$, then $U_{\mathbb{C}}$ can be identified with $L^p_{\mathbb{C}}(\mu)$ as defined in 242P, 243K, 244O. (We may call $U_{\mathbb{C}}$ the **complexification** of the Riesz space U.)

(l) Let (X, Σ, μ) be a measure space and V a Banach lattice. Write $\mathcal{L}^1_V$ for the space of Bochner integrable functions from conegligible subsets of X to V, and L^1_V for the corresponding set of equivalence classes (253Yf). (i) Show that L^1_V is a Banach lattice under the ordering defined by saying that $f^\bullet \leq g^\bullet$ iff $f(x) \leq g(x)$ in V for μ-almost every $x \in X$. (ii) Show that when $V = L^1(\nu)$, for some other measure space (Y, T, ν), then this ordering on L^1_V agrees with the ordering of $L^1(\lambda)$ where λ is the (c.l.d.) product measure on $X \times Y$ and we identify L^1_V with $L^1(\lambda)$, as in 253Yi. (iii) Show that if V has an order-continuous norm, so has L^1_V. (*Hint*: 354Yd(ii).) (iv) Show that if μ is Lebesgue measure on $[0,1]$ and $V = \ell^\infty$, then L^1_V is not Dedekind σ-complete.

354 Notes and comments Apart from some of the exercises, the material of this section is pretty strictly confined to ideas which will be useful later in this volume. The basic Banach lattices of measure theory are the L^p spaces of Chapter 24; these all have Fatou norms with the Levi property (244Ye-244Yf), and for $p < \infty$ their norms are order-continuous (244Yd). In Chapter 36 I will return to these spaces in a more abstract context. Here I am mostly concerned to establish a vocabulary in which their various properties, and the relationships between these properties, can be expressed.

In normed Riesz spaces we have a very rich mixture of structures, and must take particular care over the concepts of 'boundedness', 'convergence' and 'density', which have more than one possible interpretation. In particular, we must scrupulously distinguish between 'order-bounded' and 'norm-bounded' sets. I have not yet formally introduced any of the various concepts of order-convergence (see §367), but I think that even so it is best to get into the habit of reminding oneself, when a convergent sequence appears, that it is convergent for the norm topology, rather than in any sense related directly to the order structure.

I should perhaps warn you that for the study of M-spaces 354L is not as helpful as it may look. The trouble is that apart from a few special cases (as in 354Xs) the topological space used in the representation is actually more complicated and mysterious than the M-space it is representing.

After the introduction of M-spaces, this section becomes a natural place for 'uniformly complete' spaces (354Yi). For the moment I leave these in the exercises. But I mention them now because they offer a straightforward route towards a theory of 'complex Riesz spaces' (354Yk). In large parts of functional analysis it is natural, and in some parts it is necessary, to work with normed spaces over $\mathbb{C}$ rather than over $\mathbb{R}$, and for L^2 spaces in particular it is useful to have a proper grasp of the complex case. And while the insights offered by the theory of Riesz spaces are not especially important in such areas, I think we should always seek connexions between different topics. So it is worth remembering that uniformly complete Riesz spaces have complexifications.

I shall have a great deal more to say about L-spaces when I come to spaces of additive functionals (§362) and to L^1 spaces again (§365) and to linear operators on them (§371); and before that, there will be something in the next section on their duals, and on L-spaces which are themselves dual spaces. For the moment I just give some easy results, direct translations of the corresponding facts in §246, which have natural expressions in the language of this section, holding deeper ideas over. In particular, the characterization of uniformly integrable sets as relatively weakly compact sets (247C) is valid in general L-spaces (356Q).

For an extensive treatment of Banach lattices, going very much deeper than I have space for in this volume, see Lindenstrauss & Tzafriri 79. For a careful exposition of a great deal of useful information, see Schaefer 74.

355 Spaces of linear operators

We come now to a discussion of linear operators between Riesz spaces. Of course linear operators are central to any kind of functional analysis, and a feature of the theory of Riesz spaces is the way the order structure picks out certain classes of operators for special consideration. Here I concentrate on positive and order-continuous operators, with a brief mention of sequential order-continuity. It turns out, in fact, that we need to work with operators which are differences of positive operators or of order-continuous positive operators. I define the basic spaces $\mathsf{L}^{\sim}$, $\mathsf{L}^{\times}$ and $\mathsf{L}^{\sim}_c$ (355A, 355G), with their most important properties (355B, 355E, 355H-355I) and some remarks on the special case of Banach lattices (355C, 355K). At the same time I give an important theorem on extension of operators (355F) and a corollary (355J).

The most important case is of course that in which the codomain is $\mathbb{R}$, so that our operators become real-valued functionals; I shall come to these in the next section.

355A Definition Let U and V be Riesz spaces. A linear operator $T : U \to V$ is **order-bounded** if $T[A]$ is order-bounded in V for every order-bounded $A \subseteq U$.

I will write $\mathsf{L}^{\sim}(U;V)$ for the set of order-bounded linear operators from U to V.

355B Lemma If U and V are Riesz spaces,

(a) a linear operator $T : U \to V$ is order-bounded iff $\{Tu : 0 \le u \le w\}$ is bounded above in V for every $w \in U^+$;

(b) in particular, any positive linear operator from U to V belongs to $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$;

(c) $\mathsf{L}^\sim$ is a linear space;

(d) if W is another Riesz space and $T : U \to V$ and $S : V \to W$ are order-bounded linear operators, then $ST : U \to W$ is order-bounded.

proof (a) This is elementary. If $T \in \mathsf{L}^\sim$ and $w \in U^+$, $[0,w]$ is order-bounded, so its image must be order-bounded in V, and in particular bounded above. On the other hand, if T satisfies the condition, and A is order-bounded, then $A \subseteq [u_1,u_2]$ for some $u_1 \le u_2$, and

$$T[A] \subseteq T[u_1 + [0,u_2 - u_1]] = Tu_1 + T[[0,u_2 - u_1]]$$

is bounded above; similarly, $T[-A]$ is bounded above, so $T[A]$ is bounded below; as A is arbitrary, T is order-bounded.

(b) If T is positive then $\{Tu : 0 \le u \le w\}$ is bounded above by Tw for every $w \ge 0$, so $T \in \mathsf{L}^\sim$.

(c) If $T_1, T_2 \in \mathsf{L}^\sim$, $\alpha \in \mathbb{R}$ and $A \subseteq U$ is order-bounded, then there are $v_1, v_2 \in V$ such that $T_i[A] \subseteq [-v_i, v_i]$ for both i. Setting $v = (1+|\alpha|)v_1 + v_2$, $(\alpha T_1 + T_2)[A] \subseteq [-v,v]$; as A is arbitrary, $\alpha T_1 + T_2$ belongs to $\mathsf{L}^\sim$; as α, T_1, T_2 are arbitrary, and since the zero operator surely belongs to $\mathsf{L}^\sim$, $\mathsf{L}^\sim$ is a linear subspace of the space of all linear operators from U to V.

(d) This is immediate from the definition; if $A \subseteq U$ is order-bounded, then $T[A] \subseteq V$ and $(ST)[A] = S[T[A]] \subseteq W$ are order-bounded.

355C Theorem If U and V are Banach lattices then every order-bounded linear operator (in particular, every positive linear operator) from U to V is continuous.

proof ? Suppose, if possible, that $T : U \to V$ is an order-bounded linear operator which is not continuous. Then for each $n \in \mathbb{N}$ we can find a $u_n \in U$ such that $\|u_n\| \le 2^{-n}$ but $\|Tu_n\| \ge n$. Now $u = \sup_{n\in\mathbb{N}} |u_n|$ is defined in U (354C), and there is a $v \in V$ such that $-v \le Tw \le v$ whenever $-u \le w \le u$; but this means that $\|v\| \ge \|Tu_n\| \ge n$ for every n, which is impossible. **X**

355D Lemma Let U be a Riesz space and V any linear space over $\mathbb{R}$. Then a function $T : U^+ \to V$ extends to a linear operator from U to V iff

$$T(u+u') = Tu + Tu', \quad T(\alpha u) = \alpha Tu$$

for all u, $u' \in U^+$ and every $\alpha > 0$, and in this case the extension is unique.

proof For in this case we can, and must, set

$$T_1 u = Tu_1 - Tu_2 \text{ whenever } u_1,\, u_2 \in U^+ \text{ and } u = u_1 - u_2;$$

it is elementary to check that this defines $T_1 u$ uniquely for every $u \in U$, and that T_1 is a linear operator extending T.

355E Theorem Let U be a Riesz space and V a Dedekind complete Riesz space.

(a) The space $\mathsf{L}^\sim$ of order-bounded linear operators from U to V is a Dedekind complete Riesz space; its positive cone is the set of positive linear operators from U to V. In particular, every order-bounded linear operator from U to V is expressible as the difference of positive linear operators.

(b) For $T \in \mathsf{L}^\sim$, T^+ and $|T|$ are defined in the Riesz space $\mathsf{L}^\sim$ by the formulae

$$T^+(w) = \sup\{Tu : 0 \le u \le w\},$$

$$|T|(w) = \sup\{Tu : |u| \le w\} = \sup\{\textstyle\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$$

for every $w \in U^+$.

(c) If S, $T \in \mathsf{L}^\sim$ then

$$(S \vee T)(w) = \sup_{0 \le u \le w} Su + T(w-u), \quad (S \wedge T)(w) = \inf_{0 \le u \le w} Su + T(w-u)$$

for every $w \in U^+$.

(d) Suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed. Then A is bounded above in $\mathsf{L}^\sim$ iff $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$, and in this case $(\sup A)(u) = \sup_{T \in A} Tu$ for every $u \ge 0$.

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $\inf A = 0$ in $\mathsf{L}^\sim$ iff $\inf_{T \in A} Tu = 0$ in V for every $u \in U^+$.

proof (a)(i) Suppose that $T \in \mathsf{L}^\sim$. For $w \in U^+$ set $R_T(w) = \sup\{Tu : 0 \le u \le w\}$; this is defined because V is Dedekind complete and $\{Tu : 0 \le u \le w\}$ is bounded above in V. Then $R_T(w_1 + w_2) = R_T w_1 + R_T w_2$ for all w_1, $w_2 \in U^+$. **P** Setting $A_i = [0, w_i]$ for each i, and $A = [0, w]$, then of course $A_1 + A_2 \subseteq A$; but also $A \subseteq A_1 + A_2$, because if $u \in A$ then $u = (u \wedge w_1) + (u - w_1)^+$, and $0 \le (u - w_1)^+ \le (w - w_1)^+ = w_2$, so $u \in A_1 + A_2$. Consequently

$$\begin{aligned} R_T w = \sup T[A] &= \sup T[A_1 + A_2] = \sup(T[A_1] + T[A_2]) \\ &= \sup T[A_1] + \sup T[A_2] = R_T w_1 + R_T w_2 \end{aligned}$$

by 351Dc. **Q** Next, it is easy to see that $R_T(\alpha w) = \alpha R_T w$ for $w \in U^+$, $\alpha > 0$, just because $u \mapsto \alpha u$, $v \mapsto \alpha v$ are isomorphisms of the partially ordered linear spaces U and V. It follows from 355D that we can extend R_T to a linear operator from U to V.

Because $R_T u \ge T0 = 0$ for every $u \in U^+$, R_T is a positive linear operator. But also $R_T u \ge Tu$ for every $u \in U^+$, so $R_T - T$ is also positive, and $T = R_T - (R_T - T)$ is the difference of two positive linear operators.

(ii) This shows that every order-bounded operator is a difference of positive operators. But of course if T_1 and T_2 are positive, then $(T_1 - T_2)u \le T_1 w$ whenever $0 \le u \le w$ in U, so that $T_1 - T_2$ is order-bounded, by the criterion in 355Ba. Thus $\mathsf{L}^\sim$ is precisely the set of differences of positive operators.

(iii) Just as in 351F, $\mathsf{L}^\sim$ is a partially ordered linear space if we say that $S \le T$ iff $Su \le Tu$ for every $u \in U^+$. Now it is a Riesz space. **P** Take any $T \in \mathsf{L}^\sim$. Then R_T, as defined in (i), is an upper bound for $\{0, T\}$ in $\mathsf{L}^\sim$. If $S \in \mathsf{L}^\sim$ is any other upper bound for $\{0, T\}$, then for any $w \in U^+$ we must have $Sw \ge Su \ge Tu$ whenever $u \in [0, w]$, so that $Sw \ge R_T w$; as w is arbitrary, $S \ge R_T$; as S is arbitrary, $R_T = \sup\{0, T\}$ in $\mathsf{L}^\sim$. Thus $\sup\{0, T\}$ is defined in $\mathsf{L}^\sim$ for every $T \in \mathsf{L}^\sim$; by 352B, $\mathsf{L}^\sim$ is a Riesz space. **Q**

(I defer the proof that it is Dedekind complete to (d-ii) below.)

(b) As remarked in (a-iii), $R_T = T^+$ for each $T \in \mathsf{L}^\sim$; but this is just the formula given for T^+. Now, if $T \in \mathsf{L}^\sim$ and $w \in U^+$,

$$\begin{aligned} |T|(w) = 2T^+ w - Tw &= 2 \sup_{u \in [0,w]} Tu - Tw \\ &= \sup_{u \in [0,w]} T(2u - w) = \sup_{u \in [-w,w]} Tu, \end{aligned}$$

which is the first formula offered for $|T|$. In particular, if $|u| \le w$ then Tu, $-Tu = T(-u)$ are both less than or equal to $|T|(w)$, so that $|Tu| \le |T|(w)$. So if $u_0, \dots, u_n$ are such that $\sum_{i=0}^n |u_i| = w$, then

$$\sum_{i=0}^n |Tu_i| \le \sum_{i=0}^n |T|(|u_i|) = |T|(w).$$

Thus $B = \{\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$ is bounded above by $|T|(w)$. On the other hand, if v is an upper bound for B and $|u| \le w$, then

$$Tu \le |Tu| + |T(w - |u|)| \le v;$$

as u is arbitrary, $|T|(w) \le v$; thus $|T|(w)$ is the least upper bound for B. This completes the proof of part (ii) of the theorem.

(c) We know that $S \vee T = T + (S - T)^+$ (352D), so that

$$\begin{aligned} (S \vee T)(w) = Tw + (S - T)^+(w) &= Tw + \sup_{0 \le u \le w} (S - T)(u) \\ &= \sup_{0 \le u \le w} Tw + (S - T)(u) = \sup_{0 \le u \le w} Su + T(w - u) \end{aligned}$$

for every $w \in U^+$, by the formula in (b). Also from 352D we have $S \wedge T = S + T - T \vee S$, so that

$$(S \wedge T)(w) = Sw + Tw - \sup_{0 \le u \le w} Tu + S(w-u)$$
$$= \inf_{0 \le u \le w} Sw + Tw - Tu - S(w-u)$$

(351Db)

$$= \inf_{0 \le u \le w} Su + T(w-u)$$

for $w \in U^+$.

(d)(i) Now suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed and that $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$. In this case, because V is Dedekind complete, we may set $Ru = \sup_{T \in A} Tu$ for every $u \in U^+$. Now $R(u_1 + u_2) = Ru_1 + Ru_2$ for all u_1, $u_2 \in U^+$. **P** Set $B_i = \{Tu_i : T \in A\}$ for each i, $B = \{T(u_1 + u_2) : T \in A\}$. Then $B \subseteq B_1 + B_2$, so

$$R(u_1 + u_2) = \sup B \le \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2.$$

On the other hand, if $v_i \in B_i$ for both i, there are $T_i \in A$ such that $v_i = T_i u_i$ for each i; because A is upwards-directed, there is a $T \in A$ such that $T \ge T_i$ for both i, and now

$$R(u_1 + u_2) \ge T(u_1 + u_2) = Tu_1 + Tu_2 \ge T_1 u_1 + t_2 u_2 = v_1 + v_2.$$

As v_1, v_2 are arbitrary,

$$R(u_1 + u_2) \ge \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2. \text{ **Q**}$$

It is also easy to see that $R(\alpha u) = \alpha Ru$ for every $u \in U^+$, $\alpha > 0$. So, using 355D again, R has an extension to a linear operator from U to V.

Now if we fix any $T_0 \in A$, we have $T_0 u \le Ru$ for every $u \in U^+$, so $R - T_0$ is a positive linear operator, and $R = (R - T_0) + T_0$ belongs to $\mathsf{L}^\sim$. Again, $Tu \le Ru$ for every $T \in A$, $u \in U^+$, so R is an upper bound for A in $\mathsf{L}^\sim$; and, finally, if S is any upper bound for A in $\mathsf{L}^\sim$, then Su is an upper bound for $\{Tu : T \in A\}$, and must be greater than or equal to Ru, for every $u \in U^+$; so that $R \le S$ and $R = \sup A$ in $\mathsf{L}^\sim$.

(ii) Now $\mathsf{L}^\sim$ is Dedekind complete. **P** If $A \subseteq \mathsf{L}^\sim$ is non-empty and bounded above by S say, then $A' = \{T_0 \vee T_1 \vee \ldots \vee T_n : T_0, \ldots, T_n \in A\}$ is upwards-directed and bounded above by S, so $\{Tu : T \in A'\}$ is bounded above by Su for every $u \in U^+$; by (i) just above, A' has a supremum in $\mathsf{L}^\sim$, which will also be a supremum for A. **Q**

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $-A = \{-T : T \in A\}$ is non-empty and upwards-directed, so

$$\inf A = 0 \iff \sup(-A) = 0$$
$$\iff \sup_{T \in A}(-Tw) = 0 \text{ for every } w \in U^+$$
$$\iff \inf_{T \in A} Tw = 0 \text{ for every } w \in U^+.$$

355F Theorem Let U and V be Riesz spaces and $U_0 \subseteq U$ a Riesz subspace which is *either* order-dense *or* a solid linear subspace. Suppose that $T_0 : U_0 \to V$ is an order-continuous positive linear operator such that $Su = \sup\{T_0 w : w \in U_0, 0 \le w \le u\}$ is defined in V for every $u \in U^+$. Then

(i) T_0 has an extension to an order-continuous positive linear operator $T : U \to V$.
(ii) If T_0 is a Riesz homomorphism so is T.
(iii) If U_0 is order-dense then T is unique.
(iv) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then so is T.

proof (a) I check first that

$$S(u + u') = Su + Su', \quad S(\gamma u) = \gamma Su$$

for all u, $u' \in U^+$ and $\gamma > 0$. **P** For every $u \in U$ set $A(u) = \{v : v \in U_0,\ 0 \le v \le u\}$. Note that for any $u \in U^+$, $v \in U_0^+$

$$v \wedge u = \sup_{w \in A(u)} v \wedge w = \sup A(v \wedge u);$$

this is because $\sup A(u) = u$ if U_0 is order-dense in U, while $v \wedge u \in A(u)$ if U_0 is solid. (i) If $v \in A(u + u')$ then

$$\begin{aligned}\sup(A(v \wedge u) + A(v - u)^+) &= \sup A(v \wedge u) + \sup A(v - u)^+ \\ &= v \wedge u + (v - u)^+ = v.\end{aligned}$$

Now

$$\begin{aligned}T_0 v &= \sup T_0[A(v \wedge u) + A(v - u)^+] \\ &= \sup(T_0[A(v \wedge u)] + T_0[A(v - u)^+]) \le Su + Su'\end{aligned}$$

because $(v - u)^+ \le u'$. As v is arbitrary, $S(u + u') \le Su + Su'$. (ii) Next,

$$\begin{aligned}Su + Su' &= \sup T_0[A(u)] + \sup T_0[A(u')] = \sup T_0[A(u) + A(u')] \\ &\le \sup T_0[A(u + u')] = S(u + u')\end{aligned}$$

because $A(u) + A(u') \subseteq A(u + u')$, so $S(u + u') = Su + Su'$. (iii) Of course $A(\gamma u) = \gamma A(u)$ so $S(\gamma u) = \gamma Su$. **Q**

By 355D, S has an extension to a linear operator from U to V; call this operator T. Of course $Tu = Su \ge 0$ whenever $u \ge 0$, so T is positive. If $u \in U_0^+$ then $Tu = Su = T_0 u$, so T extends T_0.

(b) If $B \subseteq U^+$ is non-empty and upwards-directed and has a supremum $u_0 \in U$, then of course $Tu \le Tu_0$ for every $u \in B$, so $\sup T[B] \le Tu_0$. On the other hand, for any $v \in A(u_0)$ we have

$$v = \sup_{u \in B} v \wedge u = \sup_{u \in B} \sup_{w \in A(u)} v \wedge w;$$

also $\bigcup_{u \in B} A(u)$ is upwards-directed, so

$$T_0 v = \sup\{T_0(v \wedge w) : w \in \bigcup_{u \in B} A(u)\} \le \sup_{u \in B} Tu.$$

As v is arbitrary, $Tu_0 = Su_0 \le \sup_{u \in B} Tu$. As B is arbitrary, T is order-continuous (351Ga).

(c) Now suppose that T_0 is a Riesz homomorphism. If $u \in U$ then, in the language of (a) above,

$$\begin{aligned}Tu^+ \wedge Tu^- &= \sup T_0[A(u^+)] \wedge T_0[A(u^-)] \\ &= \sup\{w \wedge w' : w \in T_0[A(u^+)],\ w' \in T_0[A(u^-)]\end{aligned}$$

(352Ea)

$$= \sup\{T_0(v \wedge v') : v \in A(u^+),\ v' \in A(u^-)\}$$

(because T_0 is a Riesz homomorphism)

$$= 0.$$

So T is a Riesz homomorphism (352G(iv)).

(d) If U_0 is order-dense, any order-continuous linear operator extending T_0 agrees with S and T on U^+, so is equal to T.

(e) Finally, if U_0 is order-dense and T_0 is an injective Riesz homomorphism, then for any non-zero $u \in U$ there is a non-zero $v \in U_0$ such that $|v| \le |u|$; so that

$$|Tu| = T|u| \ge T_0|v| > 0$$

because T is a Riesz homomorphism, by (c). As u is arbitrary, T is injective.

355G Definition Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\times(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of order-continuous positive linear operators, and $\mathsf{L}^\sim_c(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of sequentially order-continuous positive linear operators.

Because a composition of (sequentially) order-continuous functions is (sequentially) order-continuous, we shall have

$$ST \in \mathsf{L}^\times(U;W) \text{ whenever } S \in \mathsf{L}^\times(V;W),\, T \in \mathsf{L}^\times(U;V),$$

$$ST \in \mathsf{L}^\sim_c(U;W) \text{ whenever } S \in \mathsf{L}^\sim_c(V;W),\, T \in \mathsf{L}^\sim_c(U;V),$$

for all Riesz spaces U and all Dedekind complete Riesz spaces V, W.

355H Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then

(i) $\mathsf{L}^\times = \mathsf{L}^\times(U;V)$ is a band in $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$, therefore a Dedekind complete Riesz space in its own right

(ii) a member T of $\mathsf{L}^\sim$ belongs to $\mathsf{L}^\times$ iff $|T|$ is order-continuous.

proof There is a fair bit to check, but each individual step is easy enough.

(a) Suppose that S, T are order-continuous positive linear operators from U to V. Then $S+T$ is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then for any u_1, $u_2 \in A$ there is a $u \in A$ such that $u \le u_1$, $u \le u_2$, and now $(S+T)(u) \le Su_1 + Tu_2$. Consequently any lower bound for $(S+T)[A]$ must also be a lower bound for $S[A]+T[A]$. But since

$$\inf(S[A]+T[A]) = \inf S[A] + \inf T[A] = 0$$

(351Dc), $\inf(S+T)[A]$ must also be 0; as A is arbitrary, $S+T$ is order-continuous, by 351Ga. **Q**

(b) Consequently $S+T \in \mathsf{L}^\times$ for all S, $T \in \mathsf{L}^\times$. Since $-T$ and αT belong to $\mathsf{L}^\times$ for every $T \in \mathsf{L}^\times$ and $\alpha \ge 0$, we see that $\mathsf{L}^\times$ is a linear subspace of $\mathsf{L}^\sim$.

(c) If $T : U \to V$ is an order-continuous linear operator, $S : U \to V$ is linear and $0 \le S \le T$, then S is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then any lower bound of $S[A]$ must also be a lower bound of $T[A]$, so $\inf S[A] = 0$; as A is arbitrary, S is order-continuous. **Q**

It follows that $\mathsf{L}^\times$ is a solid linear subspace of $\mathsf{L}^\sim$. **P** If $T \in \mathsf{L}^\times$ and $|S| \le |T|$ in $\mathsf{L}^\sim$, express T as $T_1 - T_2$ where T_1, T_2 are order-continuous positive linear operators. Then

$$S^+,\, S^- \le |S| \le |T| \le T_1 + T_2,$$

so S^+ and S^- are order-continuous and $S = S^+ - S^- \in \mathsf{L}^\times$. **Q**

Accordingly $\mathsf{L}^\times$ is a Dedekind complete Riesz space in its own right (353J(b-i)).

(d) The argument of (c) also shows that if $T \in \mathsf{L}^\times$ then $|T|$ is order-continuous; so that for $T \in \mathsf{L}^\sim$,

$$T \in \mathsf{L}^\times \iff |T| \in \mathsf{L}^\times \iff |T| \text{ is order-continuous.}$$

(e) If $C \subseteq (\mathsf{L}^\times)^+$ is non-empty, upwards-directed and has a supremum $T \in L^\sim$, then T is order-continuous, so belongs to $\mathsf{L}^\times$. **P** Suppose that $A \subseteq U^+$ is non-empty, upwards-directed and has supremum w. Then

$$Tw = \sup_{S\in C} Sw = \sup_{S\in C} \sup_{u\in A} Su = \sup_{u\in A} Tu,$$

putting 355Ed and 351G(a-iii) together. So (using 351Ga again) T is order-continuous. **Q** Consequently $\mathsf{L}^\times$ is a band in $\mathsf{L}^\sim$ (352Ob).

This completes the proof.

355I Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\sim_c(U;V)$ is a band in $\mathsf{L}^\sim(U;V)$, and a member T of $\mathsf{L}^\sim(U;V)$ belongs to $\mathsf{L}^\sim_c(U;V)$ iff $|T|$ is sequentially order-continuous.

proof Copy the arguments of 355H.

355J Proposition Let U be a Riesz space and V a Dedekind complete Riesz space. Let $U_0 \subseteq U$ be an order-dense Riesz subspace; then $T \mapsto T\restriction U_0$ is an embedding of $\mathsf{L}^\times(U;V)$ as a solid linear subspace of $\mathsf{L}^\times(U_0;V)$. In particular, any $T_0 \in \mathsf{L}^\times(U_0;V)$ has at most one extension in $\mathsf{L}^\times(U;V)$.

proof (a) Because the embedding $U_0 \subseteq U$ is positive and order-continuous (352Nb), $T\restriction U_0$ is positive and order-continuous whenever T is; so $T\restriction U_0 \in \mathsf{L}^\times(U_0;V)$ whenever $T \in \mathsf{L}^\times(U;V)$. Because the map $T \mapsto T\restriction U_0$ is linear, the image W of $\mathsf{L}^\times(U;V)$ is a linear subspace of $\mathsf{L}^\times(U_0;V)$.

(b) If $T \in \mathsf{L}^\times(U;V)$ and $T\restriction U_0 \geq 0$, then $T \geq 0$. **P?** Suppose, if possible, that there is a $u \in U^+$ such that $Tu \not\geq 0$. Because $|T| \in \mathsf{L}^\times(U;V)$ is order-continuous and $A = \{v : v \in U_0, v \leq u\}$ is an upwards-directed set with supremum u, $\inf\{|T|(u-v) : v \in A\} = 0$ and there is a $v \in A$ such that $Tu+|T|(u-v) \not\geq 0$. But $Tv = Tu + T(v-u) \leq Tu + |T|(u-v)$ so $Tv \not\geq 0$ and $T\restriction U_0 \not\geq 0$. **XQ**

This shows that the map $T \mapsto T\restriction U_0$ is an order-isomorphism between $\mathsf{L}^\times(U;V)$ and W, and in particular is injective.

(c) Now suppose that $S_0 \in W$ and that $|S| \leq |S_0|$ in $\mathsf{L}^\times(U_0;V)$. Then $S \in W$. **P** Take $T_0 \in \mathsf{L}^\times(U;V)$ such that $T_0\restriction U_0 = S_0$. Then $S_1 = |T_0|\restriction U_0$ is a positive member of W such that $S_0 \leq S_1$, $-S_0 \leq S_1$, so $S^+ \leq S_1$. Consequently, for any $u \in U^+$,

$$\sup\{S^+v : v \in U_0,\, 0 \leq v \leq u\} \leq \sup\{S_1 v : v \in U_0,\, 0 \leq v \leq u\} \leq |T_0|(u)$$

is defined in V (recall that we are assuming that V is Dedekind complete). But this means that S^+ has an extension to an order-continuous positive linear operator from U to V (355F), and belongs to W. Similarly, $S^- \in W$, so $S \in W$. **Q**

This shows that W is a solid linear subspace of $\mathsf{L}^\times(U_0;V)$, as claimed.

355K Proposition Let U be a Banach lattice with an order-continuous norm.

(a) If V is any Archimedean Riesz space and $T : U \to V$ is a positive linear operator, then T is order-continuous.

(b) If V is a Dedekind complete Riesz space then $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V)$.

proof (a) Suppose that $A \subseteq U^+$ is non-empty and downwards-directed and has infimum 0. Then for each $n \in \mathbb{N}$ there is a $u_n \in A$ such that $\|u_n\| \leq 4^{-n}$. By 354C, $u = \sup_{n\in\mathbb{N}} 2^n u_n$ is defined in U. Now $Tu_n \leq 2^{-n}Tu$ for every n, so any lower bound for $T[A]$ must also be a lower bound for $\{2^{-n}Tu : n \in \mathbb{N}\}$ and therefore (because V is Archimedean) less than or equal to 0. Thus $\inf T[A] = 0$; as A is arbitrary, T is order-continuous.

(b) This is now immediate from 355Ea and the definition of $\mathsf{L}^\times$.

355X Basic exercises >(a) Let U and V be arbitrary Riesz spaces. (i) Show that the set $\mathsf{L}(U;V)$ of all linear operators from U to V is a partially ordered linear space if we say that $S \leq T$ whenever $Su \leq Tu$ for every $u \in U^+$. (ii) Show that if U and V are Banach lattices then the set of positive operators is closed in the normed space $\mathsf{B}(U;V)$ of bounded linear operators from U to V.

>(b) If U is a Riesz space and $\|\;\|$, $\|\;\|'$ are two norms on U both rendering it a Banach lattice, show that they are equivalent, that is, give rise to the same topology.

(c) Let U be a Riesz space with a Riesz norm, V an Archimedean Riesz space with an order unit, and $T : U \to V$ a linear operator which is continuous for the given norm on U and the order-unit norm on V. Show that T is order-bounded.

(d) Let U be a Riesz space, V an Archimedean Riesz space, and $T : U^+ \to V^+$ a map such that $T(u_1 + u_2) = Tu_1 + Tu_2$ for all u_1, $u_2 \in U^+$. Show that T has an extension to a linear operator from U to V.

>(e) Show that if r, $s \geq 1$ are integers then the Riesz space $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$ can be identified with the space of real $s \times r$ matrices, saying that a matrix is positive iff every coefficient is positive, so that if $T = \langle\tau_{ij}\rangle_{1\leq i\leq s,1\leq j\leq r}$ then $|T|$, taken in $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$, is $\langle|\tau_{ij}|\rangle_{1\leq i\leq s,1\leq j\leq r}$. Show that a matrix represents a Riesz homomorphism iff each row has at most one non-zero coefficient.

>(f) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that if $T_0,\dots,T_n\in\mathsf{L}^\sim(U;V)$ then

$$(T_0\vee\ldots\vee T_n)(w)=\sup\{\sum_{i=0}^n T_iu_i : u_i\ge 0\ \forall\, i\le n,\ \sum_{i=0}^n u_i=w\}$$

for every $w\in U^+$.

>(g) Let U be a Riesz space, V a Dedekind complete Riesz space, and $A\subseteq\mathsf{L}^\sim(U;V)$ a non-empty set. Show that A is bounded above in $\mathsf{L}^\sim(U;V)$ iff $C_w=\{\sum_{i=0}^n T_iu_i : T_0,\dots,T_n\in A,\ u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=w\}$ is bounded above in V for every $w\in U^+$, and in this case $(\sup A)(w)=\sup C_w$ for every $w\in U^+$.

355Y Further exercises (a) Let U and V be Banach lattices. For $T\in\mathsf{L}^\sim=\mathsf{L}^\sim(U;V)$, set $\|T\|_\sim=\sup_{w\in U^+,\|w\|\le 1}\inf\{\|v\| : |Tu|\le v \text{ whenever } |u|\le w\}$. Show that $\|\,\|_\sim$ is a norm on $\mathsf{L}^\sim$ under which $\mathsf{L}^\sim$ is a Banach space, and that the set of positive linear operators is closed in $\mathsf{L}^\sim$.

(b) Give an example of a continuous linear operator from ℓ^2 to itself which is not order-bounded.

(c) Let U and V be Riesz spaces and $T:U\to V$ a linear operator. (i) Show that for any $w\in U^+$, $C_w=\{\sum_{i=0}^n|Tu_i| : u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=w\}$ is upwards-directed, and has the same upper bounds as $\{Tu : |u|\le w\}$. (*Hint*: induce on m and n to see that if $u_0,\dots,u_n,u_0',\dots,u_m'\in U^+$ are such that $\sum_{i=0}^n u_i=\sum_{j=0}^m u_j'$, there is a family $\langle u_{ij}\rangle_{i\le n,j\le m}$ in U^+ such that $\sum_{j=0}^m u_{ij}=u_i$ for every $i\le n$, $\sum_{i=0}^n u_{ij}=u_j'$ for every $j\le m$.) (ii) Show that if $\sup C_w$ is defined for every $w\in U^+$, then $S=T\vee(-T)$ is defined in the partially ordered linear space $\mathsf{L}^\sim(U;V)$ and $Sw=\sup C_w$ for every $w\in U^+$.

(d) Let U, V and W be Riesz spaces, of which V and W are Dedekind complete. (i) Show that for any $S\in\mathsf{L}^\times(V;W)$, the map $T\mapsto ST:L^\sim(U;V)\to\mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(\mathsf{L}^\sim(U;V);\mathsf{L}^\sim(U;W))$, and is a Riesz homomorphism if S is. (*Hint*: 355Yc.) (ii) Show that for any $T\in\mathsf{L}^\sim(U;V)$, the map $S\mapsto ST:L^\sim(V;W)\to\mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(L^\sim(V;W);\mathsf{L}^\sim(U;W))$.

(e) Let μ be the usual measure on $\{0,1\}^{\mathbb{N}}$ and $\boldsymbol{c}$ the Banach lattice of convergent sequences. Find a linear operator $T:L^2(\mu)\to\boldsymbol{c}$ which is norm-continuous, therefore order-bounded, such that 0 and T have no common upper bound in the partially ordered linear space of all linear operators from $L^2(\mu)$ to $\boldsymbol{c}$.

(f) Let U and V be Banach lattices. Let $\mathsf{L}^{\mathrm{reg}}$ be the linear space of operators from U to V expressible as the difference of positive operators. For $T\in\mathsf{L}^{\mathrm{reg}}$ let $\|T\|_{\mathrm{reg}}$ be

$$\inf\{\|T_1+T_2\| : T_1,\,T_2:U\to V \text{ are positive, } T=T_1-T_2\}.$$

Show that $\|\,\|_{\mathrm{reg}}$ is a norm under which $\mathsf{L}^{\mathrm{reg}}$ is complete.

(g) Let U and V be Riesz spaces. For this exercise only, say that $\mathsf{L}^\times(U;V)$ is to be the set of linear operators $T:U\to V$ such that whenever $A\subseteq U$ is non-empty, downwards-directed and has infimum 0 then $\{v : v\in V^+,\ \exists\, w\in A,\ |Tu|\le v \text{ whenever } |u|\le w\}$ has infimum 0 in V. (i) Show that $\mathsf{L}^\times(U;V)$ is a linear space. (ii) Show that if U is Archimedean then $\mathsf{L}^\times(U;V)\subseteq\mathsf{L}^\sim(U;V)$. (iii) Show that if U is Archimedean and V is Dedekind complete then this definition agrees with that of 355G. (iv) Show that for any Riesz spaces U, V and W, $ST\in\mathsf{L}^\times(U;W)$ for every $S\in\mathsf{L}^\times(V;W)$, $T\in\mathsf{L}^\times(U;V)$. (v) Show that if U and V are Banach lattices, then $\mathsf{L}^\times(U;V)$ is closed in $\mathsf{L}^\sim(U;V)$ for the norm $\|\,\|_\sim$ of 355Ya. (vi) Show that if V is Archimedean and U is a Banach lattice with an order-continuous norm, then $\mathsf{L}^\times(U;V)=\mathsf{L}^\sim(U;V)$.

(h) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that the band projection $P:\mathsf{L}^\sim(U;V)\to\mathsf{L}^\times(U;V)$ is given by the formula

$$(PT)(w)=\inf\{\sup_{u\in A}Tu : A\subseteq U^+ \text{ is non-empty, upwards-directed and has supremum } w\}$$

for every $w\in U^+$, $T\in(\mathsf{L}^\sim)^+$. (Cf. 362Bd.)

(i) Show that if U is a Riesz space with the countable sup property (241Yd), then $\mathsf{L}^\sim_c(U;V) = \mathsf{L}^\times(U;V)$ for every Dedekind complete Riesz space V.

(j) Let U and V be Riesz spaces, of which V is Dedekind complete, and U_0 a solid linear subspace of U. Show that the map $T \mapsto T{\restriction}U_0$ is an order-continuous Riesz homomorphism from $\mathsf{L}^\times(U;V)$ onto a solid linear subspace of $\mathsf{L}^\times(U_0;V)$.

(k) Let U be a uniformly complete Riesz space (354Yi) and V a Dedekind complete Riesz space. Let $U_{\mathbb{C}}$, $V_{\mathbb{C}}$ be their complexifications (354Yk). Show that the complexification of $\mathsf{L}^\sim(U;V)$ can be identified with the complex linear space of linear operators $T : U_{\mathbb{C}} \to V_{\mathbb{C}}$ such that $B_T(w) = \{|Tu| : |u| \le w\}$ is bounded above in V for every $w \in U^+$, and that now $|T|(w) = \sup B_T(w)$ for every $T \in \mathsf{L}^\sim(U;V)_{\mathbb{C}}$, $w \in U^+$. (*Hint*: if u, $v \in U$ and $|u+iv| = w$, then u and v can be simultaneously approximated for the order-unit norm $\| \|_w$ on the solid linear subspace generated by w by finite sums $\sum_{j=0}^n (\cos\theta_j)w_j$, $\sum_{j=0}^n (\sin\theta_j)w_j$ where $w_j \in U^+$, $\sum_{j=0}^n w_j = w$. Consequently $|T(u+iv)| \le |T|(w)$ for every $T \in \mathsf{L}^\sim_{\mathbb{C}}$.)

355 Notes and comments I have had to make some choices in the basic definitions of this chapter (355A, 355G). For Dedekind complete codomains V, there is no doubt what $\mathsf{L}^\sim(U;V)$ should be, since the order-bounded operators (in the sense of 355A) are just the differences of positive operators (335Ea). (These are sometimes called '**regular**' operators.) When V is not Dedekind complete, we have to choose between the two notions, as not every order-bounded operator need be regular (355Ye). In my previous book (Fremlin 74a) I chose the regular operators; I have still not encountered any really persuasive reason to settle definitively on either class. In 355G the technical complications in dealing with any natural equivalent of the larger space (see 355Yg) are such that I have settled for the narrower class, but explicitly restricting the definition to the case in which V is Dedekind complete. In the applications in this book, the codomains are nearly always Dedekind complete, so we can pass these questions by.

The elementary extension technique in 355D may recall the definition of the Lebesgue integral (122L-122M). In the same way, 351G may remind you of the theorem that a linear operator between normed spaces is continuous everywhere if it is continuous anywhere, or of the corresponding results about Boolean homomorphisms and additive functionals on Boolean algebras (313L, 326Ga, 326N).

Of course 355Ea is the central fact about the space $\mathsf{L}^\sim(U;V)$ for Dedekind complete V; because we get a new Riesz space from old ones, the prospect of indefinite recursion immediately presents itself. For Banach lattices, $\mathsf{L}^\sim(U;V)$ is a linear subspace of the space $\mathsf{B}(U;V)$ of bounded linear operators (355C); the question of when the two are equal will be of great importance to us. I give only the vaguest hints on how to show that they can be different (355Yb, 355Ye), but these should be enough to make it plain that equality is the exception rather than the rule. It is also very useful that we have effective formulae to describe the Riesz space operations on $\mathsf{L}^\sim(U;V)$ (355E, 355Xf-355Xg, 355Yc). You may wish to compare these with the corresponding formulae for additive functionals on Boolean algebras in 326Yj and 362B.

If we think of $\mathsf{L}^\sim$ as somehow corresponding to the space of bounded additive functionals on a Boolean algebra, the bands $\mathsf{L}^\sim_c$ and $\mathsf{L}^\times$ correspond to the spaces of countably additive and completely additive functionals. In fact (as will appear in §362) this correspondence is very close indeed. For the moment, all I have sought to establish is that $\mathsf{L}^\sim_c$ and $\mathsf{L}^\times$ are indeed bands. Of course any case in which $\mathsf{L}^\sim(U;V) = \mathsf{L}^\sim_c(U;V)$ or $\mathsf{L}^\sim_c(U;V) = \mathsf{L}^\times(U;V)$ is of interest (355Kb, 355Yi).

Between Banach lattices, positive linear operators are continuous (355C); it follows at once that the Riesz space structure determines the topology (355Xb), so that it is not to be wondered at that there are further connexions between the norm and the spaces $\mathsf{L}^\sim$ and $\mathsf{L}^\times$, as in 355K.

355F will be a basic tool in the theory of representations of Riesz spaces; if we can represent an order-dense Riesz subspace of U as a subspace of a Dedekind complete space V, we have at least some chance of expressing U also as a subspace of V. Of course it has other applications, starting with analysis of the dual spaces.

356 Dual spaces

As always in functional analysis, large parts of the theory of Riesz spaces are based on the study of linear functionals. Following the scheme of the last section, I define spaces $U^\sim$, $U_c^\sim$ and $U^\times$, the 'order-bounded', 'sequentially order-continuous' and 'order-continuous' duals of a Riesz space U (356A). These are Dedekind complete Riesz spaces (356B). If U carries a Riesz norm they are closely connected with the normed space dual U^*, which is itself a Banach lattice (356D). For each of them, we have a canonical Riesz homomorphism from U to the corresponding bidual. The map from U to $U^{\times\times}$ is particularly important (356I); when this map is an isomorphism we call U 'perfect' (356J). The last third of the section deals with L- and M-spaces and the duality between them (356N, 356P), with two important theorems on uniform integrability (356O, 356Q).

356A Definition Let U be a Riesz space.

(a) I write $U^\sim$ for the space $\mathcal{L}^\sim(U;\mathbb{R})$ of order-bounded real-valued linear functionals on U, the **order-bounded dual** of U.

(b) $U_c^\sim$ will be the space $\mathcal{L}_c^\sim(U;\mathbb{R})$ of differences of sequentially order-continuous positive real-valued linear functionals on U, the **sequentially order-continuous dual** of U.

(c) $U^\times$ will be the space $\mathcal{L}^\times(U;\mathbb{R})$ of differences of order-continuous positive real-valued linear functionals on U, the **order-continuous dual** of U.

Remark It is easy to check that the three spaces $U^\sim$, $U_c^\sim$ and $U^\times$ are in general different (356Xa-356Xc). But the examples there leave open the question: can we find a Riesz space U, for which $U_c^\sim \neq U^\times$, and which is actually Dedekind complete, rather than just Dedekind σ-complete, as in 356Xc? This leads to unexpectedly deep water; it is yet another form of the Banach-Ulam problem. Really this is a question for Volume 5, but in 363S below I collect the relevant ideas which are within the scope of the present volume.

356B Theorem For any Riesz space U, its order-bounded dual $U^\sim$ is a Dedekind complete Riesz space in which $U_c^\sim$ and $U^\times$ are bands, therefore Dedekind complete Riesz spaces in their own right. For $f \in U^\sim$, f^+ and $|f| \in U^\sim$ are defined by the formulae

$$f^+(w) = \sup\{f(u) : 0 \le u \le w\}, \quad |f|(w) = \sup\{f(u) : |u| \le w\}$$

for every $w \in U^+$. A non-empty upwards-directed set $A \subseteq U^\sim$ is bounded above iff $\sup_{f\in A} f(u)$ is finite for every $u \in U$, and in this case $(\sup A)(u) = \sup_{f\in A} f(u)$ for every $u \in U^+$.

proof 355E, 355H, 355I.

356C Proposition Let U be any Riesz space and P a band projection on U. Then its adjoint $P' : U^\sim \to U^\sim$, defined by setting $P'(f) = fP$ for every $f \in U^\sim$, is a band projection on $U^\sim$.

proof (a) Because $P : U \to U$ is a positive linear operator, $P'f \in U^\sim$ for every $f \in U^\sim$ (355Bd), and P' is a positive linear operator from $U^\sim$ to itself. Set $Q = I - P$, the complementary band projection on U; then Q' is another positive linear operator on $U^\sim$, and $P'f + Q'f = f$ for every f. Now $P'f \wedge Q'f = 0$ for every $f \ge 0$. **P** For any $w \in U^+$,

$$\begin{aligned}(P'f - Q'f)^+(w) &= \sup_{0\le u\le w} (P'f - Q'f)(u) = \sup_{0\le u\le w} f(Pu - Qu) \\ &= f(Pw) = (P'f)(w),\end{aligned}$$

so $(P'f - Q'f)^+ = P'f$, that is, $P'f \wedge Q'f = 0$. **Q** By 352Rd, P' is a band projection.

356D Proposition Let U be a Riesz space with a Riesz norm.

(a) The normed space dual U^* of U is a solid linear subspace of $U^\sim$, and in itself is a Banach lattice with a Fatou norm and has the Levi property.

(b) The norm of U is order-continuous iff $U^* \subseteq U^\times$.

(c) If U is a Banach lattice, then $U^* = U^\sim$, so that $U^\sim$, $U^\times$ and $U^\sim_c$ are all Banach lattices.

(d) If U is a Banach lattice with order-continuous norm then $U^* = U^\times = U^\sim$.

proof (a)(i) If $f \in U^*$ then

$$\sup_{|u|\le w} f(u) \le \sup_{|u|\le w} \|f\|\|u\| = \|f\|\|w\| < \infty$$

for every $w \in U^+$, so $f \in U^\sim$ (355Ba). Thus $U^* \subseteq U^\sim$.

(ii) If f, $g \in U^*$ and $|f| \le |g|$, then for any $w \in U$

$$|f(w)| \le |f|(|w|) \le |g|(|w|) = \sup_{|u|\le|w|} g(u) \le \sup_{|u|\le|w|} \|g\|\|u\| \le \|g\|\|w\|.$$

As w is arbitrary, $f \in U^*$ and $\|f\| \le \|g\|$; as f and g are arbitrary, U^* is a solid linear subspace of $U^\sim$ and the norm of U^* is a Riesz norm. Because U^* is a Banach space it is also a Banach lattice.

(iii) If $A \subseteq (U^*)^+$ is non-empty, upwards-directed and $M = \sup_{f\in A} \|f\|$ is finite, then $\sup_{f\in A} f(u) \le M\|u\|$ is finite for every $u \in U^+$, so $g = \sup A$ is defined in $U^\sim$ (355Ed). Now $g(u) = \sup_{f\in A} f(u)$ for every $u \in U^+$, as also noted in 355Ed, so

$$|g(u)| \le g(|u|) \le M\||u|\| = M\|u\|$$

for every $u \in U$, and $\|g\| \le M$. But as A is arbitrary, this simultaneously proves that the norm of $U^\sim$ is Fatou and has the Levi property.

(b)(i) Suppose that the norm is order-continuous. If $f \in U^*$ and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0, then

$$\inf_{u\in A} |f|(u) \le \inf_{u\in A} \|f\|\|u\| = 0,$$

so $|f| \in U^\times$ and $f \in U^\times$. Thus $U^* \subseteq U^\times$.

(ii) Now suppose that the norm is not order-continuous. Then there is a non-empty downwards-directed set $A \subseteq U$, with infimum 0, such that $\inf_{u\in A} \|u\| = \delta > 0$. Set

$$B = \{v : v \ge u \text{ for some } u \in A\}.$$

Then B is convex. **P** If v_1, $v_2 \in B$ and $\alpha \in [0,1]$, there are u_1, $u_2 \in A$ such that $v_i \ge u_i$ for both i; now there is a $u \in A$ such that $u \le u_1 \wedge u_2$, so that

$$u = \alpha u + (1-\alpha)u \le \alpha v_1 + (1-\alpha)v_2,$$

and $\alpha v_1 + (1-\alpha)v_2 \in B$. **Q** Also $\inf_{v\in B} \|v\| = \delta > 0$. By the Hahn-Banach theorem (3A5Cb), there is an $f \in U^*$ such that $\inf_{v\in B} f(v) > 0$. But now

$$\inf_{u\in A} |f|(u) \ge \inf_{u\in A} f(u) > 0$$

and $|f|$ is not order-continuous; so $U^* \not\subseteq U^\times$.

(c) By 355C, $U^\sim \subseteq U^*$, so $U^\sim = U^*$. Now $U^\times$ and $U^\sim_c$, being bands, are closed linear subspaces (354Bd), so are Banach lattices in their own right.

(d) Put (b) and (c) together.

356E Biduals If you have studied any functional analysis at all, it will come as no surprise that duals-of-duals are important in the theory of Riesz spaces. I start with a simple lemma.

Lemma Let U be a Riesz space and $f : U \to \mathbb{R}$ a positive linear functional. Then for any $u \in U^+$ there is a positive linear functional $g : U \to \mathbb{R}$ such that $0 \le g \le f$, $g(u) = f(u)$ and $g(v) = 0$ whenever $u \wedge v = 0$.

proof Set $g(v) = \sup_{\alpha\ge 0} f(v \wedge \alpha u)$ for every $v \in U^+$. Then it is easy to see that $g(\beta v) = \beta g(v)$ for every $v \in U^+$, $\beta \in [0,\infty[$. If v, $w \in U^+$ then

$$(v \wedge \alpha u) + (w \wedge \alpha u) \le (v + w) \wedge 2\alpha u \le (v \wedge 2\alpha u) + (w \wedge 2\alpha u)$$

for every $\alpha \ge 0$ (352Fa), so $g(v + w) = g(v) + g(w)$. Accordingly g has an extension to a linear functional from U to $\mathbb{R}$ (355D). Of course $0 \le g(v) \le f(v)$ for $v \ge 0$, so $0 \le g \le f$ in $U^\sim$. We have $g(u) = f(u)$, while if $u \wedge v = 0$ then $\alpha u \wedge v = 0$ for every $\alpha \ge 0$, so $g(v) = 0$.

356F Theorem Let U be a Riesz space and V a solid linear subspace of $U^\sim$. For $u \in U$ define $\hat{u} : V \to \mathbb{R}$ by setting $\hat{u}(f) = f(u)$ for every $f \in V$. Then $u \mapsto \hat{u}$ is a Riesz homomorphism from U to $V^\times$.

proof (a) By the definition of addition and scalar multiplication in V, $\hat{u}$ is linear for every u; also $\widehat{\alpha u} = \alpha\hat{u}$, $(u_1 + u_2)\hat{} = \hat{u}_1 + \hat{u}_2$ for all u, u_1, $u_2 \in U$ and $\alpha \in \mathbb{R}$. If $u \geq 0$ then $\hat{u}(f) = f(u) \geq 0$ for every $f \in V^+$, so $\hat{u} \geq 0$; accordingly every $\hat{u}$ is the difference of two positive functionals, and $u \mapsto \hat{u}$ is a linear operator from U to $V^\sim$.

(b) If $B \subseteq V$ is a non-empty downwards-directed set with infimum 0, then $\inf_{f\in B} f(u) = 0$ for every $u \in U^+$, by 355Ee. But this means that $\hat{u}$ is order-continuous for every $u \in U^+$, so that $\hat{u} \in V^\times$ for every $u \in U$.

(c) If $u \wedge v = 0$ in U, then for any $f \in V^+$ there is a $g \in [0, f]$ such that $g(u) = f(u)$, $g(v) = 0$ (356D). So

$$(\hat{u} \wedge \hat{v})(f) \leq \hat{u}(f - g) + \hat{v}(g) = f(u) - g(u) + g(v) = 0.$$

As f is arbitrary, $\hat{u} \wedge \hat{v} = 0$. As u and v are arbitrary, $u \mapsto \hat{u}$ is a Riesz homomorphism (352G).

356G Lemma Suppose that U is a Riesz space such that $U^\sim$ separates the points of U. Then U is Archimedean.

proof ? Otherwise, there are u, $v \in U$ such that $v > 0$ and $nv \leq u$ for every $n \in \mathbb{N}$. Now there is an $f \in U^\sim$ such that $f(v) \neq 0$; but $f(v) \leq |f|(v) \leq \frac{1}{n}|f|(u)$ for every n, so this is impossible. **X**

356H Lemma Let U be an Archimedean Riesz space and $f > 0$ in $U^\times$. Then there is a $u \in U$ such that (i) $u > 0$ (ii) $f(v) > 0$ whenever $0 < v \leq u$ (iii) $g(u) = 0$ whenever $g \wedge f = 0$ in $U^\times$. Moreover, if $u_0 \in U^+$ is such that $f(u_0) > 0$, we can arrange that $u \leq u_0$.

proof (Because $f > 0$ there certainly is some $u_0 \in U$ such that $f(u_0) > 0$.)

(a) Set $A = \{v : 0 \leq v \leq u_0,\ f(v) = 0\}$. Then $(v_1 + v_2) \wedge u_0 \in A$ for all v_1, $v_2 \in A$, so A is upwards-directed. Because $f(u_0) > 0 = \sup f[A]$ and f is order-continuous, u_0 cannot be the least upper bound of A, and there is another upper bound u_1 of A strictly less than u_0.

Set $u = u_0 - u_1 > 0$. If $0 \leq v \leq u$ and $f(v) = 0$, then

$$w \in A \Longrightarrow w \leq u_1 \Longrightarrow w + v \leq u_0 \Longrightarrow w + v \in A;$$

consequently $nv \in A$ and $nv \leq u_0$ for every $n \in \mathbb{N}$, so $v = 0$. Thus u has properties (i) and (ii).

(b) Now suppose that $g \wedge f = 0$ in $U^\times$. Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$ there is a $v_n \in [0, u]$ such that $f(v_n) + g(u - v_n) \leq 2^{-n}\epsilon$ (355Ec). If $v \leq v_n$ for every $n \in \mathbb{N}$ then $f(v) = 0$ so $v = 0$; thus $\inf_{n\in\mathbb{N}} v_n = 0$. Set $w_n = \inf_{i\leq n} v_i$ for each $n \in \mathbb{N}$; then $\langle w_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 so (because g is order-continuous) $\inf_{n\in\mathbb{N}} g(w_n) = 0$. But

$$u - w_n = \sup_{i\leq n} u - v_i \leq \sum_{i=0}^n u - v_i,$$

so

$$g(u - w_n) \leq \sum_{i=0}^n g(u - v_i) \leq 2\epsilon$$

for every n, and

$$g(u) \leq 2\epsilon + \inf_{n\in\mathbb{N}} g(w_n) = 2\epsilon.$$

As ϵ is arbitrary, $g(u) = 0$; as g is arbitrary, u has the third required property.

356I Theorem Let U be any Archimedean Riesz space. Then the canonical map from U to $U^{\times\times}$ (356F) is an order-continuous Riesz homomorphism from U onto an order-dense Riesz subspace of $U^{\times\times}$. If U is Dedekind complete, its image in $U^{\times\times}$ is solid.

proof (a) By 356F, $u \mapsto \hat{u} : U \to U^{\times\times}$ is a Riesz homomorphism.

To see that it is order-continuous, take any non-empty downwards-directed set $A \subseteq U$ with infimum 0. Then $C = \{\hat{u} : u \in A\}$ is downwards-directed, and for any $f \in (U^\times)^+$

$$\inf_{\phi\in C}\phi(f) = \inf_{u\in A} f(u) = 0$$

because f is order-continuous. As f is arbitrary, $\inf C = 0$ (355Ee); as A is arbitrary, $u \mapsto \hat{u}$ is order-continuous (351Ga).

(b) Now suppose that $\phi > 0$ in $U^{\times\times}$. By 356H, there is an $f > 0$ in $U^\times$ such that $\phi(f) > 0$ and $\phi(g) = 0$ whenever $g \wedge f = 0$. Next, there is a $u > 0$ in U such that $f(u) > 0$. Since $U \geq 0$, $\hat{u} \geq 0$; since $\hat{u}(f) > 0$, $\hat{u} \wedge \phi > 0$.

Because $U^{\times\times}$ (being Dedekind complete) is Archimedean, $\inf_{\alpha>0}\alpha\hat{u} = 0$, and there is an $\alpha > 0$ such that

$$\psi = (\hat{u} \wedge \phi - \alpha\hat{u})^+ > 0.$$

Let $g \in (U^\times)^+$ be such that $\psi(g) > 0$ and $\theta(g) = 0$ whenever $\theta \wedge \psi = 0$ in $U^{\times\times}$. Let $v \in U^+$ be such that $g(v) > 0$ and $h(v) = 0$ whenever $h \wedge g = 0$ in $U^\times$.

Because $\hat{v}(g) = g(v) > 0$, $\hat{v} \wedge \psi > 0$. As $\psi \leq \hat{u}$, $\hat{v} \wedge \hat{u} > 0$ and $\hat{v} \wedge \alpha\hat{u} > 0$. Set $w = v \wedge \alpha u$; then $\hat{w} = \hat{v} \wedge \alpha\hat{u}$, by 356F, so $\hat{w} > 0$.

? Suppose, if possible, that $\hat{w} \not\leq \phi$. Then $\theta = (\hat{w} - \phi)^+ > 0$, so there is an $h \in (U^\times)^+$ such that $\theta(h) > 0$ and $\theta(h') > 0$ whenever $0 < h' \leq h$ (356H, for the fourth and last time). Now examine

$$\theta(h \wedge g) \leq (\alpha\hat{u} - \phi \wedge \hat{u})^+(g)$$

(because $\hat{w} \leq \alpha\hat{u}$, $\phi \wedge \hat{u} \leq \phi$, $h \wedge g \leq g$)

$$= 0$$

because $(\alpha\hat{u} - \phi \wedge \hat{u})^+ \wedge \psi = 0$. So $h \wedge g = 0$ and $h(v) = 0$. But this means that

$$\theta(h) \leq \hat{w}(h) \leq \hat{v}(h) = 0,$$

which is impossible. **X**

Thus $0 < \hat{w} \leq \phi$. As ϕ is arbitrary, the image $\hat{U}$ of U is quasi-order-dense in $U^{\times\times}$, therefore order-dense (353A).

(c) Now suppose that U is Dedekind complete and that $0 \leq \phi \leq \psi \in \hat{U}$. Express ψ as $\hat{u}$ where $u \in U$, and set $A = \{v : v \in U, v \leq u^+, \hat{v} \leq \phi\}$. If $v \in U$ and $0 \leq \hat{v} \leq \phi$, then $w = v^+ \wedge u^+ \in A$ and $\hat{w} = \hat{v}$; thus $\phi = \sup\{\hat{v} : v \in A\} = \hat{v}_0$, where $v_0 = \sup A$. So $\phi \in \hat{U}$. As ϕ and ψ are arbitrary, $\hat{U}$ is solid in $U^{\times\times}$.

356J Definition A Riesz space U is **perfect** if the canonical map from U to $U^{\times\times}$ is an isomorphism.

356K Proposition A Riesz space U is perfect iff (i) it is Dedekind complete (ii) $U^\times$ separates the points of U (iii) whenever $A \subseteq U$ is non-empty and upwards-directed and $\{f(u) : u \in A\}$ is bounded for every $f \in U^\times$, then A is bounded above in U.

proof (a) Suppose that U is perfect. Because it is isomorphic to $U^{\times\times}$, which is surely Dedekind complete, U also is Dedekind complete. Because the map $u \mapsto \hat{u} : U \to U^{\times\times}$ is injective, $U^\times$ separates the points of U. If $A \subseteq U$ is non-empty and upwards-directed ad $\{f(u) : u \in A\}$ is bounded above for every $f \in U^\times$, then $B = \{\hat{u} : u \in A\}$ is non-empty and upwards-directed and $\sup_{\phi\in B}\phi(f) < \infty$ for every $f \in U^\times$, so $\sup B$ is defined in $U^{\times\sim}$ (355Ed); but $U^{\times\times}$ is a band in $U^{\times\sim}$, so $\sup B \in U^{\times\times}$ and is of the form $\hat{w}$ for some $w \in U$. Because $u \mapsto \hat{u}$ is a Riesz space isomorphism, $w = \sup A$ in U. Thus U satisfies the three conditions.

(b) Suppose that U satisfies the three conditions. We know that $u \mapsto \hat{u}$ is a Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). It is injective because $U^\times$ separates the points of U. If $\phi \geq 0$ in $U^{\times\times}$, set $A = \{u : u \in U^+, \hat{u} \leq \phi\}$. Then A is non-empty and upwards-directed and for any $f \in U^\times$

$$\sup_{u\in A} f(u) \leq \sup_{u\in A}|f|(u) \leq \sup_{u\in A}\hat{u}(|f|) \leq \phi(|f|) < \infty,$$

so by condition (iii) A has an upper bound in U. Since U is Dedekind complete, $w = \sup A$ is defined in U. Now

$$\hat{w} = \sup_{u\in A}\hat{u} = \phi.$$

As ϕ is arbitrary, the image of U includes $(U^{\times\times})^+$, therefore is the whole of $U^{\times\times}$, and $u \mapsto \hat{u}$ is a bijective Riesz homomorphism, that is, a Riesz space isomorphism.

356L Proposition (a) Any band in a perfect Riesz space is a perfect Riesz space in its own right.
(b) For any Riesz space U, $U^\sim$ is perfect; consequently $U^\sim_c$ and $U^\times$ are perfect.

proof (a) I use the criterion of 356K. Let U be a perfect Riesz space and V a band in U. Then V is Dedekind complete because U is (353Jb). If $v \in V \setminus \{0\}$ there is an $f \in U^\times$ such that $f(v) \neq 0$; but the embedding $V \subseteq U$ is order-continuous (352N), so $g = f \restriction V$ belongs to $V^\times$, and $g(v) \neq 0$. Thus $V^\times$ separates the points of V. If $A \subseteq V$ is non-empty and upwards-directed and $\sup_{v \in A} g(v)$ is finite for every $g \in V^\times$, then $\sup_{v \in A} f(v) < \infty$ for every $f \in U^\times$ (again because $f \restriction V \in V^\times$), so A has an upper bound in U; because U is Dedekind complete, $\sup A$ is defined in U; because V is a band, $\sup A \in V$ and is an upper bound for A in V. Thus V satisfies the conditions of 356K and is perfect.

(b) $U^\sim$ is Dedekind complete, by 355Ea. If $f \in U^\sim \setminus \{0\}$, there is a $u \in U$ such that $f(u) \neq 0$; now $\hat{u}(f) \neq 0$, where $\hat{u} \in U^{\sim\times}$ (356F). Thus $U^{\sim\times}$ separates the points of $U^\sim$. If $A \subseteq U^\sim$ is non-empty and upwards-directed and $\sup_{f \in A} \phi(f)$ is finite for every $\phi \in U^{\sim\times}$, then, in particular,

$$\sup_{f \in A} f(u) = \sup_{f \in A} \hat{u}(f) < \infty$$

for every $u \in U$, so A is bounded above in $U^\sim$, by 355Ed. Thus $U^\sim$ satisfies the conditions of 356K and is perfect.

By (a), it follows at once that $U^\times$ and $U^\sim_c$ are perfect.

356M Proposition If U is a Banach lattice in which the norm is order-continuous and has the Levi property, then U is perfect.

proof By 356Db, $U^* = U^\times$; since U^* surely separates the points of U, so does $U^\times$. By 354Ee, U is Dedekind complete. If $A \subseteq U$ is non-empty and upwards-directed and $f[A]$ is bounded for every $f \in U^\times$, then A is norm-bounded, by the Uniform Boundedness Theorem (3A5Hb). Because the norm is supposed to have the Levi property, A is bounded above in U. Thus U satisfies all the conditions of 356K and is perfect.

356N L- and M-spaces I come now to the duality between L-spaces and M-spaces which I hinted at in §354.

Proposition Let U be an Archimedean Riesz space with an order-unit norm.
(a) $U^* = U^\sim$ is an L-space.
(b) If e is the standard order unit of U, then $\|f\| = |f|(e)$ for every $f \in U^*$.
(c) A linear functional $f : U \to \mathbb{R}$ is positive iff it belongs to U^* and $\|f\| = f(e)$.
(d) If $e \neq 0$ there is a positive linear functional f on U such that $f(e) = 1$.

proof (a)-(b) We know already that $U^* \subseteq U^\sim$ is a Banach lattice (356Da). If $f \in U^\sim$ then

$$\sup\{|f(u)| : \|u\| \leq 1\} = \sup\{|f(u)| : |u| \leq e\} = |f|(e),$$

so $f \in U^*$ and $\|f\| = |f|(e)$; thus $U^\sim = U^*$. If f, $g \geq 0$ in U^*, then

$$\|f + g\| = (f + g)(e) = f(e) + g(e) = \|f\| + \|g\|;$$

thus U^* is an L-space.

(c) As already remarked, if f is positive then $f \in U^*$ and $\|f\| = f(e)$. On the other hand, if $f \in U^*$ and $\|f\| = f(e)$, take any $u \geq 0$. Set $v = (1 + \|u\|)^{-1} u$. Then $0 \leq v \leq e$ and $\|e - v\| \leq 1$ and

$$f(e - v) \leq |f(e - v)| \leq \|f\| = f(e).$$

But this means that $f(v) \geq 0$ so $f(u) \geq 0$. As u is arbitrary, $f \geq 0$.

(d) By the Hahn-Banach theorem (3A5Ac), there is an $f \in U^*$ such that $f(e) = \|f\| = 1$; by (c), f is positive.

356O Theorem Let U be an Archimedean Riesz space with order-unit norm. Then a set $A \subseteq U^* = U^\sim$ is uniformly integrable iff it is norm-bounded and $\lim_{n \to \infty} \sup_{f \in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U^+.

proof (a) Suppose that A is uniformly integrable. Then it is surely norm-bounded (354Ra). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U^+ bounded above by w, then for any $\epsilon>0$ we can find an $h\ge 0$ in U^* such that $\|(|f|-h)^+\|\le\epsilon$ for every $f\in A$. Now $\sum_{i=0}^n h(u_i)\le h(w)$ for every n, and $\lim_{n\to 0}h(u_n)=0$; since at the same time

$$|f(u_n)|\le|f|(u_n)\le h(u_n)+(|f|-h)^+(u_n)\le h(u_n)+\epsilon\|u_n\|\le h(u_n)+\epsilon\|w\|$$

for every $f\in A$, $n\in\mathbb{N}$, $\limsup_{n\to\infty}\sup_{f\in A}|f|(u_n)\le\epsilon\|w\|$. As ϵ is arbitrary,

$$\lim_{n\to\infty}\sup_{f\in A}|f|(u_n)=0,$$

and the conditions are satisfied.

(b)(i) Now suppose that A is norm-bounded but not uniformly integrable. Write B for the solid hull of A, M for $\sup_{f\in A}\|f\|=\sup_{f\in B}\|f\|$; then there is a disjoint sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $B\cap(U^*)^+$ which is not norm-convergent to 0 (354R(b-iv)), that is,

$$\delta=\tfrac{1}{2}\limsup_{n\to\infty}g_n(e)=\tfrac{1}{2}\limsup_{n\to\infty}\|g_n\|>0,$$

where e is the standard order unit of U.

(ii) Set

$$C=\{v:0\le v\le e,\ \sup_{g\in B}g(v)\ge\delta\},$$

$$D=\{w:0\le w\le e,\ \limsup_{n\to\infty}g_n(w)>\delta\}.$$

Then for any $u\in D$ we can find $v\in C$, $w\in D$ such that $v\wedge w=0$. **P** Set $\delta'=\limsup_{n\to\infty}g_n(u)$, $\eta=(\delta'-\delta)/(3+M)>0$; take $k\in\mathbb{N}$ so large that $k\eta\ge M$.

Because $g_n(u)\ge\delta'-\eta$ for infinitely many n, we can find a set $K\subseteq\mathbb{N}$, of size k, such that $g_i(u)\ge\delta'-\eta$ for every $i\in K$. Now we know that, for each $i\in K$, $g_i\wedge k\sum_{j\in K,j\ne i}g_j=0$, so there is a $v_i\le u$ such that $g_i(u-v_i)+k\sum_{j\in K,j\ne i}g_j(v_i)\le\eta$ (355Ec). Now

$$g_i(v_i)\ge g_i(u)-\eta\ge\delta'-2\eta,\quad g_i(v_j)\le\frac{\eta}{k}\text{ for }i,\,j\in K,\,i\ne j.$$

Set $v_i'=(v_i-\sum_{j\in K,j\ne i}v_j)^+$ for each $i\in K$; then

$$g_i(v_i')\ge g_i(v_i)-\sum_{j\in K,j\ne i}g_i(v_j)\ge\delta'-3\eta$$

for every $i\in K$, while $v_j'\wedge v_i'=0$ for distinct i, $j\in K$.

For each $n\in\mathbb{N}$,

$$\sum_{i\in K}g_n(u\wedge\frac{1}{\eta}v_i')\le g_n(u)\le\|g_n\|\le\eta k,$$

so there is some $i(n)\in K$ such that

$$g_n(u\wedge\frac{1}{\eta}v_{i(n)}')\le\eta,\quad g_n(u-\frac{1}{\eta}v_{i(n)}')^+\ge g_n(u)-\eta.$$

Since $\{n:g_n(u)\ge\delta+2\eta\}$ is infinite, there is some $m\in K$ such that $J=\{n:g_n(u)\ge\delta+2\eta,\ i(n)=m\}$ is infinite. Try

$$v=(v_m'-\eta u)^+,\quad w=(u-\frac{1}{\eta}v_m')^+.$$

Then v, $w\in[0,u]$ and $v\wedge w=0$. Next,

$$g_m(v)\ge g_m(v_m')-\eta M\ge\delta'-3\eta-\eta M=\delta,$$

so $v\in C$, while for any $n\in J$

$$g_n(w)=g_n(u-\frac{1}{\eta}v_{i(n)}')^+\ge g_n(u)-\eta\ge\delta+\eta;$$

since J is infinite,

$$\limsup_{n\to\infty}g_n(w)\ge\delta+\eta>\delta$$

and $w \in D$. **Q**

(iii) Since $e \in D$, we can choose inductively sequences $\langle w_n \rangle_{n\in\mathbb{N}}$ in D, $\langle v_n \rangle_{n\in\mathbb{N}}$ in C such that $w_0 = e$, $v_n \wedge w_{n+1} = 0$, $v_n \vee w_{n+1} \le w_n$ for every $n \in \mathbb{N}$. But in this case $\langle v_n \rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $[0,u]$, while for each $n \in \mathbb{N}$, we can find $f_n \in A$ such that $|f_n|(v_n) > \frac{2}{3}\delta$. Now there is a $u_n \in [0,v_n]$ such that $|f_n(u_n)| \ge \frac{1}{3}\delta$. **P** Set $\gamma = \sup_{0\le v\le v_n} |f_n(v)|$. Then $f_n^+(v_n)$, $f_n^-(v_n)$ are both less than or equal to γ, so $|f_n|(v_n) \le 2\gamma$ and $\gamma > \frac{1}{3}\delta$; so there is a $u_n \in [0,v_n]$ such that $|f_n(u_n)| \ge \frac{1}{3}\delta$. **Q**

Accordingly we have a disjoint sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $[0,e]$ such that $\sup_{f\in A} |f(u_n)| \ge \frac{1}{3}\delta$ for every $n \in \mathbb{N}$.

(iv) All this is on the assumption that A is norm-bounded and not uniformly integrable. So, turning it round, we see that if A is norm-bounded and $\lim_{n\to\infty} \sup_{f\in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n \rangle_{n\in\mathbb{N}}$, A must be uniformly integrable.

This completes the proof.

356P Proposition Let U be an L-space.

(a) U is perfect.

(b) $U^* = U^\sim = U^\times$ is an M-space; its standard order unit is the functional $\int$ defined by setting $\int u = \|u^+\| - \|u^-\|$ for every $u \in U$.

(c) If $A \subseteq U$ is non-empty and upwards-directed and $\sup_{u\in A} \int u$ is finite, then $\sup A$ is defined in U and $\int \sup A = \sup_{u\in A} \int u$.

proof (a) By 354N we know that the norm on U is order-continuous and has the Levi property, so 356M tells us that U is perfect.

(b) 356Dd tells us that $U^* = U^\sim = U^\times$.

The L-space property tells us that the functional $u \mapsto \|u\| : U^+ \to \mathbb{R}$ is additive; of course it is also homogeneous, so by 355D it has an extension to a linear functional $\int : U \to \mathbb{R}$ satisfying the given formula. Because $\int u = \|u\| \ge 0$ for $u \ge 0$, $\int \in (U^\sim)^+$. For $f \in U^\sim$,

$$\begin{aligned} |f| \le \int &\iff |f|(u) \le \int u \text{ for every } u \in U^+ \\ &\iff |f(v)| \le \|u\| \text{ whenever } |v| \le u \in U \\ &\iff |f(v)| \le \|v\| \text{ for every } v \in U \\ &\iff \|f\| \le 1, \end{aligned}$$

so the norm on $U^* = U^\sim$ is the order-unit norm defined from $\int$, and $U^\sim$ is an M-space, as claimed.

(c) Fix $u_0 \in A$, and set $B = \{u^+ : u \in A,\, u \ge u_0\}$. Then $B \subseteq U^+$ is upwards-directed, and

$$\begin{aligned} \sup_{v\in B} \|v\| = \sup_{u\in A, u\ge u_0} \int u^+ &= \sup_{u\in A, u\ge u_0} \int u + \int u^- \\ &\le \sup_{u\in A, u\ge u_0} \int u + \int u_0^- < \infty. \end{aligned}$$

Because $\|\;\|$ has the Levi property, B is bounded above. But (because A is upwards-directed) every member of A is dominated by some member of B, so A also is bounded above. Because U is Dedekind complete, $\sup A$ is defined in U. Finally, $\int \sup A = \sup_{u\in A} \int u$ because $\int$, being a positive member of $U^\times$, is order-continuous.

356Q Theorem Let U be any L-space. Then a subset of U is uniformly integrable iff it is relatively weakly compact.

proof (a) Let $A \subseteq U$ be a uniformly integrable set.

(i) Suppose that $\mathcal{F}$ is an ultrafilter on X containing A. Then $A \ne \emptyset$. Because A is norm-bounded, $\sup_{u\in A} |f(u)| < \infty$ and $\phi(f) = \lim_{u\to\mathcal{F}} f(u)$ is defined in $\mathbb{R}$ for every $f \in U^*$ (2A3Se).

If f, $g \in U^*$ then

$$\phi(f+g) = \lim_{u\to\mathcal{F}} f(u) + g(u) = \lim_{u\to\mathcal{F}} f(u) + \lim_{u\to\mathcal{F}} g(u) = \phi(f) + \phi(g)$$

(2A3Sf). Similarly,

$$\phi(\alpha f) = \lim_{u\to\mathcal{F}} \alpha f(u) = \alpha\phi(f)$$

for every $f \in U^*$, $\alpha \in \mathbb{R}$. Thus $\phi : U^* \to \mathbb{R}$ is linear. Also

$$|\phi(f)| \le \sup_{u\in A} |f(u)| \le \|f\| \sup_{u\in A} \|u\|,$$

so $\phi \in U^{**} = U^{*\sim}$.

(ii) Now the point of this argument is that $\phi \in U^{*\times}$. **P** Suppose that $B \subseteq U^*$ is non-empty and downwards-directed and has infimum 0. Fix $f_0 \in B$. Let $\epsilon > 0$. Then there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, which means that

$$|f(u)| \le |f|(|u|) \le |f|(w) + |f|(|u| - w)^+ \le |f|(w) + \epsilon\|f\|$$

for every $f \in U^*$ and every $u \in A$. Accordingly $|\phi(f)| \le |f|(w) + \epsilon\|f\|$ for every $f \in U^*$. Now $\inf_{f\in B} f(w) = 0$ (using 355Ee, as usual), so there is an $f_1 \in B$ such that $f_1 \le f_0$ and $f_1(w) \le \epsilon$. In this case

$$|\phi|(f_1) = \sup_{|f|\le f_1} |\phi(f)| \le \sup_{|f|\le f_1} |f|(w) + \epsilon\|f\| \le f_1(w) + \epsilon\|f_1\| \le \epsilon(1 + \|f_0\|).$$

As ϵ is arbitrary, $\inf_{f\in B} |\phi|(f) = 0$; as B is arbitrary, $|\phi|$ is order-continuous and $\phi \in U^{*\times}$. **Q**

(iii) At this point, we recall that $U^* = U^\times$ and that the canonical map from U to $U^{\times\times}$ is surjective (356P). So there is a $u_0 \in U$ such that $\hat{u}_0 = \phi$. But now we see that

$$f(u_0) = \phi(f) = \lim_{u\to\mathcal{F}} f(u)$$

for every $f \in U^*$; which is just what is meant by saying that $\mathcal{F} \to u_0$ for the weak topology on U (2A3Sd).

Accordingly every ultrafilter on U containing A has a limit in U. But because the weak topology on U is regular (3A3Be), it follows that the closure of A for the weak topology is compact (3A3De), so that A is relatively weakly compact.

(b) For the converse I use the criterion of 354R(b-iv). Suppose that $A \subseteq U$ is relatively weakly compact. Then A is norm-bounded, by the Uniform Boundedness Theorem. Now let $\langle u_n\rangle_{n\in\mathbb{N}}$ be any disjoint sequence in the solid hull of A. For each n, let U_n be the band of U generated by u_n. Let P_n be the band projection from U onto U_n (353H). Let $v_n \in A$ be such that $|u_n| \le |v_n|$; then

$$|u_n| = P_n|u_n| \le P_n|v_n| = |P_n v_n|,$$

so $\|u_n\| \le \|P_n v_n\|$ for each n. Let $g_n \in U^*$ be such that $\|g_n\| = 1$ and $g_n(P_n v_n) = \|P_n v_n\|$.

Define $T : U \to \mathbb{R}^{\mathbb{N}}$ by setting $Tu = \langle g_n(P_n u)\rangle_{n\in\mathbb{N}}$ for each $u \in U$. Then T is a continuous linear operator from U to ℓ^1. **P** For $m \ne n$, $U_m \cap U_n = \{0\}$, because $|u_m| \wedge |u_n| = 0$. So, for any $u \in U$, $\langle P_n u\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U, and

$$\sum_{i=0}^n \|P_i u\| = \|\sum_{i=0}^n |P_i u|\| = \|\sup_{i\le n} |P_i u|\| \le \|u\|$$

for every n; accordingly

$$\|Tu\|_1 = \sum_{i=0}^\infty |g_i P_i u| \le \sum_{i=0}^\infty \|P_i u\| \le \|u\|.$$

Since T is certainly a linear operator (because every coordinate functional $g_i P_i$ is linear), we have the result. **Q**

Consequently $T[A]$ is relatively weakly compact in ℓ^1, because T is continuous for the weak topologies (2A5If). But ℓ^1 can be identified with $L^1(\mu)$, where μ is counting measure on $\mathbb{N}$. So $T[A]$ is uniformly integrable in ℓ^1, by 247C, and in particular $\lim_{n\to\infty} \sup_{w\in T[A]} |w(n)| = 0$. But this means that

$$\lim_{n\to\infty} \|u_n\| \le \lim_{n\to\infty} |g_n(P_n v_n)| = \lim_{n\to\infty} |(Tv_n)(n)| = 0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A satisfies the conditions of 354R(b-iv) and is uniformly integrable.

356X Basic exercises (a) Show that if $U = \ell^\infty$ then $U^\times = U_c^\sim$ can be identified with ℓ^1, and is properly included in $U^\sim$. (*Hint*: show that if $f \in U_c^\sim$ then $f(u) = \sum_{n=0}^\infty u(n) f(e_n)$, where $e_n(n) = 1$, $e_n(i) = 0$ for $i \ne n$.)

(b) Show that if $U = C([0,1])$ then $U^\times = U_c^\sim = \{0\}$. (*Hint*: show that if $f \in (U_c^\sim)^+$ and $\langle q_n\rangle_{n\in\mathbb{N}}$ enumerates $\mathbb{Q}\cap[0,1]$, then for each $n \in \mathbb{N}$ there is a $u_n \in U^+$ such that $u_n(q_n) = 1$ and $f(u_n) \le 2^{-n}$.)

(c) Let X be an uncountable set and μ the countable-cocountable measure on X, Σ its domain (211R). Let U be the space of bounded Σ-measurable real-valued functions on X. Show that U is a Dedekind σ-complete Banach lattice if given the supremum norm $\| \|_\infty$. Show that $U^\times$ can be identified with $\ell^1(X)$ (cf. 356Xa), and that $u \mapsto \int u\,d\mu$ belongs to $U_c^\sim \setminus U^\times$.

(d) Let U be a Dedekind σ-complete Riesz space and $f \in U_c^\sim$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U which is order-convergent to $u \in U$ in the sense that $u = \inf_{n\in\mathbb{N}}\sup_{m\ge n} u_m = \sup_{n\in\mathbb{N}}\inf_{m\ge n} u_m$. Show that $\lim_{n\to\infty} f(u_n)$ exists and is equal to $f(u)$.

(e) Let U be any Riesz space. Show that the band projection $P : U^\sim \to U^\times$ is defined by the formula

$$(Pf)(u) = \inf\{\sup_{v\in A} f(v) : A \subseteq U \text{ is non-empty, upwards-directed and has supremum} u\}$$

for every $f \in (U^\sim)^+$, $u \in U^+$. (*Hint*: show that the formula for Pf always defines an order-continuous linear functional. Compare 355Yh, 356Yb and 362Bd.)

(f) Let U be any Riesz space. Show that the band projection $P : U^\sim \to U_c^\sim$ is defined by the formula

$$(Pf)(u) = \inf\{\sup_{n\in\mathbb{N}} f(v_n) : \langle v_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence with supremum } u\}$$

for every $f \in (U^\sim)^+$, $u \in U^+$.

(g) Let U be a Riesz space with a Riesz norm. Show that U^* is perfect.

(h) Let U be a Riesz space with a Riesz norm. Show that the canonical map from U to U^{**} is a Riesz homomorphism.

(i) Let V be a perfect Riesz space and U any Riesz space. Show that $\mathfrak{L}^\sim(U;V)$ is perfect. (*Hint*: show that if $u \in U$, $g \in V^\times$ then $T \mapsto g(Tu)$ belongs to $\mathfrak{L}^\sim(U;V)^\times$.)

(j) Let U be an M-space. Show that it is perfect iff it is Dedekind complete and $U^\times$ separates the points of U.

(k) Let U be a Banach lattice which, as a Riesz space, is perfect. Show that its norm has the Levi property.

(l) Write out a proof from first principles that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in ℓ^1 such that $|u_n(n)| \ge \delta > 0$ for every $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is not relatively weakly compact.

(m) Let U be an L-space and $A \subseteq U$ a non-empty set. Show that the following are equiveridical: (i) A is uniformly integrable (ii) $\inf_{f\in B}\sup_{u\in A}|f(u)|$ for every non-empty downwards-directed set $B \subseteq U^\times$ with infimum 0 (iii) $\inf_{n\in\mathbb{N}}\sup_{u\in A}|f_n(u)| = 0$ for every non-increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$ with infimum 0 (iv) A is norm-bounded and $\lim_{n\to\infty}\sup_{u\in A}|f_n(u)| = 0$ for every disjoint order-bounded sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$.

356Y Further exercises (a) Let U be a Riesz space with the countable sup property. Show that $U^\times = U_c^\sim$.

(b) Let U be a Riesz space, and $\mathcal{A}$ a family of non-empty downwards-directed subsets of U^+ all with infimum 0. (i) Show that $U_{\mathcal{A}}^\sim = \{f : f \in U^\sim, \inf_{u\in A}|f|(u) = 0 \text{ for every } A \in \mathcal{A}\}$ is a band in $U^\sim$. (ii) Set $\mathcal{A}^* = \{A_0 + \ldots + A_n : A_0, \ldots, A_n \in \mathcal{A}\}$. Sbow that $U_{\mathcal{A}}^\sim = U_{\mathcal{A}^*}^\sim$. (iii) Take any $f \in (U^\sim)^+$, and let g, h be the components of f in $U_{\mathcal{A}}^\sim$, $(U_{\mathcal{A}}^\sim)^\perp$ respectively. Show that

$$g(u) = \inf_{A\in\mathcal{A}^*}\sup_{v\in A} f(u-v)^+, \quad h(u) = \sup_{A\in\mathcal{A}^*}\inf_{v\in A} f(u\wedge v)$$

for every $u \in U^+$. (Cf. 362Xi.)

(c) Let U be a Riesz space. For any band $V \subseteq U$ write V° for $\{f : f \in U^\times, f(v) = 0$ for every $v \in V\}$. Show that $V \mapsto (V^\perp)^\circ$ is a surjective order-continuous Boolean homomorphism from the algebra of complemented bands of U onto the band algebra of $U^\times$, and that it is injective iff $U^\times$ separates the points of U.

(d) Let U be a Riesz space such that $U^\sim$ separates the points of U. For any band $V \subseteq U^\sim$ write V° for $\{x : x \in U, f(x) = 0$ for every $f \in V\}$. Show that $V \mapsto (V^\perp)^\circ$ is a surjective Boolean homomorphism from the algebra of bands of $U^\sim$ onto the band algebra of U, and that it is injective iff $U^\sim = U^\times$.

(e) Let U be a Dedekind complete Riesz space such that $U^\times$ separates the points of U and U is the solid linear subspace of itself generated by a countable set. Show that U is perfect.

(f) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U such that $\langle f(u_n)\rangle_{n\in\mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: use 356Xm(iv) to show that $\{u_n : n \in \mathbb{N}\}$ is relatively weakly compact.)

(g) Let U be a perfect Banach lattice with order-continuous norm and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U such that $\langle f(u_n)\rangle_{n\in\mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: set $\phi(f) = \lim_{n\to\infty} f_n(u)$. For any $g \in (U^*)^+$ let V_g be the solid linear subspace of U^* generated by g, $W_g = \{u : g(|u|) = 0\}^\perp$, $\|u\|_g = g(|u|)$ for $u \in W_g$. Show that the completion of W_g under $\| \|_g$ is an L-space with dual isomorphic to V_g, and hence (using 356Yf) that $\phi\restriction V_g$ belongs to $V_g^\times$; as g is arbitrary, $\phi \in V^\times$ and may be identified with an element of U.)

(h) Let U be a uniformly complete Archimedean Riesz space with complexification V (354Yk). (i) Show that the complexification of $U^\sim$ can be identified with the space of linear functionals $f : V \to \mathbb{C}$ such that $\sup_{|v|\le u} |f(v)|$ is finite for every $u \in U^+$. (ii) Show that if U is a Banach lattice, then the complexification of $U^\sim = U^*$ can be identified (as normed space) with V^*. (See 355Yk.)

356 Notes and comments The section starts easily enough, with special cases of results in §355 (356B). When U has a Riesz norm, the identification of U^* as a subspace of $U^\sim$, and the characterization of order-continuous norms (356D) are pleasingly comprehensive and straightforward. Coming to biduals, we need to think a little (356F), but there is still no real difficulty at first. In 356H-356I, however, something more substantial is happening. I have written these arguments out in what seems to be the shortest route to the main theorem, at the cost perhaps of neglecting any intuitive foundation. What I think we are really doing is matching bands in U, $U^\times$ and $U^{\times\times}$, as in 356Yc.

From now on, almost the first thing we shall ask of any new Riesz space will be whether it is perfect, and if not, which of the three conditions of 356K it fails to satisfy. For reasons which will I hope appear in the next chapter, perfect Riesz spaces are particularly important in measure theory; in particular, all L^p spaces for $p \in [1, \infty[$ are perfect (366D), as are the L^∞ spaces of localizable measure spaces (365K). Further examples will be discussed in §369 and §374. Of course we have to remember that there are also important Riesz spaces which are not perfect, of which $C([0, 1])$ and $\boldsymbol{c}_0$ are two of the simplest examples.

The duality between L- and M-spaces (356N, 356P) is natural and satisfying. We are now in a position to make a determined attempt to tidy up the notion of 'uniform integrability'. I give two major theorems. The first is yet another 'disjoint-sequence' characterization of uniformly integrable sets, to go with 246G and 354R. The essential difference here is that we are looking at disjoint sequences in a predual; in a sense, this means that the result is a sharper one, because the M-space U need not be Dedekind complete (for instance, it could be $C([0, 1])$ – this indeed is the archetype for applications of the theorem) and therefore need not have as many disjoint sequences as its dual. (For instance, in the dual of $C([0, 1])$ we have all the point masses δ_t, where $\delta_t(u) = u(t)$; these form a disjoint family in $C([0, 1])^\sim$ not corresponding to any disjoint family in $C([0, 1])$.) The essence of the proof is a device to extract a disjoint sequence in U to match approximately a subsequence of a given disjoint sequence in $U^\sim$. In the example just suggested, this would correspond, given a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of distinct points in $[0, 1]$, to finding a subsequence $\langle t_{n(i)}\rangle_{i\in\mathbb{N}}$ which is discrete, so that we can find disjoint $u_i \in C([0, 1])$ with $u_i(t_{n(i)}) = 1$ for each i.

The second theorem, 356Q, is a new version of a result already given in §247: in any L-space, uniform integrability is the same as relative weak compactness. I hope you are not exasperated by having been asked, in Volume 2, to master a complex argument (one of the more difficult sections of that volume) which was going to be superseded. Actually it is worse than that. A theorem of Kakutani (369E) tells us that every L-space is isomorphic to an L^1 space. So 356Q is itself a consequence of 247C. I do at least owe you an explanation for writing out two proofs. The first point is that the result is sufficiently important for it to be well worth while spending time in its neighbourhood, and the contrasts and similarities between the two arguments are instructive. The second is that the proof I have just given was not really accessible at the level of Volume 2. It does not rely on every single page of this chapter, but the key idea (that U is isomorphic to $U^{\times\times}$, so it will be enough if we can show that A is relatively compact in $U^{\times\times}$) depends essentially on 356I, which lies pretty deep in the abstract theory of Riesz spaces. The third is an aesthetic one: a theorem about L-spaces ought to be proved in the category of normed Riesz spaces, without calling on a large body of theory outside. Of course this is a book on measure theory, so I did the measure theory first, but if you look at everything that went into it, the proof in §247 is I believe longer, in the formal sense, than the one here, even setting aside the labour of proving Kakutani's theorem.

Let us examine the ideas in the two proofs. First, concerning the proof that uniformly integrable sets are relatively compact, the method here is very smooth and natural; the definition I chose of 'uniform integrability' is exactly adapted to showing that uniformly integrable sets are relatively compact in the order-continuous bidual; all the effort goes into the proof that L-spaces are perfect. The previous argument depended on identifying the dual of L^1 as L^∞ – and was disagreeably complicated by the fact that the identification is not always valid, so that I needed to reduce the problem to the σ-finite case (part (b-ii) of the proof of 247C). After that, the Radon-Nikodým theorem did the trick. Actually Kakutani's theorem shows that the side-step to σ-finite spaces is irrelevant. It directly represents an abstract L-space as $L^1(\mu)$ for a localizable measure μ, in which case $(L^1)^* \cong L^\infty$ exactly.

In the other direction, both arguments depend on a disjoint-sequence criterion for uniform integrability (246G(iii) or 354R(b-iv)). These criteria belong to the 'easy' side of the topic; straightforward Riesz space arguments do the job, whether written out in that language or not. (Of course the new one in this section, 356O, lies a little deeper.) I go a bit faster this time because I feel that you ought by now to be happy with the Hahn-Banach theorem and the Uniform Boundedness Theorem, which I was avoiding in Volume 2. And then of course I quote the result for ℓ^1. This looks like cheating. But ℓ^1 really is easier, as you will find if you just write out part (a) of the proof of 247C for this case. It is not exactly that you can dispense with any particular element of the argument; rather it is that the formulae become much more direct when you can write $u(i)$ in place of $\int_{F_i} u$, and 'cluster points for the weak topology' become pointwise limits of subsequences, so that the key step (the 'sliding hump', in which $u_{k(j)}(n(k(j)))$ is the only significant coordinate of $u_{k(j)}$), is easier to find.

We now have a wide enough variety of conditions equivalent to uniform integrability for it to be easy to find others; I give a couple in 356Xm, corresponding in a way to those in 246G. You may have noticed, in the proof of 247C, that in fact the full strength of the hypothesis 'relatively weakly compact' is never used; all that is demanded is that a couple of sequences should have cluster points for the weak topology. So we see that a set A is uniformly integrable iff every sequence in A has a weak cluster point. But this extra refinement is nothing to do with L-spaces; it is generally true, in any normed space U, that a set $A \subseteq U$ is relatively weakly compact iff every sequence in A has a cluster point in U for the weak topology ('Eberlein's theorem'; see 462D in Volume 4, Köthe 69, 24.2.1, or Dunford & Schwartz 57, V.6.1).

There is a very rich theory concerning weak compactness in perfect Riesz spaces, based on the ideas here; some of it is explored in Fremlin 74a. As a sample, I give one of the basic properties of perfect Banach lattices with order-continuous norms: they are 'weakly sequentially complete' (356Yg).

Chapter 36

Function Spaces

Chapter 24 of Volume 2 was devoted to the elementary theory of the 'function spaces' L^0, L^1, L^2 and L^∞ associated with a given measure space. In this chapter I return to these spaces to show how they can be related to the more abstract themes of the present volume. In particular, I develop constructions to demonstrate, as clearly as I can, the way in which all the function spaces associated with a measure space in fact depend only on its measure algebra; and how many of their features can (in my view) best be understood in terms of constructions involving measure algebras.

The chapter is very long, not because there are many essentially new ideas, but because the intuitions I seek to develop depend, for their logical foundations, on technically complex arguments. This is perhaps best exemplified by §364. If two measure spaces (X,Σ,μ) and (Y,T,ν) have isomorphic measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ then the spaces $L^0(\mu)$, $L^0(\nu)$ are isomorphic as topological f-algebras; and more: for any isomorphism between $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ there is a unique corresponding isomorphism between the L^0 spaces. The intuition involved is in a way very simple. If f, g are measurable real-valued functions on X and Y respectively, then $f^\bullet\in L^0(\mu)$ will correspond to $g^\bullet\in L^0(\nu)$ if and only if $[\![f^\bullet>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet\in\mathfrak{A}$ corresponds to $[\![g^\bullet>\alpha]\!]=\{y:g(y)>\alpha\}^\bullet\in\mathfrak{B}$ for every α. But the check that this formula is consistent, and defines an isomorphism of the required kind, involves a good deal of detailed work. It turns out, in fact, that the *measures* μ and ν do not enter this part of the argument at all, except through their ideals of negligible sets (used in the construction of $\mathfrak{A}$ and $\mathfrak{B}$). This is already evident, if you look for it, in the theory of $L^0(\mu)$; in §241, as written out, you will find that the measure of an individual set is not once mentioned, except in the exercises. Consequently there is an invitation to develop the theory with algebras $\mathfrak{A}$ which are not necessarily measure algebras. Here is another reason for the length of the chapter; substantial parts of the work are being done in greater generality than the corresponding sections of Chapter 24, necessitating a degree of repetition. Of course this is not 'measure theory' in the strict sense; but for thirty years now measure theory has been coloured by the existence of these generalizations, and I think it is useful to understand which parts of the theory apply only to measure algebras, and which can be extended to other σ-complete Boolean algebras, like the algebraic theory of L^0, or even to all Boolean algebras, like the theory of L^∞.

Here, then, are two of the objectives of this chapter: first, to express the ideas of Chapter 24 in ways making explicit their independence of particular measure spaces, by setting up constructions based exclusively on the measure algebras involved; second, to set out some natural generalizations to other algebras. But to justify the effort needed I ought to point to some mathematically significant idea which demands these constructions for its expression, and here I mention the categorical nature of the constructions. Between Boolean algebras we have a variety of natural and important classes of 'morphism'; for instance, the Boolean homomorphisms and the order-continuous Boolean homomorphisms; while between measure algebras we have in addition the measure-preserving Boolean homomorphisms. Now it turns out that if we construct the L^p spaces in the natural ways then morphisms between the underlying algebras give rise to morphisms between their L^p spaces. For instance, any Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a multiplicative norm-contractive Riesz homomorphism from $L^\infty(\mathfrak{A})$ to $L^\infty(\mathfrak{B})$; if $\mathfrak{A}$ and $\mathfrak{B}$ are Dedekind σ-complete, then any sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a sequentially order-continuous multiplicative Riesz homomorphism from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$; and if $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras, then any measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces norm-preserving Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ for every $p\in[1,\infty]$. All of these are 'functors', that is, a composition of homomorphisms between algebras gives rise to a composition of the corresponding operators between their function spaces, and are 'covariant', that is, a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ leads to an operator from $L^p(\mathfrak{A})$ to $L^p(\mathfrak{B})$. But the same constructions lead us to a functor which is 'contravariant': starting from an order-continuous Boolean homomorphism from a semi-finite measure algebra $(\mathfrak{A},\bar\mu)$ to a measure algebra $(\mathfrak{B},\bar\nu)$, we have an operator from $L^1(\mathfrak{B},\bar\nu)$ to $L^1(\mathfrak{A},\bar\mu)$. This last is in fact a kind of conditional expectation operator. In my view it is not possible to make sense of the theory of measure-preserving transformations without at least an intuitive grasp of these ideas.

Another theme is the characterization of each construction in terms of universal mapping theorems: for instance, each L^p space, for $1\le p\le\infty$, can be characterized as Banach lattice in terms of factorizations of functions of an appropriate class from the underlying algebra to Banach lattices.

Now let me try to sketch a route-map for the journey ahead. I begin with two sections on the space $S(\mathfrak{A})$; this construction applies to any Boolean algebra (indeed, any Boolean ring), and corresponds to the space of 'simple functions' on a measure space. Just because it is especially close to the algebra (or ring) $\mathfrak{A}$, there is a particularly large number of universal mapping theorems corresponding to different aspects of its structure (§361). In §362 I seek to relate ideas on additive functionals on Boolean algebras from Chapter 23 and §§326-327 to the theory of Riesz space duals in §356. I then turn to the systematic discussion of the function spaces of Chapter 24: L^∞ (§363), L^0 (§364), L^1 (§365) and other L^p (§366), followed by an account of convergence in measure (§367). While all these sections are dominated by the objectives sketched in the paragraphs above, I do include a few major theorems not covered by the ideas of Volume 2, such as the Kelley-Nachbin characterization of the Banach spaces $L^\infty(\mathfrak{A})$ for Dedekind complete $\mathfrak{A}$ (363R). In the last two sections of the chapter I turn to the use of L^0 spaces in the representation of Archimedean Riesz spaces (§368) and of Banach lattices separated by their order-continuous duals (§369).

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This is the fundamental Riesz space associated with a Boolean ring $\mathfrak{A}$. When $\mathfrak{A}$ is a ring of sets, $S(\mathfrak{A})$ can be regarded as the linear space of 'simple functions' generated by the characteristic functions of members of $\mathfrak{A}$ (361L). Its most important property is the universal mapping theorem 361F, which establishes a one-to-one correspondence between (finitely) additive functions on $\mathfrak{A}$ (361B-361C) and linear operators on $S(\mathfrak{A})$. Simple universal mapping theorems of this type can be interesting, but do not by themselves lead to new insights; what makes this one important is the fact that $S(\mathfrak{A})$ has a canonical Riesz space structure, norm and multiplication (361E). From this we can deduce universal mapping theorems for many other classes of function (361G, 361H, 361I, 361Xb). (Particularly important are countably additive and completely additive real-valued functionals, which will be dealt with in the next section.) While the exact construction of $S(\mathfrak{A})$ (and the associated map from $\mathfrak{A}$ to $S(\mathfrak{A})$) can be varied (361D, 361L, 361M, 361Ya), its structure is uniquely defined, so homomorphisms between Boolean rings correspond to maps between their $S(\ )$-spaces (361J), and (when $\mathfrak{A}$ is an algebra) $\mathfrak{A}$ can be recovered from the Riesz space $S(\mathfrak{A})$ as the algebra of its projection bands (361K).

361A Boolean rings In this section I speak of Boolean *rings* rather than *algebras*; there are ideas in §365 below which are more naturally expressed in terms of the ring of elements of finite measure in a measure algebra than in terms of the whole algebra. I should perhaps therefore recall some of the ideas of §311, which is the last time when Boolean rings without identity were mentioned, and set out some simple facts.

(a) Any Boolean ring $\mathfrak{A}$ can be represented as the ring of compact open subsets of a zero-dimensional locally compact Hausdorff space X (311I); X is just the set of surjective ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$ (311E).

(b) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a function, then the following are equiveridical: (i) π is a ring homomorphism; (ii) $\pi(a \setminus b) = \pi a \setminus \pi b$ for all a, $b \in \mathfrak{A}$; (iii) $\pi 0 = 0$ and $\pi(a \cup b) = \pi a \cup \pi b$, $\pi(a \cap b) = \pi a \cap \pi b$ for all a, $b \in \mathfrak{A}$. **P** See 312H. To prove (ii)⇒(iii), observe that if a, $b \in \mathfrak{A}$ then

$$\pi(a \cap b) = \pi a \setminus \pi(a \setminus b) = \pi a \cap \pi b. \quad \mathbf{Q}$$

(c) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a ring homomorphism, then π is order-continuous iff $\inf \pi[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$; while π is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \pi a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. (See 313L.)

(d) The following will be a particularly important type of Boolean ring for us. If $(\mathfrak{A}, \bar\mu)$ is a measure algebra, then the ideal $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$ is a Boolean ring in its own right. Now suppose that $(\mathfrak{B}, \bar\nu)$ is another measure algebra and $\mathfrak{B}^f \subseteq \mathfrak{B}$ the corresponding ring of elements of finite measure. We can say that a ring homomorphism $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is **measure-preserving** if $\bar\nu\pi a = \bar\mu a$ for every $a \in \mathfrak{A}^f$. Now in this case π is order-continuous. **P** If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a\in A} \bar\mu a = 0$, by 321F; but this means that $\inf_{a\in A} \bar\nu\pi a = 0$, and $\inf \pi[A] = 0$ in $\mathfrak{B}^f$. **Q**

361B Definition Let $\mathfrak{A}$ be a Boolean ring and U a linear space. A function $\nu : \mathfrak{A} \to U$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

361C Elementary facts We have the following immediate consequences of this definition, corresponding to 326B and 313L. Let $\mathfrak{A}$ be a Boolean ring, U a linear space and $\nu : \mathfrak{A} \to U$ an additive function.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $a_0, \dots , a_m$ are disjoint in $\mathfrak{A}$, then $\nu(\sup_{j\le m} a_j) = \sum_{j=0}^m \nu a_j$.

(c) If $\mathfrak{B}$ is another Boolean ring and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a ring homomorphism, then $\nu\pi : \mathfrak{B} \to U$ is additive. In particular, if $\mathfrak{B}$ is a subring of $\mathfrak{A}$, then $\nu{\restriction}\mathfrak{B} : \mathfrak{B} \to U$ is additive.

(d) If V is another linear space and $T : U \to V$ is a linear operator, then $T\nu : \mathfrak{A} \to V$ is additive.

(e) If U is a partially ordered linear space, then ν is order-preserving iff it is non-negative, that is, $\nu a \ge 0$ for every $a \in \mathfrak{A}$. **P** (α) If ν is order-preserving, then of course $0 = \nu 0 \le \nu a$ for every $a \in \mathfrak{A}$. (β) If ν is non-negative, and $a \subseteq b$ in $\mathfrak{A}$, then

$$\nu a \le \nu a + \nu(b \setminus a) = \nu b. \text{ **Q**}$$

(f) If U is a partially ordered linear space and ν is non-negative, then (i) ν is order-continuous iff $\inf \nu[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 (ii) ν is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. **P** (i) If ν is order-continuous, then of course $\inf \nu[A] = \nu 0 = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0. If ν satisfies the condition, and $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum c, then $\{c \setminus a : a \in \mathfrak{A}\}$ is downwards-directed with infimum 0 (313Aa), so that

$$\sup_{a\in A} \nu a = \sup_{a\in A} \nu c - \nu(c \setminus a) = \nu c - \inf_{a\in A} \nu(c \setminus a)$$

(by 351Db)

$$= \nu c.$$

Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum c, then

$$\inf_{a\in A} \nu a = \inf_{a\in A} \nu c + \nu(a \setminus c) = \nu c + \inf_{a\in A} \nu(a \setminus c) = \nu c.$$

Putting these together, ν is order-continuous. (ii) If ν is sequentially order-continuous, then of course $\inf_{n\in\mathbb{N}} \nu a_n = \nu 0 = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum c, then $\langle c \setminus a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\sup_{n\in\mathbb{N}} \nu a_n = \sup_{n\in\mathbb{N}} \nu c - \nu(c \setminus a_n) = \nu c - \inf_{n\in\mathbb{N}} \nu(c \setminus a_n) = \nu c.$$

Similarly, if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum c, then $\langle a_n \setminus c\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\inf_{n\in\mathbb{N}} \nu a_n = \inf_{n\in\mathbb{N}} \nu c + \nu(c \setminus a_n) = \nu c + \inf_{n\in\mathbb{N}} \nu(c \setminus a_n) = \nu c.$$

Thus ν is sequentially order-continuous. **Q**

361D Construction Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. For $a \in \mathfrak{A}$ write χa for the characteristic function of the open-and-compact subset $\widehat{a}$ of Z corresponding to a. Let $S(\mathfrak{A})$ be the linear subspace of $\mathbb{R}^Z$ generated by $\{\chi a : a \in \mathfrak{A}\}$. Because χa is a bounded function for every a, $S(\mathfrak{A})$ is a subspace of the space $\ell^\infty(Z)$ of all bounded real-valued functions on Z (354Ha), and $\| \|_\infty$ is a norm on $S(\mathfrak{A})$. Because $\chi a \times \chi b = \chi(a \cap b)$ for all a, $b \in \mathfrak{A}$ (writing $\times$ for pointwise multiplication of functions, as in 281B), $S(\mathfrak{A})$ is closed under $\times$.

361E I give a portmanteau proposition running through the elementary, mostly algebraic, properties of $S(\mathfrak{A})$.

Proposition Let $\mathfrak{A}$ be a Boolean ring, with Stone space Z. Write S for $S(\mathfrak{A})$.

(a) If $a_0, \dots, a_n \in \mathfrak{A}$, there are disjoint $b_0, \dots, b_m$ such that each a_i is expressible as the supremum of some of the b_j.

(b) If $u \in S$, it is expressible in the form $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \in \mathbb{R}$ for each j. If all the b_j are non-zero then $\|u\|_\infty = \sup_{j \le m} |\beta_j|$.

(c) If $u \in S$ is non-negative, it is expressible in the form $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \ge 0$ for each j, and simultaneously in the form $\sum_{j=0}^m \gamma_j \chi c_j$ where $c_0 \supseteq c_1 \supseteq \dots \supseteq c_m$ and $\gamma_j \ge 0$ for every j.

(d) If $u = \sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \in \mathbb{R}$ for each j, then $|u| = \sum_{j=0}^m |\beta_j| \chi b_j \in S$.

(e) S is a Riesz subspace of $\mathbb{R}^Z$; in its own right, it is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then S has an order unit $\chi 1$ and $\|u\|_\infty = \min\{\alpha : \alpha \ge 0,\ |u| \le \alpha \chi 1\}$ for every $u \in S$.

(f) The map $\chi : \mathfrak{A} \to S$ is injective, additive, non-negative, a lattice homomorphism and order-continuous.

(g) Suppose that $u \ge 0$ in S and $\delta \ge 0$ in $\mathbb{R}$. Then

$$[\![u > \delta]\!] = \sup\{a : a \in \mathfrak{A},\ (\delta + \eta)\chi a \le u \text{ for some } \eta > 0\}$$

is defined in $\mathfrak{A}$, and

$$\delta \chi [\![u > \delta]\!] \le u \le \delta \chi [\![u > 0]\!] \vee \|u\|_\infty [\![u > \delta]\!].$$

In particular, $u \le \|u\|_\infty \chi [\![u > 0]\!]$ and there is an $\eta > 0$ such that $\eta \chi [\![u > 0]\!] \le u$. If u, $v \ge 0$ in S then $u \wedge v = 0$ iff $[\![u > 0]\!] \cap [\![v > 0]\!] = 0$.

(h) Under $\times$, S is an f-algebra (352W) and a commutative normed algebra (2A4J).

(i) For any $u \in S$, $u \ge 0$ iff $u = v \times v$ for some $v \in S$.

proof Write $\widehat{a}$ for the open-and-compact subset of Z corresponding to $a \in \mathfrak{A}$.

(a) Induce on n. If $n = 0$ take $m = 0$, $b_0 = a_0$. For the inductive step to $n \ge 1$, take disjoint $b_0, \dots, b_m$ such that a_i is the supremum of some of the b_j for each $i < n$; now replace $b_0, \dots, b_m$ with $b_0 \cap a_n, \dots, b_m \cap a_n, b_0 \setminus a_n, \dots, b_m \setminus a_n, a_n \setminus \sup_{j \le m} b_j$ to obtain a suitable string for $a_0, \dots, a_n$.

(b) If $u = 0$ set $m = 0$, $b_0 = 0$, $\beta_0 = 0$. Otherwise, express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}$ and $\alpha_0, \dots, \alpha_n$ are real numbers. Let $b_0, \dots, b_m$ be disjoint and such that every a_i is expressible as the supremum of some of the b_j. Set $\gamma_{ij} = 1$ if $b_j \subseteq a_i$, 0 otherwise, so that, because the b_j are disjoint, $\chi a_i = \sum_{j=0}^k \gamma_{ij} \chi b_j$ for each i. Then

$$u = \sum_{i=0}^n \alpha_i \chi a_i = \sum_{i=0}^n \sum_{j=0}^m \alpha_i \gamma_{ij} \chi b_j = \sum_{j=0}^m \beta_j \chi b_j,$$

setting $\beta_j = \sum_{i=0}^n \alpha_i \gamma_{ij}$ for each $j \le k$.

The expression for $\|u\|_\infty$ is now obvious.

(c)(i) If $u \ge 0$ in (b), we must have $\beta_j = u(z) \ge 0$ whenever $z \in \widehat{b}_j$, so that $\beta_j \ge 0$ whenever $b_j \ne 0$; consequently $u = \sum_{j=0}^m |\beta_j| \chi b_j$ is in the required form.

(ii) If we suppose that every β_j is non-negative, and rearrange the terms of the sum so that $\beta_0 \le \dots \le \beta_m$, then we may set $\gamma_0 = \beta_0$, $\gamma_j = \beta_j - \beta_{j-1}$ for $1 \le j \le m$, $c_j = \sup_{i \ge j} b_i$ to get

$$\sum_{j=0}^m \gamma_j \chi c_j = \sum_{j=0}^m \sum_{i=j}^m \gamma_j \chi b_i = \sum_{i=0}^m \sum_{j=0}^i \gamma_j \chi b_i = \sum_{i=0}^m \beta_i \chi b_i = u.$$

(d) is trivial, because $\widehat{b}_0, \dots, \widehat{b}_n$ are disjoint.

(e) By (d), $|u| \in S$ for every $u \in S$, so S is a Riesz subspace of $\mathbb{R}^Z$, and in itself is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then $\chi 1$, the constant function with value 1, belongs to S, and is an order unit of S; while

$$\|u\|_\infty = \min\{\alpha : \alpha \ge 0,\ |u(z)| \le \alpha\ \forall\, z \in Z\} = \min\{\alpha : \alpha \ge 0,\ |u| \le \alpha \chi 1\}$$

for every $u \in S$.

(f) χ is injective because $\widehat{a} \neq \widehat{b}$ whenever $a \neq b$. χ is additive because $\widehat{a} \cap \widehat{b} = \emptyset$ whenever $a \cap b = 0$. Of course it is non-negative. It is a lattice homomorphism because $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ and $E \mapsto \chi E : \mathcal{P}Z \to \mathbb{R}^Z$ are. To see that ν is order-continuous, take a non-empty downwards-directed $A \subseteq \mathfrak{A}$ with infimum 0. **?** Suppose, if possible, that $\{\chi a : a \in A\}$ does not have infimum 0 in S. Then there is a $u > 0$ in S such that $u \leq \chi a$ for every $a \in \mathfrak{A}$. Now u can be expressed as $\sum_{j=0}^{m} \beta_j b_j$ where $b_0, \ldots, b_m$ are disjoint. There must be some $z_0 \in Z$ such that $u(z_0) > 0$; take j such that $z_0 \in \widehat{b}_j$, so that $b_j \neq 0$ and $\beta_j = u(z_0) > 0$. But now, for any $z \in \widehat{b}_j$, $a \in A$,

$$(\chi a)(z) \geq u(z) = \beta_j > 0$$

and $z \in \widehat{a}$. As z is arbitrary, $\widehat{b}_j \subseteq \widehat{a}$ and $b_j \subseteq a$; as a is arbitrary, b_j is a non-zero lower bound for A in $\mathfrak{A}$. **X** So $\inf \chi[A] = 0$ in S. As A is arbitrary, χ is order-continuous, by the criterion of 361C(f-i).

(g) Express u as $\sum_{j=0}^{m} \beta_j \chi b_j$ where $b_0, \ldots, b_m$ are disjoint and every $\beta_j \geq 0$. Then given $\delta \geq 0$, $\eta > 0$, $a \in \mathfrak{A}$ we have $(\delta + \eta)\chi a \leq u$ iff $a \subseteq \sup\{b_j : j \leq m, \beta_j \geq \delta + \eta\}$. So $[\![u > \delta]\!] = \sup\{b_j : j \leq m, \beta_j > \delta\}$. Writing $c = [\![u > \delta]\!]$, $d = [\![u > 0]\!] = \sup\{b_j : \beta_j > 0\}$, we have

$$\begin{aligned} u(z) &\leq \|u\|_\infty \text{ if } z \in \widehat{c}, \\ &\leq \delta \text{ if } z \in \widehat{d} \setminus \widehat{c}, \\ &= 0 \text{ if } z \notin \widehat{d}. \end{aligned}$$

So

$$\delta \chi c \leq u \leq \|u\|_\infty \chi c \vee \delta \chi d,$$

as claimed. Taking $\delta = 0$ we get $u \leq \|u\|_\infty \chi d$. Set

$$\eta = \min(\{1\} \cup \{\beta_j : j \leq m, \beta_j > 0\});$$

then $\eta > 0$ and $\eta \chi d \leq u$.

If u, $v \in S^+$ take η, $\eta' > 0$ such that

$$\eta \chi [\![u > 0]\!] \leq u, \quad \eta' \chi [\![v > 0]\!] \leq v.$$

Then

$$\min(\eta, \eta')\chi([\![u > 0]\!] \cap [\![v > 0]\!]) \leq u \wedge v \leq \max(\|u\|_\infty, \|v\|_\infty)\chi([\![u > 0]\!] \cap [\![v > 0]\!]).$$

So

$$u \wedge v = 0 \Longrightarrow [\![u > 0]\!] \cap [\![v > 0]\!] = 0 \Longrightarrow u \wedge v = 0.$$

(h) S is a commutative f-algebra and normed algebra just because it is a Riesz subspace of the f-algebra and commutative normed algebra $\ell^\infty(Z)$ and is closed under multiplication.

(i) If $u = \sum_{j=0}^{m} \beta_j \chi b_j$ where $b_0, \ldots, b_m$ are disjoint and $\beta_j \geq 0$ for every j, then $u = v \times v$ where $v = \sum_{j=0}^{m} \sqrt{\beta_j} \chi b_j$.

361F I now turn to the universal mapping theorems which really define the construction.

Theorem Let $\mathfrak{A}$ be a Boolean ring, and U any linear space. Then there is a one-to-one correspondence between additive functions $\nu : \mathfrak{A} \to U$ and linear operators $T : S(\mathfrak{A}) \to U$, given by the formula $\nu = T\chi$.

proof (a) The core of the proof is the following observation. Let $\nu : \mathfrak{A} \to U$ be additive. If $a_0, \ldots, a_n \in \mathfrak{A}$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$ are such that $\sum_{i=0}^{n} \alpha_i \chi a_i = 0$ in $S = S(\mathfrak{A})$, then $\sum_{i=0}^{n} \alpha_i \nu a_i = 0$ in U. **P** As in 361E, we can find disjoint $b_0, \ldots, b_m$ such that each a_i is the supremum of some of the b_j; set $\gamma_{ij} = 1$ if $b_j \subseteq a_i$, 0 otherwise, so that $\chi a_i = \sum_{j=0}^{m} \gamma_{ij} \chi b_j$, $\nu a_i = \sum_{j=0}^{m} \gamma_{ij} \nu b_j$ for each i. Set $\beta_j = \sum_{i=0}^{n} \alpha_i \gamma_{ij}$, so that

$$0 = \sum_{i=0}^{n} \alpha_i \chi a_i = \sum_{j=0}^{m} \beta_j \chi b_j.$$

Now $\beta_j \nu b_j = 0$ in U for each j, because either $b_j = 0$ and $\nu b_j = 0$, or there is some $z \in \widehat{b}_j$ so that β_j must be 0. Accordingly

$$0=\sum_{j=0}^m\beta_j\nu b_j=\sum_{j=0}^m\sum_{i=0}^n\alpha_i\gamma_{ij}\nu b_i=\sum_{i=0}^n\alpha_i\nu a_i.\ \mathbf{Q}$$

(b) It follows that if $u\in S$ is expressible simultaneously as $\sum_{i=0}^n\alpha_i\chi a_i=\sum_{j=0}^m\beta_j\chi b_j$, then

$$\sum_{i=0}^n\alpha_i\chi a_i+\sum_{j=0}^m(-\beta_j)\chi b_j=0\text{ in }S,$$

so that

$$\sum_{i=0}^n\alpha_i\nu a_i+\sum_{j=0}^m(-\beta_j)\nu b_j=0\text{ in }U,$$

and

$$\sum_{i=0}^n\alpha_i\nu a_i=\sum_{j=0}^m\beta_j\nu b_j.$$

We can therefore define $T:S\to U$ by setting

$$T(\sum_{i=0}^n\alpha_i\chi a_i)=\sum_{i=0}^n\alpha_i\nu a_i$$

whenever $a_0,\dots,a_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$.

(c) It is now elementary to check that T is linear, and that $T\chi a=\nu a$ for every $a\in\mathfrak{A}$. Of course this last condition uniquely defines T, because $\{\chi a:a\in\mathfrak{A}\}$ spans the linear space S.

361G Theorem Let $\mathfrak{A}$ be a Boolean ring, and U a partially ordered linear space. Let $\nu:\mathfrak{A}\to U$ be an additive function, and $T:S(\mathfrak{A})\to U$ the corresponding linear operator.

(a) ν is non-negative iff T is positive.

(b) In this case,

(i) if T is order-continuous or sequentially order-continuous, so is ν;

(ii) if U is Archimedean and ν is order-continuous or sequentially order-continuous, so is T.

(c) If U is a Riesz space, then the following are equiveridical:

(i) T is a Riesz homomorphism;

(ii) $\nu a\wedge\nu b=0$ in U whenever $a\cap b=0$ in $\mathfrak{A}$;

(iii) ν is a lattice homomorphism.

proof Write S for $S(\mathfrak{A})$.

(a) If T is positive, then surely $\nu a=T\chi a\ge0$ for every $a\in\mathfrak{A}$, so $\nu=T\chi$ is non-negative. If ν is non-negative, and $u\ge0$ in S, then u is expressible as $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m\in\mathfrak{A}$ and $\beta_j\ge0$ for every j (361Ec), so that

$$Tu=\sum_{j=0}^m\beta_j\nu b_j\ge0.$$

Thus T is positive.

(b)(i) If T is order-continuous (resp. sequentially order-continuous) then $\nu=T\chi$ is the composition of two order-continuous (resp. sequentially order-continuous) functions (361Ef), so must be order-continuous (resp. sequentially order-continuous).

(ii) Assume now that U is Archimedean.

(α) Suppose that ν is order-continuous and that $A\subseteq S$ is non-empty, downwards-directed and has infimum 0. Fix $u_0\in A$, set $\alpha=\|u\|_\infty$ and $a_0=[\![u>0]\!]$ (in the language of 361Eg). If $\alpha=0$ then of course $\inf_{u\in A}Tu=Tu_0=0$. Otherwise, take any $w\in U$ such that $w\not\le0$. Then there is some $\delta>0$ such that $w\not\le\delta\nu a_0$, because U is Archimedean. Set $A'=\{u:u\in A,\ u\le u_0\}$; because A is downwards-directed, A' has the same lower bounds as A, and $\inf A'=0$, while A' is still downwards-directed. For $u\in A'$ set $c_u=[\![u>\delta]\!]$, so that

$$\delta\chi c_u\le u\le\alpha\chi c_u+\delta\chi[\![u>0]\!]\le\alpha\chi c_u+\delta\chi a_0$$

(361Eg). If $u,v\in A'$ and $u\le v$, then $c_u\subseteq c_v$, so $C=\{c_u:u\in A'\}$ is downwards-directed; but if c is any lower bound for C in $\mathfrak{A}$, $\delta\chi c$ is a lower bound for A' in S, so is zero, and $c=0$ in $\mathfrak{A}$. Thus $\inf C=0$ in $\mathfrak{A}$, and $\inf_{u\in A'}\nu c_u=0$ in U. But this means, in particular, that $\frac{1}{\alpha}(w-\delta\nu a_0)$ is not a lower bound for $\nu[C]$, and there is some $u\in A'$ such that $\frac{1}{\alpha}(w-\delta\nu a_0)\not\le\nu c_u$, that is, $w-\delta\nu a_0\not\le\alpha\nu c_u$, that is, $w\not\le\delta\nu a_0+\alpha\nu c_u$. As $u\le\alpha\chi c_u+\delta\chi a_0$,

$$Tu \le T(\alpha\chi c_u + \delta\chi a_0) = \alpha\nu c_u + \delta\nu a_0,$$

and $w \not\le Tu$. Since w is arbitrary, this means that $0 = \inf T[A]$; as A is arbitrary, T is order-continuous.

(β) The argument for sequential order-continuity is essentially the same. Suppose that ν is sequentially order-continuous and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0. Again set $\alpha = \|u_0\|$, $a_0 = [\![u_0 > 0]\!]$; again we may suppose that $\alpha > 0$; again take any $w \in U$ such that $w \not\le 0$. As before, there is some $\delta > 0$ such that $w \not\le \delta\nu a_0$. For $n \in \mathbb{N}$ set $c_n = [\![u_n > \delta]\!]$, so that

$$\delta\chi c_n \le u_n \le \alpha\chi c_n + \delta\chi a_0.$$

The sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing because $\langle u_n\rangle_{n\in\mathbb{N}}$ is, and if $c \subseteq c_n$ for every n, then $\delta\chi c \le u_n$ for every n, so is zero, and $c = 0$ in $\mathfrak{A}$. Thus $\inf_{n\in\mathbb{N}} c_n = 0$ in $\mathfrak{A}$, and $\inf_{n\in\mathbb{N}} \nu c_n = 0$ in U, because ν is sequentially order-continuous. Replacing A', C in the argument above by $\{u_n : n \in \mathbb{N}\}$, $\{c_n : n \in \mathbb{N}\}$ we find an n such that $w \not\le Tu_n$. Since w is arbitrary, this means that $0 = \inf_{n\in\mathbb{N}} Tu_n$; as $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(c)(i)⇒(iii) If $T : S(\mathfrak{A}) \to U$ is a Riesz homomorphism, and $\nu = T\chi$, then surely ν is a lattice homomorphism because T and χ are.

(iii)⇒(ii) is trivial.

(ii)⇒(i) If $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, then for any $u \in S(\mathfrak{A})$ we have an expression of u as $\sum_{j=0}^m \beta_j\chi b_j$, where $b_0, \dots, b_m \in \mathfrak{A}$ are disjoint. Now

$$|Tu| = |\sum_{j=0}^m \beta_j\nu b_j| = \sum_{j=0}^m |\beta_j|\nu b_j = T(\sum_{j=0}^m |\beta_j|\chi b_j) = T(|u|)$$

by 352Fb and 361Ed. As u is arbitrary, T is a Riesz homomorphism (352G).

361H Theorem Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Suppose that $\nu : \mathfrak{A} \to U$ is an additive function and $T : S = S(\mathfrak{A}) \to U$ the corresponding linear operator. Then $T \in \mathsf{L}^\sim = \mathsf{L}^\sim(S;U)$ iff $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$, and in this case $|T| \in \mathsf{L}^\sim$ corresponds to $|\nu| : \mathfrak{A} \to U$, defined by setting

$$|\nu|(a) = \sup\{\sum_{j=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}$$
$$= \sup\{\nu b - \nu(a \setminus b) : b \subseteq a\}$$

for every $a \in \mathfrak{A}$.

proof (a) Suppose that $T \in \mathsf{L}^\sim$ and $a \in \mathfrak{A}$. Then for any $b \subseteq a$, we have $\chi b \le \chi a$ so

$$|\nu b| = |T\chi b| \le |T|(\chi a).$$

Accordingly $\{\nu b : b \subseteq a\}$ is order-bounded in U.

(b) Now suppose that $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ we can define $w_a = \sup\{|\nu b| : b \subseteq a\}$; in this case, $\nu b - \nu(a \setminus b) \le 2w_a$ whenever $b \subseteq a$, so $\theta a = \sup_{b\subseteq a} \nu b - \nu(a \setminus b)$ is also defined in U. Considering $b = a$, $b = 0$ we see that $\theta a \ge |\nu a|$. Next, $\theta : \mathfrak{A} \to U$ is additive. **P** Take a_1, $a_2 \in \mathfrak{A}$ such that $a_1 \cap a_2 = 0$; set $a_0 = a_1 \cup a_2$. For each $j \le 2$ set

$$A_j = \{\nu(a_j \cap b) - \nu(a_j \setminus b) : b \in \mathfrak{A}\} \subseteq U.$$

Then $A_0 \subseteq A_1 + A_2$, because

$$\nu(a_0 \cap b) - \nu(a_0 \setminus b) = \nu(a_1 \cap b) - \nu(a_1 \setminus b) + \nu(a_2 \cap b) - \nu(a_2 \setminus b)$$

for every $b \in \mathfrak{A}$. But also $A_1 + A_2 \subseteq A_0$, because if b_1, $b_2 \in \mathfrak{A}$ then

$$\nu(a_1 \cap b_1) - \nu(a_1 \setminus b_1) + \nu(a_2 \cap b_2) - \nu(a_2 \setminus b_2) = \nu(a_0 \cap b) - \nu(a_0 \setminus b)$$

where $b = (a_1 \cap b_1) \cup (a_2 \cap b_2)$. So $A_0 = A_1 + A_2$, and

$$\theta a_0 = \sup A_0 = \sup A_1 + \sup A_2 = \theta a_1 + \theta a_2$$

(351Dc). **Q**

We therefore have a corresponding positive operator $T_1 : S \to U$ such that $\theta = T_1\chi$. But we also see that $\theta a = \sup\{\sum_{i=0}^n |\nu a_i| : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}$ for every $a \in \mathfrak{A}$. **P** If $a_0, \dots , a_n$ are disjoint and included in a, then

$$\sum_{i=0}^n |\nu a_i| \le \sum_{i=0}^n \theta a_i = \theta(\sup_{i\le n} a_i) \le \theta a.$$

On the other hand,

$$\theta a \le \sup_{b\subseteq a} |\nu b| + |\nu(a \setminus b)| \le \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}. \textbf{ Q}$$

It follows that $T \in \mathsf{L}^\sim$. **P** Take any $u \ge 0$ in S. Set $a = [\![u > 0]\!]$ (361Eg) and $\alpha = \|u\|_\infty$. If $0 < |v| \le u$, then v is expressible as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n$ are disjoint and no α_i nor a_i is zero. Since $|v| \le \alpha\chi a$, we must have $|\alpha_i| \le \alpha$, $a_i \subseteq a$ for each i. So

$$|Tv| = |\sum_{i=0}^n \alpha_i\chi a_i| \le \sum_{i=0}^n |\alpha_i||\nu a_i| \le \alpha \sum_{i=0}^n |\nu a_i| = \alpha\theta a.$$

Thus $\{|Tv| : |v| \le u\}$ is bounded above by $\alpha\theta a$. As u is arbitrary, $T \in \mathsf{L}^\sim$. **Q**

(c) Thus $T \in \mathsf{L}^\sim$ iff ν is order-bounded on the sets $\{b : b \subseteq a\}$, and in this case the two formulae offered for $|\nu|$ are consistent and make $|\nu| = \theta$. Finally, $\theta = |T|\chi$. **P** Take $a \in \mathfrak{A}$. If $a_0, \dots , a_n \subseteq a$ are disjoint, then

$$\sum_{i=0}^n |\nu a_i| = \sum_{i=0}^n |T\chi a_i| \le \sum_{i=0}^n |T|(\chi a_i) \le |T|(\chi a);$$

so $\theta a \le |T|(\chi a)$. On the other hand, the argument at the end of (b) above shows that $|T|(\chi a) \le \theta a$ for every a. Thus $|T|(\chi a) = \theta a$ for every $a \in \mathfrak{A}$, as required. **Q**

361I Theorem Let $\mathfrak{A}$ be a Boolean ring, U a normed space and $\nu : \mathfrak{A} \to U$ an additive function. Give $S = S(\mathfrak{A})$ its norm $\| \|_\infty$, and let $T : S \to U$ be the linear operator corresponding to ν. Then T is a bounded linear operator iff $\{\nu a : a \in \mathfrak{A}\}$ is bounded, and in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

proof (a) If T is bounded, then

$$\|\nu a - \nu b\| = \|T(\chi a - \chi b)\| \le \|T\|\|\chi a - \chi b\|_\infty \le \|T\|$$

for every $a \in \mathfrak{A}$, so ν is bounded and $\sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\| \le \|T\|$.

(b)(i) For the converse, we need a refinement of an idea in 361Ec. If $u \in S$ and $u \ge 0$ and $\|u\|_\infty \le 1$, then u is expressible as $\sum_{i=0}^m \gamma_i\chi c_i$ where $\gamma_i \ge 0$ and $\sum_{i=0}^m \gamma_i = 1$. **P** If $u = 0$, take $n = 0$, $c_0 = 0$, $\gamma_0 = 1$. Otherwise, start from an expression $u = \sum_{j=0}^n \gamma_j\chi c_j$ where $c_0 \supseteq \dots \supseteq c_n$ and every γ_j is non-negative, as in 361Ec. We may suppose that $c_n \ne 0$, in which case

$$\sum_{j=0}^n \gamma_j = u(z) \le 1$$

for every $z \in \widehat{c}_n \subseteq Z$, the Stone space of $\mathfrak{A}$. Set $m = n+1$, $c_m = 0$ and $\gamma_m = 1-\sum_{j=0}^n \gamma_j$ to get the required form. **Q**

(ii) The next fact we need is an elementary property of real numbers: if $\gamma_0, \dots , \gamma_m, \gamma'_0, \dots , \gamma'_n \ge 0$ and $\sum_{i=0}^m \gamma_i = \sum_{j=0}^n \gamma'_j$, then there are $\delta_{ij} \ge 0$ such that $\gamma_i = \sum_{j=0}^m \delta_{ij}$ for every $i \le m$ and $\gamma'_j = \sum_{i=0}^n \delta_{ij}$ for every $j \le n$. **P** This is just the case $U = \mathbb{R}$ of 352Fd. **Q**

(iii) Now suppose that ν is bounded; set $\alpha_0 = \sup_{a\in\mathfrak{A}} \|\nu a\| < \infty$. Then

$$\alpha = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\| \le 2\alpha_0$$

is also finite. If $u \in S$ and $\|u\|_\infty \le 1$, then we can express u as $u^+ - u^-$ where u^+, u^- are non-negative and also of norm at most 1. By (i), we can express these as

$$u^+ = \sum_{i=0}^m \gamma_i\chi c_i, \quad u^- = \sum_{j=0}^n \gamma'_j\chi c'_j$$

where all the γ_i, γ'_j are non-negative and $\sum_{i=0}^m \gamma_i = \sum_{j=0}^n \gamma'_j = 1$. Take $\langle\delta_{ij}\rangle_{i\le m,j\le n}$ from (ii). Set $c_{ij} = c_i$, $c'_{ij} = c'_j$ for all i, j, so that

$$u^+ = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}\chi c_{ij}, \quad u^- = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}\chi c'_{ij},$$

$$u = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}(\chi c_{ij} - \chi c'_{ij}),$$

$$Tu = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}(\nu c_{ij} - \nu c'_{ij}),$$

$$\|Tu\| \le \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}\|\nu c_{ij} - \nu c'_{ij}\| \le \sum_{i=0}^m \sum_{j=0}^n \delta_{ij}\alpha = \alpha.$$

As u is arbitrary, T is a bounded linear operator and $\|T\| \le \alpha$, as required.

361J The last few paragraphs describe the properties of $S(\mathfrak{A})$ in terms of universal mapping theorems. The next theorem looks at the construction as a functor which converts Boolean algebras into Riesz spaces and ring homomorphisms into Riesz homomorphisms.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism.

(a) We have a Riesz homomorphism $T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$ given by the formula

$$T_\pi(\chi a) = \chi(\pi a) \text{ for every } a \in \mathfrak{A}.$$

For any $u \in S(\mathfrak{A})$, $\|T_\pi u\|_\infty = \min\{\|u'\|_\infty : u' \in S(\mathfrak{A}), T_\pi u' = T_\pi u\}$; in particular, $\|T_\pi u\|_\infty \le \|u\|_\infty$. Moreover, $T_\pi(u \times u') = T_\pi u \times T_\pi u'$ for all u, $u' \in S(\mathfrak{A})$.

(b) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : u \in S(\mathfrak{A}), T_\pi u = v\}$ for every $v \in S(\mathfrak{B})$.

(c) The kernel of T_π is just the set of those $u \in S(\mathfrak{A})$ such that $\pi[\![|u| > 0]\!] = 0$, defining $[\![\ldots > \ldots]\!]$ as in 361Eg.

(d) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in S(\mathfrak{A})$.

(e) T_π is order-continuous iff π is order-continuous.

(f) T_π is sequentially order-continuous iff π is sequentially order-continuous.

(g) If $\mathfrak{C}$ is another Boolean ring and $\phi : \mathfrak{B} \to \mathfrak{C}$ is another ring homomorphism, then $T_{\phi\pi} = T_\phi T_\pi : S(\mathfrak{A}) \to S(\mathfrak{C})$.

proof (a) The map $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$ is additive (361Cc), so corresponds to a linear operator $T = T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$, by 361F. χ and π are both lattice homomorphisms, so $\chi\pi$ also is, and T is a Riesz homomorphism (361Gc). If $u = \sum_{i=0}^n \alpha_i \chi a_i$, where $a_0, \ldots, a_n$ are disjoint, then look at $I = \{i : i \le n, \pi a_i \ne 0\}$. We have

$$Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i) = \sum_{i \in I} \alpha_i \chi(\pi a_i)$$

and $\pi a_0, \ldots, \pi a_n$ are disjoint, so that

$$\|Tu\|_\infty = \sup_{i \in I} |\alpha_i| = \|u'\|_\infty \le \sup_{a_i \ne 0} |\alpha_i| \le \|u\|_\infty,$$

where $u' = \sum_{i \in I} \alpha_i \chi a_i$, so that $Tu' = Tu$. If a, $a' \in \mathfrak{A}$, then

$$T(\chi a \times \chi a') = T\chi(a \cap a') = \chi\pi(a \cap a') = \chi\pi a \times \chi\pi a' = T\chi a \times T\chi a',$$

so T is multiplicative.

(b) If π is surjective, then $T[S(\mathfrak{A})]$ must be the linear span of

$$\{T(\chi a) : a \in \mathfrak{A}\} = \{\chi(\pi a) : a \in \mathfrak{A}\} = \{\chi b : b \in \mathfrak{B}\},$$

so is the whole of $S(\mathfrak{B})$. If T is surjective, and $b \in \mathfrak{B}$, then there must be a $u \in \mathfrak{A}$ such that $Tu = \chi b$. We can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \ldots, a_n$ are disjoint; now

$$\chi b = Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i),$$

and $\pi a_0, \ldots, \pi a_n$ are disjoint in $\mathfrak{B}$, so we must have

$$b = \sup_{i \in I} \pi a_i = \pi(\sup_{i \in I} a_i) \in \pi[\mathfrak{A}],$$

where $I = \{i : \alpha_i = 1\}$. As b is arbitrary, π is surjective. Of course the formula for $\|v\|_\infty$ is a consequence of the formula for $\|Tu\|_\infty$ in (a).

(c)(i) If $\pi[\![|u| > 0]\!] = 0$ then $|u| \le \alpha\chi a$, where $\alpha = \|u\|_\infty$, $a = [\![|u| > 0]\!]$, so

$$|Tu| = T|u| \le \alpha T(\chi a) = \alpha\chi(\pi a) = 0,$$

and $Tu = 0$. **(ii)** If $u \in S(\mathfrak{A})$ and $Tu = 0$, express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \ldots, a_n$ are disjoint and every α_i is non-zero (361Eb). In this case

$$0 = |Tu| = T|u| = \sum_{i=0}^n |\alpha_i| \chi(\pi a_i),$$

so $\pi a_i = 0$ for every i, and

$$\pi[\![|u| > 0]\!] = \pi(\sup_{i\le n} a_i) = \sup_{i\le n} \pi a_i = 0.$$

(d) If T is injective and $a \in \mathfrak{A} \setminus \{0\}$, then $\chi(\pi a) = T(\chi a) \ne 0$, so $\pi a \ne 0$; as a is arbitrary, π is injective. If π is injective then $\pi[\![|u| > 0]\!] \ne 0$ for every non-zero $u \in S(\mathfrak{A})$, so T is injective, by (c). Now the formula in (a) shows that T is norm-preserving.

(e)(i) If T is order-continuous and $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, let b be any lower bound for $\pi[A]$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a) = T(\chi a)$$

for any $a \in A$. But $T\chi$ is order-continuous, by 361Ef, so $\inf_{a\in A} T(\chi a) = 0$, and b must be 0. As b is arbitrary, $\inf_{a\in A} \pi a = 0$; as A is arbitrary, π is order-continuous. **(ii)** If π is order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$, using 361Ef again; but now by 361G(b-ii) T must be order-continuous.

(f)(i) If T is sequentially order-continuous, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, let b be any lower bound for $\{\pi a_n : n \in \mathbb{N}\}$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a_n) = T(\chi a_n)$$

for any $a \in A$. But $T\chi$ is sequentially order-continuous so $\inf_{n\in\mathbb{N}} T(\chi a_n) = 0$, and b must be 0. As b is arbitrary, $\inf_{n\in\mathbb{N}} \pi a_n = 0$; as A is arbitrary, π is sequentially order-continuous. **(ii)** If π is sequentially order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$; but now T must be sequentially order-continuous.

(g) We need only check that

$$T_{\phi\pi}(\chi a) = \chi(\phi(\pi a)) = T_\phi(\chi(\pi a)) = T_\phi T(\chi a)$$

for every $a \in \mathfrak{A}$.

361K Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $S(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $S(\mathfrak{A})$.

proof Write S for $S(\mathfrak{A})$.

(a) The point is that, for any $a \in \mathfrak{A}$,
 (i) $|u| \wedge |v| = 0$ whenever $u \in V_a$, $v \in V_{1\setminus a}$,
 (ii) $V_a + V_{1\setminus a} = S$.
P (i) is just because $\chi a \wedge \chi(1 \setminus a) = 0$. As for (ii), if $w \in S$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}. \textbf{ Q}$$

(b) Accordingly $V_a + V_a^\perp \supseteq V_a + V_{1\setminus a} = U$ and V_a is a projection band (352R). Next, any projection band $U \subseteq S$ is of the form V_a. **P** We know that $\chi 1 = u + v$ where $u \in U$, $v \in U^\perp$. Since $|u| \wedge |v| = 0$, u and v must be the characteristic functions of complementary subsets of Z, the Stone space of $\mathfrak{A}$. But $\{z : u(z) \ne 0\} = \{z : u(z) \ge 1\}$ must be of the form $\widehat{a}$, where $a = [\![u > 0]\!]$, in which case $u = \chi a$ and $v = \chi(1 \setminus a)$. Accordingly $U \supseteq V_a$ and $U^\perp \supseteq V_{1\setminus a}$. But this means that U must be V_a precisely. **Q**

(c) Thus $a \mapsto V_a$ is a surjective function from $\mathfrak{A}$ onto the algebra of projection bands in S. Now

$$a \subseteq b \iff \chi a \in V_b \iff V_a \subseteq V_b,$$

so $a \mapsto V_a$ is order-preserving and bijective. By 312L it is a Boolean isomorphism.

361L Proposition Let X be a set, and Σ a ring of subsets of X, that is, a subring of the Boolean ring $\mathcal{P}X$. Then $S(\Sigma)$ can be identified, as ordered linear space, with the linear subspace of $\ell^\infty(X)$ generated by the characteristic functions of members of Σ, which is a Riesz subspace of $\ell^\infty(X)$. The norm of $S(\Sigma)$ corresponds to the uniform norm on $\ell^\infty(X)$, and its multiplication to pointwise multiplication of functions.

proof Let Z be the Stone space of Σ, and for $E \in \Sigma$ write χE for the characteristic function of E as a subset of X, $\hat{\chi} E$ for the characteristic function of the open-and-compact subset of Z corresponding to E.

Of course $\chi:\Sigma\to\ell^\infty(X)$ is additive, so by 361F there is a linear operator $T:S\to\ell^\infty(X)$, writing S for $S(\Sigma)$, such that $T(\hat\chi E)=\chi E$ for every $E\in\Sigma$.

If $u\in S$, $Tu\ge 0$ iff $u\ge 0$. **P** Express u as $\sum_{j=0}^m\beta_j\hat\chi E_j$ where $E_0,\dots,E_m$ are disjoint. Then $Tu=\sum_{j=0}^m\beta_j\chi E_j$, so

$$u\ge 0\iff\beta_j\ge 0\text{ whenever }E_j\ne\emptyset\iff Tu\ge 0.\ \mathbf{Q}$$

But this means (α) that

$$Tu=0\iff Tu\ge 0\ \&\ T(-u)\ge 0\iff u\ge 0\ \&\ -u\ge 0\iff u=0,$$

so that T is injective and is a linear space isomorphism between S and its image $\mathcal{S}$, which must be the linear space spanned by $\{\chi E:E\in\Sigma\}$ (β) that T is an order-isomorphism between S and $\mathcal{S}$.

Because $\chi E\wedge\chi F=0$ whenever E, $F\in\Sigma$ and $E\cap F=\emptyset$, T is a Riesz homomorphism and $\mathcal{S}$ is a Riesz subspace of $\ell^\infty(X)$ (361Gc). Now

$$\|u\|_\infty=\inf\{\alpha:|u|\le\alpha\hat\chi X\}=\inf\{\alpha:|Tu|\le\alpha\chi X\}=\|Tu\|_\infty$$

for every $u\in S$. Finally,

$$T(\hat\chi E\times\hat\chi F)=T(\hat\chi(E\cap F))=\chi(E\cap F)=T(\hat\chi E)\times T(\hat\chi F)$$

for all E, $F\in\Sigma$, so $\mathcal{S}$ is closed under pointwise multiplication and the multiplications of S, $\mathcal{S}$ are identified under T.

361M Proposition Let X be a set, Σ a ring of subsets of X, and $\mathcal{I}$ an ideal of Σ; write $\mathfrak{A}$ for the quotient ring $\Sigma/\mathcal{I}$. Let $\mathcal{S}$ be the linear span of $\{\chi E:E\in\Sigma\}$ in $\mathbb{R}^X$, and write

$$V=\{f:f\in\mathcal{S},\ \{x:f(x)\ne 0\}\in\mathcal{I}\}.$$

Then V is a solid linear subspace of $\mathcal{S}$. Now $S(\mathfrak{A})$ becomes identified with the quotient Riesz space $\mathcal{S}/V$, if for every $E\in\Sigma$ we identify $\chi(E^\bullet)\in S(\mathfrak{A})$ with $(\chi E)^\bullet\in\mathcal{S}/V$. If we give $\mathcal{S}$ its uniform norm inherited from $\ell^\infty(X)$, V is a closed linear subspace of $\mathcal{S}$, and the quotient norm on $\mathcal{S}/V$ corresponds to the norm of $S(\mathfrak{A})$:

$$\|f^\bullet\|=\min\{\alpha:\{x:|f(x)|>\alpha\}\in\mathcal{I}\}.$$

If we write $\times$ for pointwise multiplication on $\mathcal{S}$, then V is an ideal of the ring $(\mathcal{S},+,\times)$, and the multiplication induced on $\mathcal{S}/V$ corresponds to the multiplication of $S(\mathfrak{A})$.

proof Use 361J and 361L. We can identify $\mathcal{S}$ with $S(\Sigma)$. Now the canonical ring homomorphism $E\mapsto E^\bullet$ corresponds to a surjective Riesz homomorphism T from $S(\Sigma)$ to $S(\mathfrak{A})$ which takes χE to $\chi(E^\bullet)$. For $f\in\mathcal{S}$, $[\![|f|>0]\!]$ is just $\{x:f(x)\ne 0\}$, so the kernel of T is just the set of those $f\in\mathcal{S}$ such that $\{x:f(x)\ne 0\}\in\mathcal{I}$, which is V. So

$$S(\mathfrak{A})=T[\mathcal{S}]\cong\mathcal{S}/V.$$

As noted in 361Ja, $T(f\times g)=Tf\times Tg$ for all f, $g\in\mathcal{S}$, so the multiplications of $\mathcal{S}/V$ and $S(\mathfrak{A})$ match. As for the norms, the norm of $S(\mathfrak{A})$ corresponds to the norm of $\mathcal{S}/V$ by the formulae in 361Ja or 361Jb. (To see that V is closed in $\mathcal{S}$, we need note only that if $f\in\overline{V}$ then

$$\|Tf\|_\infty=\inf_{g\in V}\|f+g\|_\infty=\inf_{g\in V}\|f-g\|_\infty=0,$$

so that $Tf=0$ and $f\in V$.) To check the formula for $\|f^\bullet\|$, take any $f\in\mathcal{S}$. Express it as $\sum_{i=0}^n\alpha_i\chi E_i$ where $E_0,\dots,E_n\in\Sigma$ are disjoint. Set $I=\{i:E_i\notin\mathcal{I}\}$; then

$$\|Tf\|_\infty=\max_{i\in I}|\alpha_i|=\min\{\alpha:\{x:|f(x)|>\alpha\}\in\mathcal{I}\}.$$

361X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring and U a linear space. Show that a function $\nu:\mathfrak{A}\to U$ is additive iff $\nu 0=0$ and $\nu(a\cup b)+\nu(a\cap b)=\nu a+\nu b$ for all a, $b\in\mathfrak{A}$.

>(b) Let U be an **algebra over** $\mathbb{R}$, that is, a real linear space endowed with a multiplication $\times$ such that $(U,+,\times)$ is a ring and $\alpha(w\times z)=(\alpha w)\times z=w\times(\alpha z)$ for all w, $z\in U$ and all $\alpha\in\mathbb{R}$. Let $\mathfrak{A}$ be a Boolean ring, $\nu:\mathfrak{A}\to U$ an additive function and $T:S(\mathfrak{A})\to U$ the corresponding linear operator. Show that T is multiplicative iff $\nu(a\cap b)=\nu a\times\nu b$ for all a, $b\in\mathfrak{A}$.

>(c) Let $\mathfrak{A}$ be a Boolean ring, and U a Dedekind complete Riesz space. Suppose that $\nu : \mathfrak{A} \to U$ is an additive function such that the corresponding linear operator $T : S(\mathfrak{A}) \to U$ belongs to $\mathsf{L}^\sim = \mathsf{L}^\sim(S(\mathfrak{A}); U)$. Show that $T^+ \in \mathsf{L}^\sim$ corresponds to $\nu^+ : \mathfrak{A} \to U$, where $\nu^+ a = \sup_{b \subseteq a} \nu b$ for every $a \in \mathfrak{A}$.

(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that there is a natural one-to-one correspondence between Boolean homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and Riesz homomorphisms $T : S(\mathfrak{A}) \to S(\mathfrak{B})$ such that $T(\chi 1_{\mathfrak{A}}) = \chi 1_{\mathfrak{B}}$, given by setting $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(e) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings and $T : S(\mathfrak{A}) \to S(\mathfrak{B})$ a linear operator such that $T(u \times v) = Tu \times Tv$ for all u, $v \in S(\mathfrak{A})$. Show that there is a ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings. Show that any isomorphism of the algebras $S(\mathfrak{A})$ and $S(\mathfrak{B})$ (using the word 'algebra' in the sense of 361Xb) must be a Riesz space isomorphism, and therefore corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T : S(\mathfrak{A}) \to S(\mathfrak{B})$ a Riesz homomorphism. Show that there are a ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ and a non-negative $v \in S(\mathfrak{B})$ such that $T(\chi a) = v \times \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(h) Let $\mathfrak{A}$ be a Boolean ring. Show that for any $u \in S(\mathfrak{A})$ the solid linear subspace of $S(\mathfrak{A})$ generated by u is a projection band in $S(\mathfrak{A})$. Show that the set of such bands is an ideal in the algebra of all projection bands, and is isomorphic to $\mathfrak{A}$.

>(i) Let X be a set, Σ a σ-algebra of subsets of X. Show that the linear span $\mathcal{S}$ in $\mathbb{R}^X$ of $\{\chi E : E \in \Sigma\}$ is just the set of Σ-measurable functions $f : X \to \mathbb{R}$ which take only finitely many values.

(j) For any Boolean ring $\mathfrak{A}$, we may define its 'complex S-space' $S_{\mathbb{C}}(\mathfrak{A})$ as the linear span in $\mathbb{C}^X$ of the characteristic functions of open-and-compact subsets of the Stone space Z of $\mathfrak{A}$. State and prove results corresponding to 361Ea-361Ed, 361Eh, 361F, 361L and 361M.

361Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring. Let V be the linear space of all formal sums of the form $\sum_{i=0}^n \alpha_i a_i$ where $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ and $a_0, \dots, a_n \in \mathfrak{A}$. Let $W \subseteq V$ be the linear subspace spanned by members of V of the form $(a \cup b) - a - b$ where a, $b \in \mathfrak{A}$ are disjoint. Define $\chi' : \mathfrak{A} \to V/W$ by taking $\chi' a$ to be the image in V/W of $a \in V$. Show, without using the axiom of choice, that the pair $(V/W, \chi')$ has the universal mapping property of $(S(\mathfrak{A}), \chi)$ as described in 361F and that V/W has a Riesz space structure, a norm and a multiplicative structure as described in 361D-361E. Prove results corresponding to 361E-361M.

(b) Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Let $A \subseteq \mathsf{L}^\sim = \mathsf{L}^\sim(S(\mathfrak{A}); U)$ be a non-empty set. Suppose that $\tilde{T} = \sup A$ is defined in $\mathsf{L}^\sim$, and that $\tilde{\nu} = \tilde{T}\chi$. Show that for any $a \in \mathfrak{A}$,

$$\tilde{\nu} a = \sup\{\textstyle\sum_{i=0}^n T_i(\chi a_i) : T_0, \dots, T_n \in A,\ a_0, \dots, a_n \subseteq a \text{ are disjoint, } \sup_{i \le n} a_i = a\}.$$

(c) Let $\mathfrak{A}$ be a Boolean algebra. Show that the algebra of all bands of $S(\mathfrak{A})$ can be identified with the Dedekind completion of $\mathfrak{A}$ (314U).

(d) Let $\mathfrak{A}$ be a Boolean ring, and U a complex normed space. Let $\nu : \mathfrak{A} \to U$ be an additive function and $T : S_{\mathbb{C}}(\mathfrak{A}) \to U$ the corresponding linear operator (cf. 361Xj). Show that (giving $S_{\mathbb{C}}(\mathfrak{A})$ its usual norm $\| \|_\infty$)

$$\|T\| = \sup\{\|\textstyle\sum_{j=0}^n \zeta_j \nu a_j\| : a_0, \dots, a_n \in \mathfrak{A} \text{ are disjoint, } |\zeta_j| = 1 \text{ for every } j\}$$

if either is finite.

(e) Let U be a Riesz space. Show that it is isomorphic to $S(\mathfrak{A})$, for some Boolean algebra $\mathfrak{A}$, iff it has an order unit and every solid linear subspace of U is a projection band.

(f) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j\in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that U is a linear space and $\theta : C \to U$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Show that there is a unique additive function $\nu : \mathfrak{A} \to U$ extending θ. (*Hint*: 326Q.)

361 Notes and comments The space $S(\mathfrak{A})$ corresponds of course to the idea of 'simple function' which belongs to the very beginnings of the theory of integration (122A). All that 361D is trying to do is to set up a logically sound description of this obvious concept which can be derived from the Boolean ring $\mathfrak{A}$ itself. To my eye, there is a defect in the construction there. It relies on the axiom of choice, since it uses the Stone space; but none of the elementary properties of $S(\mathfrak{A})$ have anything to do with the axiom of choice. In 361Ya I offer an alternative construction which is in a formal sense more 'elementary'. If you work through the suggestion there you will find, however, that the technical details become significantly more complicated, and would be intolerable were it not for the intuition provided by the Stone space construction. Of course this intuition is chiefly valuable in the finitistic arguments used in 361E, 361F and 361I; and for these arguments we really need the Stone representation only for finite Boolean rings, which does not depend on the axiom of choice.

It is quite true that in most of this volume (and in most of this chapter) I use the axiom of choice without scruple and without comment. I mention it here only because I find myself using arguments dependent on choice to prove theorems of a type to which the axiom cannot be relevant.

The linear space structure of $S(\mathfrak{A})$, together with the map χ, are uniquely determined by the first universal mapping theorem here, 361F. This result says nothing about the order structure, which needs the further refinement in 361Ga. What is striking is that the partial order defined by 361Ga is actually a lattice ordering, so that we can have a universal mapping theorem for functions to Riesz spaces, as in 361Gc and 361Ja. Moreover, the same ordering provides a happy abundance of results concerning order-continuous functions (361Gb, 361Je-361Jf). When the codomain is a Dedekind complete Riesz space, so that we have a Riesz space $\mathrm{L}^\sim(S;U)$, and a corresponding modulus function $T \mapsto |T|$ for linear operators, there are reasonably natural formulae for $|T|\chi$ in terms of $T\chi$ (361H); see also 361Xc and 361Yb. The multiplicative structure of $S(\mathfrak{A})$ is defined by 361Xb, and the norm by 361I.

The Boolean ring $\mathfrak{A}$ cannot be recovered from the linear space structure of $S(\mathfrak{A})$ alone (since this tells us only the cardinality of $\mathfrak{A}$), but if we add either the ordering or the multiplication of $S(\mathfrak{A})$ then $\mathfrak{A}$ is easy to identify (361K, 361Xf).

The most important Boolean algebras of measure theory arise either as algebras of sets or as their quotients, so it is a welcome fact that in such cases the spaces $S(\mathfrak{A})$ have straightforward representations in terms of the construction of $\mathfrak{A}$ (361L-361M).

In Chapter 24 I offered a paragraph in each section to sketch a version of the theory based on the field of complex numbers rather than the field of real numbers. This was because so many of the most important applications of these ideas involve complex numbers, even though (in my view) the ideas themselves are most clearly and characteristically expressed in terms of real numbers. In the present chapter we are one step farther away from these applications, and I therefore relegate complex numbers to the exercises, as in 361Xj and 361Yd.

362 $S^\sim$

The next stage in our journey is the systematic investigation of linear functionals on spaces $S = S(\mathfrak{A})$. We already know that these correspond to additive real-valued functionals on the algebra $\mathfrak{A}$ (361F). My purpose here is to show how the structure of the Riesz space dual $S^\sim$ and its bands is related to the classes of additive functionals introduced in §§326-327. The first step is just to check the identification of the linear and order structures of $S^\sim$ and the space M of bounded finitely additive functionals (362A); all the ideas needed for this have already been set out, and the basic properties of $S^\sim$ are covered by the general results in §356. Next, we need to be able to describe the operations on M corresponding to the Riesz space operations

$|\,|$, $\vee$, $\wedge$ on $S^\sim$, and the band projections from $S^\sim$ onto $S_c^\sim$ and $S^\times$; these are dealt with in 362B, with a supplementary remark in 362D. In the case of measure algebras, we have some further important bands which present themselves in M, rather than in $S^\sim$, and which are treated in 362C. Since all these spaces are L-spaces, it is worth taking a moment to identify their uniformly integrable subsets; I do this in 362F.

While some of the ideas here have interesting extensions to the case in which $\mathfrak{A}$ is a Boolean ring without identity, these can I think be left to one side; the work of this section will be done on the assumption that every $\mathfrak{A}$ is a Boolean algebra.

362A Theorem Let $\mathfrak{A}$ be a Boolean algebra. Write S for $S(\mathfrak{A})$.

(a) The partially ordered linear space of all finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the partially ordered linear space of all real-valued linear functionals on S.

(b) The linear space of bounded finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim$ of order-bounded linear functionals on S. If $f \in S^\sim$ corresponds to $\nu : \mathfrak{A} \to \mathbb{R}$, then $f^+ \in S^\sim$ corresponds to ν^+, where

$$\nu^+ a = \sup_{b \subseteq a} \nu b$$

for every $a \in \mathfrak{A}$, and

$$\|f\| = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a).$$

(c) The linear space of bounded countably additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S_c^\sim$.

(d) The linear space of completely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\times$.

proof By 361F, we have a canonical one-to-one correspondence between linear functionals $f : S \to \mathbb{R}$ and additive functionals $\nu_f : \mathfrak{A} \to \mathbb{R}$, given by setting $\nu_f = f\chi$.

(a) Now it is clear that $\nu_{f+g} = \nu_f + \nu_g$, $\nu_{\alpha f} = \alpha\nu_f$ for all f, g and α, so this one-to-one correspondence is a linear space isomorphism. To see that it is also an order-isomorphism, we need note only that ν_f is non-negative iff f is, by 361Ga.

(b) Recall from 356N that, because S is a Riesz space with order unit (361Ee), $S^\sim$ has a corresponding norm under which it is an L-space.

(i) If $f \in S^\sim$, then

$$\sup_{b \in \mathfrak{A}} |\nu_f b| = \sup_{b \in \mathfrak{A}} |f(\chi b)| \le \sup\{|f(u)| : u \in S,\ |u| \le \chi 1\}$$

is finite, and ν_f is bounded.

(ii) Now suppose that ν_f is bounded and that $v \in S^+$. Then there is an $\alpha \ge 0$ such that $v \le \alpha\chi 1$ (361Ee). If $u \in S$ and $|u| \le v$, then we can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint (361Eb); now $|\alpha_i| \le \alpha$ whenever $a_i \ne 0$, so

$$|f(u)| = |\sum_{i=0}^n \alpha_i \nu_f a_i| \le \alpha \sum_{i=0}^n |\nu_f a_i| = \alpha(\nu_f c_1 - \nu_f c_2) \le 2\alpha \sup_{b \in \mathfrak{A}} |\nu_f b|,$$

setting $c_1 = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$, $c_2 = \sup\{a_i : i \le n,\ \nu_f a_i < 0\}$. This shows that $\{f(u) : |u| \le v\}$ is bounded. As v is arbitrary, $f \in S^\sim$ (356Aa).

(iii) To check the correspondence between f^+ and ν_f^+, refine the arguments of (i) and (ii) as follows. Take any $f \in S^\sim$. If $a \in \mathfrak{A}$,

$$\nu_f^+ a = \sup_{b \subseteq a} \nu_f b = \sup_{b \subseteq a} f(\chi b) \le \sup\{f(u) : u \in S,\ 0 \le u \le \chi a\} = f^+(\chi a).$$

On the other hand, if $u \in S$ and $0 \le u \le \chi a$, then we can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint; now $0 \le \alpha_i \le 1$ whenever $a_i \ne 0$, so

$$f(u) = \sum_{i=0}^n \alpha_i \nu_f a_i \le \nu_f c \le \nu_f^+ a,$$

where $c = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$. As u is arbitrary, $f^+(\chi a) \le \nu_f^+ a$. This shows that $\nu_f^+ = f^+\chi$ is finitely additive, and that $\nu_f^+ = \nu_{f^+}$, as claimed.

(iv) Now, for any $f \in S^\sim$,

$$\|f\| = |f|(\chi 1)$$

(356N)

$$= (2f^+ - f)(\chi 1)$$

(352D)

$$= 2\nu_f^+ 1 - \nu_f 1$$

(by (iii) just above)

$$= \sup_{a \in \mathfrak{A}} 2\nu_f a - \nu_f 1 = \sup_{a \in \mathfrak{A}} \nu_f a - \nu_f(1 \setminus a).$$

(c) If $f \geq 0$ in $S^\sim$, then f is sequentially order-continuous iff ν_f is sequentially order-continuous (361Gb), that is, iff ν_f is countably additive (326Gc). Generally, an order-bounded linear functional belongs to $S_c^\sim$ iff it is expressible as the difference of two sequentially order-continuous positive linear functionals (356Ab), while a bounded finitely additive functional is countably additive iff it is expressible as the difference of two non-negative countably additive functionals (326H); so in the present context $f \in S_c^\sim$ iff ν_f is bounded and countably additive.

(d) If $f \geq 0$ in $S^\sim$, then f is order-continuous iff ν_f is order-continuous (361Gb), that is, iff ν_f is completely additive (326Kc). Generally, an order-bounded linear functional belongs to $S^\times$ iff it is expressible as the difference of two order-continuous positive linear functionals (356Ac), while a finitely additive functional is completely additive iff it is expressible as the difference of two non-negative completely additive functionals (326M); so in the present context $f \in S^\times$ iff ν_f is completely additive.

362B Spaces of finitely additive functionals The identifications in the last theorem mean that we can relate the Riesz space structure of $S(\mathfrak{A})^\sim$ to constructions involving finitely additive functionals. I have already set out the most useful facts as exercises (326Yj, 326Ym, 326Yn, 326Yp, 326Yq); it is now time to repeat them more formally.

Theorem Let $\mathfrak{A}$ be a Boolean algebra. Write $S = S(\mathfrak{A})$, and let M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$, $M_\sigma \subseteq M$ the space of bounded countably additive functionals, and $M_\tau \subseteq M_\sigma$ the space of completely additive functionals.

(a) For any μ, $\nu \in M$, $\mu \vee \nu$, $\mu \wedge \nu$ and $|\nu|$ are defined by the formulae

$$(\mu \vee \nu)(a) = \sup_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$(\mu \wedge \nu)(a) = \inf_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b) = \sup_{b,c \subseteq a} \nu b - \nu c$$

for every $a \in \mathfrak{A}$. Setting

$$\|\nu\| = |\nu|(1) = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a),$$

M becomes an L-space.

(b) M_σ and M_τ are projection bands in M, therefore L-spaces in their own right. In particular, $|\nu| \in M_\sigma$ for every $\nu \in M_\sigma$, and $|\nu| \in M_\tau$ for every $\nu \in M_\tau$.

(c) The band projection $P_\sigma : M \to M_\sigma$ is defined by the formula

$$(P_\sigma \nu)(c) = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(d) The band projection $P_\tau : M \to M_\tau$ is defined by the formula

$$(P_\tau \nu)(c) = \inf\{\sup_{a \in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(e) If $A \subseteq M$ is upwards-directed, then A is bounded above in M iff $\{\nu 1 : \nu \in A\}$ is bounded above in $\mathbb{R}$, and in this case (if $A \neq \emptyset$) $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup_{\nu\in A} \nu a \text{ for every } a \in \mathfrak{A}.$$

(f) Suppose that μ, $\nu \in M$.

(i) The following are equiveridical:

(α) ν belongs to the band in M generated by μ;

(β) for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \le \epsilon$ whenever $|\mu|a \le \delta$;

(γ) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{n\to\infty} |\mu|(a_n) = 0$.

(ii) Now suppose that μ, $\nu \ge 0$, and let ν_1, ν_2 be the components of ν in the band generated by μ and its complement. Then

$$\nu_1 c = \sup_{\delta>0} \inf_{\mu a\le\delta} \nu(c \setminus a), \quad \nu_2 c = \inf_{\delta>0} \sup_{a\subseteq c,\mu a\le\delta} \nu a$$

for every $c \in \mathfrak{A}$.

proof (a) Of course $\mu \vee \nu = \nu + (\mu - \nu)^+$, $\mu \wedge \nu = \nu - (\nu - \mu)^+$, $|\nu| = \nu \vee (-\nu)$ (352D), so the formula of 362Ab gives

$$\begin{aligned}
(\mu \vee \nu)(a) &= \nu a + \sup_{b\subseteq a} \mu b - \nu b = \sup_{b\subseteq a} \mu b + \nu(a \setminus b),\\
(\mu \wedge \nu)(a) &= \nu a - \sup_{b\subseteq a} \nu b - \mu b = \inf_{b\subseteq a} \mu b + \nu(a \setminus b),\\
|\nu|(a) &= \sup_{b\subseteq a} \nu b - \nu(a \setminus b) \le \sup_{b,c\subseteq a} \nu b - \nu c = \sup_{b,c\subseteq a} \nu(b \setminus c) - \nu(c \setminus b)\\
&\le \sup_{b,c\subseteq a} |\nu|(b \setminus c) + |\nu|(c \setminus b) = \sup_{b,c\subseteq a} |\nu|(b \triangle c) \le |\nu|(a).
\end{aligned}$$

The formula offered for $\|\nu\|$ corresponds exactly to the formula in 362Ab for the norm of the associated member of $S(\mathfrak{A})^\sim$; because $S(\mathfrak{A})^\sim$ is an L-space under its norm, so is M.

(b) By 362Ac-362Ad, M_σ and M_τ may be identified with $S(\mathfrak{A})^\sim_c$ and $S(\mathfrak{A})^\times$, which are projection bands in $S(\mathfrak{A})^\sim$ (356B); so that M_σ and M_τ are projection bands in M, and are L-spaces in their own right (354O).

(c) Take any $\nu \ge 0$ in M. Set

$$\nu_\sigma c = \inf\{\sup_{n\in\mathbb{N}} \nu a_n : \langle a_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

for every $c \in \mathfrak{A}$. Then of course $0 \le \nu_\sigma c \le \nu c$ for every c. The point is that ν_σ is countably additive. **P** Let $\langle c_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$, with supremum c. Then for any $\epsilon > 0$ we have non-decreasing sequences $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle a_{in}\rangle_{n\in\mathbb{N}}$, for $i \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} a_n = c, \quad \sup_{n\in\mathbb{N}} a_{in} = c_i \text{ for } i \in \mathbb{N},$$

$$\sup_{n\in\mathbb{N}} \nu a_n \le \nu_\sigma c + \epsilon,$$

$$\sup_{n\in\mathbb{N}} \nu a_{in} \le \nu_\sigma c_i + 2^{-i}\epsilon \text{ for every } i \in \mathbb{N}.$$

Set $b_n = \sup_{i\le n} a_{in}$ for each n; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and

$$\sup_{n\in\mathbb{N}} b_n = \sup_{i,n\in\mathbb{N}} a_{in} = \sup_{i\in\mathbb{N}} c_i = c,$$

so

$$\begin{aligned}
\nu_\sigma c &\le \sup_{n\in\mathbb{N}} \nu b_n = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \nu a_{in}\\
&= \sum_{i=0}^{\infty} \sup_{n\in\mathbb{N}} \nu a_{in} \le \sum_{i=0}^{\infty} \nu_\sigma c_i + 2^{-i}\epsilon = \sum_{i=0}^{\infty} \nu_\sigma c_i + 2\epsilon.
\end{aligned}$$

On the other hand, $\langle a_n \cap c_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $c \cap c_i = c_i$ for each i, so $\nu_\sigma c_i \le \sup_{n\in\mathbb{N}} \nu(a_n \cap c_i)$, and

$$\sum_{i=0}^{\infty}\nu_\sigma c_i \le \sum_{i=0}^{\infty}\sup_{n\in\mathbb{N}}\nu(a_n\cap c_i) = \sup_{n\in\mathbb{N}}\sum_{i=0}^{\infty}\nu(a_n\cap c_i)$$

(because $\langle a_n\rangle_{n\in\mathbb{N}}$ is non-decreasing)

$$\le \sup_{n\in\mathbb{N}}\nu a_n$$

(because $\langle c_i\rangle_{i\in\mathbb{N}}$ is disjoint)

$$\le \nu_\sigma c + \epsilon.$$

As ϵ is arbitrary, $\nu_\sigma c = \sum_{i=0}^{\infty}\nu_\sigma c_i$; as $\langle c_i\rangle_{i\in\mathbb{N}}$ is arbitrary, ν_σ is countably additive. **Q**

Thus $\nu_\sigma \in M_\sigma$. On the other hand, if $\nu' \in M_\sigma$ and $0 \le \nu' \le \nu$, then whenever $c \in \mathfrak{A}$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum c,

$$\nu' c = \sup_{n\in\mathbb{N}}\nu' a_n \le \sup_{n\in\mathbb{N}}\nu a_n.$$

So we must have $\nu' c \le \nu_\sigma c$. This means that

$$\nu_\sigma = \sup\{\nu' : \nu' \in M_\sigma,\ \nu' \le \nu\} = P_\sigma\nu,$$

as claimed.

(d) The same ideas, with essentially elementary modifications, deal with the completely additive part. Take any $\nu \ge 0$ in M. Set

$$\nu_\tau c = \inf\{\sup_{a\in A}\nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

for every $c \in \mathfrak{A}$. Then of course $0 \le \nu_\tau c \le \nu c$ for every c. The point is that ν_τ is completely additive. **P** Note first that if $c \in \mathfrak{A}$, $\epsilon > 0$ there is a non-empty upwards-directed A, with supremum c, such that $\sup_{a\in A}\nu a \le \nu_\tau c + \epsilon\nu c$; for if $\nu c = 0$ we can take $A = \{c\}$. Now let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$. Then for any $\epsilon > 0$ we have non-empty upwards-directed sets A, A_i, for $i \in I$, such that

$$\sup A = 1, \quad \sup A_i = c_i \text{ for } i \in I, \quad \sup_{a\in A}\nu a \le \nu_\tau 1 + \epsilon\nu 1,$$

$$\sup_{a\in A_i}\nu a \le \nu_\tau c_i + \epsilon\nu c_i \text{ for every } i \in I.$$

Set

$$B = \{\sup_{i\in J} a_i : J \subseteq I \text{ is finite, } a_i \in A_i \text{ for every } i \in J\};$$

then B is non-empty and upwards-directed, and

$$\sup B = \sup(\bigcup_{i\in I} A_i) = 1,$$

so

$$\nu_\tau 1 \le \sup_{b\in B}\nu b = \sup\{\sum_{i\in J}\nu a_i : J \subseteq I \text{ is finite}, a_i \in A_i\ \forall\ i \in J\}$$
$$\le \sum_{i\in I}\nu_\tau c_i + \epsilon\nu c_i \le \epsilon\nu 1 + \sum_{i\in I}\nu_\tau c_i.$$

On the other hand, $A_i' = \{a \cap c_i : a \in A\}$ is a non-empty upwards-directed set with supremum c_i for each i, so $\nu_\tau c_i \le \sup_{a\in A_i'}\nu a$, and

$$\sum_{i\in I}\nu_\tau c_i \le \sum_{i\in I}\sup_{a\in A}\nu(a\cap c_i) = \sup_{a\in A}\sum_{i\in I}\nu(a\cap c_i)$$
$$\le \sup_{a\in A}\nu a \le \nu_\tau 1 + \epsilon\nu 1.$$

As ϵ is arbitrary, $\nu_\tau c = \sum_{i\in I}\nu_\tau c_i$; as $\langle c_i\rangle_{i\in I}$ is arbitrary, ν_τ is completely additive, by 326N. **Q**

Thus $\nu_\tau \in M_\tau$. On the other hand, if $\nu' \in M_\tau$ and $0 \le \nu' \le \nu$, then whenever $c \in \mathfrak{A}$ and A is a non-empty upwards-directed set with supremum c,

$$\nu' c = \sup_{a\in A}\nu' a \le \sup_{a\in A}\nu a$$

(using 326Kc). So we must have $\nu' c \le \nu_\tau c$. This means that

$$\nu_\tau = \sup\{\nu' : \nu' \in M_\tau,\ \nu' \le \nu\} = P_\tau\nu,$$

as claimed.

(e) If A is empty, of course it is bounded above in M, and $\{\nu 1 : \nu\in A\} = \emptyset$ is bounded above in $\mathbb{R}$; so let us suppose that A is not empty. In this case, if $\lambda_0 \in M$ is an upper bound for A, then $\lambda_0 1$ is an upper bound for $\{\nu 1 : \nu \in A\}$. On the other hand, if $\sup_{\nu\in A}\nu 1 = \gamma$ is finite, $\gamma^* = \sup\{\nu a : \nu\in A,\ a\in\mathfrak{A}\}$ is finite. **P** Fix $\nu_0 \in A$. Set $\gamma_1 = \sup_{a\in\mathfrak{A}}|\nu_0 a| < \infty$. Then for any $\nu\in A$, $a\in\mathfrak{A}$ there is a $\nu'\in A$ such that $\nu_0\vee\nu \le \nu'$, so that

$$\nu a \le \nu' a = \nu' 1 - \nu'(1\setminus a) \le \gamma - \nu_0(1\setminus a) \le \gamma + \gamma_1.$$

So

$$\gamma^* \le \gamma + \gamma_1 < \infty. \text{ **Q**}$$

Set $\lambda a = \sup_{\nu\in A}\nu a$ for every $a\in\mathfrak{A}$. Then $\lambda : \mathfrak{A}\to\mathbb{R}$ is additive. **P** If a, $b\in\mathfrak{A}$ are disjoint, then

$$\lambda(a\cup b) = \sup_{\nu\in A}\nu(a\cup b) = \sup_{\nu\in A}\nu a + \nu b = \sup_{\nu\in A}\nu a + \sup_{\nu\in A}\nu b$$

(because A is upwards-directed)

$$= \lambda a + \lambda b. \text{ **Q**}$$

Also $\lambda a \le \gamma^*$ for every a, so

$$|\lambda a| = \max(\lambda a, -\lambda a) = \max(\lambda a, \lambda(1\setminus a) - \lambda 1) \le \gamma^* + |\lambda 1|$$

for every $a\in\mathfrak{A}$, and λ is bounded.

This shows that $\lambda\in M$, so that A is bounded above in M. Of course λ must be actually the least upper bound of A in M.

(f)(i)(α)$\Rightarrow$(β) Suppose that ν belongs to the band in M generated by μ, that is, $|\nu| = \sup_{n\in\mathbb{N}}|\nu|\wedge n|\mu|$ (352Vb). Let $\epsilon > 0$. Then there is an $n\in\mathbb{N}$ such that $|\nu|(1) \le \frac{1}{2}\epsilon + (|\nu|\wedge n|\mu|)(1)$ ((e) above). Set $\delta = \frac{1}{2n+1}\epsilon > 0$. If $|\mu|(a)\le\delta$, then

$$\begin{aligned}|\nu a| &\le |\nu|(a) = (|\nu|\wedge n|\mu|)(a) + (|\nu| - |\nu|\wedge n|\mu|)(a)\\ &\le n|\mu|(a) + (|\nu| - |\nu|\wedge n|\mu|)(1) \le n\delta + \frac{1}{2}\epsilon \le \epsilon.\end{aligned}$$

So (β) is satisfied.

(β)$\Rightarrow$(α) Suppose that ν does not belong to the band in M generated by $|\mu|$. Then there is a $\nu_1 > 0$ such that $\nu_1 \le |\nu|$ and $\nu_1\wedge|\mu| = 0$ (353C). For any $\delta > 0$, there is an $a\in\mathfrak{A}$ such that $\nu_1(1\setminus a) + |\mu|(a) \le \min(\delta, \frac{1}{2}\nu_1 1)$ ((a) above); now $|\mu|(a)\le\delta$ but

$$|\nu|(a) \ge \nu_1 a = \nu_1 1 - \nu_1(1\setminus a) \ge \nu_1 1 - \tfrac{1}{2}\nu_1 1 = \tfrac{1}{2}\nu_1 1.$$

Thus μ, ν do not satisfy (β) (with $\epsilon = \frac{1}{2}\nu_1 1$).

(β)$\Rightarrow$(γ) is trivial.

(γ)$\Rightarrow$(α) Observe first that if $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{k\to\infty}\mu c_k = 0$, then $\lim_{k\to\infty}\nu^+ c_k = 0$. **P** Let $\epsilon > 0$. Because $\nu^+\wedge\nu^- = 0$, there is a $b\in\mathfrak{A}$ such that $\nu^+ b + \nu^-(1\setminus b)\le\epsilon$, by part (a). Now $\langle c_k\setminus b\rangle_{k\in\mathbb{N}}$ is non-increasing and $\lim_{k\to\infty}\mu(c_k\setminus b) = 0$, so $\lim_{k\to\infty}\nu(c_k\setminus b) = 0$ and

$$\begin{aligned}\limsup_{k\to\infty}\nu^+ c_k &= \limsup_{k\to\infty}\nu^+(c_k\cap b) + \nu(c_k\setminus b) + \nu^-(c_k\setminus b)\\ &\le \nu^+ b + \nu^-(1\setminus b) \le \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{k\to\infty}\nu^+ c_k = 0$. **Q**

? Now suppose, if possible, that ν^+ does not belong to the band generated by μ. Then there is a $\nu_1 > 0$ such that $\nu_1 \le \nu^+$ and $\nu_1 \wedge |\mu| = 0$. Set $\epsilon = \frac{1}{4}\nu_1 1 > 0$. For each $n \in \mathbb{N}$, we can choose $a_n \in \mathfrak{A}$ such that $|\mu|a_n + \nu_1(1 \setminus a_n) \le 2^{-n}\epsilon$, by part (a) again. For $n \ge k$, set $b_{kn} = \sup_{k \le i \le n} a_i$; then

$$|\mu|b_{kn} \le \sum_{i=k}^{n} |\mu|a_i \le 2^{-k+1}\epsilon,$$

and $\langle b_{kn}\rangle_{n\ge k}$ is non-decreasing. Set $\gamma_k = \sup_{n\ge k} \nu_1 b_{kn}$ and choose $m(k) \ge k$ such that $\nu_1 b_{k,m(k)} \ge \gamma_k - 2^{-k}\epsilon$. Setting $b_k = b_{k,m(k)}$, we see that $b_k \cup b_{k+1} = b_{kn}$ where $n = \max(m(k), m(k+1))$, so that

$$\nu_1(b_k \cup b_{k+1}) \le \gamma_k \le \nu_1 b_k + 2^{-k}\epsilon$$

and $\nu_1(b_{k+1} \setminus b_k) \le 2^{-k}\epsilon$. Set $c_k = \inf_{i\le k} b_i$ for each k; then

$$\nu_1(b_{k+1} \setminus c_{k+1}) = \nu_1(b_{k+1} \setminus c_k) \le \nu_1(b_{k+1} \setminus b_k) + \nu_1(b_k \setminus c_k) \le 2^{-k}\epsilon + \nu_1(b_k \setminus c_k)$$

for each k; inducing on k, we see that

$$\nu_1(b_k \setminus c_k) \le \sum_{i=0}^{k-1} 2^{-i}\epsilon \le 2\epsilon$$

for every k. This means that

$$\nu^+ c_k \ge \nu_1 c_k \ge \nu_1 b_k - 2\epsilon \ge \nu_1 a_k - 2\epsilon = \nu_1 1 - \nu_1(1 \setminus a_k) - 2\epsilon \ge 4\epsilon - \epsilon - 2\epsilon = \epsilon$$

for every $k \in \mathbb{N}$. On the other hand, $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence and

$$|\mu|c_k \le |\mu|b_k \le 2^{-k+1}\epsilon$$

for every k, which contradicts the paragraph just above. **X**

This means that ν^+ must belong to the band generated by μ. Similarly $\nu^- = (-\nu)^+$ belongs to the band generated by μ and $\nu = \nu^+ + \nu^-$ also does.

(ii) Take $c \in \mathfrak{A}$. Set

$$\beta_1 = \sup_{\delta>0} \inf_{\mu a\le\delta} \nu(c \setminus a), \quad \beta_2 = \inf_{\delta>0} \sup_{a\subseteq c,\mu a\le\delta} \nu a.$$

Then

$$\beta_1 = \sup_{\delta>0} \inf_{a\subseteq c,\mu a\le\delta} \nu(c \setminus a) = \nu c - \beta_2.$$

Take any $\epsilon > 0$. Because ν_1 belongs to the band generated by μ, part (i) tells us that there is a $\delta > 0$ such that $\nu_1 a \le \epsilon$ whenever $\mu a \le \delta$. In this case, if $\mu a \le \delta$,

$$\nu(c \setminus a) = \nu c - \nu(c \cap a) \ge \nu c - \epsilon \ge \nu_1 c - \epsilon;$$

thus

$$\beta_1 \ge \inf_{\mu a\le\delta} \nu(c \setminus a) \ge \nu_1 c - \epsilon.$$

As ϵ is arbitrary, $\beta_1 \ge \nu_1 c$. On the other hand, given ϵ, $\delta > 0$, there is an $a \subseteq c$ such that $\mu a + \nu_2(c \setminus a) \le \min(\delta, \epsilon)$, because $\mu \wedge \nu_2 = 0$ (using (a) again). In this case, of course, $\mu a \le \delta$, while

$$\nu a \ge \nu_2 a = \nu_2 c - \nu_2(c \setminus a) \ge \nu_2 c - \epsilon.$$

Thus $\sup_{a\subseteq c,\mu a\le\delta} \nu a \ge \nu_2 c - \epsilon$. As δ is arbitrary, $\beta_2 \ge \nu_2 c - \epsilon$. As ϵ is arbitrary, $\beta_2 \ge \nu_2 c$; but as

$$\beta_1 + \beta_2 = \nu c = \nu_1 c + \nu_2 c,$$

$\beta_i = \nu_i c$ for both i, as claimed.

362C The formula in 362B(f-i) has, I hope, already reminded you of the concept of 'absolutely continuous' additive functional from the Radon-Nikodým theorem (Chapter 23, §327). The expressions in 362Bf are limited by the assumption that μ, like ν, is finite-valued. If we relax this we get an alternative version of some of the same ideas.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write $S = S(\mathfrak{A})$, and let M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$. Write

$$M_{ac} = \{\nu : \nu \in M \text{ is absolutely continuous with respect to } \bar\mu\}$$

(see 327A),

$$M_{tc} = \{\nu : \nu \in M \text{ is continuous with respect to the measure-algebra topology on } \mathfrak{A}\},$$

$$M_t = \{\nu : \nu \in M,\ |\nu|1 = \sup_{\bar{\mu}a<\infty} |\nu|a\}.$$

Then M_{ac}, M_{tc} and M_t are bands in M.

proof (a)(i) It is easy to check that M_{ac} is a linear subspace of M.

(ii) If $\nu \in M_{ac}$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then $\nu' \in M_{ac}$. **P** Given $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}a \le \delta$. Now

$$|\nu' a| \le |\nu'|(a) \le |\nu|(a) \le 2\sup_{c \subseteq a} |\nu c| \le \epsilon$$

(using the formula for $|\nu|$ in 362Ba) whenever $\bar{\mu}a \le \delta$. As ϵ is arbitrary, ν' is absolutely continuous. **Q**

(iii) If $A \subseteq M_{ac}$ is non-empty and upwards-directed and $\nu = \sup A$ in M, then $\nu \in M_{ac}$. **P** Let $\epsilon > 0$. Then there is a $\nu' \in A$ such that $\nu 1 \le \nu' 1 + \frac{1}{2}\epsilon$ (362Be). Now there is a $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}a \le \delta$. If now $\bar{\mu}a \le \delta$,

$$|\nu a| \le |\nu' a| + (\nu - \nu')(a) \le \tfrac{1}{2}\epsilon + (\nu - \nu')(1) \le \epsilon.$$

As ϵ is arbitrary, ν is absolutely continuous with respect to $\bar{\mu}$. **Q**

Putting these together, we see that M_{ac} is a band.

(b)(i) We know that M_{tc} consists just of those $\nu \in M$ which are continuous at 0 (327Bc). Of course this is a linear subspace of M.

(ii) If $\nu \in M_{tc}$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then $|\nu| \in M_{tc}$. **P** Write $\mathfrak{A}^f = \{d : d \in \mathfrak{A},\ \bar{\mu}d < \infty\}$. Given $\epsilon > 0$ there are $d \in \mathfrak{A}^f$, $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}(a \cap d) \le \delta$. Now

$$|\nu' a| \le |\nu'|(a) \le |\nu|(a) \le 2\sup_{c \subseteq a} |\nu c| \le \epsilon$$

whenever $\bar{\mu}(a \cap d) \le \delta$. As ϵ is arbitrary, ν' is continuous at 0 and belongs to M_{tc}. **Q**

(iii) If $A \subseteq M_{tc}$ is non-empty and upwards-directed and $\nu = \sup A$ in M, then $\nu \in M_{tc}$. **P** Let $\epsilon > 0$. Then there is a $\nu' \in A$ such that $\nu 1 \le \nu' 1 + \frac{1}{2}\epsilon$ (362Be). There are $d \in \mathfrak{A}^f$, $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}(a \cap d) \le \delta$. If now $\bar{\mu}(a \cap d) \le \delta$,

$$|\nu a| \le |\nu' a| + (\nu - \nu')(a) \le \tfrac{1}{2}\epsilon + (\nu - \nu')(1) \le \epsilon.$$

As ϵ is arbitrary, ν is continuous at 0, therefore belongs to M_{tc}. **Q**

Putting these together, we see that M_{tc} is a band.

(c)(i) M_t is a linear subspace of M. **P** Suppose that ν_1, $\nu_2 \in M_t$ and $\alpha \in \mathbb{R}$. Given $\epsilon > 0$, there are a_1, $a_2 \in \mathfrak{A}^f$ such that $|\nu_1|(1 \setminus a_1) \le \frac{\epsilon}{1+|\alpha|}$, $|\nu_2|(1 \setminus a_2) \le \epsilon$. Set $a = a_1 \cup a_2$; then $\bar{\mu}a < \infty$ and

$$|\nu_1 + \nu_2|(1 \setminus a) \le 2\epsilon, \quad |\alpha\nu_1|(1 \setminus a) \le \epsilon.$$

As ϵ is arbitrary, $\nu_1 + \nu_2$ and $\alpha\nu_1$ belong to M_t; as ν_1, ν_2 and α are arbitrary, M_t is a linear subspace of M. **Q**

(ii) If $\nu \in M_t$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then

$$\inf_{\bar{\mu}a<\infty} |\nu'|(1 \setminus a) \le \inf_{\bar{\mu}a<\infty} |\nu|(1 \setminus a) = 0,$$

so $\nu' \in M_t$. Thus M_t is a solid linear subspace of M.

(iii) If $A \subseteq M_t^+$ is non-empty and upwards-directed and $\nu = \sup A$ is defined in M, then $\nu \in M_t$. **P**

$$|\nu|1 = \nu 1 = \sup_{\nu' \in A} \nu' 1 = \sup_{\nu' \in A, \bar{\mu}a<\infty} \nu' a = \sup_{\bar{\mu}a<\infty} \nu a.$$

As A is arbitrary, $\nu \in M_t$. **Q** Thus M_t is a band in M.

362D For semi-finite measure algebras, among others, the formula of 362Bd takes a special form.

Proposition Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Boolean algebra. Let M be the space of bounded finitely additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the space of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection, as in 362B. Then for any $\nu \in M^+$ and $c \in \mathfrak{A}$ there is a non-empty upwards-directed set $A \subseteq \mathfrak{A}$ with supremum c such that $(P_\tau\nu)(c) = \sup_{a\in A} \nu a$; that is, the 'inf' in 362Bd can be read as 'min'.

proof By 362Bd, we can find for each n a non-empty upwards-directed A_n, with supremum c, such that $\sup_{a\in A_n} \nu a \le (P_\tau\nu)(c) + 2^{-n}$. Set $B_n = \{c \setminus a : a \in A_n\}$ for each n, so that B_n is downwards-directed and has infimum 0, and

$$B = \{b : \text{for every } n \in \mathbb{N} \text{ there is a } b' \in B_n \text{ such that } b \supseteq b'\}$$

is also a downwards-directed set with infimum 0 (316H). Consequently $A = \{c \setminus b : b \in B\}$ is upwards-directed and has supremum c. Moreover, for any $n \in \mathbb{N}$ and $a \in A$, there is an $a' \in A_n$ such that $a \subseteq a'$; so, using 362Bd again and referring to the choice of A_n,

$$(P_\tau\nu)(c) \le \sup_{a\in A} \nu a \le \sup_{a'\in A_n} \nu a' \le (P_\tau\nu)(c) + 2^{-n}.$$

As n is arbitrary, A has the required property.

362E Uniformly integrable sets The spaces $S^\sim$, $S^\sim_c$ and $S^\times$ of 362A, or, if you prefer, the spaces M, M_σ, M_τ, M_{ac}, M_{tc}, M_t of 362B-362C, are all L-spaces, and any serious study of them must involve a discussion of their uniformly integrable (= relatively weakly compact) subsets. The basic work has been done in 356O; I spell out its application in this context.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Then a norm-bounded set $C \subseteq M$ is uniformly integrable iff $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

proof Write $\tilde C$ for the set $\{f : f \in S(\mathfrak{A})^\sim,\ f\chi \in C\}$. Because the map $f \mapsto f\chi$ is a normed Riesz space isomorphism between $S^\sim$ and M, $\tilde C$ is uniformly integrable in M iff C is uniformly integrable in $S^\sim$.

(a) Suppose that C is uniformly integrable and that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. Then $\langle \chi a_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $S^\sim$, while $\tilde C$ is uniformly integrable, so $\lim_{n\to\infty} \sup_{f\in\tilde C} |f(\chi a_n)| = 0$, by 356O; but this means that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$. Thus the condition is satisfied.

(b) Now suppose that C is not uniformly integrable. By 356O, in the other direction, there is a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S such that $0 \le u_n \le \chi 1$ for each n and $\limsup_{n\to\infty} \sup_{f\in\tilde C} |f(u_n)| > 0$. For each n, take $c_n = [\![u_n > 0]\!]$ (361Eg); then $0 \le u_n \le \chi c_n$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint. Now

$$\begin{aligned}\limsup_{n\to\infty} \sup_{\nu\in C} |\nu|(c_n) &= \limsup_{n\to\infty} \sup_{f\in\tilde C} |f|(\chi c_n)\\ &\ge \limsup_{n\to\infty} \sup_{f\in\tilde C} |f(u_n)| > 0.\end{aligned}$$

So if we choose $\nu_n \in C$ such that $|\nu_n|(c_n) \ge \frac{1}{2}\sup_{\nu\in C} |\nu|(c_n)$, we shall have $\limsup_{n\to\infty} |\nu_n|(c_n) > 0$. Next, for each n, we can find $a_n \subseteq c_n$ such that $|\nu_n a_n| \ge \frac{1}{2}|\nu_n|(c_n)$, so that

$$\limsup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n| \ge \limsup_{n\to\infty} |\nu_n a_n| > 0.$$

Since $\langle a_n\rangle_{n\in\mathbb{N}}$, like $\langle c_n\rangle_{n\in\mathbb{N}}$, is disjoint, the condition is not satisfied. This completes the proof.

362X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν_1, ν_2 two countably additive functionals on $\mathfrak{A}$. Show that $|\nu_1| \wedge |\nu_2| = 0$ in the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$ iff there is a $c \in \mathfrak{A}$ such that $\nu_1 a = \nu_1(a \cap c)$ and $\nu_2 a = \nu_2(a \setminus c)$ for every $a \in \mathfrak{A}$.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write M, M_{ac} as in 362C. Show that for any non-negative $\nu \in M$, the component ν_{ac} of ν in M_{ac} is given by the formula

$$\nu_{ac} c = \sup_{\delta>0} \inf_{\bar\mu a\le\delta} \nu(c \setminus a).$$

(c) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write M, M_t as in 362C. (i) Show that M_t is just the set of those $\nu\in M$ such that $\nu a=\lim_{b\to\mathcal{F}}\nu(a\cap b)$ for every $a\in\mathfrak{A}$, where $\mathcal{F}$ is the filter on $\mathfrak{A}$ generated by the sets $\{b:b\in\mathfrak{A}^f,\,b\supseteq b_0\}$ as b_0 runs through the elements of $\mathfrak{A}$ of finite measure. (ii) Show that the complementary band $M_t^\perp$ of M_t in M is just the set of $\nu\in M$ such that $\nu a=0$ whenever $\bar{\mu}a<\infty$. (iii) Show that for any $\nu\in M$, its component ν_t in M_t is given by the formula $\nu_t a=\lim_{b\to\mathcal{F}}\nu(a\cap b)$ for every $a\in\mathfrak{A}$.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write M, M_σ, M_τ, M_{ac}, M_{tc} and M_t as in 362B-362C. Show that (i) $M_\sigma\subseteq M_{ac}$ (ii) $M_{ac}\cap M_t=M_{tc}\subseteq M_\tau$ (iii) if $(\mathfrak{A},\bar{\mu})$ is σ-finite, then $M_\sigma=M_t\cap M_{ac}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a non-zero finitely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ is an **atom** if whenever a, $b\in\mathfrak{A}$ and $a\cap b=0$ then at least one of νa, νb is zero. Show that for a non-zero finitely additive functional ν the following are equiveridical: (i) ν is an atom; (ii) ν is bounded and $|\nu|$ is an atom; (iii) ν is bounded and the corresponding linear functional $f_{|\nu|}=|f_\nu|\in S(\mathfrak{A})^\sim$ is a Riesz homomorphism; (iv) there are a multiplicative linear functional $f:S(\mathfrak{A})\to\mathbb{R}$ and an $\alpha\in\mathbb{R}$ such that $\nu a=\alpha f(\chi a)$ for every $a\in\mathfrak{A}$. Show that a completely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ is an atom iff there are $a\in\mathfrak{A}$, $\alpha\in\mathbb{R}\setminus\{0\}$ such that a is an atom in $\mathfrak{A}$ and $\nu b=\alpha$ when $a\subseteq b$, 0 when $a\cap b=0$.

(f) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a bounded finitely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ is atomless if for every $\epsilon>0$ there is a finite partition C of unity in $\mathfrak{A}$ such that $|\nu|c\le\epsilon$ for every $c\in C$ (cf. 326Ya). (i) Show that the atomless functionals form a band M_c in the Riesz space M of all bounded finitely additive functionals on $\mathfrak{A}$. (ii) Show that the complementary band $M_c^\perp$ consists of just those $\nu\in M$ expressible as a sum $\sum_{i\in I}\nu_i$ of countably many atoms $\nu_i\in M$. (iii) Show that if $\mathfrak{A}$ is purely atomic then an atomless completely additive functional on $\mathfrak{A}$ must be 0.

(g) Let X be a set and Σ an algebra of subsets of X. Let M be the Riesz space of bounded finitely additive functionals on Σ, M_τ the space of completely additive functionals and M_p the space of functionals expressible in the form $\nu E=\sum_{x\in E}\alpha_x$ for some absolutely summable family $\langle\alpha_x\rangle_{x\in X}$ of real numbers. (i) Show that M_p is a band in M. (ii) Show that if all singleton subsets of X belong to Σ then $M_p=M_\tau$. (iii) Show that if Σ is a σ-algebra then every member of M_p is countably additive. (iv) Show that if X is a compact zero-dimensional Hausdorff space and Σ is the algebra of open-and-closed subsets of X then the complementary band $M_p^\perp$ of M_p in M is the band M_c of atomless functionals described in 362Xf.

(h) Let (X,Σ,μ) be a measure space. Let M be the Riesz space of bounded finitely additive functionals on Σ and M_σ the space of bounded countably additive functionals. Let M_{tc}, M_{ac} be the spaces of truly continuous and bounded absolutely continuous additive functionals as defined in 232A. Show that M_{tc} and M_{ac} are bands in M and that $M_{tc}\subseteq M_\sigma\cap M_{ac}$. Show that if μ is σ-finite then $M_{tc}=M_\sigma\cap M_{ac}$.

(i) Let $\mathfrak{A}$ be a Boolean algebra and M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$. (i) For any non-empty downwards-directed set $A\subseteq\mathfrak{A}$ set $N_A=\{\nu:\nu\in M,\,\inf_{a\in A}|\nu|a=0\}$. Show that N_A is a band in M. (ii) For any non-empty set $\mathcal{A}$ of non-empty downwards-directed sets in $\mathfrak{A}$ set $M_{\mathcal{A}}=\{\nu:\nu\in M,\,\inf_{a\in A}|\nu|a=0\ \forall\ A\in\mathcal{A}\}$. Show that $M_{\mathcal{A}}$ is a band in M. (iii) Explain how to represent as such $M_{\mathcal{A}}$ the bands M_σ, M_τ, M_t, M_{ac}, M_{tc} described above, and also any band generated by a single element of M. (iv) Suppose, in (ii), that $\mathcal{A}$ has the property that for any A, $A'\in\mathcal{A}$ there is a $B\in\mathcal{A}$ such that for every $b\in B$ there are $a\in A$, $a'\in A'$ such that $a\cup a'\subseteq b$. Show that for any non-negative $\nu\in M$, the component ν_1 of ν in $M_{\mathcal{A}}$ is given by the formula $\nu_1 c=\inf_{A\in\mathcal{A}}\sup_{a\in A}\nu(c\setminus a)$, so that the component ν_2 of ν in $M_{\mathcal{A}}^\perp$ is given by the formula $\nu_2 c=\sup_{A\in\mathcal{A}}\inf_{a\in A}\nu(c\cap a)$. (Cf. 356Yb.)

362Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$ (353B). Show that the bands M_σ, M_τ, M_c (362B, 362Xe, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most six atoms. Give an example in which $\mathfrak{C}_0$ has six atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $\mathfrak{A}$ is Dedekind σ-complete?

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$. Show that the bands M_σ, M_τ, M_c, M_{ac}, M_{tc}, M_t (362B, 362C, 362Xe, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most twelve atoms. Give an example in which $\mathfrak{C}_0$ has twelve atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $(\mathfrak{A},\bar{\mu})$ is semi-finite (iv) $(\mathfrak{A},\bar{\mu})$ is localizable (v) $(\mathfrak{A},\bar{\mu})$ is σ-finite (vi) $(\mathfrak{A},\bar{\mu})$ is totally finite?

(c) Give an example of a set X, a σ-algebra Σ of subsets of X, and a functional in M_p (as defined in 362Xg) which is not completely additive.

(d) Let U be a Riesz space and f, $g \in U^\sim$. Show that the following are equiveridical: (α) g is in the band of $U^\sim$ generated by f; (β) for every $u \in U^+$, $\epsilon > 0$ there is a $\delta > 0$ such that $|g(v)| \le \epsilon$ whenever $0 \le v \le u$ and $|f|(v) \le \delta$; (γ) $\lim_{n\to\infty} g(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U^+ and $\lim_{n\to\infty} f(u_n) = 0$. (*Hint*: 362B(f-i).)

(e) Let $\mathfrak{A}$ be a weakly σ-distributive Boolean algebra (316Yg). Show that the 'inf' in the formula for $P_\sigma\nu$ in 362Bc can be replaced by 'min'.

(f) Let $\mathfrak{A}$ be any Boolean algebra and M the space of bounded finitely additive functionals on $\mathfrak{A}$. Let $C \subseteq M$ be such that $\sup_{\nu\in C} |\nu a| < \infty$ for every $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n|$ is finite for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(g) Let $\mathfrak{A}$ be a Boolean algebra and M_τ the space of completely additive functionals on $\mathfrak{A}$. Let $C \subseteq M_\tau$ be such that $\sup_{\nu\in C} |\nu a| < \infty$ for every atom $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n|$ is finite for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle \nu_n\rangle_{n\in\mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$. Suppose that $\nu a = \lim_{n\to\infty} \nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive and that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly integrable. (*Hint*: 246Yg.) Show that if every ν_n is completely additive, so is ν.

(i) Let $\mathfrak{A}$ be a Boolean algebra, M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$, and $M_c \subseteq M$ the space of atomless functionals (362Xf). Show that for a non-negative $\nu \in M$ the component ν_c of ν in M_c is given by the formula

$$\nu_c a = \inf_{\delta>0} \sup\{\textstyle\sum_{i=0}^n \nu a_i : a_0, \dots , a_n \subseteq a \text{ are disjoint}, \nu a_i \le \delta \text{ for every } i\}$$

for each $a \in \mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded additive real-valued functionals on $\mathfrak{A}$. Show that the complexification of M, as defined in 354Yk, can be identified with the Banach space of bounded additive functionals $\nu : \mathfrak{A} \to \mathbb{C}$, writing

$$\|\nu\| = \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots , a_n \text{ are disjoint elements of } \mathfrak{A}\}$$

for such ν.

362 Notes and comments The Boolean algebras most immediately important in measure theory are of course σ-algebras of measurable sets and their quotient measure algebras. It is therefore natural to begin any investigation by concentrating on Dedekind σ-complete algebras. Nevertheless, in this section and the last (and in §326), I have gone to some trouble not to specialize to σ-complete algebras except when necessary. Partly this is just force of habit, but partly it is because I wish to lay a foundation for a further step forward: the investigation of the ways in which additive functionals on general Boolean algebras reflect the concepts of measure theory, and indeed can generate them. Some of the results in this direction can be surprising. I do not think it obvious that the condition (γ) in 362B(f-i), for instance, is sufficient in the absence of any hypothesis of Dedekind σ-completeness or countable additivity.

Given a Boolean algebra $\mathfrak{A}$ with the associated Riesz space $M \cong S(\mathfrak{A})^\sim$ of bounded additive functionals on $\mathfrak{A}$, we now have a substantial list of bands in M: M_σ, M_τ, M_c (362Xf), and for a measure algebra the further bands M_{ac}, M_{tc} and M_t; for an algebra of sets we also have M_p (362Xg). These bands can be used to generate finite subalgebras of the band algebra of M (362Ya-362Yb), and for any such finite subalgebra we have a corresponding decomposition of M as a direct sum of the bands which are the atoms of the subalgebra (352Tb). This decomposition of M can be regarded as a recipe for decomposing its members into finite sums of functionals with special properties. What I called the 'Lebesgue decomposition' in 232I is just such a recipe. In that context I had a measure space (X, Σ, μ) and was looking at the countably additive functionals from Σ to $\mathbb{R}$, that is, at M_σ in the language of this section, and the bands involved in the decomposition were M_p, M_{ac} and M_{tc}. But I hope that it will be plain that these ideas can be refined indefinitely, as we refine the classification of additive functionals. At each stage, of course, the exact enumeration of the subalgebra of bands generated by the classification (as in 362Ya-362Yb) is a necessary check that we have understood the relationships between the classes we have described.

These decompositions are of such importance that it is worth examining the corresponding band projections. I give formulae for the action of band projections on (non-negative) functionals in 362Bc, 362Bd, 362B(f-ii), 362Xb, 362Xc(iii), 362Xi(iv) and 362Yi. Of course these are readily adapted to give formulae for the projections onto the complementary bands, as in 362Bf and 362Xi.

If we have an algebra of sets, the completely additive functionals are (usually) of relatively minor importance; in the standard examples, they correspond to functionals defined as weighted sums of point masses (362Xg(ii)). The point is that measure algebras $\mathfrak{A}$ appear as quotients of σ-algebras Σ of sets by σ-ideals $\mathcal{I}$; consequently the countably additive functionals on $\mathfrak{A}$ correspond exactly to the countably additive functionals on Σ which are zero on $\mathcal{I}$; but the canonical homomorphism from Σ to $\mathfrak{A}$ is hardly ever order-continuous, so completely additive functionals on $\mathfrak{A}$ rarely correspond to completely additive functionals on Σ. On the other hand, when we are looking at countably additive functionals on Σ, we have to consider the possibility that they are singular in the sense that they are carried on some member of $\mathcal{I}$; in the measure algebra context this possibility disappears, and we can often be sure that every countably additive functional is absolutely continuous, as in 327Bb.

For any Boolean algebra $\mathfrak{A}$, we can regard it as the algebra of open-and-closed subsets of its Stone space Z; the points of Z correspond to Boolean homomorphisms from $\mathfrak{A}$ to $\{0, 1\}$, which are the fundamental 'atoms' in the space of additive functionals on $\mathfrak{A}$ (362Xe, 362Xg(iv)). It is the case that all non-negative additive functionals on a Boolean algebra $\mathfrak{A}$ can be represented by appropriate measures on its Stone space (see 416P in Volume 4), but I prefer to hold this result back until it can take its place among other theorems on representing functionals by measures and integrals.

It is one of the leitmotivs of this chapter, that Boolean algebras and Riesz spaces are Siamese twins; again and again, matching results are proved by the application of identical ideas. A typical example is the pair 362B(f-i) and 362Yd. Many of us have been tempted to try to describe something which would provide a common generalization of Boolean algebras and Riesz spaces (and lattice-ordered groups). I have not yet seen any such structure which was worth the trouble. Most of the time, in this chapter, I shall be using ideas from the general theory of Riesz spaces to suggest and illuminate questions in measure theory; but if you pursue this subject you will surely find that intuitions often come to you first in the context of Boolean algebras, and the applications to Riesz spaces are secondary.

In 362E I give a condition for uniform integrability in terms of disjoint sequences, following the pattern established in 246G and repeated in 354R and 356O. The condition of 362E assumes that the set is norm-bounded; but if you have 246G to hand, you will see that it can be done with weaker assumptions involving atoms, as in 362Yf-362Yg.

I mention once again the Banach-Ulam problem: if $\mathfrak{A}$ is Dedekind complete, can $S(\mathfrak{A})^\sim_c$ be different from $S(\mathfrak{A})^\times$? This is obviously equivalent to the form given in the notes to §326 above. See 363S below.

363 L^∞

In this section I set out to describe an abstract construction for L^∞ spaces on arbitrary Boolean algebras, corresponding to the $L^\infty(\mu)$ spaces of §243. I begin with the definition of $L^\infty(\mathfrak{A})$ (363A) and elementary facts concerning its own structure and the embedding $S(\mathfrak{A}) \subseteq L^\infty(\mathfrak{A})$ (363B-363D). I give the basic universal mapping theorems which define the Banach lattice structure of L^∞ (363E) and a description of the action of Boolean homomorphisms on L^∞ spaces (363F-363G) before discussing the representation of $L^\infty(\Sigma)$ and $L^\infty(\Sigma/\mathcal{I})$ for σ-algebras Σ and ideals $\mathcal{I}$ of sets (363H). This leads at once to the identification of $L^\infty(\mu)$, as defined in Volume 2, with $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ (363I). Like $S(\mathfrak{A})$, $L^\infty(\mathfrak{A})$ determines the algebra $\mathfrak{A}$ (363J). I briefly discuss the dual spaces of L^∞; they correspond exactly to the duals of S described in §362 (363K). Linear functionals on L^∞ can for some purposes be treated as 'integrals' (363L).

In the second half of the section I present some of the theory of Dedekind complete and σ-complete algebras. First, $L^\infty(\mathfrak{A})$ is Dedekind (σ-)complete iff $\mathfrak{A}$ is (363M). The spaces $L^\infty(\mathfrak{A})$, for Dedekind σ-complete $\mathfrak{A}$, are precisely the Dedekind σ-complete Riesz spaces with order unit (363N-363P). The spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are precisely the normed spaces which may be put in place of $\mathbb{R}$ in the Hahn-Banach theorem (363R). Finally, I mention some equivalent forms of the Banach-Ulam problem (363S).

363A Definition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z. I will write $L^\infty(\mathfrak{A})$ for the space $C(Z) = C_b(Z)$ of continuous real-valued functions from Z to $\mathbb{R}$, endowed with the linear structure, order structure, norm and multiplication of $C(Z) = C_b(Z)$. (Recall that because Z is compact (311I), $\{u(z) : z \in Z\}$ is bounded for every $u \in L^\infty(\mathfrak{A}) = C(Z)$ (2A3N(b-iii)), that is, $C(Z) = C_b(Z)$. Of course if $\mathfrak{A} = \{0\}$, so that $Z = \emptyset$, then $C(Z)$ has just one member, the empty function.)

363B Theorem Let $\mathfrak{A}$ be any Boolean algebra; write L^∞ for $L^\infty(\mathfrak{A})$.

(a) L^∞ is an M-space; its standard order unit is the constant function taking the value 1 at each point; in particular, it is a Banach lattice with a Fatou norm and the Levi property.

(b) L^∞ is a commutative Banach algebra and an f-algebra.

(c) If $u \in L^\infty$ then $u \geq 0$ iff there is a $v \in L^\infty$ such that $u = v \times v$.

proof (a) See 354Hb and 354J.

(b)-(c) are obvious from the definitions of Banach algebra (2A4J) and f-algebra (352W) and the ordering of $L^\infty = C(Z)$.

363C Proposition Let $\mathfrak{A}$ be any Boolean algebra. Then $S(\mathfrak{A})$ is a norm-dense, order-dense Riesz subspace of $L^\infty(\mathfrak{A})$, closed under multiplication.

proof Let Z be the Stone space of $\mathfrak{A}$. Using the definition of $S = S(\mathfrak{A})$ set out in 361D, it is obvious that S is a linear subspace of $L^\infty = L^\infty(\mathfrak{A})$ closed under multiplication. Because S, like L^∞, is a Riesz subspace of $\mathbb{R}^X$ (361Ee), S is a Riesz subspace of L^∞. By the Stone-Weierstrass theorem (in either of the forms given in 281A and 281E), S is norm-dense in L^∞. Consequently it is order-dense (354I).

363D Proposition Let $\mathfrak{A}$ be a Boolean algebra. If we regard $\chi a \in S(\mathfrak{A})$ (361D) as a member of $L^\infty(\mathfrak{A})$ for each $a \in \mathfrak{A}$, then $\chi : \mathfrak{A} \to L^\infty(\mathfrak{A})$ is additive, order-preserving, order-continuous and a lattice homomorphism.

proof Because the embedding $S = S(\mathfrak{A}) \subseteq L^\infty(\mathfrak{A}) = L^\infty$ is a Riesz homomorphism, $\chi : \mathfrak{A} \to L^\infty$ is additive and a lattice homomorphism (361F-361G). Because S is order-dense in L^∞ (363C), the embedding $S \subseteq L^\infty$ is order-continuous (352Nb), so $\chi : \mathfrak{A} \to L^\infty$ is order-continuous (361Gb).

363E Theorem Let $\mathfrak{A}$ be a Boolean algebra, and U a Banach space. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function.

(a) There is a unique bounded linear operator $T : L^\infty(\mathfrak{A}) \to U$ such that $T\chi = \nu$; in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

(b) If U is a Banach lattice then T is positive iff ν is non-negative; and in this case T is order-continuous iff ν is order-continuous, and sequentially order-continuous iff ν is sequentially order-continuous.

(c) If U is a Banach lattice then T is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) By 361I there is a unique bounded linear operator $T_0 : S \to U$ such that $T_0\chi = \nu$, and $\|T_0\| = \sup\{\|\nu a - \nu b\| : a,\ b \in \mathfrak{A}\}$. But because U is a Banach space and S is dense in L^∞, T_0 has a unique extension to a bounded linear operator $T : L^\infty \to U$ with the same norm (2A4I).

(b)(i) If T is positive then T_0 is positive so ν is non-negative, by 361Ga.

(ii) If ν is non-negative then T_0 is positive, by 361Ga in the other direction. But if $u \in L^{\infty+}$ and $\epsilon > 0$, then by 354I there is a $v \in S^+$ such that $\|u - v\|_\infty \le \epsilon$; now $\|Tu - Tv\| \le \epsilon\|T\|$. But $Tv = T_0 v$ belongs to the positive cone U^+ of U. As ϵ is arbitrary, Tu belongs to the closure of U^+, which is U^+ (354Bc). As u is arbitrary, T is positive.

(iii) Now suppose that ν is order-continuous as well as non-negative, and that $A \subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0. Set

$$B = \{v : v \in S, \text{ there is some } u \in A \text{ such that } v \ge u\}.$$

Then B is downwards-directed (indeed, $v_1 \wedge v_2 \in B$ for every v_1, $v_2 \in B$), and $u = \inf\{v : v \in B,\ u \le v\}$ for every $u \in A$ (354I again), so B has the same lower bounds as A and $\inf B = 0$ in L^∞ and in S. But we know from 361Gb that T_0 is order-continuous, while any lower bound for $\{Tu : u \in A\}$ in U must also be a lower bound for $\{Tv : v \in B\} = \{T_0 v : v \in B\}$, so $\inf_{u\in A} Tu = \inf_{v\in B} T_0 v = 0$ in U. As A is arbitrary, T is order-continuous (351Ga).

(iv) Suppose next that ν is only sequentially order-continuous, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^∞ with infimum 0. For each n, k choose $w_{nk} \in S$ such that $u_n \le w_{nk}$ and $\|w_{nk} - u_n\|_\infty \le 2^{-k}$ (354I once more), and set $w'_n = \inf_{j,k\le n} w_{jk}$ for each n. Then $\langle w'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S, and any lower bound of $\{w'_n : n \in \mathbb{N}\}$ is also a lower bound of $\{u_n : n \in \mathbb{N}\}$, so $0 = \inf_{n\in\mathbb{N}} w'_n$ in S and L^∞. Since $T_0 : S \to U$ is sequentially order-continuous (361Gb),

$$\inf_{n\in\mathbb{N}} Tu_n \le \inf_{n\in\mathbb{N}} Tw'_n = \inf_{n\in\mathbb{N}} T_0 w'_n = 0$$

in U. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(v) On the other hand, if T is order-continuous or sequentially order-continuous, so is $\nu = T\chi$, because χ is order-continuous (363D).

(c) We know that $T_0 : S \to U$ is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, by 361Gc. But T_0 is a Riesz homomorphism iff T is. **P** If T is a Riesz homomorphism so is T_0, because the embedding $S \subseteq L^\infty$ is a Riesz homomorphism. On the other hand, if T_0 is a Riesz homomorphism, then the functions $u \mapsto u^+ \mapsto T(u^+)$, $u \mapsto Tu \mapsto (Tu)^+$ are continuous (by 354Bb) and agree on S, so agree on L^∞, and T is a Riesz homomorphism, by 352G. **Q**

363F Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) There is an associated multiplicative Riesz homomorphism $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$, of norm at most 1, defined by saying that $T_\pi(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(b) For any $u \in L^\infty(\mathfrak{A})$, there is a $u' \in L^\infty(\mathfrak{A})$ such that $T_\pi u = T_\pi u'$ and $\|u'\|_\infty = \|T_\pi u\|_\infty \le \|u\|_\infty$.

(c)(i) The kernel of T_π is the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A},\ \pi a = 0\}$.

(ii) The set of values of T_π is the closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(d) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : T_\pi u = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$.

(f) T_π is order-continuous, or sequentially order-continuous, iff π is.

(g) If $\mathfrak{C}$ is another Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ is another Boolean homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{C})$.

proof Let Z and W be the Stone spaces of $\mathfrak{A}$ and $\mathfrak{B}$. By 312P there is a continuous function $\phi : W \to Z$ such that $\widehat{\pi a} = \phi^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. Write T for T_π.

(a) For $u \in L^\infty(\mathfrak{A}) = C(Z)$, set $Tu = u\phi : W \to \mathbb{R}$. Then $Tu \in C(W) = L^\infty(\mathfrak{B})$. It is obvious, or at any rate very easy to check, that $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ is linear, multiplicative, a Riesz homomorphism and of norm 1 unless $\mathfrak{B} = \{0\}$, $W = \emptyset$. If $a \in \mathfrak{A}$, then

$$T(\chi a) = (\chi a)\phi = (\chi\widehat{a})\phi = \chi(\phi^{-1}[\widehat{a}]) = \chi(\pi a),$$

identifying $\chi a \in L^\infty(\mathfrak{A})$ with the characteristic function $\chi\widehat{a} : Z \to \{0, 1\}$ of the set $\widehat{a}$. Of course $T_\pi = T$ is the only continuous linear operator with these properties, by 363Ea.

(b) Set $\alpha = \|Tu\|_\infty$, $u'(z) = \max(-\alpha, \min(u(z), \alpha))$ for $z \in Z$; that is, $u' = (-\alpha e) \vee (u \wedge \alpha e)$ in $L^\infty(\mathfrak{A})$, where e is the standard order unit of $L^\infty(\mathfrak{A})$. Then Te is the standard order unit of $L^\infty(\mathfrak{B})$, so

$$Tu' = (-\alpha Te) \vee (Tu \wedge \alpha Te) = Tu,$$

while

$$\|u'\|_\infty \le \alpha = \|Tu\|_\infty = \|Tu'\|_\infty \le \|u'\|_\infty \le \|u\|_\infty.$$

(c)(i) Let U be the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : \pi a = 0\}$, and U_0 the kernel of T. Because T is continuous and linear, U_0 is a closed linear subspace, and $T(\chi a) = \chi 0 = 0$ whenever $\pi a = 0$; so $U \subseteq U_0$. Now take any $u \in U_0$ and $\epsilon > 0$. Then $T(u^+) = (Tu)^+ = 0$, so $u^+ \in U_0$. By 354I there is a $u' \in S(\mathfrak{A})$ such that $0 \le u' \le u^+$ and $\|u^+ - u'\|_\infty \le \epsilon$. Now $0 \le Tu' \le Tu^+ = 0$, so $Tu' = 0$. Express u' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $\alpha_i \ge 0$ for each i. For each i, $\alpha_i \chi(\pi a_i) = T(\alpha_i \chi a_i) = 0$, so $\pi a_i = 0$ or $\alpha_i = 0$; in either case $\alpha_i \chi a_i \in U$. Consequently $u' \in U$. As ϵ is arbitrary and U is closed, $u^+ \in U$. Similarly, $u^- = (-u)^+ \in U$ and $u = u^+ - u^- \in U$. As u is arbitrary, $U_0 \subseteq U$ and $U_0 = U$.

(ii) Let V be the closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$, and $V_0 = T[L^\infty(\mathfrak{A})]$. Then $T[S(\mathfrak{A})] \subseteq V$, so

$$V_0 = T[\overline{S(\mathfrak{A})}] \subseteq \overline{T[S(\mathfrak{A})]} \subseteq \overline{V} = V.$$

On the other hand, V_0 is a closed linear subspace in $L^\infty(\mathfrak{B})$. **P** It is a linear subspace because T is a linear operator. To see that it is closed, take any $v \in \overline{V}_0$. Then there is a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V_0 such that $\|v - v_n\|_\infty \le 2^{-n}$ for every $n \in \mathbb{N}$. Choose $u_n \in L^\infty(\mathfrak{A})$ such that $Tu_0 = v_0$, while $Tu_n = v_n - v_{n-1}$ and $\|u_n\|_\infty = \|v_n - v_{n-1}\|_\infty$ for $n \ge 1$ (using (b) above). Then

$$\sum_{n=1}^\infty \|u_n\|_\infty \le \sum_{n=1}^\infty \|v - v_n\|_\infty + \|v - v_{n-1}\|_\infty$$

is finite, so $u = \lim_{n\to\infty} \sum_{i=0}^n u_i$ is defined in the Banach space $L^\infty(\mathfrak{A})$, and

$$Tu = \lim_{n\to\infty} \sum_{i=0}^n Tu_i = \lim_{n\to\infty} v_n = v.$$

As v is arbitrary, V_0 is closed. **Q** Since $\chi(\pi a) = T(\chi a) \in V_0$ for every $a \in \mathfrak{A}$, $V \subseteq V_0$ and $V = V_0$, as required.

(d) If π is surjective, then T is surjective, by (c-ii). If T is surjective and $b \in \mathfrak{B}$, then there is a $u \in L^\infty(\mathfrak{A})$ such that $Tu = \chi b$. Now there is a $u' \in S(\mathfrak{A})$ such that $\|u - u'\|_\infty \le \frac{1}{3}$, so that $\|Tu' - \chi b\|_\infty \le \frac{1}{3}$. Taking $a \in \mathfrak{A}$ such that $\{z : u'(z) \ge \frac{1}{2}\} = \widehat{a}$, we must have $\pi a = b$, since

$$\widehat{b} = \{w : (Tu')(w) \ge \tfrac{1}{2}\} = \phi^{-1}[\widehat{a}] = \widehat{\pi a}.$$

As b is arbitrary, π is surjective.

Now (b) tells us that in this case $\|v\|_\infty = \min\{\|u\|_\infty : Tu = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) By (c-i), T is injective iff π is injective. In this case, for any $u \in L^\infty(\mathfrak{A})$,

$$\|Tu\|_\infty = \|T|u|\|_\infty$$

(because T is a Riesz homomorphism)

$$\ge \sup\{\|Tu'\|_\infty : u' \in S(\mathfrak{A}),\ u' \le |u|\}$$
$$= \sup\{\|u'\|_\infty : u' \in S(\mathfrak{A}),\ u' \le |u|\}$$

(by 361Jd)

$$= \|u\|_\infty$$

(by 354I)

$$\ge \|Tu\|_\infty,$$

and $\|Tu\|_\infty = \|u\|_\infty$.

(f) If T is (sequentially) order-continuous then $\pi = T\chi$ is (sequentially) order-continuous, by 363D. If π is (sequentially) order-continuous then $\chi\pi : \mathfrak{A} \to L^\infty(\mathfrak{B})$ is (sequentially) order-continuous, so T is (sequentially) order-continuous, by 363Eb.

(g) This is elementary, in view of the uniqueness of $T_{\theta\pi}$.

363G Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, then $L^\infty(\mathfrak{C})$ can be identified, as Banach lattice and as Banach algebra, with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi c : c \in \mathfrak{C}\}$.

(b) If $\mathcal{I}$ is an ideal of $\mathfrak{A}$, then $L^\infty(\mathfrak{A}/\mathcal{I})$ can be identified, as Banach lattice and as Banach algebra, with the quotient space $L^\infty(\mathfrak{A})/V$, where V is the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathcal{I}\}$.

proof Apply 363Fc to the identity map from $\mathfrak{C}$ to $\mathfrak{A}$ and the canonical map from $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{I}$.

363H Representations of $L^\infty(\mathfrak{A})$ Much of the importance of the concept of $L^\infty(\mathfrak{A})$ arises from the way it is naturally represented in the contexts in which the most familiar Boolean algebras appear.

Proposition Let X be a set and Σ a σ-algebra of subsets of X.

(a) $L^\infty(\Sigma)$ can be identified, as Banach algebra and Banach lattice, with the space $\mathcal{L}^\infty$ of bounded Σ-measurable real-valued functions on X, with the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$ for $f \in \mathcal{L}^\infty$; this identification matches $\chi E \in L^\infty(\Sigma)$ with the characteristic function of E as a subset of X, for every $E \in \Sigma$. In particular, for any set X, $L^\infty(\mathcal{P}X)$ can be identified with $\ell^\infty(X)$.

(b) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ is a surjective sequentially order-continuous Boolean homomorphism with kernel $\mathcal{I}$, then $L^\infty(\mathfrak{A})$ can be identified, as Banach algebra and Banach lattice, with $\mathcal{L}^\infty/\mathcal{W}$, where $\mathcal{W} = \{f : f \in \mathcal{L}^\infty,\ \{x : f(x) \neq 0\} \in \mathcal{I}\}$ is a closed ideal and solid linear subspace of $\mathcal{L}^\infty$. For $f \in \mathcal{L}^\infty$,

$$\|f^\bullet\|_\infty = \min\{\alpha : \alpha \geq 0,\ \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

(c) In particular, if $\mathcal{I}$ is any σ-ideal of Σ and $E \mapsto E^\bullet$ is the canonical homomorphism from Σ onto $\mathfrak{A} = \Sigma/\mathcal{I}$, then we have an identification of $L^\infty(\mathfrak{A})$ with a quotient of $\mathcal{L}^\infty$, and for any $E \in \Sigma$ we can identify $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$ with the equivalence class $(\chi E)^\bullet \in \mathcal{L}^\infty/\mathcal{W}$ of the characteristic function χE.

proof (a) For the elementary properties of the space of Σ-measurable functions, see §121. In particular, it is easy to check that $\mathcal{L}^\infty$ is a Riesz space, with a Riesz norm, and a normed algebra. To check that it is a Banach space, observe that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^\infty$, then $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \to 0$ as m, $n \to \infty$, so $f(x) = \lim_{n\to\infty} f_n(x)$ is defined for every $x \in X$; now f is Σ-measurable, by 121Fa. Of course $\|f\|_\infty \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$, so $f \in \mathcal{L}^\infty$, while

$$\|f - f_n\|_\infty \leq \sup_{m\geq n} \|f_m - f_n\|_\infty \to 0$$

as $n \to \infty$, so $f = \lim_{n\to\infty} f_n$ in $\mathcal{L}^\infty$. As $\langle f_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{L}^\infty$ is complete.

By 361L we can identify $S(\Sigma)$, as Riesz space and normed algebra, with the linear span $\mathcal{S}$ of $\{\chi E : E \in \Sigma\}$, which is a subspace of $\mathcal{L}^\infty$. Now the point is that it is dense. **P** If $f \in \mathcal{L}^\infty$ and $\epsilon > 0$, then for each $n \in \mathbb{Z}$ set $E_n = \{x : n\epsilon \leq f(x) < (n+1)\epsilon\} \in \Sigma$; then $E_n = \emptyset$ if $|n| > 1 + \frac{1}{\epsilon}\|f\|_\infty$, so $g = \sum_{n\in\mathbb{Z}} n\epsilon\chi E_n$ belongs to $\mathcal{S}$, and of course $\|f - g\|_\infty \leq \epsilon$. **Q**

Consequently the canonical normed space isomorphism between $S(\Sigma)$ and $\mathcal{S}$ extends (uniquely) to a normed space isomorphism between $L^\infty(\Sigma)$ and $\mathcal{L}^\infty$ (use 2A4I). Because the operations $\vee$ and $\times$ are continuous in both $L^\infty(\Sigma)$ and $\mathcal{L}^\infty$, and their actions on $S(\Sigma)$ and $\mathcal{S}$ are identified by our isomorphism, the isomorphism between $L^\infty(\Sigma)$ and $\mathcal{L}^\infty$ identifies their lattice and algebra structures.

(b)(i) By 363F, we have a multiplicative Riesz homomorphism $T = T_\pi$ from $L^\infty(\Sigma)$ to $L^\infty(\mathfrak{A})$ which is surjective (363Fd) and has kernel the closed linear subspace W of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \mathcal{I}\}$. Now under the isomorphism described in (a), W corresponds to $\mathcal{W}$. **P** (α) $\mathcal{W}$ is a linear subspace of $\mathcal{L}^\infty$ because

$$\{x : (f+g)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\} \in \mathcal{I},$$

$$\{x : (\alpha f)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \in \mathcal{I}$$

whenever f, $g \in \mathcal{W}$ and $\alpha \in \mathbb{R}$. (β) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$ converging to $f \in \mathcal{L}^\infty$, then

$$\{x : f(x) \neq 0\} \subseteq \bigcup_{n\in\mathbb{N}}\{x : f_n(x) \neq 0\} \in \mathcal{I},$$

so $f \in \mathcal{W}$. Thus $\mathcal{W}$ is a closed linear subspace of $\mathcal{L}^\infty$. (γ) If $E \in \mathcal{I}$, then χE, taken in $S(\Sigma)$ or $L^\infty(\Sigma)$, corresponds to the function $\chi E : X \to \{0,1\}$, which belongs to $\mathcal{W}$; so that W must correspond to the closed linear span in $\mathcal{L}^\infty$ of such characteristic functions, which is a subspace of $\mathcal{W}$. (δ) On the other hand, if $f \in \mathcal{W}$ and $\epsilon > 0$, set

$$E_n = \{x : n\epsilon \le f(x) \le (n+1)\epsilon\}, \quad E'_n = \{x : -(n+1)\epsilon \le f(x) \le -n\epsilon\}$$

for $n \ge 1$; all these belong to $\mathcal{I}$, so $g = \sum_{n=1}^{\infty} n(\chi E_n - \chi E'_n) \in \mathcal{W}$ corresponds to a member of W, while $\|f - g\|_\infty \le \epsilon$. As W is closed, f also must correspond to some member of W. As f is arbitrary, W and $\mathcal{W}$ match exactly. **Q**

(ii) Because T is a multiplicative Riesz homomorphism, $L^\infty(\mathfrak{A}) \cong L^\infty(\Sigma)/W$ is matched canonically, in its linear, order and multiplicative structures, with $\mathcal{L}^\infty/\mathcal{W}$. We know also that

$$\|v\|_\infty = \inf\{\|u\|_\infty : u \in L^\infty(\Sigma),\, Tu = v\}$$

for every $v \in L^\infty(\mathfrak{A})$ (363Fd), that is, that the norm of $L^\infty(\mathfrak{A})$ corresponds to the quotient norm on $L^\infty(\Sigma)/W$.

As for the given formula for the norm, take any $f \in \mathcal{L}^\infty$. There is a $g \in \mathcal{L}^\infty$ such that $Tf = Tg$ and $\|Tf\|_\infty = \|g\|_\infty$. (Here I am treating T as an operator from $\mathcal{L}^\infty$ onto $L^\infty(\mathfrak{A})$.) In this case

$$\{x : |f(x)| > \|Tf\|_\infty\} \subseteq \{x : f(x) \ne g(x)\} \in \mathcal{I}.$$

On the other hand, if $\{x : |f(x)| > \alpha\} \in \mathcal{I}$, and we set $h = -\alpha\mathbf{1} \vee (f \wedge \mathbf{1})$, then $Th = Tf$, so $\|Tf\|_\infty \le \|h\|_\infty \le \alpha$.

(c) Put (a) and (b) together.

363I Corollary Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Then $L^\infty(\mu)$ can be identified, as Banach lattice and Banach algebra, with $L^\infty(\mathfrak{A})$; the identification matches $(\chi E)^\bullet \in L^\infty(\mu)$ with $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$, for every $E \in \Sigma$.

Remark The space I called $\mathcal{L}^\infty(\mu)$ in Chapter 24 is not strictly speaking the space $\mathcal{L}^\infty \cong L^\infty(\Sigma)$ of 363H; I took $\mathcal{L}^\infty(\mu) \subseteq \mathcal{L}^0(\mu)$ to be the set of essentially bounded, virtually measurable functions defined almost everywhere on X, and in general this is larger. But, as remarked in the notes to §243, $L^\infty(\mu)$ can equally well be regarded as a quotient of what I there called $\mathcal{L}^\infty_{\text{strict}}$, which is the $\mathcal{L}^\infty$ above, because every function in $\mathcal{L}^\infty(\mu)$ is equal almost everywhere to some member of $\mathcal{L}^\infty_{\text{strict}}$.

363J Recovering the algebra $\mathfrak{A}$: Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $L^\infty(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $L^\infty(\mathfrak{A})$.

proof The proof is nearly identical to that of 361K. If $a \in \mathfrak{A}$, $u \in V_a$ and $v \in V_{1\setminus a}$, then $|u| \wedge |v| = 0$ because $\chi a \wedge \chi(1 \setminus a) = 0$; and if $w \in L^\infty(\mathfrak{A})$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}$$

because $|w \times \chi a| \le \|w\|_\infty \chi a$ and $|w \times \chi(1 \setminus a)| \le \|w\|_\infty \chi(1 \setminus a)$. So V_a and $V_{1\setminus a}$ are complementary projection bands in $L^\infty = L^\infty(\mathfrak{A})$. Next, if $U \subseteq L^\infty$ is a projection band, then $\chi 1$ is expressible as $u + v$ where $u \in U$, $v \in U^\perp$; thinking of L^∞ as the space of continuous real-valued functions on the Stone space Z of $\mathfrak{A}$, u and v must be the characteristic functions of complementary subsets E, F of Z, which must be open-and-closed, so that $E = \widehat{a}$, $F = \widehat{1 \setminus a}$. In this case $V_a \subseteq U$ and $V_{1\setminus a} \subseteq U^\perp$, so U must be V_a precisely. Thus $a \mapsto V_a$ is surjective. Finally, just as in 361K, $a \subseteq b \iff V_a \subseteq V_b$, so we have a Boolean isomorphism.

363K Dual spaces of L^∞ The questions treated in §362 yield nothing new in the present context. I spell out the details.

Proposition Let $\mathfrak{A}$ be a Boolean algebra. Let M, M_σ and M_τ be the L-spaces of bounded finitely additive functionals, bounded countably additive functionals and completely additive functionals on $\mathfrak{A}$. Then the embedding $S(\mathfrak{A}) \subseteq L^\infty(\mathfrak{A})$ induces Riesz space isomorphisms between $S(\mathfrak{A})^\sim \cong M$ and $L^\infty(\mathfrak{A})^\sim = L^\infty(\mathfrak{A})^*$, $S(\mathfrak{A})^\sim_c \cong M_\sigma$ and $L^\infty(\mathfrak{A})^\sim_c$, and $S(\mathfrak{A})^\times \cong M_\tau$ and $L^\infty(\mathfrak{A})^\times$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) For the identifications $S^\sim \cong M$, $S^\sim_c \cong M_\sigma$ and $S^\times \cong M_\tau$ see 362A.

(b) $L^{\infty *} = L^{\infty\sim}$ either because L^∞ is a Banach lattice (356Dc) or because L^∞ has an order-unit norm, so that a linear functional on L^∞ is order-bounded iff it is bounded on the unit ball.

(c) If f is a positive linear functional on L^∞, then $f\restriction S$ is a positive linear functional. Because S is order-dense in L^∞ (363C), the embedding is order-continuous (352Nb); so if f is (sequentially) order-continuous, so is $f\restriction S$. Accordingly the restriction operator $f \mapsto f\restriction S$ gives maps from $L^{\infty\sim}$ to $S^\sim$, $(L^\infty)^\sim_c$ to $S^\sim_c$ and $L^{\infty\times}$ to $S^\times$. If $f \in L^{\infty\sim}$ and $f\restriction S \ge 0$, then $f(u^+) \ge 0$ for every $u \in S$ and therefore for every $u \in L^\infty$, and $f \ge 0$; so all these restriction maps are injective positive linear operators.

(d) I need to show that they are surjective.

(i) If $g \in S^\sim$, then g is bounded on the unit ball $\{u : u \in S, \|u\|_\infty \le 1\}$, so has an extension to a continuous linear $f : L^\infty \to \mathbb{R}$ (2A4I); thus $S^\sim = \{f\restriction S : f \in L^{\infty\sim}\}$. This means that $f \mapsto f\restriction S$ is actually a Riesz space isomorphism between $L^{\infty\sim}$ and $S^\sim$. In particular, $|f|\restriction S = |f\restriction S|$ for any $f \in L^{\infty\sim}$.

(ii) If $f : L^\infty \to \mathbb{R}$ is a positive linear operator and $f\restriction S \in S^\sim_c$, let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in L^∞ with infimum 0. For each n, $k \in \mathbb{N}$ there is a $v_{nk} \in S$ such that $u_n \le v_{nk} \le u_n + 2^{-k}e$, where e is the standard order unit of L^∞ (354I, as usual); set $w_n = \inf_{i,k\le n} v_{ik}$; then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0, so

$$0 \le \inf_{n\in\mathbb{N}} f(u_n) \le \inf_{n\in\mathbb{N}} f(w_n) = 0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f \in (L^\infty)^\sim_c$. Consequently, for general $f \in L^{\infty\sim}$,

$$f \in (L^\infty)^\sim_c \iff |f| \in (L^\infty)^\sim_c \iff |f\restriction S| \in S^\sim_c \iff f\restriction S \in S^\sim_c,$$

and the map $f \mapsto f\restriction S : (L^\infty)^\sim_c \to S^\sim_c$ is a Riesz space isomorphism.

(iii) Similarly, if $f \in L^{\infty\sim}$ is non-negative and $f\restriction S \in S^\times$, then whenever $A \subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, $B = \{w : w \in S, \exists\, u \in A, w \ge u\}$ has infimum 0, so $\inf_{u\in A} f(u) \le \inf_{w\in B} f(w) \le 0$ and $f \in L^{\infty\times}$. As in (ii), it follows that $f \mapsto f\restriction S$ is a surjection from $L^{\infty\times}$ onto $S^\times$.

***363L Integration with respect to a finitely additive functional (a)** If $\mathfrak{A}$ is a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is a bounded additive functional, then by 363K we have a corresponding functional $f_\nu \in L^\infty(\mathfrak{A})^*$ defined by saying that $f_\nu(\chi a) = \nu a$ for every $a \in \mathfrak{A}$. There are contexts in which it is convenient, and even helpful, to use the formula $\int u\,d\nu$ in place of $f_\nu(u)$ for $u \in L^\infty = L^\infty(\mathfrak{A})$. When doing so, we must of course remember that we may have lost some of the standard properties of 'integration'. But enough of our intuitions (including, for instance, the idea of stochastic independence) remain valid to make the formula a guide to interesting ideas.

(b) Let M be the L-space of bounded finitely additive functionals on $\mathfrak{A}$ (362B). Then we have a function $(u,\nu) \mapsto \int u\,d\nu : L^\infty \times M \to \mathbb{R}$. Now this map is bilinear. **P** For μ, $\nu \in M$, u, $v \in L^\infty$ and $\alpha \in \mathbb{R}$,

$$\int u + v\,d\nu = \int u\,d\nu + \int v\,d\nu, \quad \int \alpha u\,d\nu = \alpha \int u\,d\nu$$

just because f_ν is linear. On the other side, we have

$$(f_\mu + f_\nu)(\chi a) = f_\mu(\chi a) + f_\nu(\chi a) = \mu a + \nu a = (\mu+\nu)(a) = f_{\mu+\nu}(\chi a)$$

for every $a \in \mathfrak{A}$, so that $f_\mu + f_\nu$ and $f_{\mu+\nu}$ must agree on $S(\mathfrak{A})$ and therefore on L^∞. But this means that $\int u\,d(\mu+\nu) = \int u\,d\mu + \int u\,d\nu$. Similarly, $\int u\,d(\alpha\mu) = \alpha\int u\,d\mu$. **Q**

(c) If ν is non-negative, we have $\int u\,d\nu \ge 0$ whenever $u \ge 0$, as in part (c) of the proof of 363K. Consequently, for any $\nu \in M$ and $u \in L^\infty$,

$$\begin{aligned} |\int u\,d\nu| &= |\int u^+\,d\nu^+ - \int u^- d\nu^+ - \int u^+ d\nu^- + \int u^- d\nu^-| \\ &\le \int u^+\,d\nu^+ + \int u^- d\nu^+ + \int u^+ d\nu^- + \int u^- d\nu^- \\ &= \int |u| d|\nu| \le \int \|u\|_\infty \chi 1\,d|\nu| = \|u\|_\infty |\nu|(1) = \|u\|_\infty \|\nu\|. \end{aligned}$$

So $(u,\nu)\mapsto\int u\,d\nu$ has norm (as defined in 253Ab) at most 1. If $\mathfrak{A}\ne 0$, the norm is exactly 1. (For this we need to know that there is a $\nu\in M^+$ such that $\nu 1=1$. Take any z in the Stone space of $\mathfrak{A}$ and set $\nu a=1$ if $z\in\widehat{a}$, 0 otherwise.)

(d) We do not have any result corresponding to B.Levi's theorem in this language, because (even if ν is non-negative and countably additive) there is no reason to suppose that $\sup_{n\in\mathbb{N}}u_n$ is defined in L^∞ just because $\sup_{n\in\mathbb{N}}\int u_n d\nu$ is finite. But if ν is countably additive and $\mathfrak{A}$ is Dedekind σ-complete, we have something corresponding to Lebesgue's Dominated Convergence Theorem (363Yh).

(e) One formula which we can imitate in the present context is that of 252O, where the ordinary integral is represented in the form

$$\int f d\mu=\int_0^\infty \mu\{x:f(x)\ge t\}dt.$$

In the context of general Boolean algebras, we cannot directly represent the set $[\![f\ge t]\!]=\{x:f(x)\ge t\}$ (though in the next section I will show that in Dedekind σ-complete Boolean algebras there is an effective expression of this idea, and I will use it in the principal definition of §365). But what we can say is the following. If $\mathfrak{A}$ is any Boolean algebra, and $\nu:\mathfrak{A}\to[0,\infty[$ is a non-negative additive functional, and $u\in L^\infty(\mathfrak{A})^+$, then

$$\int u\,d\nu=\int_0^\infty \sup\{\nu a:t\chi a\le u\}dt,$$

where the right-hand integral is taken with respect to Lebesgue measure. **P** (i) For $t\ge 0$ set $h(t)=\sup\{\nu a:t\chi a\le u\}$. Then h is non-increasing and zero for $t>\|u\|_\infty$, so $\int_0^\infty h(t)dt$ is defined in $\mathbb{R}$. If we set $h_n(t)=h(2^{-n}(k+1))$ whenever k, $n\in\mathbb{N}$ and $2^{-n}k\le t<2^{-n}(k+1)$, then $\langle h_n(t)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence which converges to $h(t)$ whenever h is continuous at t, which is almost everywhere (222A, or otherwise); so $\int_0^\infty h(t)dt=\lim_{n\to\infty}\int_0^\infty h_n(t)dt$. Next, given $n\in\mathbb{N}$ and $\epsilon>0$, we can choose for each $k\le k^*=\lfloor 2^n\|u\|_\infty\rfloor$ an a_k such that $2^{-n}(k+1)\chi a_k\le u$ and $\nu a_k\ge h(2^{-n}(k+1))-\epsilon$. In this case $\sum_{k=0}^{k^*}2^{-n}\chi a_k\le u$, so

$$\begin{aligned}\int_0^\infty h_n(t)dt&=2^{-n}\sum_{k=0}^{k^*}h(2^{-n}(k+1))\le\|u\|_\infty\epsilon+2^{-n}\sum_{k=0}^{k^*}\nu a_k\\&=\|u\|_\infty\epsilon+\int\sum_{k=0}^{k^*}2^{-n}\chi a_k d\nu\le\|u\|_\infty\epsilon+\int u\,d\nu.\end{aligned}$$

As n and ϵ are arbitrary, $\int_0^\infty h(t)dt\le\int u\,d\nu$. (ii) In the other direction, there is for any $\epsilon>0$ a $v\in S(\mathfrak{A})$ such that $v\le u\le v+\epsilon\chi 1$. If we express v as $\sum_{j=0}^m\gamma_j\chi c_j$ where $c_0\supseteq\ldots\supseteq c_m$ and $\gamma_j\ge 0$ for every j (361Ec), then we shall have $h(t)\ge\nu c_k$ whenever $t\le\sum_{j=0}^k\gamma_j$, so

$$\int_0^\infty h(t)dt\ge\sum_{k=0}^m\gamma_k\nu c_k=\int v\,d\nu\ge\int u\,d\nu-\epsilon\nu 1.$$

As ϵ is arbitrary, $\int_0^\infty h(t)dt\ge\int u\,d\nu$ and the two 'integrals' are equal. **Q**

(f) The formula $\int d\nu$ is especially natural when $\mathfrak{A}$ is an algebra of sets, so that L^∞ can be directly interpreted as a space of functions (363Yf); better still, when $\mathfrak{A}$ is actually a σ-algebra of subsets of a set X, L^∞ can be identified with the space of bounded $\mathfrak{A}$-measurable functions on X, as in 363Ha. So in such contexts I may write $\int g\,d\nu$ when $g:X\to\mathbb{R}$ is bounded and $\mathfrak{A}$-measurable, and $\nu:\mathfrak{A}\to\mathbb{R}$ is an additive functional. But I will try to take care to signal any such deviation from the normal principle that the symbol $\int$ refers to the sequentially order-continuous integral defined in §122 with the minor modifications introduced in §§133 and 135. My purpose in this paragraph has been only to indicate something of what can be done with *finite* additivity alone.

363M Now I come to a fundamental fact underlying a number of theorems in both this volume and the last.

Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathfrak{A}$ is Dedekind σ-complete iff $L^\infty(\mathfrak{A})$ is Dedekind σ-complete.

(b) $\mathfrak{A}$ is Dedekind complete iff $L^\infty(\mathfrak{A})$ is Dedekind complete.

proof (a)(i) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. By 314M, we may identify $\mathfrak{A}$ with a quotient $\Sigma/\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of the Stone space Z of $\mathfrak{A}$, and $\Sigma = \{E\triangle A : E \in \mathcal{E}, A \in \mathcal{M}\}$, writing $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ for the algebra of open-and-closed subsets of Z. By 363H, $L^\infty = L^\infty(\mathfrak{A})$ can be identified with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$, and $\mathcal{V}$ is the space of functions zero except on a member of $\mathcal{I}$.

Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^∞ with an upper bound $u \in L^\infty$. Express u_n, u as $f_n^\bullet$, $f^\bullet$ where f_n, $f \in \mathcal{L}^\infty$. Set $g(z) = \sup_{n\in\mathbb{N}} \min(f_n(z), f(z))$ for every $z \in Z$; then $g \in \mathcal{L}^\infty$ (121F), so we have a corresponding member $v = g^\bullet$ of L^∞. For each $n \in \mathbb{N}$, $u \ge u_n$ so $(f_n - f)^+ \in \mathcal{V}$,

$$\{z : f_n(z) > g(z)\} \subseteq \{z : f_n(z) > f(z)\} \in \mathcal{M}$$

and $v \ge u_n$. If $w \in L^\infty$ and $w \ge u_n$ for every n, then express w as $h^\bullet$ where $h \in \mathcal{L}^\infty$; we have $(f_n - h)^+ \in \mathcal{V}$ for every n, so

$$\{z : g(z) > h(z)\} \subseteq \bigcup_{n\in\mathbb{N}}\{z : f_n(z) > h(z)\} \in \mathcal{M}$$

because $\mathcal{M}$ is a σ-ideal, and $(g - h)^+ \in \mathcal{V}$, $w \ge v$. Thus $v = \sup_{n\in\mathbb{N}} u_n$ in L^∞. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, L^∞ is Dedekind σ-complete (using 353G).

(ii) Now suppose that L^∞ is Dedekind σ-complete, and that A is a countable non-empty set in $\mathfrak{A}$. In this case $\{\chi a : a \in A\}$ has a least upper bound u in L^∞. Take a $v \in S(\mathfrak{A})$ such that $0 \le v \le u$ and $\|u - v\|_\infty \le \frac{1}{3}$; set $b = [\![v > \frac{1}{3}]\!]$, as defined in 361Eg. If $a \in A$, then $\|(\chi a - v)^+\|_\infty \le \|u - v\|_\infty \le \frac{1}{3}$, so $\frac{2}{3}\chi a \le v$ and $a \subseteq b$. If $c \in \mathfrak{A}$ is any upper bound for A, then $v \le u \le \chi c$ so $b \subseteq c$. Thus $b = \sup A$ in $\mathfrak{A}$. As A is arbitrary, $\mathfrak{A}$ is Dedekind σ-complete.

(b)(i) For the second half of this theorem I use an argument which depends on joining the representation described in (a-i) above with the original definition of L^∞ in 363A. The point is that $C(Z) \subseteq \mathcal{L}^\infty$, and for any $f \in C(Z) = L^\infty(\mathfrak{A})$ its equivalence class $f^\bullet$ in $\mathcal{L}^\infty/\mathcal{V}$ corresponds to f itself. **P** Perhaps it will help to give a name T to the canonical isomorphism from $\mathcal{L}^\infty/\mathcal{V}$ to L^∞. Then $V = \{f : Tf^\bullet = f\}$ is a closed linear subspace of $C(Z)$, because $f \mapsto f^\bullet$ and T are continuous linear operators. But if $a \in \mathfrak{A}$, then $(\widehat{a})^\bullet$, the equivalence class of $\widehat{a} \in \Sigma$ in $\Sigma/\mathcal{M}$, corresponds to a (see the proof of 314M), so $(\chi\widehat{a})^\bullet \in \mathcal{L}^\infty/\mathcal{V}$ corresponds to χa; that is, $T(\chi\widehat{a})^\bullet = \chi\widehat{a}$, if we identify $\chi a \in L^\infty$ with $\chi\widehat{a} : Z \to \{0, 1\}$. So V contains $\chi\widehat{a}$ for every $a \in \mathfrak{A}$; because V is a linear subspace, $S(\mathfrak{A}) \subseteq V$; because V is closed, $L^\infty \subseteq V$. **Q**

For a general $f \in \mathcal{L}^\infty$, $g = Tf^\bullet$ must be the unique member of $C(Z)$ such that $g^\bullet = f^\bullet$, that is, such that $\{z : g(z) \ne f(z)\}$ is meager.

(ii) Suppose now that $\mathfrak{A}$ is actually Dedekind complete. In this case Z is extremally disconnected (314S). Consequently every open set belongs to Σ. **P** If G is open, then $\overline{G}$ is open-and-closed; but $A = \overline{G}\setminus G$ is a closed set with empty interior, so is meager, and $G = \overline{G}\triangle A \in \Sigma$. **Q**

Let $A \subseteq L^\infty = C(Z)$ be any non-empty set with an upper bound in $C(Z)$. For each $z \in Z$ set $g(z) = \sup_{u\in A} u(z)$. Then

$$G_\alpha = \{z : g(z) > \alpha\} = \bigcup_{u\in A}\{z : u(z) > \alpha\}$$

is open for every $\alpha \in \mathbb{R}$ (that is, g is lower semi-continuous). Thus $G_\alpha \in \Sigma$ for every α, so $g \in \mathcal{L}^\infty$, and $v = Tg^\bullet$ is defined in $C(Z)$. For any $u \in A$, $g \ge u$ in $\mathcal{L}^\infty$, so

$$v = Tg^\bullet \ge Tu^\bullet = u$$

in L^∞; thus v is an upper bound for A in L^∞. On the other hand, if w is any upper bound for A in $L^\infty = C(Z)$, then surely $w(z) \ge u(z)$ for every $z \in Z$, $u \in A$, so $w \ge g$ and

$$w = Tw^\bullet \ge Tg^\bullet = v.$$

This means that v is the least upper bound of A. As A is arbitrary, L^∞ is Dedekind complete.

(iii) Finally, if L^∞ is Dedekind complete, then the argument of (b-ii), applied to arbitrary non-empty subsets A of $\mathfrak{A}$, shows that $\mathfrak{A}$ is also Dedekind complete.

363N Much of the importance of L^∞ spaces in the theory of Riesz spaces arises from the next result.

Proposition Let U be a Dedekind σ-complete Riesz space with an order unit. Then U is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands in U.

proof (a) By 353M, U is isomorphic to a norm-dense Riesz subspace of $C(X)$ for some compact Hausdorff space X; for the rest of this argument, therefore, we may suppose that U actually is such a subspace.

(b) Now $U = C(X)$. **P** If $g \in C(X)$ then by 354I there are sequences $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle f'_n\rangle_{n\in\mathbb{N}}$ in U such that $f_n \le g \le g_n$ and $\|g_n - f_n\|_\infty \le 2^{-n}$ for every n. Now $\{f_n : n \in \mathbb{N}\}$ has a least upper bound f in U; since we must have $f_n \le f \le g_n$ for every n, $f = g$ and $g \in U$. **Q**

(c) Next, X is zero-dimensional. **P** Suppose that $G \subseteq X$ is open and $x \in G$. Then there is an open set G_1 such that $x \in G_1 \subseteq \overline{G}_1 \subseteq G$ (3A3Bc). There is an $f \in C(X)$ such that $0 \le f \le \chi G_1$ and $f(x) > 0$ (also by 3A3Bc); write H for $\{y : f(y) > 0\}$. Set $g = \sup_{n\in\mathbb{N}}(nf \wedge \chi X)$, the supremum being taken in $U = C(X)$. For each $y \in H$, we must have $g(y) \ge \min(1, nf(y))$ for every n, so that $g(y) = 1$. On the other hand, if $y \in X \setminus \overline{H}$, there is an $h \in C(X)$ such that $h(y) > 0$ and $0 \le h \le \chi(X \setminus \overline{H})$; now $h \wedge f = 0$ so $h \wedge g = 0$ and $g(y) = 0$. Thus $\chi H \le g \le \chi\overline{H}$. The set $\{y : g(y) \in \{0, 1\}\}$ is closed and includes $H \cup (X \setminus \overline{H})$ so must be the whole of X; thus $G_2 = \{y : g(y) > \frac{1}{2}\} = \{y : g(y) \ge \frac{1}{2}\}$ is open-and-closed, and we have

$$x \in H \subseteq G_2 \subseteq \overline{H} \subseteq \overline{G}_1 \subseteq G.$$

As x, G are arbitrary, the set of open-and-closed subsets of X is a base for the topology of X, and X is zero-dimensional. **Q**

(d) We can therefore identify X with the Stone space of its algebra $\mathcal{E}$ of open-and-closed sets (311J). But in this case 363A immediately identifies $U = C(X)$ with $L^\infty(\mathcal{E})$. By 363J, $\mathcal{E}$ is isomorphic to $\mathfrak{A}$, so $U \cong L^\infty(\mathfrak{A})$.

Remark Note that in part (c) of the argument above, we have to take great care over the interpretation of 'sup'. In the space of all real-valued functions on X, the supremum of $\{nf \wedge \chi X : n \in \mathbb{N}\}$ is just χH. But g is supposed to be the least *continuous* function greater than or equal to $nf \wedge \chi X$ for every n, and is therefore likely to be strictly greater than χH, even though sandwiched between χH and $\chi\overline{H}$.

363O Corollary Let U be a Dedekind σ-complete M-space. Then U is isomorphic, as Banach lattice, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands of U.

proof This is merely the special case of 363N in which U is known from the start to be complete under an order-unit norm.

363P Corollary Let U be any Dedekind σ-complete Riesz space and $e \in U^+$. Then the solid linear subspace U_e of U generated by e is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$; and if U is Dedekind complete, so is $\mathfrak{A}$.

proof Because U is Dedekind σ-complete, so is U_e (353J(a-i)). Apply 363N to U_e to see that $U_e \cong L^\infty(\mathfrak{A})$ for some $\mathfrak{A}$. Because U_e is Dedekind σ-complete, so is $\mathfrak{A}$, by 363Ma; while if U is Dedekind complete, so are U_e and $\mathfrak{A}$, by 353J(b-i) and 363Mb.

363Q The next theorem will be a striking characterization of the Dedekind complete L^∞ spaces as normed spaces. As a warming-up exercise I give a much simpler result concerning their nature as Banach lattices.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Then for any Banach lattice U, a linear operator $T : U \to L^\infty = L^\infty(\mathfrak{A})$ is continuous iff it is order-bounded, and in this case $\|T\| = \||T|\|$, where the modulus $|T|$ is taken in $\mathrm{L}^\sim(U; L^\infty)$.

proof It is generally true that order-bounded operators between Banach lattices are continuous (355C). If $T:U\to L^\infty$ is continuous, then for any $w\in U^+$

$$|u|\le w\Longrightarrow\|u\|\le\|w\|\Longrightarrow\|Tu\|_\infty\le\|T\|\|w\|\Longrightarrow|Tu|\le\|T\|\|w\|\chi 1.$$

So T is order-bounded. As L^∞ is Dedekind complete (363Mb), $|T|$ is defined in $\mathrm{L}^\sim(U;L^\infty)$ (355Ea). For any $w\in U$,

$$|T||w|=\sup\{|Tu|:|u|\le|w|\}\le\|T\|\|w\|\chi 1,$$

so $\||T|(w)\|\le\|T\|\|w\|$; accordingly $\||T|\|\le\|T\|$. On the other hand, of course,

$$|Tw|\le|T||w|\le\||T|\|\|w\|\chi 1$$

for every $w\in U$, so $\|T\|\le\||T|\|$ and the two norms are equal.

Remark Of course what is happening here is that the spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are just the Dedekind complete M-spaces; this is an elementary consequence of 363N and 363M.

363R Now for something much deeper.

Theorem Let U be a normed space over $\mathbb{R}$. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic, as normed space, to $L^\infty(\mathfrak{A})$;

(ii) whenever V is a normed space, V_0 a linear subspace of V, and $T_0:V_0\to U$ is a bounded linear operator, there is an extension of T_0 to a bounded linear operator $T:V\to U$ with $\|T\|=\|T_0\|$.

proof For the purposes of the argument below, let us say that a normed space U satisfying the condition (ii) has the 'Hahn-Banach property'.

Part A: (i)⇒(ii) I have to show that $L^\infty(\mathfrak{A})$ has the Hahn-Banach property for every Dedekind complete Boolean algebra $\mathfrak{A}$. Let V be a normed space, V_0 a linear subspace of V, and $T_0:V_0\to L^\infty=L^\infty(\mathfrak{A})$ a bounded linear operator. Set $\gamma=\|T_0\|$.

Let $\mathfrak{P}$ be the set of all functions T such that $\operatorname{dom}T$ is a linear subspace of V including V_0 and $T:\operatorname{dom}T\to U$ is a bounded linear operator extending T_0 and with norm at most γ. Order $\mathfrak{P}$ by saying that $T_1\le T_2$ if T_2 extends T_1. Then any non-empty totally ordered subset $\mathfrak{Q}$ of $\mathfrak{P}$ has an upper bound in $\mathfrak{P}$. **P** Set $\operatorname{dom}T=\bigcup\{\operatorname{dom}T_1:T_1\in\mathfrak{Q}\}$, $Tv=T_1v$ whenever $T_1\in\mathfrak{Q}$ and $v\in\operatorname{dom}T_1$; it is elementary to check that $T\in\mathfrak{P}$, so that T is an upper bound for $\mathfrak{Q}$ in $\mathfrak{P}$. **Q**

By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\tilde T$. Now $\operatorname{dom}\tilde T=V$. **P?** Suppose, if possible, otherwise. Write $\tilde V=\operatorname{dom}\tilde T$ and take any $\tilde v\in V\setminus\tilde V$; let V_1 be the linear span of $\tilde V\cup\{\tilde v\}$, that is, $\{v+\alpha\tilde v:v\in\tilde V,\,\alpha\in\mathbb{R}\}$.

If v_1, $v_2\in\tilde V$ then, writing e for the standard order unit of L^∞,

$$\begin{aligned}\tilde Tv_1+\tilde Tv_2=\tilde T(v_1+v_2)&\le\|\tilde T(v_1+v_2)\|_\infty e\\&\le\gamma\|v_1+v_2\|e\le\gamma\|v_1-\tilde v\|e+\gamma\|v_2+\tilde v\|e,\end{aligned}$$

so

$$\tilde Tv_1-\gamma\|v_1-\tilde v\|e\le\gamma\|v_2+\tilde v\|e-\tilde Tv_2.$$

Because L^∞ is Dedekind complete (363Mb),

$$\tilde u=\sup_{v_1\in\tilde V}\tilde Tv_1-\gamma\|v_1-\tilde v\|e$$

is defined in L^∞ and $\tilde u\le\gamma\|v_2+\tilde v\|e-Tv_2$ for every $v_2\in\tilde V$. Putting these together, we have

$$\tilde Tv-\tilde u\le\gamma\|v-\tilde v\|e,\quad\tilde Tv+\tilde u\le\gamma\|v+\tilde v\|e$$

for all $v\in\tilde V$. Consequently, if $v\in\tilde V$, then for $\alpha>0$

$$\tilde Tv+\alpha\tilde u=\alpha(\tilde T(\tfrac{1}{\alpha}v)+\tilde u)\le\alpha\gamma\|\tfrac{1}{\alpha}v+\tilde v\|e=\gamma\|v+\alpha\tilde v\|e,$$

while for $\alpha<0$

$$\tilde Tv+\alpha\tilde u=|\alpha|(\tilde T(-\tfrac{1}{\alpha}v)-\tilde u)\le|\alpha|\gamma\|-\tfrac{1}{\alpha}v-\tilde v\|e=\gamma\|v+\alpha\tilde v\|e,$$

and of course

$$\tilde{T}v \le \|\tilde{T}v\|_\infty e \le \gamma\|v\|e.$$

So we have

$$\tilde{T}v + \alpha\tilde{u} \le \gamma\|v + \alpha\tilde{v}\|e$$

for every $v \in \tilde{V}$, $\alpha \in \mathbb{R}$.

Define $T_1 : V_1 \to L^\infty$ by setting $T_1(v + \alpha\tilde{v}) = \tilde{T}v + \alpha\tilde{u}$ for every $v \in \tilde{V}$, $\alpha \in \mathbb{R}$. (This is well-defined because $\tilde{v} \notin \tilde{V}$, so any member of V_1 is uniquely expressible as $v + \alpha\tilde{v}$ where $v \in \tilde{V}$ and $\alpha \in \mathbb{R}$.) Then T_1 is a linear operator, extending T_0, from a linear subspace of V to L^∞. But from the calculations above we know that $T_1 v \le \gamma\|v\|e$ for every $v \in V_1$; since we also have

$$T_1 v = -T_1(-v) \ge -\gamma\|-v\|e = -\gamma\|v\|e,$$

$\|T_1 v\|_\infty \le \gamma\|v\|$ for every $v \in V_1$, and $T_1 \in \mathfrak{P}$. But now T_1 is a member of $\mathfrak{P}$ properly extending $\tilde{T}$, which is supposed to be impossible. **XQ**

Accordingly $\tilde{T} : V \to L^\infty$ is an extension of T_0 to the whole of V, with the same norm as T_0. As V and T_0 are arbitrary, L^∞ has the Hahn-Banach property.

Part B: (ii)⇒(i) Now let U be a normed space with the Hahn-Banach property. If $U = \{0\}$ then of course it is isomorphic to $L^\infty(\mathfrak{A})$, where $\mathfrak{A} = \{0\}$, so henceforth I will take it for granted that $U \ne \{0\}$.

(a) Let Z be the unit ball of the dual U^* of U, with the weak* topology. Then Z is a compact Hausdorff space (3A5F). For $u \in U$ set $Z_u = \{f : f \in Z, |f(u)| = \|u\|\}$; then Z_u is a closed subset of Z (because $f \mapsto f(u)$ is continuous), and is non-empty, by the Hahn-Banach theorem (3A5Ab, or Part A above!) Now let $\mathfrak{P}$ be the set of those closed sets $X \subseteq Z$ such that $X \cap Z_u \ne \emptyset$ for every $u \in U$. If $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and totally ordered, then $\bigcap\mathfrak{Q} \in \mathfrak{P}$, because for any $u \in U$

$$\{X \cap Z_u : X \in \mathfrak{Q}\}$$

is a downwards-directed family of non-empty compact sets, so must have non-empty intersection. By Zorn's Lemma, upside down, $\mathfrak{P}$ has a minimal element X; with its relative topology, X is a compact Hausdorff space.

(b) We have a linear operator $R : U \to C(X)$ given by setting $(Ru)(x) = x(u)$ for every $u \in U$, $x \in X$; because $X \subseteq Z$, $\|Ru\|_\infty \le \|u\|$, and because $X \in \mathfrak{P}$, $\|Ru\|_\infty = \|u\|$, for every $u \in U$. Moreover, if $G \subseteq X$ is a non-empty open set (in the relative topology of X) then $X \setminus G$ cannot belong to $\mathfrak{P}$, because X is minimal, so there is a (non-zero) $u \in U$ such that $|x(u)| < \|u\|$ for every $x \in X \setminus G$. Replacing u by $\|u\|^{-1}u$ if need be, we may suppose that $\|u\| = 1$.

What this means is that $W = R[U]$ is a linear subspace of $C(X)$ which is isomorphic, as normed space, to U, and has the property that whenever $G \subseteq X$ is a non-empty relatively open set there is an $f \in W$ such that $\|f\|_\infty = 1$ and $|f(x)| < 1$ for every $x \in X \setminus G$. Observe that, because $X \setminus G$ is compact, there is now some $\alpha < 1$ such that $|f(u)| \le \alpha$ for every $f \in X \setminus G$.

Because W is isomorphic to U, it has the Hahn-Banach property.

(c) Now consider $V = \ell^\infty(X)$, $V_0 = W$, $T_0 : V_0 \to W$ the identity map. Because W has the Hahn-Banach property, there is a linear operator $T : \ell^\infty(X) \to W$, extending T_0, and of norm $\|T_0\| = 1$.

(d) If $h \in \ell^\infty(X)$ and $x_0 \in X \setminus \overline{\{x : h(x) \ne 0\}}$, then $(Th)(x_0) = 0$. **P?** Otherwise, set $G = \{y : y \in X \setminus \overline{\{x : h(x) \ne 0\}}, (Th)(y) \ne 0\}$. This is a non-empty open set in X, so there are $f \in W$, $\alpha < 1$ such that $\|f\|_\infty = 1$, $|f(x)| \le \alpha$ for every $x \in X \setminus G$.

Because $\|f\|_\infty = 1$, there must be some $x_1 \in X$ such that $|f(x_1)| = 1$, and of course $x_1 \in G$, so that $(Th)(x_1) \ne 0$. But let $\delta > 0$ be such that $\delta\|h\|_\infty \le 1 - \alpha$. Then, because $h(x) = 0$ for $x \in G$, $|f(x)| + |\delta h(x)| \le 1$ for every $x \in X$, and $\|f + \delta h\|_\infty$, $\|f - \delta h\|_\infty$ are both less than or equal to 1. As $Tf = f$ and $\|T\| = 1$, this means that

$$\|f - \delta Th\|_\infty \le 1, \quad \|f + \delta Th\|_\infty \le 1;$$

consequently

$$|f(x_1)| + \delta|(Th)(x_1)| = \max(|(f + \delta Th)(x_1)|, |(f - \delta Th)(x_1)|) \le 1.$$

But $|f(x_1)| = 1$ and $\delta(Th)(x_1) \ne 0$, so this is impossible. **XQ**

(e) It follows that $Th = h$ for every $h \in C(X)$. **P?** Suppose, if possible, otherwise. Then there is a $\delta > 0$ such that $G = \{x : |(Th)(x) - h(x)| > \delta\}$ is not empty. Let $f \in W$ be such that $\|f\| = 1$ but $|f(x)| < 1$ for every $x \in X \setminus G$. Then there is an $x_0 \in X$ such that $|f(x_0)| = 1$; of course x_0 must belong to G. Set $f_1 = \frac{h(x_0)}{f(x_0)} f$, so that $f_1 \in W$ and $f_1(x_0) = h(x_0)$. Set

$$h_1(x) = \max(h(x) - \delta, \min(h(x) + \delta, f_1(x)))$$

for $x \in X$. Then $h_1 \in C(X)$. Setting

$$H = \{x : |h(x) - h(x_0)| + |f_1(x) - f_1(x_0)| < \delta\},$$

H is an open set containing x_0 and

$$|f_1(x) - h(x)| \le |f_1(x_0) - h(x_0)| + \delta = \delta, \quad h_1(x) = f_1(x)$$

for every $x \in H$. Consequently $x_0 \notin \overline{\{x : (f_1 - h_1)(x) \ne 0\}}$, and $T(f_1 - h_1)(x_0) = 0$, by (d). But this means that

$$(Th_1)(x_0) = (Tf_1)(x_0) = f_1(x_0) = h(x_0),$$

so that

$$|h(x_0) - (Th)(x_0)| = |T(h_1 - h)(x_0)| \le \|T(h_1 - h)\|_\infty \le \|h_1 - h\|_\infty \le \delta,$$

which is impossible, because $x_0 \in G$. **XQ**

(f) This tells us at once that $W = C(X)$. But (d) also tells us that X is extremally disconnected. **P** Let $G \subseteq X$ be any open set. Then $\chi X = \chi G + \chi(X \setminus G)$, so

$$\chi X = T(\chi X) = h_1 + h_2,$$

where $h_1 = T(\chi G)$, $h_2 = T(\chi(X \setminus G))$. Now from (d) we see that h_1 must be zero on $X \setminus \overline{G}$ while h_2 must be zero on G. Thus we have $h_1(x) = 1$ for $x \in G$; as h_1 is continuous, $h_1(x) = 1$ for $x \in \overline{G}$, and $h_1 = \chi\overline{G}$. Of course it follows that $\overline{G}$ is open. As G is arbitrary, X is extremally disconnected. **Q**

(g) Being also compact and Hausdorff, therefore regular (3A3Bc), X is zero-dimensional (3A3Bd). We may therefore identify X with the Stone space of its regular open algebra $\mathfrak{A}$ (314S), and $W = C(X)$ with $L^\infty(\mathfrak{A})$. Thus $R : U \to C(X)$ is a Banach space isomorphism between U and $C(X) \cong L^\infty(\mathfrak{A})$; so U is of the type declared.

363S The Banach-Ulam problem At a couple of points already (232Hc, the notes to §326) I have remarked on a problem which was early recognised as a fundamental question in abstract measure theory. I now set out some formulations of the problem which arise naturally from the work done so far. I will do this by writing down a list of statements which are equiveridical in the sense that if one of them is true, so are all the others; the 'Banach-Ulam problem' asks whether they are indeed true.

I should remark that this is not generally counted as an 'open' problem. It is in fact believed by most of us that these statements are independent of the usual axioms of Zermelo-Fraenkel set theory, including the axiom of choice and even the continuum hypothesis. As such, this problem belongs to Volume 5 rather than anywhere earlier, but its manifestations will become steadily more obtrusive as we continue through this volume and the next, and I think it will be helpful to begin collecting them now. The ideas needed to show that the statements here imply each other are already accessible; in particular, they involve no set theory beyond Zorn's Lemma. These implications constitute the following theorem, derived from Luxemburg 67a.

Theorem The following statements are equiveridical.

(i) There are a set X and a probability measure ν, with domain $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$.

(ii) There are a localizable measure space (X, Σ, μ) and an absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ which is not truly continuous, so has no Radon-Nikodým derivative (definitions: 232Ab, 232Hf).

(iii) There are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ which is not completely additive.

(iv) There is a Dedekind complete Riesz space U such that $U_c^\sim \ne U^\times$.

proof (a)(i)⇒(ii) Let X be a set with a probability measure ν, defined on $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$. Let μ be counting measure on X. Then $(X, \mathcal{P}X, \mu)$ is strictly localizable, and $\nu : \mathcal{P}X \to \mathbb{R}$ is countably additive; also $\nu E = 0$ whenever μE is finite, so ν is absolutely continuous with respect to μ. But if $\mu E < \infty$ then E is finite and $\nu(X \setminus E) = 1$, so ν is not truly continuous, and has no Radon-Nikodým derivative (232D).

(b)(ii)⇒(iii) Let (X, Σ, μ) be a localizable measure space and $\nu : \Sigma \to \mathbb{R}$ an absolutely continuous countably additive functional which is not truly continuous. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ; then we have an absolutely continuous countably additive functional $\bar\nu : \mathfrak{A} \to \mathbb{R}$ defined by setting $\bar\nu E^\bullet = \nu E$ for every $E \in \Sigma$ (327C). Since ν is not truly continuous, $\bar\nu$ is not completely additive (327Ce). Also $\mathfrak{A}$ is Dedekind complete, because μ is localizable, so $\mathfrak{A}$ and $\bar\nu$ witness (iii).

(c)(iii)⇒(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional which is not completely additive. Because ν is bounded (326I), therefore expressible as the difference of non-negative countably additive functionals (326H), there must be a non-negative countably additive functional ν' on $\mathfrak{A}$ which is not completely additive.

By 326N, there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\sum_{i\in I} \nu' a_i < \nu' 1$. Set $K = \{i : i \in I,\, \nu' a_i > 0\}$; then K must be countable, so

$$\nu'(\sup_{i\in I\setminus K} a_i) = \nu' 1 - \nu'(\sup_{i\in K} a_i) = \nu' 1 - \textstyle\sum_{i\in K} \nu' a_i > 0.$$

For $J \subseteq I$ set $\mu J = \nu'(\sup_{i\in J\setminus K} a_i)$; the supremum is always defined because $\mathfrak{A}$ is Dedekind complete. Because ν' is countably additive and non-negative, so is μ; because $\nu' a_i = 0$ for $i \in J \setminus K$, $\mu\{i\} = 0$ for every $i \in I$. Multiplying μ by a suitable scalar, if need be, $(I, \mathcal{P}I, \mu)$ witnesses that (i) is true.

(d)(iii)⇒(iv) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a countably additive functional which is not completely additive, then $U = L^\infty(\mathfrak{A})$ is a Dedekind complete Riesz space (363Mb) and $U_c^\sim \ne U^\times$, by 363K (recalling, as in (c) above, that the functional must be bounded).

(e)(iv)⇒(iii) Let U be a Dedekind complete Riesz space such that $U^\times \ne U_c^\sim$. Take $f \in U_c^\sim \setminus U^\times$; replacing f by $|f|$ if need be, we may suppose that $f \ge 0$ is sequentially order-continuous but not order-continuous (355H, 355I). Let A be a non-empty downwards-directed set in U, with infimum 0, such that $\inf_{u\in A} f(u) > 0$ (351Ga). Take $e \in A$, and consider the solid linear subspace U_e of U generated by e; write g for the restriction of f to U_e. Because the embedding of U_e in U is order-continuous, $g \in (U_e)_c^\sim$; because $A \cap U_e$ is downwards-directed and has infimum 0, and

$$\inf_{u\in A\cap U_e} g(u) = \inf_{u\in A} f(u) > 0,$$

$g \notin U_e^\times$. But U_e is a Riesz space with order unit e, and is Dedekind complete because U is; so it can be identified with $L^\infty(\mathfrak{A})$ for some Boolean algebra $\mathfrak{A}$ (363N), and $\mathfrak{A}$ is Dedekind complete, by 363M.

Accordingly we have a Dedekind complete Boolean algebra $\mathfrak{A}$ such that $L^\infty(\mathfrak{A})_c^\sim \ne L^\infty(\mathfrak{A})^\times$. By 363K, there is a (bounded) countably additive functional on $\mathfrak{A}$ which is not completely additive, and (iii) is true.

363X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and U a Banach algebra. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function and $T : L^\infty(\mathfrak{A}) \to U$ the corresponding bounded linear operator. Show that T is multiplicative iff $\nu(a \cap b) = \nu a \times \nu b$ for all a, $b \in \mathfrak{A}$.

>(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{A})$ a linear operator. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $T = T_\pi$ (ii) $T(u \times v) = Tu \times Tv$ for all u, $v \in L^\infty(\mathfrak{A})$ (iii) T is a Riesz homomorphism and $Te_{\mathfrak{A}} = e_{\mathfrak{B}}$, where $e_{\mathfrak{A}}$ is the standard order unit of $L^\infty(\mathfrak{A})$.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ a Riesz homomorphism. Show that there are a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ and a $v \ge 0$ in $L^\infty(\mathfrak{B})$ such that $Tu = v \times T_\pi u$ for every $u \in L^\infty(\mathfrak{A})$, where T_π is the operator associated with π (363F).

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Show that $L^\infty(\mathfrak{C})$, regarded as a subspace of $L^\infty(\mathfrak{A})$ (363Ga), is order-dense in $L^\infty(\mathfrak{A})$ iff $\mathfrak{C}$ is order-dense in $\mathfrak{A}$.

>(e) Let (X,Σ,μ) be a measure space with measure algebra $\mathfrak{A}$, and $\mathcal{L}^\infty$ the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** of μ is a positive linear operator $T:L^\infty(\mathfrak{A})\to\mathcal{L}^\infty$ such that $T(\chi 1_{\mathfrak{A}})=\chi X$ and $(Tu)^\bullet=u$ for every $u\in L^\infty(\mathfrak{A})$, writing $f\mapsto f^\bullet$ for the canonical map from $\mathcal{L}^\infty$ to $L^\infty(\mathfrak{A})$ (363H-363I). (i) Show that if $\theta:\mathfrak{A}\to\Sigma$ is a lifting in the sense of 341A then T_θ, as defined in 363F, is a linear lifting. (ii) Show that if $T:L^\infty(\mathfrak{A})\to\mathcal{L}^\infty$ is a linear lifting, then there is a corresponding lower density $\underline{\theta}:\mathfrak{A}\to\Sigma$ defined by setting $\underline{\theta}a=\{x:T(\chi a)(x)=1\}$ for each $a\in\mathfrak{A}$. (iii) Show that $\underline{\theta}$, as defined in (ii), is a lifting iff T is a Riesz homomorphism.

(f) Let U be any commutative ring with multiplicative identity 1. Show that the set A of **idempotents** in U (that is, elements $a\in U$ such that $a^2=a$) is a Boolean algebra with identity 1, writing $a\cap b=ab$, $1\setminus a=1-a$ for a, $b\in A$.

(g) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Boolean algebras of idempotents of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that for any $u\in L^\infty(\mathfrak{A})$, $\alpha\in\mathbb{R}$ there are elements $[\![u\ge\alpha]\!]$, $[\![u>\alpha]\!]\in\mathfrak{A}$, where $[\![u\ge\alpha]\!]$ is the largest $a\in\mathfrak{A}$ such that $u\times\chi a\ge\alpha\chi a$, and $[\![u>\alpha]\!]=\sup_{\beta>\alpha}[\![u\ge\beta]\!]$. (ii) Show that in the context of 363H, if u corresponds to $f^\bullet$ for $f\in\mathcal{L}^\infty$, then $[\![u\ge\alpha]\!]=\{x:f(x)\ge\alpha\}^\bullet$, $[\![u>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet$. (iii) Show that if $A\subseteq L^\infty$ is non-empty and $v\in L^\infty$, then $v=\sup A$ iff $[\![v>\alpha]\!]=\sup_{u\in A}[\![u>\alpha]\!]$ for every $\alpha\in\mathbb{R}$; in particular, $v=u$ iff $[\![v>\alpha]\!]=[\![u>\alpha]\!]$ for every α. (iv) Show that a function $\phi:\mathbb{R}\to\mathfrak{A}$ is of the form $\phi(\alpha)=[\![u>\alpha]\!]$ iff (α) $\phi(\alpha)=\sup_{\beta>\alpha}\phi(\beta)$ for every $\alpha\in\mathbb{R}$ (β) there is an M such that $\phi(M)=0$, $\phi(-M)=1$. (v) Put (iii) and (iv) together to give a proof that L^∞ is Dedekind σ-complete if $\mathfrak{A}$ is.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $U\subseteq L^\infty(\mathfrak{A})$ a (sequentially) order-closed Riesz subspace containing $\chi 1$. Show that U can be identified with $L^\infty(\mathfrak{B})$ for some (sequentially) order-closed subalgebra $\mathfrak{B}\subseteq\mathfrak{A}$. (*Hint*: set $\mathfrak{B}=\{b:\chi b\in U\}$ and use 363N.)

(j) For a Boolean algebra $\mathfrak{A}$, with Stone space Z, write $L^\infty_{\mathbb{C}}(\mathfrak{A})$ for the Banach algebra $C(Z;\mathbb{C})$ of continuous complex-valued functions on Z. Prove results corresponding to 363C, 363Ea, 363F-363I for the complex case.

363Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Given the linear structure, ordering, multiplication and norm of $S(\mathfrak{A})$ as described in §361, show that a norm completion of $S(\mathfrak{A})$ will serve for $L^\infty(\mathfrak{A})$ in the sense that all the results of 363B-363Q can be proved with no use of the axiom of choice except an occasional appeal to countably many choices in sequential forms of the theorems.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^\infty(\mathfrak{A})$ has the countable sup property (241Yd, 353Ye).

(c) Let X be an extremally disconnected topological space, and $\mathfrak{G}$ its regular open algebra. Show that there is a natural isomorphism between $L^\infty(\mathfrak{G})$ and $C_b(X)$.

(d) Let X be a compact Hausdorff space. Let us say that a linear subspace U of $C(X)$ is **ℓ^∞-complemented** in $C(X)$ if there is a linear subspace V such that $C(X)=U\oplus V$ and $\|u+v\|_\infty=\max(\|u\|_\infty,\|v\|_\infty)$ for all $u\in U$, $v\in V$. Show that there is a one-to-one correspondence between such subspaces U and open-and-closed subsets E of X, given by setting $U=\{f:f\in C(X),\ f(x)=0\ \forall\ x\in X\setminus E\}$. Hence show that if $\mathfrak{A}$ is any Boolean algebra, there is a canonical isomorphism between $\mathfrak{A}$ and the partially ordered set of ℓ^∞-complemented subspaces of $L^\infty(\mathfrak{A})$.

(e) Let $\mathfrak{A}$ be a Boolean algebra. (i) If $u\in L^\infty=L^\infty(\mathfrak{A})$, show that $|u|=e$, the standard order unit of L^∞, iff $\max(\|u+v\|_\infty,\|u-v\|_\infty)>1$ whenever $v\in L^\infty\setminus\{0\}$. (ii) Show that if u, $v\in L^\infty$ then $|u|\wedge|v|=0$ iff $\|\alpha u+v+w\|_\infty\le\max(\|\alpha u+w\|_\infty,\|v+w\|_\infty)$ whenever $\alpha=\pm1$ and $w\in L^\infty$. (iii) Show that if $T:L^\infty\to L^\infty$ is a normed space automorphism then there are a Boolean automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ and a $w\in L^\infty$ such that $|w|=e$ and $Tu=w\times T_\pi u$ for every $u\in L^\infty$.

(f) Let X be a set, Σ an algebra of subsets of X, and $\mathcal{I}$ an ideal in Σ. (i) Show that $L^\infty(\Sigma)$ can be identified, as Banach lattice and Banach algebra, with the space $\mathcal{L}^\infty$ of bounded functions $f : X \to \mathbb{R}$ such that whenever $\alpha < \beta$ in $\mathbb{R}$ there is an $E \in \Sigma$ such that $\{x : f(x) \le \alpha\} \subseteq E \subseteq \{x : f(x) \le \beta\}$. (ii) Show that $\mathcal{L}^\infty = \{g\phi : g \in C(Z)\}$, where Z is the Stone space of Σ and $\phi : X \to Z$ is a function (to be described). (iii) Show that $L^\infty(\Sigma/\mathcal{I})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{V}$ is the set of those functions $f \in \mathcal{L}^\infty$ such that for every $\epsilon > 0$ there is a member of $\mathcal{I}$ including $\{x : |f(x)| \ge \epsilon\}$.

(g) Let (X, Σ, μ) be a complete probability space with measure algebra $\mathfrak{A}$. Let $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and set $\Sigma_n = \{F : F^\bullet \in \mathfrak{B}_n\}$ for each n. Let $P_n : L^1(\mu) \to L^1(\mu\restriction\Sigma_n)$ be the conditional expectation operator for each n, so that $P_n\restriction L^\infty(\mu)$ is a positive linear operator from $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$ to $L^\infty(\mu\restriction\Sigma_n) \cong L^\infty(\mathfrak{B}_n)$. Suppose that we are given for each n a lifting $\theta_n : \mathfrak{B}_n \to \Sigma_n$ and that $\theta_{n+1}b = \theta_n b$ whenever $n \in \mathbb{N}$, $b \in \mathfrak{B}_n$. Let $T_n : L^\infty(\mathfrak{B}_n) \to \mathcal{L}^\infty$ be the corresponding linear liftings (363Xe), and $\mathcal{F}$ any non-principal ultrafilter on $\mathbb{N}$. (i) Show that for any $u \in L^\infty(\mathfrak{A})$, $\langle T_n P_n u \rangle_{n\in\mathbb{N}}$ converges almost everywhere. (ii) For $u \in L^\infty(\mathfrak{A})$ set $(Tu)(x) = \lim_{n\to\mathcal{F}}(T_n P_n u)(x)$ for $x \in X$, $u \in L^\infty(\mathfrak{A})$. Show that T is a linear lifting of μ. (iii) Use 363Xe(ii) and 341J to show that there is a lifting θ of μ extending every θ_n. (iv) Use this as the countable-cofinality inductive step in a proof of the Lifting Theorem (using partial liftings rather than partial lower densities, as suggested in 341Li).

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in $L^\infty(\mathfrak{A})$ such that $\inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m$ and $\sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m$ are equal to u say. Show that $\int u\,d\nu = \lim_{n\to\infty} \int u_n d\nu$.

(i) Let Σ be the family of those sets $E \subseteq [0,1]$ such that $\mu(\operatorname{int} E) = \mu\overline{E}$, where μ is Lebesgue measure. (i) Show that Σ is an algebra of subsets of $[0,1]$ and that every member of Σ is Lebesgue measurable. (ii) Show that if we identify $L^\infty(\Sigma)$ with a set of real-valued functions on $[0,1]$, as in 363Yf, then we get just the space of Riemann integrable functions. (iii) Show that if we write ν for $\mu\restriction\Sigma$, then $\int d\nu$, as defined in 363L, is just the Riemann integral.

(j) Show that a normed space over $\mathbb{C}$ has the Hahn-Banach property of 363R for complex spaces iff it is isomorphic to $L^\infty_{\mathbb{C}}(\mathfrak{A})$, as described in 363Xj, for some Dedekind complete Boolean algebra $\mathfrak{A}$.

363 Notes and comments As with $S(\mathfrak{A})$, I have chosen a definition of $L^\infty(\mathfrak{A})$ in terms of the Stone space of $\mathfrak{A}$; but as with $S(\mathfrak{A})$, this is optional (363Ya). By and large the basic properties of L^∞ are derived very naturally from those of S. The spaces $L^\infty(\mathfrak{A})$, for general Boolean algebras $\mathfrak{A}$, are not in fact particularly important; they have too few properties not shared by all the spaces $C(X)$ for compact Hausdorff X. The point at which it becomes helpful to interpret $C(X)$ as $L^\infty(\mathfrak{A})$ is when $C(X)$ is Dedekind σ-complete. The spaces X for which this is true are difficult to picture, and alternative representations of L^∞ along the lines of 363H-363I can be easier on the imagination.

For Dedekind σ-complete $\mathfrak{A}$, there is an alternative description of members of $L^\infty(\mathfrak{A})$ in terms of objects '$[\![u > \alpha]\!]$' (363Xh); I will return to this idea in the next section. For the moment I remark only that it gives an alternative approach to 363M not necessarily depending on the representation of L^∞ as a quotient $\mathcal{L}^\infty/\mathcal{V}$ nor on an analysis of a Stone space. I used a version of such an argument in the proof of 363M which I gave in Fremlin 74a, 43D.

I spend so much time on 363M not only because Dedekind completeness is one of the basic properties of any lattice, but because it offers an abstract expression of one of the central results of Chapter 24. In 243H I showed that $L^\infty(\mu)$ is always Dedekind σ-complete, and that it is Dedekind complete if μ is localizable. We can now relate this to the results of 321H and 322Be: the measure algebra of any measure is Dedekind σ-complete, and the measure algebra of a localizable measure is Dedekind complete.

The ideas of the proof of 363M can of course be rearranged in various ways. One uses 353Yb: for completely regular spaces X, $C(X)$ is Dedekind complete iff X is extremally disconnected; while for compact Hausdorff spaces, X is extremally disconnected iff it is the Stone space of a Dedekind complete algebra. With the right modification of the concept 'extremally disconnected' (314Yf), the same approach works for Dedekind σ-completeness.

363R is the 'Nachbin-Kelley theorem'; it is commonly phrased 'a normed space U has the Hahn-Banach extension property iff it is isomorphic, as normed space, to $C(X)$ for some compact extremally disconnected Hausdorff space X', but the expression in terms of L^∞ spaces seems natural in the present context. The implication in one direction (Part A of the proof) calls for nothing but a check through one of the standard proofs of the Hahn-Banach theorem to make sure that the argument applies in the generalized form. Part B of the proof has ideas in it; I have tried to set it out in a way suggesting that if you can remember the construction of the set X the rest is just a matter of a little ingenuity.

One way of trying to understand the multiple structures of L^∞ spaces is by looking at the corresponding automorphisms. We observe, for instance, that an operator T from $L^\infty(\mathfrak{A})$ to itself is a Banach algebra automorphism iff it is a Banach lattice automorphism preserving the standard order unit iff it corresponds to an automorphism of the algebra $\mathfrak{A}$ (363Xb). Of course there are Banach space automorphisms of L^∞ which do not respect the order or multiplicative structure; but they have to be closely related to algebra isomorphisms (363Ye).

I devote a couple of exercises (363Xe, 363Yg) to indications of how the ideas here are relevant to the Lifting Theorem. If you found the formulae of the proof of 341G obscure it may help to work through the parallel argument.

A lecture by W.A.J.Luxemburg on the equivalence between (i) and (iv) in 363S was one of the turning points in my mathematical apprenticeship. I introduce it here, even though the real importance of the Banach-Ulam problem lies in the metamathematical ideas it has nourished, because these formulations provide a focus for questions which arise naturally in this volume and which otherwise might prove distracting. The next group of significant ideas in this context will appear in §438.

364 L^0

My next objective is to develop an abstract construction corresponding to the $L^0(\mu)$ spaces of §241. These generalized L^0 spaces will form the basis of the work of the rest of this chapter and also the next; partly because their own properties are remarkable, but even more because they form a framework for the study of Archimedean Riesz spaces in general (see §368). There seem to be significant new difficulties, and I take the space to describe an approach which can be made essentially independent of the route through Stone spaces used in the last three sections. I embark directly on a definition in the new language (364A), and relate it to the constructions of §241 (364C-364E, 364J) and §§361-363 (364K). The ideas of Chapter 27 can also be expressed in this language; I make a start on developing the machinery for this in 364G, with the formula '$[\![u \in E]\!]$', 'the region in which u belongs to E', and some exercises (364Xd-364Xf). Following through the questions addressed in §363, I discuss Dedekind completeness in L^0 (364M-364O), properties of its multiplication (364P), the expression of the original algebra in terms of L^0 (364Q), the action of Boolean homomorphisms on L^0 (364R) and product spaces (364S). In 364T-364W I describe representations of the L^0 space of a regular open algebra.

364A Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. I will write $L^0(\mathfrak{A})$ for the set of all functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{A}$ such that

(α) $[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$,

(β) $\inf_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 0$,

(γ) $\sup_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 1$.

364B Remarks (a) My reasons for using the notation '$[\![u > \alpha]\!]$' rather than '$u(\alpha)$' will I hope become clear in the next few paragraphs. For the moment, if you think of $\mathfrak{A}$ as a σ-algebra of sets and of $L^0(\mathfrak{A})$ as the family of $\mathfrak{A}$-measurable real-valued functions, then $[\![u > \alpha]\!]$ corresponds to the set $\{x : u(x) > \alpha\}$ (364Ja).

(b) Some readers will recognise the formula '$[\![\ldots]\!]$' as belonging to the language of forcing, so that $[\![u > \alpha]\!]$ could be read as 'the Boolean value of the proposition "$u > \alpha$"'. But a vocalisation closer to my intention might be 'the region where $u > \alpha$'.

(c) Note that condition (α) of 364A automatically ensures that $[\![u > \alpha]\!] \subseteq [\![u > \alpha']\!]$ whenever $\alpha' \leq \alpha$ in $\mathbb{R}$.

(d) In fact it will sometimes be convenient to note that the conditions of 364A can be replaced by

(α') $[\![u > \alpha]\!] = \sup_{q \in \mathbb{Q}, q > \alpha} [\![u > q]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$,

(β') $\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1$;

the point being that we need look only at suprema and infima of countable subsets of $\mathfrak{A}$.

(e) In order to make sense of this construction we need to match it with an alternative route to the same object, based on σ-algebras and σ-ideals of sets, as follows.

364C Proposition Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ.

(a) Write $\mathcal{L}^0$ for the space of all Σ-measurable functions from X to $\mathbb{R}$. Then $\mathcal{L}^0$, with its linear structure, ordering and multiplication inherited from $\mathbb{R}^X$, is a Dedekind σ-complete f-algebra with multiplicative identity.

(b) Set

$$\mathcal{W} = \{f : f \in \mathcal{L}^0, \{x : f(x) \neq 0\} \in \mathcal{I}\}.$$

Then

(i) $\mathcal{W}$ is a sequentially order-closed solid linear subspace and ideal of $\mathcal{L}^0$;

(ii) the quotient space $\mathcal{L}^0/\mathcal{W}$, with its inherited linear, order and multiplicative structures, is a Dedekind σ-complete Riesz space and an f-algebra with a multiplicative identity;

(iii) for f, $g \in \mathcal{L}^0$, $f^\bullet \leq g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) > g(x)\} \in \mathcal{I}$, and $f^\bullet = g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.

proof (Compare 241A-241H.)

(a) The point is just that $\mathcal{L}^0$ is a sequentially order-closed Riesz subspace and subalgebra of $\mathbb{R}^X$. The facts we need to know – that constant functions belong to $\mathcal{L}^0$, that $f+g$, αf, $f \times g$, $\sup_{n \in \mathbb{N}} f_n$ belong to $\mathcal{L}^0$ whenever f, g, f_n do and $\{f_n : n \in \mathbb{N}\}$ is bounded above – are all covered by 121E-121F. Its multiplicative identity is of course the constant function χX.

(b)(i) The necessary verifications are all elementary.

(ii) Because $\mathcal{W}$ is a solid linear subspace of the Riesz space $\mathcal{L}^0$, the quotient inherits a Riesz space structure (352Jb); because $\mathcal{W}$ is an ideal of the ring $(\mathcal{L}^0, +, \times)$, $\mathcal{L}^0/\mathcal{W}$ inherits a multiplication; it is a commutative algebra because $\mathcal{L}^0$ is; and has a multiplicative identity $e = \chi X^\bullet$ because χX is the identity of $\mathcal{L}^0$.

To check that $\mathcal{L}^0/\mathcal{W}$ is an f-algebra it is enough to observe that, for any non-negative f, g , $h \in \mathcal{L}^0$,

$$f^\bullet \times g^\bullet = (f \times g)^\bullet \geq 0,$$

and if $f^\bullet \wedge g^\bullet = 0$ then $\{x : f(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$, so that $\{x : f(x)h(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$ and

$$(f^\bullet \times h^\bullet) \wedge g^\bullet = (h^\bullet \times f^\bullet) \wedge g^\bullet = 0.$$

Finally, $\mathcal{L}^0/\mathcal{W}$ is Dedekind σ-complete, by 353J(a-iii).

(iii) For f, $g \in \mathcal{L}^0$,

$$f^\bullet \leq g^\bullet \iff (f-g)^+ \in \mathcal{W} \iff \{x : f(x) > g(x)\} = \{x : (f-g)^+(x) \neq 0\} \in \mathcal{I}$$

(using the fact that the canonical map from $\mathcal{L}^0$ to $\mathcal{L}^0/\mathcal{W}$ is a Riesz homomorphism, so that $((f-g)^+)^\bullet = (f^\bullet - g^\bullet)^+$). Similarly

$$f^\bullet = g^\bullet \iff f - g \in \mathcal{W} \iff \{x : f(x) \neq g(x)\} = \{x : (f-g)(x) \neq 0\} \in \mathcal{I}.$$

364D Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi:\Sigma\to\mathfrak{A}$ a surjective Boolean homomorphism, with kernel a σ-ideal $\mathcal{I}$; define $\mathcal{L}^0$ and $\mathcal{W}$ as in 364C, so that $U=\mathcal{L}^0/\mathcal{W}$ is a Dedekind σ-complete f-algebra with multiplicative identity.

(a) We have a canonical bijection $T:U\to L^0=L^0(\mathfrak{A})$ defined by the formula

$$[\![Tf^\bullet>\alpha]\!]=\pi\{x:f(x)>\alpha\}$$

for every $f\in\mathcal{L}^0$, $\alpha\in\mathbb{R}$.

(b)(i) For any u, $v\in U$,

$$[\![T(u+v)>\alpha]\!]=\sup_{q\in\mathbb{Q}}[\![Tu>q]\!]\cap[\![Tv>\alpha-q]\!]$$

for every $\alpha\in\mathbb{R}$.

(ii) For any $u\in U$, $\gamma>0$,

$$[\![T(\gamma u)>\alpha]\!]=[\![Tu>\tfrac{\alpha}{\gamma}]\!]$$

for every $\alpha\in\mathbb{R}$.

(iii) For any u, $v\in U$,

$$u\le v\iff[\![Tu>\alpha]\!]\subseteq[\![Tv>\alpha]\!]\text{ for every }\alpha\in\mathbb{R}.$$

(iv) For any u, $v\in U^+$,

$$[\![T(u\times v)>\alpha]\!]=\sup_{q\in\mathbb{Q},q>0}[\![Tu>q]\!]\cap[\![Tv>\tfrac{\alpha}{q}]\!]$$

for every $\alpha\ge0$.

(v) Writing $e=(\chi X)^\bullet$ for the multiplicative identity of U, we have

$$[\![Te>\alpha]\!]=1\text{ if }\alpha<1,\ 0\text{ if }\alpha\ge1.$$

proof (a)(i) Given $f\in\mathcal{L}^0$, set $\zeta_f(\alpha)=\pi\{x:f(x)>\alpha\}$ for $\alpha\in\mathbb{R}$. Then it is easy to see that ζ_f satisfies the conditions $(\alpha)'$-$(\gamma)'$ of 364Bd, because π is sequentially order-continuous (313Qb). Moreover, if $f^\bullet=g^\bullet$ in U, then

$$\zeta_f(\alpha)\bigtriangleup\zeta_g(\alpha)=\pi(\{x:f(x)>\alpha\}\triangle\{x:g(x)>\alpha\})=0$$

for every $\alpha\in\mathbb{R}$, because

$$\{x:f(x)>\alpha\}\triangle\{x:g(x)>\alpha\}\subseteq\{x:f(x)\ne g(x)\}\in\mathcal{I},$$

and $\zeta_f=\zeta_g$. So we have a well-defined member Tu of L^0 defined by the given formula, for any $u\in U$.

(ii) Next, given $w\in L^0$, there is a $u\in\mathcal{L}^0/\mathcal{W}$ such that $Tu=w$. **P** For each $q\in\mathbb{Q}$, choose $F_q\in\Sigma$ such that $\pi F_q=[\![w>q]\!]$ in $\mathfrak{A}$. Note that if $q'\ge q$ then

$$\pi(F_{q'}\setminus F_q)=[\![u>q']\!]\setminus[\![u>q]\!]=0,$$

so $F_{q'}\setminus F_q\in\mathcal{I}$. Set

$$H=\bigcup_{q\in\mathbb{Q}}F_q\setminus\bigcap_{n\in\mathbb{N}}\bigcup_{q\in\mathbb{Q},q\ge n}F_q\in\Sigma,$$

and for $x\in X$ set

$$\begin{aligned}f(x)&=\sup\{q:q\in\mathbb{Q},\,x\in F_q\}\text{ if }x\in H,\\&=0\text{ otherwise.}\end{aligned}$$

(H is chosen just to make the formula here give a finite value for every x.) We have

$$\begin{aligned}\pi H&=\sup_{q\in\mathbb{Q}}[\![w>q]\!]\setminus\inf_{n\in\mathbb{N}}\sup_{q\in\mathbb{Q},q\ge n}[\![w>q]\!]\\&=1_{\mathfrak{A}}\setminus\inf_{n\in\mathbb{N}}[\![w>n]\!]=1_{\mathfrak{A}}\setminus0_{\mathfrak{A}}=1_{\mathfrak{A}},\end{aligned}$$

so $X\setminus H\in\mathcal{I}$. Now, for any $\alpha\in\mathbb{R}$,

$$\{x : f(x) > \alpha\} = \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \cup (X \setminus H) \text{ if } \alpha < 0$$
$$= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \setminus (X \setminus H) \text{ if } \alpha \geq 0,$$

and in either case belongs to Σ; so that $f \in \mathcal{L}^0$ and $f^\bullet$ is defined in L^0. Next, for any $\alpha \in \mathbb{R}$,

$$[\![Tf^\bullet > \alpha]\!] = \pi\{x : f(x) > \alpha\} = \pi(\bigcup_{q\in\mathbb{Q},q>\alpha} F_q)$$
$$= \sup_{q\in\mathbb{Q},q>\alpha} [\![w > q]\!] = [\![w > \alpha]\!],$$

and $Tf^\bullet = w$. **Q**

(iii) Thus T is surjective. To see that it is injective, observe that if f, $g \in \mathcal{L}^0$, then

$$Tf^\bullet = Tg^\bullet \Longrightarrow [\![Tf^\bullet > \alpha]\!] = [\![Tg^\bullet > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}$$
$$\Longrightarrow \pi\{x : f(x) > \alpha\} = \pi\{x : g(x) > \alpha\} \text{ for every } \alpha \in \mathbb{R}$$
$$\Longrightarrow \{x : f(x) > \alpha\}\triangle\{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R}$$
$$\Longrightarrow \{x : f(x) \neq g(x)\} = \bigcup_{q\in\mathbb{Q}}(\{x : f(x) > q\}\triangle\{x : g(x) > q\}) \in \mathcal{I}$$
$$\Longrightarrow f^\bullet = g^\bullet.$$

So we have the claimed bijection.

(b)(i) Let f, $g \in \mathcal{L}^0$ be such that $u = f^\bullet$, $v = g^\bullet$, so that $u + v = (f + g)^\bullet$. For any $\alpha \in \mathbb{R}$,

$$[\![T(u + v) > \alpha]\!] = \pi\{x : f(x) + g(x) > \alpha\}$$
$$= \pi(\bigcup_{q\in\mathbb{Q}}\{x : f(x) > q\} \cap \{x : g(x) > \alpha - q\})$$
$$= \sup_{q\in\mathbb{Q}} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \alpha - q\}$$

(because π is a sequentially order-continuous Boolean homomorphism)

$$= \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!].$$

(ii) Let $f \in \mathcal{L}^0$ be such that $f^\bullet = u$, so that $(\gamma f)^\bullet = \gamma u$. For any $\alpha \in \mathbb{R}$,

$$[\![T(\gamma u) > \alpha]\!] = \pi\{x : \gamma f(x) > \alpha\} = \pi\{x : f(x) > \tfrac{\alpha}{\gamma}\} = [\![Tu > \tfrac{\alpha}{\gamma}]\!].$$

(iii) Let f, $g \in \mathcal{L}^0$ be such that $f^\bullet = u$ and $g^\bullet = v$. Then

$$u \leq v \iff \{x : f(x) > g(x)\} \in \mathcal{I}$$

(see 364C(b-iii))

$$\iff \bigcup_{q\in\mathbb{Q}}\{x : f(x) > q \geq g(x)\} \in \mathcal{I}$$
$$\iff \{x : f(x) > \alpha\} \setminus \{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R}$$
$$\iff \pi\{x : f(x) > \alpha\} \setminus \pi\{x : g(x) > \alpha\} = 0 \text{ for every } \alpha$$
$$\iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha.$$

(iv) Now suppose that u, $v \geq 0$, so that they can be expressed as $f^\bullet$, $g^\bullet$ where f, $g \geq 0$ in $\mathcal{L}^0$ (351J), and $u \times v = (f \times g)^\bullet$. If $\alpha \geq 0$, then

$$\begin{aligned}[\![T(u \times v) > \alpha]\!] &= \pi\Bigl(\bigcup_{q \in \mathbb{Q}, q > 0} \{x : f(x) > q\} \cap \{x : g(x) > \frac{\alpha}{q}\}\Bigr) \\ &= \sup_{q \in \mathbb{Q}, q > 0} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \frac{\alpha}{q}\} \\ &= \sup_{q \in \mathbb{Q}, q > 0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!].\end{aligned}$$

(v) This is trivial, because

$$\begin{aligned}[\![T(\chi X)^\bullet > \alpha]\!] &= \pi\{x : (\chi X)(x) > \alpha\} \\ &= \pi X = 1 \text{ if } \alpha < 1, \\ &= \pi\emptyset = 0 \text{ if } \alpha \geq 1.\end{aligned}$$

364E Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then $L^0 = L^0(\mathfrak{A})$ has the structure of a Dedekind σ-complete f-algebra with multiplicative identity e, defined by saying

$$[\![u + v > \alpha]\!] = \sup_{q \in \mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u > \alpha]\!] = [\![u > \tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in \,]0, \infty[$ and $\alpha \in \mathbb{R}$,

$$u \leq v \iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$[\![u \times v > \alpha]\!] = \sup_{q \in \mathbb{Q}, q > 0} [\![u > q]\!] \cap [\![v > \tfrac{\alpha}{q}]\!]$$

whenever u, $v \geq 0$ in L^0 and $\alpha \geq 0$,

$$[\![e > \alpha]\!] = 1 \text{ if } \alpha < 1, \ 0 \text{ if } \alpha \geq 1.$$

proof (a) By the Loomis-Sikorski theorem (314M), we can find a set Z (the Stone space of $\mathfrak{A}$), a σ-algebra Σ of subsets of Z (the algebra generated by the open-and-closed sets and the ideal $\mathcal{M}$ of meager sets) and a surjective sequentially order-continuous Boolean homomorphism $\pi : \Sigma \to \mathfrak{A}$ (corresponding to the identification between $\mathfrak{A}$ and the quotient $\Sigma/\mathcal{M}$). Consequently, defining $\mathcal{L}^0$ and $\mathcal{W}$ as in 364C, we have a bijection between the Dedekind σ-complete f-algebra $\mathcal{L}^0/\mathcal{W}$ and L^0 (364Da). Of course this endows L^0 itself with the structure of a Dedekind σ-complete f-algebra; and 364Db tells us that the description of the algebraic operations above is consistent with this structure.

(b) In fact the f-algebra structure is completely defined by the description offered. For while scalar multiplication is not described for $\gamma \leq 0$, the assertion that L^0 is a Riesz space implies that $0u = 0$ and that $\gamma u = (-\gamma)(-u)$ for $\gamma < 0$; so if we have formulae to describe $u + v$ and γu for $\gamma > 0$, this suffices to define the linear structure of L^0. Note that we have an element $\underline{0}$ in L^0 defined by setting

$$[\![\underline{0} > \alpha]\!] = 0 \text{ if } \alpha \geq 0, \ 1 \text{ if } \alpha < 0,$$

and the formula for $u + v$ shows us that

$$[\![\underline{0} + u > \alpha]\!] = \sup_{q \in \mathbb{Q}} [\![\underline{0} > q]\!] \cap [\![u > \alpha - q]\!] = \sup_{q \in \mathbb{Q}, q < 0} [\![u > \alpha - q]\!] = [\![u > \alpha]\!]$$

for every α, so that $\underline{0}$ is the zero of L^0. As for multiplication, if L^0 is to be an f-algebra we must have

$$[\![u \times v > \alpha]\!] \supseteq [\![\underline{0} > \alpha]\!] = 1$$

whenever u, $v \in (L^0)^+$ and $\alpha < 0$, because $u \times v \geq \underline{0}$. So the formula offered is sufficient to determine $u \times v$ for non-negative u and v; and for others we know that

$$u\times v=(u^+\times v^+)-(u^+\times v^-)-(u^-\times v^+)+(u^-\times v^-),$$

so the whole of the multiplication of L^0 is defined.

364F The rest of this section will be devoted to understanding the structure just established. I start with a pair of elementary facts.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) If u, $v\in L^0=L^0(\mathfrak{A})$ and α, $\beta\in\mathbb{R}$,

$$[\![u+v>\alpha+\beta]\!]\subseteq[\![u>\alpha]\!]\cup[\![v>\beta]\!].$$

(b) If u, $v\ge 0$ in L^0 and α, $\beta\ge 0$ in $\mathbb{R}$,

$$[\![u\times v>\alpha\beta]\!]\subseteq[\![u>\alpha]\!]\cup[\![v>\beta]\!].$$

proof (a) For any $q\in\mathbb{Q}$, either $q\ge\alpha$ and $[\![u>q]\!]\subseteq[\![u>\alpha]\!]$, or $q\le\alpha$ and $[\![v>\alpha+\beta-q]\!]\subseteq[\![v>\beta]\!]$; thus in all cases

$$[\![u>q]\!]\cap[\![v>\alpha+\beta-q]\!]\subseteq[\![u>\alpha]\!]\cup[\![v>\beta]\!];$$

taking the supremum over q, we have the result.

(b) The same idea works, replacing $\alpha+\beta-q$ by $\alpha\beta/q$ for $q>0$.

364G Yet another description of L^0 is sometimes appropriate, and leads naturally to an important construction (364I).

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0=L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u\in L^0$ corresponds to $\phi\in\Phi$ iff $[\![u>\alpha]\!]=\phi(]\alpha,\infty[)$ for every $\alpha\in\mathbb{R}$.

proof (a) If $\phi\in\Phi$, then the map $\alpha\mapsto\phi(]\alpha,\infty[)$ satisfies the conditions of 364Bd, so corresponds to an element u_ϕ of L^0.

(b) If ϕ, $\psi\in\Phi$ and $u_\phi=u_\psi$, then $\phi=\psi$. **P** Set $\mathcal{A}=\{E:E\in\mathcal{B},\,\phi(E)=\psi(E)\}$. Then $\mathcal{A}$ is a σ-subalgebra of $\mathcal{B}$, because ϕ and ψ are both sequentially order-continuous Boolean homomorphisms, and contains $]\alpha,\infty[$ for every $\alpha\in\mathbb{R}$. Now $\mathcal{A}$ contains $]-\infty,\alpha]$ for every α, and therefore includes $\mathcal{B}$ (121J). But this means that $\phi=\psi$. **Q**

(c) Thus $\phi\mapsto u_\phi$ is injective. But it is also surjective. **P** As in 364E, take a set Z, a σ-algebra Σ of subsets of Z and a surjective sequentially order-continuous Boolean homomorphism $\pi:\Sigma\to\mathfrak{A}$; let $T:\mathcal{L}^0/\mathcal{W}\to L^0$ be the bijection described in 364D. If $u\in L^0$, there is an $f\in\mathcal{L}^0$ such that $Tf^\bullet=u$. Now consider $\phi E=\pi f^{-1}[E]$ for $E\in\mathcal{B}$. $f^{-1}[E]$ always belongs to Σ (121Ef), so ϕE is always well-defined; $E\mapsto f^{-1}[E]$ and π are sequentially order-continuous, so ϕ also is; and

$$\phi(]\alpha,\infty[)=\pi\{z:f(z)>\alpha\}=[\![u>\alpha]\!]$$

for every α, so $u=u_\phi$. **Q**

Thus we have the declared bijection.

364H Definition In the context of 364G, I will write $[\![u\in E]\!]$, 'the region where u takes values in E', for $\phi(E)$, where $\phi:\mathcal{B}\to\mathfrak{A}$ is the homomorphism corresponding to $u\in L^0$. Thus $[\![u>\alpha]\!]=[\![u\in\,]\alpha,\infty[\,]\!]$. In the same spirit I write $[\![u\ge\alpha]\!]$ for $[\![u\in[\alpha,\infty[\,]\!]=\inf_{\beta<\alpha}[\![u>\beta]\!]$, $[\![u\ne 0]\!]=[\![|u|>0]\!]=[\![u>0]\!]\cup[\![u<0]\!]$ and so on, so that (for instance) $[\![u=\alpha]\!]=[\![u\in\{\alpha\}]\!]=[\![u\ge\alpha]\!]\setminus[\![u>\alpha]\!]$ for $u\in L^0$, $\alpha\in\mathbb{R}$.

364I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $E\subseteq\mathbb{R}$ a Borel set, and $h:E\to\mathbb{R}$ a Borel measurable function. Then whenever $u\in L^0=L^0(\mathfrak{A})$ is such that $[\![u\in E]\!]=1$, there is an element $\bar h(u)$ of L^0 defined by saying that $[\![\bar h(u)\in F]\!]=[\![u\in h^{-1}[F]]\!]$ for every Borel set $F\subseteq\mathbb{R}$.

proof All we have to observe is that $F\mapsto[\![u\in h^{-1}[F]]\!]$ is a sequentially order-continuous Boolean homomorphism. (The condition '$[\![u\in E]\!]=1$' ensures that $[\![u\in h^{-1}[\mathbb{R}]]\!]=1$.)

364J Examples Perhaps I should spell out the most important contexts in which we apply these ideas, even though they have in effect already been mentioned.

(a) Let X be a set and Σ a σ-algebra of subsets of X. Then we may identify $L^0(\Sigma)$ with the space $\mathcal{L}^0$ of Σ-measurable real-valued functions on X. (This is the case $\mathfrak{A} = \Sigma$ of 364D.) For $f \in \mathcal{L}^0$, $[\![f \in E]\!]$ (364H) is just $f^{-1}[E]$, for any Borel set $E \subseteq \mathbb{R}$; and if h is a Borel measurable function, $\bar{h}(f)$ (364I) is just the composition hf, for any $f \in \mathcal{L}^0$.

(b) Now suppose that $\mathcal{I}$ is a σ-ideal of Σ and that $\mathfrak{A} = \Sigma/\mathcal{I}$. Then, as in 364D, we identify $L^0(\mathfrak{A})$ with a quotient $\mathcal{L}^0/\mathcal{W}$. For $f \in \mathcal{L}^0$, $[\![f^\bullet \in E]\!] = f^{-1}[E]^\bullet$, and $\bar{h}(f^\bullet) = (hf)^\bullet$, for any Borel set E and any Borel measurable function $h : \mathbb{R} \to \mathbb{R}$.

(c) In particular, if (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, then $L^0(\mathfrak{A})$ becomes identified with $L^0(\mu)$ as defined in §241.

The same remarks as in 363I apply here; the space $\mathcal{L}^0(\mu)$ of §241 is larger than the space $\mathcal{L}^0$ of the present section. But for every $f \in \mathcal{L}^0(\mu)$ there is a $g \in \mathcal{L}^0$ such that $g = f$ a.e. (241Bk), so that $L^0(\mu)$ can be identified with $\mathcal{L}^0/\mathcal{N}$, where $\mathcal{N}$ is the set of functions in $\mathcal{L}^0$ which are zero almost everywhere (241Yh).

364K Embedding S and L^∞ in L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) We have a canonical embedding of $L^\infty = L^\infty(\mathfrak{A})$ as an order-dense solid linear subspace of $L^0 = L^0(\mathfrak{A})$; it is the solid linear subspace generated by the multiplicative identity e of L^0. Consequently $S = S(\mathfrak{A})$ is also embedded as an order-dense Riesz subspace and subalgebra of L^0.

(b) This embedding respects the linear, lattice and multiplicative structures of L^∞ and S.

(c) For $a \in \mathfrak{A}$, χa, when regarded as a member of L^0, can be described by the formula

$$\begin{aligned}[\![\chi a > \alpha]\!] &= 1 \text{ if } \alpha < 0,\\ &= a \text{ if } 0 \le \alpha < 1,\\ &= 0 \text{ if } 1 \le \alpha.\end{aligned}$$

The function $\chi : \mathfrak{A} \to L^0$ is additive, injective, order-continuous and a lattice homomorphism.

(d) For every $u \in (L^0)^+$ there is a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in S such that $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$.

proof Let Z, Σ, $\mathcal{L}^0$, $\mathcal{W}$ and π be as in the proof of 364E. I defined L^∞ to be the space $C(Z)$ of continuous real-valued functions on Z (363A); but because $\mathfrak{A}$ is Dedekind σ-complete, there is an alternative representation as $\mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$ (363Hb). Put like this, we clearly have an embedding of $L^\infty \cong \mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$ in $L^0 \cong \mathcal{L}^0/\mathcal{W}$; and this embedding represents L^∞ as a Riesz subspace and subalgebra of L^0, because $\mathcal{L}^\infty$ is a Riesz subspace and subalgebra of $\mathcal{L}^0$. L^∞ becomes the solid linear subspace of L^0 generated by $(\chi Z)^\bullet = e$, because $\mathcal{L}^\infty$ is the solid linear subspace of $\mathcal{L}^0$ generated by χZ. To see that L^∞ is order-dense in L^0, we have only to note that $f = \sup_{n\in\mathbb{N}} f \wedge n\chi Z$ in $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$, and therefore (because the map $f \mapsto f^\bullet$ is sequentially order-continuous) $u = \sup_{n\in\mathbb{N}} u \wedge ne$ in L^0 for every $u \in L^0$.

To identify χa, we have the formula $\chi(\pi E) = (\chi E)^\bullet$, as in 363Hc; but this means that, if $a = \pi E$,

$$\begin{aligned}[\![\chi a > \alpha]\!] = \pi\{z : \chi E(z) > \alpha\} &= \pi Z = 1 \text{ if } \alpha < 0,\\ &= \pi E = a \text{ if } 0 \le \alpha < 1,\\ &= \pi\emptyset = 0 \text{ if } \alpha \ge 1,\end{aligned}$$

using the formula in 364Da. Evidently χ is injective.

Because S is an order-dense Riesz subspace and subalgebra of L^∞ (363C), the same embedding represents it as an order-dense Riesz subspace and subalgebra of L^0. (For 'order-dense', use 352Nc.)

Because $\chi : \mathfrak{A} \to L^\infty$ is additive, order-continuous and a lattice homomorphism (363D), and the embedding map $L^\infty \subseteq L^0$ also is, $\chi : \mathfrak{A} \to L^0$ has the same properties.

Finally, if $u \ge 0$ in L^0, we can represent it as $f^\bullet$ where $f \ge 0$ in $\mathcal{L}^0$. For $n \in \mathbb{N}$ set

$$f_n(z) = 2^{-n}k \text{ if } 2^{-n}k \le f(z) < 2^{-n}(k+1) \text{ where } 0 \le k < 4^n,$$
$$= 0 \text{ if } f(z) \ge 2^n;$$

then $\langle f_n^\bullet \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in S^+ with supremum u.

364L Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Then $S(\mathfrak{A}^f)$ can be embedded as a Riesz subspace of $L^0(\mathfrak{A})$, which is order-dense iff $(\mathfrak{A}, \bar\mu)$ is semi-finite.

proof (Recall that $\mathfrak{A}^f$ is the ring $\{a : \bar\mu a < \infty\}$.) The embedding $\mathfrak{A}^f \subseteq \mathfrak{A}$ is an injective ring homomorphism, so induces an embedding of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S(\mathfrak{A})$, by 361J. Now $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ iff $(\mathfrak{A}, \bar\mu)$ is semi-finite. **P** (i) If $(\mathfrak{A}, \bar\mu)$ is semi-finite and $v > 0$ in $S(\mathfrak{A})$, then v is expressible as $\sum_{j=0}^n \beta_j \chi b_j$ where $\beta_j \ge 0$ for each j and some $\beta_j\chi b_j$ is non-zero; now there is a non-zero $a \in \mathfrak{A}^f$ such that $a \subseteq b_j$, so that $0 < \beta_j\chi a \in S(\mathfrak{A}^f)$ and $\beta_j \chi a \le v$. As v is arbitrary, $S(\mathfrak{A}^f)$ is quasi-order-dense, therefor order-dense (353A). (ii) If $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ and $b \in \mathfrak{A} \setminus \{0\}$, there is a $u > 0$ in $S(\mathfrak{A}^f)$ such that $u \le \chi b$; now there are $\alpha > 0$, $a \in \mathfrak{A}^f \setminus \{0\}$ such that $\alpha\chi a \le u$, in which case $a \subseteq b$. **Q**

Now because $S(\mathfrak{A}^f) \subseteq S(\mathfrak{A})$ and $S(\mathfrak{A})$ is order-dense in $L^0(\mathfrak{A})$, we must have

$$S(\mathfrak{A}^f) \text{ is order-dense in } L^0(\mathfrak{A}) \iff S(\mathfrak{A}^f) \text{ is order-dense in } S(\mathfrak{A})$$
$$\iff (\mathfrak{A}, \bar\mu) \text{ is semi-finite.}$$

364M Suprema and infima in L^0 We know that any $L^0(\mathfrak{A})$ is a Dedekind σ-complete partially ordered set. There is a useful description of suprema for this ordering, as follows.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and A a subset of $L^0 = L^0(\mathfrak{A})$.

(a) A is bounded above in L^0 iff there is a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, with infimum 0, such that $[\![u > n]\!] \subseteq c_n$ for every $u \in A$.

(b) If A is non-empty, then A has a supremum in L^0 iff $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$ and $\inf_{n\in\mathbb{N}} c_n = 0$; and in this case $c_\alpha = [\![\sup A > \alpha]\!]$ for every α.

(c) If A is non-empty and bounded above, then A has a supremum in L^0 iff $\sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$.

proof (a)(i) If A has an upper bound u_0, set $c_n = [\![u_0 > n]\!]$ for each n; then $\langle c_n\rangle_{n\in\mathbb{N}}$ satisfies the conditions.

(ii) If $\langle c_n\rangle_{n\in\mathbb{N}}$ satisfies the conditions, set

$$\phi(\alpha) = 1 \text{ if } \alpha < 0,$$
$$= \inf_{i\le n} c_i \text{ if } n \in \mathbb{N},\ \alpha \in [n, n+1[\,.$$

Then it is easy to check that ϕ satisfies the conditions of 364A, since $\inf_{n\in\mathbb{N}} c_n = 0$. So there is a $u_0 \in L^0$ such that $\phi(\alpha) = [\![u_0 > \alpha]\!]$ for each α. Now, given $u \in A$ and $\alpha \in \mathbb{R}$,

$$[\![u > \alpha]\!] \subseteq 1 = [\![u_0 > \alpha]\!] \text{ if } \alpha < 0,$$
$$\subseteq \inf_{i\le n} [\![u > i]\!] \subseteq \inf_{i\le n} c_i = [\![u_0 > \alpha]\!] \text{ if } n \in \mathbb{N},\ \alpha \in [n, n+1[\,.$$

Thus u_0 is an upper bound for A in L^0.

(b)(i) Suppose that $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every α, and that $\inf_{n\in\mathbb{N}} c_n = 0$. Then, for any α,

$$\sup_{q\in\mathbb{Q},q>\alpha} c_q = \sup_{u\in A,q\in\mathbb{Q},q>\alpha} [\![u > q]\!] = \sup_{u\in A} [\![u > q]\!] = c_\alpha.$$

Also, we are supposing that A contains some u_0, so that

$$\sup_{n\in\mathbb{N}} c_{-n} \supseteq \sup_{n\in\mathbb{N}} [\![u_0 > -n]\!] = 1.$$

Accordingly there is a $u^* \in L^0$ such that $[\![u^* > \alpha]\!] = c_\alpha$ for every $\alpha \in \mathbb{R}$. But now, for $v \in L^0$,

$$\begin{aligned} v \text{ is an upper bound for } A &\iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } u \in A,\ \alpha \in \mathbb{R} \\ &\iff [\![u^* > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \\ &\iff u^* \le v, \end{aligned}$$

so that $u^* = \sup A$ in L^0.

(ii) Now suppose that $u^* = \sup A$ is defined in L^0. Of course $[\![u^* > \alpha]\!]$ must be an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$ for every α. **?** Suppose we have an α for which it is not the least upper bound, that is, there is a $c \subset [\![u^* > \alpha]\!]$ which is an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$. Define $\phi : \mathbb{R} \to \mathfrak{A}$ by setting

$$\begin{aligned} \phi(\beta) &= c \cap [\![u^* > \beta]\!] \text{ if } \beta \ge \alpha, \\ &= [\![u^* > \beta]\!] \text{ if } \beta < \alpha. \end{aligned}$$

It is easy to see that ϕ satisfies the conditions of 364A (we need the distributive law 313Ba to check that $\phi(\beta) = \sup_{\gamma>\beta} \phi(\gamma)$ if $\beta \ge \alpha$), so corresponds to a member v of L^0. But we now find that v is an upper bound for A (because if $u \in A$, $\beta \ge \alpha$ then

$$[\![u > \beta]\!] \subseteq [\![u > \alpha]\!] \cap [\![u^* > \beta]\!] \subseteq c \cap [\![u^* > \beta]\!] = [\![v > \beta]\!],)$$

that $v \le u^*$ and that $v \ne u^*$ (because $[\![v > \alpha]\!] = c \ne [\![u^* > \alpha]\!]$); but this is impossible, because u^* is supposed to be the supremum of A. **X** Thus if $u^* = \sup A$ is defined in L^0, then $\sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$. Also, of course,

$$\inf_{n\in\mathbb{N}} \sup_{u\in A} [\![u > n]\!] = \inf_{n\in\mathbb{N}} [\![u^* > n]\!] = 0.$$

(c) This is now easy. If A has a supremum, then surely it satisfies the condition, by (b). If A satisfies the condition, then we have a family $\langle c_\alpha\rangle_{\alpha\in\mathbb{R}}$ as required in (b); but also, by (a) or otherwise, there is a sequence $\langle c'_n\rangle_{n\in\mathbb{N}}$ such that $c_n \subseteq c'_n$ for every n and $\inf_{n\in\mathbb{N}} c'_n = 0$, so $\inf_{n\in\mathbb{N}} c_n$ is also 0, and both conditions in (b) are satisfied, so A has a supremum.

364N We do not have such a simple formula for general infima (though see 364Xl), but the following facts are useful.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) If u, $v \in L^0 = L^0(\mathfrak{A})$, then $[\![u \wedge v > \alpha]\!] = [\![u > \alpha]\!] \cap [\![v > \alpha]\!]$ for every $\alpha \in \mathbb{R}$.

(b) If A is a non-empty subset of $(L^0)^+$, then $\inf A = 0$ in L^0 iff $\inf_{u\in A} [\![u > \alpha]\!] = 0$ in $\mathfrak{A}$ for every $\alpha > 0$.

proof (a) Take Z, $\mathcal{L}^0$ and π as in the proof of 364E. Express u as $f^\bullet$, v as $g^\bullet$ where f, $g \in \mathcal{L}^0$, so that $u \wedge v = (f \wedge g)^\bullet$, because the canonical map from $\mathcal{L}^0$ to L^0 is a Riesz homomorphism (351J). Then

$$\begin{aligned} [\![u \wedge v > \alpha]\!] &= \pi\{z : \min(f(z), g(z)) > \alpha\} = \pi(\{z : f(z) > \alpha\} \cap \{z : g(z) > \alpha\}) \\ &= \pi\{z : f(z) > \alpha\} \cap \pi\{z : g(z) > \alpha\} = [\![u > \alpha]\!] \cap [\![v > \alpha]\!] \end{aligned}$$

for every α.

(b)(i) If $\inf_{u\in A} [\![u > \alpha]\!] = 0$ for every $\alpha > 0$, and v is any lower bound for A, then $[\![v > \alpha]\!]$ must be 0 for every $\alpha > 0$, so that $[\![v > 0]\!] = 0$; now since $[\![0 > \alpha]\!] = 0$ for $\alpha \ge 0$, 1 for $\alpha < 0$, $v \le 0$. As v is arbitrary, $\inf A = 0$.

(ii) If $\alpha > 0$ is such that $\inf_{u\in A} [\![u > \alpha]\!]$ is undefined, or not equal to 0, let $c \in \mathfrak{A}$ be such that $0 \ne c \subseteq [\![u > \alpha]\!]$ for every $u \in A$, and consider $v = \alpha\chi c$. Then $[\![v > \beta]\!] = [\![\chi c > \frac{\beta}{\alpha}]\!]$ is 1 if $\beta < 0$, c if $0 \le \beta < \alpha$ and 0 if $\beta \ge \alpha$. If $u \in A$ then $[\![u > \beta]\!]$ is 1 if $\beta < 0$ (since $u \ge 0$), at least $[\![u > \alpha]\!] \supseteq c$ if $0 \le \beta < \alpha$, and always includes 0; so that $v \le u$. As u is arbitrary, $\inf A$ is either undefined in L^0 or not 0.

364O Now we have a reward for our labour, in that the following basic theorem is easy.

Theorem For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0 = L^0(\mathfrak{A})$ is Dedekind complete iff $\mathfrak{A}$ is.

proof The description of suprema in 364Mc makes it obvious that if $\mathfrak{A}$ is Dedekind complete, so that $\sup_{u\in A} [\![u > \alpha]\!]$ is always defined, then L^0 must be Dedekind complete. On the other hand, if L^0 is Dedekind complete, then so is $L^\infty(\mathfrak{A})$ (by 364K and 353J(b-i)), so that $\mathfrak{A}$ is also Dedekind complete, by 363Mb.

364P The multiplication of L^0 I have already observed that L^0 is always an f-algebra with identity; in particular (because L^0 is surely Archimedean) the map $u \mapsto u \times v$ is order-continuous for every $v \geq 0$ (353Oa), and multiplication is commutative (353Ob, or otherwise). The multiplicative identity is $\chi 1$ (364E, 364Kc). By 353Pb, or otherwise, $u \times v = 0$ iff $|u| \wedge |v| = 0$. There is one special feature of multiplication in L^0 which I can mention here.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then an element u of L^0 has a multiplicative inverse in L^0 iff $|u|$ is a weak order unit in L^0 iff $[\![|u| > 0]\!] = 1$.

proof If u is invertible, then $|u|$ is a weak order unit, by 353Pc or otherwise. In this case, setting $c = 1 \setminus [\![|u| > 0]\!]$, we see that

$$[\![|u| \wedge \chi c > 0]\!] = [\![|u| > 0]\!] \cap c = 0$$

(364Na), so that $|u| \wedge \chi c \leq 0$ and $\chi c = 0$, that is, $c = 0$; so $[\![|u| > 0]\!]$ must be 1. To complete the circuit, suppose that $[\![|u| > 0]\!] = 1$. Let Z, Σ, $\mathcal{L}^0$, π, $\mathcal{M}$ be as in the proof of 364E, and $S : \mathcal{L}^0 \to L^0$ the canonical map, so that $[\![Sh > \alpha]\!] = \pi\{z : h(z) > \alpha\}$ for every $h \in \mathcal{L}^0$, $\alpha \in \mathbb{R}$. Express u as Sf where $f \in \mathcal{L}^0$. Then $\pi\{z : |f(z)| > 0\} = [\![Sf > 0]\!] = 1$, so $\{z : f(z) = 0\} \in \mathcal{M}$. Set

$$g(z) = \frac{1}{f(z)} \text{ if } f(z) \neq 0, \quad g(z) = 0 \text{ if } f(z) = 0.$$

Then $\{z : f(z)g(z) \neq 1\} \in \mathcal{M}$ so

$$u \times Sg = S(f \times g) = S(\chi Z) = \chi 1$$

and u is invertible.

Remark The repeated phrase 'by 353x or otherwise' reflects the fact that the abstract methods there can all be replaced in this case by simple direct arguments based on the construction in 364C-364E.

364Q Recovering the algebra: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the band of $L^0 = L^0(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands of $L^0(\mathfrak{A})$.

proof I copy from the argument for 363J, itself based on 361K. If $a \in \mathfrak{A}$, $w \in L^0$ then $w \times \chi a \in V_a$. **P** If $v \in V_a^\perp$ then $|\chi a| \wedge |v| = 0$, so $\chi a \times v = 0$, so $(w \times \chi a) \times v = 0$, so $|w \times \chi a| \wedge |v| = 0$; thus $w \times \chi a \in V_a^{\perp\perp}$, which is equal to V_a because L^0 is Archimedean (353B). **Q** Now, if $a \in \mathfrak{A}$, $u \in V_a$ and $v \in V_{1\setminus a}$, then $|u| \wedge |v| = 0$ because $\chi a \wedge \chi(1 \setminus a) = 0$; and if $w \in L^0(\mathfrak{A})$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_1 + V_{1\setminus a}.$$

So V_a and $V_{1\setminus a}$ are complementary projection bands in L^0. Next, if $U \subseteq L^0$ is a projection band, then $\chi 1$ is expressible as $u + v = u \vee v$ where $u \in U$, $v \in U^\perp$. Setting $a = [\![u > \frac{1}{2}]\!]$, $a' = [\![v > \frac{1}{2}]\!]$ we must have $a \cup a' = 1$, $a \cap a' = 0$ (using 364M and 364Na), so that $a' = 1 \setminus a$; also $\frac{1}{2}\chi a \leq u$, so that $\chi a \in U$, and similarly $\chi(1 \setminus a) \in U^\perp$. In this case $V_a \subseteq U$ and $V_{1\setminus a} \subseteq U^\perp$, so U must be V_a precisely. Thus $a \mapsto V_a$ is surjective. Finally, just as in 361K, $a \subseteq b \iff V_a \subseteq V_b$, so we have a Boolean isomorphism.

364R I come at last to the result corresponding to 361J and 363F.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism.

(a) We have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$$

for every $\alpha \in \mathbb{R}$, $u \in L^0(\mathfrak{A})$.

(b) Defining $\chi a \in L^0(\mathfrak{A})$ as in 364K, $T_\pi(\chi a) = \chi(\pi a)$ in $L^0(\mathfrak{B})$ for every $a \in \mathfrak{A}$. If we regard $L^\infty(\mathfrak{A})$ and $L^\infty(\mathfrak{B})$ as embedded in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, then T_π, as defined here, agrees on $L^\infty(\mathfrak{A})$ with T_π as defined in 363F.

(c) T_π is order-continuous iff π is order-continuous, injective iff π is injective, surjective iff π is surjective.

(d) $[\![T_\pi u \in E]\!] = \pi[\![u \in E]\!]$ for every $u \in L^0(\mathfrak{A})$ and every Borel set $E \subseteq \mathbb{R}$; consequently $\bar{h}T_\pi = T_\pi\bar{h}$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$, writing $\bar{h}$ indifferently for the associated maps from $L^0(\mathfrak{A})$ to itself and from $L^0(\mathfrak{B})$ to itself (364I).

(e) If $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ another sequentially order-continuous Boolean homomorphism then $T_{\theta\pi} = T_\theta T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$.

proof I write T for T_π.

(a)(i) To see that Tu is well-defined in $L^0(\mathfrak{B})$ for every $u \in L^0(\mathfrak{A})$, all we need to do is to check that the map $\alpha \mapsto \pi[\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{B}$ satisfies the conditions of 364Bd, and this is easy, because π preserves all countable suprema and infima.

(ii) To see that T is linear and order-preserving and multiplicative, we can use the formulae of 364E. For instance, if u, $v \in L^0(\mathfrak{A})$, then

$$[\![Tu + Tv > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!] = \sup_{q\in\mathbb{Q}} \pi[\![u > q]\!] \cap \pi[\![v > \alpha - q]\!]$$
$$= \pi(\sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!]) = \pi[\![u + v > \alpha]\!] = [\![T(u + v) > \alpha]\!]$$

for every $\alpha \in \mathbb{R}$, so that $Tu + Tv = T(u + v)$. In the same way,

$$T(\gamma u) = \gamma Tu \text{ whenever } \gamma > 0,$$

$$Tu \le Tv \text{ whenever } u \le v,$$

$$Tu \times Tv = T(u \times v) \text{ whenever } u,\, v \ge 0,$$

so that, using the distributive laws, T is linear and multiplicative.

To see that T is a sequentially order-continuous Riesz homomorphism, suppose that $A \subseteq L^0(\mathfrak{A})$ is a countable non-empty set with a supremum $u^* \in L^0(\mathfrak{A})$; then $T[A]$ is a non-empty subset of $L^0(\mathfrak{B})$ with an upper bound Tu^*, and

$$\sup_{u\in A} [\![Tu > \alpha]\!] = \sup_{u\in A} \pi[\![u > \alpha]\!] = \pi(\sup_{u\in A} [\![u > \alpha]\!]) = \pi[\![u^* > \alpha]\!]$$

(using 364M)

$$= [\![Tu^* > \alpha]\!]$$

for every $\alpha \in \mathbb{R}$. So using 364M again, $Tu^* = \sup_{u\in A} Tu$. Now this is true, in particular, for doubleton sets A, so that T is a Riesz homomorphism; and also for non-decreasing sequences, so that T is sequentially order-continuous.

(b) The identification of $T(\chi a)$ with $\chi(\pi a)$ is another almost trivial verification. It follows that T agrees with the map of 363F on $S(\mathfrak{A})$, if we think of $S(\mathfrak{A})$ as a subspace of $L^0(\mathfrak{A})$. Next, if $u \in L^\infty(\mathfrak{A}) \subseteq L^0(\mathfrak{A})$, and $\gamma = \|u\|_\infty$, then $|u| \le \gamma\chi 1_{\mathfrak{A}}$, so that $|Tu| \le \gamma\chi 1_{\mathfrak{B}}$, and $Tu \in L^\infty(\mathfrak{B})$, with $\|Tu\|_\infty \le \gamma$. Thus $T{\restriction}L^\infty(\mathfrak{A})$ has norm at most 1. As it agrees with the map of 363F on $S(\mathfrak{A})$, which is $\| \|_\infty$-dense in $L^\infty(\mathfrak{A})$ (363C), and both are continuous, they must agree on the whole of $L^\infty(\mathfrak{A})$.

(c)(i)($\boldsymbol{\alpha}$) Suppose that π is order-continuous, and that $A \subseteq L^0(\mathfrak{A})$ is a non-empty set with a supremum $u^* \in L^0(\mathfrak{A})$. Then for any $\alpha \in \mathbb{R}$,

$$[\![Tu^* > \alpha]\!] = \pi[\![u^* > \alpha]\!] = \pi(\sup_{u\in A} [\![u > \alpha]\!])$$

(by 364M)

$$= \sup_{u\in A} \pi[\![u > \alpha]\!]$$

(because π is order-continuous)

$$= \sup_{u\in A} [\![Tu > \alpha]\!].$$

As α is arbitrary, $Tu^* = \sup T[A]$, by 364M again. As A is arbitrary, T is order-continuous (351Ga).

(β) Now suppose that T is order-continuous and that $A \subseteq \mathfrak{A}$ is a non-empty set with supremum c in $\mathfrak{A}$. Then $\chi c = \sup_{a\in A} \chi a$ (364Kc) so

$$\chi(\pi c) = T(\chi c) = \sup_{a\in A} T(\chi a) = \sup_{a\in A} \chi(\pi a).$$

But now

$$\pi c = [\![\chi(\pi c) > 0]\!] = \sup_{a\in A} [\![\chi(\pi a) > 0]\!] = \sup_{a\in A} \pi a.$$

As A is arbitrary, π is order-continuous.

(ii)(α) If π is injective and u, v are distinct elements of $L^0(\mathfrak{A})$, then there must be some α such that $[\![u > \alpha]\!] \neq [\![v > \alpha]\!]$, in which case $[\![Tu > \alpha]\!] \neq [\![Tv > \alpha]\!]$ and $Tu \neq Tv$.

(β) Now suppose that T is injective. It is easy to see that $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ is injective, so that $T\chi : \mathfrak{A} \to L^0(\mathfrak{B})$ is injective; but this is the same as $\chi\pi$ (by (b)), so π must also be injective.

(iii)(α) Suppose that π is surjective. Let Σ be a σ-algebra of sets such that there is a sequentially order-continuous Boolean surjection $\phi : \Sigma \to \mathfrak{A}$. Then $\pi\phi : \Sigma \to \mathfrak{B}$ is surjective. So given $w \in L^0(\mathfrak{B})$, there is an $f \in \mathcal{L}^0(\Sigma)$ such that $[\![w > \alpha]\!] = \pi\phi\{x : f(x) > \alpha\}$ for every $\alpha \in \mathbb{R}$ (364D). But, also by 364D, there is a $u \in L^0(\mathfrak{A})$ such that $[\![u > \alpha]\!] = \phi\{x : f(x) > \alpha\}$ for every α. And now of course $Tu = w$. As w is arbitrary, T is surjective.

(β) If T is surjective, and $b \in \mathfrak{B}$, there must be some $u \in L^0(\mathfrak{A})$ such that $Tu = \chi b$. Now set $a = [\![u > 0]\!]$ and see that $\pi a = [\![\chi b > 0]\!] = b$. As b is arbitrary, π is surjective.

(d) The map $E \mapsto \pi[\![u \in E]\!]$ is a sequentially order-continuous Boolean homomorphism, equal to $[\![Tu \in E]\!]$ when E is of the form $]\alpha, \infty[$; so by 364G the two are equal for all Borel sets E.

If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, $u \in L^0(\mathfrak{A})$ and $E \subseteq \mathbb{R}$ is a Borel set, then

$$\begin{aligned}[\![\bar{h}(Tu) \in E]\!] &= [\![Tu \in h^{-1}[E]]\!] = \pi[\![u \in h^{-1}[E]]\!] \\ &= \pi[\![\bar{h}(u) \in E]\!] = [\![T(\bar{h}(u)) \in E]\!].\end{aligned}$$

As E and u are arbitrary, $T\bar{h} = \bar{h}T$.

(e) This is immediate from the definitions.

364S Products: Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Dedekind σ-complete Boolean algebras, with simple product $\mathfrak{A}$. If $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ is the coordinate map for each i, and $T_i : L^0(\mathfrak{A}) \to L^0(\mathfrak{A}_i)$ the corresponding homomorphism, then $u \mapsto Tu = \langle T_i u\rangle_{i\in I} : L^0(\mathfrak{A}) \to \prod_{i\in I} L^0(\mathfrak{A}_i)$ is a multiplicative Riesz space isomorphism, so $L^0(\mathfrak{A})$ may be identified with the f-algebra product $\prod_{i\in I} L^0(\mathfrak{A}_i)$ (352Wc).

proof Because each π_i is a surjective order-continuous Boolean homomorphism, 364R assures us that there are corresponding surjective multiplicative Riesz homomorphisms T_i. So all we need to check is that the multiplicative Riesz homomorphism $T : L^0(\mathfrak{A}) \to \prod_{i\in I} L^0(\mathfrak{A}_i)$ is a bijection.

If u, $v \in L^0(\mathfrak{A})$ are distinct, there must be some $\alpha \in \mathbb{R}$ such that $[\![Tu > \alpha]\!] \neq [\![Tv > \alpha]\!]$. In this case there must be an $i \in I$ such that $\pi_i[\![Tu > \alpha]\!] \neq \pi_i[\![Tv > \alpha]\!]$, that is, $[\![T_i u > \alpha]\!] \neq [\![T_i v > \alpha]\!]$. So $T_i u \neq T_i v$ and $Tu \neq Tv$. As u, v are arbitrary, T is injective.

If $w = \langle w_i\rangle_{i\in I}$ is any member of $\prod_{i\in I} L^0(\mathfrak{A}_i)$, then for $\alpha \in \mathbb{R}$ set

$$\phi(\alpha) = \langle[\![w_i > \alpha]\!]\rangle_{i\in I} \in \mathfrak{A}.$$

It is easy to check that ϕ satisfies the conditions of 364A, because, for instance,

$$\sup_{\beta>\alpha} \pi_i\phi(\beta) = \sup_{\beta>\alpha} [\![w_i > \beta]\!] = [\![w_i > \alpha]\!] = \pi_i\phi(\alpha)$$

for every i, so that $\sup_{\beta>\alpha} \phi(\beta) = \phi(\alpha)$, for every $\alpha \in \mathbb{R}$; and the other two conditions are also satisfied because they are satisfied coordinate-by-coordinate. So there is a $u \in L^0(\mathfrak{A})$ such that $\phi(\alpha) = [\![u > \alpha]\!]$ for every α, that is, $\pi_i[\![u > \alpha]\!] = [\![w_i > \alpha]\!]$ for all α, i, that is, $T_i u = w_i$ for every i, that is, $Tu = w$. As w is arbitrary, T is surjective and we are done.

***364T Regular open algebras** I noted in 314P that for every topological space X there is a corresponding Dedekind complete Boolean algebra $\mathfrak{G}$ of regular open sets. We have an identification of $L^0(\mathfrak{G})$ as a space of equivalence classes of functions, different in kind from the representations above, as follows. This is hard work (especially if we do it in full generality), but instructive. I start with a temporary definition.

Definition Let $(X, \mathfrak{T})$ be a topological space, $f : X \to \mathbb{R}$ a function. For $x \in X$ write

$$\omega(f, x) = \inf_{G\in\mathfrak{T}, x\in G} \sup_{y,z\in G} |f(y) - f(z)|$$

(allowing ∞).

***364U Theorem** Let X be any topological space, and $\mathfrak{G}$ its regular open algebra. Let U be the set of functions $f : X \to \mathbb{R}$ such that $\{x : \omega(f, x) < \epsilon\}$ is dense in X for every $\epsilon > 0$. Then U is a Riesz subspace of $\mathbb{R}^X$, closed under multiplication, and we have a surjective multiplicative Riesz homomorphism $T : U \to L^0(\mathfrak{G})$ defined by writing

$$[\![Tf > \alpha]\!] = \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}},$$

the supremum being taken in $\mathfrak{G}$, for every $\alpha \in \mathbb{R}$, $f \in U$. The kernel of T is the set W of functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : |f(x)| \le \epsilon\}$ is dense for every $\epsilon > 0$, so $L^0(\mathfrak{G})$ can be identified, as f-algebra, with the quotient space U/W.

proof (a)(i)($\boldsymbol{\alpha}$) The first thing to observe is that for any $f \in \mathbb{R}^X$, $\epsilon > 0$ the set

$$\{x : \omega(f, x) < \epsilon\} = \bigcup\{G : G \subseteq X \text{ is open and non-empty} \\ \text{and } \sup_{y,z\in G} |f(y) - f(z)| < \epsilon\}$$

is open.

($\boldsymbol{\beta}$) Next, it is easy to see that

$$\omega(f + g, x) \le \omega(f, x) + \omega(g, x),$$

$$\omega(\gamma f, x) = |\gamma|\omega(f, x),$$

$$\omega(|f|, x) \le \omega(f, x),$$

for all f, $g \in \mathbb{R}^X$ and $\gamma \in \mathbb{R}$.

($\boldsymbol{\gamma}$) Thirdly, it is useful to know that if $f \in U$ and $G \subseteq X$ is a non-empty open set, then there is a non-empty open set $G' \subseteq G$ on which f is bounded. **P** Take any $x_0 \in G$ such that $\omega(f, x_0) < 1$; then there is a non-empty open set G' containing x_0 such that $|f(y) - f(z)| < 1$ for all y, $z \in G'$, and we may suppose that $G' \subseteq G$. But now $|f(x)| \le 1 + |f(x_0)|$ for every $x \in G'$. **Q**

(ii) So if f, $g \in U$ and $\gamma \in \mathbb{R}$ then

$$\{x : \omega(f + g, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \tfrac{1}{2}\epsilon\} \cap \{x : \omega(g, x) < \tfrac{1}{2}\epsilon\}$$

is the intersection of two dense open sets and is therefore dense, while

$$\{x : \omega(\gamma f, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \tfrac{\epsilon}{1+|\gamma|}\},$$

$$\{x : \omega(|f|, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \epsilon\}$$

are also dense. As ϵ is arbitrary, $f + g$, γf and $|f|$ all belong to U; as f, g and γ are arbitrary, U is a Riesz subspace of $\mathbb{R}^X$.

(iii) If f, $g \in U$ then $f \times g \in U$. **P** Take $\epsilon > 0$ and let G_0 be a non-empty open subset of X. By the last remark in (i) above, there is a non-empty open set $G_1 \subseteq G_0$ such that $|f| \vee |g|$ is bounded on G_1; say $\max(|f(x)|, |g(x)|) \le \gamma$ for every $x \in G_1$.

Set $\delta=\min(1,\frac{\epsilon}{2(1+\gamma)})>0$. Then there is an $x\in G_1$ such that $\omega(f,x)<\delta$ and $\omega(g,x)<\delta$. Let H, H' be open sets containing x such that $|f(y)-f(z)|<\delta$ for all y, $z\in H$ and $|g(y)-g(z)|<\delta$ for all y, $z\in H'$. Consider $G=G_1\cap H\cap H'$. This is an open set containing x, and if y, $z\in G$ then

$$\begin{aligned}|f(y)g(y)-f(z)g(z)|&\le|f(y)-f(z)||g(y)-g(z)|\\&\qquad+|f(y)-f(z)||g(z)|+|f(z)||g(y)-g(z)|\\&\le\delta^2+\delta\gamma+\gamma\delta.\end{aligned}$$

Accordingly

$$\omega(f\times g,x)\le\delta(1+2\gamma)<\epsilon,$$

while $x\in G_0$. As G_0 is arbitrary, $\{x:\omega(f\times g,x)<\epsilon\}$ is dense; as ϵ is arbitrary, $f\times g\in U$. **Q**

Thus U is a subalgebra of $\mathbb{R}^X$.

(b) Now, for $f\in U$, consider the map $\phi_f:\mathbb{R}\to\mathfrak{G}$ defined by setting

$$\phi_f(\alpha)=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}$$

for every $\alpha\in\mathbb{R}$. Then ϕ_f satisfies the conditions of 364A. **P** (See 314P for the calculation of suprema and infima in $\mathfrak{G}$.) (i) If $\alpha\in\mathbb{R}$ then

$$\begin{aligned}\phi_f(\alpha)&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}=\sup_{\gamma>\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\gamma\}}\\&=\sup_{\beta>\alpha}\sup_{\gamma>\beta}\operatorname{int}\overline{\{x:f(x)>\gamma\}}=\sup_{\beta>\alpha}\phi_f(\alpha).\end{aligned}$$

(ii) If $G_0\subseteq X$ is a non-empty open set, then there is a non-empty open set $G_1\subseteq G_0$ such that f is bounded on G_1; say $|f(x)|<\gamma$ for every $x\in G_1$. If $\beta>\gamma$ then G_1 does not meet $\{x:f(x)>\beta\}$, so $G_1\cap\operatorname{int}\overline{\{x:f(x)>\gamma\}}=\emptyset$; as β is arbitrary, $G_1\cap\phi_f(\gamma)=\emptyset$ and $G_0\not\subseteq\inf_{\alpha\in\mathbb{R}}\phi_f(\alpha)$. On the other hand, $G_1\subseteq\{x:f(x)>-\gamma\}$, so

$$G_1\subseteq\operatorname{int}\overline{\{x:f(x)>-\gamma\}}\subseteq\phi_f(-\gamma)$$

and $G_0\cap\sup_{\alpha\in\mathbb{R}}\phi_f(\alpha)\ne\emptyset$. As G_0 is arbitrary, $\inf_{\alpha\in\mathbb{R}}\phi_f(\alpha)=\emptyset$ and $\sup_{\alpha\in\mathbb{R}}\phi_f(\alpha)=X$. **Q**

(c) Thus we have a map $T:U\to L^0=L^0(\mathfrak{G})$ defined by setting $[\![Tf>\alpha]\!]=\phi_f(\alpha)$ for every $\alpha\in\mathbb{R}$, $f\in U$.

It is worth noting that

$$\{x:f(x)>\alpha+\omega(f,x)\}\subseteq[\![Tf>\alpha]\!]\subseteq\{x:f(x)+\omega(f,x)\ge\alpha\}$$

for every $f\in U$, $\alpha\in\mathbb{R}$. **P** (i) If $f(x)>\alpha+\omega(f,x)$, set $\delta=\frac{1}{2}(f(x)-\alpha-\omega(f,x))>0$. Then there is an open set G containing x such that $|f(y)-f(z)|<\omega(f,x)+\delta$ for every y, $z\in G$, so that $f(y)>\alpha+\delta$ for every $y\in G$, and

$$x\in\operatorname{int}\{y:f(y)>\alpha+\delta\}\subseteq[\![Tf>\alpha]\!].$$

(ii) If $f(x)+\omega(f,x)<\alpha$, set $\delta=\frac{1}{2}(\alpha-f(x)-\omega(f,x))>0$; then there is an open neighbourhood G of x such that $|f(y)-f(z)|<\omega(f,x)+\delta$ for every y, $z\in G$, so that $f(y)<\alpha$ for every $y\in G$. Accordingly G does not meet $\{y:f(y)>\beta\}$ nor $\overline{\{y:f(y)>\beta\}}$ for any $\beta>\alpha$, $G\cap[\![Tf>\alpha]\!]=\emptyset$ and $x\notin[\![Tf>\alpha]\!]$. **Q**

(d) T is additive. **P** Let f, $g\in U$ and $\alpha<\beta\in\mathbb{R}$. Set $\delta=\frac{1}{5}(\beta-\alpha)>0$, $H=\{x:\omega(f,x)<\delta,\ \omega(g,x)<\delta\}$; then H is the intersection of two dense open sets, so is itself dense and open.

(i) If $x\in H\cap[\![T(f+g)>\beta]\!]$, then $(f+g)(x)+\omega(f+g,x)\ge\beta$; but $\omega(f+g,x)\le2\delta$ (see (a-i-β) above), so $f(x)+g(x)\ge\beta-2\delta>\alpha+2\delta$ and there is a $q\in\mathbb{Q}$ such that

$$f(x)>q+\delta\ge q+\omega(f,x),\quad g(x)>\alpha-q+\delta\ge\alpha-q+\omega(g,x).$$

Accordingly

$$x\in[\![Tf>q]\!]\cap[\![Tg>\alpha-q]\!]\subseteq[\![Tf+Tg>\alpha]\!].$$

Thus $H\cap[\![T(f+g)>\beta]\!]\subseteq[\![Tf+Tg>\alpha]\!]$. Because H is dense, $[\![T(f+g)>\beta]\!]\subseteq[\![Tf+Tg>\alpha]\!]$.

(ii) If $x\in H$, then

$$\begin{aligned}x\in\bigcup_{q\in\mathbb{Q}}&([\![Tf>q]\!]\cap[\![Tg>\beta-q]\!])\\&\Longrightarrow\exists\,q\in\mathbb{Q},\ f(x)+\omega(f,x)\ge q,\ g(x)+\omega(g,x)\ge\beta-q\\&\Longrightarrow f(x)+g(x)+2\delta\ge\beta\\&\Longrightarrow(f+g)(x)\ge\alpha+3\delta>\alpha+\omega(f+g,x)\\&\Longrightarrow x\in[\![T(f+g)>\alpha]\!].\end{aligned}$$

Thus

$$H\cap\textstyle\bigcup_{q\in\mathbb{Q}}([\![Tf>q]\!]\cap[\![Tg>\beta-q]\!])\subseteq[\![T(f+g)>\alpha]\!].$$

Because H is dense and $\bigcup_{q\in\mathbb{Q}}([\![Tf>q]\!]\cap[\![Tg>\beta-q]\!])$ is open,

$$\begin{aligned}[\![Tf+Tg>\beta]\!]&=\operatorname{int}\overline{\bigcup_{q\in\mathbb{Q}}[\![Tf>q]\!]\cap[\![Tg>\beta-q]\!]}\\&\subseteq\operatorname{int}\overline{[\![T(f+g)>\alpha]\!]}=[\![T(f+g)>\alpha]\!].\end{aligned}$$

(iii) Now let $\beta\downarrow\alpha$; we have

$$\begin{aligned}[\![T(f+g)>\alpha]\!]&=\sup_{\beta>\alpha}[\![T(f+g)>\beta]\!]\subseteq[\![Tf+Tg>\alpha]\!]\\&=\sup_{\beta>\alpha}[\![Tf+Tg>\beta]\!]\subseteq[\![T(f+g)>\alpha]\!],\end{aligned}$$

so $[\![T(f+g)>\alpha]\!]=[\![Tf+Tg>\alpha]\!]$. As α is arbitrary, $T(f+g)=Tf+Tg$; as f and g are arbitrary, T is additive. **Q**

(e) It is now easy to see that T is linear. **P** If $\gamma>0$, $f\in U$, $\alpha\in\mathbb{R}$ then

$$\begin{aligned}[\![T(\gamma f)>\alpha]\!]&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:\gamma f(x)>\beta\}}=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\frac{\beta}{\gamma}\}}\\&=\sup_{\beta>\alpha/\gamma}\operatorname{int}\overline{\{x:f(x)>\beta\}}=[\![Tf>\frac{\alpha}{\gamma}]\!]=[\![\gamma Tf>\alpha]\!].\end{aligned}$$

As α is arbitrary, $T(\gamma f)=\gamma Tf$; because we already know that T is additive, this is enough to show that T is linear. **Q**

(f) In fact T is a Riesz homomorphism. **P** If $f\in U$, $\alpha\ge 0$ then

$$\begin{aligned}[\![T(f^+)>\alpha]\!]&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f^+(x)>\beta\}}=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}\\&=[\![Tf>\alpha]\!]=[\![(Tf)^+>\alpha]\!].\end{aligned}$$

If $\alpha<0$ then

$$\begin{aligned}[\![T(f^+)>\alpha]\!]&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f^+(x)>\beta\}}\\&=X=[\![(Tf)^+>\alpha]\!].\ \mathbf{Q}\end{aligned}$$

(g) Of course the constant function χX belongs to U, and is its multiplicative identity; and $T(\chi X)$ is the multiplicative identity of L^0, because

$$
\begin{aligned}
[\![T(\chi X) > \alpha]\!] &= \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : (\chi X)(x) > \beta\}} \\
&= X \text{ if } \alpha < 1,\ \emptyset \text{ if } \alpha \ge 1.
\end{aligned}
$$

By 353Pd, or otherwise, T is multiplicative.

(h) The kernel of T is W. **P** (i) For $f \in U$,

$$
\begin{aligned}
Tf = 0 &\Longrightarrow [\![T|f| > 0]\!] = [\![|Tf| > 0]\!] = \emptyset \\
&\Longrightarrow \{x : |f(x)| > \omega(|f|, x)\} = \emptyset \\
&\Longrightarrow \operatorname{int}\{x : |f(x)| \le \epsilon\} \supseteq \{x : \omega(|f|, x) < \epsilon\} \\
&\qquad\qquad\qquad\qquad \text{is dense, for every } \epsilon > 0 \\
&\Longrightarrow f \in W.
\end{aligned}
$$

(ii) If $f \in W$, then, first,

$$\{x : \omega(f, x) < \epsilon\} \supseteq \operatorname{int}\{x : |f(x)| \le \frac{1}{3}\epsilon\}$$

is dense for every $\epsilon > 0$, so $f \in U$; and next, for any $\beta > 0$, $\overline{\{x : |f(x)| > \beta\}}$ does not meet the dense open set $\operatorname{int}\{x : |f(x)| \le \beta\}$, so

$$[\![|Tf| > 0]\!] = [\![T|f| > 0]\!] = \sup_{\beta>0} \operatorname{int} \overline{\{x : |f(x)| > \beta\}} = \emptyset$$

and $Tf = 0$. **Q**

(i) Finally, T is surjective. **P** Take any $u \in L^0$. Define $\tilde{f} : X \to [-\infty, \infty]$ by setting $\tilde{f}(x) = \sup\{\alpha : x \in [\![u > \alpha]\!]\}$ for each x, counting $\inf \emptyset$ as $-\infty$. Then

$$\{x : \tilde{f}(x) > \alpha\} = \bigcup_{\beta>\alpha} [\![u > \beta]\!]$$

is open, for every $\alpha \in \mathbb{R}$. The set

$$\{x : \tilde{f}(x) = \infty\} = \bigcap_{\alpha\in\mathbb{R}} [\![u > \alpha]\!]$$

is nowhere dense, because $\inf_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = \emptyset$ in $\mathfrak{G}$; while

$$\{x : \tilde{f}(x) = -\infty\} = X \setminus \bigcup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!]$$

is also nowhere dense, because $\sup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = X$ in $\mathfrak{G}$. Accordingly $E = \operatorname{int}\{x : \tilde{f}(x) \in \mathbb{R}\}$ is dense. Set $f(x) = \tilde{f}(x)$ for $x \in E$, 0 for $x \in X \setminus E$.

Let $\epsilon > 0$. If $G \subseteq X$ is a non-empty open set, there is an $\alpha \in \mathbb{R}$ such that $G \not\subseteq [\![u > \alpha]\!]$, so $G_1 = G \setminus \overline{[\![u > \alpha]\!]} \neq \emptyset$, and $\tilde{f}(x) \le \alpha$ for every $x \in G_1$. Set

$$\alpha' = \sup_{x\in G_1} \tilde{f}(x) \le \alpha < \infty.$$

Because E meets G_1, $\alpha' > -\infty$. Then $G_2 = G_1 \cap [\![u > \alpha' - \frac{1}{2}\epsilon]\!]$ is a non-empty open subset of G and $\alpha' - \frac{1}{2}\epsilon \le \tilde{f}(x) \le \alpha'$ for every $x \in G_2$. Accordingly $|f(y) - f(z)| \le \frac{1}{2}\epsilon$ for all y, $z \in G_2$, and $\omega(f, x) < \epsilon$ for all $x \in G_2$. As G is arbitrary, $\{x : \omega(f, x) < \epsilon\}$ is dense; as ϵ is arbitrary, $f \in U$.

Take $\alpha < \beta$ in $\mathbb{R}$, and set $\delta = \frac{1}{2}(\beta - \alpha)$. Then $H = E \cap \{x : \omega(f, x) < \delta\}$ is a dense open set, and

$$
\begin{aligned}
H \cap [\![Tf > \beta]\!] &\subseteq H \cap \{x : f(x) + \omega(f, x) \ge \beta\} \subseteq E \cap \{x : f(x) > \alpha\} \\
&\subseteq \{x : \tilde{f}(x) > \alpha\} \subseteq [\![u > \alpha]\!].
\end{aligned}
$$

As H is dense, $[\![Tf > \beta]\!] \subseteq [\![u > \alpha]\!]$. In the other direction

$$
\begin{aligned}
H \cap [\![u > \beta]\!] &\subseteq H \cap \{x : \tilde{f}(x) \ge \beta\} = H \cap \{x : f(x) \ge \beta\} \\
&\subseteq \{x : f(x) > \alpha + \omega(f, x)\} \subseteq [\![Tf > \alpha]\!],
\end{aligned}
$$

so $[\![u > \beta]\!] \subseteq [\![Tf > \alpha]\!]$. Just as in (d) above, this is enough to show that $Tf = u$. As u is arbitrary, T is surjective. **Q**

This completes the proof.

***364V Compact spaces** Suppose now that X is a compact Hausdorff topological space. In this case the space U of 364U is just the space of functions $f : X \to \mathbb{R}$ such that $\{x : f \text{ is continuous at } x\}$ is dense in X. **P** It is easy to see that

$$\{x : f \text{ is continuous at } x\} = \{x : \omega(f, x) = 0\} = \bigcap_{n\in\mathbb{N}} H_n$$

where $H_n = \{x : \omega(f, x) < 2^{-n}\}$ for each n. Each H_n is an open set (see part (a-i-α) of the proof of 364U), so by Baire's theorem (3A3G) $\bigcap_{n\in\mathbb{N}} H_n$ is dense iff every H_n is dense, that is, iff $f \in U$. **Q**

Now W becomes $\{f : f \in U, \{x : f(x) = 0\} \text{ is dense}\}$. **P** (i) If $f \in W$, then $T|f| = 0$, so (by the formula in (c) of the proof of 364U) $|f(x)| \le \omega(|f|, x)$ for every x. But $\{x : \omega(f, x) = 0\}$ is dense, because $f \in U$, so $\{x : f(x) = 0\}$ is also dense. (ii) If $f \in U$ and $\{x : f(x) = 0\}$ is dense, then

$$\omega(f, x) \ge \inf_{x\in G \text{ is open}} \sup_{y\in G} |f(y) - f(x)| \ge |f(x)|$$

for every $x \in X$. So for any $\epsilon > 0$, $\operatorname{int}\{x : |f(x)| \le \epsilon\} \supseteq \{x : \omega(f, x) < \epsilon\}$ is dense, and $f \in W$. **Q**

In the case of extremally disconnected spaces, we can go farther.

***364W Theorem** Let X be a compact Hausdorff extremally disconnected space, and $\mathfrak{G}$ its regular open algebra. Write $C^\infty = C^\infty(X)$ for the space of continuous functions $g : X \to [-\infty, \infty]$ such that $\{x : g(x) = \pm\infty\}$ is nowhere dense. Then we have a bijection $S : C^\infty \to L^0 = L^0(\mathfrak{G})$ defined by saying that

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. Addition and multiplication in L^0 correspond to the operations $\dot{+}$, $\dot{\times}$ on C^∞ defined by saying that $g\dot{+}h$, $g\dot{\times}h$ are the unique elements of C^∞ agreeing with $g + h$, $g \times h$ on $\{x : g(x), h(x) \text{ are both finite}\}$. Scalar multiplication in L^0 corresponds to the operation

$$(\gamma g)(x) = \gamma g(x) \text{ for } x \in X,\ g \in C^\infty,\ \gamma \in \mathbb{R}$$

on C^∞ (counting $0 \cdot \infty$ as 0), while the ordering of L^0 corresponds to the relation

$$g \le h \iff g(x) \le h(x) \text{ for every } x \in X.$$

proof (a) For $g \in C^\infty$, set $H_g = \{x : g(x) \in \mathbb{R}\}$, so that H_g is a dense open set, and define $Rg : X \to \mathbb{R}$ by setting $(Rg)(x) = g(x)$ if $x \in H_g$, 0 if $x \in X \setminus H_g$. Then Rg is continuous at every point of H_g, so belongs to the space U of 364U-364V. Set $Sg = T(Rg)$, where $T : U \to L^0$ is the map of 364U. Then

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. **P** (i) $\omega(g, x) = 0$ for every $x \in H_g$, so, if $\beta > \alpha$,

$$H_g \cap [\![Sg > \beta]\!] \subseteq \{x : x \in H_g,\ (Rg)(x) \ge \beta\} \subseteq \{x : g(x) \ge \beta\}$$

by the formula in part (c) of the proof of 364U. As $[\![Sg > \beta]\!]$ is open and H_g is dense,

$$[\![Sg > \beta]\!] \subseteq \overline{H_g \cap [\![Sg > \beta]\!]} \subseteq \{x : g(x) \ge \beta\} \subseteq \overline{\{x : g(x) > \alpha\}}.$$

Now

$$[\![Sg > \alpha]\!] = \sup_{\beta>\alpha} [\![Sg > \beta]\!] = \operatorname{int}\overline{\bigcup_{\beta>\alpha} [\![Sg > \beta]\!]} \subseteq \overline{\{x : g(x) > \alpha\}}.$$

(ii) In the other direction, $H_g \cap \{x : g(x) > \alpha\} \subseteq [\![Sg > \alpha]\!]$, by the other half of the formula in the proof of 364U. Again because $\{x : g(x) > \alpha\}$ is open and H_g is dense,

$$\overline{\{x : g(x) > \alpha\}} \subseteq \overline{[\![Sg > \alpha]\!]} = [\![Sg > \alpha]\!]$$

because X is extremally disconnected (see 314S). **Q**

(b) Thus we have a function S defined by the formula offered. Now if g, $h \in C^\infty$ and $g \le h$, we surely have $\{x : g(x) > \alpha\} \subseteq \{x : h(x) > \alpha\}$ for every α, so $[\![Sg > \alpha]\!] \subseteq [\![Sh > \alpha]\!]$ for every α and $Sg \le Sh$. On the other hand, if $g \not\le h$ then $Sg \not\le Sh$. **P** Take x_0 such that $g(x_0) > h(x_0)$, and $\alpha \in \mathbb{R}$ such that $g(x_0) > \alpha > h(x_0)$; set $H = \{x : g(x) > \alpha > h(x)\}$; this is a non-empty open set and $H \subseteq [\![Sg > \alpha]\!]$. On the other hand, $H \cap \{x : h(x) > \alpha\} = \emptyset$ so $H \cap [\![Sh > \alpha]\!] = \emptyset$. Thus $[\![Sg > \alpha]\!] \not\subseteq [\![Sh > \alpha]\!]$ and $Sg \not\le Sh$. **Q** In particular, S is injective.

(c) S is surjective. **P** If $u \in L^0$, set

$$g(x)=\sup\{\alpha : x\in[\![u>\alpha]\!]\}\in[-\infty,\infty]$$

for every $x\in X$, taking $\sup\emptyset=-\infty$. Then, for any $\alpha\in\mathbb{R}$, $\{x : g(x)>\alpha\}=\bigcup_{\beta>\alpha}[\![u>\alpha]\!]$ is open. On the other hand,

$$\{x : g(x)<\alpha\}=\bigcup_{\beta<\alpha}\{x : x\notin[\![u>\beta]\!]\}$$

is also open, because all the sets $[\![u>\beta]\!]$ are open-and-closed. So $g : X\to[-\infty,\infty]$ is continuous. Also

$$\{x : g(x)>-\infty\}=\bigcup_{\alpha\in\mathbb{R}}[\![u>\alpha]\!],$$

$$\{x : g(x)<\infty\}=\bigcup_{\alpha\in\mathbb{R}}X\setminus[\![u>\alpha]\!]$$

are dense, so $g\in C^\infty$. Now, for any $\alpha\in\mathbb{R}$,

$$\begin{aligned}[\![Sg>\alpha]\!]&=\overline{\{x : g(x)>\alpha\}}=\overline{\bigcup_{\beta>\alpha}[\![u>\beta]\!]}\\ &=\operatorname{int}\overline{\bigcup_{\beta>\alpha}[\![u>\beta]\!]}=\sup_{\beta>\alpha}[\![u>\beta]\!]=[\![u>\alpha]\!].\end{aligned}$$

So $Sg=u$. As u is arbitrary, S is surjective. **Q**

(d) Accordingly S is a bijection. I have already checked (in part (b)) that it is an isomorphism of the order structures. For the algebraic operations, observe that if g, $h\in C^\infty$ then there are f_1, $f_2\in C^\infty$ such that $Sg+Sh=Sf_1$ and $Sg\times Sh=Sf_2$, that is,

$$T(Rg+Rh)=TRg+TRh=TRf_1,\quad T(Rg\times Rh)=TRg\times TRh=TRf_2.$$

But this means that

$$T(Rg+Rh-Rf_1)=T((Rg\times Rh)-Rf_2)=0,$$

so that $Rg+Rh-Rf_1$, $(Rg\times Rh)-Rf_2$ belong to W and are zero on dense sets (364V). Since we know also that the set $G=\{x : g(x), h(x)\text{ are both finite}\}$ is a dense open set, while g, h, f_1 and f_2 are all continuous, we must have $f_1(x)=g(x)+h(x)$, $f_2(x)=g(x)h(x)$ for every $x\in G$. And of course this uniquely specifies f_1 and f_2 as members of C^∞.

Thus we do have operations $\dot{+}$, $\dot{\times}$ as described, rendering S additive and multiplicative. As for scalar multiplication, it is easy to check that $R(\gamma g)=\gamma Rg$ (at least, unless $\gamma=0$, which is trivial), so that $S(\gamma g)=\gamma Sg$ for every $g\in C^\infty$.

364X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For u, $v\in L^0=L^0(\mathfrak{A})$ set $[\![u<v]\!]=[\![v>u]\!]=[\![v-u>0]\!]$, $[\![u\le v]\!]=[\![v\ge u]\!]=1\setminus[\![v<u]\!]$, $[\![u=v]\!]=[\![u\le v]\!]\cap[\![v\le u]\!]$. (i) Show that $([\![u<v]\!],[\![u=v]\!],[\![u>v]\!])$ is always a partition of unity in $\mathfrak{A}$. (ii) Show that for any u, u', v, $v'\in L^0$, $[\![u\le u']\!]\cap[\![v\le v']\!]\subseteq[\![u+v\le u'+v']\!]$ and $[\![u=u']\!]\cap[\![v=v']\!]\subseteq[\![u\times v=u'\times v']\!]$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that if u, $v\in L^0=L^0(\mathfrak{A})$ and α, $\beta\in\mathbb{R}$ then $[\![u+v\ge\alpha+\beta]\!]\subseteq[\![u\ge\alpha]\!]\cup[\![v\ge\beta]\!]$. (ii) Show that if u, $v\in(L^0)^+$ and α, $\beta\ge0$ then $[\![u\times v\ge\alpha\beta]\!]\subseteq[\![u\ge\alpha]\!]\cup[\![v\ge\beta]\!]$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u\in L^0(\mathfrak{A})$. Show that $\{[\![u\in E]\!] : E\subseteq\mathbb{R}$ is Borel$\}$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{[\![u>\alpha]\!] : \alpha\in\mathbb{R}\}$.

>(d) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Show that for any $u\in L^0(\mathfrak{A})$ there is a unique Radon probability measure ν on $\mathbb{R}$ (the **distribution** of u) such that $\nu E=\bar\mu[\![u\in E]\!]$ for every Borel set $E\subseteq\mathbb{R}$. (*Hint*: 271B.)

(e) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\langle u_i\rangle_{i\in I}$ any family in $L^0(\mathfrak{A})$; for each $i\in I$ let $\mathfrak{B}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![u_i>\alpha]\!] : \alpha\in\mathbb{R}\}$. Show that the following are equiveridical: (i) $\bar\mu(\inf_{i\in J}[\![u_i>\alpha_i]\!])=\prod_{i\in J}\bar\mu[\![u_i>\alpha_i]\!]$ whenever $J\subseteq I$ is finite and $\alpha_i\in\mathbb{R}$ for each $i\in J$ (ii) $\langle\mathfrak{B}_i\rangle_{i\in I}$ is independent in the sense of 325L. (In this case we may call $\langle u_i\rangle_{i\in I}$ **(stochastically) independent**.)

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and u, v two stochastically independent members of $L^0(\mathfrak{A})$. Show that the distribution of their sum is the convolution of their distributions. (*Hint*: 272S.)

>(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and g, $h : \mathbb{R} \to \mathbb{R}$ Borel measurable functions. (i) Show that $\bar{g}\bar{h} = \overline{gh}$, where $\bar{g}$, $\bar{h} : L^0 \to L^0$ are defined as in 364I. (ii) Show that $\overline{g+h}(u) = \bar{g}(u) + \bar{h}(u)$, $\overline{g \times h}(u) = \bar{g}(u) \times \bar{h}(u)$ for every $u \in L^0 = L^0(\mathfrak{A})$. (iii) Show that if $\langle h_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel measurable functions on $\mathbb{R}$ and $\sup_{n \in \mathbb{N}} h_n = h$, then $\sup_{n \in \mathbb{N}} \bar{h}_n(u) = \bar{h}(u)$ for every $u \in L^0$. (iv) Show that if h is non-decreasing and continuous on the left, then $\bar{h}(\sup A) = \sup \bar{h}[A]$ whenever $A \subseteq L^0$ is a non-empty set with a supremum in L^0.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that $S(\mathfrak{A})$ can be identified (α) with the set of those $u \in L^0 = L^0(\mathfrak{A})$ such that $\{[\![u > \alpha]\!] : \alpha \in \mathbb{R}\}$ is finite (β) with the set of those $u \in L^0$ such that $[\![u \in I]\!] = 1$ for some finite $I \subseteq \mathbb{R}$. (ii) Show that $L^\infty(\mathfrak{A})$ can be identified with the set of those $u \in L^0$ such that $[\![u \in [-\alpha, \alpha]]\!] = 1$ for some $\alpha \ge 0$, and that $\|u\|_\infty$ is the smallest such α.

(i) Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and $u \in L^0(\mathfrak{A})$, then for any $\alpha \in \mathbb{R}$

$$[\![u > \alpha]\!] = \inf_{\beta > \alpha} \sup\{a : a \in \mathfrak{A},\, u \times \chi a \ge \beta \chi a\}$$

(compare 363Xh).

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u \ge 0$ in $L^0 = L^0(\mathfrak{A})$. Show that $u = \sup_{q \in \mathbb{Q}} q\chi[\![u > q]\!]$ in L^0.

(k) (i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A \subseteq L^0(\mathfrak{A})$ a non-empty countable set with supremum w. Show that $[\![w \in E]\!] \subseteq \sup_{u \in A} [\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$. (ii) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra and $A \subseteq L^0(\mathfrak{A})$ a non-empty set with supremum w. Show that $[\![w \in E]\!] \subseteq \sup_{u \in A} [\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$.

(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A \subseteq L^0 = L^0(\mathfrak{A})$ a non-empty set which is bounded below in L^0. Suppose that $\phi_0(\alpha) = \inf_{u \in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$. Show that $v = \inf A$ is defined in L^0, and that $[\![v > \alpha]\!] = \sup_{\beta > \alpha} \phi_0(\beta)$ for every $\alpha \in \mathbb{R}$.

(m) Let (X, Σ, μ) be a measure space and $f : X \to [0, \infty[$ a function; set $A = \{g^\bullet : g \in \mathcal{L}^0(\mu),\, g \le f$ a.e.$\}$. (i) Show that if (X, Σ, μ) *either* is localizable *or* has the measurable envelope property (213Xl), then $\sup A$ is defined in $L^0(\mu)$. (ii) Show that if (X, Σ, μ) is complete and locally determined and $w = \sup A$ is defined in $L^0(\mu)$, then $w \in A$.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that if u, $v \in L^0 = L^0(\mathfrak{A})$ then the following are equiveridical: (α) $[\![|v| > 0]\!] \subseteq [\![|u| > 0]\!]$ (β) v belongs to the band of L^0 generated by u (γ) there is a $w \in L^0$ such that $u \times w = v$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $L^0(\mathfrak{A}_a)$ can be identified, as f-algebra, with the band of $L^0(\mathfrak{A})$ generated by χa.

(p) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the corresponding Riesz homomorphism (364R). Show that (i) the kernel of T is the sequentially order-closed solid linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A},\, \pi a = 0\}$ (ii) the set of values of T is the sequentially order-closed Riesz subspace of $L^0(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism, with $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ the associated operator. Suppose that h is a Borel measurable real-valued function defined on a Borel subset of $\mathbb{R}$. Show that $\bar{h}(Tu) = T\bar{h}(u)$ whenever $u \in L^0(\mathfrak{A})$ and $\bar{h}(u)$ is defined in the sense of 364I.

>(r) Let X and Y be sets, Σ, T σ-algebras of subsets of X, Y respectively, and $\mathcal{I}$, $\mathcal{J}$ σ-ideals of Σ, T. Set $\mathfrak{A} = \Sigma/\mathcal{I}$, $\mathfrak{B} = \mathrm{T}/\mathcal{J}$. Suppose that $\phi : X \to Y$ is a function such that $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$,

$\phi^{-1}[F] \in \mathcal{I}$ for every $F \in \mathcal{J}$. Show that there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by saying that $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$. Let $T : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ be the Riesz homomorphism corresponding to π. Show that if we identify $L^0(\mathfrak{B})$ with $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}}$ and $L^0(\mathfrak{A})$ with $\mathcal{L}^0_{\Sigma}/\mathcal{W}_{\mathcal{I}}$ in the manner of 364D, then $T(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0_{\mathrm{T}}$.

(s) Use the ideas of part (d) of the proof of 364U to show that the operator T there is multiplicative, without appealing to 353P.

364Y Further exercises (a) Show directly, without using the Stone representation, that if $\mathfrak{A}$ is any Dedekind σ-complete Boolean algebra then the formulae of 364E define a group operation $+$ on $L^0(\mathfrak{A})$, and generally an f-algebra structure.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^0(\mathfrak{A})$ has the countable sup property.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u_1, \dots , u_n$ members of $L^0(\mathfrak{A})$. Write $\mathcal{B}_n$ for the algebra of Borel sets in $\mathbb{R}^n$. (i) Show that there is a unique sequentially order-continuous Boolean homomorphism $E \mapsto [\![(u_1, \dots , u_n) \in E]\!] : \mathcal{B}_n \to \mathfrak{A}$ such that $[\![(u_1, \dots , u_n) \in E]\!] = \inf_{i\le n} [\![u_i > \alpha_i]\!]$ when $E = \prod_{i\le n}]\alpha_i, \infty[$. (ii) Show that for every sequentially order-continuous Boolean homomorphism $\phi : \mathcal{B}_n \to \mathfrak{A}$ there are $u_1, \dots , u_n \in L^0(\mathfrak{A})$ such that $\phi E = [\![(u_1, \dots , u_n) \in E]\!]$ for every $E \in \mathcal{B}_n$.

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $n \ge 1$ and $h : \mathbb{R}^n \to \mathbb{R}$ a Borel measurable function. Show that we have a corresponding function $\bar h : L^0(\mathfrak{A})^n \to L^0(\mathfrak{A})$ defined by saying that $[\![\bar h(u_1, \dots , u_n) \in E]\!] = [\![(u_1, \dots , u_n) \in h^{-1}[E]]\!]$ for every Borel set $E \subseteq \mathbb{R}$.

(e) Suppose that $h_1(x, y) = x + y$, $h_2(x, y) = xy$, $h_3(x, y) = \max(x, y)$ for all x, $y \in \mathbb{R}$. Show that, in the language of 364Yd, $\bar h_1(u, v) = u + v$, $\bar h_2(u, v) = u \times v$, $\bar h_3(u, v) = u \vee v$ for all u, $v \in L^0$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a Riesz homomorphism such that $Te = e'$, where e, e' are the multiplicative identities of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Show that there is a unique sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $T = T_\pi$ in the sense of 364R. (*Hint*: use 353Pd. Compare 375A below.)

(g) Suppose, in 364U, that $X = \mathbb{Q}$. (i) Show that there is an $f \in W$ such that $f(q) > 0$ for every $q \in \mathbb{Q}$. (ii) Show that there is a $u \in L^0$ such that no $f \in U$ representing u can be continuous at any point of $\mathbb{Q}$.

(h) Let X and Y be topological spaces and $\phi : X \to Y$ a continuous function such that $\phi^{-1}[M]$ is nowhere dense in X for every nowhere dense subset M of Y. (Cf. 313R.) (i) Show that we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathfrak{G}_Y$ of Y to the regular open algebra $\mathfrak{G}_X$ of X defined by the formula $\pi G = \operatorname{int} \overline{\phi^{-1}[G]}$ for every $G \in \mathfrak{G}_Y$. (ii) Show that if U_X, U_Y are the function spaces of 364U then $g\phi \in U_X$ for every $g \in U_Y$. (iii) Show that if $T_X : U_X \to L^0(\mathfrak{G}_X)$, $T_Y : U_Y \to L^0(\mathfrak{G}_Y)$ are the canonical surjections, and $T : L^0(\mathfrak{G}_Y) \to L^0(\mathfrak{G}_X)$ is the homomorphism corresponding to π, then $T(T_Y g) = T_X(g\phi)$ for every $g \in U_Y$. (iv) Rewrite these ideas for the special case in which X is a dense subset of Y and ϕ is the identity map, showing that in this case π and T are isomorphisms.

(i) Let X be a Baire space, $\mathfrak{G}$ its algebra of regular open sets, $\mathcal{M}$ its ideal of meager sets, and $\widehat{\mathcal{B}}$ the Baire property σ-algebra $\{G \triangle A : G \subseteq X \text{ is open}, A \in \mathcal{M}\}$, so that $\mathfrak{G}$ can be identified with $\widehat{\mathcal{B}}/\mathcal{M}$ (314Yd). (i) Repeat the arguments of 364V in this context. (ii) Show that the space U of 364U-364V is a subspace of $\mathcal{L}^0(\widehat{\mathcal{B}})$, and that $W = U \cap \mathcal{W}$ where $\mathcal{W} = \{f : f \in \mathbb{R}^X, \{x : f(x) \ne 0\} \in \mathcal{M}\}$, so that the representations of $L^0(\mathfrak{G})$ as U/W, $\mathcal{L}^0/\mathcal{W}$ are consistent.

(j) Work through the arguments of 364U and 364Yi for the case of compact Hausdorff X, seeking simplifications based on 364V.

(k) Let X be an extremally disconnected compact Hausdorff space with regular open algebra $\mathfrak{G}$. Let U_0 be the space of real-valued functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : f \text{ is continuous at } x\}$ is dense. Show that U_0 is a Riesz subspace of the space U of 364U, and that every member of $L^0(\mathfrak{G})$ is represented by a member of U_0.

(l) Let X be a Baire space. Let Q be the set of all continuous real-valued functions defined on subsets of X, and Q^* the set of all members of Q which are maximal in the sense that there is no member of Q properly extending them. (i) Show that the domain of any member of Q^* is a dense G_δ set. (ii) Show that we can define addition and multiplication and scalar multiplication on Q^* by saying that $f\dot{+}g$, $f\dot{\times}g$, $\gamma.f$ are to be the unique members of Q^* extending the partially-defined functions $f+g$, $f\times g$, γf, and that these definitions render Q^* an f-algebra if we say that $f \le g$ iff $f(x) \le g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$. (iii) Show that every member of Q^* has an extension to a member of U, as defined in 364U, and that these extensions define an isomorphism between Q^* and $L^0(\mathfrak{G})$, where $\mathfrak{G}$ is the regular open algebra of X. (iv) Show that if X is compact, Hausdorff and extremally disconnected, then every member of Q^* has a unique extension to a member of $C^\infty(X)$, as defined in 364W.

(m) Let X be an extremally disconnected Hausdorff space, and Z any compact Hausdorff space. Show that if $D \subseteq X$ is dense and $f : D \to Z$ is continuous, there is a continuous $g : X \to Z$ extending f.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Write $L^0_{\mathbb{C}} = \{u+iv : u,\, v \in L^0\}$ for the complexification of $L^0 = L^0(\mathfrak{A})$ as defined in 354Yk. (i) Writing $\mathcal{B}(\mathbb{C})$ for the Borel σ-algebra of $\mathbb{C}$, show that there is a one-to-one correspondence between sequentially order-continuous Boolean homomorphisms $\phi : \mathcal{B}(\mathbb{C}) \to \mathfrak{A}$ and members $w = u + iv$ of $L^0_{\mathbb{C}}$ defined by saying that $[\![u > \alpha]\!] \cap [\![v > \beta]\!] = \phi\{z : \mathcal{R}\mathrm{e}\, z > \alpha, \mathcal{I}\mathrm{m}\, z > \beta\}$. (ii) Show that if Σ is a σ-algebra of subsets of a set X and $\pi : \Sigma \to \mathfrak{A}$ is a surjective sequentially order-continuous Boolean homomorphism with kernel $\mathcal{I}$, then we can identify $L^0_{\mathbb{C}}$ with $\mathcal{L}^0_{\mathbb{C}}/\mathcal{W}$, where $\mathcal{L}^0_{\mathbb{C}}$ is the set of Σ-measurable functions from X to $\mathbb{C}$, and $\mathcal{W}$ is the set of those $f \in \mathcal{L}^0_{\mathbb{C}}$ such that $\{x : f(x) \ne 0\} \in \mathcal{I}$.

364 Notes and comments This has been a long section, and so far all we have is a supposedly thorough grasp of the construction of L^0 spaces; discussion of their properties still lies ahead. The difficulties seem to stem from a variety of causes. First, L^0 spaces have a rich structure, being linear ordered spaces with multiplications; consequently all the main theorems have to check rather a lot of different aspects. Second, unlike L^∞ spaces, they are not accessible by means of the theory of normed spaces, so I must expect to do more of the work here rather than in an appendix. But this is in fact a crucial difference, because it affects the proof of the central theorem 364E. The point is that a given algebra $\mathfrak{A}$ will be expressible in the form $\Sigma/\mathcal{I}$ for a variety of algebras Σ of sets. Consequently any definition of $L^0(\mathfrak{A})$ as a quotient $\mathcal{L}^0(\Sigma)/\mathcal{W}$ must include a check that the structure produced is independent of the particular pair Σ, $\mathcal{I}$ chosen.

The same question arises with $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$. But in the case of S, I was able to use a general theory of additive functions on $\mathfrak{A}$ (see the proof of 361L), while in the case of L^∞ I could quote the result for S and a little theory of normed spaces (see the proof of 363H). The theorems of §368 will show, among other things, that a similar approach (describing L^0 as a special kind of extension of S or L^∞) can be made to work in the present situation. I have chosen, however, an alternative route using a novel technique. The price is the time required to develop skill in the technique, and to relate it to the earlier approach (364D, 364E, 364K). The reward is a construction which is based directly on the algebra $\mathfrak{A}$, independent of any representation (364A), and methods of dealing with it which are complementary to those of the previous three sections. In particular, they can be used in the absence of the full axiom of choice (364Ya).

I have deliberately chosen the notation $[\![u > \alpha]\!]$ from the theory of Forcing. I do not propose to try to explain myself here, but I remark that much of the labour of this section is a necessary basis for understanding real analysis in Boolean-valued models of set theory. The idea is that just as a function $f : X \to \mathbb{R}$ can be described in terms of the sets $\{x : f(x) > \alpha\}$, so can an element u of $L^0(\mathfrak{A})$ be described in terms of the elements $[\![u > \alpha]\!]$ of $\mathfrak{A}$ where in some sense u is greater than α. This description is well adapted to discussion of the order struction of $L^0(\mathfrak{A})$ (see 364M-364O), but rather ill-adapted to discussion of its linear and multiplicative structures, which leads to a large part of the length of the work above. Once we have succeeded in describing the algebraic operations on L^0 in terms of the values of $[\![u > \alpha]\!]$, however, as in 364E, the fundamental result on the action of Boolean homomorphisms (364R) is elegant and reasonably straightforward.

The concept '$[\![u > \alpha]\!]$' can be dramatically generalized to the concept '$[\![(u_1, \dots, u_n) \in E]\!]$', where E is a Borel subset of $\mathbb{R}^n$ and $u_1, \dots, u_n \in L^0(\mathfrak{A})$ (364H, 364Yc). This is supposed to recall the notation $\Pr(X \in E)$, already used in Chapter 27. If, as sometimes seems reasonable, we wish to regard a random variable as a member of $L^0(\mu)$ rather than of $\mathcal{L}^0(\mu)$, then '$[\![u \in E]\!]$' is the appropriate translation of '$X^{-1}[E]$'.

The reasons why we can reach all Borel sets E here, but then have to stop, seem to lie fairly deep. We see that we have here another potential definition of $L^0(\mathfrak{A})$, as the set of sequentially order-continuous Boolean homomorphisms from the Borel σ-algebra of $\mathbb{R}$ to $\mathfrak{A}$. This is suitably independent of realizations of $\mathfrak{A}$, but makes the f-algebra structure of L^0 difficult to elucidate, unless we move to a further level of abstraction in the definitions, as in 364Ye.

I take the space to describe the L^0 spaces of general regular open algebras in detail (364U) partly to offer a demonstration of an appropriate technique, and partly to show that we are not limited to σ-algebras of sets and their quotients. This really is a new representation; for instance, it does not meld in any straightforward way with the constructions of 364G-364I. Of course the most important examples are compact Hausdorff spaces, for which alternative methods are available (364V-364W, 364Yk, 364Yi, 364Yl); from the point of view of applications, indeed, it is worth sorting out compact Hausdorff spaces in general (364Yj). The version in 364W is derived from VULIKH 67. But I have starred everything from 364T on, because I shall not rely on this work later for anything essential.

365 L^1

Continuing my programme of developing the ideas of Chapter 24 at a deeper level of abstraction, I arrive at last at L^1. As usual, the first step is to establish a definition which can be matched both with the constructions of the previous sections and with the definition of $L^1(\mu)$ in §242 (365A-365C, 365F). Next, I give what I regard as the most characteristic internal properties of L^1 spaces, including versions of the Radon-Nikodým theorem (365E), before turning to abstract versions of theorems in §235 (365H, 365T) and the duality between L^1 and L^∞ (365I-365K). As in §§361 and 363, the construction is associated with universal mapping theorems (365L-365N) which define the Banach lattice structure of L^1. As in §§361, 363 and 364, homomorphisms between measure algebras correspond to operators between their L^1 spaces; but now the duality theory gives us two types of operators (365O-365Q), of which one class can be thought of as abstract conditional expectations (365R). For localizable measure algebras, the underlying algebra can be recovered from its L^1 space (365S), but the measure cannot.

365A Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. For $u \in L^0(\mathfrak{A})$, write

$$\|u\|_1 = \int_0^\infty \bar{\mu}[\![|u| > \alpha]\!]\,d\alpha,$$

the integral being with respect to Lebesgue measure on $\mathbb{R}$, and allowing ∞ as a value of the integral. (Because the integrand is monotonic, it is certainly measurable.) Set $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu}) = \{u : u \in L^0(\mathfrak{A}),\ \|u\|_1 < \infty\}$.

It is convenient to note at once that if $u \in L^1(\mathfrak{A}, \bar{\mu})$, then $\mu[\![|u| > \alpha]\!]$ must be finite for almost every $\alpha > 0$, and therefore for every $\alpha > 0$, since it is a non-increasing function of α; so that $[\![u > \alpha]\!]$ also belongs to the Boolean ring $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ for every $\alpha > 0$.

365B Theorem Let (X, Σ, μ) be a measure space with measure algebra $(\mathfrak{A}, \bar{\mu})$. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$, defined in 364D and 364J, matches $L^1(\mu) \subseteq L^0(\mu)$, defined in §242, with $L^1_{\bar{\mu}} \subseteq L^0(\mathfrak{A})$, and the standard norm of $L^1(\mu)$ with $\|\,\|_1 : L^1_{\bar{\mu}} \to [0, \infty[$, as defined in 365A.

proof Take any $f \in \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ (364C); write $f^\bullet$ for its equivalence class in $L^0(\mu)$, and u for the corresponding element of $L^0(\mathfrak{A})$, so that $[\![|u| > \alpha]\!] = \{x : |f(x)| > \alpha\}^\bullet$ in $\mathfrak{A}$, for every $\alpha \in \mathbb{R}$, and

$$\|u\|_1 = \int_0^\infty \mu\{x : |f(x)| > \alpha\}\,d\alpha.$$

(a) If f is a non-negative simple function, it is expressible as $\sum_{i=0}^n \alpha_i \chi E_i$ where $E_0, \dots, E_n$ are disjoint measurable sets of finite measure and $\alpha_i \geq 0$ for each i; re-enumerating if necessary, we may suppose that $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_n$. In this case, setting $\alpha_{n+1} = 0$,

$$\|u\|_1 = \int \mu\{x : f(x) > \alpha\}\, d\alpha = \sum_{j=0}^{n}(\sum_{i=0}^{j} \mu E_i)(\alpha_j - \alpha_{j+1})$$
$$= \sum_{i=0}^{n}\sum_{j=i}^{n} \mu E_i(\alpha_j - \alpha_{j+1}) = \sum_{i=0}^{n} \alpha_i \mu E_i = \int f d\mu.$$

(b) If f is integrable, then there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $|f(x)| = \sup_{n\in\mathbb{N}} f_n(x)$ a.e.; now

$$\mu\{x : |f(x)| > \alpha\} = \sup_{n\in\mathbb{N}} \mu\{x : f_n(x) > \alpha\}$$

for every $\alpha \in \mathbb{R}$, so

$$\int |f| = \sup_{n\in\mathbb{N}} \int f_n = \sup_{n\in\mathbb{N}} \int_0^\infty \mu\{x : f_n(x) > \alpha\} d\alpha$$
$$= \int_0^\infty \mu\{x : |f(x)| > \alpha\} d\alpha = \|u\|_1.$$

Thus in this case $u \in L^1(\mathfrak{A}, \bar\mu)$ and $\|u\|_1 = \|f^\bullet\|_1$.

It follows that every member of $L^1(\mu)$ corresponds to a member of $L^1(\mathfrak{A}, \bar\mu)$, and that the norm of $L^1(\mu)$ corresponds to $\| \|_1$ on $L^1(\mathfrak{A}, \bar\mu)$.

(c) On the other hand, if $u \in L^1(\mathfrak{A}, \bar\mu)$, then $\mu\{x : |f(x)| > \alpha\} = \bar\mu[\![|u| > \alpha]\!]$ is finite for every $\alpha > 0$ (365A). Also, if g is any simple function with $0 \le g \le |f|$,

$$\int g\, d\mu = \int_0^\infty \mu\{x : g(x) > \alpha\} d\alpha \le \int_0^\infty \mu\{x : f(x) > \alpha\} d\alpha = \|u\|_1 < \infty.$$

So f is integrable (122J). This shows that every member of $L^1(\mathfrak{A}, \bar\mu)$ corresponds to a member of $L^1(\mu)$.

365C Accordingly we can apply everything we know about $L^1(\mu)$ spaces to $L^1_{\bar\mu}$ spaces. For instance:

Theorem For any measure algebra $(\mathfrak{A}, \bar\mu)$, $L^1_{\bar\mu} = L^1(\mathfrak{A}, \bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$, and $\| \|_1$ is a norm on $L^1_{\bar\mu}$ under which $L^1_{\bar\mu}$ is an L-space. Consequently $L^1_{\bar\mu}$ is a perfect Riesz space with an order-continuous norm which has the Levi property, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $L^1_{\bar\mu}$ then it converges for $\| \|_1$ to $\sup_{n\in\mathbb{N}} u_n$.

proof $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra of some measure space (X, Σ, μ) (321J). $L^1(\mu)$ is a solid linear subspace of $L^0(\mu)$ (242Cb), so $L^1_{\bar\mu}$ is a solid linear subspace of $L^0(\mathfrak{A})$. $L^1(\mu)$ is an L-space (354M), so $L^1_{\bar\mu}$ also is. The rest of the properties claimed are general features of L-spaces (354N, 354E, 356P).

365D Integration Let $(\mathfrak{A}, \bar\mu)$ be any measure algebra.

(a) If $u \in L^1 = L^1(\mathfrak{A}, \bar\mu)$, then u^+, u^- belong to L^1, and we may set

$$\int u = \|u^+\|_1 - \|u^-\|_1 = \int_0^\infty \bar\mu[\![u > \alpha]\!]\, d\alpha - \int_0^\infty \bar\mu[\![-u > \alpha]\!]\, d\alpha.$$

Now $\int : L^1 \to \mathbb{R}$ is an order-continuous positive linear functional (356P), and under the translation of 365B matches the integral on $L^1(\mu)$ as defined in 242Ab.

(b) Of course $\|u\|_1 = \int |u| \ge |\int u|$ for every $u \in L^1$.

(c) If $u \in L^1$, $a \in \mathfrak{A}$ we may set $\int_a u = \int u \times \chi a$. (Compare 242Ac.) If $\gamma > 0$ and $0 \ne a \subseteq [\![u > \gamma]\!]$ then there is a $\delta > \gamma$ such that $a' = a \cap [\![u > \delta]\!] \ne 0$, so that

$$\int_a u = \int_0^\infty \bar\mu(a \cap [\![u > \alpha]\!]) d\alpha \ge \int_0^\gamma \bar\mu a\, d\alpha + \int_\gamma^\delta \bar\mu a' > \gamma \bar\mu a.$$

In particular, setting $a = [\![u > \gamma]\!]$, $\bar\mu[\![u > \gamma]\!]$ must be finite.

(d) If $u \in L^1$ then $u \ge 0$ iff $\int_a u \ge 0$ for every $a \in \mathfrak{A}^f$, writing $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$, as usual. **P** If $u \ge 0$ then $u \times \chi a \ge 0$, $\int_a u \ge 0$ for every $a \in \mathfrak{A}$. If $u \not\ge 0$, then $[\![u^- > 0]\!] \ne 0$ and there is an $\alpha > 0$ such that $a = [\![u^- > \alpha]\!] \ne 0$. But now $\bar{\mu}a$ is finite ((c) above) and

$$\int u \times \chi a = -\int u^- \times \chi a = -\int \bar{\mu}(a \cap [\![u^- \ge \beta]\!])d\beta \le -\alpha\bar{\mu}a < 0,$$

so $\int_a u < 0$. **Q** If u, $v \in L^1$ and $\int_a u = \int_a v$ for every $a \in \mathfrak{A}^f$ then $u = v$ (cf. 242Ce). If $u \ge 0$ in L^1 then $\int u = \sup\{\int_a u : a \in \mathfrak{A}^f\}$. **P** Of course $u \times \chi a \le u$ so $\int_a u \le u$ for every $a \in \mathfrak{A}$. On the other hand, setting $a_n = [\![u > 2^{-n}]\!]$, $\langle u \times \chi a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so $\int u = \lim_{n\to\infty} \int_{a_n} u$, while $\bar{\mu}a_n$ is finite for every n. **Q**

(e) If $u \in L^1$, $u \ge 0$ and $\int u = 0$ then $u = 0$ (122Rc). If $u \in L^1$, $u \ge 0$ and $\int_a u = 0$ then $u \times \chi a = 0$, that is, $a \cap [\![u > 0]\!] = 0$.

(f) If $C \subseteq L^1$ is non-empty and upwards-directed and $\sup_{v\in C} \int v$ is finite, then $\sup C$ is defined in L^1 and $\int \sup C = \sup_{v\in C} \int v$ (356P).

(g) It will occasionally be convenient to adapt the conventions of §133 to the new context; so that I may write $\int u = \infty$ if $u \in L^0(\mathfrak{A})$, $u^- \in L^1$, $u^+ \notin L^1$, while $\int u = -\infty$ if $u^+ \in L^1$ and $u^- \notin L^1$.

(h) On this convention, we can restate (f) as follows: if $C \subseteq (L^0)^+$ is non-empty and upwards-directed and has a supremum u in L^0, then $\int u = \sup_{v\in C} \int v$ in $[0, \infty]$. **P** For if $\sup_{v\in C} \int v$ is infinite, then surely $\int u = \infty$; while otherwise we can apply (f). **Q**

365E The Radon-Nikodým theorem again (a) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ an additive functional. Then the following are equiveridical:

(i) there is a $u \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}$;

(ii) ν is additive and continuous for the measure-algebra topology on $\mathfrak{A}$;

(iii) ν is completely additive.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra, and $\nu : \mathfrak{A}^f \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is additive and bounded and $\inf_{a\in A} |\nu a| = 0$ whenever $A \subseteq \mathfrak{A}^f$ is downwards-directed and has infimum 0;

(ii) there is a $u \in L^1$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}^f$.

proof (a) The equivalence of (ii) and (iii) is 327Bd. The equivalence of (i) and (iii) is just a translation of 327D into the new context.

(b)(i)$\Rightarrow$(ii)(α) Set $M = \sup_{a\in\mathfrak{A}^f} |\nu a| < \infty$.

Let $D \subseteq \mathfrak{A}^f$ be a maximal disjoint set. For each $d \in D$, write $\mathfrak{A}_d$ for the principal ideal of $\mathfrak{A}$ generated by d, and $\bar{\mu}_d$ for the restriction of $\bar{\mu}$ to $\mathfrak{A}_d$, so that $(\mathfrak{A}_d, \bar{\mu}_d)$ is a totally finite measure algebra. Set $\nu_d = \nu\restriction\mathfrak{A}_d$; then $\nu_d : \mathfrak{A}_d \to \mathbb{R}$ is completely additive. By (a), there is a $u_d \in L^1(\mathfrak{A}_d, \bar{\mu}_d)$ such that $\int_a u_d = \nu_d a = \nu a$ for every $a \subseteq d$.

Now $u_d^+ \in L^0(\mathfrak{A}_d)$ corresponds to a member $\tilde{u}_d^+$ of $L^0(\mathfrak{A})^+$ defined by saying

$$\begin{aligned}[\![\tilde{u}_d^+ > \alpha]\!] &= [\![u_d^+ > \alpha]\!] = [\![u_d > \alpha]\!] \text{ if } \alpha \ge 0,\\ &= 1 \text{ if } \alpha \le 0.\end{aligned}$$

Set $d^+ = [\![u_d > 0]\!]$. If $a \in \mathfrak{A}$, then

$$\int_a \tilde{u}_d^+ = \int_0^\infty \bar{\mu}(a \cap [\![\tilde{u}_d^+ > \alpha]\!])d\alpha = \int_0^\infty \bar{\mu}(a \cap [\![u_d^+ > \alpha]\!])d\alpha = \int_{a\cap d} u_d^+;$$

taking $a = 1$, we see that $\|\tilde{u}_d^+\|_1 = \nu d^+$ is finite, so that $\tilde{u}_d^+ \in L^1_{\bar{\mu}}$.

(β) For any finite $I \subseteq D$, set $v_I = \sum_{d\in I} \tilde{u}_d^+$. Then

$$\int v_I = \nu(\sup_{d\in I} d^+) \le M;$$

consequently the upwards-directed set $A = \{v_I : I \subseteq D \text{ is finite}\}$ is bounded above in $L^1_{\bar{\mu}}$, and we can set $v = \sup A$ in $L^1_{\bar{\mu}}$. If $a \in \mathfrak{A}$, then $\int_a v_I = \sum_{d\in I} \int_{a\cap d} u_d^+$ for each finite $I \subseteq D$, so $\int_a v = \sum_{d\in D} \int_{a\cap d} u_d^+$.

Applying the same arguments to $-\nu$, there is a $w \in L^1_{\bar{\mu}}$ such that

$$\int_a w = \sum_{d \in D} \int_{a \cap d} u_d^-$$

for every $a \in \mathfrak{A}$. Try $u = v - w$; then

$$\int_a u = \sum_{d \in D} \int_{a \cap d} u_d^+ - \int_{a \cap d} u_d^- = \sum_{d \in D} \int_{a \cap d} u_d = \sum_{d \in D} \nu(a \cap d)$$

for every $a \in \mathfrak{A}$.

(γ) Now take any $a \in \mathfrak{A}^f$. For $J \subseteq D$ set $a_J = \sup_{d \in J} a \cap d$. Let $\epsilon > 0$. Then there is a finite $I \subseteq D$ such that

$$|\int_a u - \nu a_J| = |\sum_{d \in D} \nu(a \cap d) - \sum_{d \in J} \nu(a \cap d)| \le \epsilon$$

whenever $I \subseteq J \subseteq D$ and J is finite. But now consider

$$A = \{a \setminus a_J : I \subseteq J \subseteq D,\ J \text{ is finite}\}.$$

Then $\inf A = 0$, so there is a finite J such that $I \subseteq J \subseteq D$ and

$$|\nu a - \nu a_J| = |\nu(a \setminus a_J)| \le \epsilon.$$

Consequently

$$|\nu a - \int_a u| \le |\nu a - \nu a_J| + |\int_a u - \nu a_J| \le 2\epsilon.$$

As ϵ is arbitrary, $\nu a = \int_a u$. As a is arbitrary, (ii) is proved.

(ii)$\Rightarrow$(i) From where we now are, this is nearly trivial. Thinking of νa as $\int u \times \chi a$, ν is surely additive and bounded. Also $|\nu a| \le \int |u| \times \chi a$. If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, the same is true of $\{|u| \times \chi a : a \in A\}$, because $a \mapsto |u| \times \chi a$ is order-continuous, so

$$\inf_{a \in A} |\nu a| \le \inf_{a \in A} \int |u| \times \chi a = \inf_{a \in A} \||u| \times \chi a\|_1 = 0.$$

365F It will be useful later to have spelt out the following elementary facts.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Write S^f for the intersection $S(\mathfrak{A}) \cap L^1(\mathfrak{A}, \bar{\mu})$. Then S^f is a norm-dense and order-dense Riesz subspace of $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, and can be identified with $S(\mathfrak{A}^f)$. The function $\chi : \mathfrak{A}^f \to S^f \subseteq L^1$ is an injective order-continuous additive lattice homomorphism. If $u \ge 0$ in L^1, there is a non-decreasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $(S^f)^+$ such that $u = \sup_{n \in \mathbb{N}} u_n = \lim_{n \to \infty} u_n$.

proof (a) As in 364L, we can think of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S = S(\mathfrak{A})$, embedded in $L^0(\mathfrak{A})$. If $u \in S$, it is expressible as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n \in \mathfrak{A}$ are disjoint and no α_i is zero. Now $|u| = \sum_{i=0}^n |\alpha_i| \chi a_i$, so $u \in L^1$ iff $\bar{\mu} a_i < \infty$ for every i, that is, iff $u \in S(\mathfrak{A}^f)$; thus $S^f = S(\mathfrak{A}^f)$.

Now suppose that $u \ge 0$ in L^1. By 364Kd, there is a non-decreasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $S(\mathfrak{A})^+$ such that $u_0 \ge 0$ and $u = \sup_{n \in \mathbb{N}} u_n$ in L^0. Because L^1 is a solid linear subspace of L^0, every u_n belongs to L^1 and therefore to S^f. By 365C, $\langle u_n \rangle_{n \in \mathbb{N}}$ is norm-convergent to u. This shows also that S^f is order-dense in L^1.

The map $\chi : \mathfrak{A}^f \to S^f$ is an injective order-continuous additive lattice homomorphism; because S^f is regularly embedded in L^1, χ has the same properties when regarded as a map into L^1.

For general $u \in L^1$, there are sequences in S^f converging to u^+ and to u^-, so that their difference is a sequence in S^f converging to u, and u belongs to the closure of S^f; thus S^f is norm-dense in L^1.

Remark Of course S^f here corresponds to the space of (equivalence classes of) simple functions.

365G Semi-finite algebras: Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ is order-dense in $L^0 = L^0(\mathfrak{A})$.

(b) In this case, writing $S^f = S(\mathfrak{A}) \cap L^1$ (as in 365F), $\int u = \sup\{\int v : v \in S^f,\ 0 \le v \le u\}$ in $[0, \infty]$ for every $u \in (L^0)^+$.

proof (a) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite then S^f is order-dense in L^0 (364L), so L^1 must also be. If L^1 is order-dense in L^0, then so is S^f, by 365F and 352Nc, so $(\mathfrak{A}, \bar{\mu})$ is semi-finite, by the other half of 364L.

(b) Set $C=\{v: v\in S^f,\ 0\le v\le u\}$. Then C is an upwards-directed set with supremum u, because S^f is order-dense in L^0. So $\int u=\sup_{v\in C}\int v$ by 365Dh.

365H Measurable transformations We have a generalization of the ideas of §235 in this abstract context.

Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ be the sequentially order-continuous Riesz homomorphism associated with π (364R).

(a) Suppose that $w\ge 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a}w\,d\bar\nu=\bar\mu a$ whenever $a\in\mathfrak{A}$ and $\bar\mu a<\infty$. Then for any $u\in L^1(\mathfrak{A},\bar\mu)$ and $a\in\mathfrak{A}$, $\int_{\pi a}Tu\times w\,d\bar\nu$ is defined and equal to $\int u\,d\bar\mu$.

(b) Suppose that $w'\ge 0$ in $L^0(\mathfrak{A})$ is such that $\int_a w'd\bar\mu=\bar\nu(\pi a)$ for every $a\in\mathfrak{A}$. Then $\int Tu\,d\bar\nu=\int u\times w'\,d\bar\mu$ whenever $u\in L^0(\mathfrak{A})$ and either integral is defined in $[-\infty,\infty]$.

Remark I am using the convention of 365Dg concerning '∞' as the value of an integral, and the notation '$\int u\,d\bar\mu$' is supposed to indicate that I am considering the integral in $L^1(\mathfrak{A},\bar\mu)$.

proof (a) If $u\in S^f=L^1(\mathfrak{A},\bar\mu)\cap S(\mathfrak{A})$ then u is expressible as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n$ have finite measure, so that $Tu=\sum_{i=0}^n\alpha_i\chi(\pi a_i)$ and

$$\int Tu\times w\,d\bar\nu=\sum_{i=0}^n\alpha_i\int_{\pi a_i}w=\sum_{i=0}^n\alpha_i\bar\mu a_i=\int u\,d\bar\mu.$$

If $u\ge 0$ in $L^1(\mathfrak{A},\bar\mu)$ there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S^f with supremum u, so that $Tu=\sup_{n\in\mathbb{N}}Tu_n$ and $w\times Tu=\sup_{n\in\mathbb{N}}w\times Tu_n$ in $L^0(\mathfrak{B})$, and

$$\int Tu\times w=\sup_{n\in\mathbb{N}}\int Tu_n\times w=\sup_{n\in\mathbb{N}}\int u_n=\int u.$$

(365C tells us that in this context $Tu\times w\in L^1(\mathfrak{B},\bar\nu)$.) Finally, for general $u\in L^1(\mathfrak{A},\bar\mu)$,

$$\int Tu\times w=\int Tu^+\times w-\int Tu^-\times w=\int u^+-\int u^-=\int u.$$

(b) The argument follows the same lines: start with $u=\chi a$ for $a\in\mathfrak{A}$, then with $u\in S(\mathfrak{A})$, then with $u\in L^0(\mathfrak{A})^+$ and conclude with general $u\in L^0(\mathfrak{A})$. The point is that T is a Riesz homomorphism, so that at the last step

$$\begin{aligned}\int Tu&=\int(Tu)^+-\int(Tu)^-=\int T(u^+)-\int T(u^-)\\&=\int u^+\times w'-\int u^-\times w'=\int(u\times w')^+-\int(u\times w')^-=\int u\times w'\end{aligned}$$

whenever either side is defined in $[-\infty,\infty]$.

365I The duality between L^1 and L^∞ Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. If we identify $L^\infty=L^\infty(\mathfrak{A})$ with the solid linear subspace of $L^0=L^0(\mathfrak{A})$ generated by $e=\chi 1_{\mathfrak{A}}$ (364K), then we have a bilinear map $(u,v)\mapsto u\times v:L^1\times L^\infty\to L^1$, because $|u\times v|\le\|v\|_\infty|u|$ and $L^1=L^1(\mathfrak{A},\bar\mu)$ is a solid linear subspace of L^0. Note that $\|u\times v\|_1\le\|u\|_1\|v\|_\infty$, so that the bilinear map $(u,v)\mapsto u\times v$ has norm at most 1 (253A). Consequently we have a bilinear functional $(u,v)\mapsto\int u\times v:L^1\times L^\infty\to\mathbb{R}$, which also has norm at most 1, corresponding to linear operators $S:L^1\to(L^\infty)^*$ and $T:L^\infty\to(L^1)^*$, both of norm at most 1. Because L^1 and L^∞ are both Banach lattices, we have $(L^1)^*=(L^1)^\sim$, $(L^\infty)^*=(L^\infty)^\sim$ (356Dc). Because the norm of L^1 is order-continuous, $(L^1)^*=(L^1)^\times$ (356Dd).

365J Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and set $L^1=L^1(\mathfrak{A},\bar\mu)$, $L^\infty=L^\infty(\mathfrak{A})$. Let $S:L^1\to(L^\infty)^*=(L^\infty)^\sim$, $T:L^\infty\to(L^1)^*=(L^1)^\sim=(L^1)^\times$ be the canonical maps defined by the duality between L^1 and L^∞, as in 365I. Then

(a) S and T are order-continuous Riesz homomorphisms, $S[L^1]\subseteq(L^\infty)^\times$, and S is norm-preserving;

(b) $(\mathfrak{A},\bar\mu)$ is semi-finite iff T is injective, and in this case T is norm-preserving, while S is a normed Riesz space isomorphism between L^1 and $(L^\infty)^\times$;

(c) $(\mathfrak{A},\bar\mu)$ is localizable iff T is bijective, and in this case T is a normed Riesz space isomorphism between L^∞ and $(L^1)^*=(L^1)^\times$.

proof Take a measure space (X,Σ,μ) such that $(\mathfrak{A},\bar\mu)$ is isomorphic to its measure algebra. Then we can identify L^1 with $L^1(\mu)$, as in 365B, and L^∞ with $L^\infty(\mu)$, as in 363I. Moreover, these identifications are based on the canonical embeddings of L^1 and L^∞ in $L^0(\mu)$, so that the duality described in 365I corresponds to the familiar duality $(u,v)\mapsto\int u\times v$ already used in 243F.

(a)(i) If $u\ge 0$ in L^1 and $v\ge 0$ in L^∞ then $u\times v\ge 0$ and

$$(Tv)(u)=\int u\times v\ge 0.$$

As u is arbitrary, $Tv\ge 0$ in $(L^1)^\times$; as v is arbitrary, T is a positive linear operator.

If $v\in L^\infty$, set $a=[\![v>0]\!]\in\mathfrak{A}$. (Remember that we are identifying $L^0(\mu)$, as defined in §241, with $L^0(\mathfrak{A})$, as defined in §364.) Then $v^+=v\times\chi a$, so for any $u\ge 0$ in L^1

$$(Tv^+)(u)=\int u\times v\times\chi a=(Tv)(u\times\chi a)\le(Tv)^+(u).$$

As u is arbitrary, $Tv^+\le(Tv)^+$. On the other hand, because T is a positive linear operator, $Tv^+\ge Tv$ and $Tv^+\ge 0$, so $Tv^+\ge(Tv)^+$. Thus $Tv^+=(Tv)^+$. As v is arbitrary, T is a Riesz homomorphism (352G).

(ii) Exactly the same arguments show that S is a Riesz homomorphism.

(iii) Given $u\in L^1$, set $a=[\![u>0]\!]$; then

$$\|Su\|\ge(Su)(\chi a-\chi(1\setminus a))=\int_a u-\int_{1\setminus a}u=\int|u|=\|u\|_1\ge\|Su\|.$$

So S is norm-preserving.

(iv) By 355Ka, S is order-continuous.

(v) If $A\subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0, and $u\in(L^1)^+$, then $\inf_{v\in A}u\times v=0$ for every $u\in(L^1)^+$, because $v\mapsto u\times v:L^0\to L^0$ is order-continuous. So

$$\inf_{v\in A}(Tv)(u)=\inf_{v\in A}\int u\times v=\inf_{v\in A}\|u\times v\|_1=0$$

and the only possible non-negative lower bound for $T[A]$ in $(L^1)^\times$ is 0. As A is arbitrary, T is order-continuous.

(vi) The ideas of (v) show also that $S[L^1]\subseteq(L^\infty)^\times$. **P** If $u\in(L^1)^+$ and $A\subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{v\in A}(Su)(v)=\inf_{v\in A}\int u\times v=0.$$

As A is arbitrary, Su is order-continuous. For general $u\in L^1$, $Su=Su^+-Su^-$ belongs to $(L^\infty)^\times$. **Q**

(b)(i) If $(\mathfrak{A},\bar\mu)$ is not semi-finite, let $a\in\mathfrak{A}$ be such that $\bar\mu a=\infty$ and $\bar\mu b=\infty$ whenever $0\ne b\subseteq a$. If $u\in L^1$, then $[\![|u|>\frac{1}{n}]\!]$ has finite measure for every $n\ge 1$, so must be disjoint from a; accordingly

$$a\cap[\![|u|>0]\!]=\sup_{n\ge 1}a\cap[\![|u|>\tfrac{1}{n}]\!]=0.$$

This means that $\int u\times\chi a=0$ for every $u\in L^1$. Accordingly $T(\chi a)=0$ and T is not injective.

(ii) If $(\mathfrak{A},\bar\mu)$ is semi-finite, take any $v\in L^\infty$. Then if $0\le\delta<\|v\|_\infty$, $a=[\![|v|>\delta]\!]\ne 0$. Let $b\subseteq a$ be such that $0<\bar\mu b<\infty$. Then $\chi b\in L^1$, and

$$\|Tv\|=\||Tv|\|=\|T|v|\|\ge(T|v|)(\chi b)/\|\chi b\|_1\ge\delta$$

because $|v|\times\chi b\ge\delta\chi b$, so

$$(T|v|)(\chi b)\ge\delta\bar\mu b=\delta\|\chi b\|_1.$$

As δ is arbitrary, $\|Tv\|\ge\|v\|_\infty$. But we already know that $\|Tv\|\le\|v\|_\infty$, so the two are equal. As v is arbitrary, T is norm-preserving (and, in particular, is injective).

(iii) Still supposing that $(\mathfrak{A},\bar\mu)$ is semi-finite, $S[L^1]=(L^\infty)^\times$. **P** Take any $h\in(L^\infty)^\times$. For $a\in\mathfrak{A}$, set $\nu a=h(\chi a^\bullet)$. By 363K, $\nu:\mathfrak{A}\to\mathbb{R}$ is completely additive. By 365Ea, there is a $u\in L^1$ such that

$$(Su)(\chi a) = \int u \times \chi a = \int_a u = \nu a = h(\chi a)$$

for every $a \in \mathfrak{A}$. Because Su and h are both linear functionals on L^∞, they must agree on $S(\mathfrak{A})$; because they are continuous and $S(\mathfrak{A})$ is dense in L^∞ (363C), $Su = h$. As h is arbitrary, S is surjective. **Q**

(c) Using (b), we know that if either T is bijective or $(\mathfrak{A}, \bar{\mu})$ is localizable, then $(\mathfrak{A}, \bar{\mu})$ is semi-finite, so that (X, Σ, μ) is also semi-finite (322Bd). Given this, 243G tells us that T is bijective iff (X, Σ, μ) is localizable, and in this case T is norm-preserving; but of course (X, Σ, μ) is localizable iff $(\mathfrak{A}, \bar{\mu})$ is (322Be).

365K Corollary If $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra, $L^\infty(\mathfrak{A})$ is perfect.

proof By 365J(b)-(c), we can identify L^∞ with $(L^1)^\times \cong (L^\infty)^{\times\times}$.

365L Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and U a Banach space. Let $\nu : \mathfrak{A}^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator $T : L^1 \to U$, where $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, such that $\nu a = T(\chi a)$ for every $a \in \mathfrak{A}^f$;

(ii)(α) ν is additive (β) there is an $M \ge 0$ such that $\|\nu a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

proof (a)(i)⇒(ii) If $T : L^1 \to U$ is a continuous linear operator, then $\chi a \in L^1$ for every $a \in \mathfrak{A}^f$, so $\nu = T\chi$ is a function from $\mathfrak{A}^f$ to U. If a, $b \in \mathfrak{A}^f$ and $a \cap b = 0$, then $\chi(a \cup b) = \chi a + \chi b$ in $L^0 = L^0(\mathfrak{A})$ and therefore in L^1, so

$$\nu(a \cup b) = T\chi(a \cup b) = T(\chi a + \chi b) = T(\chi a) + T(\chi b) = \nu a + \nu b.$$

If $a \in \mathfrak{A}^f$ then $\|\chi a\|_1 = \bar{\mu}a$ (using the formula in 365A, or otherwise), so

$$\|\nu a\| = \|T(\chi a)\| \le \|T\|\|\chi a\|_1 = \|T\|\bar{\mu}a.$$

(b)(ii)⇒(i) Now suppose that $\nu : \mathfrak{A}^f \to U$ is additive and that $\|\nu a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$. Let $S^f = L^1 \cap S(\mathfrak{A})$, as in 365F. Then there is a linear operator $T_0 : S^f \to U$ such that $T_0(\chi a) = \nu a$ for every $a \in \mathfrak{A}^f$ (361F). Next, $\|T_0 u\| \le M\|u\|_1$ for every $u \in S^f$. **P** If $u \in S^f \cong S(\mathfrak{A}^f)$, then u is expressible as $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m \in \mathfrak{A}^f$ are disjoint (361Eb). So

$$\|T_0 u\| = \| \sum_{j=0}^m \beta_j \nu b_j\| \le M \sum_{j=0}^m |\beta_j|\bar{\mu}b_j = M\|u\|_1. \text{ **Q**}$$

There is therefore a continuous linear operator $T : L^1 \to U$, extending T_0, and with $\|T\| \le \|T_0\| \le M$ (2A4I). Of course we still have $\nu = T\chi$.

(c) The argument in (b) shows that $T_0 = T\restriction S^f$ and T are uniquely defined from ν. We have also seen that if T, ν correspond to each other then

$$\|\nu a\| \le \|T\|\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f,$$

$$\|T\| \le M \text{ whenever } \|\nu a\| \le M\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f,$$

so that

$$\|T\| = \min\{M : M \ge 0,\ \|\nu a\| \le M\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f\}.$$

365M Corollary Let (X, Σ, μ) be a measure space and U any Banach space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$. Let $\nu : \Sigma^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator $T : L^1 \to U$, where $L^1 = L^1(\mu)$, such that $\nu E = T(\chi E)^\bullet$ for every $E \in \Sigma^f$;

(ii)(α) $\nu(E \cup F) = \nu E + \nu F$ whenever E, $F \in \Sigma^f$ and $E \cap F = 0$ (β) there is an $M \ge 0$ auch that $\|\nu E\| \le M\mu E$ for every $E \in \Sigma^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

proof This is a direct translation of 365L. The only point to note is that if ν satisfies the conditions of (ii), and E, $F \in \Sigma^f$ are such that $E^\bullet = F^\bullet$ in the measure algebra $(\mathfrak{A}, \bar{\mu})$ of (X, Σ, μ), then $\mu(E \setminus F) = \mu(F \setminus E) = 0$, so that $\nu(E \setminus F) = \nu(F \setminus E) = 0$ (using condition (ii-β)) and

$$\nu E = \nu(E \cap F) + \nu(E \setminus F) = \nu(E \cap F) + \nu(F \setminus E) = \nu F.$$

This means that we have a function $\bar{\nu} : \mathfrak{A}^f \to U$, where

$$\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\} = \{E^\bullet : E \in \Sigma^f\},$$

defined by setting $\bar{\nu}E^\bullet = \nu E$ for every $E \in \Sigma^f$. Of course we now have $\bar{\nu}(a \cup b) = \bar{\nu}a + \bar{\nu}b$ whenever a, $b \in \mathfrak{A}^f$ and $a \cap b = 0$ (since we can express them as $a = E^\bullet$, $b = F^\bullet$ with $E \cap F = \emptyset$), and $\|\bar{\nu}a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$. Thus we have a one-to-one correspondence between functions $\nu : \Sigma^f \to U$ satisfying the conditions (ii) here, and functions $\bar{\nu} : \mathfrak{A}^f \to U$ satisfying the conditions (ii) of 365L. The rest of the argument is covered by the identification between $L^1(\mu)$ and $L^1(\mathfrak{A}, \bar{\mu})$ in 365B.

365N Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, U a Banach lattice, and $T : L^1 \to U$ a bounded linear operator, where $L^1 = L^1(\mathfrak{A}, \bar{\mu})$. Let $\nu : \mathfrak{A}^f \to U$ be the corresponding additive function, as in 365L.

(a) T is a positive linear operator iff $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$; in this case, T is order-continuous.

(b) If U is Dedekind complete and $T \in \mathrm{L}^\sim(L^1; U)$, then $|T| : L^1 \to U$ corresponds to $|\nu| : \mathfrak{A}^f \to U$, where

$$|\nu|(a) = \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}$$

for every $a \in \mathfrak{A}^f$.

(c) T is a Riesz homomorphism iff ν is a lattice homomorphism.

proof As in 365F, let S^f be $L^1 \cap S(\mathfrak{A})$, identified with $S(\mathfrak{A}^f)$.

(a)(i) If T is a positive linear operator and $a \in \mathfrak{A}^f$, then $\chi a \ge 0$ in L^1, so $\nu a = T(\chi a) \ge 0$ in U.

(ii) Now suppose that $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$, and let $u \ge 0$ in L^1, $\epsilon > 0$ in $\mathbb{R}$. Then there is a $v \in S^f$ such that $0 \le v \le u$ and $\|u - v\|_1 \le \epsilon$ (365F). Express v as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$, $\alpha_i \ge 0$ for each i. Now

$$\|Tu - Tv\| \le \|T\|\|u - v\|_1 \le \epsilon\|T\|.$$

On the other hand,

$$Tv = \textstyle\sum_{i=0}^n \alpha_i \nu a_i \in U^+.$$

As U^+ is norm-closed in U, and ϵ is arbitrary, $Tu \in U^+$. As u is arbitrary, T is a positive linear operator.

(iii) By 355Ka, T is order-continuous.

(b) The point is that $|T{\restriction}S^f| = |T|{\restriction}S^f$. **P** (i) Because the embedding $S^f \subseteq L^1$ is positive, the map $P \mapsto P{\restriction}S^f$ is a positive linear operator from $\mathrm{L}^\sim(L^1; U)$ to $\mathrm{L}^\sim(S^f; U)$ (see 355Bd). So $|T{\restriction}S^f| \le |T|{\restriction}S^f$. (ii) There is a positive linear operator $T_1 : L^1 \to U$ extending $|T{\restriction}S^f|$, by 365M and (a) above, and now $T_1{\restriction}S^f$ dominates both $T{\restriction}S^f$ and $-T{\restriction}S^f$; since $(S^f)^+$ is dense in $(L^1)^+$, $T_1 \ge T$ and $T_1 \ge -T$, so that $T_1 \ge |T|$ and

$$|T{\restriction}S^f| = T_1{\restriction}S^f \ge |T|{\restriction}S^f. \ \mathbf{Q}$$

Now 361H tells us that

$$|T|(\chi a) = |T{\restriction}S^f|(\chi a) = |\nu|a$$

for every $a \in \mathfrak{A}^f$.

(c)(i) If T is a lattice homomorphism, then so is $\nu = T\chi$, because $\chi : \mathfrak{A}^f \to S^f$ is a lattice homomorphism.

(ii) Now suppose that χ is a lattice homomorphism. In this case $T{\restriction}S^f$ is a Riesz homomorphism (361Gc), that is, $|Tv| = T|v|$ for every $v \in S^f$. Because S^f is dense in L^1 and the map $u \mapsto |u|$ is continuous both in L^1 and in U (354Bb), $|Tu| = T|u|$ for every $u \in L^1$, and T is a Riesz homomorphism.

365O Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a) There is a unique order-continuous norm-preserving Riesz homomorphism $T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ such that $T_\pi(\chi a) = \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$. We have $T_\pi(u \times \chi a) = T_\pi u \times \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$ and $u \in L^1_{\bar{\mu}}$.

(b) $\int T_\pi u = \int u$, $\int_{\pi a} T_\pi u = \int_a u$ for every $u \in L^1_{\bar{\nu}}$, $a \in \mathfrak{A}^f$.

(c) $[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in L^1(\mathfrak{A}, \mu)$, $\alpha > 0$.
(d) T_π is surjective iff π is.
(e) If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B}^f \to \mathfrak{C}$ another measure-preserving Boolean homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^1_{\bar\mu} \to L^1(\mathfrak{C}, \bar\lambda)$.

proof Throughout the proof I will write T for T_π and S^f for $S(\mathfrak{A}) \cap L^1_{\bar\mu} \cong S(\mathfrak{A}^f)$ (see 365F).

(a)(i) We have a map $\psi : \mathfrak{A}^f \to L^1_{\bar\nu}$ defined by writing $\psi a = \chi(\pi a)$ for $a \in \mathfrak{A}^f$. Because

$$\chi\pi(a \cup b) = \chi(\pi a \cup \pi b) = \chi\pi a + \chi\pi b, \quad \|\chi(\pi a)\|_1 = \bar\nu(\pi a) = \bar\mu a$$

whenever a, $b \in \mathfrak{A}^f$ and $a \cap b = 0$, we get a (unique) corresponding bounded linear operator $T : L^1_{\bar\mu} \to L^1_{\bar\nu}$ such that $T\chi = \chi\pi$ on $\mathfrak{A}^f$ (365L). Because $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ and $\chi : \mathfrak{B}^f \to L^1_{\bar\nu}$ are lattice homomorphisms, so is ψ, and T is a Riesz homomorphism (365Nc).

(ii) If $u \in S^f$, express u as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n$ are disjoint in $\mathfrak{A}^f$. Then $Tu = \sum_{i=0}^n \alpha_i\chi(\pi a_i)$ and $\pi a_0, \dots , \pi a_n$ are disjoint in $\mathfrak{B}^f$, so

$$\|Tu\|_1 = \sum_{i=0}^n |\alpha_i|\bar\nu(\pi a_i) = \sum_{i=0}^n |\alpha_i|\bar\mu a_i = \|u\|_1.$$

Because S^f is dense in $L^1_{\bar\mu}$ and $u \mapsto \|u\|_1$ is continuous (in both $L^1_{\bar\mu}$ and $L^1_{\bar\nu}$), $\|Tu\|_1 = \|u\|_1$ for every $u \in L^1_{\bar\mu}$, that is, T is norm-preserving. As noted in 365Na, T is order-continuous.

(iii) If a, $b \in \mathfrak{A}^f$ then

$$T(\chi a \times \chi b) = T(\chi(a \cap b)) = \chi\pi(a \cap b) = \chi(\pi a \cap \pi b) = \chi\pi a \times \chi\pi b = \chi\pi a \times T(\chi b).$$

Because T is linear and $\times$ is bilinear, $T(\chi a \times u) = \chi\pi a \times Tu$ for every $u \in S^f$. Because the maps $u \mapsto u \times \chi a : L^1_{\bar\mu} \to L^1_{\bar\mu}$, $T : L^1_{\bar\mu} \to L^1_{\bar\nu}$ and $v \mapsto v \times \chi\pi a : L^1_{\bar\nu} \to L^1_{\bar\nu}$ are all continuous, $Tu \times \chi\pi a = T(u \times \chi a)$ for every $u \in L^1_{\bar\mu}$.

(iv) T is unique because the formula $T(\chi a) = \chi\pi a$ defines T on the norm-dense and order-dense subspace S^f.

(b) Because T is positive,

$$\int Tu = \|Tu^+\|_1 - \|Tu^-\|_1 = \|u^+\|_1 - \|u^-\|_1 = \int u.$$

For $a \in \mathfrak{A}^f$,

$$\int_{\pi a} Tu = \int Tu \times \chi\pi a = \int T(u \times \chi a) = \int u \times \chi a = \int_a u.$$

(c) If $u \in S^f$, express it as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n$ are disjoint; then

$$\pi[\![u > \alpha]\!] = \pi(\sup_{i\in I} a_i) = \sup_{i\in I} \pi a_i = [\![Tu > \alpha]\!]$$

where $I = \{i : i \le n, \alpha_i > \alpha\}$. For $u \in (L^1)^+$, take a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S^f with supremum u; then $\sup_{n\in\mathbb{N}} Tu_n = Tu$, so

$$\pi[\![u > \alpha]\!] = \pi(\sup_{n\in\mathbb{N}} [\![u_n > \alpha]\!])$$

(364Mb; $[\![u > \alpha]\!] \in \mathfrak{A}^f$ by 365A)

$$= \sup_{n\in\mathbb{N}} \pi[\![u_n > \alpha]\!]$$

(because π is order-continuous, see 361Ad)

$$= \sup_{n\in\mathbb{N}} [\![Tu_n > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is order-continuous. For general $u \in L^1$,

$$\pi[\![u > \alpha]\!] = \pi[\![u^+ > \alpha]\!] = [\![T(u^+) > \alpha]\!] = [\![(Tu)^+ > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is a Riesz homomorphism.

(d)(i) Suppose that T is surjective and that $b \in \mathfrak{B}^f$. Then there is a $u \in L^1_{\bar\mu}$ such that $Tu = \chi b$. Now

$$b = [\![Tu > \tfrac{1}{2}]\!] = \pi[\![u > \tfrac{1}{2}]\!] \in \pi[\mathfrak{A}^f];$$

as b is arbitrary, π is surjective.

(ii) Suppose now that π is surjective. Then $T[L^1_{\bar\mu}]$ is a linear subspace of $L^1_{\bar\nu}$ containing χb for every $b \in \mathfrak{B}^f$, so includes $S(\mathfrak{B}^f)$. If $v \in (L^1_{\bar\nu})^+$ there is a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{B}^f)^+$ with supremum v. For each n, choose u_n such that $Tu_n = v_n$. Setting $u'_n = \sup_{i\le n} u_i$, we get a non-decreasing sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ such that $v_n \le Tu'_n \le v$ for every $n \in \mathbb{N}$. So

$$\sup_{n\in\mathbb{N}} \|u'_n\|_1 = \sup_{n\in\mathbb{N}} \|Tu'_n\|_1 \le \|v\|_1 < \infty$$

and $u = \sup_{n\in\mathbb{N}} u'_n$ is defined in $L^1_{\bar\mu}$, with

$$Tu = \sup_{n\in\mathbb{N}} Tu'_n = v.$$

Thus $(L^1_{\bar\nu})^+ \subseteq T[L^1_{\bar\mu}]$; consequently $L^1_{\bar\nu} \subseteq T[L^1_{\bar\mu}]$ and T is surjective.

(e) This is an immediate consequence of the 'uniqueness' assertion in (i), because for any $a \in \mathfrak{A}^f$

$$T_\theta T_\pi(\chi a) = T_\theta \chi(\pi a) = \chi(\theta\pi a),$$

so that $T_\theta T_\pi : L^1_{\bar\mu} \to L^1_{\bar\lambda}$ is a bounded linear operator taking the right values at elements χa, and must therefore be equal to $T_{\theta\pi}$.

365P Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}$ an order-continuous ring homomorphism.

(a) There is a unique positive linear operator $P_\pi : L^1_{\bar\nu} \to L^1_{\bar\mu}$ such that $\int_a P_\pi v = \int_{\pi a} v$ for every $v \in L^1_{\bar\nu}$, $a \in \mathfrak{A}^f$.

(b) P_π is order-continuous and norm-continuous, and $\|P_\pi\| \le 1$.

(c) If $a \in \mathfrak{A}^f$, $v \in L^1_{\bar\nu}$ then $P_\pi(v \times \chi\pi a) = P_\pi v \times \chi a$.

(d) If $\pi[\mathfrak{A}^f]$ is order-dense in $\mathfrak{B}$ then P_π is a norm-preserving Riesz homomorphism; in particular, P_π is injective.

(e) If $(\mathfrak{B}, \bar\nu)$ is semi-finite and π is injective, then P_π is surjective, and there is for every $u \in L^1_{\bar\mu}$ a $v \in L^1_{\bar\nu}$ such that $P_\pi v = u$ and $\|v\|_1 = \|u\|_1$.

(f) Suppose again that $(\mathfrak{B}, \bar\nu)$ is semi-finite. If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ an order-continuous Boolean homomorphism, then $P_{\theta\pi} = P_\pi P_{\theta'} : L^1(\mathfrak{C}, \bar\lambda) \to L^1_{\bar\mu}$, where I write θ' for the restriction of θ to $\mathfrak{B}^f$.

proof I write P for P_π.

(a)-(b) For $v \in L^1_{\bar\nu}$, $a \in \mathfrak{A}^f$ set $\nu_v(a) = \int_{\pi a} v$. Then $\nu_v : \mathfrak{A}^f \to \mathbb{R}$ is additive, bounded (by $\|v\|_1$) and if $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{a\in A} |\nu_v(a)| \le \inf_{a\in A} \int |v| \times \chi\pi a = 0$$

because $a \mapsto \int |v| \times \chi\pi a$ is a composition of order-continuous functions, therefore order-continuous. So 365Eb tells us that there is a $Pv \in L^1_{\bar\mu}$ such that $\int_a Pv = \nu_v(a) = \int_{\pi a} v$ for every $a \in \mathfrak{A}^f$. By 365Dd, this formula defines Pv uniquely. Consequently P must be linear (since $Pv_1 + Pv_2$, αPv will always have the properties defining $P(v_1 + v_2)$, $P(\alpha v)$).

If $v \ge 0$ in $L^1_{\bar\nu}$, then $\int_a Pv = \int_{\pi a} v \ge 0$ for every $a \in \mathfrak{A}^f$, so $Pv \ge 0$ (365Dd); thus P is positive. It must therefore be norm-continuous and order-continuous (355C, 355Ka).

Again supposing that $v \ge 0$, we have

$$\|Pv\|_1 = \int Pv = \sup_{a\in\mathfrak{A}^f} \int_a Pv = \sup_{a\in\mathfrak{A}^f} \int_{\pi a} v \le \|v\|_1$$

(using 365Dd). For general $v \in L^1_{\bar\nu}$,

$$\|Pv\|_1 = \||Pv|\|_1 \le \|P|v|\|_1 \le \|v\|_1.$$

(c) For any $c \in \mathfrak{A}^f$,

$$\int_c Pv\times\chi a=\int_{c\cap a}Pv=\int_{\pi(c\cap a)}v=\int_{\pi c}v\times\chi\pi a=\int_c P(v\times\chi\pi a).$$

(d) Now suppose that $\pi[\mathfrak{A}^f]$ is order-dense. Take any v, $v'\in L^1_{\bar\nu}$ such that $v\wedge v'=0$. **?** Suppose, if possible, that $u=Pv\wedge Pv'>0$. Take $\alpha>0$ such that $a=[\![u>\alpha]\!]$ is non-zero. Since

$$\int_{\pi a}v=\int_a Pv\ge\int_a u>0,$$

$b=\pi a\cap[\![v>0]\!]\ne 0$. Let $c\in\mathfrak{A}^f$ be such that $0\ne\pi c\subseteq b$; then $\pi(a\cap c)=\pi c\ne 0$, so $a\cap c\ne 0$, and

$$0<\int_{a\cap c}u\le\int_{a\cap c}Pv'\le\int_{\pi c}v'.$$

But $\pi c\subseteq[\![v>0]\!]$ and $v\wedge v'=0$ so $\int_{\pi c}v'=0$. **X**

So $Pv\wedge Pv'=0$. As v, v' are arbitrary, P is a Riesz homomorphism (352G).

Next, if $v\ge 0$ in $L^1_{\bar\nu}$,

$$\int Pv=\sup_{a\in\mathfrak{A}^f}\int_a Pv=\sup_{a\in\mathfrak{A}^f}\int_{\pi a}v=\int v$$

because $\pi[\mathfrak{A}^f]$ is upwards-directed and has supremum 1 in $\mathfrak{B}$. So, for general $v\in L^1_{\bar\nu}$,

$$\|Pv\|_1=\int|Pv|=\int P|v|=\int|v|=\|v\|_1,$$

and P is norm-preserving.

(e) Next suppose that $(\mathfrak{B},\bar\nu)$ is semi-finite and that π is injective.

(i) If $u>0$ in $L^1_{\bar\mu}$, there is a $v>0$ in $L^1_{\bar\nu}$ such that $Pv\le u$ and $\int Pv\ge\int v$. **P** Let $\delta>0$ be such that $a=[\![u>\delta]\!]\ne 0$. Then $\pi a\ne 0$. Because $(\mathfrak{B},\bar\nu)$ is semi-finite, there is a non-zero $b\in\mathfrak{B}^f$ such that $b\subseteq\pi a$. Set $u_1=P(\chi b)$. Then $u_1\ge 0$, and also $\int_a u_1=\bar\nu b>0$,

$$\int_{1\setminus a}u_1=\sup_{c\in\mathfrak{A}^f}\int_{c\setminus a}u_1=\sup_{c\in\mathfrak{A}^f}\int_{\pi c\setminus\pi a}\chi b=0.$$

So $u_1\times\chi(1\setminus a)=0$ and $0\ne[\![u_1>0]\!]\subseteq a$. Let $\gamma>0$ be such that $[\![u_1>\gamma]\!]\ne[\![u_1>0]\!]$, and set $a_1=a\setminus[\![u_1>\gamma]\!]$, $v=\delta\gamma^{-1}\chi(b\cap\pi a_1)$. Then

$$Pv=\delta\gamma^{-1}P(\chi b\times\chi(\pi a_1))=\delta\gamma^{-1}P(\chi b)\times\chi a_1=\delta\gamma^{-1}u_1\times\chi a_1\le\delta\chi a\le u$$

because

$$[\![u_1\times\chi a_1>\gamma]\!]\subseteq[\![u_1>\gamma]\!]\cap a_1=0$$

so

$$u_1\times\chi a_1\le\gamma\chi[\![u_1>0]\!]\le\gamma\chi a.$$

Also $Pv>0$ because $a_1\cap[\![u_1>0]\!]\ne 0$, so $v\ne 0$; and

$$\int Pv\ge\int_{a_1}Pv=\int_{\pi a_1}v=\int v.\ \mathbf{Q}$$

(ii) Now take any $u\ge 0$ in $L^1_{\bar\mu}$, and set $B=\{v:v\in L^1_{\bar\nu},\ v\ge 0,\ Pv\le u,\ \int v\le\int Pv\}$. Let $C\subseteq B$ be a maximal upwards-directed set (applying Zorn's Lemma to the family $\mathfrak{P}$ of all upwards-directed subsets of B). We have

$$\sup_{v\in C}\int v\le\sup_{v\in C}\int Pv\le\|u\|_1,$$

so $v_0=\sup C$ is defined in $L^1_{\bar\nu}$ (365Df). Because P is order-continuous, $Pv_0=\sup P[C]\le u$, and

$$\int Pv_0=\sup_{v\in C}\int Pv\le\sup_{v\in C}\int v=\int v_0.$$

? Suppose, if possible, that $Pv_0\ne u$. In this case, by (α), there is a $v_1>0$ such that $Pv_1\le u-Pv_0$, $\int v_1\le\int Pv_1$. In this case, $v_0+v_1\in B$, so $C'=C\cup\{v_0+v_1\}$ is an upwards-directed subset of B strictly larger than C, which is impossible. **X** Thus $Pv_0=u$; also

$$\|v_0\|_1=\int v_0\le\int Pv_0=\|Pv_0\|_1.$$

(iii) Now take any $u\in L^1_{\bar\mu}$. By (ii), there are non-negative v_1, $v_2\in L^1_{\bar\nu}$ such that $Pv_1=u^+$, $Pv_2=u^-$, $\|v_1\|_1\le\|u^+\|_1$, $\|v_2\|_1\le\|u^-\|_1$. Setting $v=v_1-v_2$, we have $Pv=u$. Also we must have

$$\|v\|_1 \le \|v_1\|_1 + \|v_2\|_1 \le \|u^+\|_1 + \|u^-\|_1 = \|u\|_1 \le \|v\|_1,$$

so $\|v\|_1 = \|u\|_1$, as required.

(f) As usual, this is a consequence of the uniqueness of P. However (because I do not assume that $\pi[\mathfrak{A}^f] \subseteq \mathfrak{B}^f$) there is an extra refinement: we need to know that $\int_b P_{\theta'}w = \int_{\theta b} w$ for every $b \in \mathfrak{B}$, $w \in L^1_{\bar\lambda}$. **P** Because θ is order-continuous and $(\mathfrak{B}, \bar\nu)$ is semi-finite, $\theta b = \sup\{\theta b' : b' \in \mathfrak{B}^f,\ b' \subseteq b\}$, so if $w \ge 0$ then

$$\int_{\theta b} w = \sup_{b' \in \mathfrak{B}^f, b' \subseteq b} \int_{\theta b'} w = \sup_{b' \in \mathfrak{B}^f, b' \subseteq b} \int_{b'} P_{\theta'} w = \int_b P_{\theta'} w.$$

Expressing w as $w^+ - w^-$, we see that the same is true for every $w \in L^1_{\bar\nu}$. **Q**

Now we can say that $PP_{\theta'}$ is a positive linear operator from $L^1_{\bar\lambda}$ to $L^1_{\bar\mu}$ such that

$$\int_a PP_{\theta'}w = \int_{\pi a} P_{\theta'}w = \int_{\theta\pi a} w = \int_a P_{\theta\pi} w$$

whenever $a \in \mathfrak{A}^f$, $w \in L^1_{\bar\lambda}$.

365Q Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\mu)$ be measure algebras and $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ a measure-preserving ring homomorphism.

(a) In the language of 365O-365P above, $P_\pi T_\pi$ is the identity operator on $L^1(\mathfrak{A}, \bar\mu)$.

(b) If π is surjective (so that it is an isomorphism between $\mathfrak{A}^f$ and $\mathfrak{B}^f$) then $P_\pi = T_\pi^{-1} = T_{\pi^{-1}}$, $T_\pi = P_\pi^{-1} = P_{\pi^{-1}}$.

proof (a) If $u \in L^1_{\bar\mu}$, $a \in \mathfrak{A}^f$ then

$$\int_a P_\pi T_\pi u = \int_{\pi a} T_\pi u = \int_a u.$$

So $u = P_\pi T_\pi u$, by 365Dd.

(b) From 365Od, we know that T_π is surjective, while $P_\pi T_\pi$ is the identity, so that $P_\pi = T_\pi^{-1}$, $T_\pi = P_\pi^{-1}$. As for $T_{\pi^{-1}}$, 365Oe tells us that $T_{\pi^{-1}} = T_\pi^{-1}$; so

$$P_{\pi^{-1}} = T_{\pi^{-1}}^{-1} = T_\pi.$$

365R Conditional expectations It is a nearly universal rule that any investigation of L^1 spaces must include a look at conditional expectations. In the present context, they take the following form.

(a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra; write $\bar\nu$ for the restriction $\bar\mu\restriction\mathfrak{B}$. The identity map from $\mathfrak{B}$ to $\mathfrak{A}$ induces operators $T : L^1_{\bar\nu} \to L^1_{\bar\mu}$ and $P : L^1_{\bar\mu} \to L^1_{\bar\nu}$. If we take $L^0(\mathfrak{A})$ to be defined as the set of functions from $\mathbb{R}$ to $\mathfrak{A}$ described in 364A, then $L^0(\mathfrak{B})$ becomes a subset of $L^0(\mathfrak{A})$ in the literal sense, and T is actually the identity operator associated with the subset $L^1_{\bar\nu} \subseteq L^1_{\bar\mu}$; $L^1_{\bar\nu}$ is a norm-closed and order-closed Riesz subspace of $L^1_{\bar\mu}$. P is a positive linear operator, while PT is the identity, so P is a projection from $L^1_{\bar\mu}$ onto $L^1_{\bar\nu}$. P is defined by the familiar formula

$$\int_b Pu = \int_b u \text{ for every } u \in L^1_{\bar\mu},\ b \in \mathfrak{B},$$

so is the conditional expectation operator in the sense of 242J.

(b) Just as in 233I-233J and 242K, we have a fundamental inequality concerning convex functions: if $h : \mathbb{R} \to \mathbb{R}$ is a convex function and $u \in L^1_{\bar\mu}$, then $h(\int u) \le \int \bar h(u)$; and if $\bar h(u) \in L^1_{\bar\mu}$ (364I), then $\bar h(Pu) \le P(\bar h(u))$. **P** I repeat the proof of 233I-233J. For each $q \in \mathbb{Q}$, take $\beta_q \in \mathbb{R}$ such that $h(t) \ge h_q(t) = h(q) + \beta_q(t - q)$ for every $t \in \mathbb{R}$, so that $h(t) = \sup_{q\in\mathbb{Q}} h_q(t)$ for every $t \in \mathbb{R}$, and $\bar h(u) = \sup_{q\in\mathbb{Q}} \bar h_q(u)$ for every $u \in L^0 = L^0(\mathfrak{A})$. (This is because

$$\begin{aligned}[\![\bar h(u) > \alpha]\!] &= [\![u \in h^{-1}[\,]\alpha,\infty[\,]]\!] = [\![u \in \textstyle\bigcup_{q\in\mathbb{Q}} h_q^{-1}[\,]\alpha,\infty[\,]]\!] \\ &= \sup_{q\in\mathbb{Q}} [\![u \in h_q^{-1}[\,]\alpha,\infty[\,]]\!] = \sup_{q\in\mathbb{Q}} [\![\bar h_q(u) > \alpha]\!]\end{aligned}$$

for every $\alpha \in \mathbb{R}$.) But setting $e = \chi 1$, we see that $\bar h_q(u) = h(q)e + \beta_q(u - qe)$ for every $u \in L^0$, so that

$$\int h_q(u) = h(q) + \beta_q(\int u - q) = h_q(\int u),$$

$$P(\bar{h}_q(u)) = h(q)e + \beta_q(Pu - qe) = \bar{h}_q(Pu)$$

because $\int e = 1$ and $Pe = e$. Taking the supremum over q, we get

$$h(\int u) = \sup_{q\in\mathbb{Q}} h_q(\int u) = \sup_{q\in\mathbb{Q}} \int \bar{h}_q(u) \le \int \bar{h}(u),$$

and if $\bar{h}(u) \in L^1_{\bar{\mu}}$ then

$$\bar{h}(Pu) = \sup_{q\in\mathbb{Q}} \bar{h}_q(Pu) = \sup_{q\in\mathbb{Q}} P(\bar{h}_q(u)) \le P(\bar{h}(u)). \textbf{ Q}$$

Of course the result in this form can also be deduced from 233I-233J if we represent $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a probability space (X, Σ, μ) and set $\mathrm{T} = \{E : E \in \Sigma,\ E^\bullet \in \mathfrak{B}\}$.

(c) I note here a fact which is occasionally useful. If $u \in L^1_{\bar{\mu}}$ is non-negative, then $[\![Pu > 0]\!] = \mathrm{upr}([\![u > 0]\!], \mathfrak{C})$, the upper envelope of $[\![u > 0]\!]$ in $\mathfrak{C}$ as defined in 314V. **P** We have only to observe that, for $c \in \mathfrak{C}$,

$$c \cap [\![Pu > 0]\!] = 0 \iff \chi c \times Pu = 0 \iff \int_c Pu = 0 \iff \int_c u = 0 \iff c \cap [\![u > 0]\!] = 0.$$

Taking complements, $c \supseteq [\![Pu > 0]\!]$ iff $c \supseteq [\![u > 0]\!]$. **Q**

365S Recovering the algebra: Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Then $\mathfrak{A}$ is isomorphic to the band algebra of $L^1 = L^1(\mathfrak{A}, \bar{\mu})$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\bar{\mu}$, $\bar{\nu}$ two measures on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}, \bar{\nu})$ are both semi-finite measure algebras. Then $L^1(\mathfrak{A}, \bar{\mu})$ is isomorphic, as Banach lattice, to $L^1(\mathfrak{A}, \bar{\nu})$.

proof (a) Because $(\mathfrak{A}, \bar{\mu})$ is semi-finite, L^1 is order-dense in $L^0 = L^0(\mathfrak{A})$ (365G). Consequently, L^1 and L^0 have isomorphic band algebras (353D). But the band algebra of L^0 is just its algebra of projection bands (because $\mathfrak{A}$ and therefore L^0 are Dedekind complete, see 364O and 353I), which is isomorphic to $\mathfrak{A}$ (364Q).

(b) Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the identity map. Regarding π as an order-continuous Boolean homomorphism from $\mathfrak{A}^f_{\bar{\mu}} = \{a : \bar{\mu}a < \infty\}$ to $(\mathfrak{A}, \bar{\nu})$, we have an associated positive linear operator $P = P_\pi : L^1_{\bar{\nu}} \to L^1_{\bar{\mu}}$; similarly, we have $Q = P_{\pi^{-1}} : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$, and both P and Q have norm at most 1 (365Pb). Now 365Pf assures us that QP is the identity operator on $L^1_{\bar{\nu}}$ and PQ is the identity operator on $L^1_{\bar{\mu}}$. So P and Q are the two halves of a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1_{\bar{\nu}}$.

365T Having opened the question of varying measures on a single Boolean algebra, this seems an appropriate moment for a general description of how they interact.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\bar{\mu} : \mathfrak{A} \to [0, \infty]$, $\bar{\nu} : \mathfrak{A} \to [0, \infty]$ two functions such that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}, \bar{\nu})$ are both semi-finite (therefore localizable) measure algebras.

(a) There is a unique $u \in L^0 = L^0(\mathfrak{A})$ such that (if we allow ∞ as a value of the integral) $\int_a u\, d\bar{\mu} = \bar{\nu}a$ for every $a \in \mathfrak{A}$.

(b) For every $v \in L^1_{\bar{\nu}}$, $\int v\, d\bar{\nu} = \int u \times v\, d\bar{\mu}$.

(c) u is strictly positive (i.e., $[\![u > 0]\!] = 1$) and, writing $\frac{1}{u}$ for the multiplicative inverse of u, $\int_a \frac{1}{u} d\bar{\nu} = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

proof (a) Because $(\mathfrak{A}, \bar{\nu})$ is semi-finite, there is a partition of unity $D \subseteq \mathfrak{A}$ such that $\bar{\nu}d < \infty$ for every $d \in D$. For each $d \in D$, the functional $a \mapsto \bar{\nu}(a \cap d) : \mathfrak{A} \to \mathbb{R}$ is completely additive, so there is a $u_d \in L^1_{\bar{\mu}}$ such that $\int_a u_d d\bar{\mu} = \bar{\nu}(a \cap d)$ for every $a \in \mathfrak{A}$. Because $\int_a u_d d\bar{\mu} \ge 0$ for every a, $u_d \ge 0$. Because $\int_{1\setminus d} u_d = 0$, $[\![u_d > 0]\!] \subseteq d$. Now $u = \sup_{d\in D} u_d$ is defined in L^0. **P** (See 368K below.) For $n \in \mathbb{N}$, set $c_n = \sup_{d\in D} [\![u_d > n]\!]$. If d, $d' \in D$ are distinct, then $d \cap [\![u_{d'} > n]\!] = 0$, so $d \cap c_n = [\![u_d > n]\!]$. Set $c = \inf_{n\in\mathbb{N}} c_n$. If $d \in D$, then

$$d \cap c = \inf_{n\in\mathbb{N}} d \cap c_n = \inf_{n\in\mathbb{N}} [\![u_d > n]\!] = 0.$$

But $c \subseteq c_0 \subseteq \sup D$, so $c = 0$. By 364Ma, $\{u_d : d \in D\}$ is bounded above in L^0, so has a supremum, because L^0 is Dedekind complete, by 364O. **Q**

For finite $I \subseteq D$ set $\tilde{u}_I = \sum_{d\in I} u_d = \sup_{d\in I} u_d$ (because $u_d \wedge u_c = 0$ for distinct c, $d \in D$). Then $u = \sup\{\tilde{u}_I : I \subseteq D,\ I \text{ is finite}\}$. So, for any $a \in \mathfrak{A}$,

$$\int_a u\,d\bar\mu = \sup_{I\subseteq D \text{ is finite}} \int_a \tilde u_I\,d\bar\mu$$

(365Dh)

$$= \sup_{I\subseteq D \text{ is finite}} \sum_{d\in I}\int_a u_d d\bar\mu = \sup_{I\subseteq D \text{ is finite}} \sum_{d\in I}\bar\nu(a\cap d) = \bar\nu a.$$

Note that if $a\in\mathfrak A$ is non-zero, then $\bar\nu a>0$, so $a\cap[\![u>0]\!]\ne0$; consequently $[\![u>0]\!]=1$.

To see that u is unique, observe that if u' has the same property then for any $d\in D$

$$\int_a(u\times\chi d)d\bar\mu = \bar\nu(a\cap d) = \int_a(u'\times\chi d)$$

for every $a\in\mathfrak A$, so that $u\times\chi d = u'\times\chi d$; because $\sup D=1$ in $\mathfrak A$, u must be equal to u'.

(b) Use 365Hb, with π and T the identity maps.

(c) In the same way, there is a $w\in L^0$ such that $\int_a w\,d\bar\nu = \bar\mu a$ for every $a\in\mathfrak A$. To relate u and w, observe that applying 365Hb we get

$$\int w\times\chi a\times u\,d\bar\mu = \int w\times\chi a\,d\bar\nu$$

for every $a\in\mathfrak A$, that is, $\int_a w\times u\,d\bar\mu=\bar\mu a$ for every a. But from this we see that $w\times u\times\chi b=\chi b$ at least when $\bar\mu b<\infty$, so that $w\times u=\chi 1$ is the multiplicative identity of L^0, and $w=\frac{1}{u}$.

365X Basic exercises (a) Let $(\mathfrak A,\bar\mu)$ be a measure algebra, and $u\in L^1_{\bar\mu}$. Show that

$$\int u = \int_0^\infty \bar\mu[\![u>\alpha]\!]\,d\alpha - \int_{-\infty}^0 \bar\mu(1\setminus[\![u>\alpha]\!])\,d\alpha.$$

>(b) Let $(\mathfrak A,\bar\mu)$ be any measure algebra, and $u\in L^1_{\bar\mu}$. (i) Show that $\|u\|_1\le 2\sup_{a\in\mathfrak A^f}|\int_a u|$. (*Hint*: 246F.) (ii) Show that for any $\epsilon>0$ there is an $a\in\mathfrak A^f$ such that $|\int u-\int_b u|\le\epsilon$ whenever $a\subseteq b\in\mathfrak A$.

>(c) Let U be an L-space. If $\langle u_n\rangle_{n\in\mathbb N}$ is any norm-bounded sequence in U^+, show that $\liminf_{n\to\infty}u_n = \sup_{n\in\mathbb N}\inf_{m\ge n}u_m$ is defined in U, and that $\int\liminf_{n\to\infty}u_n\le\liminf_{n\to\infty}\int u_n$.

(d) Let U be an L-space. Let $\mathcal F$ be a filter on U^+ such that $\{u:u\ge0,\ \|u\|\le k\}$ belongs to $\mathcal F$ for some $k\in\mathbb N$. Show that $u_0=\sup_{F\in\mathcal F}\inf F$ is defined in U, and that $\int u_0\le\sup_{F\in\mathcal F}\inf_{u\in F}\int u$.

(e) Let $(\mathfrak A,\bar\mu)$ be a measure algebra and $A\subseteq L^1_{\bar\mu}$ a non-empty set. Show that A is bounded above in $L^1_{\bar\mu}$ iff

$$\sup\{\sum_{i=0}^n\int_{a_i}u_i : a_0,\dots,a_n \text{ is a partition of unity in } \mathfrak A,\ u_0,\dots,u_n\in A\}$$

is finite, and that in this case the supremum is $\int\sup A$. (*Hint*: given $u_0,\dots,u_n\in A$, set $b_{ij}=[\![u_i\ge u_j]\!]$, $b_i=\sup_{j\ne i}b_{ij}$, $a_i=b_i\setminus\sup_{j<i}b_j$, and show that $\int\sup_{i\le n}u_i=\sum_{i=0}^n\int_{a_i}u_i$.)

(f) Let $(\mathfrak A,\bar\mu)$ be any measure algebra and $\nu:\mathfrak A^f\to\mathbb R$ a bounded additive function. Show that the following are equiveridical: (i) there is a $u\in L^1_{\bar\mu}$ such that $\nu a=\int_a u$ for every $a\in\mathfrak A^f$; (ii) for every $\epsilon>0$ there is a $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $\bar\mu a\le\delta$; (iii) for every $\epsilon>0$, $c\in\mathfrak A^f$ there is a $\delta>0$ such that $\nu a\le\epsilon$ whenever $a\subseteq c$ and $\bar\mu a\le\delta$; (iv) for every $\epsilon>0$ there are $c\in\mathfrak A^f$, $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $a\in\mathfrak A^f$ and $\bar\mu(a\cap c)\le\delta$; (v) $\lim_{n\to\infty}\nu a_n=0$ whenever $\langle a_n\rangle_{n\in\mathbb N}$ is a non-increasing sequence in $\mathfrak A^f$ with infimum 0.

(g) Let $(\mathfrak A,\bar\mu)$ and $(\mathfrak B,\bar\nu)$ be measure algebras, and $\pi:\mathfrak A\to\mathfrak B$ a sequentially order-continuous Boolean homomorphism. Let $T:L^0(\mathfrak A)\to L^0(\mathfrak B)$ be the Riesz homomorphism associated with π (364R). Suppose that $w\ge0$ in $L^0(\mathfrak B)$ is such that $\int_{\pi a}w\,d\bar\nu=\bar\mu a$ whenever $a\in\mathfrak A$. Show that for any $u\in L^0(\mathfrak A,\bar\mu)$, $\int Tu\times w\,d\bar\nu=\int u\,d\bar\mu$ whenever either is defined in $[-\infty,\infty]$.

>(h) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $a\in\mathfrak{A}$; write $\mathfrak{A}_a$ for the principal ideal it generates. Show that if π is the identity embedding of $\mathfrak{A}^f\cap\mathfrak{A}_a$ into $\mathfrak{A}^f$, then T_π, as defined in 365O, identifies $L^1(\mathfrak{A}_a,\bar{\mu}\restriction\mathfrak{A}_a)$ with a band in $L^1(\mathfrak{A},\bar{\mu})$.

>(i) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$. Let $\phi: X\to Y$ be an inverse-measure-preserving function and $\pi:\mathfrak{B}\to\mathfrak{A}$ the corresponding measure-preserving homomorphism (324M). Show that $T_\pi: L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ (365O) corresponds to the map $g^\bullet\mapsto(g\phi)^\bullet: L^1(\nu)\to L^1(\mu)$ of 242Xf.

(j) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Let $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ be a ring homomorphism such that, for some $\gamma>0$, $\bar{\nu}(\pi a)\le\gamma\bar{\mu}a$ for every $a\in\mathfrak{A}^f$. (i) Show that there is a unique order-continuous Riesz homomorphism $T: L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ such that $T(\chi a)=\chi(\pi a)$ whenever $a\in\mathfrak{A}^f$, and that $\|T\|\le\gamma$. (ii) Show that $[\![Tu>\alpha]\!]=\pi[\![u>\alpha]\!]$ for every $u\in L^1(\mathfrak{A},\mu)$, $\alpha>0$. (iii) Show that T is surjective iff π is, injective iff π is. (iv) Show that T is norm-preserving iff $\bar{\nu}(\pi a)=\bar{\mu}a$ for every $a\in\mathfrak{A}^f$.

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $T: L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ and $P: L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ be the operators corresponding to $\pi\restriction\mathfrak{A}^f$, as described in 365O-365P, and $\tilde{T}: L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ the operator corresponding to π, as described in 363F. (i) Show that $T(u\times v)=Tu\times\tilde{T}v$ for every $u\in L^1_{\bar{\mu}}$, $v\in L^\infty(\mathfrak{A})$. (ii) Show that if π is order-continuous, then $\int Pv\times u=\int v\times\tilde{T}u$ for every $u\in L^\infty(\mathfrak{A})$, $v\in L^1_{\bar{\nu}}$.

>(l) Let (X,Σ,μ) be a probability space, with measure algebra $(\mathfrak{A},\bar{\mu})$, and let T be a σ-subalgebra of Σ. Set $\nu=\mu\restriction\mathrm{T}$, $\mathfrak{B}=\{F^\bullet:F\in\mathrm{T}\}\subseteq\mathfrak{A}$, $\bar{\nu}=\bar{\mu}\restriction\mathfrak{B}$, so that $(\mathfrak{B},\bar{\nu})$ is a measure algebra. Let $\pi:\mathfrak{B}\to\mathfrak{A}$ be the identity homomorphism. Show that $T_\pi: L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ (365O) corresponds to the canonical embedding of $L^1(\nu)$ in $L^1(\mu)$ described in 242Jb, while $P_\pi: L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ (365P) corresponds to the conditional expectation operator described in 242Jd.

(m) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\hat{\mu})$ its localization (322P). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1(\widehat{\mathfrak{A}},\hat{\mu})$, so that the band algebra of $L^1_{\bar{\mu}}$ can be identified with the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functions such that $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{A},\bar{\nu})$ are measure algebras. Show that $L^1_{\bar{\mu}}\subseteq L^1_{\bar{\nu}}$ (as subsets of $L^0(\mathfrak{A})$) iff there is a $\gamma>0$ such that $\bar{\nu}a\le\gamma\bar{\mu}a$ for every $a\in\mathfrak{A}$. (*Hint*: show that the identity operator from $L^1_{\bar{\mu}}$ to $L^1_{\bar{\nu}}$ is bounded.)

(o) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, I_∞ the ideal of 'purely infinite' elements of $\mathfrak{A}$, $\bar{\mu}_{sf}$ the measure on $\mathfrak{B}=\mathfrak{A}/I_\infty$ (322Xa). Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be the canonical map. Show that T_π, as defined in 365O, is a Banach lattice isomorphism between $L^1(\mathfrak{A},\bar{\mu})$ and $L^1(\mathfrak{B},\bar{\mu}_{sf})$.

(p) Let (X,Σ,μ) be a a semi-finite measure space. Show that $L^1(\mu)$ is separable iff μ is σ-finite and has countable Maharam type.

365Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$. Show that the topological density of $L^1(\mathfrak{A},\bar{\mu})$ (331Yf) is $\max(\omega,\tau(\mathfrak{A}),c(\mathfrak{A}))$, where $\tau(\mathfrak{A})$, $c(\mathfrak{A})$ are the Maharam type and cellularity of $\mathfrak{A}$.

365 Notes and comments You should not suppose that L^1 spaces appear in the second half of this chapter because they are of secondary importance. Indeed I regard them as the most important of all function spaces. I have delayed the discussion of them for so long because it is here that for the first time we need measure algebras in an essential way.

The actual definition of $L^1_{\bar{\mu}}$ which I give is designed for speed rather than illumination; I seek only a formula, visibly independent of any particular representation of $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space, for which I can prove 365B. I use a relatively primitive argument in 365B, not appealing to Fubini's theorem; of course the formula $\int_0^\infty\mu\{x:|f(x)|>\alpha\}d\alpha=\int|f(x)|dx$ can also be regarded as a reversal of order of integration in a double integral (252N, 252O). 365C-365D and 365Ea are now elementary. In 365Eb I take a page to describe a form of the Radon-Nikodým theorem which is applicable to arbitrary measure algebras, at the cost of dealing with functionals on the ring $\mathfrak{A}^f$ rather than on the whole algebra $\mathfrak{A}$. This is less for the sake of applications than to emphasize one of the central properties of L^1: it depends only on $\mathfrak{A}^f$ and $\bar{\mu}\restriction\mathfrak{A}^f$. For alternative versions of the condition 365Eb(i) see 365Xf.

The convergence theorems (B.Levi's theorem, Fatou's lemma and Lebesgue's dominated convergence theorem) are so central to the theory of integrable functions that it is natural to look for versions in the language here. Corresponding to B.Levi's theorem is the Levi property of a norm in an L-space; note how the abstract formulation makes it natural to speak of general upwards-directed families rather than of non-decreasing sequences, though the sequential form is so often used that I have spelt it out (365C). In the same way, the integral becomes order-continuous rather than just sequentially order-continuous (365Da). Corresponding to Fatou's lemma we have 365Xc-365Xd. For abstract versions of Lebesgue's theorem I will wait until §367.

In 365H I have deliberately followed the hypotheses of 235A and 235T. Of course 365H can be deduced from these if we use the Stone representations of $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$, so that π can be represented by a function between the Stone spaces (312P). But 365H is essentially simpler, because the technical problems concerning measurability which took up so much of §235 have been swept under the carpet. In the same way, 365Xg corresponds to 235E. Here we have a fair example of the way in which the abstract expression in terms of measure algebras can be tidier than the expression in terms of measure spaces. But in my view this is because here, at least, some of the mathematics has been left out.

365J is a re-run of 243G, but with the additional refinement that I examine the action of L^1 on L^∞ (the operator S) as well as the action of L^∞ on L^1 (the operator T). Note that (b-iii) and (c) of the proof of 365J depend on the Radon-Nikodým theorem; these parts lie a step deeper than the rest. 365L-365N correspond closely to 361F-361H and 363E.

Theorems 365O-365Q lie at the centre of my picture of L^1 spaces, and are supposed to show their dual nature. Starting from a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ we have two essentially different routes to the L^1-space: we can either build it up from characteristic functions of elements of finite measure, so that it is naturally embedded in $L^0(\mathfrak{A})$, or we can think of it as the order-continuous dual of $L^\infty(\mathfrak{A})$. The first is a 'covariant' construction (signalled by the formula $T_{\theta\pi}=T_\theta T_\pi$ in 365Oe) and the second is 'contravariant' (so that $P_{\theta\pi}=P_\pi P_{\theta'}$ in 365Pf). The first construction is the natural one if we are seeking to copy the ideas of §242, but the second arises inevitably if we follow the ordinary paths of functional analysis and study dual spaces whenever they appear. The link between them is the Radon-Nikodým theorem.

I have deliberately written out 365O and 365P with different hypotheses on the homomorphism π in the hope of showing that the two routes to L^1 really are different, and can be expected to tell us different things about it. I use the letter P in 365P in order to echo the language of 242J; in the most important context, in which $\mathfrak{A}$ is actually a subalgebra of $\mathfrak{B}$ and π is the identity map, P is a kind of conditional expectation operator (365R). I note that in the proof of 365Pe I have returned to first principles, using some of the ideas of the Radon-Nikodým theorem (232E), but a different approach to the exhaustion step (converting 'for every $u>0$ there is a $v>0$ such that $Pv\le u$' into 'P is surjective'). I chose the somewhat cruder method in 232E (part (c) of the proof) in order to use the weakest possible form of the axiom of choice. In the present context such scruples seem absurd.

I used the words 'covariant' and 'contravariant' above; of course this distinction depends on the side of the mirror on which we are standing; if our measure-preserving homomorphism is derived (contravariantly) from an inverse-measure-preserving transformation, then the T's become contravariant (365Xi). An important component of this work, for me, is the fact that not all measure-preserving homomorphisms between measure algebras can be represented by inverse-measure-preserving functions (343Jb, 343M).

I have already remarked (in the notes to §244) that the properties of $L^1(\mu)$ are not much affected by peculiarities in a measure space (X,Σ,μ), because (unlike L^0 or L^∞) they really depend only on $\mathfrak{A}^f$, the ring of elements of finite measure in the measure algebra. (See 365O-365Q and 365Xm-365Xn.) Note that while the algebra $\mathfrak{A}$ is uniquely determined (given that $(\mathfrak{A},\bar{\mu})$ is localizable, 365Sa), the measure $\bar{\mu}$ is not; if $\mathfrak{A}$ is any algebra carrying two non-isomorphic semi-finite measures, the corresponding L^1 spaces are still isomorphic (365Sb). For instance, the L^1-spaces of Lebesgue measure μ on $\mathbb{R}$, and the subspace measure $\mu_{[0,1]}$ on $[0,1]$, are isomorphic, though their measure algebras are not.

I make no attempt here to add to the results in §§246, 247, 354 and 356 concerning uniform integrability and weak compactness. Once we have left measure spaces behind, these ideas belong to the theory of Banach lattices, and there is little to relate them to the questions dealt with in this section. But see 373Xj and 373Xn below.

366 L^p

In this section I apply the methods of this chapter to L^p spaces, where $1 < p < \infty$. The constructions proceed without surprises up to 366E, translating the ideas of §244 by the methods used in §365. Turning to the action of Boolean homomorphisms on L^p spaces, I introduce a space M^0, which can be regarded as the part of L^0 that can be determined from the ring $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure (366F), and which includes L^p whenever $1 \le p < \infty$. Now a measure-preserving ring homomorphism from $\mathfrak{A}^f$ to $\mathfrak{B}^f$ acts on the M^0 spaces in a way which includes injective Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ and surjective positive linear operators from $L^p(\mathfrak{B},\bar\nu)$ to $L^p(\mathfrak{A},\bar\mu)$ (366H). The latter may be regarded as conditional expectation operators (366J). The case $p = 2$ (366K-366L) is of course by far the most important.

366A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and suppose that $1 < p < \infty$. For $u \in L^0(\mathfrak{A})$, define $|u|^p \in L^0(\mathfrak{A})$ by setting

$$\begin{aligned}[\![|u|^p > \alpha]\!] &= [\![|u| > \alpha^{1/p}]\!] \text{ if } \alpha \ge 0\\ &= 1 \text{ if } \alpha < 0.\end{aligned}$$

(In the language of 364I, $|u|^p = \bar h(u)$, where $h(t) = |t|^p$ for $t \in \mathbb{R}$.) Set

$$L^p_{\bar\mu} = L^p(\mathfrak{A},\bar\mu) = \{u : u \in L^0(\mathfrak{A}),\ |u|^p \in L^1(\mathfrak{A},\bar\mu)\},$$

and for $u \in L^p_{\bar\mu}$ set

$$\|u\|_p = (\int |u|^p)^{1/p} = \||u|^p\|_1^{1/p}.$$

366B Theorem Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ (364Jc) makes $L^p(\mu)$, as defined in §244, correspond to $L^p(\mathfrak{A},\bar\mu)$.

proof What we really have to check is that if $w \in L^0(\mu)$ corresponds to $u \in L^0(\mathfrak{A})$, then $|w|^p$, as defined in 244A, corresponds to $|u|^p$ as defined in 366A. But this was noted in 364Jb.

Now, because the isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ matches $L^1(\mu)$ with $L^1(\mathfrak{A},\bar\mu)$ (365B), we can be sure that $|w|^p \in L^1(\mu)$ iff $|u|^p \in L^1(\mathfrak{A},\bar\mu)$, and that in this case

$$\|w\|_p = \bigl(\int |w|^p\bigr)^{1/p} = \bigl(\int |u|^p\bigr)^{1/p} = \|u\|_p,$$

as required.

366C Corollary For any measure algebra $(\mathfrak{A},\bar\mu)$ and $p \in]1,\infty[$, $L^p = L^p(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$. It is a Dedekind complete Banach lattice under its norm $\|\,\|_p$. Setting $q = p/(p-1)$, $(L^p)^*$ is identified with $L^q(\mathfrak{A},\bar\mu)$ by the duality $(u,v) \mapsto \int u \times v$. Writing $\mathfrak{A}^f$ for the ring $\{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in L^p.

proof Because we can find a measure space (X,Σ,μ) such that $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of μ (321J), this is just a digest of the results in 244B, 244E, 244F, 244G, 244H and 244K. (Of course $S(\mathfrak{A}^f)$ corresponds to the space S of equivalence classes of simple functions in 244Ha, just as in 365F.)

366D I can add a little more, corresponding to 365C and 365J.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $p \in]1,\infty[$.

(a) The norm $\|\,\|_p$ on $L^p = L^p(\mathfrak{A},\bar\mu)$ is order-continuous.

(b) L^p has the Levi property.

(c) Setting $q = p/(p-1)$, the canonical identification of $L^q = L^q(\mathfrak{A}, \bar\mu)$ with $(L^p)^*$ is a Riesz space isomorphism between L^q and $(L^p)^\sim = (L^p)^\times$.

(d) L^p is a perfect Riesz space.

proof (a) Suppose that $A \subseteq L^p$ is non-empty, downwards-directed and has infimum 0. For $u, v \ge 0$ in L^p, $u \le v \Rightarrow u^p \le v^p$ (by the definition in 366A, or otherwise), so $B = \{u^p : u \in A\}$ is downwards-directed. If $v_0 = \inf B$ in $L^1 = L^1(\mathfrak{A}, \bar\mu)$, then $v_0^{1/p}$ (defined by the formula in 366A, or otherwise) is less than or equal to every member of A, so must be 0, and $v_0 = 0$. Accordingly $\inf B = 0$ in L^1. Because $\| \|_1$ is order-continuous (365C),

$$\inf_{u \in A} \|u\|_p = \inf_{u \in A} \|u^p\|_1^{1/p} = (\inf_{v \in B} \|v\|_1)^{1/p} = 0.$$

As A is arbitrary, $\| \|_p$ is order-continuous.

(b) Now suppose that $A \subseteq (L^p)^+$ is non-empty, upwards-directed and norm-bounded. Then $B = \{u^p : u \in A\}$ is non-empty, upwards-directed and norm-bounded in L^1. So $v_0 = \sup B$ is defined in L^1, and $v_0^{1/p}$ is an upper bound for A in L^p.

(c) By 356Dd, $(L^p)^* = (L^p)^\sim = (L^p)^\times$. The extra information we need is that the identification of L^q with $(L^p)^*$ is an order-isomorphism. **P** (α) If $w \in (L^q)^+$ and $u \in (L^p)^+$ then $u \times w \ge 0$ in L^1, so $(Tw)(u) = \int u \times w \ge 0$, writing $T : L^q \to (L^p)^*$ for the canonical bijection. As u is arbitrary, $Tw \ge 0$. As w is arbitrary, T is a positive linear operator. (β) If $w \in L^q$ and $Tw \ge 0$, consider $u = (w^-)^{q/p}$. Then $u \ge 0$ in L^p and $w^+ \times u = 0$ (because $[\![w^+ > 0]\!] \cap [\![u > 0]\!] = [\![w^+ > 0]\!] \cap [\![w^- > 0]\!] = 0$), so

$$0 \le (Tw)(u) = \int w \times u = -\int w^- \times u = -\int (w^-)^p \le 0,$$

and $\int (w^-)^p = 0$. But as $(w^-)^p \ge 0$ in L^1, this means that $(w^-)^p$ and w^- must be 0, that is, $w \ge 0$. As w is arbitrary, T^{-1} is positive and T is an order-isomorphism. **Q**

(d) This is an immediate consequence of (c), since $p = q/(q-1)$, so that L^p can be identified with $(L^q)^* = (L^q)^\times$. From 356M we see that it is also a consequence of (a) and (b).

366E Proposition Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, and $p \in [1, \infty]$. Set $q = p/(p-1)$ if $1 < p < \infty$, and $q = \infty$ if $p = 1$, $q = 1$ if $p = \infty$. Then

$$L^q(\mathfrak{A}, \bar\mu) = \{u : u \in L^0(\mathfrak{A}),\ u \times v \in L^1(\mathfrak{A}, \bar\mu) \text{ for every } v \in L^p(\mathfrak{A}, \bar\mu)\}.$$

proof (a) We already know that if $u \in L^p$ and $v \in L^q$ then $u \times v \in L^1$; this is elementary if $p \in \{1, \infty\}$ and otherwise is covered by 366C.

(b) So suppose that $u \in L^0 \setminus L^p$. If $p = 1$ then of course $\chi 1 \in L^\infty = L^q$ and $u \times \chi 1 \notin L^1$. If $p > 1$ set

$$A = \{w : w \in S(\mathfrak{A}^f),\ 0 \le w \le |u|\}.$$

Because $\bar\mu$ is semi-finite, $S(\mathfrak{A}^f)$ is order-dense in L^0 (364L), and $|u| = \sup A$. Because the norm on L^p has the Levi property (366Db, 363Ba) and A is not bounded above in L^p, $\sup_{w \in A} \|w\|_p = \infty$.

For each $n \in \mathbb{N}$ choose $w_n \in A$ with $\|w_n\|_p > 4^n$. Then there is a $v_n \in L^q$ such that $\|v_n\|_q = 1$ and $\int w_n \times v_n \ge 4^n$. **P** (α) If $p < \infty$ this is covered by 366C, since $\|w_n\|_p = \sup\{\int w_n \times v : \|v\|_q \le 1\}$. (β) If $p = \infty$ then $[\![w_n > 4^n]\!] \ne 0$; because $\bar\mu$ is semi-finite, there is a $b \subseteq [\![w_n > 4^n]\!]$ such that $0 < \bar\mu b < \infty$, and $\|\frac{1}{\bar\mu b}\chi b\|_1 = 1$, while $\int w_n \times \frac{1}{\bar\mu b}\chi b \ge 4^n$. **Q**

Because L^q is complete (363Ba, 366C), $v = \sum_{n=0}^\infty 2^{-n}|v_n|$ is defined in L^q. But now

$$\int |u| \times v \ge 2^{-n} \int w_n \times v_n \ge 2^n$$

for every n, so $u \times v \notin L^1$.

Remark This result is characteristic of perfect subspaces of L^0; see 369C and 369J.

366F The next step is to look at the action of Boolean homomorphisms, as in 365O. It will be convenient to be able to deal with all L^p spaces at once by introducing names for a pair of spaces which include all of them.

Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Write

$$M^0_{\bar{\mu}} = M^0(\mathfrak{A}, \bar{\mu}) = \{u : u \in L^0(\mathfrak{A}),\ \bar{\mu}[\![|u| > \alpha]\!] < \infty \text{ for every } \alpha > 0\},$$

$$M^{1,0}_{\bar{\mu}} = M^{1,0}(\mathfrak{A}, \bar{\mu}) = \{u : u \in M^0_{\bar{\mu}},\ u \times \chi a \in L^1(\mathfrak{A}, \bar{\mu}) \text{ whenever } \bar{\mu}a < \infty\}.$$

366G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra. Write $M^0 = M^0(\mathfrak{A}, \bar{\mu})$, etc.

(a) M^0 and $M^{1,0}$ are Dedekind complete solid linear subspaces of L^0 which include L^p for every $p \in [1, \infty[$; moreover, M^0 is closed under multiplication.

(b) If $u \in M^0$ and $u \geq 0$, there is a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u = \sup_{n\in\mathbb{N}} u_n$.

(c) $M^{1,0} = \{u : u \in L^0,\ (|u| - \epsilon\chi 1)^+ \in L^1 \text{ for every } \epsilon > 0\} = L^1 + (L^\infty \cap M^0)$.

(d) If u, $v \in M^{1,0}$ and $\int_a u \leq \int_a v$ whenever $\bar{\mu}a < \infty$, then $u \leq v$; so if $\int_a u = \int_a v$ whenever $\bar{\mu}a < \infty$, $u = v$.

proof (a) If u, $v \in M^0$ and $\gamma \in \mathbb{R}$, then for any $\alpha > 0$

$$[\![|u + v| > \alpha]\!] \subseteq [\![|u| > \tfrac{1}{2}\alpha]\!] \cup [\![|v| > \tfrac{1}{2}\alpha]\!],$$

$$[\![|\gamma u| > \alpha]\!] \subseteq [\![|u| > \tfrac{\alpha}{1+|\gamma|}]\!],$$

$$[\![|u \times v| > \alpha]\!] \subseteq [\![|u| > \sqrt{\alpha}\,]\!] \cup [\![|v| > \sqrt{\alpha}\,]\!]$$

(364F) are of finite measure. So $u + v$, γu and $u \times v$ belong to M^0. Thus M^0 is a linear subspace of L^0 closed under multiplication. If $u \in M^0$, $|v| \leq |u|$ and $\alpha > 0$, then $[\![|v| > \alpha]\!] \subseteq [\![|u| > \alpha]\!]$ is of finite measure; thus $v \in M^0$ and M^0 is a solid linear subspace of L^0. It follows that $M^{1,0}$ also is. If $u \in L^p = L^p(\mathfrak{A}, \bar{\mu})$, where $p < \infty$, and $\alpha > 0$, then $[\![|u| > \alpha]\!] = [\![|u|^p > \alpha^p]\!]$ is of finite measure, so $u \in M^0$; moreover, if $\bar{\mu}a < \infty$, then $\chi a \in L^q$, where $q = p/(p-1)$, so $u \times \chi a \in L^1$; thus $u \in M^{1,0}$.

To see that M^0 is Dedekind complete, observe that if $A \subseteq (M^0)^+$ is non-empty and bounded above by $u_0 \in M^0$, and $\alpha > 0$, then $\{[\![u > \alpha]\!] : u \in A\}$ is bounded above by $[\![u_0 > \alpha]\!] \in \mathfrak{A}^f$, so has a supremum in $\mathfrak{A}$ (321C). Accordingly $\sup A$ is defined in L^0 (364Mc) and belongs to M^0. Finally, $M^{1,0}$, being a solid linear subspace of M^0, must also be Dedekind complete.

(b) If $u \geq 0$ in M^0, then there is a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $S = S(\mathfrak{A})$ such that $u = \sup_{n\in\mathbb{N}} u_n$ and $u_0 \geq 0$ (364Kd). But now every u_n belongs to $S \cap M^0 = S(\mathfrak{A}^f)$, just as in 365F.

(c)(i) If $u \in M^{1,0}$ and $\epsilon > 0$, then $a = [\![|u| > \epsilon]\!] \in \mathfrak{A}^f$, so $u \times \chi a \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$; but $(|u| - \epsilon\chi 1)^+ \leq |u| \times \chi a$, so $(|u| - \epsilon\chi 1)^+ \in L^1$.

(ii) Suppose that $u \in L^0$ and $(|u| - \epsilon\chi 1)^+ \in L^1$ for every $\epsilon > 0$. Then, given $\epsilon > 0$, $v = (|u| - \frac{1}{2}\epsilon\chi 1)^+ \in L^1$, and $\bar{\mu}[\![v > \frac{1}{2}\epsilon]\!] < \infty$; but $[\![|u| > \epsilon]\!] \subseteq [\![v > \frac{1}{2}\epsilon]\!]$, so also has finite measure. Thus $u \in M^0$. Next, if $a \in \mathfrak{A}^f$, then $|u \times \chi a| \leq \chi a + (|u| - \chi 1)^+ \in L^1$, so $u \in M^{1,0}$.

(iii) Of course L^1 and $L^\infty \cap M^0$ are included in $M^{1,0}$, so their linear sum also is. On the other hand, if $u \in M^{1,0}$, then

$$u = (u^+ - \chi 1)^+ - (u^- - \chi 1)^+ + (u^+ \wedge \chi 1) - (u^- \wedge \chi 1) \in L^1 + (L^\infty \cap M^0).$$

(d) Take $\alpha > 0$ and set $a = [\![u - u' > \alpha]\!]$. Because both u and u' belong to $M^{1,0}_{\bar{\mu}}$, $\bar{\mu}a < \infty$ and $\int_a u \leq \int_a u'$, that is, $\int_a u - u' \leq 0$; so a must be 0 (365Dc). As α is arbitrary, $u - u' \leq 0$ and $u \leq u'$. If $\int_a u = \int_a u'$ for every $a \in \mathfrak{A}^f$, then $u' \leq u$ so $u = u'$.

366H Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras. For $p \in [1, \infty]$, write $L^p_{\bar{\mu}}$, $L^p_{\bar{\nu}}$ for $L^p(\mathfrak{A}, \bar{\mu})$, $L^p(\mathfrak{B}, \bar{\nu})$; similarly, write $M^0_{\bar{\mu}}$ for $M^0(\mathfrak{A}, \bar{\mu})$, etc. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a)(i) We have a unique order-continuous Riesz homomorphism $T = T_\pi : M^0_{\bar{\mu}} \to M^0_{\bar{\nu}}$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}^f$.

(ii) $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in M^0_{\bar{\mu}}$ and $\alpha > 0$.

(iii) T is injective and multiplicative.

(iv) For $p \in [1,\infty]$ and $u \in M^0_{\bar\mu}$, $Tu \in L^p_{\bar\nu}$ iff $u \in L^p_{\bar\mu}$, and in this case $\|Tu\|_p = \|u\|_p$. In particular, $\int Tu = \int u$ whenever $u \in L^1_{\bar\mu}$.

(v) For $u \in M^0_{\bar\mu}$, $Tu \in M^{1,0}_{\bar\nu}$ iff $u \in M^{1,0}_{\bar\mu}$.

(b)(i) We have a unique order-continuous positive linear operator $P = P_\pi : M^{1,0}_{\bar\nu} \to M^{1,0}_{\bar\mu}$ such that $\int_a Pv = \int_{\pi a} v$ whenever $v \in M^{1,0}_{\bar\nu}$ and $a \in \mathfrak{A}^f$.

(ii) If $u \in M^0_{\bar\nu}$, $v \in M^{1,0}_{\bar\nu}$ and $v \times Tu \in M^{1,0}_{\bar\nu}$, then $P(v \times Tu) = u \times Pv$.

(iii) If $q \in [1,\infty[$ and $v \in L^q_{\bar\nu}$, then $Pv \in L^q_{\bar\mu}$ and $\|Pv\|_q \le \|v\|_q$; if $v \in L^\infty_{\bar\nu} \cap M^0_{\bar\nu}$, then $Pv \in L^\infty_{\bar\mu}$ and $\|Pv\|_\infty \le \|v\|_\infty$.

(iv) $PTu = u$ for every $u \in M^{1,0}_{\bar\mu}$; in particular, $P[L^p_{\bar\nu}] = L^p_{\bar\mu}$ for every $p \in [1,\infty[$.

(c) If $(\mathfrak{C},\lambda)$ is another measure algebra and $\theta : \mathfrak{B}^f \to \mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : M^0_{\bar\mu} \to M^0_{\bar\lambda}$ and $P_{\theta\pi} = P_\pi P_\theta : M^{1,0}_{\bar\lambda} \to M^{1,0}_{\bar\mu}$.

(d) Now suppose that $\pi[\mathfrak{A}^f] = \mathfrak{B}^f$, so that π is a measure-preserving isomorphism between the rings $\mathfrak{A}^f$ and $\mathfrak{B}^f$. Then

(i) T is a Riesz space isomorphism between $M^0_{\bar\mu}$ and $M^0_{\bar\nu}$, and its inverse is $T_{\pi^{-1}}$.

(ii) P is a Riesz space isomorphism between $M^{1,0}_{\bar\nu}$ and $M^{1,0}_{\bar\nu}$, and its inverse is $P_{\pi^{-1}}$.

(iii) The restriction of T to $M^{1,0}_{\bar\mu}$ is $P^{-1} = P_{\pi^{-1}}$; the restriction of $T^{-1} = T_{\pi^{-1}}$ to $M^{1,0}_{\bar\nu}$ is P.

(iv) For any $p \in [1,\infty[$, $T{\restriction}L^p_{\bar\mu} = P_{\pi^{-1}}{\restriction}L^p_{\bar\mu}$ and $P{\restriction}L^p_{\bar\nu} = T_{\pi^{-1}}{\restriction}L^p_{\bar\nu}$ are the two halves of a Banach lattice isomorphism between $L^p_{\bar\mu}$ and $L^p_{\bar\nu}$.

proof (a)(i) By 361J, π induces a multiplicative Riesz homomorphism $T_0 : S(\mathfrak{A}^f) \to S(\mathfrak{B}^f)$ which is order-continuous because π is (361Ad, 361Je). If $u \in S(\mathfrak{A}^f)$ and $\alpha > 0$, then $[\![T_0u > \alpha]\!] = \pi[\![u > \alpha]\!]$. **P** Express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0,\dots,a_n$ are disjoint in $\mathfrak{A}^f$; then $T_0 u = \sum_{i=0}^n \alpha_i\chi(\pi a_i)$, so

$$[\![T_0u > \alpha]\!] = \sup\{\pi a_i : i \le n,\, \alpha_i > \alpha\} = \pi(\sup\{a_i : i \le n,\, \alpha_i > \alpha\}) = \pi[\![u > \alpha]\!]. \quad \mathbf{Q}$$

Now if $u_0 \ge 0$ in $M^0_{\bar\mu}$, $\sup\{T_0u : u \in S(\mathfrak{A}^f),\, 0 \le u \le u_0\}$ is defined in $M^0_{\bar\nu}$. **P** Set $A = \{u : u \in S(\mathfrak{A}^f),\, 0 \le u \le u_0\}$. Because $u_0 = \sup A$ (366Gb),

$$\sup_{u\in A}[\![Tu > \alpha]\!] = \sup_{u\in A}\pi[\![u > \alpha]\!] = \pi(\sup_{u\in A}[\![u > \alpha]\!]) = \pi[\![u_0 > \alpha]\!]$$

is defined and belongs to $\mathfrak{B}^f$ for any $\alpha > 0$. Also

$$\inf_{n\ge1}\sup_{u\in A}[\![Tu > n]\!] = \pi(\inf_{n\ge1}[\![u_0 > n]\!]) = 0.$$

By 364Mb, $v_0 = \sup T_0[A]$ is defined in $L^0(\mathfrak{B})$, and $[\![v_0 > \alpha]\!] = \pi[\![u_0 > \alpha]\!] \in \mathfrak{B}^f$ for every $\alpha > 0$, so $v_0 \in M^0_{\bar\nu}$, as required. **Q**

Consequently T_0 has a unique extension to an order-continuous Riesz homomorphism $T : M^0_{\bar\mu} \to M^0_{\bar\nu}$ (355F).

(ii) If $u_0 \in M^0_{\bar\mu}$ and $\alpha > 0$, then

$$[\![Tu_0 > \alpha]\!] = [\![Tu_0^+ > \alpha]\!]$$

(because T is a Riesz homomorphism)

$$= \sup_{u\in S(\mathfrak{A}^f),0\le u\le u_0^+}[\![Tu > \alpha]\!]$$

(because T is order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar\mu}$)

$$= \pi[\![u_0 > \alpha]\!]$$

by the argument used in (i).

(iii) I have already remarked, at the beginning of the proof of (i), that $T(u \times u') = Tu \times Tu'$ for u, $u' \in S(\mathfrak{A}^f)$. Because both T and $\times$ are order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar\mu}$,

$$\begin{aligned} T(u_0 \times u_1) &= \sup\{T(u \times u') : u,\, u' \in S(\mathfrak{A}^f),\, 0 \le u \le u_0,\, 0 \le u' \le u_1\} \\ &= \sup_{u,u'} Tu \times Tu' = Tu_0 \times Tu_1 \end{aligned}$$

whenever u_0, $u_1 \ge 0$ in $M^0_{\bar\mu}$. Because T is linear and $\times$ is bilinear, it follows that T is multiplicative on $M^0_{\bar\mu}$.

To see that it is injective, observe that if $u \ne 0$ in $M^0_{\bar\mu}$ then there is some $\alpha > 0$ such that $a = [\![|u| > \alpha]\!] \ne 0$, so that $0 < \chi\pi a \le T|u| = |Tu|$ and $Tu \ne 0$.

(iv)(α) Suppose that $p \in [1,\infty[$ and that $u \in L^p_{\bar\mu}$. Then for any $\alpha > 0$,

$$[\![|Tu|^p > \alpha]\!] = [\![T|u| > \alpha^{1/p}]\!] = \pi[\![|u| > \alpha^{1/p}]\!] = \pi[\![|u|^p > \alpha]\!].$$

So

$$\||Tu|^p\|_1 = \int_0^\infty \bar\nu[\![|Tu|^p > \alpha]\!]\,d\alpha = \int_0^\infty \bar\mu[\![|u|^p > \alpha]\!]\,d\alpha = \||u|^p\|_1 < \infty$$

and $Tu \in L^p_{\bar\nu}$, with $\|Tu\|_p = \|u\|_p$.

(β) As for the case $p = \infty$, if $u \in L^\infty_{\bar\mu}$ and $\gamma = \|u\|_\infty > 0$ then $[\![|u| > \gamma]\!] = 0$, so $[\![|Tu| > \gamma]\!] = \pi[\![|u| > \gamma]\!] = 0$. This shows that $\|Tu\|_\infty \le \gamma$. On the other hand, if $0 < \alpha < \gamma$ then $a = [\![|u| > \alpha]\!] \ne 0$, and $\chi a \le |u|$ so $\chi(\pi a) \le |Tu|$; as $\pi a \ne 0$ (because $\bar\nu(\pi a) = \bar\mu a > 0$), $\|Tu\|_\infty > \alpha$. This shows that $\|Tu\|_\infty = \|u\|_\infty$, at least when $u \ne 0$; but the case $u = 0$ is trivial.

(γ) Now take any $p \in [1,\infty]$, and suppose that $u \in M^0_{\bar\mu}$ and that $Tu \in L^p_{\bar\nu}$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}^f)$ with supremum $|u|$ and $u_0 \ge 0$ (366Gb). Then $Tu_n \le Tu$ so $\|u_n\|_p = \|Tu_n\|_p \le \|Tu\|_p$. But this means that $\langle u_n\rangle_{n\in\mathbb{N}}$ is bounded above in $L^p_{\bar\mu}$ (366Db), so that $|u|$ and u belong to $L^p_{\bar\mu}$.

(δ) If $u \in L^1_{\bar\mu}$, then

$$\int Tu = \|(Tu)^+\|_1 - \|(Tu)^-\|_1 = \|Tu^+\|_1 - \|Tu^-\|_1 = \|u^+\|_1 - \|u^-\|_1 = \int u.$$

(v) If $u \in M^{1,0}_{\bar\mu}$ and $\epsilon > 0$, then $T(|u| \wedge \epsilon\chi 1_{\mathfrak{A}}) = |Tu| \wedge \epsilon\chi 1_{\mathfrak{B}}$. **P** Set $a = [\![|u| > \epsilon]\!] \in \mathfrak{A}^f$. Then $|u| \wedge \epsilon\chi 1_{\mathfrak{A}} = \epsilon\chi a + |u| - |u| \times \chi a$ and $[\![|Tu| > \epsilon]\!] = \pi a$. So

$$\begin{aligned} T(|u| \wedge \epsilon\chi 1_{\mathfrak{A}}) &= T(\epsilon\chi a) + T|u| - T(|u| \times \chi a) \\ &= \epsilon\chi(\pi a) + |Tu| - |Tu| \times \chi(\pi a) = |Tu| \wedge \epsilon\chi 1_{\mathfrak{B}}. \text{ **Q**} \end{aligned}$$

Consequently

$$T(|u| - \epsilon\chi 1_{\mathfrak{A}})^+ = T(|u| - |u| \wedge \epsilon\chi 1_{\mathfrak{A}}) = (|Tu| - \epsilon\chi 1_{\mathfrak{B}})^+.$$

But this means that $(|u| - \epsilon\chi 1_{\mathfrak{A}})^+ \in L^1_{\bar\mu}$ iff $(|Tu| - \epsilon\chi 1_{\mathfrak{B}})^+ \in L^1_{\bar\nu}$. Since this is true for every $\epsilon > 0$, 366Gc tells us that $u \in M^{1,0}_{\bar\mu}$ iff $Tu \in M^{1,0}_{\bar\nu}$.

(b)(i)(α) By 365Pa, we have an order-continuous positive linear operator $P_0 : L^1_{\bar\nu} \to L^1_{\bar\mu}$ such that $\int_a P_0 v = \int_{\pi a} v$ for every $v \in L^1_{\bar\nu}$ and $a \in \mathfrak{A}^f$.

(β) We now find that if $v_0 \ge 0$ in $M^{1,0}_{\bar\nu}$ and $B = \{v : v \in L^1_{\bar\nu},\ 0 \le v \le v_0\}$, then $P_0[B]$ has a supremum in $L^0(\mathfrak{A})$ which belongs to $M^{1,0}_{\bar\mu}$. **P** Because B is upwards-directed and P_0 is order-preserving, $P_0[B]$ is upwards-directed. If $\alpha > 0$ and $v \in B$ and $a = [\![P_0 v > \alpha]\!]$, then

$$v \le (v_0 - \tfrac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \frac{\alpha}{2}\chi 1_{\mathfrak{B}},$$

so

$$\begin{aligned} \alpha\bar\mu a \le \int_a P_0 v = \int_{\pi a} v &\le \int (v_0 - \frac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \frac{\alpha}{2}\bar\nu(\pi a) \\ &= \int (v_0 - \frac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \frac{\alpha}{2}\bar\mu a \end{aligned}$$

and

$$\bar\mu[\![P_0 v > \alpha]\!] \le \frac{2}{\alpha}\int (v_0 - \tfrac{\alpha}{2}\chi 1_{\mathfrak{B}})^+.$$

Thus $\{[\![P_0 v > \alpha]\!] : v \in B\}$ is an upwards-directed set in $\mathfrak{A}^f$ with measures bounded above in $\mathbb{R}$, and $c_\alpha = \sup_{v\in B} [\![P_0 v > \alpha]\!]$ is defined in $\mathfrak{A}^f$. Also

$$\inf_{n\ge1} c_n \le \inf_{n\ge1} \frac{2}{n}\int (v_0 - \tfrac{n}{2}\chi 1_{\mathfrak{B}})^+ = 0.$$

So, by 364Mb, $P_0[B]$ has a supremum $u_0 \in L^0(\mathfrak{A})$, and $[\![u_0 > \alpha]\!] = c_\alpha \in \mathfrak{A}^f$ for every $\alpha > 0$, so $u_0 \in M^0_{\bar\mu}$. If $c \in \mathfrak{A}^f$, then

$$\int_c u_0 = \sup_{v\in B}\int_c P_0 v = \sup_{v\in B}\int_{\pi c} v \le \int_{\pi c} v_0 < \infty,$$

so $u_0 \in M^{1,0}_{\bar\mu}$. **Q**

(γ) Now 355F tells us that P_0 has a unique extension to an order-continuous positive linear operator $P : M^{1,0}_{\bar\nu} \to M^{1,0}_{\bar\mu}$. If $v_0 \ge 0$ in $M^{1,0}_{\bar\nu}$ and $a \in \mathfrak{A}^f$, then, as remarked above,

$$\int_a Pv_0 = \sup\{\int_a P_0 v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\} = \sup\{\int_{\pi a} v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\} = \int_{\pi a} v_0;$$

because P is linear, $\int_a Pv = \int_{\pi a} v$ for every $v \in M^{1,0}_{\bar\nu}$, $a \in \mathfrak{A}^f$.

(δ) By 366Gd, P is uniquely defined by the formula

$$\int_a Pv = \int_{\pi a} v \text{ whenever } v \in M^{1,0}_{\bar\nu},\ a \in \mathfrak{A}^f.$$

(ii) Because $M^0_{\bar\mu}$ is closed under multiplication, $u \times Pv$ certainly belongs to $M^0_{\bar\mu}$.

(α) Suppose that u, $v \ge 0$. Fix $c \in \mathfrak{A}^f$ for the moment. Suppose that $u' \in S(\mathfrak{A}^f)$. Then we can express u' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$ for every $i \le n$. Accordingly

$$\int_c u' \times Pv = \sum_{i=0}^n \alpha_i \int_{c\cap a_i} Pv = \sum_{i=0}^n \alpha_i \int v \times \chi(\pi a_i) \times \chi(\pi c) = \int_{\pi c} v \times Tu'.$$

Next, we can find a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)^+$ with supremum u, and

$$\begin{aligned}\sup_{n\in\mathbb{N}}\int_c u_n \times Pv &= \sup_{n\in\mathbb{N}}\int_{\pi c} v \times Tu_n = \int_{\pi c}\sup_{n\in\mathbb{N}} v \times Tu_n \\ &= \int_{\pi c} v \times \sup_{n\in\mathbb{N}} Tu_n = \int_{\pi c} v \times Tu,\end{aligned}$$

using the order-continuity of T, $\int$ and $\times$. But this means that $u \times Pv = \sup_{n\in\mathbb{N}} u_n \times Pv$ is integrable over c and that $\int_c u \times Pv = \int_{\pi c} v \times Tu$. As c is arbitrary, $u \times Pv = P(v \times Tu) \in M^{1,0}_{\bar\mu}$.

(β) For general u, v,

$$v^+ \times Tu^+ + v^+ \times Tu^- + v^- \times Tu^+ + v^- \times Tu^- = |v| \times T|u| = |v \times Tu| \in M^{1,0}_{\bar\nu}$$

(because T is a Riesz homomorphism), so we may apply (α) to each of the four products; combining them, we get $P(v \times Tu) = u \times Pv$, as required.

(iii) Because P is a positive operator, we surely have $|Pv| \le P|v|$, so it will be enough to show that $\|Pv\|_q \le \|v\|_q$ for $v \ge 0$ in $L^q_{\bar\nu}$.

(α) I take the case $q = 1$ first. In this case, for any $a \in \mathfrak{A}^f$, we have $\int_a Pv = \int_{\pi a} v \le \|v\|_1$. In particular, setting $a_n = [\![Pv > 2^{-n}]\!]$, $\int_{a_n} Pv \le \|v\|_1$. But $Pv = \sup_{n\in\mathbb{N}} Pv \times \chi a_n$, so

$$\|Pv\|_1 = \sup_{n\in\mathbb{N}} \int_{a_n} Pv \le \|v\|_1.$$

(β) Next, suppose that $q = \infty$, so that $v \in (L^\infty_{\bar\nu})^+$; say $\|v\|_\infty = \gamma$. **?** If $\gamma > 0$ and $a = [\![Pv > \gamma]\!] \ne 0$, then

$$\gamma\bar\mu a < \int_a Pv = \int_{\pi a} v \le \gamma\bar\nu(\pi a) = \gamma\bar\mu a. \ \mathbf{X}$$

So $[\![Pv > \gamma]\!] = 0$ and $Pv \in L^\infty_{\bar\mu}$, with $\|Pv\|_\infty \le \|v\|_\infty$, at least when $\|v\|_\infty > 0$; but the case $\|v\|_\infty = 0$ is trivial.

(γ) I come at last to the ‘general’ case $q \in\,]1,\infty[$, $v \in L^q_{\bar\nu}$. In this case set $p = q/(q-1)$. If $u \in L^p_{\bar\mu}$ then $Tu \in L^p_{\bar\nu}$ so $Tu \times v \in L^1_{\bar\nu}$ and

$$|\int u \times Pv| \le \|u \times Pv\|_1$$
$$= \|P(Tu \times v)\|_1$$

(by (ii))

$$\le \|Tu \times v\|_1$$

(by (α) just above)

$$= \int |Tu| \times |v| \le \|Tu\|_p\|v\|_q = \|u\|_p\|v\|_q$$

by (a-iii) of this theorem. But this means that $u \mapsto \int u \times Pv$ is a bounded linear functional on $L^p_{\bar\mu}$, and is therefore represented by some $w \in L^q_{\bar\mu}$ with $\|w\|_q \le \|v\|_q$. If $a \in \mathfrak{A}^f$ then $\chi a \in L^p_{\bar\mu}$, so $\int_a w = \int_a Pv$; accordingly Pv is actually equal to w (by 366Gd) and $\|Pv\|_q = \|w\|_q \le \|v\|_q$, as claimed.

(iv) If $u \in M^{1,0}_{\bar\mu}$ and $a \in \mathfrak{A}^f$, we must have

$$\int_a PTu = \int_{\pi a} Tu = \int T(\chi a) \times Tu = \int T(\chi a \times u) = \int \chi a \times u = \int_a u,$$

using (a-iv) to see that $\int \chi a \times u$ is defined and equal to $\int T(\chi a \times u)$. As a is arbitrary, $u \in M^{1,0}_{\bar\mu}$ and $PTu = u$.

(c) As usual, in view of the uniqueness of $T_{\theta\pi}$ and $P_{\theta\pi}$, all we have to check is that

$$T_\theta T(\chi a) = T_\theta \chi(\pi a) = \chi(\theta\pi a) = T_{\theta\pi}(\chi a),$$

$$\int_a PP_\theta w = \int_{\pi a} P_\theta w = \int_{\theta\pi a} w = \int_a P_{\theta\pi} w$$

whenever $a \in \mathfrak{A}^f$, $w \in M^{1,0}_{\bar\lambda}$.

(d)(i) By (c), $T_{\pi^{-1}}T = T_{\pi^{-1}\pi}$ must be the identity operator on $M^0_{\bar\mu}$; similarly, $TT_{\pi^{-1}}$ is the identity operator on $M^0_{\bar\nu}$. Because T and $T_{\pi^{-1}}$ are Riesz homomorphisms, they must be the two halves of a Riesz space isomorphism.

(ii) In the same way, P and $P_{\pi^{-1}}$ must be the two halves of an ordered linear space isomorphism between $M^{1,0}_{\bar\mu}$ and $M^{1,0}_{\bar\nu}$, and are therefore both Riesz homomorphisms.

(iii) By (b-iv), $PTu = u$ for every $u \in M^{1,0}_{\bar\mu}$, so $T\restriction M^{1,0}_{\bar\mu}$ must be P^{-1}. Similarly $P = P^{-1}_{\pi^{-1}}$ is the restriction of $T^{-1} = T_{\pi^{-1}}$ to $M^{1,0}_{\bar\nu}$.

(iv) Because $T^{-1}[L^p_{\bar\nu}] = L^p_{\bar\mu}$ (by (a-iv)), and T is a bijection between $M^0_{\bar\mu}$ and $M^0_{\bar\nu}$, $T\restriction L^p_{\bar\mu}$ must be a Riesz space isomorphism between $L^p_{\bar\mu}$ and $L^p_{\bar\nu}$; (a-iv) also tells us that it is norm-preserving. Now its inverse is $P\restriction L^p_{\bar\nu}$, by (iii) here.

366I Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Then, for any $p \in [1, \infty[$, $L^p(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ can be identified, as Banach lattice, with the closed linear subspace of $L^p(\mathfrak{A}, \bar\mu)$ generated by $\{\chi b : b \in \mathfrak{B}, \bar\mu b < \infty\}$.

proof The identity map $b \mapsto b : \mathfrak{B} \to \mathfrak{A}$ induces an injective Riesz homomorphism $T : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ (364R) such that $Tu \in L^p_{\mathfrak{A}} = L^p(\mathfrak{A}, \bar\mu)$ and $\|Tu\|_p = \|u\|_p$ whenever $p \in [1, \infty[$ and $u \in L^p_{\mathfrak{B}} = L^p(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ (366H(a-iv)). Because $S(\mathfrak{B}^f)$, the linear span of $\{\chi b : b \in \mathfrak{B}, \bar\mu b < \infty\}$, is dense in $L^p_{\mathfrak{B}}$ (366C), the image of $L^p_{\mathfrak{B}}$ in $L^p_{\mathfrak{A}}$ must be the closure of the image of $S(\mathfrak{B}^f)$ in $L^p_{\mathfrak{A}}$, that is, the closed linear span of $\{\chi b : b \in \mathfrak{B}^f\}$ interpreted as a subset of $L^p_{\mathfrak{A}}$.

366J Corollary If $(\mathfrak{A}, \bar\mu)$ is a probability algebra, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ is the conditional expectation operator (365R), then $\|Pu\|_p \le \|u\|_p$ whenever $p \in [1, \infty]$ and $u \in L^p(\mathfrak{A}, \bar\mu)$.

proof Because $(\mathfrak{A}, \bar\mu)$ is totally finite, $M^0(\mathfrak{A}, \bar\mu) = L^1(\mathfrak{A}, \bar\mu)$, so that the operator P of 366Hb can be identified with the conditional expectation operator of 365R. Now 366H(b-iii) gives the result.

Remark Of course this is also covered by 244M.

366K Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism. Let $T:L^2_{\bar\mu}\to L^2_{\bar\nu}$ and $P:L^2_{\bar\nu}\to L^2_{\bar\mu}$ be the corresponding operators, as in 366H. Then $TP:L^2_{\bar\nu}\to L^2_{\bar\nu}$ is an orthogonal projection, its range $TP[L^2_{\bar\nu}]$ being isomorphic, as Banach lattice, to $L^2_{\bar\mu}$. The kernel of TP is just

$$\{v:v\in L^2_{\bar\nu},\ \int_{\pi a}v=0\text{ for every }a\in\mathfrak{A}^f\}.$$

proof Most of this is simply because T is a norm-preserving Riesz homomorphism (so that $T[L^2_{\bar\mu}]$ is isomorphic to $L^2_{\bar\mu}$), PT is the identity on $L^2_{\bar\mu}$ (so that $(TP)^2=TP$) and $\|P\|\le 1$ (so that $\|TP\|\le 1$). These are enough to ensure that TP is a projection of norm at most 1, that is, an orthogonal projection. Also

$$TPv=0\iff Pv=0\iff\int_a Pv=0\text{ for every }a\in\mathfrak{A}^f$$
$$\iff\int_{\pi a}v=0\text{ for every }a\in\mathfrak{A}^f.$$

366L Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving ring automorphism. Then there is a corresponding Banach lattice isomorphism T of $L^2=L^2(\mathfrak{A},\bar\mu)$ defined by writing $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$. Its inverse is defined by the formula

$$\int_a T^{-1}u=\int_{\pi a}u\text{ for every }u\in L^2,\ a\in\mathfrak{A}^f.$$

proof In the language of 366H, $T=T_\pi$ and $T^{-1}=P$.

366X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in\,]1,\infty[$. Show that, for $u\in L^0(\mathfrak{A})$, $u\in L^p(\mathfrak{A},\bar\mu)$ iff $\gamma=p\int_0^\infty\alpha^{p-1}\bar\mu[\![|u|>\alpha]\!]\,d\alpha$ is finite, and that in this case $\|u\|_p=\gamma^{1/p}$. (Cf. 263Xa.)

>(b) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $p\in[1,\infty]$. Show that the band algebra of $L^p(\mathfrak{A},\bar\mu)$ is isomorphic to $\mathfrak{A}$. (Cf. 365S.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in\,]1,\infty[$. Show that $L^p(\mathfrak{A},\bar\mu)$ is separable iff $L^1(\mathfrak{A},\bar\mu)$ is.

(d) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. (i) Show that $L^\infty(\mathfrak{A})\cap M^0(\mathfrak{A},\bar\mu)$ and $L^\infty(\mathfrak{A})\cap M^{1,0}(\mathfrak{A},\bar\mu)$, as defined in 366G, are equal. (ii) Call this intersection $M^{\infty,0}(\mathfrak{A},\bar\mu)$. Show that it is a norm-closed solid linear subspace of $L^\infty(\mathfrak{A})$, therefore a Banach lattice in its own right.

(e) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $(\widehat{\mathfrak{A}},\hat\mu)$ its localization (322P). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^p(\mathfrak{A},\bar\mu)$ and $L^p(\widehat{\mathfrak{A}},\hat\mu)$ for every $p\in[1,\infty[$, so that the band algebra of $L^p(\mathfrak{A},\bar\mu)$ can be identified with $\widehat{\mathfrak{A}}$.

(f) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra which is not localizable (cf. 211Ya, 216D), and $(\widehat{\mathfrak{A}},\hat\mu)$ its localization. Let $\pi:\mathfrak{A}\to\widehat{\mathfrak{A}}$ be the identity embedding, so that π is an order-continuous measure-preserving Boolean homomorphism. Show that if we set $v=\chi b$ where $b\in\widehat{\mathfrak{A}}\setminus\mathfrak{A}$, then there is no $u\in L^\infty(\mathfrak{A})$ such that $\int_a u=\int_{\pi a}v$ whenever $\bar\mu a<\infty$.

(g) In 366H, show that $[\![Tu\in E]\!]=\pi[\![u\in E]\!]$ whenever $u\in M^0_{\bar\mu}$ and $E\subseteq\mathbb{R}$ is a Borel set such that $0\notin\overline{E}$.

>(h) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and let G be the group of all measure-preserving ring automorphisms of $\mathfrak{A}^f$. Let H be the group of all Banach lattice automorphisms of $L^2(\mathfrak{A},\bar\mu)$. Show that the map $\pi\mapsto T$ of 366L is an injective group homomorphism from G to H, so that G is represented as a subgroup of H.

(i) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be any family of measure algebras, with simple product $(\mathfrak{A},\bar\mu)$ (322K). Show that for any $p\in[1,\infty[$, $L^p(\mathfrak{A},\bar\mu)$ can be identified, as normed Riesz space, with the solid linear subspace

$$\{u:\|u\|=\bigl(\sum_{i\in I}\|u(i)\|^p\bigr)^{1/p}<\infty\}$$

of $\prod_{i\in I}L^p(\mathfrak{A}_i,\bar\mu_i)$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functionals rendering $\mathfrak{A}$ a semi-finite measure algebra. Show that for any $p \in [1,\infty[$, $L^p(\mathfrak{A},\bar{\mu})$ and $L^p(\mathfrak{A},\bar{\nu})$ are isomorphic as normed Riesz spaces. (*Hint*: use 366Xe to reduce to the case in which $\mathfrak{A}$ is Dedekind complete. Take $w \in L^0(\mathfrak{A})$ such that $\int_a w\,d\bar{\mu} = \bar{\nu}a$ for every $a \in \mathfrak{A}$ (365T). Set $Tu = w^{1/p} \times u$ for $u \in L^p(\mathfrak{A},\bar{\mu})$.)

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, and $p \in [1,\infty[$. Show that the following are equiveridical: (i) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Banach lattices; (ii) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Riesz spaces; (iii) $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic Dedekind completions.

366Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and suppose that $0 < p < 1$. Write $L^p = L^p(\mathfrak{A},\bar{\mu})$ for $\{u : u \in L^0(\mathfrak{A}),\ |u|^p \in L^1(\mathfrak{A},\bar{\mu})\}$, and for $u \in L^p$ set $\tau(u) = \int |u|^p$. (i) Show that τ defines a Hausdorff linear space topology on L^p (see 2A5B). (ii) Show that if $A \subseteq L^p$ is non-empty, downwards-directed and has infimum 0 then $\inf_{u\in A}\tau(u) = 0$. (iii) Show that if $A \subseteq L^p$ is non-empty, upwards-directed and bounded in the linear topological space sense (see 245Yf) then A is bounded above. (iv) Show that $(L^p)^\sim = (L^p)^\times$ is just the set of continuous linear functionals from L^p to $\mathbb{R}$, and is $\{0\}$ iff $\mathfrak{A}$ is atomless.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that $M^0(\mathfrak{A},\bar{\mu})$ has the countable sup property.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and define $M^{\infty,0}(\mathfrak{A},\bar{\mu})$ as in 366Xd. Show that $(M^{\infty,0}(\mathfrak{A},\bar{\mu}))^\times$ can be identified with $L^1(\mathfrak{A},\bar{\mu})$.

(d) In 366H, show that if $\tilde{T} : M^0(\mathfrak{A},\bar{\mu}) \to M^0(\mathfrak{B},\bar{\nu})$ is any positive linear operator such that $\tilde{T}(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}^f$, then $\tilde{T}$ is order-continuous, so that it is equal to T_π.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. (i) Show that there is a natural one-to-one correspondence between $M^{1,0}(\mathfrak{A},\bar{\mu})$ and the set of additive functionals $\nu : \mathfrak{A}^f \to \mathbb{R}$ such that $\nu \ll \mu$ in the double sense that for every $\epsilon > 0$ there are δ, $M > 0$ such that $|\nu a| \le \epsilon$ whenever $\mu a \le \delta$ and $|\nu a| \le \epsilon\mu a$ whenever $\mu a \ge M$. (ii) Use this description of $M^{1,0}$ to prove 366H(b-i).

(f) In 366H, show that the following are equiveridical: (α) $\pi[\mathfrak{A}^f] = \mathfrak{B}^f$; ($\beta$) $T = T_\pi$ is surjective; (γ) $P = P_\pi$ is injective; (δ) P is a Riesz homomorphism; (ϵ) there is some $q \in [1,\infty]$ such that $\|Pv\|_q = \|v\|_q$ for every $v \in L^q_{\bar{\nu}}$; (ζ) $TPv = v$ for every $v \in M^{1,0}_{\bar{\nu}}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and suppose that $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is a measure-preserving ring homomorphism, as in 366H; let $T : M^0(\mathfrak{A},\bar{\mu}) \to M^0(\mathfrak{B},\bar{\nu})$ be the associated linear operator. Show that if $0 < p < 1$ (as in 366Ya) then $L^p(\mathfrak{A},\bar{\mu}) \subseteq M^0(\mathfrak{A},\bar{\mu})$ and $T^{-1}[L^p(\mathfrak{B},\bar{\nu})] = L^p(\mathfrak{A},\bar{\mu})$.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra. (i) For each Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$, let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the associated Riesz space isomorphism (364R), and let $w_\pi \in L^1(\mathfrak{A})^+$ be such that $\int_a w_\pi = \mu(\pi^{-1}a)$ for every $a \in \mathfrak{A}$ (365Ea). Set $Q_\pi u = T_\pi u \times \sqrt{w_\pi}$ for $u \in L^0(\mathfrak{A})$. Show that $\|Q_\pi u\|_2 = \|u\|_2$ for every $u \in L^2(\mathfrak{A})$. (ii) Show that if π, $\phi : \mathfrak{A} \to \mathfrak{A}$ are Boolean automorphisms then $Q_{\pi\phi} = Q_\pi Q_\phi$.

366 Notes and comments The L^p spaces, for $1 \le p \le \infty$, constitute the most important family of leading examples for the theory of Banach lattices, and it is not to be wondered at that their properties reflect a wide variety of general results. Thus 366Dd and 366E can both be regarded as special cases of theorems about perfect Riesz spaces (356M and 369D). In a different direction, the concept of 'Orlicz space' (369Xd below) generalizes the L^p spaces if they are regarded as normed subspaces of L^0 invariant under measure-preserving automorphisms of the underlying algebra. Yet another generalization looks at the (non-locally-convex) spaces L^p for $0 < p < 1$ (366Ya).

In 366H and its associated results I try to emphasize the way in which measure-preserving homomorphisms of the underlying algebras induce both 'direct' and 'dual' operators on L^p spaces. We have already seen the phenomenon in 365P. I express this in a slightly different form in 366H, noting that we really do need the homomorphisms to be measure-preserving, for the dual operators as well as the direct operators, so we no longer have the shift in the hypotheses which appears between 365O and 365P. Of course all these refinements in the hypotheses are irrelevant to the principal applications of the results, and they make substantial demands on the reader; but I believe that the demands are actually demands to expand one's imagination, to encompass the different ways in which the spaces depend on the underlying measure algebras.

In the context of 366H, L^∞ is set apart from the other L^p spaces, because $L^\infty(\mathfrak{A})$ is not in general determined by the ideal $\mathfrak{A}^f$, and the hypotheses of 366H do not look outside $\mathfrak{A}^f$. 366H(a-iv) and 366H(b-iii) reach only the space $M^{\infty,0}$ as defined in 366Xd. To deal with L^∞ we need slightly stronger hypotheses. If we are given a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, rather than from $\mathfrak{A}^f$ to $\mathfrak{B}^f$, then of course the direct operator T has a natural version acting on $L^\infty(\mathfrak{A})$ and indeed on $M^{1,\infty}(\mathfrak{A},\bar{\mu})$, as in 363F and 369Xm. If we know that $(\mathfrak{A},\bar{\mu})$ is localizable, then $\mathfrak{A}$ can be recovered from $\mathfrak{A}^f$, and the dual operator P acts on $L^\infty(\mathfrak{B})$, as in 369Xm. But in general we can't expect this to work (366Xf).

Of course 366H can be applied to many other spaces; for reasons which will appear in §§371 and 374, the archetypes are not really L^p spaces at all, but the spaces $M^{1,0}$ (366F) and $M^{1,\infty}$ (369N).

I include 366L and 366Yh as pointers to one of the important applications of these ideas: the investigation of properties of a measure-preserving homomorphism in terms of its action on L^p spaces. The case $p=2$ is the most useful because the group of unitary operators (that is, the normed space automorphisms) of L^2 has been studied intensively.

367 Convergence in measure

Continuing through the ideas of Chapter 24, I come to 'convergence in measure'. The basic results of §245 all translate easily into the new language (367M-367N, 367Q). The associated concept of (sequential) order-convergence can also be expressed in abstract terms (367A), and I take the trouble to do this in the context of general lattices (367A-367B), since the concept can be applied in many ways (367C-367F, 367L, 367Xa-367Xm). In the particular case of L^0 spaces, which are the first aim of this section, the idea is most naturally expressed by 367G. It enables us to express some of the fundamental theorems from Volumes 1 and 2 in the language of this chapter (367J-367K).

In 367O and 367P I give two of the most characteristic properties of the topology of convergence in measure on L^0; it is one of the fundamental types of topological Riesz space. Another striking fact is the way it is determined by the Riesz space structure (367T). In 367U I set out a theorem which is the basis of many remarkable applications of the concept; for the sake of a result in §369 I give one such application (367V).

367A Order*-convergence As I have remarked before, the function spaces of measure theory have three interdependent structures: they are linear spaces, they have a variety of interesting topologies, and they are ordered spaces. Ordinary elementary functional analysis studies interactions between topologies and linear structures, in the theory of normed spaces and, more generally, of linear topological spaces. Chapter 35 in this volume looked at interactions between linear and order structures. It is natural to seek to complete the triangle with a theory of topological ordered spaces. The relative obscurity of any such theory is in part due to the difficulty of finding convincing definitions; that is, isolating concepts which lead to in elegant and useful general theorems. Among the many rival ideas, however, I believe it is possible to identify one which is particularly important in the context of measure theory.

In its natural home in the theory of L^0 spaces, this notion of 'order*-convergence' has a very straightforward expression (367G). But, suitably interpreted, the same idea can be applied in other contexts, some of which will be very useful to us, and I therefore begin with a definition which is applicable in any lattice.

Definition Let P be a lattice and $\langle p_n\rangle_{n\in\mathbb{N}}$ a sequence in P, p an element of P. I will say that $\langle p_n\rangle_{n\in\mathbb{N}}$ **order*-converges** to p, or that p is the **order*-limit** of $\langle p_n\rangle_{n\in\mathbb{N}}$, if

$$\begin{aligned} p &= \inf\{q : \exists\, n\in\mathbb{N},\ q \ge (p'\vee p_i)\wedge p''\ \forall\, i\ge n\} \\ &= \sup\{q : \exists\, n\in\mathbb{N},\ q \le p'\vee(p_i\wedge p'')\ \forall\, i\ge n\} \end{aligned}$$

whenever $p'\le p\le p''$ in P.

367B Lemma Let P be a lattice.

(a) A sequence in P can order*-converge to at most one point.

(b) A constant sequence order*-converges to its constant value.

(c) Any subsequence of an order*-convergent sequence is order*-convergent, with the same limit.

(d) If $\langle p_n\rangle_{n\in\mathbb{N}}$ and $\langle p'_n\rangle_{n\in\mathbb{N}}$ both order*-converge to p, and $p_n \le q_n \le p'_n$ for every n, then $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) If $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$\begin{aligned} p &= \inf\{q : \exists\, n \in \mathbb{N},\, q \ge p_i\ \forall\, i \ge n\} \\ &= \sup\{q : \exists\, n \in \mathbb{N},\, q \ge p_i\ \forall\, i \ge n\}. \end{aligned}$$

(f) If P is a Dedekind σ-complete lattice (314A) and $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$p = \sup_{n\in\mathbb{N}} \inf_{i\ge n} p_i = \inf_{n\in\mathbb{N}} \sup_{i\ge n} p_i.$$

proof (a) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to both p and $\tilde{p}$. Set $p' = p \wedge \tilde{p}$, $p'' = p \vee \tilde{p}$; then

$$p = \inf\{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p_i) \wedge p''\ \forall\, i \ge n\} = \tilde{p}.$$

(b) is trivial.

(c) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p, and that $\langle p'_n\rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle p_n\rangle_{n\in\mathbb{N}}$. Take p', p'' such that $p' \le p \le p''$, and set

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p_i \wedge p'')\ \forall\, i \ge n\},$$

$$B' = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p'_i \wedge p'')\ \forall\, i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p_i) \wedge p''\ \forall\, i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p'_i) \wedge p''\ \forall\, i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all sufficiently large i

$$q \le p' \vee (p'_i \wedge p'') \le (p' \vee p'_i) \wedge p'' \le q'.$$

As $p = \inf C$, we must have $q \le p$; thus p is an upper bound for B'. On the other hand, $\{p'_i : i \ge n\} \subseteq \{p_i : i \ge n\}$ for every n, so $B \subseteq B'$ and p must be the least upper bound of B', since $p = \sup B$.

Similarly, $p = \inf C'$. As p' and p'' are arbitrary, $\langle p'_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(d) Take p', p'' such that $p' \le p \le p''$, and set

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p_i \wedge p'')\ \forall\, i \ge n\},$$

$$B' = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (q_i \wedge p'')\ \forall\, i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p'_i) \wedge p''\ \forall\, i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee q_i) \wedge p''\ \forall\, i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all sufficiently large i

$$q \le p' \vee (q_i \wedge p'') \le (p' \vee p'_i) \wedge p'' \le q'.$$

As $p = \inf C$, we must have $q \le p$; thus p is an upper bound for B'. On the other hand, $p' \vee (p_i \wedge p'') \le p' \vee (q_i \wedge p'')$ for every i, so $B \subseteq B'$ and $p = \sup B'$. Similarly, $p = \inf C'$. As p' and p'' are arbitrary, $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) Set

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p_i\ \forall\, i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\, q \ge p_i\ \forall\, i \ge n\}.$$

(i) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to p. Let p', p'' be such that $p'\le p_n\le p''$ for every $n\in\mathbb{N}$ and $p'\le p\le p''$. Then

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

so $\sup B=p$. Similarly, $\inf C=p$, so the condition is satisfied.

(ii) Suppose that $\sup B=\inf C=p$. Take any p', p'' such that $p'\le p\le p''$ and set

$$B'=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

$$C'=\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p_i)\wedge p''\;\forall\,i\ge n\}.$$

If $q\in B'$ and $q'\in C$, then for all large enough i

$$q\le p'\vee(p_i\wedge p'')\le p'\vee q'=q'$$

because $p\le q'$. As $\inf C=p$, p is an upper bound for B'. On the other hand, if $q\in B$, then $q\le p$, so $q\le p'\vee(p_i\wedge p'')$ whenever $q\le p_i$, which is so for all sufficiently large i, and $q\in B'$. Thus $B'\supseteq B$ and p must be the supremum of B'. Similarly, $p=\inf C'$; as p' and p'' are arbitrary, $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(f) This follows at once from (e). Setting

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p_i\;\forall\,i\ge n\},$$

$$B'=\{\inf_{i\ge n}p_i:i\in\mathbb{N}\},$$

then $B'\subseteq B$ and for every $q\in B$ there is a $q'\in B'$ such that $q\le q'$; so $\sup B=\sup B'$ if either is defined. Similarly,

$$\inf\{q:\exists\,n\in\mathbb{N},\,q\ge p_i\;\forall\,i\ge n\}=\inf_{n\in\mathbb{N}}\sup_{i\ge n}p_i$$

if either is defined.

367C Proposition Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ two sequences in U order*-converging to u, v respectively.

(a) If $w\in U$, $\langle u_n+w\rangle_{n\in\mathbb{N}}$ order*-converges to $u+w$, and αu_n order*-converges to αu for every $\alpha\in\mathbb{R}$.

(b) $\langle u_n\vee v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\vee v$.

(c) If $\langle w_n\rangle_{n\in\mathbb{N}}$ is any sequence in U, then it order*-converges to $w\in U$ iff $\langle|w_n-w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

(d) $\langle u_n+v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u+v$.

(e) If U is Archimedean, and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to $\alpha\in\mathbb{R}$, then $\langle\alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu.

(f) Again suppose that U is Archimedean. Then a sequence $\langle w_n\rangle_{n\in\mathbb{N}}$ in U^+ is *not* order*-convergent to 0 iff there is a $\tilde w>0$ such that $\tilde w=\sup_{i\ge n}\tilde w\wedge w_i$ for every $n\in\mathbb{N}$.

proof (a)(i) $\langle u_n+w\rangle_{n\in\mathbb{N}}$ order*-converges to $u+w$ because the ordering of U is translation-invariant; the map $w'\mapsto w'+w$ is an order-isomorphism.

(ii)(α) If $\alpha>0$, then the map $w'\mapsto\alpha w'$ is an order-isomorphism, so $\langle\alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu.

(β) If $\alpha=0$ then $\langle\alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\alpha u=0$ by 367Bb.

(γ) If $w'\le -u\le w''$ then $-w''\le u\le w'$ so

$$\begin{aligned}u&=\inf\{w:\exists\,n\in\mathbb{N},\,w\ge((-w'')\vee u_i)\wedge(-w')\;\forall\,i\ge n\}\\&=\sup\{w:\exists\,n\in\mathbb{N},\,w\le(-w'')\vee(u_i\wedge(-w'))\;\forall\,i\ge n\}.\end{aligned}$$

Turning these formulae upside down,

$$\begin{aligned}-u&=\sup\{w:\exists\,n\in\mathbb{N},\,w\le(w''\wedge(-u_i))\vee w'\;\forall\,i\ge n\}\\&=\inf\{w:\exists\,n\in\mathbb{N},\,w\ge w''\wedge((-u_i)\vee w')\;\forall\,i\ge n\}.\end{aligned}$$

As w' and w'' are arbitrary, $\langle -u_n\rangle_{n\in\mathbb{N}}$ order*-converges to $-u$.

(δ) Putting (α) and (γ) together, $\langle \alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu for every $\alpha < 0$.

(b) Suppose that $w' \le u \vee v \le w''$. Set

$$B = \{w : \exists\, n \in \mathbb{N},\ w \le w' \vee ((u_i \vee v_i) \wedge w'')\ \forall\, i \ge n\},$$

$$C = \{w : \exists\, n \in \mathbb{N},\ w \ge (w' \vee (u_i \vee v_i)) \wedge w''\ \forall\, i \ge n\},$$

$$B_1 = \{w : \exists\, n \in \mathbb{N},\ w \le (w' \wedge u) \vee (u_i \wedge w'')\ \forall\, i \ge n\},$$

$$B_2 = \{w : \exists\, n \in \mathbb{N},\ w \le (w' \wedge v) \vee (v_i \wedge w'')\ \forall\, i \ge n\},$$

$$C_1 = \{w : \exists\, n \in \mathbb{N},\ w \ge ((w' \wedge u) \vee u_i) \wedge w''\ \forall\, i \ge n\},$$

$$C_2 = \{w : \exists\, n \in \mathbb{N},\ w \ge ((w' \wedge v) \vee v_i) \wedge w''\ \forall\, i \ge n\},$$

If $w_1 \in B_1$ and $w_2 \in B_2$ then $w_1 \vee w_2 \in B$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \le (w' \wedge u) \vee (u_i \wedge w'')$ for every $i \ge n$, while $w_2 \le (w' \wedge v) \vee (v_i \wedge w'')$ for every $i \ge n$. So

$$\begin{aligned} w_1 \vee w_2 &\le (w' \wedge u) \vee (w' \wedge v) \vee (u_i \wedge w'') \vee (v_i \wedge w'') \\ &= (w' \wedge (u \vee v)) \vee ((u_i \vee v_i) \wedge w'') \end{aligned}$$

(by the distributive law 352Ec)

$$= w' \vee ((u_i \vee v_i) \wedge w'')$$

for every $i \ge n$, and $w_1 \vee w_2 \in B$. **Q**

Similarly, if $w_1 \in C_1$ and $w_2 \in C_2$ then $w_1 \vee w_2 \in C$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \ge ((w' \wedge u) \vee u_i) \wedge w''$, $w_2 \ge ((w' \wedge v) \vee v_i) \wedge w''$ for every $i \ge n$. So

$$\begin{aligned} w_1 \vee w_2 &\ge (((w' \wedge u) \vee u_i) \wedge w'') \vee (((w' \wedge v) \vee v_i) \wedge w'') \\ &= ((w' \wedge u) \vee u_i \vee (w' \wedge v) \vee v_i) \wedge w'' \\ &= ((w' \wedge (u \vee v)) \vee (u_i \vee v_i)) \wedge w'' \\ &= (w' \vee (u_i \vee v_i)) \wedge w'' \end{aligned}$$

for every $i \ge n$, so $w_1 \vee w_2 \in C$. **Q**

At the same time, of course, $w \le \tilde{w}$ whenever $w \in B$, $\tilde{w} \in C$, since there is some $i \in \mathbb{N}$ such that

$$w \le w' \vee ((u_i \vee v_i) \wedge w'') \le (w' \vee (u_i \vee v_i)) \wedge w'' \le \tilde{w}.$$

Since

$$\sup\{w_1 \vee w_2 : w_1 \in B_1,\ w_2 \in B_2\} = (\sup B_1) \vee (\sup B_2) = u \vee v,$$

$$\inf\{w_1 \vee w_2 : w_1 \in C_1,\ w_2 \in C_2\} = (\inf C_1) \vee (\inf C_2) = u \vee v$$

(using the generalized distributive laws in 352E), we must have $\sup B = \inf C = u \vee v$. As w' and w'' are arbitrary, $\langle u_n \vee v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $u \vee v$.

(c) The hard parts are over. (i) If $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to w, then $\langle w_n - w\rangle_{n\in\mathbb{N}}$, $\langle w - w_n\rangle_{n\in\mathbb{N}}$ and $\langle |w_n - w|\rangle_{n\in\mathbb{N}} = \langle (w_n - w) \vee (w - w_n)\rangle_{n\in\mathbb{N}}$ all order*-converge to 0, putting (a) and (b) together. (ii) If $\langle |w_n - w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, then so do $\langle -|w_n - w|\rangle_{n\in\mathbb{N}}$ and $\langle w_n - w\rangle_{n\in\mathbb{N}}$, by (a) and 367Bd; so $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (a) again.

(d) $\langle |u_n - u|\rangle_{n\in\mathbb{N}}$ and $\langle |v_n - v|\rangle_{n\in\mathbb{N}}$ order*-converge to 0, by (c), so $\langle 2(|u_n - u| \vee |v_n - v|)\rangle_{n\in\mathbb{N}}$ also order*-converges to 0, by (b) and (a). But

$$0 \le |(u_n + v_n) - (u + v)| \le |u_n - u| + |v_n - v| \le 2(|u_n - u| \vee |v_n - v|)$$

for every n, so $\langle |(u_n + v_n) - (u + v)|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Bb and 367Bd, and $\langle u_n + v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u + v$.

(e) Set $\beta_n = \sup_{i\ge n}|\alpha_i-\alpha|$ for each n. Then $\langle\beta_n\rangle_{n\in\mathbb{N}}\to 0$, so $\inf_{n\in\mathbb{N}}\beta_n|u|=0$, because U is Archimedean. Consequently $\langle\beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be. But we also have $\beta_0|u_n-u|$ order*-converging to 0, by (c) and (a), so $\langle\beta_0|u_n-u|+\beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (d). As $|\alpha_n u_n-\alpha u|\le\beta_0|u_n-u|+\beta_n|u|$ for every n, $\langle\alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu, as required.

(f)(i) Suppose that $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0. Then there are w', w'' such that $w'\le 0\le w''$ and either

$$B=\{w:\exists\, n\in\mathbb{N},\ w\le w'\vee(w_i\wedge w'')\ \forall\, i\ge n\}$$

does not have supremum 0, or

$$C=\{w:\exists\, n\in\mathbb{N},\ w\ge(w'\vee w_i)\wedge w''\ \forall\, i\ge n\}$$

does not have infimum 0. Now $0\in B$, because every $w_i\ge 0$, and every member of B is a lower bound for C; so 0 cannot be the greatest lower bound of C. Let $\tilde{w}>0$ be a lower bound for C.

Let $n\in\mathbb{N}$, and set

$$C_n=\{w:w\ge(w'\vee w_i)\wedge w''\ \forall\, i\ge n\}=\{w:w\ge w_i\wedge w''\ \forall\, i\ge n\}.$$

Because U is Archimedean, we know that $\inf(C_n-A_n)=0$, where $A_n=\{w_i\wedge w'':i\ge n\}$ (353F). Now $\tilde{w}$ is a lower bound for C_n, so

$$\begin{aligned}\inf_{i\ge n}(\tilde{w}-w_i)^+&\le\inf\{(w-w_i)^+:w\in C,\ i\ge n\}\\&\le\inf\{(w-(w_i\wedge w''))^+:w\in C,\ i\ge n\}\\&=\inf\{w-(w_i\wedge w''):w\in C,\ i\ge n\}=\inf(C_n-A_n)=0.\end{aligned}$$

As this is true for every $n\in\mathbb{N}$, $\tilde{w}$ has the property declared.

(ii) If $\tilde{w}>0$ is such that $\tilde{w}=\sup_{i\ge n}\tilde{w}\wedge w_i$ for every $n\in\mathbb{N}$, then

$$\{w:\exists\, n\in\mathbb{N},\ w\ge(0\vee w_i)\wedge\tilde{w}\ \forall\, i\ge n\}$$

cannot have infimum 0, and $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0.

367D As an example of the use of this concept in a relatively abstract setting, I offer the following.

Proposition Let U be a Banach lattice and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U which is norm-convergent to $u\in U$. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is order-bounded and order*-convergent to u. So if $\langle u_n\rangle_{n\in\mathbb{N}}$ itself is order*-convergent, its order*-limit is u.

proof Let $\langle u'_n\rangle_{n\in\mathbb{N}}$ be a subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $\|u'_n-u\|\le 2^{-n}$ for each $n\in\mathbb{N}$. Then $v_n=\sup_{i\ge n}|u'_i-u|$ is defined in U, and $\|v_n\|\le 2^{-n+1}$, for each n (354C). Because $\inf_{n\in\mathbb{N}}\|v_n\|=0$, $\inf_{n\in\mathbb{N}}v_n$ must be 0, while $u-v_n\le u'_i\le u+v_n$ whenever $i\ge n$; so $\langle u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to u, by 367Be.

Now if $\langle u_n\rangle_{n\in\mathbb{N}}$ has an order*-limit, this must be u, by 367Ba and 367Bc.

367E Proposition Let U be a Riesz space with an order-continuous norm. Then any order-bounded order*-convergent sequence is norm-convergent.

proof Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u. Then $\langle|u_n-u|\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (367Cc), so

$$C=\{v:\exists\, n\in\mathbb{N},\ v\ge|u_i-u|\ \forall\, i\ge n\}$$

has infimum 0 (367Be). Because U is a lattice, C is downwards-directed, so $\inf_{v\in C}\|v\|=0$. But

$$\inf_{v\in C}\|v\|\ge\inf_{n\in\mathbb{N}}\sup_{i\ge n}\|u_i-u\|,$$

so $\lim_{n\to\infty}\|u_n-u\|=0$, that is, $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u.

367F One of the fundamental obstacles to the development of any satisfying general theory of ordered topological spaces is the erratic nature of the relations between subspace topologies of order topologies and order topologies on subspaces. The particular virtue of order*-convergence in the context of function spaces is that it is relatively robust when transferred to the subspaces we are interested in.

Proposition Let U be an Archimedean Riesz space and V a regularly embedded Riesz subspace. (For instance, V might be either solid or order-dense.) If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in V and $v \in V$, then $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v when regarded as a sequence in V, iff it order*-converges to v when regarded as a sequence in U.

proof (a) Since, in either V or U, $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v iff $\langle |v_n - v|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 (367Cc), it is enough to consider the case $v_n \ge 0$, $v = 0$.

(b) If $\langle v_n\rangle_{n\in\mathbb{N}}$ is *not* order*-convergent to 0 in U, then, by 367Cf, there is a $u > 0$ in U such that $u = \sup_{i\ge n} u \wedge v_i$ for every $n \in \mathbb{N}$ (the supremum being taken in U, of course). In particular, there is a $k \in \mathbb{N}$ such that $u \wedge v_k > 0$. Now consider the set

$$C = \{w : w \in V, \exists\, n \in \mathbb{N}, w \ge (0 \vee v_i) \wedge v_k \;\forall\, i \ge n\}.$$

Then for any $w \in C$,

$$u \wedge v_k = \sup_{i\ge n} u \wedge v_i \wedge v_k \le w,$$

using the generalized distributive law in U, so 0 is not the greatest lower bound of C in U. But as the embedding of V in U is order-continuous, 0 is not the greatest lower bound of C in V, and $\langle v_n\rangle_{n\in\mathbb{N}}$ cannot be order*-convergent to 0 in V.

(c) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in V. Because V also is Archimedean (351Rc), there is a $w > 0$ in V such that $w = \sup_{i\ge n} w \wedge v_i$ for every $n \in \mathbb{N}$, the suprema being taken in V. Again because V is regularly embedded in U, we have the same suprema in U, so, by 367Cf in the other direction, $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in U.

367G I now spell out the connexion between the definition above and the concepts introduced in 245C.

Proposition Let X be a set, Σ a σ-algebra of subsets of X, $\mathfrak{A}$ a Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ a sequentially order-continuous surjective Boolean homomorphism; let $\mathcal{I}$ be its kernel. Write $\mathcal{L}^0$ for the space of Σ-measurable real-valued functions on X, and let $T : \mathcal{L}^0 \to L^0 = L^0(\mathfrak{A})$ be the canonical Riesz homomorphism (364D, 364R). Then for any $\langle f_n\rangle_{n\in\mathbb{N}}$, f in $\mathcal{L}^0$, $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf in L^0 iff $X \setminus \{x : f(x) = \lim_{n\to\infty} f_n(x)\} \in \mathcal{I}$.

proof Set $H = \{x : \lim_{n\to\infty} f_n(x) \text{ exists} = f(x)\}$; of course $H \in \Sigma$. Set $g_n(x) = |f_n(x) - f(x)|$ for $n \in \mathbb{N}$, $x \in X$.

(a) If $X \setminus H \in \mathcal{I}$, set $h_n(x) = \sup_{i\ge n} g_i(x)$ for $x \in H$ and $h_n(x) = 0$ for $x \in X \setminus H$. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0 in $\mathcal{L}^0$, so $\inf_{n\in\mathbb{N}} Th_n = 0$ in L^0, because T is sequentially order-continuous (364Ra). But as $\setminus H \in \mathcal{I}$, $Th_n \ge Tg_i = |Tf_i - Tf|$ whenever $i \ge n$, so $\langle |Tf_n - Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be or 367Bf, and $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf, by 367Cc.

(b) Now suppose that $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf. Set $g'_n(x) = \min(1, g_n(x))$ for $n \in \mathbb{N}$, $x \in X$; then $\langle Tg'_n\rangle_{n\in\mathbb{N}} = \langle e \wedge |Tf_n - Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, where $e = T(\chi X)$. By 367Bf, $\inf_{n\in\mathbb{N}} \sup_{i\ge n} Tg'_i = 0$ in L^0. But T is a sequentially order-continuous Riesz homomorphism, so $T(\inf_{n\in\mathbb{N}} \sup_{i\ge n} g'_i) = 0$, that is,

$$X \setminus H = \{x : \inf_{n\in\mathbb{N}} \sup_{i\ge n} g'_i > 0\}$$

belongs to $\mathcal{I}$.

367H Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.
(a) Any order*-convergent sequence in $L^0 = L^0(\mathfrak{A})$ is order-bounded.
(b) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, then it is order*-convergent to $u \in L^0$ iff

$$u = \inf_{n\in\mathbb{N}} \sup_{i\ge n} u_i = \sup_{n\in\mathbb{N}} \inf_{i\ge n} u_i.$$

proof (a) We can express $\mathfrak{A}$ as a quotient $\Sigma/\mathcal{I}$ of a σ-algebra of sets, in which case L^0 can be identified with the canonical image of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ (364D). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in L^0, then it is expressible as $\langle Tf_n\rangle_{n\in\mathbb{N}}$, where $T : \mathcal{L}^0 \to L^0$ is the canonical map, and 367G tells us that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges for every $x \in H$, where $X \setminus H \in \mathcal{I}$. If we set $h(x) = \sup_{n\in\mathbb{N}} |f_n(x)|$ for $x \in H$, 0 for $x \in X \setminus H$, then we see that $|u_n| \le Th$ for every $n \in \mathbb{N}$, so that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^0.

(b) This now follows from 367Be, because L^0 is Dedekind σ-complete.

367I Proposition Suppose that $E \subseteq \mathbb{R}$ is a Borel set and $h : E \to \mathbb{R}$ is a continuous function. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and set $Q_E = \{u : u \in L^0, [\![u \in E]\!] = 1\}$, where $L^0 = L^0(\mathfrak{A})$. Let $\bar{h} : Q_E \to L^0$ be the function defined by h (364I). Then $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in Q_E order*-converging to $u \in Q_E$.

proof This is an easy consequence of 367G. We can represent $\mathfrak{A}$ as $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of subsets of some set X and $\mathcal{I}$ is a σ-ideal of Σ (314M); let $T : \mathcal{L}^0 \to L^0(\mathfrak{A})$ be the corresponding homomorphism (364D, 367G). Now we can find Σ-measurable functions $\langle f_n\rangle_{n\in\mathbb{N}}$, f such that $Tf_n = u_n$, $Tf = u$, as in 367G; and the hypothesis $[\![u_n \in E]\!] = 1$, $[\![u \in E]\!] = 1$ means just that, adjusting f_n and f on a member of $\mathcal{I}$ if necessary, we can suppose that $f_n(x)$, $f(x) \in E$ for every $x \in X$. (I am passing over the trivial case $E = \emptyset$, $X \in \mathcal{I}$, $\mathfrak{A} = \{0\}$.) Accordingly $\bar{h}(u_n) = T(hf_n)$, $\bar{h}(u) = T(hf)$, and (because h is continuous)

$$\{x : h(f(x)) \ne \lim_{n\to\infty} h(f_n(x))\} \subseteq \{x : f(x) \ne \lim_{n\to\infty} f_n(x)\} \in \mathcal{I},$$

so $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$.

367J Dominated convergence We now have a suitable language in which to express an abstract version of Lebesgue's Dominated Convergence Theorem.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ which is order-bounded and order*-convergent in L^1, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u in L^1; in particular, $\int u = \lim_{n\to\infty} \int u_n$.

proof The norm of L^1 is order-continuous (365C), so $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u, by 367E. As $\int$ is norm-continuous, $\int u = \lim_{n\to\infty} \int u_n$.

367K The Martingale Theorem In the same way, we can re-write theorems from §275 in this language.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$. For each $n \in \mathbb{N}$ let $P_n : L^1 = L^1_{\bar{\mu}} \to L^1 \cap L^0(\mathfrak{B}_n)$ be the conditional expectation operator (365R); let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and P the conditional expectation operator onto $L^1 \cap L^0(\mathfrak{B})$.

(a) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-bounded sequence in L^1 such that $P_n(u_{n+1}) = u_n$ for every $n \in \mathbb{N}$, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^1.

(b) If $u \in L^1$ then $\langle P_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to Pu.

proof If we represent $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a probability space, these become mere translations of 275G and 275I. (Note that this argument relies on the description of order*-convergence in L^0 in terms of a.e. convergence of functions, as in 367G; so that we need to know that order*-convergence in L^1 matches order*-convergence in L^0, which is what 367F is for.)

367L Some of the most important applications of these ideas concern spaces of continuous functions. I do not think that this is the time to go very far along this road, but one particular fact will be useful in §376.

Proposition Let X be a locally compact Hausdorff space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$, the space of continuous real-valued functions on X. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : x \in X, \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager. In particular, $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 if $\lim_{n\to\infty} u_n(x) = 0$ for every x.

proof (a) The following elementary fact is worth noting: if $A \subseteq C(X)^+$ is non-empty and $\inf A = 0$ in $C(X)$, then $G = \bigcup_{u \in A}\{x : u(x) < \epsilon\}$ is dense for every $\epsilon > 0$. **P?** If not, take $x_0 \in X \setminus \overline{G}$. Because X is completely regular (3A3Bb), there is a continuous function $w : X \to [0,1]$ such that $w(x_0) = 1$ and $w(x) = 0$ for every $x \in \overline{G}$. But in this case $0 < \epsilon w \le u$ for every $u \in A$, which is impossible. **XQ**

(b) Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0. Set $v_n = |u_n| \wedge \chi X$, so that $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 (using 367C, as usual). Set

$$B = \{v : v \in C(X),\, \exists\, n \in \mathbb{N},\, v_i \le v\ \forall\, i \ge n\},$$

so that $\inf B = 0$ in $C(X)$ (367Be). For each $k \in \mathbb{N}$, set $G_k = \bigcup_{v\in B}\{x : v(x) < 2^{-k}\}$; then G_k is dense, by (a), and of course is open. So $H = \bigcup_{k\in\mathbb{N}} X \setminus G_k$ is a countable union of nowhere dense sets and is meager. But this means that

$$\begin{aligned}\{x : \limsup_{n\to\infty} |u_n(x)| > 0\} &= \{x : \limsup_{n\to\infty} v_n(x) > 0\}\\ &\subseteq \{x : \inf_{v\in B} v(x) > 0\} \subseteq H\end{aligned}$$

is meager.

(c) Now suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ does not order*-converge to 0. By 367Cf, there is a $w > 0$ in $C(X)$ such that $w = \sup_{i\ge n} w \wedge |u_i|$ for every $n \in \mathbb{N}$; that is, $\inf_{i\ge n}(w - |u_i|)^+ = 0$ for every n. Set

$$G_n = \{x : \inf_{i\ge n}(w - |u_i|)^+(x) < 2^{-n}\} = \{x : \sup_{i\ge n} |u_i(x)| > w(x) - 2^{-n}\}$$

for each n. Then

$$H = \bigcap_{n\in\mathbb{N}} G_n = \{x : \limsup_{n\to\infty} u_n(x) \ge w(x)\}$$

is the intersection of a sequence of dense open sets, and its complement is meager.

Let G be the non-empty open set $\{x : w(x) > 0\}$. Then G is not meager, by Baire's theorem (3A3Ha); so $G \cap H$ cannot be meager. But $\{x : \limsup_{n\to\infty} |u_n(x)| > 0\}$ includes $G \cap H$, so is also not meager.

Remark Unless the topology of X is discrete, $C(X)$ is not regularly embedded in $\mathbb{R}^X$, and we expect to find sequences in $C(X)$ which order*-converge to 0 in $C(X)$ but not in $\mathbb{R}^X$. But the proposition tells us that if we have a sequence in $C(X)$ which order*-converges in $\mathbb{R}^X$ to a member of $C(X)$, then it order*-converges in $C(X)$.

367M Everything above concerns a particular notion of sequential convergence. There is inevitably a suggestion that there ought to be a topological interpretation of this convergence (see 367Yc, 367Yl), but I have taken care to avoid spelling one out. I come now to something which really is a topology, and is as closely involved with order-convergence as any.

Convergence in measure Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. For $a \in \mathfrak{A}^f = \{a : \bar\mu a < \infty\}$ and $u \in L^0 = L^0(\mathfrak{A})$ set $\tau_a(u) = \int |u| \wedge \chi a$, $\tau_{a\epsilon}(u) = \bar\mu(a \cap [\![|u| > \epsilon]\!])$. Then the **topology of convergence in measure** on L^0 is defined *either* as the topology generated by the pseudometrics $(u, v) \mapsto \tau_a(u - v)$ *or* by saying that $G \subseteq L^0$ is open iff for every $u \in G$ there are $a \in \mathfrak{A}^f$, $\epsilon > 0$ such that $v \in G$ whenever $\tau_{a\epsilon}(u - v) \le \epsilon$.

Remark The sentences above include a number of assertions which need proving. But at this point, rather than write out any of the relevant arguments, I refer you to §245. Since we know that $L^0(\mathfrak{A})$ can be identified with $L^0(\mu)$ for a suitable measure space (X, Σ, μ) (321J, 364Jc), everything we know about general spaces $L^0(\mu)$ can be applied directly to $L^0(\mathfrak{A})$ for measure algebras $(\mathfrak{A}, \bar\mu)$; and that is what I will do for the next few paragraphs. So far, all I have done is to write τ_a in place of the $\bar\tau_F$ of 245A, and call on the remarks in 245Bb and 245F.

367N Theorem (a) For any measure algebra $(\mathfrak{A}, \bar\mu)$, the topology $\mathfrak{T}$ of convergence in measure on $L^0 = L^0(\mathfrak{A})$ is a linear space topology, and any order*-convergent sequence in L^0 is $\mathfrak{T}$-convergent to the same limit.

(b) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(c) $(\mathfrak{A}, \bar\mu)$ is localizable iff $\mathfrak{T}$ is Hausdorff and L^0 is complete under the uniformity corresponding to $\mathfrak{T}$.

(d) $(\mathfrak{A}, \bar\mu)$ is σ-finite iff $\mathfrak{T}$ is metrizable.

proof 245D, 245Cb, 245E. Of course we need 322B to assure us that the phrases 'semi-finite', 'localizable', 'σ-finite' here correspond to the same phrases used in §245, and 367G to identify order*-convergence in L^0 with the order-convergence studied in §245.

367O Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. If $A \subseteq L^0$ is a non-empty, downwards-directed set with infimum 0, then for every neighbourhood G of 0 in L^0 there is a $u \in A$ such that $v \in G$ whenever $|v| \le u$.

proof Let $a \in \mathfrak{A}^f$, $\epsilon > 0$ be such that $u \in G$ whenever $\int |u| \wedge \chi a \le \epsilon$ (see 245Bb). Since $\{u \wedge \chi a : u \in A\}$ is a downwards-directed set in $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ with infimum 0 in L^1, there must be a $u \in A$ such that $\int u \wedge \chi a \le \epsilon$ (365Da). But now $[-u, u] \subseteq G$, as required.

367P Theorem Let U be a Banach lattice and $(\mathfrak{A}, \bar{\mu})$ a measure algebra. Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. If $T : U \to L^0 = L^0(\mathfrak{A})$ is a positive linear operator, then it is continuous.

proof Take any open set $G \subseteq L^0$. **?** Suppose, if possible, that $T^{-1}[G]$ is not open. Then we can find u, $\langle u_n \rangle_{n \in \mathbb{N}} \in U$ such that $Tu \in G$ and $\|u_n - u\| \le 2^{-n}$, $Tu_n \notin G$ for every n. Set $H = G - Tu$; then H is an open set containing 0 but not $T(u_n - u)$, for any $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} \|u_n - u\| < \infty$, $v = \sum_{n=0}^{\infty} n|u_n - u|$ is defined in U, and $|T(u_n - u)| \le \frac{1}{n} Tv$ for every $n \ge 1$. But by 367N (or otherwise) we know that there is some n such that $w \in H$ whenever $|w| \le \frac{1}{n} Tv$, so that $T(u_n - u) \in H$ for some n, which is impossible. **X**

367Q Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra.

(a) A sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $L^0 = L^0(\mathfrak{A})$ converges in measure to $u \in L^0$ iff every subsequence of $\langle u_n \rangle_{n \in \mathbb{N}}$ has a sub-subsequence which order*-converges to u.

(b) A set $F \subseteq L^0$ is closed for the topology of convergence in measure iff $u \in F$ whenever there is a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in F order*-converging to $u \in L^0$.

proof 245K, 245L.

367R It will be useful later to be able to quote the following straightforward facts.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Give $\mathfrak{A}$ its measure-algebra topology (323A) and $L^0 = L^0(\mathfrak{A})$ the topology of convergence in measure. Then the map $\chi : \mathfrak{A} \to L^0$ is a homeomorphism between $\mathfrak{A}$ and its image in L^0.

proof Of course χ is injective (364Kc). The measure-algebra topology of $\mathfrak{A}$ is defined by the pseudometrics $\rho_a(b, c) = \bar{\mu}(a \cap (b \triangle c))$, while the topology of L^0 is defined by the pseudometrics $\tau_a(u, v) = \int |u - v| \wedge \chi a$, in both cases taking a to run over elements of $\mathfrak{A}$ of finite measure; as $\tau_a(\chi b, \chi c)$ is always equal to $\rho_a(b, c)$, we have the result.

367S Proposition Let $E \subseteq \mathbb{R}$ be a Borel set, and $h : E \to \mathbb{R}$ a continuous function. Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\bar{h} : Q_E \to L^0 = L^0(\mathfrak{A})$ the associated function, where $Q_E = \{u : u \in L^0, [\![u \in E]\!] = 1\}$ (364I). Then $\bar{h}$ is continuous for the topology of convergence in measure.

proof (Compare 245Dd.) Express $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a measure space (X, Σ, μ), and write $f^\bullet$ for the element of L^0 corresponding to any $f \in \mathcal{L}^0$. Take any $u \in Q_E$, any $a \in \mathfrak{A}$ such that $\bar{\mu} a < \infty$, and any $\epsilon > 0$. Express u as $f^\bullet$ where $f : X \to \mathbb{R}$ is a measurable function, and a as $F^\bullet$ where $F \in \Sigma$. Then $f(x) \in E$ a.e.(x). For each $n \in \mathbb{N}$, write E_n for

$$\{t : t \in E,\ |h(s) - h(t)| \le \tfrac{1}{3}\epsilon \text{ whenever } s \in E \text{ and } |s - t| \le 2^{-n}\}.$$

Then $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of Borel sets with union E, so there is an n such that $\mu\{x : x \in F, f(x) \notin E_n\} \le \frac{1}{3}\epsilon$.

Now suppose that $v \in Q_E$ is such that $\int |v - u| \wedge \chi a \le \frac{1}{3}\epsilon / 2^n$. Express v as $g^\bullet$ where $g : X \to \mathbb{R}$ is a measurable function. Then $g(x) \in E$ for almost every x, and

$$\int_F \min(1, |g(x) - f(x)|) \mu(dx) \le \frac{1}{3}\epsilon / 2^n,$$

so $\mu\{x : x \in F, |f(x) - g(x)| > 2^{-n}\} \le \frac{1}{3}\epsilon$, and

$$\begin{aligned}\{x : x \in F, |h(g(x)) - h(f(x))| > \frac{1}{3}\epsilon\} \\ \subseteq \{x : x \in F,\ f(x) \notin E_n\} \cup \{x : g(x) \notin E\} \\ \cup \{x : x \in F,\ |f(x) - g(x)| > 2^{-n}\}\end{aligned}$$

has measure at most $\frac{2}{3}\epsilon$. But this means that

$$\int |\bar{h}(v) - \bar{h}(u)| \wedge \chi a = \int_F \min(1, |hg(x) - hf(x)|)\mu(dx) \le \epsilon.$$

As u, a and ϵ are arbitrary, $\bar{h}$ is continuous.

367T Intrinsic description of convergence in measure It is a remarkable fact that the topology of convergence in measure, not only on L^0 but on its order-dense Riesz subspaces, can be described in terms of the Riesz space structure alone, without referring at all to the underlying measure algebra or to integration. (Compare 324H.) There is more than one way of doing this. As far as I know, none is outstandingly convincing; I present a formulation which seems to me to exhibit some, at least, of the essence of the phenomenon.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and U an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. Suppose that $A \subseteq U$ and $u^* \in U$. Then u^* belongs to the closure of A for the topology of convergence in measure iff

there is an order-dense Riesz subspace V of U such that

for every $v \in V^+$ there is a non-empty downwards-directed $B \subseteq U$, with infimum 0, such that

for every $w \in B$ there is a $u \in A$ such that

$$|u - u^*| \wedge v \le w.$$

proof (a) Suppose first that $u^* \in \overline{A}$. Take V to be $U \cap L^1(\mathfrak{A}, \bar{\mu})$; then V is an order-dense Riesz subspace of L^0, by 352Nc, and is therefore order-dense in U. (This is where I use the hypothesis that $(\mathfrak{A}, \bar{\mu})$ is semi-finite, so that L^1 is order-dense in L^0, by 365Ga.)

Take any $v \in V^+$. For each $n \in \mathbb{N}$, set $a_n = [\![v > 2^{-n}]\!] \in \mathfrak{A}^f$. Because $u^* \in \overline{A}$, there is a $u_n \in A$ such that $\bar{\mu} b_n \le 2^{-n}$, where

$$b_n = a_n \cap [\![|u_n - u^*| > 2^{-n}]\!] = [\![|u_n - u^*| \wedge v > 2^{-n}]\!].$$

Set $c_n = \sup_{i \ge n} b_i$; then $\bar{\mu} c_n \le 2^{-n+1}$ for each n, so $\inf_{n\in\mathbb{N}} c_n = 0$ and $\inf_{n\in\mathbb{N}} w_n = 0$ in L^0, where $w_n = v \times \chi c_n + 2^{-n}\chi 1$. Also $|u_n - u^*| \wedge v \le w_n$ for each n.

The w_n need not belong to U, so we cannot set $B = \{w_n : n \in \mathbb{N}\}$. But if instead we write

$$B = \{w : w \in U,\ w \ge v \wedge w_n \text{ for some } n \in \mathbb{N}\},$$

then B is non-empty and downwards-directed (because $\langle w_n\rangle_{n\in\mathbb{N}}$ is non-increasing); and

$$\begin{aligned}\inf B &= v - \sup\{v - w : w \in B\} \\ &= v - \sup\{w : w \in U,\ w \le (v - w_n)^+ \text{ for some } n \in \mathbb{N}\} \\ &= v - \sup_{n\in\mathbb{N}}(v - w_n)^+\end{aligned}$$

(because U is order-dense in L^0)

$$= 0.$$

Since for every $w \in B$ there is an n such that $v \wedge |u_n - u^*| \le v \wedge w_n \le w$, B witnesses that the condition is satisfied.

(b) Now suppose that the condition is satisfied. Fix $a \in \mathfrak{A}^f$, $\epsilon > 0$. Because V is order-dense in U and therefore in L^0, there is a $v \in V$ such that $0 \le v \le \chi a$ and $\int v \ge \bar{\mu} a - \epsilon$. Let B be a downwards-directed set, with infimum 0, such that for every $w \in B$ there is a $u \in A$ with $v \wedge |u - u^*| \le w$. Then there is a $w \in B$ such that $\int w \wedge v \le \epsilon$. Now there is a $u \in A$ such that $|u - u^*| \wedge v \le w$, so that

$$\int |u - u^*| \wedge \chi a \le \epsilon + \int |u - u^*| \wedge v \le \epsilon + \int w \wedge v \le 2\epsilon.$$

As a and ϵ are arbitrary, $u^* \in \overline{A}$.

***367U Theorem** Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra; write L^1 for $L^1(\mathfrak{A}, \bar{\mu})$. Let $P : (L^1)^{**} \to L^1$ be the linear operator corresponding to the band projection from $(L^1)^{**} = (L^1)^{\times\sim}$ onto $(L^1)^{\times\times}$ and the canonical isomorphism between L^1 and $(L^1)^{\times\times}$. For $A \subseteq L^1$ write A^* for the weak* closure of the image of A in $(L^1)^{**}$. Then for every $A \subseteq L^1$

$$P[A^*] \subseteq \overline{\Gamma(A)},$$

where $\Gamma(A)$ is the convex hull of A and $\overline{\Gamma(A)}$ is the closure of $\Gamma(A)$ in $L^0 = L^0(\mathfrak{A})$ for the topology of convergence in measure.

proof (a) The statement of the theorem includes a number of assertions: that $(L^1)^* = (L^1)^\times$; that $(L^1)^{**} = ((L^1)^*)^\sim$; that the natural embedding of L^1 into $(L^1)^{**} = (L^1)^{\times\sim}$ identifies L^1 with $(L^1)^{\times\times}$; and that $(L^1)^{\times\times}$ is a band in $(L^1)^{\times\sim}$. For proofs of these see 365C, 356B, 356D, 356N and 356P.

Now for the new argument. First, observe that the statement of the theorem involves the measure algebra $(\mathfrak{A}, \bar{\mu})$ and the space L^0 only in the definition of 'convergence in measure'; everything else depends only on the Banach lattice structure of L^1. And since we are concerned only with the question of whether members of $P[A^*]$, which is surely a subset of L^1, belong to $\overline{\Gamma(A)}$, 367T shows that this also can be answered in terms of the Riesz space structure of L^1. What this means is that we can suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable. **P** Let $(\widehat{\mathfrak{A}}, \tilde{\mu})$ be the localization of $(\mathfrak{A}, \bar{\mu})$ (322P). The natural expression of $\mathfrak{A}$ as an order-dense subalgebra of $\widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$ with $\widehat{\mathfrak{A}}^f$ (322O), so that $L^1(\mathfrak{A}, \bar{\mu})$ becomes identified with $L^1(\widehat{\mathfrak{A}}, \tilde{\mu})$, by 365Od. Thus we can think of L^1 as $L^1(\widehat{\mathfrak{A}}, \tilde{\mu})$, and $(\widehat{\mathfrak{A}}, \tilde{\mu})$ is localizable. **Q**

(b) Take $\phi \in A^*$ and set $u_0 = P\phi$; I have to show that $u_0 \in \overline{\Gamma(A)}$. Write R for the canonical map from L^1 to $(L^1)^{**}$, so that ϕ belongs to the weak* closure of $R[A]$.

Consider first the case $u_0 = 0$. Take any $c \in \mathfrak{A}^f$ and $\epsilon > 0$. We know that $(L^1)^* = (L^1)^\sim = (L^1)^\times$ can be identified with $L^\infty = L^\infty(\mathfrak{A})$ (365Jc), so that $\phi \in (L^\infty)^* = (L^\infty)^\sim$ must be in the band orthogonal to $(L^\infty)^\times$. Now we can identify $(L^\infty)^\sim$ with the Riesz space M of bounded additive functionals on $\mathfrak{A}$, and if we do so then $(L^\infty)^\times$ corresponds to the space M_τ of completely additive functionals (363K). Writing $P_\tau : M \to M_\tau$ for the band projection, we must have $P_\tau(\nu) = 0$, where $\nu \in M$ is defined by setting $\nu a = \phi(\chi a)$ for each $a \in \mathfrak{A}$; consequently $P_\tau(|\nu|) = 0$ and there is an upwards-directed family $C \subseteq \mathfrak{A}$, with supremum 1, such that $|\nu|(a) = 0$ for every $a \in C$ (362D). Since $\bar{\mu}c = \sup_{a \in C} \bar{\mu}(a \cap c)$, there is an $a \in C$ such that $\bar{\mu}(c \setminus a) \le \epsilon$.

Consider the map $Q : L^1 \to L^1$ defined by setting $Qw = w \times \chi a$ for every $w \in L^1$. Then its adjoint $Q' : L^\infty \to L^\infty$ (3A5Ed) can be defined by the same formula: $Q'v = v \times \chi a$ for every $v \in L^\infty$. Since $|\phi| \in (L^\infty)^\sim$ corresponds to $|\nu| \in M$, we have

$$|\phi(Q'v)| \le \|v\|_\infty |\phi|(\chi a) = \|v\|_\infty |\nu|(a) = 0$$

for every $v \in L^\infty$, and $Q''\phi = 0$, where $Q'' : (L^\infty)^* \to (L^\infty)^*$ is the adjoint of Q'. Since Q'' is continuous for the weak* topology on $(L^\infty)^*$, $0 \in \overline{Q''[R[A]]}$, where $\overline{Q''R[A]}$ is the closure for the weak* topology of $(L^\infty)^*$. But of course $Q''R = RQ$, while the weak* topology of $(L^\infty)^*$ corresponds, on the image $R[L^1]$ of L^1, to the weak topology of L^1; so that 0 belongs to the closure of $Q[A]$ for the weak topology of L^1.

Because Q is linear, $Q[\Gamma(A)]$ is convex. Since 0 belongs to the closure of $Q[\Gamma(A)]$ for the weak topology of L^1, it belongs to the closure of $Q[\Gamma(A)]$ for the norm topology (3A5Ee). So 0 belongs to the closure of $Q[\Gamma(A)]$ for the norm topology, and there is a $w \in \Gamma(A)$ such that $\|w \times \chi a\|_1 \le \epsilon^2$. But this means that $\bar{\mu}(a \cap [\![|w| \ge \epsilon]\!]) \le \epsilon$ and $\bar{\mu}(c \cap [\![|w| \ge \epsilon]\!]) \le 2\epsilon$. Since c and ϵ are arbitrary, $0 \in \overline{\Gamma(A)}$.

(c) This deals with the case $u_0 = 0$. Now the general case follows at once if we set $B = A - u_0$ and observe that $\phi - Ru_0 \in B^*$, so

$$0 = P(\phi - Ru_0) \in \overline{\Gamma(B)} = \overline{\Gamma(A) - u_0} = \overline{\Gamma(A)} - u_0.$$

Remark This is a version of a theorem from Bukhvalov 95.

***367V Corollary** Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Let $\mathcal{C}$ be a family of convex subsets of $L^0 = L^0(\mathfrak{A})$, all closed for the topology of convergence in measure, with the finite intersection property, and suppose that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $C \in \mathcal{C}$ such that $\sup_{u\in C} \int_b |u| < \infty$. Then $\bigcap \mathcal{C} \neq \emptyset$.

proof Because $\mathcal{C}$ has the finite intersection property, there is an ultrafilter $\mathcal{F}$ on L^0 including $\mathcal{C}$. Set

$$I = \{a : a \in \mathfrak{A},\ \inf_{F\in\mathcal{F}} \sup_{u\in F} \int_a |u| < \infty\};$$

because $\mathcal{F}$ is a filter, I is an ideal in $\mathfrak{A}$, and the condition on $\mathcal{C}$ tells us that I is order-dense. For each $a \in I$, define $Q_a : L^0 \to L^0$ by setting $Q_a u = u \times \chi a$. Then there is an $F \in \mathcal{F}$ such that $Q_a[F]$ is a norm-bounded set in L^1, so $\phi_a = \lim_{u\to\mathcal{F}} RQ_a u$ is defined in $(L^\infty)^*$ for the weak* topology on $(L^\infty)^*$, writing R for the canonical map from L^1 to $(L^\infty)^* \cong (L^1)^{**}$. If $P : (L^\infty)^* \to L^1$ is the map corresponding to the band projection $\tilde{P}$ from $(L^\infty)^\sim$ onto $(L^\infty)^\times$, as in 367U, and $C \in \mathcal{C}$, then 367U tells us that $P(\phi_a)$ must belong to the closure of the convex set $Q_a[C]$ for the topology of convergence in measure. Moreover, if $a \subseteq b \in I$, so that $Q_a = Q_a Q_b$, then $P(\phi_a) = Q_a P(\phi_b)$. **P** $Q_a \restriction L^1$ is a band projection on L^1, so its adjoint Q_a' is a band projection on $L^\infty \cong (L^1)^\sim$ (356C) and Q_a'' is a band projection on $(L^\infty)^* \cong (L^\infty)^\sim$. This means that Q_a'' will commute with $\tilde{P}$ (352Sb). But also Q_a'' is continuous for the weak* topology of $(L^\infty)^*$, so

$$Q_a''(\phi_b) = \lim_{u\to\mathcal{F}} Q_a'' RQ_b u = \lim_{u\to\mathcal{F}} RQ_a Q_b u = \phi_a,$$

and

$$P(\phi_a) = R^{-1}\tilde{P}(\phi_a) = R^{-1}\tilde{P}Q_a''(\phi_b) = R^{-1}Q_a''\tilde{P}(\phi_b) = Q_a R^{-1}\tilde{P}(\phi_b) = Q_a P(\phi_b). \textbf{ Q}$$

What this means is that if we take a partition D of unity included in I (313K), so that $L^0 \cong \prod_{d\in D} L^0(\mathfrak{A}_d)$ (315F(iii), 364S), and define $w \in L^0$ by saying that $w \times \chi d = P(\phi_d)$ for every $d \in D$, then we shall have $w \times \chi a = P(\phi_a)$ for every $a \in I$. But now, given $a \in \mathfrak{A}^f$ and $\epsilon > 0$ and $C \in \mathcal{C}$, there is a $b \in I$ such that $\bar{\mu}(a \setminus b) \le \epsilon$; $w \times \chi b \in \overline{Q_b[C]}$, so there is a $u \in C$ such that $\bar{\mu}(b \cap [\![|w-u| \ge \epsilon]\!]) \le \epsilon$; and $\bar{\mu}(a \cap [\![|w-u| \ge \epsilon]\!]) \le 2\epsilon$. As a, ϵ are arbitrary and C is closed, $w \in C$; as C is arbitrary, $w \in \bigcap \mathcal{C}$ and $\bigcap \mathcal{C} \neq \emptyset$.

367X Basic exercises >(a) Let P be a lattice. (i) Show that if $p \in P$ and $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in P, then $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p iff $p = \sup_{n\in\mathbb{N}} p_n$. (ii) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ a sequence in P order*-converging to $p \in P$. Show that $p = \sup_{n\in\mathbb{N}} p \wedge p_n = \inf_{n\in\mathbb{N}} p \vee p_n$. (iii) Let $\langle p_n\rangle_{n\in\mathbb{N}}$, $\langle q_n\rangle_{n\in\mathbb{N}}$ be two sequences in P which are order*-convergent to p, q respectively. Show that if $p_n \le q_n$ for every n then $p \le q$. (iv) Let $\langle p_n\rangle_{n\in\mathbb{N}}$ be a sequence in P. Show that $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to $p \in P$ iff $\langle p_n \vee p\rangle_{n\in\mathbb{N}}$ and $\langle p_n \wedge p\rangle_{n\in\mathbb{N}}$ order*-converge to p.

(b) Let P and Q be lattices, and $f : P \to Q$ an order-preserving function. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence which order*-converges to p in P. Show that $\langle f(p_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $f(p)$ in Q if *either* f is order-continuous *or* P is Dedekind σ-complete and f is sequentially order-continuous.

(c) Let P be *either* a Boolean algebra *or* a Riesz space. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P such that $\langle p_{2n}\rangle_{n\in\mathbb{N}}$ and $\langle p_{2n+1}\rangle_{n\in\mathbb{N}}$ are both order*-convergent to $p \in P$. Show that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p. (*Hint*: 313B, 352E.)

>(d) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ two sequences in $\mathfrak{A}$ order*-converging to a, b respectively. Show that $\langle a_n \cup b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n \cap b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n \setminus b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n \triangle b_n\rangle_{n\in\mathbb{N}}$ order*-converge to $a \cup b$, $a \cap b$, $a \setminus b$ and $a \triangle b$ respectively.

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Show that $\langle a_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0 iff there is a non-zero $a \in \mathfrak{A}$ such that $a = \sup_{i\ge n} a \wedge a_i$ for every $n \in \mathbb{N}$.

>(f) (i) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order*-convergent sequence in U^+ with limit u. Show that $h(u) \le \liminf_{n\to\infty} h(u_n)$ for every $h \in (U^\times)^+$. (ii) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order-bounded order*-convergent sequence in U with limit u. Show that $h(u) = \lim_{n\to\infty} h(u_n)$ for every $h \in U^\times$. (Compare 356Xd.)

>(g) Let U be a Riesz space with a Fatou norm $\| \|$. (i) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in U with limit u, then $\|u\| \le \liminf_{n\to\infty}\|u_n\|$. (*Hint*: $\langle |u_n| \wedge |u|\rangle_{n\in\mathbb{N}}$ is order*-convergent to $|u|$.) (ii) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-convergent sequence in U it has an order*-convergent subsequence. (*Hint*: if $\sum_{n=0}^{\infty}\|u_n\| < \infty$ then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.)

(h) Let U and V be Archimedean Riesz spaces and $T : U \to V$ an order-continuous Riesz homomorphism. Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U which order*-converges to $u \in U$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tu in V.

(i) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra. Show that if $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $b \in \mathfrak{B}$, then $\langle b_n\rangle_{n\in\mathbb{N}}$ order*-converges to b in $\mathfrak{B}$ iff it order*-converges to b in $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ two sequences in $L^0(\mathfrak{A})$ which are order*-convergent to u, v respectively. Show that $\langle u_n \times v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u \times v$. Show that if u, u_n all have multiplicative inverses u^{-1}, u_n^{-1} then $\langle u_n^{-1}\rangle_{n\in\mathbb{N}}$ order*-converges to u^{-1}.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Show that for any $\langle a_n\rangle_{n\in\mathbb{N}}$, $a \in \mathfrak{A}$, $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ order*-converges to $a^\bullet$ in $\mathfrak{A}/\mathcal{I}$ iff $\inf_{n\in\mathbb{N}}\sup_{m\ge n} a_m \triangle a \in \mathcal{I}$.

>(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle h_n\rangle_{n\in\mathbb{N}}$ a sequence of Borel measurable functions from $\mathbb{R}$ to itself such that $h(t) = \lim_{n\to\infty} h_n(t)$ is defined for every $t \in \mathbb{R}$. Show that $\langle \bar{h}_n(u)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$ for every $u \in L^0 = L^0(\mathfrak{A})$, where $\bar{h}_n$, $\bar{h} : L^0 \to L^0$ are defined as in 364I.

(m) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ which is order*-convergent to $u \in L^1$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u iff $\{u_n : n \in \mathbb{N}\}$ is uniformly integrable iff $\|u\|_1 = \lim_{n\to\infty}\|u_n\|_1$. (*Hint*: 245H, 246J.)

(n) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a norm-bounded sequence in U. Show that there are a $v \in U$ and a subsequence $\langle v_n\rangle_{n\in\mathbb{N}}$ of $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $\langle \frac{1}{n+1}\sum_{i=0}^{n} w_i\rangle_{n\in\mathbb{N}}$ order*-converges to v for every subsequence $\langle w_n\rangle_{n\in\mathbb{N}}$ of $\langle v_n\rangle_{n\in\mathbb{N}}$. (*Hint*: 276H.)

>(o) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. Show that $u \mapsto |u|$, $(u, v) \mapsto u \vee v$, $(u, v) \mapsto u \times v$ are continuous.

(p) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $p \in [1, \infty[$. For $v \in (L^p)^+ = L^p(\mathfrak{A}, \bar{\mu})^+$ define $\rho_v : L^0 \times L^0 \to [0, \infty[$ by setting $\rho_v(u_1, u_2) = \||u_1 - u_2| \wedge v\|_p$ for all u_1, $u_2 \in U$. Show that each ρ_v is a pseudometric and that the topology on $L^0 = L^0(\mathfrak{A})$ defined by $\{\rho_v : v \in (L^p)^+\}$ is the topology of convergence in measure.

(q) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra. Suppose we have a double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $L^0 = L^0(\mathfrak{A})$ such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in L^0 for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(r) Let (X, Σ, μ) be a semi-finite measure space. Show that $L^0(\mu)$ is separable for the topology of convergence in measure iff μ is σ-finite and has countable Maharam type. (Cf. 365Xp.)

(s) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. (i) Show that if $\langle a_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $a \in \mathfrak{A}$, then $\langle a_n\rangle_{n\in\mathbb{N}} \to a$ for the measure-algebra topology. (ii) Show that if $(\mathfrak{A}, \bar{\mu})$ is σ-finite, then (α) a sequence converges to a for the topology of $\mathfrak{A}$ iff every subsequence has a sub-subsequence which is order*-convergent to to a (β) a set $F \subseteq \mathfrak{A}$ is closed for the topology of $\mathfrak{A}$ iff $a \in F$ whenever there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in F which is order*-convergent to $a \in \mathfrak{A}$.

(t) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra which is not σ-finite. Show that there is a set $A \subseteq L^0(\mathfrak{A})$ such that the limit of any order*-convergent sequence in A belongs to A, but A is not closed for the topology of convergence in measure.

(u) Let U be a Banach lattice with an order-continuous norm. (i) Show that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to $u \in U$ iff every subsequence has a sub-subsequence which is order-bounded and order*-convergent to u. (ii) Show that a set $F \subseteq U$ is closed for the norm topology iff $u \in F$ whenever there is an order-bounded sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in F order*-converging to $u \in U$.

(v) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. For $u \in L^0 = L^0(\mathfrak{A})$ let ν_u be the distribution of u (364Xd). Show that $u \mapsto \nu_u$ is continuous when L^0 is given the topology of convergence in measure and the space of probability distributions on $\mathbb{R}$ is given the vague topology (274Ld).

(w) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle u_n \rangle_{n\in\mathbb{N}}$ a stochastically independent sequence in L^0, all with the Cauchy distribution $\nu_{C,1}$ with centre 0 and scale parameter 1 (285Xm). For each n let C_n be the convex hull of $\{u_i : i \ge n\}$, and $\overline{C_n}$ its closure for the topology of convergence in measure. Show that every $u \in \overline{C_0}$ has distribution $\nu_{C,1}$. (*Hint*: consider first $u \in C_0$.) Show that $\overline{C_0}$ is bounded for the topology of convergence in measure. Show that $\bigcap_{n\in\mathbb{N}} \overline{C_n} = \emptyset$.

(x) If U is a linear space and $C \subseteq U$ is a convex set, a function $f : C \to \mathbb{R}$ is **convex** if $f(\alpha x+(1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$ whenever x, $y \in C$ and $\alpha \in [0,1]$. Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra and $C \subseteq L^1(\mathfrak{A}, \bar{\mu})$ a non-empty convex norm-bounded set which is closed in $L^0(\mathfrak{A})$ for the topology of convergence in measure. Show that any convex function $f : C \to \mathbb{R}$ which is lower semi-continuous for the topology of convergence in measure is bounded below and attains its infimum.

367Y Further exercises (a) Give an example of an Archimedean Riesz space U and an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in U which is order*-convergent to 0, but such that there is no non-increasing sequence $\langle v_n \rangle_{n\in\mathbb{N}}$, with infimum 0, such that $u_n \le v_n$ for every $n \in \mathbb{N}$.

(b) Let P be a distributive lattice. Show that if $\langle p_n \rangle_{n\in\mathbb{N}}$ is a sequence in P order*-converging to $p \in P$, then $\langle p_n \vee q \rangle_{n\in\mathbb{N}}$, $\langle p_n \wedge q \rangle_{n\in\mathbb{N}}$ order*-converge to $p \vee q$, $p \wedge q$ respectively for any $q \in P$.

(c) Let P be any lattice. (i) Show that there is a topology on P for which a set $A \subseteq P$ is closed iff $p \in A$ whenever there is a sequence in A which is order*-convergent to p. Show that any closed set for this topology is sequentially order-closed. (ii) Now let Q be another lattice, with the topology defined in the same way, and $f : P \to Q$ an order-preserving function. Show that if f is topologically continuous it is sequentially order-continuous.

(d) Let us say that a lattice P is $(2,\infty)$**-distributive** if (α) whenever A, $B \subseteq P$ are non-empty sets with infima p, q respectively, then $\inf\{a \vee b : a \in A,\ b \in B\} = p \vee q$ (β) whenever A, $B \subseteq P$ are non-empty sets with suprema p, q respectively, then $\sup\{a \wedge b : a \in A,\ b \in B\} = p \wedge q$. Show that, in this case, if $\langle p_n \rangle_{n\in\mathbb{N}}$ order*-converges to p and $\langle q_n \rangle_{n\in\mathbb{N}}$ order*-converges to q, $\langle p_n \vee q_n \rangle_{n\in\mathbb{N}}$ order*-converges to $p \vee q$.

(e) Give an example of a lattice P with two sequences $\langle p_n \rangle_{n\in\mathbb{N}}$, $\langle q_n \rangle_{n\in\mathbb{N}}$, both order*-convergent to p, such that $\langle p_n \vee q_n \rangle_{n\in\mathbb{N}}$ is not order*-convergent to p.

(f) (i) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V such that $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V but does not order*-converge in U. (ii) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V, order-bounded in V, such that $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in U but does not order*-converge in V.

(g) Let U be an Archimedean f-algebra. Show that if $\langle u_n \rangle_{n\in\mathbb{N}}$, $\langle v_n \rangle_{n\in\mathbb{N}}$ are sequences in U order*-converging to u, v respectively, then $\langle u_n \times v_n \rangle_{n\in\mathbb{N}}$ order*-converges to $u \times v$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \ge 1$. Let $E \subseteq \mathbb{R}^r$ be a Borel set and write $Q_E = \{(u_1, \dots, u_r) : [\![(u_1, \dots, u_r) \in E]\!] = 1\} \subseteq L^0(\mathfrak{A})^r$ (364Yc). Let $h : E \to \mathbb{R}$ be a continuous function and $\bar{h} : Q_E \to L^0$ the corresponding map (364Yd). Show that if $\langle \mathbf{u}_n \rangle_{n\in\mathbb{N}}$ is a sequence in Q_E which is order*-convergent to $\mathbf{u} \in Q_E$ (in the lattice $(L^0)^r$), then $\langle \bar{h}(\mathbf{u}_n) \rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar{h}(\mathbf{u})$.

(i) Let X be a completely regular Baire space (definition: 314Yd), and $\langle u_n \rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$. Show that $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager in X.

(j) (i) Give an example of a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C([0,1])$ such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in [0,1]$, but $\{u_n : n \in \mathbb{N}\}$ is not order-bounded in $C([0,1])$. (ii) Give an example of an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C(\mathbb{Q})$ such that $\lim_{n\to\infty} u_n(q) = 0$ for every $q \in \mathbb{Q}$, but $\sup_{i\ge n} u_i = \chi\mathbb{Q}$ in $C(\mathbb{Q})$ for every $n \in \mathbb{N}$. (iii) Give an example of a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C([0,1])$ such that $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C([0,1])$, but $\lim_{n\to\infty} u_n(q) > 0$ for every $q \in \mathbb{Q} \cap [0,1]$.

(k) Write out an alternative proof of 367K/367Yi based on the fact that, for a Baire space X, $C(X)$ can be identified with an order-dense Riesz subspace of a quotient of the space of Σ-measurable functions, where Σ is the algebra of subsets of X with the Baire property, as in 364Yi.

(l) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra. Show that there is a topology on $\mathfrak{A}$ such that the closure of any $A \subseteq \mathfrak{A}$ is precisely the set of limits of order*-convergent sequences in A.

(m) Give an example of a set X and a double sequence $\langle u_{mn}\rangle_{m,n\in\mathbb{N}}$ in $\mathbb{R}^X$ such that $\lim_{n\to\infty} u_{mn}(x) = u_m(x)$ exists for every $m \in \mathbb{N}$ and $x \in X$, $\lim_{m\to\infty} u_m(x) = 0$ for every $x \in X$, but there is no sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ in $\{u_{mn} : m,\, n \in \mathbb{N}\}$ such that $\lim_{k\to\infty} v_k(x) = 0$ for every x.

(n) Let U be any Banach lattice with an order-continuous norm. For $v \in U^+$ define $\rho_v : U \times U \to [0,\infty[$ by setting $\rho_v(u_1,u_2) = \||u_1 - u_2| \wedge v\|$ for all u_1, $u_2 \in U$. Show that every ρ_v is a pseudometric on U, and that $\{\rho_v : v \in U^+\}$ defines a Hausdorff linear space topology on U.

(o) Let U be any Riesz space. For $h \in (U_c^\sim)^+$ (356Ab), $v \in U^+$ define $\rho_{vh} : U \times U \to [0,\infty[$ by setting $\rho_{vh}(u_1,u_2) = h(|u_1 - u_2| \wedge v)$ for all u_1, $u_2 \in U$. Show that each ρ_{vh} is a pseudometric on U, and that $\{\rho_{vh} : h \in (U_c^\sim)^+,\, v \in U^+\}$ defines a linear space topology on U.

(p) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra. Show that the function $(\alpha,u) \mapsto [\![u > \alpha]\!] : \mathbb{R} \times L^0 \to \mathfrak{A}$ is Borel measurable when $L^0 = L^0(\mathfrak{A})$ is given the topology of convergence in measure and $\mathfrak{A}$ is given its measure-algebra topology. (*Hint*: if $a \in \mathfrak{A}$, $\gamma \ge 0$ then $\{(\alpha,u) : \bar\mu(a \cap [\![u > \alpha]\!]) > \gamma\}$ is open.)

(q) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$. Show that there is no Hausdorff topology $\mathfrak{T}$ on $L^0(\mathfrak{G})$ such that $\langle u_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}$-convergent to u whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u. (*Hint*: Let H be any $\mathfrak{T}$-open set containing 0. Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$. Find inductively a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $\chi G_n \in H$, $q_n \in G_n$ for every n. Conclude that $\chi\mathbb{R} \in \overline{H}$.)

(r) Give an example of a Banach lattice with a norm which is not order-continuous, but in which every order-bounded order*-convergent sequence is norm-convergent.

(s) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \ge 1$. Let $E \subseteq \mathbb{R}^r$ be a Borel set and write $Q_E = \{(u_1,\dots,u_r) : [\![(u_1,\dots,u_r) \in E]\!] = 1\} \subseteq L^0(\mathfrak{A})^r$ (364Yc). Let $h : E \to \mathbb{R}$ be a continuous function and $\bar h : Q_E \to L^0$ the corresponding map (364Yd). Show that if $\bar h$ is continuous if L^0 is given its topology of convergence in measure and $(L^0)^r$ the product topology.

(t) Show that 367U is true for all measure algebras, whether semi-finite or not.

367 Notes and comments I have given a very general definition of 'order*-convergence'. The general theory of convergence structures on ordered spaces is complex and full of traps for the unwary. I have tried to lay out a safe path to the results which are important in the context of this book. But the propositions here are necessarily full of little conditions (e.g., the requirement that U should be Archimedean, in 367F) whose significance may not be immediately obvious. In particular, the definition is very much better adapted to distributive lattices than to others (367Yb, 367Yd, 367Ye). It is useful in the study of Riesz spaces and Boolean algebras largely because these satisfy strong distributive laws (313B, 352E). The special feature which distinguishes the definition here from other definitions of order-convergence is the fact that it can be applied to sequences which are not order-bounded. For order-bounded sequences there are useful simplifications (367Be-f), but the Martingale Theorem (357J), for instance, if we want to express it in terms of its natural home in the Riesz space L^1, refers to sequences which are hardly ever order-bounded.

The * in the phrase 'order*-convergent' is supposed to be a warning that it may not represent exactly the concept you expect. I think nearly any author using the phrase 'order-convergent' would accept sequences fulfilling the conditions of 367Bf; but beyond this no standard definitions have taken root.

The fact that order*-convergent sequences in an L^0 space are order-bounded (367H) is actually one of the characteristic properties of L^0. Related ideas will be important in the next section (368A, 368M).

It is one of the outstanding characteristics of measure algebras in this context that they provide non-trivial linear space topologies on their L^0 spaces, related in striking ways to the order structure. Not all L^0

spaces have such topologies (367Yq). It is not known whether a topology corresponding to 'convergence in measure' can be defined on $L^0(\mathfrak{A})$ for any $\mathfrak{A}$ which is not a measure algebra; this is the 'control measure problem', which I will discuss in §393 (see 393L).

367T shows that the topology of convergence in measure on $L^0(\mathfrak{A})$ is (at least for semi-finite measure algebras) determined by the Riesz space structure of L^0; and that indeed the same is true of its order-dense Riesz subspaces. This fact is important for a full understanding of the representation theorems in §369 below. If a Riesz space U can be embedded as an order-dense subspace of any such L^0, then there is already a 'topology of convergence in measure' on U, independent of the embedding. It is therefore not surprising that there should be alternative descriptions of the topology of convergence in measure on the important subspaces of L^0 (367Xp, 367Yn).

For σ-finite measure algebras, the topology of convergence in measure is easily described in terms of order-convergence (367Q). For other measure algebras, the formula fails (367Xt). 367Yq shows that trying to apply the same ideas to Riesz spaces in general gives rise to some very curious phenomena.

367V enables us to prove results which would ordinarily be associated with some form of compactness. Of course compactness is indeed involved, as the proof through 367U makes clear; but it is weak* compactness in $(L^1)^{**}$, rather than in the space immediately to hand.

I hardly mention 'uniform integrability' in this section, not because it is uninteresting, but because I have nothing to add at this point to 246J and the exercises in §246. But I do include translations of Lebesgue's Dominated Convergence Theorem (367I) and the Martingale Theorem (367J) to show how these can be expressed in the language of this chapter.

368 Embedding Riesz spaces in L^0

In this section I turn to the representation of Archimedean Riesz spaces as function spaces. Any Archimedean Riesz space U can be represented as an order-dense subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E). Consequently we get representations of Archimedean Riesz spaces as quotients of subspaces of $\mathbb{R}^X$ (368F) and as subspaces of $C^\infty(X)$ (368G), and a notion of 'Dedekind completion' (368I-368J). Closely associated with these is the fact that we have a very general extension theorem for order-continuous Riesz homomorphisms into L^0 spaces (368B). I give a characterization of L^0 spaces in terms of lateral completeness (368M, 368Yd), and I discuss weakly (σ,∞)-distributive Riesz spaces (368N-368S).

368A Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $A \subseteq (L^0)^+$ a set with no upper bound in L^0, where $L^0 = L^0(\mathfrak{A})$. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, there is a $v > 0$ in L^0 such that $nv = \sup_{u\in A} u \wedge nv$ for every $n \in \mathbb{N}$.

proof The hypothesis 'A is countable or $\mathfrak{A}$ is Dedekind complete' ensures that $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined for each α. By 364Ma, $c = \inf_{n\in\mathbb{N}} c_n = \inf_{\alpha\in\mathbb{R}} c_\alpha$ is non-zero. Now for any $n \ge 1$, $\alpha \in \mathbb{R}$

$$[\![\sup_{u\in A}(u \wedge n\chi c) > \alpha]\!] = \sup_{u\in A} [\![u > \alpha]\!] \cap [\![\chi c > \tfrac{\alpha}{n}]\!] = [\![\chi c > \tfrac{\alpha}{n}]\!],$$

because if $0 \le \alpha \le k \in \mathbb{N}$ then

$$\sup_{u\in A} [\![u > \alpha]\!] \supseteq c_k \supseteq c \supseteq [\![\chi c > \tfrac{\alpha}{n}]\!],$$

while if $\alpha < 0$ then (because A is a non-empty subset of $(L^0)^+$)

$$\sup_{u\in A} [\![u > \alpha]\!] = 1 = [\![\chi c > \tfrac{\alpha}{n}]\!].$$

So $\sup_{u\in A} u \wedge n\chi c = n\chi c$ for every $n \ge 1$, and we can take $v = \chi c$. (The case $n = 0$ is of course trivial.)

368B Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, U an Archimedean Riesz space, V an order-dense Riesz subspace of U and $T : V \to L^0 = L^0(\mathfrak{A})$ an order-continuous Riesz homomorphism. Then T has a unique extension to an order-continuous Riesz homomorphism $\tilde{T} : U \to L^0$.

proof (a) The key to the proof is the following: if $u \ge 0$ in U, then $\{Tv : v \in V, 0 \le v \le u\}$ is bounded above in L^0. **P?** Suppose, if possible, otherwise. Then by 368A there is a $w > 0$ in L^0 such that

$nw = \sup_{v\in A} nw \wedge Tv$ for every $n \in \mathbb{N}$, where $A = \{v : v \in V,\ 0 \le v \le u\}$. In particular, there is a $v_0 \in A$ such that $w_0 = w \wedge Tv_0 > 0$. Because U is Archimedean, $\inf_{k\ge1} \frac{1}{k}u = 0$, so $v_0 = \sup_{k\ge1}(v_0 - \frac{1}{k}u)^+$. Because V is order-dense in U, $v_0 = \sup B$ where

$$B = \{v : v \in V,\ 0 \le v \le (v_0 - \frac{1}{k}u)^+ \text{ for some } k \ge 1\}.$$

Because T is order-continuous, $Tv_0 = \sup T[B]$ in L^0, and there is a $v_1 \in B$ such that $w_1 = w_0 \wedge Tv_1 > 0$. Let $k \ge 1$ be such that $v_1 \le (v_0 - \frac{1}{k}u)^+$. Then for any $m \in \mathbb{N}$,

$$mv_1 \wedge u \le (mv_1 \wedge kv_0) + (mv_1 \wedge (u - kv_0)^+)$$

(352Fa)

$$\le kv_0 + (m+k)(v_1 \wedge (\frac{1}{k}u - v_0)^+) = kv_0.$$

So for any $v \in A$, $m \in \mathbb{N}$,

$$mw_1 \wedge Tv = mw_1 \wedge mTv_1 \wedge Tv \le T(mv_1 \wedge v) \le T(mv_1 \wedge u) \le T(kv_0) = kTv_0.$$

But this means that, for $m \in \mathbb{N}$,

$$mw_1 = mw_1 \wedge mw = \sup_{v\in A} mw_1 \wedge (mw \wedge Tv) = \sup_{v\in A} mw_1 \wedge Tv \le kTv_0,$$

which is impossible because L^0 is Archimedean and $w_1 > 0$. **XQ**

(b) Because L^0 is Dedekind complete, $\sup\{Tv : v \in V,\ 0 \le v \le u\}$ is defined in L^0 for every $u \in U$. By 355F, T has a unique extension to an order-continuous Riesz homomorphism from U to L^0.

368C Corollary Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and U, V order-dense Riesz subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Then any Riesz space isomorphism between U and V extends uniquely to a Riesz space isomorphism between $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$; and in this case $\mathfrak{A}$ and $\mathfrak{B}$ must be isomorphic as Boolean algebras.

proof If $T : U \to V$ is a Riesz space isomorphism, then 368B tells us that we have (unique) order-continuous Riesz homomorphisms $S : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ and $S' : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ extending T, T^{-1} respectively. Now $S'S : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is an order-continuous Riesz homomorphism agreeing with the identity on U, so must be the identity on $L^0(\mathfrak{A})$; similarly SS' is the identity on $L^0(\mathfrak{B})$, and S is a Riesz space isomorphism. To see that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, recall that by 364Q they can be identified with the algebras of projection bands of $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, which must be isomorphic.

368D Corollary Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and that U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$ which is isomorphic, as Riesz space, to $L^0(\mathfrak{B})$ for some Dedekind complete Boolean algebra $\mathfrak{B}$. Then $U = L^0(\mathfrak{A})$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ (so, in particular, is Dedekind complete).

proof The identity mapping $U \to U$ is surely an order-continuous Riesz homomorphism, so by 368B extends to an order-continuous Riesz homomorphism $\tilde{T} : L^0(\mathfrak{A}) \to U$. Now $\tilde{T}$ must be injective, because if $u \ne 0$ in $L^0(\mathfrak{A})$ there is a $u' \in U$ such that $0 < u' \le |u|$, so that $0 < u' \le |\tilde{T}u|$. So we must have $U = L^0(\mathfrak{A})$ and $\tilde{T}$ the identity map. By 368C, or otherwise, $\mathfrak{A} \cong \mathfrak{B}$.

368E Theorem Let U be any Archimedean Riesz space, and $\mathfrak{A}$ its band algebra (353B). Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$.

proof (a) If $U = \{0\}$ then $\mathfrak{A} = \{0\}$, $L^0 = L^0(\mathfrak{A}) = \{0\}$ and the result is trivial; I shall therefore suppose henceforth that U is non-trivial. Note that by 352Q $\mathfrak{A}$ is Dedekind complete.

Let $C \subseteq U^+ \setminus \{0\}$ be a maximal disjoint set (in the sense of 352C); to obtain such a set apply Zorn's lemma to the family of all disjoint subsets of $U^+ \setminus \{0\}$. Now I can write down the formula for the embedding $T : U \to L^0$ immediately, though there will be a good deal of work to do in justification: for $u \in U$ and $\alpha \in \mathbb{R}$, $[\![Tu > \alpha]\!]$ will be the band in U generated by

$$\{e \wedge (u - \alpha e)^+ : e \in C\}.$$

(For once, I allow myself to use the formula $[\![\ldots]\!]$ without checking immediately that it represents a member of L^0; all I claim for the moment is that $[\![Tu > \alpha]\!]$ is a member of $\mathfrak{A}$ determined by u and α.)

(b) Before getting down to the main argument, I make some remarks which will be useful later.

(i) If $u > 0$ in U, then there is some $e \in C$ such that $u \wedge e > 0$, since otherwise we ought to have added u to C. Thus $C^\perp = \{0\}$.

(ii) If $u \in U$ and $e \in C$ and $\alpha \in \mathbb{R}$, then $v = e \wedge (\alpha e - u)^+$ belongs to $[\![Tu > \alpha]\!]^\perp$. **P** If $e' \in C$, then either $e' \neq e$ so

$$v \wedge e' \wedge (u - \alpha e')^+ \le e \wedge e' = 0,$$

or $e' = e$ and

$$v \wedge e' \wedge (u - \alpha e')^+ \le (\alpha e - u)^+ \wedge (u - \alpha e)^+ = 0.$$

Accordingly $[\![Tu > \alpha]\!]$ is included in the band $\{v\}^\perp$ and $v \in [\![Tu > \alpha]\!]^\perp$. **Q**

(c) Now I must confirm that the formula given for $[\![Tu > \alpha]\!]$ is consistent with the conditions laid down in 364A. **P** Take $u \in U$.

(i) If $\alpha \le \beta$ then

$$0 \le e \wedge (u - \beta e)^+ \le e \wedge (u - \alpha e)^+ \in [\![Tu > \alpha]\!]$$

so $e \wedge (u - \beta e)^+ \in [\![Tu > \alpha]\!]$ for every $e \in C$ and $[\![Tu > \beta]\!] \subseteq [\![Tu > \alpha]\!]$.

(ii) Given $\alpha \in \mathbb{R}$, set $W = \sup_{\beta > \alpha} [\![Tu > \beta]\!]$ in $\mathfrak{A}$, that is, the band in U generated by $\{e \wedge (u - \beta e)^+ : e \in C, \beta > \alpha\}$. Then for each $e \in C$,

$$\sup_{\beta > \alpha} e \wedge (u - \beta e)^+ = e \wedge (u - \inf_{\beta > \alpha} e)^+ = e \wedge (u - \alpha e)^+$$

using the general distributive laws in U (352E), the translation-invariance of the order (351D) and the fact that U is Archimedean (to see that $\alpha e = \inf_{\beta > \alpha} \beta e$). So $e \wedge (u - \alpha e)^+ \in W$; as e is arbitrary, $[\![Tu > \alpha]\!] \subseteq W$ and $[\![Tu > \alpha]\!] = W$.

(iii) Now set $W = \inf_{n \in \mathbb{N}} [\![Tu > n]\!]$. For any $e \in C$, $n \in \mathbb{N}$ we have

$$e \wedge (ne - u)^+ \in [\![Tu > n]\!]^\perp \subseteq W^\perp,$$

so that

$$e \wedge (e - \tfrac{1}{n} u^+)^+ \le e \wedge (e - \tfrac{1}{n} u)^+ \in W^\perp$$

for every $n \ge 1$ and

$$e = \sup_{n \ge 1} e \wedge (e - \tfrac{1}{n} u^+)^+ \in W^\perp.$$

Thus $C \subseteq W^\perp$ and $W \subseteq C^\perp = \{0\}$. So we have $\inf_{n \in \mathbb{N}} [\![Tu > n]\!] = 0$.

(iv) Finally, set $W = \sup_{n \in \mathbb{N}} [\![Tu > -n]\!]$. Then

$$e \wedge (e - \tfrac{1}{n} u^-)^+ \le e \wedge (e + \tfrac{1}{n} u)^+ \le e \wedge (u + ne)^+ \in W$$

for every $n \ge 1$ and $e \in C$, so

$$e = \sup_{n \ge 1} e \wedge (e - \tfrac{1}{n} u^-)^+ \in W$$

for every $e \in C$ and $W^\perp = \{0\}$, $W = U$. Thus all three conditions of 364A are satisfied. **Q**

(d) Thus we have a well-defined map $T : U \to L^0$. I show next that $T(u+v) = Tu + Tv$ for all $u, v \in U$. **P** I rely on the formulae in 364E and 364Fa, and on partitions of unity in $\mathfrak{A}$, constructed as follows. Fix $n \ge 1$ for the moment. Then we know that

$$\sup_{i \in \mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 1, \quad \inf_{i \in \mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 0.$$

So setting

$$V_i = [\![Tu > \tfrac{i}{n}]\!] \setminus [\![Tu > \tfrac{i+1}{n}]\!] = [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tu > \tfrac{i+1}{n}]\!]^\perp,$$

$\langle V_i\rangle_{i\in\mathbb{Z}}$ is a partition of unity in $\mathfrak{A}$. Similarly, $\langle W_i\rangle_{i\in\mathbb{Z}}$ is a partition of unity, where

$$W_i = [\![Tv > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{i+1}{n}]\!]^\perp.$$

Now, for any i, j, $k \in \mathbb{Z}$ such that $i+j \ge k$,

$$V_i \cap W_j \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{i+j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!];$$

thus

$$[\![Tu + Tv > \tfrac{k}{n}]\!] \supseteq \sup_{i+j\ge k} V_i \cap W_j.$$

On the other hand, if $q \in \mathbb{Q}$ and $k \in \mathbb{Z}$, there is an $i \in \mathbb{Z}$ such that $\frac{i}{n} \le q < \frac{i+1}{n}$, so that

$$[\![Tu > q]\!] \cap [\![Tv > \tfrac{k+1}{n} - q]\!] \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{k-i}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j;$$

thus for any $k \in \mathbb{Z}$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!].$$

Also, if $0 < w \in V_i \cap W_j$, $e \in C$ then

$$w \wedge e \wedge (u - \frac{i+1}{n}e)^+ = w \wedge e \wedge (v - \frac{j+1}{n}e)^+ = 0,$$

so that

$$w \wedge e \wedge (u + v - \frac{i+j+2}{n}e)^+ = 0$$

because

$$(u + v - \frac{i+j+2}{n}e)^+ \le (u - \frac{i+1}{n}e)^+ + (v - \frac{j+1}{n}e)^+$$

by 352Fc. But this means that $V_i \cap W_j \cap [\![T(u+v) > \frac{i+j+2}{n}]\!] = \{0\}$. Turning this round,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \cap \sup_{i+j\le k-1} V_i \cap W_j = 0,$$

and because $\sup_{i,j\in\mathbb{Z}} V_i \cap W_j = U$ in $\mathfrak{A}$,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j.$$

Finally, if $i + j \ge k$ and $0 < w \in V_i \cap V_j$, then there is an $e \in C$ such that $w_1 = w \wedge e \wedge (u - \frac{i}{n}e)^+ > 0$; there is an $e' \in C$ such that $w_2 = w_1 \wedge e' \wedge (v - \frac{j}{n}e')^+ > 0$; of course $e = e'$, and

$$\begin{aligned} 0 < w_2 \le e \wedge (u - \frac{i}{n}e)^+ \wedge (v - \frac{j}{n})^+ &\le e \wedge (u + v - \frac{i+j}{n}e)^+ \\ &\in [\![T(u+v) > \frac{i+j}{n}]\!] \subseteq [\![T(u+v) > \frac{k}{n}]\!] \end{aligned}$$

using 352Fc. This shows that $w \notin [\![T(u+v) > \frac{k}{n}]\!]^\perp$; as w is arbitrary, $V_i \cap W_j \subseteq [\![T(u+v) > \frac{k}{n}]\!]$; so we get

$$\sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!].$$

Putting all these four together, we see that

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!],$$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!]$$

for all $n \ge 1$, $k \in \mathbb{Z}$. But this means that we must have

$$[\![Tu + Tv > \beta]\!] \subseteq [\![T(u+v) > \alpha]\!], \quad [\![T(u+v) > \beta]\!] \subseteq [\![Tu + Tv > \alpha]\!]$$

whenever $\alpha < \beta$. Consequently

$$\begin{aligned}[\![Tu+Tv>\alpha]\!] &= \sup_{\beta>\alpha}[\![Tu+Tv>\beta]\!] \subseteq [\![T(u+v)>\alpha]\!]\\ &= \sup_{\beta>\alpha}[\![T(u+v)>\beta]\!] \subseteq [\![Tu+Tv>\alpha]\!]\end{aligned}$$

and $[\![Tu+Tv>\alpha]\!] = [\![T(u+v)>\alpha]\!]$ for every α, that is, $T(u+v)=Tu+Tv$. **Q**

(e) The hardest part is over. If $u \in U$, $\gamma>0$ and $\alpha \in \mathbb{R}$, then for any $e \in C$

$$\min(1,\frac{1}{\gamma})(e \wedge (\gamma u-\alpha e)^+) \le e \wedge (u-\frac{\alpha}{\gamma}e)^+ \le \max(1,\frac{1}{\gamma})(e \wedge (\gamma u-\alpha e)^+),$$

so

$$[\![T(\gamma u)>\alpha]\!] = [\![Tu>\tfrac{\alpha}{\gamma}]\!] = [\![\gamma Tu>\alpha]\!];$$

as α is arbitrary, $\gamma Tu = T(\gamma u)$; as γ and u are arbitrary, T is linear. (We need only check linearity for $\gamma>0$ because we know from the additivity of T that $T(-u)=-Tu$ for every u.)

(f) To see that T is a Riesz homomorphism, take any $u \in U$ and $\alpha \in \mathbb{R}$ and consider the band $[\![Tu>\alpha]\!] \cup [\![-Tu>\alpha]\!] = [\![|Tu|>\alpha]\!]$ (by 364Mb). This is the band generated by $\{e \wedge (u-\alpha e)^+ : e \in C\} \cup \{e \wedge (-u-\alpha e)^+ : e \in C\}$. But this must also be the band generated by

$$\{(e \wedge (u-\alpha e)^+) \vee (e \wedge (-u-\alpha e)^+) : e \in C\} = \{e \wedge (|u|-\alpha e)^+ : e \in C\},$$

which is $[\![T|u|>\alpha]\!]$. Thus $[\![|Tu|>\alpha]\!] = [\![T|u|>\alpha]\!]$ for every α and $|Tu|=T|u|$. As u is arbitrary, T is a Riesz homomorphism.

(g) To see that T is injective, take any non-zero $u \in U$. Then there must be some $e \in C$ such that $|u| \wedge e \ne 0$, and some $\alpha>0$ such that $|u| \wedge e \not\le \alpha e$, so that $e \wedge (|u|-\alpha e)^+ \ne 0$ and $[\![T|u|>\alpha]\!] \ne \{0\}$ and $T|u| \ne 0$ and $Tu \ne 0$.

Thus T embeds U as a Riesz subspace of L^0.

(h) Finally, I must check that $T[U]$ is order-dense in L^0. **P** Let $p>0$ in L^0. Then there is some $\alpha>0$ such that $V=[\![p>\alpha]\!] \ne 0$. Take $u>0$ in V. Let $e \in C$ be such that $u \wedge e>0$. Then $v=u \wedge \alpha e>0$. Now $e \wedge (v-\alpha e)^+=0$; but also $e' \wedge v=0$ for every $e' \in C$ distinct from e, so that $[\![Tv>\alpha]\!]=\{0\}$. Also $v \in V$, so $e' \wedge (v-\beta e')^+ \in V$ whenever $e' \in C$ and $\beta \ge 0$, and $[\![Tv>\beta]\!] \subseteq V$ for every $\beta \ge 0$. Accordingly we have

$$\begin{aligned}[\![Tv>\beta]\!] &= \{0\} \subseteq [\![p>\beta]\!] \text{ if } \beta \ge \alpha,\\ &\subseteq V \subseteq [\![p>\beta]\!] \text{ if } 0 \le \beta<\alpha,\\ &= U = [\![p>\beta]\!] \text{ if } \beta<0,\end{aligned}$$

and $Tv \le p$. Also $Tv>0$, by (g). As p is arbitrary, $T[U]$ is order-dense in L^0. **Q**

368F Corollary A Riesz space U is Archimedean iff it is isomorphic to a Riesz subspace of some reduced power $\mathbb{R}^X|\mathcal{F}$, where X is a set and $\mathcal{F}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

proof (a) If U is an Archimedean Riesz space, then by 368E there is a space of the form $L^0=L^0(\mathfrak{A})$ such that U can be embedded into L^0. As in the proof of 364E, L^0 is isomorphic to some space of the form $\mathcal{L}^0(\Sigma)/\mathcal{W}$, where Σ is a σ-algebra of subsets of some set X and $\mathcal{W}=\{f : f \in \mathcal{L}^0,\ \{x : f(x) \ne 0\} \in \mathcal{I}\}$, $\mathcal{I}$ being a σ-ideal of Σ. But now $\mathcal{F}=\{A : A \cup E = X \text{ for some } E \in \mathcal{I}\}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ for every sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. (I am passing over the trivial case $X \in \mathcal{I}$, since then U must be $\{0\}$.) And $\mathcal{L}^0/\mathcal{W}$ is (isomorphic to) the image of $\mathcal{L}^0$ in $\mathbb{R}^X|\mathcal{F}$, since $\mathcal{W}=\{f : f \in \mathcal{L}^0,\ \{x : f(x)=0\} \in \mathcal{F}\}$. Thus U is isomorphic to a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$.

(b) On the other hand, if $\mathcal{F}$ is a filter on X closed under countable intersections, then $\mathcal{W}=\{f : f \in \mathbb{R}^X,\ \{x : f(x)=0\} \in \mathcal{F}\}$ is a sequentially order-closed solid linear subspace of the Dedekind σ-complete Riesz space $\mathbb{R}^X$, so that $\mathbb{R}^X|\mathcal{F} = \mathbb{R}^X/\mathcal{W}$ is Dedekind σ-complete (353J(a-iii)) and all its Riesz subspaces must be Archimedean (353H, 351Rc).

368G Corollary Every Archimedean Riesz space U is isomorphic to an order-dense Riesz subspace of some space $C^\infty(X)$, where X is an extremally disconnected compact Hausdorff space.

proof Let Z be the Stone space of the band algebra $\mathfrak{A}$ of U. Because $\mathfrak{A}$ is Dedekind complete (352Q), Z is extremally disconnected and $\mathfrak{A}$ can be identified with the regular open algebra $\mathfrak{G}$ of Z (314S). By 364W, $L^0(\mathfrak{G})$ can be identified with $C^\infty(Z)$. So the embedding of U as an order-dense Riesz subspace of $L^0(\mathfrak{A})$ (368E) can be regarded as an embedding of U as an order-dense Riesz subspace of $C^\infty(Z)$.

368H Corollary Any Dedekind complete Riesz space U is isomorphic to an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

proof Embed U in $L^0 = L^0(\mathfrak{A})$ as in 368E; because U is order-dense in L^0 and (in itself) Dedekind complete, it is solid (353K).

368I Corollary Let U be an Archimedean Riesz space. Then U can be embedded as an order-dense Riesz subspace of a Dedekind complete Riesz space V in such a way that the solid linear subspace of V generated by U is V itself, and this can be done in essentially only one way. If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, there is a unique positive linear operator $\tilde{T} : V \to W$ extending T.

proof By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is a Dedekind complete Boolean algebra. In this case, we can take V to be the solid linear subspace generated by U, that is, $\{v : |v| \le u \text{ for some } u \in U\}$; being a solid linear subspace of the Dedekind complete Riesz space $L^0(\mathfrak{A})$, V is Dedekind complete, and of course U is order-dense in V.

If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, then for any $v \in V^+$ there is a $u_0 \in U$ such that $v \le u_0$, so that Tu_0 is an upper bound for $\{Tu : u \in U,\, 0 \le u \le v\}$; as W is Dedekind complete, $\sup_{u\in U, 0\le u\le v} Tu$ is defined in W. By 355F, T has a unique extension to an order-continuous positive linear operator from V to W.

In particular, if V_1 is another Dedekind complete Riesz space in which U can be embedded as an order-dense Riesz subspace, this embedding of U extends to an embedding of V; since V is Dedekind complete, its copy in V_1 must be a solid linear subspace, so if V_1 is the solid linear subspace of itself generated by U, we get an identification between V and V_1, uniquely determined by the embeddings of U in V and V_1.

368J Definition If U is an Archimedean Riesz space, a **Dedekind completion** of U is a Dedekind complete Riesz space V together with an embedding of U in V as an order-dense Riesz subspace of V such that the solid linear subspace of V generated by U is V itself. 368I tells us that every Archimedean Riesz space U has an essentially unique Dedekind completion, so that we may speak of 'the' Dedekind completion of U.

368K This is a convenient point at which to give a characterization of the Riesz spaces $L^0(\mathfrak{A})$.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that $A \subseteq L^0(\mathfrak{A})^+$ is disjoint. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, A is bounded above in $L^0(\mathfrak{A})$.

proof If $A = \emptyset$, this is trivial; suppose that A is not empty. For $n \in \mathbb{N}$, set $a_n = \sup_{u\in A} [\![u > n]\!]$; this is always defined; set $a = \inf_{n\in\mathbb{N}} a_n$. Now $a = 0$. **P?** Otherwise, there must be a $u \in A$ such that $a' = a \cap [\![u > 0]\!] \ne 0$, since $a \subseteq a_0$. But now, for any n, and any $v \in A \setminus \{u\}$,

$$a' \cap [\![v > n]\!] \subseteq [\![u > 0]\!] \cap [\![v > 0]\!] = 0,$$

so that $a' \subseteq [\![u > n]\!]$. As n is arbitrary, $\inf_{n\in\mathbb{N}} [\![u > n]\!] \ne 0$, which is impossible. **XQ**

By 364Ma, A is bounded above.

368L Definition A Riesz space U is called **laterally complete** or **universally complete** if A is bounded above whenever $A \subseteq U^+$ is disjoint.

368M Theorem Let U be an Archimedean Riesz space. Then the following are equiveridical:
(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic to $L^0(\mathfrak{A})$;
(ii) U is Dedekind σ-complete and laterally complete;
(iii) whenever V is an Archimedean Riesz space, V_0 is an order-dense Riesz subspace of V and $T_0 : V_0 \to U$ is an order-continuous Riesz homomorphism, there is a positive linear operator $T : V \to U$ extending T_0.

proof (a)(i)⇒(ii) and **(i)⇒(iii)** are covered by 368K and 368B.

(b)(ii)⇒(i) Assume (ii). By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ for a Dedekind complete Boolean algebra $\mathfrak{A}$.

(α) If $u \in U^+$ and $a \in \mathfrak{A}$ then $u \times \chi a \in U$. **P** Set $A = \{v : v \in U,\, 0 \le v \le \chi a\}$, and let $C \subseteq A$ be a maximal disjoint set; then $w = \sup C$ is defined in U, and is also the supremum in L^0. Set $b = [\![w > 0]\!]$. As $w \le \chi a$, $b \subseteq a$. **?** If $b \ne a$, then $\chi(a \setminus b) > 0$, and there is a $v' \in U$ such that $0 < v' \le \chi(a \setminus b)$; but now $v' \in A$ and $v' \wedge w = 0$, so $v' \wedge v = 0$ for every $v \in C$, and we ought to have added v' to C. **X** Thus $[\![w > 0]\!] = a$.

Now consider $u' = \sup_{n\in\mathbb{N}} u \wedge nw$; as U is Dedekind σ-complete, $u' \in U$. Since $[\![u' > 0]\!] \subseteq a$, $u' \le u \times \chi a$. On the other hand,

$$u \times \chi[\![w > \tfrac{1}{n}]\!] \times \chi[\![u \le n]\!] \le u \wedge n^2 w \le u'$$

for every $n \ge 1$, so, taking the supremum over n, $u \times \chi a \le u'$. Accordingly

$$u \times \chi a = u' \in U,$$

as required. **Q**

(β) If $w \ge 0$ in L^0, there is a $u \in U$ such that $\frac{1}{2}w \le u \le w$. **P** Set

$$A = \{u : u \in U,\, 0 \le u \le w\},$$

$$C = \{a : a \in \mathfrak{A},\, a \subseteq [\![u - \tfrac{1}{2}w \ge 0]\!] \text{ for some } u \in A\}.$$

Then $\sup A = w$, so C is order-dense in $\mathfrak{A}$. (If $a \in \mathfrak{A} \setminus \{0\}$, either $a \cap [\![w > 0]\!] = 0$ and $a \subseteq [\![0 - \frac{1}{2}w \ge 0]\!]$, so $a \in C$, or there is a $u \in U$ such that $0 < u \le w \times \chi a$. In the latter case there is some n such that $2^n u \le w$ and $2^{n+1}u \not\le w$, and now $c = a \cap [\![2^n u - \frac{1}{2}w \ge 0]\!]$ is a non-zero member of C included in a.) Let $D \subseteq C$ be a partition of unity and for each $d \in D$ choose $u_d \in A$ such that $d \subseteq [\![u_d - \frac{1}{2}w \ge 0]\!]$. By ($\alpha$), $u_d \times \chi d \in U$ for every $d \in D$, so $u = \sup_{d\in D} u_d \times \chi d \in U$. Now $u \le w$, but also $[\![u - \frac{1}{2}w \ge 0]\!] \supseteq d$ for every $d \in D$, so is equal to 1, and $u \ge \frac{1}{2}w$, as required. **Q**

(γ) Given $w \ge 0$ in L^0, we can therefore choose $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ inductively such that $v_0 = 0$ and

$$u_n \in U, \quad \frac{1}{2}(w - v_n) \le u_n \le w - v_n, \quad v_{n+1} = v_n + u_n$$

for every $n \in \mathbb{N}$. Now $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U and $w - v_n \le 2^{-n}w$ for every n, so $w = \sup_{n\in\mathbb{N}} v_n \in U$.

As w is arbitrary, $(L^0)^+ \subseteq U$ and $U = L^0$ is of the right form.

(c)(iii)⇒(i) As in (b), we may suppose that U is an order-dense Riesz subspace of L^0. But now apply condition (iii) with $V = L^0$, $V_0 = U$ and T_0 the identity operator. There is an extension $T : L^0 \to U$. If $v \ge 0$ in L^0, $Tv \ge T_0 = u$ whenever $u \in U$ and $u \le v$, so $Tv \ge v$, since $v = \sup\{u : u \in U,\, 0 \le u \le v\}$ in L^0. Similarly, $T(Tv - v) \ge Tv - v$. But as $Tv \in U$, $T(Tv) = Tv$ and $T(Tv - v) = 0$, so $v = Tv \in U$. As v is arbitrary, $U = L^0$.

368N Weakly (σ,∞)-distributive Riesz spaces We are now ready to look at the class of Riesz spaces corresponding to the weakly (σ,∞)-distributive Boolean algebras of §316.

Definition Let U be a Riesz space. Then U is **weakly (σ,∞)-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{n\in\mathbb{N}} A_n$ has an upper bound in U, then

$$\{u : u \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \le u\}$$

has infimum 0 in U.

Remark Because the definition looks only at sequences $\langle A_n\rangle_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is order-bounded, we can invert it, as follows: a Riesz space U is weakly (σ,∞)-distributive iff whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty upwards-directed subsets of U^+, all with supremum u_0, then

$$\{u : u \in U^+, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } u \le v\}$$

also has supremum u_0.

368O Lemma Let U be an Archimedean Riesz space. Then the following are equiveridical:
(i) U is not weakly (σ,∞)-distributive;
(ii) there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for every $n \in \mathbb{N}$.

proof (ii)⇒(i) is immediate from the definition of 'weakly (σ,∞)-distributive'. For (i)⇒(ii), suppose that U is not weakly (σ,∞)-distributive. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\bigcup_{n\in\mathbb{N}} A_n$ is bounded above, but

$$A = \{w : w \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \le w\}$$

does not have infimum 0. Let $u > 0$ be a lower bound for A, and set $A'_n = \{u \wedge v : v \in A_n\}$ for each $n \in \mathbb{N}$. Then each A'_n is a non-empty downwards-directed set with infimum 0. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence such that $u_n \in A'_n$ for every n. Express each u_n as $u \wedge v_n$ where $v_n \in A_n$. Let B be the set of upper bounds of $\{v_n : n \in \mathbb{N}\}$. Then $\inf_{w\in B, n\in\mathbb{N}} w - v_n = 0$, because U is Archimedean (353F), while $B \subseteq A$, so $u \le w$ for every $w \in B$. If u' is any upper bound for $\{u_n : n \in \mathbb{N}\}$, then

$$u - u' \le u - u \wedge v_n = (u - v_n)^+ \le (w - v_n)^+ = w - v_n$$

for every $n \in \mathbb{N}$, $w \in B$. So $u' \ge u$. Thus $u = \sup_{n\in\mathbb{N}} u_n$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, u and $\langle A'_n\rangle_{n\in\mathbb{N}}$ witness that (ii) is true.

368P Proposition (a) A regularly embedded Riesz subspace of an Archimedean weakly (σ,∞)-distributive Riesz space is weakly (σ,∞)-distributive.
(b) An Archimedean Riesz space with a weakly (σ,∞)-distributive order-dense Riesz subspace is weakly (σ,∞)-distributive.
(c) If U is a Riesz space such that $U^\times$ separates the points of U, then U is weakly (σ,∞)-distributive; in particular, $U^\sim$ and $U^\times$ are weakly (σ,∞)-distributive for every Riesz space U.

proof (a) Suppose that U is an Archimedean Riesz space and that $V \subseteq U$ is a regularly embedded Riesz subspace which is not weakly (σ,∞)-distributive. Then 368O tells us that there are a $v > 0$ in V and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of V, all with infimum 0 in V, such that $\sup_{n\in\mathbb{N}} v_n = v$ in V whenever $v_n \in A_n$ for every $n \in \mathbb{N}$. Because V is regularly embedded in U, $\inf A_n = 0$ in U for every n and $\sup_{n\in\mathbb{N}} v_n = v$ in U for every sequence $\langle v_n\rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$, so U is not weakly (σ,∞)-distributive. Turning this round, we have (a).

(b) Let U be an Archimedean Riesz space which is not weakly (σ,∞)-distributive, and V an order-dense Riesz subspace of U. By 368O again, there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets in U, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for every n. Let $v \in V$ be such that $0 < v \le u$. Set

$$B_n = \{w : w \in V, \text{ there is some } u \in A_n \text{ such that } v \wedge u \le w \le v\}$$

for each $n \in \mathbb{N}$. Because A_n is downwards-directed, $w \wedge w' \in B_n$ for all w, $w' \in B_n$; $v \in B_n$, so $B_n \ne \emptyset$; and $\inf B_n = 0$ in V. **P** Setting

$$C = \{w : w \in V^+, \text{ there is some } u \in A_n \text{ such that } w \le (v - u)^+\},$$

then (because V is order-dense) any upper bound for C in U is also an upper bound of $\{(v - u)^+ : u \in A_n\}$. But

$$\sup_{u\in A_n} (v - u)^+ = (v - \inf A_n)^+ = v,$$

so $v = \sup C$ in U and $\inf B_n = \inf\{v - w : w \in C\} = 0$ in U and in V. **Q**

Now if $v_n \in B_n$ for every $n \in \mathbb{N}$, we can choose $u_n \in A_n$ such that $v \wedge u_n \le v_n \le v$ for every n, so that

$$v = v \wedge u = v \wedge \sup_{n\in\mathbb{N}} u_n = \sup_{n\in\mathbb{N}} v \wedge u_n \le \sup_{n\in\mathbb{N}} v_n \le v,$$

and $v = \sup_{n\in\mathbb{N}} v_n$. Thus $\langle B_n\rangle_{n\in\mathbb{N}}$ witnesses that V is not weakly (σ,∞)-distributive.

(c) Now suppose that $U^\times$ separates the points of U. In this case U is surely Archimedean (356G). **?** If U is not weakly (σ,∞)-distributive, there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for each n. Take $f \in U^\times$ such that $f(u) \ne 0$; replacing f by $|f|$ if necessary, we may suppose that $f > 0$. Set $\delta = f(u) > 0$. For each $n \in \mathbb{N}$, there is a $u_n \in A_n$ such that $f(u_n) \le 2^{-n-2}\delta$. But in this case $\langle \sup_{i<n} u_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so

$$f(u) = \lim_{n\to\infty} f(\sup_{i\le n} u_i) \le \sum_{i=0}^\infty f(u_i) \le \tfrac{1}{2}\delta < f(u),$$

which is absurd. **X** Thus U is weakly (σ,∞)-distributive.

For any Riesz space U, U acts on $U^\sim$ as a subspace of $U^{\sim\times}$ (356F); as U surely separates the points of $U^\sim$, so does $U^{\sim\times}$. So $U^\sim$ is weakly (σ,∞)-distributive. Now $U^\times$ is a band in $U^\sim$ (356B), so is regularly embedded, and must also be weakly (σ,∞)-distributive, by (a) above.

368Q Theorem (a) For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $S(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $L^\infty(\mathfrak{A})$ is weakly (σ,∞)-distributive.

(b) For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is weakly (σ,∞)-distributive.

proof (a)(i) ? Suppose, if possible, that $\mathfrak{A}$ is weakly (σ,∞)-distributive but $S = S(\mathfrak{A})$ is not. By 368O, as usual, we have a $u > 0$ in S and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets in S, all with infimum 0, such that $u = \sup_{n\in\mathbb{N}} u_n$ whenever $u_n \in A_n$ for every n. Let $\alpha > 0$ be such that $c = [\![u > \alpha]\!] \ne 0$ (361Eg), and consider

$$B_n = \{[\![v > \alpha]\!] : v \in A_n\} \subseteq \mathfrak{A}$$

for each $n \in \mathbb{N}$. Then each B_n is downwards-directed (because A_n is), and $\inf B_n = 0$ in $\mathfrak{A}$ (because if b is a lower bound of B_n, $\alpha\chi b \le v$ for every $v \in A_n$). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there must be some $a \in \mathfrak{A}$ such that $a \not\supseteq c$ but there is, for every $n \in \mathbb{N}$, a $b_n \in B_n$ such that $a \supseteq b_n$. Take $v_n \in A_n$ such that $b_n = [\![v_n > \alpha]\!]$, so that

$$v_n \le \alpha\chi 1 \vee \|v_n\|_\infty \chi b_n \le \alpha\chi 1 \vee \|u\|_\infty \chi a.$$

Since $u = \sup_{n\in\mathbb{N}} v_n$, $u \le \alpha\chi 1 \vee \|u\|_\infty \chi a$. But in this case

$$c = [\![u > \alpha]\!] \subseteq a,$$

contradicting the choice of a. **X**

Thus S must be weakly (σ,∞)-distributive if $\mathfrak{A}$ is.

(ii) Now suppose that S is weakly (σ,∞)-distributive, and let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set $A_n = \{\chi b : b \in B_n\}$ for each n; then $A_n \subseteq S$ is non-empty, downwards-directed and has infimum 0 in S, because $\chi : \mathfrak{A} \to S$ is order-continuous (361Ef). Set

$$A = \{v : v \in S, \text{ for every } n \in \mathbb{N} \text{ there is a } u \in A_n \text{ such that } u \le v\},$$

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in B_n \text{ such that } a \subseteq b\}.$$

? If 0 is not the greatest lower bound of B, take a non-zero lower bound c. Because S is weakly (σ,∞)-distributive, $\inf A = 0$, and there is a $v \in A$ such that $\chi c \not\le v$. Express v as $\sum_{i=0}^n \alpha_i\chi a_i$, where $\langle a_i\rangle_{i\le n}$ is disjoint, and set $a = \sup\{a_i : i \le n, \alpha_i \ge 1\}$; then $\chi a \le v$, so $c \not\subseteq a$. For each n there is a $b_n \in B_n$ such that $\chi b_n \le v$. But in this case $b_n \subseteq a$ for each $n \in \mathbb{N}$, so that $a \in B$; which means that c is not a lower bound for B. **X**

Thus $\inf B = 0$ in $\mathfrak{A}$. As $\langle B_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(iii) Thus S is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is. But S is order-dense in $L^\infty = L^\infty(\mathfrak{A})$ (363C), therefore regularly embedded (352Ne), so 368Pa-b tell us that S is weakly (σ,∞)-distributive iff L^∞ is.

(b) In the same way, because S can be regarded as an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ (364K), L^0 is weakly (σ,∞)-distributive iff S is, that is, iff $\mathfrak{A}$ is.

368R Corollary An Archimedean Riesz space is weakly (σ,∞)-distributive iff its band algebra is weakly (σ,∞)-distributive.

proof Let U be an Archimedean Riesz space and $\mathfrak{A}$ its band algebra. By 368E, U is isomorphic to an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. By 368P, U is weakly (σ,∞)-distributive iff L^0 is; and by 368Qb L^0 is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is.

368S Corollary If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, any regularly embedded Riesz subspace (in particular, any solid linear subspace and any order-dense Riesz subspace) of $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive.

proof By 322F, $\mathfrak{A}$ is weakly (σ,∞)-distributive; by 368Qb, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive; by 368Pa, any regularly embedded Riesz subspace is weakly (σ,∞)-distributive.

368X Basic exercises (a) Let X be an uncountable set and Σ the countable-cocountable σ-algebra of subsets of X. Show that there is a family $A \subseteq L^0 = L^0(\Sigma)$ such that $u \wedge v = 0$ for all distinct u, $v \in A$ but A has no upper bound in L^0. Show moreover that if $w > 0$ in L^0 then there is an $n \in \mathbb{N}$ such that $nw \ne \sup_{u\in A} u \wedge nw$.

(b) Let $\mathfrak{A}$ be any Boolean algebra, and $\widehat{\mathfrak{A}}$ its Dedekind completion (314U). Show that $L^\infty(\widehat{\mathfrak{A}})$ can be identified with the Dedekind completions of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(c) Explain how to prove 368K from 368A.

(d) Show that any product of weakly (σ,∞)-distributive Riesz spaces is weakly (σ,∞)-distributive.

(e) Let $\mathfrak{A}$ be a Dedekind complete weakly (σ,∞)-distributive Boolean algebra. Show that a set $A \subseteq L^0 = L^0(\mathfrak{A})$ is order-bounded iff $\langle 2^{-n}u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in A. (*Hint*: use 368A. If $v > 0$ and $v = \sup_{u\in A} v \wedge 2^{-n}u$ for every n, we can find a $w > 0$ and a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A such that $w \le 2^{-n}u_n$ for every n.)

(f) Give a direct proof of 368S, using the ideas of 322F, but not relying on it or on 368Q.

368Y Further exercises (a) (i) Use 364U-364V to show that if X is any compact Hausdorff space then $C(X)$ can be regarded as an order-dense Riesz subspace of $L^0(\mathfrak{G})$, where $\mathfrak{G}$ is the regular open algebra of X. (ii) Use 353M to show that any Archimedean Riesz space with order unit can be embedded as an order-dense Riesz subspace of some $L^0(\mathfrak{G})$. (iii) Let U be an Archimedean Riesz space and $C \subseteq U^+$ a maximal disjoint set, as in part (a) of the proof of 368E. For $e \in C$ let U_e be the solid linear subspace of U generated by e, and let V be the solid linear subspace of U generated by C. Show that V can be embedded as an order-dense Riesz subspace of $\prod_{e\in C} U_e$ and therefore in $\prod_{e\in C} L^0(\mathfrak{G}_e) \cong L^0(\prod_{e\in C}\mathfrak{G}_e)$ for a suitable family of regular open algebras $\mathfrak{G}_e$. (iv) Now use 368B to complete a proof of 368E.

(b) Let U be any Archimedean Riesz space. Let $\mathcal{V}$ be the family of pairs (A,B) of non-empty subsets of U such that B is the set of upper bounds of A and A is the set of lower bounds of B. Show that $\mathcal{V}$ can be given the structure of a Dedekind complete Riesz space defined by the formulae

$$(A_1,B_1)+(A_2,B_2) = (A,B) \text{ iff } A_1+A_2 \subseteq A,\ B_1+B_2 \subseteq B,$$

$$\alpha(A,B) = (\alpha A,\alpha B) \text{ if } \alpha > 0,$$

$$(A_1,B_1) \le (A_2,B_2) \text{ iff } A_1 \subseteq A_2.$$

Show that $u \mapsto (\left]-\infty,u\right], \left[u,\infty\right[)$ defines an embedding of U as an order-dense Riesz subspace of $\mathcal{V}$, so that $\mathcal{V}$ may be identified with the Dedekind completion of U.

(c) Work through the proof of 364U when X is compact, Hausdorff and extremally disconnected, and show that it is easier than the general case. Hence show that 368Yb can be used to shorten the proof of 368E sketched in 368Ya.

(d) Let U be a Riesz space. Show that the following are equiveridical: (i) U is isomorphic, as Riesz space, to $L^0(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$ (ii) U is Dedekind σ-complete and has a weak order unit and whenever $A \subseteq U^+$ is countable and disjoint then A is bounded above in U.

(e) Let U be a weakly (σ,∞)-distributive Riesz space and V a Riesz subspace of U which is *either* solid *or* order-dense. Show that V is weakly (σ,∞)-distributive.

(f) Show that $C([0,1])$ is not weakly (σ,∞)-distributive. (Compare 316K.)

(g) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra. Suppose we have a double sequence $\langle a_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to a_i in $\mathfrak{A}$ for each i, while $\langle a_i\rangle_{i\in\mathbb{N}}$ order*-converges to a. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle a_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to a.

(h) Let U be a weakly (σ,∞)-distributive Riesz space with the countable sup property. Suppose we have an order-bounded double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in U such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in U for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(i) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Show that there is a topology on $L^0 = L^0(\mathfrak{A})$ such that the closure of any $A \subseteq L^0$ is precisely the set of order*-limits of sequences in A.

(j) Let U be a weakly (σ,∞)-distributive Riesz space and $f : U \to \mathbb{R}$ a positive linear functional; write f_τ for the component of f in $U^\times$. (i) Show that for any $u \in U^+$ there is an upwards-directed $A \subseteq [0,u]$, with supremum u, such that $f_\tau(u) = \sup_{v\in A} f(v)$. (See 356Xe, 362D.) (ii) Show that if f is strictly positive, so is f_τ. (Compare 391D.)

368 Notes and comments 368A-368B are manifestations of a principle which will reappear in §375: Dedekind complete L^0 spaces are in some sense 'maximal'. If we have an order-dense subspace U of such an L^0, then any Archimedean Riesz space including U as an order-dense subspace can itself be embedded in L^0 (368B). In fact this property characterizes Dedekind complete L^0 spaces (368M). Moreover, any Archimedean Riesz space U can be embedded in this way (368E); by 368C, the L^0 space (though not the embedding) is unique up to isomorphism. If U and V are Archimedean Riesz spaces, each embedded as an order-dense Riesz subspace of a Dedekind complete L^0 space, then any order-continuous Riesz homomorphism from U to V extends uniquely to the L^0 spaces (368B). If one Dedekind complete L^0 space is embedded as an order-dense Riesz subspace of another, they must in fact be the same (368D). Thus we can say that every Archimedean Riesz space U can be extended to a Dedekind complete L^0 space, in a way which respects order-continuous Riesz homomorphisms, and that this extension is maximal, in that U cannot be order-dense in any larger space.

The proof of 368E which I give is long because I am using a bare-hands approach. Alternative methods shift the burdens. For instance, if we take the trouble to develop a direct construction of the 'Dedekind completion' of a Riesz space (368Yb), then we need prove the theorem only for Dedekind complete Riesz spaces. A more substantial aid is the representation theorem for Archimedean Riesz spaces with order unit (353M); I sketch an argument in 368Ya. The drawback to this approach is the proof of Theorem 364U, which seems to be quite as long as the direct proof of 368E which I give here. Of course we need 364U only for compact Hausdorff spaces, which are usefully easier than the general case (364V, 368Yc).

368G is a version of Ogasawara's representation theorem for Archimedean Riesz spaces. Both this and 368F can be regarded as expressions of the principle that an Archimedean Riesz space is 'nearly' a space of functions.

I have remarked before on the parallels between the theories of Boolean algebras and Archimedean Riesz spaces. The notion of 'weak (σ,∞)-distributivity' is one of the more striking correspondences. (Compare, for instance, 316Xm with 368Pa.) What is really important to us, of course, is the fact that the function spaces of measure theory are mostly weakly (σ,∞)-distributive, by 368S. Of course this is easy to prove directly (368Xf), but I think that the argument through 368Q gives a better idea of what is really happening here. Some of the features of 'order*-convergence', as defined in §367, are related to weak (σ,∞)-distributivity; in 368Yi I describe a topology which can be thought of as an abstract version of the topology of convergence in measure on the L^0 space of a σ-finite measure algebra (367N).

369 Banach function spaces

In this section I continue the work of §368 with results which involve measure algebras. The first step is a modification of the basic representation theorem for Archimedean Riesz spaces. If U is any Archimedean Riesz space, it can be represented as a subspace of $L^0=L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E); now if $U^\times$ separates the points of U, there is a measure rendering $\mathfrak{A}$ a localizable measure algebra (369A, 369Xa). Moreover, we get a simultaneous representation of $U^\times$ as a subspace of L^0 (369C-369D), the duality between them corresponding exactly to the familiar duality between L^p and L^q.

Still drawing inspiration from the classical L^p spaces, we have a general theory of 'associated Fatou norms' (369F-369M, 369R). I include notes on the spaces $M^{1,\infty}$, $M^{\infty,1}$ and $M^{1,0}$ (369N-369Q), which will be particularly useful in the next chapter.

369A Theorem Let U be a Riesz space such that $U^\times$ separates the points of U. Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$.

proof (a) Consider the canonical map $S:U\to U^{\times\times}$. We know that this is a Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). Because $U^\times$ separates the points of U, S is injective. Let $\mathfrak{A}$ be the band algebra of $U^{\times\times}$ and $T:U^{\times\times}\to L^0=L^0(\mathfrak{A})$ an injective Riesz homomorphism onto an order-dense Riesz subspace V of L^0, as in 368E. The composition $TS:U\to L^0$ is now an injective Riesz homomorphism, so embeds U as a Riesz subspace of L^0, which is order-dense because V is order-dense in L^0 and $TS[U]$ is order-dense in V (352Nc). Thus all that we need to find is a measure $\bar\mu$ on $\mathfrak{A}$ rendering it a localizable measure algebra.

(b) Note that V is isomorphic, as Riesz space, to $U^{\times\times}$, which is Dedekind complete (356B), so V must be solid in L^0 (353K). Also $V^\times$ must separate the points of V (356L).

Let D be the set of those $d\in\mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ is measurable in the sense that there is some $\bar\nu$ for which $(\mathfrak{A}_d,\bar\nu)$ is a totally finite measure algebra. Then D is order-dense in $\mathfrak{A}$. **P** Take any non-zero $a\in\mathfrak{A}$. Because V is order-dense, there is a non-zero $v\in V$ such that $v\le\chi a$. Take $h\ge 0$ in $V^\times$ such that $h(v)>0$. Then there is a v' such that $0<v'\le v$ and $h(w)>0$ whenever $0<w\le v'$ in V (356H). Let $\alpha>0$ be such that $d=[\![v'>\alpha]\!]\ne 0$. Then $\chi b\le\frac{1}{\alpha}v'\in V$ whenever $b\in\mathfrak{A}_d$. Set $\bar\nu b=h(\chi b)\in[0,\infty[$ for $b\in\mathfrak{A}_d$. Because the map $b\mapsto\chi b:\mathfrak{A}\to L^0$ is additive and order-continuous, the map $b\mapsto\chi b:\mathfrak{A}_d\to V$ also is, and $\bar\nu=h\chi$ must be additive and order-continuous; in particular, $\bar\nu(\sup_{n\in\mathbb{N}}b_n)=\sum_{n=0}^\infty\bar\nu b_n$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_d$. Moreover, if $b\in\mathfrak{A}_d$ is non-zero, then $0<\alpha\chi b\le v'$, so $\bar\nu b=h(\chi b)>0$. Thus $(\mathfrak{A}_d,\bar\nu)$ is a totally finite measure algebra, and $d\in D$, while $0\ne d\subseteq a$. As a is arbitrary, D is order-dense. **Q**

(c) By 313K, there is a partition of unity $C\subseteq D$. For each $c\in C$, let $\bar\nu_c:\mathfrak{A}_c\to[0,\infty[$ be a functional such that $(\mathfrak{A}_c,\bar\nu_c)$ is a totally finite measure algebra. Define $\bar\mu:\mathfrak{A}\to[0,\infty]$ by setting $\bar\mu a=\sum_{c\in C}\bar\nu_c(a\cap c)$ for every $a\in\mathfrak{A}$. Then $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra. **P** (i) $\bar\mu 0=\sum_{c\in C}\bar\nu 0=0$. (ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, then

$$\bar\mu a=\sum_{c\in C}\bar\nu_c(a\cap c)=\sum_{c\in C,n\in\mathbb{N}}\bar\nu_c(a_n\cap c)=\sum_{n=0}^\infty\bar\mu a_n.$$

(iii) If $a\in\mathfrak{A}\setminus\{0\}$, then there is a $c\in C$ such that $a\cap c\ne 0$, so that $\bar\mu a\ge\bar\nu_c(a\cap c)>0$. Thus $(\mathfrak{A},\bar\mu)$ is a measure algebra. (iv) Moreover, in (iii), $\bar\mu(a\cap c)=\bar\nu_c(a\cap c)$ is finite. So $(\mathfrak{A},\bar\mu)$ is semi-finite. (v) $\mathfrak{A}$ is Dedekind complete, being a band algebra (352Q), so $(\mathfrak{A},\bar\mu)$ is localizable. **Q**

369B Corollary Let U be a Banach lattice with order-continuous norm. Then U can be embedded as an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A}, \bar{\mu})$.

proof By 356Dd, $U^\times = U^*$, which separates the points of U, by the Hahn-Banach theorem (3A5Ae). So 369A tells us that U can be embedded as an order-dense Riesz subspace of an appropriate $L^0(\mathfrak{A})$. But also U is Dedekind complete (354Ee), so its copy in $L^0(\mathfrak{A})$ must be solid, as in 368H.

369C The representation in 369A is complemented by the following result, which is a kind of generalization of 365J and 366Dc.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace. Set

$$V = \{v : v \in L^0,\, v \times u \in L^1 \text{ for every } u \in U\},$$

writing L^1 for $L^1(\mathfrak{A}, \bar{\mu}) \subseteq L^0$. Then V is a solid linear subspace of L^0, and we have an order-continuous injective Riesz homomorphism $T : V \to U^\times$ defined by setting

$$(Tv)(u) = \int u \times v \text{ for all } u \in U,\, v \in V.$$

The image of V is order-dense in $U^\times$. If $(\mathfrak{A}, \bar{\mu})$ is localizable, then T is surjective, so is a Riesz space isomorphism between V and $U^\times$.

proof (a)(i) Because $\times : L^0 \times L^0 \to L^0$ is bilinear and L^1 is a linear subspace of L^0, V is a linear subspace of L^0. If $u \in U$, $v \in V$, $w \in L^0$ and $|w| \le |v|$, then

$$|w \times u| = |w| \times |u| \le |v| \times |u| = |v \times u| \in L^1;$$

as L^1 is solid, $w \times u \in L^1$; as u is arbitrary, $w \in V$; this shows that V is solid.

(ii) By the definition of V, $(Tv)(u)$ is defined in $\mathbb{R}$ for all $u \in U$, $v \in V$. Because $\times$ is bilinear and $\int$ is linear, $Tv : U \to \mathbb{R}$ is linear for every $v \in V$, and T is a linear functional from V to the space of linear operators from U to $\mathbb{R}$.

(iii) If $u \ge 0$ in U and $v \ge 0$ in V, then $u \times v \ge 0$ in L^1 and $(Tv)(u) = \int u \times v \ge 0$. This shows that T is a positive linear operator from V to $U^\sim$.

(iv) If $v \ge 0$ in V and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U, then $\inf A = 0$ in L^0, because U is order-dense (352Nb). Consequently $\inf_{u \in A} u \times v = 0$ in L^0 and in L^1 (364P), and

$$\inf_{u \in A} (Tv)(u) = \inf_{u \in A} \int u \times v = 0$$

(because $\int$ is order-continuous). As A is arbitrary, Tv is order-continuous. As v is arbitrary, $T[V] \subseteq U^\times$.

(v) If $v \in V$ and $u_0 \ge 0$ in U, set $a = [\![v > 0]\!]$. Then $v^+ = v \times \chi a$. Set $A = \{u : u \in U,\, 0 \le u \le u_0 \times \chi a\}$. Because U is order-dense in L^0, $u_0 \times \chi a = \sup A$ in L^0. Because $\times$ and $\int$ are order-continuous,

$$\begin{aligned} (Tv)^+(u_0) &\ge \sup_{u \in A} (Tv)(u) = \sup_{u \in A} \int v \times u \\ &= \int v \times u_0 \times \chi a = \int v^+ \times u_0 = (Tv^+)(u_0). \end{aligned}$$

As u_0 is arbitrary, $(Tv)^+ \ge Tv^+$. But because T is a positive linear operator, we must have $Tv^+ \ge (Tv)^+$, so that $Tv^+ = (Tv)^+$. As v is arbitrary, T is a Riesz homomorphism.

(vi) Now T is injective. **P** If $v \ne 0$ in V, there is a $u > 0$ in U such that $u \le |v|$, because U is order-dense. In this case $u \times |v| > 0$ so $\int u \times |v| > 0$. Accordingly $|Tv| = T|v| \ne 0$ and $Tv \ne 0$. **Q**

(b) Putting (a-i) to (a-vi) together, we see that T is an injective Riesz homomorphism from V to $U^\times$. All this is easy. The point of the theorem is the fact that $T[V]$ is order-dense in $U^\times$.

P Take $h > 0$ in $U^\times$. Let U_1 be the solid linear subspace of L^0 generated by U. Then U is an order-dense Riesz subspace of U_1, $h : U \to \mathbb{R}$ is an order-continuous positive linear functional, and $\sup\{h(u) : u \in U,\, 0 \le u \le v\}$ is defined in $\mathbb{R}$ for every $v \ge 0$ in U_1; so we have an extension $\tilde{h}$ of h to U_1 such that $\tilde{h} \in U_1^\times$ (355F).

Set $S_1 = S(\mathfrak{A}) \cap U_1$; then S_1 is an order-dense Riesz subspace of U_1, because $S(\mathfrak{A})$ is order-dense in L^0 and U_1 is solid in L^0. Note that S_1 is the linear span of $\{\chi c : c \in I\}$, where $I = \{c : c \in \mathfrak{A},\ \chi c \in U_1\}$, and that I is an ideal in $\mathfrak{A}$.

Because $h \neq 0$, $\tilde{h} \neq 0$; there must therefore be a $u_0 \in S_1$ such that $\tilde{h}(u_0) > 0$, and a $d \in I$ such that $\tilde{h}(\chi d) > 0$. For $a \in \mathfrak{A}$, set $\nu a = \tilde{h}\chi(d \cap a)$. Because $\cap$, χ and $\tilde{h}$ are all order-continuous, so is ν, and $\nu : \mathfrak{A} \to \mathbb{R}$ is a non-negative completely additive functional.

By 365Ea, there is a $v \in L^1(\mathfrak{A}, \bar{\mu})$ such that

$$\int_a v = \nu a = \tilde{h}\chi(d \cap a)$$

for every $a \in \mathfrak{A}$; of course $v \geq 0$. We have $\int u \times v \leq \tilde{h}(u)$ whenever $u = \chi a$ for $a \in I$, and therefore for every $u \in (S_1)^+$. If $u \in U^+$, then $A = \{u' : u' \in S_1,\ 0 \leq u' \leq u\}$ is upwards-directed, $\sup A = u$ and

$$\sup_{u' \in A} \int v \times u' \leq \sup_{u' \in A} \tilde{h}(u') = \tilde{h}(u) = h(u)$$

is finite, so $v \times u = \sup_{u \in A'} v \times u'$ belongs to L^1 (365Df) and $\int v \times u \leq h(u)$. As u is arbitrary, $v \in V$ and $Tv \leq h$. At the same time, $\int_d v = \tilde{h}(\chi d) > 0$, so $Tv > 0$. As h is arbitrary, $T[V]$ is order-dense. **Q**

It follows that T is order-continuous (352Nb), as can also be easily proved by the argument of (a-iv) above.

(c) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable, that is, that $\mathfrak{A}$ is Dedekind complete. $T^{-1} : T[V] \to V$ is a Riesz space isomorphism, so certainly an order-continuous Riesz homomorphism; because V is a solid linear subspace of L^0, T^{-1} is still an injective order-continuous Riesz homomorphism when regarded as a map from $T[V]$ to L^0. Since $T[V]$ is order-dense in $U^\times$, T^{-1} has an extension to an order-continuous Riesz homomorphism $Q : U^\times \to L^0$ (368B). But $Q[U^\times] \subseteq V$. **P** Take $h \geq 0$ in $U^\times$ and $u \geq 0$ in U. Then $B = \{g : g \in T[V],\ 0 \leq g \leq h\}$ is upwards-directed and has supremum h. For $g \in B$, we know that $u \times T^{-1}g \in L^1$ and $\int u \times T^{-1}g = g(u)$, by the definition of T. But this means that

$$\sup_{g \in B} \int u \times T^{-1}g = \sup_{g \in B} g(u) = h(u) < \infty.$$

Since $\{u \times T^{-1}g : g \in B\}$ is upwards-directed, it follows that

$$u \times Qh = \sup_{g \in B} u \times Qg = \sup_{g \in B} u \times T^{-1}g \in L^1$$

by 365Df again. As u is arbitrary, $Qh \in V$. As h is arbitrary (and Q is linear), $Q[U^\times] \subseteq V$. **Q**

Also Q is injective. **P** If $h \in U^\times$ is non-zero, there is a $v \in V$ such that $0 < Tv \leq |h|$, so that

$$|Qh| = Q|h| \geq QTv = v > 0$$

and $Qh \neq 0$. **Q** Since QT is the identity on V, Q and T must be the two halves of a Riesz space isomorphism between V and $U^\times$.

369D Corollary Let U be any Riesz space such that $U^\times$ separates the points of U. Then there is a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ such that the pair $(U, U^\times)$ can be represented by a pair (V, W) of order-dense Riesz subspaces of $L^0 = L^0(\mathfrak{A})$ such that $W = \{w : w \in L^0,\ v \times w \in L^1 \text{ for every } v \in V\}$, writing L^1 for $L^1(\mathfrak{A}, \bar{\mu})$. In this case, $U^{\times\times}$ becomes represented by $\tilde{V} = \{v : v \in L^0,\ v \times w \in L^1 \text{ for every } w \in W\} \supseteq V$.

proof Put 369A and 369C together. The construction of 369A finds $(\mathfrak{A}, \bar{\mu})$ and an order-dense V which is isomorphic to U, and 369C identifies W with $V^\times$. To check that W is order-dense, take any $u > 0$ in L^0. There is a $v \in V$ such that $0 < v \leq u$. There is an $h \in (V^\times)^+$ such that $h(v) > 0$, so there is a $w \in W^+$ such that $w \times v \neq 0$, that is, $w \wedge v \neq 0$. But now $w \wedge v \in W$, because W is solid, and $0 < w \wedge v \leq u$.

Remark Thus the canonical embedding of U in $U^{\times\times}$ (356I) is represented by the embedding $V \subseteq \tilde{V}$; U, or V, is 'perfect' iff $V = \tilde{V}$.

369E Kakutani's Theorem (Kakutani 41) If U is any L-space, there is a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ such that U is isomorphic, as Banach lattice, to $L^1 = L^1(\mathfrak{A}, \bar{\mu})$.

proof U is a perfect Riesz space, and $U^\times = U^*$ has an order unit $\int$ defined by saying that $\int u = \|u\|$ for $u \geq 0$ (356P). By 369D, we can find a localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ and an identification of the pair $(U, U^\times)$, as dual Riesz spaces, with a pair (V, W) of subspaces of $L^0 = L^0(\mathfrak{A})$; and V will be $\{v : v \times w \in L^1$ for every $w \in W\}$. But W, like $U^\times$, must have an order unit; call it e. Because W is order-dense, $[\![e > 0]\!]$ must be 1 and e must have a multiplicative inverse $\frac{1}{e}$ in L^0 (364P). This means that V must be $\{v : v \times e \in L^1\}$, so that $v \mapsto v \times e$ is a Riesz space isomorphism between V and L^1, which gives a Riesz space isomorphism between U and L^1. Moreover, if we write $\| \|'$ for the norm on V corresponding to the norm of U, we have

$$\|u\| = \int |u| \text{ for } u \in U, \quad \|v\|' = \int |v| \times e = \int |v \times e| \text{ for } v \in V.$$

Thus the Riesz space isomorphism between U and L^1 is norm-preserving, and U and L^1 are isomorphic as Banach lattices.

369F The L^p spaces are leading examples for a general theory of normed subspaces of L^0, which I proceed to sketch in the rest of the section.

Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. An **extended Fatou norm** on $L^0 = L^0(\mathfrak{A})$ is a function $\tau : L^0 \to [0, \infty]$ such that

(i) $\tau(u + v) \le \tau(u) + \tau(v)$ for all u, $v \in L^0$;

(ii) $\tau(\alpha u) = |\alpha|\tau(u)$ for all $u \in L^0$, $\alpha \in \mathbb{R}$ (counting $0 \cdot \infty$ as 0, as usual);

(iii) $\tau(u) \le \tau(v)$ whenever $|u| \le |v|$ in L^0;

(iv) $\sup_{u \in A} \tau(u) = \tau(v)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with supremum v in L^0;

(v) $\tau(u) > 0$ for every non-zero $u \in L^0$;

(vi) whenever $u > 0$ in L^0 there is a $v \in L^0$ such that $0 < v \le u$ and $\tau(v) < \infty$.

369G Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a Dedekind σ-complete Boolean algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then $L^\tau = \{u : u \in L^0, \tau(u) < \infty\}$ is an order-dense solid linear subspace of L^0, and τ, restricted to L^τ, is a Fatou norm under which L^τ is a Banach lattice. If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$, then it has a supremum in L^τ; if $\mathfrak{A}$ is Dedekind complete, then L^τ has the Levi property.

proof (a) By (i), (ii) and (iii) of 369F, L^τ is a solid linear subspace of L^0; by (vi), it is order-dense. Hypotheses (i), (ii), (iii) and (v) show that τ is a Riesz norm on L^τ, while (iv) shows that it is a Fatou norm.

(b)(i) Suppose that $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$. Then $u = \sup_{n \in \mathbb{N}} u_n$ is defined in L^0. **P?** Otherwise, there is a $v > 0$ in L^0 such that $kv = \sup_{n \in \mathbb{N}} kv \wedge u_n$ for every $k \in \mathbb{N}$ (368A). By (v)-(vi) of 369F, there is a v' such that $0 < v' \le v$ and $0 < \tau(v') < \infty$. Now $kv' = \sup_{n \in \mathbb{N}} kv' \wedge u_n$ for every k, so

$$k\tau(v') = \tau(kv') = \sup_{n \in \mathbb{N}} \tau(kv' \wedge u_n) \le \sup_{n \in \mathbb{N}} \tau(u_n)$$

for every k, using 369F(iv), and $\sup_{n \in \mathbb{N}} \tau(u_n) = \infty$, contrary to hypothesis. **XQ** By 369F(iv) again, $\tau(u) = \sup_{n \in \mathbb{N}} \tau(u_n) < \infty$, so that $u \in L^\tau$ and $u = \sup_{n \in \mathbb{N}} u_n$ in L^τ.

(ii) It follows that L^τ is complete under τ. **P** Let $\langle u_n \rangle_{n \in \mathbb{N}}$ be a sequence in L^τ such that $\tau(u_{n+1} - u_n) \le 2^{-n}$ for every $n \in \mathbb{N}$. Set $v_{mn} = \sum_{i=m}^{n} |u_{i+1} - u_i|$ for $m \le n$; then $\tau(v_{mn}) \le 2^{-m+1}$ for every n, so by (i) just above $v_m = \sup_{n \in \mathbb{N}} v_{mn}$ is defined in L^τ, and $\tau(v_m) \le 2^{-m+1}$. Now $v_m = |u_{m+1} - u_m| + v_{m+1}$ for each m, so $\langle u_m - v_m \rangle_{m \in \mathbb{N}}$ is non-decreasing and $\langle u_m + v_m \rangle_{m \in \mathbb{N}}$ is non-increasing, while $u_m - v_m \le u_m \le u_m + v_m$ for every m. Accordingly $u = \sup_{m \in \mathbb{N}} u_m - v_m$ is defined in L^τ and $|u - u_m| \le v_m$ for every m. But this means that $\lim_{m \to \infty} \tau(u - u_m) \le \lim_{m \to \infty} \tau(v_m) = 0$ and $u = \lim_{m \to \infty} u_m$ in L^τ. As $\langle u_n \rangle_{n \in \mathbb{N}}$ is arbitrary, L^τ is complete. **Q**

(c) Now suppose that $\mathfrak{A}$ is Dedekind complete and $A \subseteq (L^\tau)^+$ is a non-empty upwards-directed norm-bounded set in L^τ. By the argument of (b-i) above, using the other half of 368A, $\sup A$ is defined in L^0 and belongs to L^τ. As A is arbitrary, L^τ has the Levi property.

369H Associate norms: Definition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Define $\tau' : L^0 \to [0, \infty]$ by setting

$$\tau'(u) = \sup\{\|u \times v\|_1 : v \in L^0, \tau(v) \le 1\}$$

for every $u\in L^0$; then τ' is the **associate** of τ. (The word suggests a symmetric relationship; it is justified by the next theorem.)

369I Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Then

(i) its associate τ' is also an extended Fatou norm on L^0;

(ii) τ is the associate of τ';

(iii) $\|u\times v\|_1\le\tau(u)\tau'(v)$ for all u, $v\in L^0$.

proof (a) Before embarking on the proof that τ' is an extended Fatou seminorm on L^0, I give the greater part of the argument needed to show that $\tau=\tau''$, where

$$\tau''(u)=\sup\{\|u\times w\|_1:w\in L^0,\,\tau'(w)\le 1\}$$

for every $u\in L^0$.

(i) Set

$$B=\{u:u\in L^1,\,\tau(u)\le 1\}.$$

Then B is a convex set in $L^1=L^1(\mathfrak{A},\bar\mu)$ and is closed for the norm topology of L^1. **P** Suppose that u belongs to the closure of B in L^1. Then for each $n\in\mathbb{N}$ we can choose $u_n\in B$ such that $\|u-u_n\|_1\le 2^{-n}$. Set $v_{mn}=\inf_{m\le i\le n}|u_i|$ for $m\le n$, and

$$v_m=\inf_{n\ge m}v_{mn}=\inf_{n\ge m}|u_n|\le|u|$$

for $m\in\mathbb{N}$. The sequence $\langle v_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, $\tau(v_m)\le\tau(u_m)\le 1$ for every m, and

$$\||u|-v_m\|_1\le\sup_{n\ge m}\||u|-v_{mn}\|_1\le\sum_{i=m}^\infty\||u|-|u_i|\|_1\le\sum_{i=m}^\infty\|u-u_i\|_1\to 0$$

as $m\to\infty$. So $|u|=\sup_{m\in\mathbb{N}}v_m$ in L^0,

$$\tau(u)=\tau(|u|)=\sup_{m\in\mathbb{N}}\tau(v_m)\le 1$$

and $u\in B$. **Q**

(ii) Now take any $u_0\in L^0$ such that $\tau(u_0)>1$. Then, writing $\mathfrak{A}^f$ for $\{a:\bar\mu a<\infty\}$,

$$A=\{u:u\in S(\mathfrak{A}^f),\,0\le u\le u_0\}$$

is an upwards-directed set with supremum u_0 (this is where I use the hypothesis that $(\mathfrak{A},\bar\mu)$ is semi-finite, so that $S(\mathfrak{A}^f)$ is order-dense in L^0), and $\sup_{u\in A}\tau(u)=\tau(u_0)>1$. Take $u_1\in A$ such that $\tau(u_1)>1$, that is, $u_1\notin B$. By the Hahn-Banach theorem (3A5Cc), there is a continuous linear functional $f:L^1\to\mathbb{R}$ such that $f(u_1)>1$ but $f(u)\le 1$ for every $u\in B$. Because $(L^1)^*=(L^1)^\sim$ (356Dc), $|f|$ is defined in $(L^1)^*$, and of course

$$|f|(u_1)\ge f(u_1)>1,\quad |f|(u)=\sup\{f(v):|v|\le u\}\le 1$$

whenever $u\in B$ and $u\ge 0$. Set $c=[\![u_1>0]\!]$, so that $\bar\mu c<\infty$, and define

$$\nu a=|f|(\chi(a\cap c))$$

for every $a\in\mathfrak{A}$. Then ν is a completely additive real-valued functional on $\mathfrak{A}$, so there is a $w\in L^1$ such that $\nu a=\int_a w$ for every $a\in\mathfrak{A}$ (365Ea). Because $\nu a\ge 0$ for every a, $w\ge 0$. Now

$$\int_a w=|f|(\chi a\times\chi c)$$

for every $a\in\mathfrak{A}$, so

$$\int w\times u=|f|(u\times\chi c)\le|f|(u)\le 1$$

for every $u\in S(\mathfrak{A})^+\cap B$. But if $\tau(v)\le 1$, then

$$A_v=\{u:u\in S(\mathfrak{A})^+\cap B,\,u\le|v|\}$$

is an upwards-directed set with supremum $|v|$, so that

$$\|w\times v\|_1=\sup_{u\in A_v}\int w\times u\le 1.$$

Thus $\tau'(w) \le 1$. On the other hand,

$$\|w \times u_0\|_1 \ge \int w \times u_0 \ge \int w \times u_1 = |f|(u_1) > 1,$$

so $\tau''(u_0) > 1$.

(iii) This shows that, for $u \in L^0$,

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

(c) Now I return to the proof that τ' is an extended Fatou norm. It is easy to check that it satisfies conditions (i)-(iv) of 369F; in effect, these depend only on the fact that $\| \|_1$ is an extended Fatou norm. For (v)-(vi), take $v > 0$ in L^0. Then there is a u such that $0 \le u \le v$ and $0 < \tau(u) < \infty$; set $\alpha = 1/\tau(u)$. Then $\tau(2\alpha u) > 1$, so that $\tau''(2\alpha u) > 1$ and there is a $w \in L^0$ such that $\tau'(w) \le 1$, $\|2\alpha u \times w\|_1 > 1$. But now set $v_1 = v \wedge |w|$; then

$$v \ge v_1 \ge u \wedge |w| > 0,$$

while $\tau'(v_1) < \infty$. Also $v \wedge \alpha u \ne 0$ so

$$\tau'(v) \ge \|v_1 \times \alpha u\|_1 > 0.$$

As v is arbitrary, τ' satisfies 369F(v)-(vi).

(d) Accordingly τ'' is also an extended Fatou norm. Now in (a) I showed that

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

It follows easily that $\tau(u) \le \tau''(u)$ for every u (since otherwise there would be some α such that

$$\tau''(\alpha u) = \alpha\tau''(u) < 1 < \alpha\tau(u) = \tau(\alpha u).)$$

On the other hand, we surely have

$$\tau(u) \le 1 \Longrightarrow \|u \times v\|_1 \le 1 \text{ whenever } \tau'(v) \le 1 \Longrightarrow \tau''(u) \le 1,$$

so we must also have $\tau''(u) \le \tau(u)$ for every u. Thus $\tau'' = \tau$, as claimed.

(e) Of course we have $\|u \times v\|_1 \le 1$ whenever $\tau(u) \le 1$ and $\tau'(v) \le 1$. It follows easily that $\|u \times v\|_1 \le \tau(u)\tau'(v)$ whenever u, $v \in L^0$ and both $\tau(u)$, $\tau'(v)$ are non-zero. But if one of them is zero, then $u \times v = 0$, because both τ and τ' satisfy (v) of 369F, so the result is trivial.

369J Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then

$$L^\theta = \{v : v \in L^0,\ u \times v \in L^1 \text{ for every } u \in L^\tau\}.$$

proof (a) If $v \in L^\theta$ and $u \in L^\tau$, then $\|u \times v\|_1$ is finite, by 369I(iii), so $u \times v \in L^1$.

(b) If $v \notin L^\theta$ then for every $n \in \mathbb{N}$ there is a u_n such that $\tau(u_n) \le 1$ and $\|u_n \times v\|_1 \ge 2^n$. Set $w_n = \sum_{i=0}^n 2^{-i}|u_i|$ for each n. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence and $\tau(w_n) \le 2$ for each n, so $w = \sup_{n\in\mathbb{N}} w_n$ is defined in L^τ, by 369G; now $\int w \times |v| \ge n+1$ for every n, so $w \times v \notin L^1$.

369K Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then L^θ may be identified, as normed Riesz space, with $(L^\tau)^\times \subseteq (L^\tau)^*$, and L^τ is a perfect Riesz space.

proof Putting 369J and 369C together, we have an identification between L^θ and $(L^\tau)^\times$. Now 369I tells us that τ is the associate of θ, so that we can identify L^τ with $(L^\theta)^\times$, and L^τ is perfect, as in 369D.

By the definition of θ, we have, for any $v \in L^\theta$,

$$\begin{aligned}\theta(v) &= \sup_{\tau(u)\le 1} \|u \times v\|_1 \\ &= \sup_{\tau(u)\le 1, \|w\|_\infty \le 1} \int u \times v \times w = \sup_{\tau(u)\le 1} \int u \times v,\end{aligned}$$

which is the norm of the linear functional on L^τ corresponding to v.

369L L^p I remarked above that the L^p spaces are leading examples for this theory; perhaps I should spell out the details. Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and $p\in[1,\infty]$. Extend the functional $\|\,\|_p$, as defined in 364K, 365A and 366A, by saying that $\|u\|_p=\infty$ if $u\in L^0\setminus L^p$. (For $p=1$ this is already done by the convention in 365A.) To see that $\|\,\|_p$ is now an extended Fatou norm, as defined in 369F, we note that conditions (i)-(iii) and (v) there are true just because L^p is a solid linear subspace on which $\|\,\|_p$ is a Riesz norm, (iv) is true because $\|\,\|_p$ is a Fatou norm with the Levi property (363Ba, 365C, 366D), and (vi) is true because $S(\mathfrak{A}^f)$ is included in L^p and order-dense in L^0 (364L).

As usual, set $q=p/(p-1)$ if $1<p<\infty$, ∞ if $p=1$, and 1 if $p=\infty$. Then $\|\,\|_q$ is the associate extended Fatou norm of $\|\,\|_p$. **P** By 365Jb and 366C, $\|v\|_q=\sup\{\|u\times v\|_1:\|u\|_p\le 1\}$ for every $v\in L^q$. But as L^q is order-dense in L^0,

$$\begin{aligned}\|v\|_q&=\sup_{w\in L^q,|w|\le v}\|w\|_q\\&=\sup\{\int|u|\times|w|:w\in L^q,\,w\le|v|,\,\|u\|_p\le 1\}=\sup\{\int|u|\times|v|:\|u\|_p\le 1\}\end{aligned}$$

for every $v\in L^0$. **Q**

369M Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0=L^0(\mathfrak{A},\bar{\mu})$. Then

(a) the embedding $L^\tau\subseteq L^0$ is continuous for the norm topology of L^τ and the topology of convergence in measure on L^0;

(b) $\tau:L^0\to[0,\infty]$ is lower semi-continuous, that is, all the balls $\{u:\tau(u)\le\gamma\}$ are closed for the topology of convergence in measure;

(c) if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0 which is order*-convergent to $u\in L^0$ (definition: 367A), then $\tau(u)\le\liminf_{n\to\infty}\tau(u_n)$.

proof (a) This is a special case of 367P.

(b) Set $B_\gamma=\{u:\tau(u)\le\gamma\}$. If $u\in L^0\setminus B_\gamma$, then

$$A=\{|u|\times\chi a:a\in\mathfrak{A}^f\}$$

is an upwards-directed set with supremum $|u|$, so there is an $a\in\mathfrak{A}^f$ such that $\tau(u\times\chi a)>\gamma$. **?** If u is in the closure of B_γ for the topology of convergence in measure, then for every $k\in\mathbb{N}$ there is a $v_k\in B_\gamma$ such that $\bar{\mu}(a\cap[\![|u-v_k|>2^{-k}]\!])\le 2^{-k}$ (see the formulae in 367M). Set

$$v'_k=|u|\wedge\inf_{i\ge k}|v_i|$$

for each k, and $v^*=\sup_{k\in\mathbb{N}}v'_k$. Then $\tau(v'_k)\le\tau(v_k)\le\gamma$ for each k, and $\langle v_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, so $\tau(v^*)\le\gamma$. But

$$a\cap[\![|u|-v^*>2^{-k}]\!]\subseteq a\cap\sup_{i\ge k}[\![|u-v_i|>2^{-k}]\!]$$

has measure at most $\sum_{i=k}^\infty 2^{-i}$ for each k, so $a\cap[\![|u|-v^*>0]\!]$ must be 0, that is, $|u|\times\chi a\le v^*$ and $\tau(|u|\times\chi a)\le\gamma$; contrary to the choice of a. **X** Thus u cannot belong to the closure of B_γ. As u is arbitrary, B_γ is closed.

(c) If $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to u, it converges in measure (367Na). If $\gamma>\liminf_{n\to\infty}\tau(u_n)$, there is a subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ in B_γ, and $\tau(u)\le\gamma$, by (b). As γ is arbitary, $\tau(u)\le\liminf_{n\to\infty}\tau(u_n)$.

369N I now turn to another special case which we have already had occasion to consider in other contexts.

Definition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Set

$$M^{\infty,1}_{\bar{\mu}}=M^{\infty,1}(\mathfrak{A},\bar{\mu})=L^1(\mathfrak{A},\bar{\mu})\cap L^\infty(\mathfrak{A}),$$

$$M^{1,\infty}_{\bar{\mu}}=M^{1,\infty}(\mathfrak{A},\bar{\mu})=L^1(\mathfrak{A},\bar{\mu})+L^\infty(\mathfrak{A}),$$

and

$$\|u\|_{\infty,1} = \max(\|u\|_1, \|u\|_\infty)$$

for $u \in L^0(\mathfrak{A})$.

Remark I hope that the notation I have chosen here will not completely overload your short-term memory. The idea is that in $M^{p,q}$ the symbol p is supposed to indicate the 'local' nature of the space, that is, the nature of $u \times \chi a$ where $u \in M^{p,q}$ and $\bar{\mu}a < \infty$, while q indicates the nature of $|u| \wedge \chi 1$ for $u \in M^{p,q}$. Thus $M^{1,\infty}$ is the space of u such that $u \times \chi a \in L^1$ for every $a \in \mathfrak{A}^f$, $|u| \wedge \chi 1 \in L^\infty$; in $M^{1,0}$ we demand further that $|u| \wedge \chi 1 \in M^0$ (366F); while in $M^{\infty,1}$ we ask that $|u| \wedge \chi 1 \in L^1$, $u \times \chi a \in L^\infty$ for every $a \in \mathfrak{A}^f$.

369O Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra.

(a) $\| \|_{\infty,1}$ is an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$.

(b) Its associate $\| \|_{1,\infty}$ may be defined by the formulae

$$\begin{aligned}\|u\|_{1,\infty} &= \min\{\|v\|_1 + \|w\|_\infty : v \in L^1,\ w \in L^\infty,\ v + w = u\}\\ &= \min\{\alpha + \int (|u| - \alpha\chi 1)^+ : \alpha \geq 0\}\\ &= \int_0^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!]) d\alpha\end{aligned}$$

for every $u \in L^0$, writing $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $L^\infty = L^\infty(\mathfrak{A})$.

(c)

$$\{u : u \in L^0,\ \|u\|_{1,\infty} < \infty\} = M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu}),$$

$$\{u : u \in L^0,\ \|u\|_{\infty,1} < \infty\} = M^{\infty,1} = M^{\infty,1}(\mathfrak{A}, \bar{\mu}).$$

(d) Writing $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in $M^{\infty,1}$ and $S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$.

(e) For any $p \in [1, \infty]$,

$$\|u\|_{1,\infty} \leq \|u\|_p \leq \|u\|_{\infty,1}$$

for every $u \in L^0$.

Remark By writing 'min' rather than 'inf' in the formulae of part (b) I mean to assert that the infima are attained.

proof (a) This is easy; all we need to know is that $\| \|_1$ and $\| \|_\infty$ are extended Fatou norms.

(b) We have four functionals on L^0 to look at; let me give them names:

$$\tau_1(u) = \sup\{\|u \times v\|_1 : \|v\|_{1,\infty} \leq 1\},$$

$$\tau_2(u) = \inf\{\|u'\|_1 + \|u''\|_\infty : u = u' + u''\},$$

$$\tau_3(u) = \inf_{\alpha \geq 0}(\alpha + \int (|u| - \alpha\chi 1)^+),$$

$$\tau_4(u) = \int_0^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!]) d\alpha.$$

(I write 'inf' here to avoid the question of attainment for the moment.) Now we have the following.

(i) $\tau_1(u) \leq \tau_2(u)$. **P** If $\|v\|_{1,\infty} \leq 1$ and $u = u' + u''$, then

$$\|u \times v\|_1 \leq \|u' \times v\|_1 + \|u'' \times v\|_1 \leq \|u'\|_1 \|v\|_\infty + \|u''\|_\infty \|v\|_1 \leq \|u'\|_1 + \|u''\|_\infty.$$

Taking the supremum over v and the infinum over u' and u'', $\tau_1(u) \leq \tau_2(u)$. **Q**

(ii) $\tau_2(u) \leq \tau_4(u)$. **P** If $\tau_4(u) = \infty$ this is trivial. Otherwise, take w such that $\|w\|_\infty \leq 1$ and $u = |u| \times w$. Set $\alpha_0 = \inf\{\alpha : \bar{\mu}[\![|u| > \alpha]\!] \leq 1\}$, and try

$$u' = w \times (|u| - \alpha_0\chi 1)^+, \quad u'' = w \times (|u| \wedge \alpha_0\chi 1).$$

Then $u = u' + u''$,

$$\|u'\|_1 = \int_0^\infty \bar\mu[\![|u'| > \alpha]\!]d\alpha$$
$$= \int_0^\infty \bar\mu[\![|u| > \alpha + \alpha_0]\!]d\alpha$$
$$= \int_{\alpha_0}^\infty \bar\mu[\![|u| > \alpha]\!]d\alpha = \int_{\alpha_0}^\infty \min(1, \bar\mu[\![|u| > \alpha]\!])d\alpha,$$

$$\|u''\|_\infty \le \alpha_0 = \int_0^{\alpha_0} \min(1, [\![|u| > \alpha]\!])d\alpha,$$

so

$$\tau_2(u) \le \|u'\|_1 + \|u''\|_\infty \le \tau_4(u). \text{ } \mathbf{Q}$$

(iii) $\tau_4(u) \le \tau_3(u)$. **P** For any $\alpha \ge 0$,

$$\tau_4(u) = \int_0^\alpha \min(1, \bar\mu[\![|u| > \beta]\!])d\beta + \int_\alpha^\infty \min(1, \bar\mu[\![|u| > \beta]\!])d\beta$$
$$\le \alpha + \int_0^\infty \bar\mu[\![|u| > \alpha + \beta]\!]d\beta$$
$$= \alpha + \int_0^\infty \bar\mu[\![|u| > \alpha + \beta]\!]d\beta$$
$$= \alpha + \int_0^\infty \bar\mu[\![(|u| - \alpha\chi 1)^+ > \beta]\!]d\beta = \alpha + \int (|u| - \alpha\chi 1)^+.$$

Taking the infimum over α, $\tau_4(u) \le \tau_3(u)$. **Q**

(iv) $\tau_3(u) \le \tau_1(u)$.

P$(\boldsymbol{\alpha})$ It is enough to consider the case $0 < \tau_1(u) < \infty$, because if $\tau_1(u) = 0$ then $u = 0$ and evidently $\tau_3(0) = 0$, while if $\tau_1(u) = \infty$ the required inequality is trivial. Furthermore, since $\tau_3(u) = \tau_3(|u|)$ and $\tau_1(u) = \tau_1(|u|)$, it is enough to consider the case $u \ge 0$.

$(\boldsymbol{\beta})$ Note next that if $\bar\mu a < \infty$, then $\|\frac{1}{\max(1,\bar\mu a)}\chi a\|_{\infty,1} \le 1$, so that $\int_a u \le \max(1, \bar\mu a)\tau_1(u)$.

$(\boldsymbol{\gamma})$ Set $c = [\![u > 2\tau_1(u)]\!]$. If $a \subseteq c$ and $\bar\mu a < \infty$, then

$$2\tau_1(u)\bar\mu a \le \int_a u \le \max(1, \bar\mu a)\tau_1(u),$$

so $\bar\mu a \le \frac{1}{2}$. As $(\mathfrak{A}, \bar\mu)$ is semi-finite, it follows that $\bar\mu c \le \frac{1}{2}$ (322Eb).

$(\boldsymbol{\delta})$ I may therefore write

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0,\ \bar\mu[\![u > \alpha]\!] \le 1\}.$$

Now $[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} [\![u > \alpha]\!]$, so

$$\bar\mu[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} \bar\mu[\![u > \alpha]\!] \le 1.$$

$(\boldsymbol{\epsilon})$ If $\alpha \ge \alpha_0$ then

$$(u - \alpha_0\chi 1)^+ \le (\alpha - \alpha_0)\chi[\![u > \alpha_0]\!] + (u - \alpha\chi 1)^+,$$

so

$$\alpha_0 + \int(u - \alpha_0\chi 1)^+ \le \alpha_0 + (\alpha - \alpha_0)\bar\mu[\![u > \alpha_0]\!] + \int(u - \alpha\chi 1)^+ \le \alpha + \int(u - \alpha\chi 1)^+.$$

If $0 \le \alpha < \alpha_0$ then, for every $\beta \in [0, \alpha_0 - \alpha[$,

$$(u - \alpha_0\chi 1)^+ + \beta[\![u > \alpha + \beta]\!] \le (u - \alpha\chi 1)^+,$$

while $\bar\mu[\![u > \alpha + \beta]\!] > 1$, so

$$\int(u - \alpha_0\chi 1)^+ + \beta + \alpha \le \alpha + \int(u - \alpha\chi 1)^+;$$

taking the supremum over β,

$$\alpha_0 + \int (u - \alpha_0\chi 1)^+ \le \alpha + \int (u - \alpha\chi 1)^+.$$

Thus $\alpha_0 + \int (u - \alpha_0\chi 1)^+ = \tau_3(u)$.

(ζ) If $\alpha_0 = 0$, take $v = \chi[\![u > 0]\!]$; then $\|v\|_{\infty,1} = \bar\mu[\![u > 0]\!] \le 1$ and

$$\tau_3(u) = \int u = \|u \times v\|_1 \le \tau_1(u).$$

(η) If $\alpha_0 > 0$, set $\gamma = \bar\mu[\![u > \alpha_0]\!]$. Take any $\beta \in [0, \alpha_0[$. Then $\bar\mu([\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]) > 1 - \gamma$, so there is a $b \subseteq [\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]$ such that $1 - \gamma < \bar\mu b < \infty$. Set $v = \chi[\![u > \alpha_0]\!] + \frac{1-\gamma}{\bar\mu b}\chi b$. Then $\|v\|_{\infty,1} = 1$ so

$$\tau_1(u) \ge \int u \times v \ge \int (u - \alpha_0\chi 1)^+ + \alpha_0\gamma + \beta\frac{1-\gamma}{\bar\mu b}\bar\mu b = \tau_3(u) - (1-\gamma)(\alpha_0 - \beta).$$

As β is arbitrary, $\tau_1(u) \ge \tau_3(u)$ in this case also. **Q**

(v) Thus $\tau_1(u) = \tau_2(u) = \tau_3(u) = \tau_4(u)$ for every $u \in L^0$, and I may write $\|u\|_{1,\infty}$ for their common value; as the associate of $\| \|_{\infty,1}$, $\| \|_{1,\infty}$ is an extended Fatou norm. As for the attainment of the infima, the argument of (iv-ϵ) above shows that, at least when $0 < \|u\|_{1,\infty} < \infty$, there is an α_0 such that $\alpha_0 + \int (|u| - \alpha_0)^+ = \|u\|_{1,\infty}$. This omits the cases $\|u\|_{1,\infty} \in \{0, \infty\}$; but in either of these cases we can set $\alpha_0 = 0$ to see that the infimum is attained for trivial reasons. For the other infimum, observe that the argument of (ii) produces u', u'' such that $u = u' + u''$ and $\|u'\|_1 + \|u''\|_\infty \le \tau_4(u)$.

(c) This is now obvious from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in terms of $\| \|_1$ and $\| \|_\infty$.

(d) To see that $S = S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$, we need only note that S is dense in L^∞ and $S \cap L^1$ is dense in L^1; so that given $v \in L^1$, $w \in L^\infty$ and $\epsilon > 0$ there are v', $w' \in S$ such that

$$\|(v + w) - (v' + w')\|_{1,\infty} \le \|v - v'\|_1 + \|w - w'\|_\infty \le \epsilon.$$

As for $M^{\infty,1}$, if $u \ge 0$ in $M^{\infty,1}$ and $r \in \mathbb{N}$, set $v_r = \sup_{k\in\mathbb{N}} 2^{-r}k\chi[\![u > 2^{-r}k]\!]$; then each $v_r \in S^f = S(\mathfrak{A}^f)$, $\|u - v_r\|_\infty \le 2^{-r}$, and $\langle v_r\rangle_{r\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so that $\lim_{r\to\infty} \int v_r = \int u$ and $\lim_{r\to\infty} \|u - v_r\|_{\infty,1} = 0$. Thus $(S^f)^+$ is dense in $(M^{\infty,1})^+$. As usual, it follows that $S^f = (S^f)^+ - (S^f)^+$ is dense in $M^{\infty,1} = (M^{\infty,1})^+ - (M^{\infty,1})^+$.

(e)(i) If $p = 1$ or $p = \infty$ this is immediate from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in (b). So suppose henceforth that $1 < p < \infty$.

(ii) If $\|u\|_{\infty,1} \le 1$ then $\|u\|_p \le 1$. **P** Because $\|u\|_\infty \le 1$, $|u|^p \le |u|$, so that $\int |u|^p \le \|u\|_1 \le 1$ and $\|u\|_p \le 1$. **Q**

On considering scalar multiples of u, we see at once that $\|u\|_p \le \|u\|_{\infty,1}$ for every $u \in L^0$.

(ii) Now set $q = p/(p-1)$. Then

$$\|u\|_p = \sup\{\|u \times v\|_1 : \|v\|_q \le 1\}$$

(369L)

$$\ge \sup\{\|u \times v\|_1 : \|v\|_{\infty,1} \le 1\} = \|u\|_{1,\infty}$$

because $\| \|_{1,\infty}$ is the associate of $\| \|_{\infty,1}$. This completes the proof.

369P In preparation for some ideas in §372, I go a little farther with $M^{1,0}$, as defined in 366F.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar\mu)$ is a norm-closed solid linear subspace of $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar\mu)$.

(b) The norm $\| \|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) $S(\mathfrak{A}^f)$ and $L^1(\mathfrak{A}, \bar\mu)$ are norm-dense and order-dense in $M^{1,0}$.

proof (a) Of course $M^{1,0}$, being a solid linear subspace of L^0 included in $M^{1,\infty}$, is a solid linear subspace of $M^{1,\infty}$. To see that it is norm-closed, take any point u of its closure. Then for any $\epsilon > 0$ there is a $v \in M^{1,0}$

such that $\|u-v\|_{1,\infty}\le\epsilon$; now $(|u-v|-\epsilon\chi 1)^+\in L^1$, so $[\![|u-v|>2\epsilon]\!]$ has finite measure; also $[\![|v|>\epsilon]\!]$ has finite measure, so

$$[\![|u|>3\epsilon]\!]\subseteq[\![|u-v|>2\epsilon]\!]\cup[\![|v|>\epsilon]\!]$$

(364Fa) has finite measure. As ϵ is arbitrary, $u\in M^{1,0}$; as u is arbitrary, $M^{1,0}$ is closed.

(b) Suppose that $A\subseteq M^{1,0}$ is non-empty and downwards-directed and has infimum 0. Let $\epsilon>0$. Set $B=\{(u-\epsilon\chi 1)^+ : u\in A\}$. Then $B\subseteq L^1$ (by 366Gc); B is non-empty and downwards-directed and has infimum 0. Because $\|\,\|_1$ is order-continuous (365C), $\inf_{v\in B}\|v\|_1=0$ and there is a $u\in A$ such that $\|(u-\epsilon\chi 1)^+\|_1\le\epsilon$, so that $\|u\|_{1,\infty}\le 2\epsilon$. As ϵ is arbitrary, $\inf_{u\in A}\|u\|_{1,\infty}=0$; as A is arbitrary, $\|\,\|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) By 366Gb, $S(\mathfrak{A}^f)$ is order-dense in $M^{1,0}$. Because the norm of $M^{1,0}$ is order-continuous, $S(\mathfrak{A}^f)$ is also norm-dense (354Ef). Now $S(\mathfrak{A}^f)\subseteq L^1\subseteq M^{1,0}$, so L^1 must also be norm-dense and order-dense.

369Q Corollary Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Set $M^{1,\infty}=M^{1,\infty}(\mathfrak{A},\bar\mu)$, etc.
(a) $(M^{1,\infty})^\times$ and $(M^{1,0})^\times$ can both be identified with $M^{\infty,1}$.
(b) $(M^{\infty,1})^\times$ can be identified with $M^{1,\infty}$; $M^{1,\infty}$ and $M^{\infty,1}$ are perfect Riesz spaces.

proof Everything is covered by 369O and 369K except the identification of $(M^{1,0})^\times$ with $M^{\infty,1}$. For this I return to 369C. Of course $M^{1,0}$ is order-dense in L^0, because it includes L^1, or otherwise. Setting

$$V=\{v : v\in L^0,\ u\times v\in L^1 \text{ for every } u\in M^{1,0}\},$$

369C identifies V with $(M^{1,0})^\times$. Of course $M^{\infty,1}\subseteq V$ just because $M^{1,0}\subseteq M^{1,\infty}$.

Also $V\subseteq M^{\infty,1}$. **P** Let $v\in V$. (i) **?** If $v\notin L^\infty$, then $a_n=[\![|v|>4^n]\!]\ne 0$ for every n. For each n, choose non-zero $b_n\subseteq a_n$ such that $\bar\mu b_n<\infty$ (using the fact that $(\mathfrak{A},\bar\mu)$ is semi-finite). Set $u=\sum_{n=0}^\infty 2^{-n}(\bar\mu b_n)^{-1}\chi b_n$; then $u\in L^1\subseteq M^{1,0}$, but $\int|v|\times u\ge 2^n$ for every $n\in\mathbb{N}$, so $|v|\times u\notin L^1$, which is impossible, because $v\in V$. **X** Thus $v\in L^\infty$. (ii) **?** If $v\notin L^1$, then (again because $(\mathfrak{A},\bar\mu)$ is semi-finite, so that $|v|=\sup_{a\in\mathfrak{A}^f}|v|\times\chi a$) $\sup_{a\in\mathfrak{A}^f}\int_a|v|=\infty$. For each $n\in\mathbb{N}$ choose $a_n\in\mathfrak{A}^f$ such that $\int_{a_n}|v|\ge 4^n$, and set $u=\sup_{n\in\mathbb{N}}2^{-n}\chi a_n\in M^{1,0}$; then $\int u\times|v|\ge 2^n$ for each n, so again $v\notin V$. **X** Thus $v\in L^1$. (iii) Putting these together, $v\in M^{\infty,1}$; as v is arbitrary, $V\subseteq M^{\infty,1}$. **Q**

So $M^{\infty,1}=V$ can be identified with $(M^{1,0})^\times$.

369R The detailed formulae of 369O are of course special to the norms $\|\,\|_1$, $\|\,\|_\infty$, but the general phenomenon is not.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra, and τ_1, τ_2 two extended Fatou norms on $L^0=L^0(\mathfrak{A})$ with associates τ_1', τ_2'. Then we have an extended Fatou norm τ defined by the formula

$$\tau(u)=\min\{\tau_1(v)+\tau_2(w) : v,\,w\in L^0,\ v+w=u\}$$

for every $u\in L^0$, and its associate τ' is given by the formula

$$\tau'(u)=\max(\tau_1'(u),\tau_2'(u))$$

for every $u\in L^0$. Moreover, the corresponding function spaces are

$$L^\tau=L^{\tau_1}+L^{\tau_2},\quad L^{\tau'}=L^{\tau_1'}\cap L^{\tau_2'}.$$

proof (a) For the moment, define τ by setting

$$\tau(u)=\inf\{\tau_1(v)+\tau_2(w) : v+w=u\}$$

for $u\in L^0$. It is easy to check that, for u, $u'\in L^0$ and $\alpha\in\mathbb{R}$,

$$\tau(u+u')\le\tau(u)+\tau(u'),\quad \tau(\alpha u)=|\alpha|\tau(u),\quad \tau(u)\le\tau(u') \text{ if } |u|\le|u'|.$$

(For the last, remember that in this case $u=u'\times z$ where $\|z\|_\infty\le 1$.)

(b) Take any non-empty, upwards-directed set $A\subseteq(L^0)^+$, with supremum u_0. Suppose that $\gamma=\sup_{u\in A}\tau(u)<\infty$. For $u\in A$, $n\in\mathbb{N}$ set

$$C_{un}=\{v : v\in L^0,\ 0\le v\le u_0,\ \tau_1(v)+\tau_2(u-v)^+\le\gamma+2^{-n}\}.$$

Then

(i) every C_{un} is non-empty (because $\tau(u) \le \gamma$);

(ii) every C_{un} is convex (because if v_1, $v_2 \in C_{un}$ and $\alpha \in [0,1]$ and $v = \alpha v_1 + (1-\alpha)v_2$, then

$$(u-v)^+ = (\alpha(u-v_1) + (1-\alpha)(u-v_2))^+ \le \alpha(u-v_1)^+ + (1-\alpha)(u-v_2)^+,$$

so

$$\begin{aligned}\tau_1(v) + \tau_2(u-v)^+ &\le \alpha\tau_1(v_1) + (1-\alpha)\tau_1(v_2) + \alpha\tau_2(u-v_1)^+ + (1-\alpha)\tau_2(u-v_2)^+ \\ &\le \gamma + 2^{-n});\end{aligned}$$

(iii) if u, $u' \in A$, m, $n \in \mathbb{N}$ and $u \le u'$, $m \le n$ then $C_{u'n} \subseteq C_{um}$;

(iv) every C_{un} is closed for the topology of convergence in measure. **P?** Suppose otherwise. Then we can find a v in the closure of C_{un} for the topology of convergence in measure, but such that $\tau_1(v)+\tau_2(u-v)^+ > \gamma + 2^{-n}$. In this case

$$\tau_1(v) = \sup\{\tau_1(v \times \chi a) : a \in \mathfrak{A}^f\}, \quad \tau_2(u-v)^+ = \sup\{\tau_2((u-v)^+ \times \chi a) : a \in \mathfrak{A}^f\},$$

so there is an $a \in \mathfrak{A}^f$ such that

$$\tau_1(v \times \chi a) + \tau_2((u-v)^+ \times \chi a) > \gamma + 2^{-n}.$$

Now there is a sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ in C_{un} such that $\bar\mu(a \cap [\![|v - v_k| \ge 2^{-k}]\!]) \le 2^{-k}$ for every k. Setting

$$v'_k = \inf_{i\ge k} v_i, \quad w_k = \inf_{i\ge k}(u-v_i)^+$$

we have

$$\tau_1(v'_k) + \tau_2(w_k) \le \tau_1(v_k) + \tau_2(u-v_k)^+ \le \gamma + 2^{-n}$$

for each k, and $\langle v'_k\rangle_{k\in\mathbb{N}}$, $\langle w_k\rangle_{k\in\mathbb{N}}$ are non-decreasing. So setting $v^* = \sup_{k\in\mathbb{N}} v \wedge v'_k$, $w^* = \sup_{k\in\mathbb{N}}(u-v)^+ \wedge w_k$, we get

$$\tau_1(v^*) + \tau_2(w^*) \le \gamma + 2^{-n}.$$

But $v^* \ge v \times \chi a$ and $w^* \ge (u-v)^+ \times \chi a$, so

$$\tau_1(v \times \chi a) + \tau_2((u-v)^+ \times \chi a) \le \gamma + 2^{-n},$$

contrary to the choice of a. **XQ**

Applying 367V, we find that $\bigcap_{u\in A,n\in\mathbb{N}} C_{un}$ is non-empty. If v belongs to the intersection, then

$$\tau_1(v) + \tau_2(u-v)^+ \le \gamma$$

for every $u \in A$; since $\{(u-v)^+ : u \in A\}$ is an upwards-directed set with supremum $(u_0 - v)^+$, and τ_2 is an extended Fatou norm,

$$\tau_1(v) + \tau_2(u_0 - v)^+ \le \gamma.$$

(c) This shows both that the infimum in the definition of $\tau(u)$ is always attained (since this is trivial if $\tau(u) = \infty$, and otherwise we consider $A = \{|u|\}$), and also that $\tau(\sup A) = \sup_{u\in A}\tau(u)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with a supremum. Thus τ satisfies conditions (i)-(iv) of 369F. Condition (vi) there is trivial, since (for instance) $\tau(v) \le \tau_1(v)$ for every v. As for 369F(v), suppose that $u > 0$ in L^0. Take u_1 such that $0 < u_1 \le u$ and $\tau'_1(u_1) \le 1$, u_2 such that $0 < u_2 \le u_1$ and $\tau'_2(u_2) \le 1$. In this case, if $u_2 = v + w$, we must have

$$\tau_1(v) + \tau_2(w) \ge \|v \times u_1\|_1 + \|w \times u_2\|_1 \ge \|u_2 \times u_2\|_1;$$

so that

$$\tau(u) \ge \|u_2 \times u_2\|_1 > 0.$$

Thus all the conditions of 369F are satisfied, and τ is an extended Fatou norm on L^0.

(d) The calculation of τ' is now very easy. Since surely we have $\tau \le \tau_i$ for both i, we must have $\tau' \ge \tau'_i$ for both i. On the other hand, if u, $z \in L^0$, then there are v, w such that $u = v+w$ and $\tau(u) = \tau_1(v)+\tau_2(w)$, so that

$$\|u\times z\|_1\le\|v\times z\|_1+\|w\times z\|_1\le\tau_1(v)\tau_1'(z)+\tau_2(w)\tau_2'(z)\le\tau(u)\max(\tau_1'(z),\tau_2'(z));$$

as u is arbitrary, $\tau'(z)\le\max(\tau_1'(z),\tau_2'(z))$. So $\tau'=\max(\tau_1',\tau_2')$, as claimed.

(e) Finally, it is obvious that

$$L^{\tau'}=\{z:\tau'(z)<\infty\}=\{z:\tau_1'(z)<\infty,\ \tau_2'(z)<\infty\}=L^{\tau_1'}\cap L^{\tau_2'},$$

while the fact that the infimum in the definition of τ is always attained means that $L^\tau\subseteq L^{\tau_1}+L^{\tau_2}$, so that we have equality here also.

369X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that the following are equiveridical: (i) there is a function $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra; (ii) $(L^\infty)^\times$ separates the points of $L^\infty=L^\infty(\mathfrak{A})$; (iii) for every non-zero $a\in\mathfrak{A}$ there is a completely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ such that $\nu a\ne0$; (iv) there is some order-dense Riesz subspace U of $L^0=L^0(\mathfrak{A})$ such that $U^\times$ separates the points of U; (v) for every order-dense Riesz subspace U of L^0 there is an order-dense Riesz subspace V of U such that $V^\times$ separates the points of V.

(b) Let us say that a function $\phi:\mathbb{R}\to\,]-\infty,\infty]$ is **convex** if $\phi(\alpha x+(1-\alpha)y)\ge\alpha\phi(x)+(1-\alpha)\phi(y)$ for all x, $y\in I$ and $\alpha\in[0,1]$, interpreting $0\cdot\infty$ as 0, as usual. For any convex function $\phi:\mathbb{R}\to\,]-\infty,\infty]$ which is not always infinite, set $\phi^*(y)=\sup_{x\in\mathbb{R}}xy-\phi(x)$ for every $y\in I$. (i) Show that $\phi^*:\mathbb{R}\to\,]-\infty,\infty]$ is convex and lower semi-continuous. (*Hint*: 233Xh.) (ii) Show that if ϕ is lower semi-continuous then $\phi=\phi^{**}$. (*Hint*: It is easy to check that $\phi^{**}\le\phi$. For the reverse inequality, set $I=\{x:\phi(x)<\infty\}$, and consider $x\in\operatorname{int}I$, $x\in I\setminus\operatorname{int}I$ and $x\notin I$ separately; 233Ha is useful for the first.)

>(c) For the purposes of this exercise and the next, say that a **Young's function** is a non-negative non-constant lower semi-continuous convex function $\phi:[0,\infty[\,\to[0,\infty]$ such that $\phi(0)=0$ and $\phi(x)$ is finite for some $x>0$. (**Warning!** the phrase 'Young's function' has other meanings.) (i) Show that in this case ϕ is non-decreasing and continuous on the left and ϕ^*, defined by saying that $\phi^*(y)=\sup_{x\ge0}xy-\phi(x)$ for every $y\ge0$, is again a Young's function. (ii) Show that $\phi^{**}=\phi$. Say that ϕ and ϕ^* are **complementary**. (iii) Compute ϕ^* in the cases (α) $\phi(x)=x$ (β) $\phi(x)=\max(0,x-1)$ (γ) $\phi(x)=x^2$ (δ) $\phi(x)=x^p$ where $1<p<\infty$.

>(d) Let ϕ, $\psi=\phi^*$ be complementary Young's functions in the sense of 369Xc, and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra. Set

$$B=\{u:u\in L^0,\ \textstyle\int\bar\phi(|u|)\le1\},\quad C=\{v:v\in L^0,\ \textstyle\int\bar\psi(|v|)\le1\}.$$

(For finite-valued ϕ, $\bar\phi:(L^0)^+\to L^0$ is given by 364I. Devise an appropriate convention for the case in which ϕ takes the value ∞.) (i) Show that B and C are order-closed solid convex sets, and that $\int|u\times v|\le2$ for all $u\in B$, $v\in C$. (*Hint*: for 'order-closed', use 364Xg(iv).) (ii) Show that there is a unique extended Fatou norm τ_ϕ on L^0 for which B is the unit ball. (iii) Show that if $u\in L^0\setminus B$ there is a $v\in C$ such that $\int|u\times v|>1$. (*Hint*: start with the case in which $u\in S(\mathfrak{A})^+$.) (iv) Show that $\tau_\psi\le\tau_\phi'\le2\tau_\psi$, where τ_ψ is the extended Fatou norm corresponding to ψ and τ_ϕ' is the associate of τ_ϕ, so that τ_ψ and τ_ϕ' can be interpreted as equivalent norms on the same Banach space.

(U and V are complementary **Orlicz spaces**; I will call τ_ϕ, τ_ψ **Orlicz norms**.)

(e) Let U be a Riesz space such that $U^\times$ separates the points of U, and suppose that $\|\,\|$ is a Fatou norm on U. (i) Show that there is a localizable measure algebra $(\mathfrak{A},\bar\mu)$ with an extended Fatou norm τ on $L^0(\mathfrak{A})$ such that U can be identified, as normed Riesz space, with an order-dense Riesz subspace of L^τ. (ii) Hence, or otherwise, show that $\|u\|=\sup_{f\in U^\times,\|f\|\le1}|f(u)|$ for every $u\in U$. (iii) Show that if U is Dedekind complete and has the Levi property, then U becomes identified with L^τ itself, and in particular is a Banach lattice (cf. 354Xn).

(f) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0(\mathfrak{A})$. Show that the norm of L^τ is order-continuous iff the norm topology of L^τ agrees with the topology of convergence in measure on any order-bounded subset of L^τ.

(g) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra of countable Maharam type, and τ an extended Fatou norm on $L^0(\mathfrak{A})$ such that the norm of L^τ is order-continuous. Show that L^τ is separable in its norm topology.

(h) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Show that $\|u\|_{1,\infty}=\max\{\int_a|u|:a\in\mathfrak{A},\ \bar\mu a\le 1\}$ for every $u\in L^0(\mathfrak{A})$. (*Hint*: take $a\supseteq [\![|u|>\alpha_0]\!]$ in part (b-iv) of the proof of 369O.)

(i) Let $(\mathfrak{A},\bar\mu)$ be any semi-finite measure algebra. Show that if τ_ϕ is any Orlicz norm (369Xd), then there is a $\gamma>0$ such that $\|u\|_{1,\infty}\le\gamma\tau_\phi(u)\le\gamma^2\|u\|_{\infty,1}$ for every $u\in L^0(\mathfrak{A})$, so that $M^{\infty,1}_{\bar\mu}\subseteq L^{\tau_\phi}\subseteq M^{1,\infty}_{\bar\mu}$.

(j) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Show that the subspaces $M^{1,\infty}_{\bar\mu}$, $M^{\infty,1}_{\bar\mu}$ of $L^0(\mathfrak{A})$ can be expressed as a complementary pair of Orlicz spaces, and that the norm $\|\;\|_{\infty,1}$ can be represented as an Orlicz norm, but $\|\;\|_{1,\infty}$ cannot.

>(k) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and U a Banach space. (i) Suppose that $\nu:\mathfrak{A}\to U$ is an additive function such that $\|\nu a\|\le\min(1,\bar\mu a)$ for every $a\in\mathfrak{A}$. Show that there is a unique bounded linear operator $T:M^{1,\infty}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}$. (ii) Suppose that $\nu:\mathfrak{A}^f\to U$ is an additive function such that $\|\nu a\|\le\max(1,\bar\mu a)$ for every $a\in\mathfrak{A}^f$. Show that there is a unique bounded linear operator $T:M^{\infty,1}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}^f$.

(l) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism, as in 366H, with associated maps $T:M^0_{\bar\mu}\to M^0_{\bar\nu}$, $P:M^{1,0}_{\bar\nu}\to M^{1,0}_{\bar\mu}$. Show that $\|Tu\|_{\infty,1}=\|u\|_{\infty,1}$ for every $u\in M^{\infty,1}_{\bar\mu}$, $\|Pv\|_{\infty,1}\le\|v\|_{\infty,1}$ for every $v\in M^{\infty,1}_{\bar\nu}$.

(m) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism. (i) Show that there is a unique Riesz homomorphism $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$ and $\|Tu\|_{1,\infty}=\|u\|_{1,\infty}$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable and π is order-continuous. Show that there is a unique positive linear operator $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ such that $\int_a Pv=\int_{\pi a}v$ for every $a\in\mathfrak{A}^f$, $v\in M^{1,\infty}_{\bar\nu}$, and that $\|Pv\|_\infty\le\|v\|_\infty$ for every $v\in L^\infty(\mathfrak{B})$, $\|Pv\|_{1,\infty}\le\|v\|_{1,\infty}$ for every $v\in M^{1,\infty}_{\bar\nu}$. (Compare 365P.)

(n) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $\phi:[0,\infty[\;\to[0,\infty]$ a Young's function, as in 369Xd; write τ_ϕ for the corresponding Orlicz norm on either $L^0(\mathfrak{A})$ or $L^0(\mathfrak{B})$. Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be a measure-preserving Boolean homomorphism, with associated map $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$, as in 369Xm. (i) Show that $\tau_\phi(Tu)=\tau_\phi(u)$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is localizable, π is order-continuous and $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ is the map of 369Xm(ii), then $\tau_\phi(Pv)\le\tau_\phi(v)$ for every $v\in M^{1,\infty}_{\bar\nu}$. (*Hint*: 365R.)

>(o) Let $(\mathfrak{A},\bar\mu)$ be any semi-finite measure algebra and τ_1, τ_2 two extended Fatou norms on $L^0(\mathfrak{A})$. Show that $u\mapsto\max(\tau_1(u),\tau_2(u))$ is an extended Fatou norm.

(p) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\tilde\mu)$ its localization (322P). Show that the Dedekind completion of $M^{1,\infty}(\mathfrak{A},\bar\mu)$ can be identified with $M^{1,\infty}(\widehat{\mathfrak{A}},\tilde\mu)$.

(q) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. (i) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B},\ \bar\mu b<\infty\}=1$ in $\mathfrak{A}$, we have an order-continuous positive linear operator $P_{\mathfrak{B}}:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\mu\restriction\mathfrak{B}}$ such that $\int_b P_{\mathfrak{B}}u=\int_b u$ whenever $u\in M^{1,\infty}_{\bar\mu}$, $b\in\mathfrak{B}$ and $\bar\mu b<\infty$. (ii) Show that if $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B}_0,\ \bar\mu b<\infty\}=1$ in $\mathfrak{A}$, and $\mathfrak{B}$ is the closure of $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, then $\langle P_{\mathfrak{B}_n}u\rangle_{n\in\mathbb{N}}$ is order*-convergent to $P_{\mathfrak{B}}u$ for every $u\in M^{1,\infty}_{\bar\mu}$. (Cf. 367K.)

(r) Let ϕ_1 and ϕ_2 be Young's functions (369Xc) and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra. Set $\phi(x)=\max(\phi_1(x),\phi_2(x))$ for $x\in[0,\infty[$. (i) Show that ϕ is a Young's function. (ii) Writing τ_{ϕ_1} τ_{ϕ_2}, τ_ϕ for the corresponding extended Fatou norms on $L^0(\mathfrak{A})$ (369Xd), show that $\tau_\phi\ge\max(\tau_{\phi_1},\tau_{\phi_2})\ge\frac{1}{2}\tau_\phi$, so that $L^{\tau_\phi}=L^{\tau_{\phi_1}}\cap L^{\tau_{\phi_2}}$ and $L^{\tau_{\phi^*}}=L^{\tau_{\phi_1^*}}+L^{\tau_{\phi_2^*}}$, writing ϕ^* for the Young's function complementary to ϕ. (iii) Repeat with $\psi=\phi_1+\phi_2$ in place of ϕ.

369Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $A \subseteq L^0 = L^0(\mathfrak{A})$ a countable set. Show that the solid linear subspace U of L^0 generated by A is a perfect Riesz space. (*Hint*: reduce to the case in which U is order-dense. If $A = \{u_n : n \in \mathbb{N}\}$, $w \in (L^0)^+ \setminus U$ find $v_n \in (L^0)^+$ such that $\int v_n \times u \ge 2^n \ge 4^n \int v_n \times |u_i|$ for every $i \le n$. Show that $v = \sup_{n\in\mathbb{N}} v_n$ is defined in L^0 and corresponds to a member of $U^\times$.)

(b) Let U be a Banach lattice and suppose that $p \in [1,\infty[$ is such that $\|u+v\|^p = \|u\|^p + \|v\|^p$ whenever u, $v \in U$ and $|u| \wedge |v| = 0$. Show that U is isomorphic, as Banach lattice, to $L^p_{\bar{\mu}}$ for some localizable measure algebra $(\mathfrak{A},\bar{\mu})$. (*Hint*: start by using 354Yd to show that the norm of U is order-continuous, as in 354Yj.)

(c) Let $\phi : [0,\infty[\to [0,\infty[$ be a strictly increasing Young's function (369Xc) such that $\phi(0) = 0$ and $\sup_{t>0} \phi(2t)/\phi(t)$ is finite. Show that the associated Orlicz norms τ_ϕ (369Xd) are always order-continuous on their function spaces.

(d) Let $\phi : [0,\infty[\to [0,\infty]$ be a Young's function, and suppose that the corresponding Orlicz norm on $L^0(\mathfrak{A}_L)$, where $(\mathfrak{A}_L,\bar{\mu}_L)$ is the measure algebra of Lebesgue measure on $\mathbb{R}$, is order-continuous on its function space L^{τ_ϕ}. Show that there is an $M \ge 0$ such that $\phi(2t) \le M\phi(t)$ for every $t \ge 0$.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and $\phi : [0,\infty[\to [0,\infty[$ a Young's function such that the Orlicz norm τ_ϕ is order-continuous on L^{τ_ϕ}. Show that if $\mathcal{F}$ is a filter on L^{τ_ϕ}, then $\mathcal{F} \to u \in L^{\tau_\phi}$ for the norm τ_ϕ iff (i) $\mathcal{F} \to u$ for the topology of convergence in measure (ii) $\limsup_{v\to\mathcal{F}} \tau_\phi(v) \le \tau_\phi(u)$. (Compare 245Xk.)

(f) Give an example of an extended Fatou norm τ on $L^0(\mathfrak{A}_L)$, where $(\mathfrak{A}_L,\bar{\mu}_L)$ is the measure algebra of Lebesgue measure on $[0,1]$, such that (i) τ gives rise to an order-continuous norm on its function space L^τ (ii) there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^τ, converging in measure to $u \in L^\tau$, such that $\lim_{n\to\infty} \tau(u_n) = \tau(u)$ but $\langle u_n\rangle_{n\in\mathbb{N}}$ does not converge to u for the norm on L^τ.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and τ an Orlicz norm on $L^0(\mathfrak{A})$. Show that L^τ has the Levi property, whether or not $\mathfrak{A}$ is Dedekind complete.

(h) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra. Show that $(M^{1,0}_{\bar{\mu}})^\times$ can be identified with $M^{\infty,1}_{\bar{\mu}}$. (*Hint*: show that neither $M^{1,0}$ nor $M^{\infty,1}$ is changed by moving first to the semi-finite version of $(\mathfrak{A},\bar{\mu})$, as described in 322Xa, and then to its localization.)

(i) Give an example to show that the result of 369R may fail if $(\mathfrak{A},\bar{\mu})$ is only semi-finite, not localizable.

369 Notes and comments The representation theorems 369A-369D give a very concrete form to the notion of 'perfect' Riesz space: it is just one which can be expressed as a subspace of $L^0(\mathfrak{A})$, for some localizable measure algebra $(\mathfrak{A},\bar{\mu})$, in such a way that it is its own second dual, where the duality here is between subspaces of L^0, taking $U' = \{v : u \times v \in L^1$ for every $u \in U\}$. (I see that in this expression I ought somewhere to mention that both U and U' are assumed to be order-dense in L^0.) Indeed I believe that the original perfect spaces were the 'vollkommene Räume' of G.Köthe, which were subspaces of $\mathbb{R}^\mathbb{N}$, corresponding to the measure algebra $\mathcal{P}\mathbb{N}$ with counting measure, so that U' or $U^\times$ was $\{v : u \times v \in \ell^1$ for every $u \in U\}$.

I have presented Kakutani's theorem on the representation of L-spaces as a corollary of 369A and 369C. As usual in such things, this is a reversal of the historical relationship; Kakutani's theorem was one of the results which led to the general theory. If we take the trouble to re-work the argument of 369A in this context, we find that the L-space condition '$\|u+v\| = \|u\| + \|v\|$ whenever u, $v \ge 0$' can be relaxed to '$\|u+v\| = \|u\| + \|v\|$ whenever $u \wedge v = 0$' (369Yb). The complete list of localizable measure algebras provided by Maharam's theorem (332B, 332J) now gives us a complete list of L-spaces.

Just as perfect Riesz spaces come in dual pairs, so do some of the most important Banach lattices: those with Fatou norms and the Levi property for which the order-continuous dual separates the points. (Note that the dual of any space with a Riesz norm has these properties; see 356Da.) I leave the details of representing such spaces to you (369Xe). The machinery of 369F-369K gives a solid basis for studying such pairs.

Among the extended Fatou norms of 369F the Orlicz norms (369Xd, 369Yc-369Ye) form a significant subfamily. Because they are defined in a way which is to some extent independent of the measure algebra involved, these spaces have some of the same properties as L^p spaces in relation to measure-preserving homomorphisms (369Xm-369Xn). In §§373-374 I will elaborate on these ideas. Among the Orlicz spaces, we have a largest and a smallest; these are just $M^{1,\infty} = L^1 + L^\infty$ and $M^{\infty,1} = L^1 \cap L^\infty$ (369N-369O, 369Xi, 369Xj). Of course these two are particularly important.

There is an interesting phenomenon here. It is easy to see that $\| \,\|_{\infty,1} = \max(\| \,\|_1, \| \,\|_\infty)$ is an extended Fatou norm and that the corresponding Banach lattice is $L^1 \cap L^\infty$; and that the same ideas work for any pair of extended Fatou norms (369Xo). To check that the dual of $L^1 \cap L^\infty$ is precisely the linear sum $L^\infty + L^1$ a little more is needed, and the generalization of this fact to other extended Fatou norms (369Q) seems to go quite deep. In view of our ordinary expectation that properties of these normed function spaces should be reflected in perfect Riesz spaces in general, I mention that I believe I have found an example, dependent on the continuum hypothesis, of two perfect Riesz subspaces U, V of $\mathbb{R}^{\mathbb{N}}$ such that their linear sum $U + V$ is not perfect.

Chapter 37

Linear operators between function spaces

As everywhere in functional analysis, the function spaces of measure theory cannot be properly understood without investigating linear operators between them. In this chapter I have collected a number of results which rely on, or illuminate, the measure-theoretic aspects of the theory. §371 is devoted to a fundamental property of linear operators on L-spaces, if considered abstractly, that is, of L^1-spaces, if considered in the language of Chapter 36, and to an introduction to the class $\mathcal{T}$ of operators which are norm-decreasing for both $\| \|_1$ and $\| \|_\infty$. This makes it possible to prove a version of (Birkhoff's) Ergodic Theorem for operators which need not be positive (372D). In §372 I give various forms of this theorem, for linear operators between function spaces, for measure-preserving Boolean homomorphisms between measure algebras, and for inverse-measure-preserving functions between measure spaces, with an excursion into the theory of continued fractions. In §373 I make a fuller analysis of the class $\mathcal{T}$, with a complete characterization of those u, v such that $v = Tu$ for some $T \in \mathcal{T}$. Using this we can describe 'rearrangement-invariant' function spaces and extended Fatou norms (§374). Returning to ideas left on one side in §§364 and 368, I investigate positive linear operators defined on L^0 spaces (§375). In the final section of the chapter (§376), I look at operators which can be defined in terms of kernels on product spaces.

371 The Chacon-Krengel theorem

The first topic I wish to treat is a remarkable property of L-spaces: if U and V are L-spaces, then every continuous linear operator $T : U \to V$ is order-bounded, and $\||T|\| = \|T\|$ (371D). This generalizes in various ways to other V (371B, 371C). I apply the result to a special type of operator between $M^{1,0}$ spaces which will be conspicuous in the next section (371F-371H).

371A Lemma Let U be an L-space, V a Banach lattice and $T : U \to V$ a bounded linear operator. Take $u \ge 0$ in U and set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots , u_n \in U^+,\ \sum_{i=0}^n u_i = u\} \subseteq V^+.$$

Then B is upwards-directed and $\sup_{v\in B} \|v\| \le \|T\|\|u\|$.

proof (a) Suppose that v, $v' \in B$. Then we have $u_0, \dots , u_m, u'_0, \dots , u'_n \in U^+$ such that $\sum_{i=0}^m u_i = \sum_{j=0}^n u'_j = u$, $v = \sum_{i=0}^m |Tu_i|$ and $v' = \sum_{j=0}^n |Tu'_j|$. Now there are $v_{ij} \ge 0$ in U, for $i \le m$ and $j \le n$, such that $u_i = \sum_{j=0}^n v_{ij}$ for $i \le m$ and $u'_j = \sum_{i=0}^m v_{ij}$ for $j \le n$ (352Fd). We have $u = \sum_{i=0}^m \sum_{j=0}^n v_{ij}$, so that $v'' = \sum_{i=0}^m \sum_{j=0}^n |Tv_{ij}| \in B$. But

$$v = \sum_{i=0}^m |Tu_i| = \sum_{i=0}^m |T(\sum_{j=0}^n v_{ij})| \le \sum_{i=0}^m \sum_{j=0}^m |Tv_{ij}| = v'',$$

and similarly $v' \le v''$. As v and v' are arbitrary, B is upwards-directed.

(b) The other part is easy. If $v \in B$ is expressed as $\sum_{i=0}^m |Tu_i|$ where $u_i \ge 0$, $\sum_{i=0}^m u_i = u$ then

$$\|v\| \le \sum_{i=0}^m \|Tu_i\| \le \|T\| \sum_{i=0}^m \|u_i\| = \|T\|\|u\|$$

because U is an L-space.

371B Theorem Let U be an L-space and V a Dedekind complete Banach lattice U with a Fatou norm. Then the Riesz space $\mathsf{L}^\sim(U; V) = \mathsf{L}^\times(U; V)$ is a closed linear subspace of the Banach space $\mathsf{B}(U; V)$ and is in itself a Banach lattice with a Fatou norm.

proof (a) I start by noting that $\mathsf{L}^\sim(U; V) = \mathsf{L}^\times(U; V) \subseteq \mathsf{B}(U; V)$ just because V has a Riesz norm and U is a Banach lattice with an order-continuous norm (355C, 355Kb).

(b) The first new step is to check that $\||T|\| \le \|T\|$ for any $T \in \mathsf{L}^\sim(U; V)$. **P** Start with any $u \in U^+$. Set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots , u_n \in U^+,\ \sum_{i=0}^n u_i = u\} \subseteq V^+,$$

as in 371A. If $u_0,\dots,u_n\ge 0$ are such that $\sum_{i=0}^n u_i=u$, then $|Tu_i|\le|T|u_i$ for each i, so that $\sum_{i=0}^n|Tu_i|\le\sum_{i=0}^n|T|u_i=|T|u$; thus B is bounded above by $|T|u$ and $\sup B\le|T|u$. On the other hand, if $|v|\le u$ in U, then $v^++v^-+(u-|v|)=u$, so $|Tv^+|+|Tv^-|+|T(u-|v|)|\in B$ and

$$|Tv|=|Tv^++Tv^-|\le|Tv^+|+|Tv^-|\le\sup B.$$

As v is arbitrary, $|T|u\le\sup B$ and $|T|u=\sup B$. Consequently

$$\||T|u\|\le\|\sup B\|=\sup_{w\in B}\|w\|\le\|T\|\|u\|$$

because V has a Fatou norm and B is upwards-directed.

For general $u\in U$,

$$\||T|u\|\le\||T||u|\|\le\||T|\|\||u|\|=\|T\|\|u\|.$$

This shows that $\||T|\|\le\|T\|$. **Q**

(c) Now if $|S|\le|T|$ in $\mathsf{L}^\sim(U;V)$, and $u\in U$, we must have

$$\|Su\|\le\||S||u|\|\le\||T||u|\|\le\||T|\|\||u|\|\le\|T\|\|u\|;$$

as u is arbitrary, $\|S\|\le\|T\|$. This shows that the norm of $\mathsf{L}^\sim(U;V)$, inherited from $\mathsf{B}(U;V)$, is a Riesz norm.

(d) Suppose next that $T\in\mathsf{B}(U;V)$ belongs to the norm-closure of $\mathsf{L}^\sim(U;V)$. For each $n\in\mathbb{N}$ choose $T_n\in\mathsf{L}^\sim(U;V)$ such that $\|T-T_n\|\le 2^{-n}$. Set $S_n=|T_{n+1}-T_n|\in\mathsf{L}^\sim(U;V)$ for each n. Then

$$\|S_n\|=\|T_{n+1}-T_n\|\le 3\cdot 2^{-n-1}$$

for each n, so $S=\sum_{n=0}^\infty S_n$ is defined in the Banach space $\mathsf{B}(U;V)$. But if $u\in U^+$, we surely have

$$Su=\sum_{n=0}^\infty S_nu\ge 0$$

in V. Moreover, if $u\in U^+$ and $|v|\le u$, then for any $n\in\mathbb{N}$

$$|T_{n+1}v-T_0v|=|\sum_{i=0}^n(T_{i+1}-T_i)v|\le\sum_{i=0}^n S_iu\le Su,$$

and $T_0v-Su\le T_{n+1}v\le T_0v+Su$; letting $n\to\infty$, we see that

$$-|T_0|u-Su\le T_0v-Su\le Tv\le T_0v+Su\le|T_0|u+Su.$$

So $|Tv|\le|T_0|u+Su$ whenever $|v|\le u$. As u is arbitrary, $T\in\mathsf{L}^\sim(U;V)$.

This shows that $\mathsf{L}^\sim(U;V)$ is closed in $\mathsf{B}(U;V)$ and is therefore a Banach space in its own right; putting this together with (b), we see that it is a Banach lattice.

(e) Finally, the norm of $\mathsf{L}^\sim(U;V)$ is a Fatou norm. **P** Let $A\subseteq\mathsf{L}^\sim(U;V)^+$ be a non-empty, upwards-directed set with supremum $T_0\in\mathsf{L}^\sim(U;V)$. For any $u\in U$,

$$\|T_0u\|=\||T_0u|\|\le\|T_0|u|\|=\|\sup_{T\in A}T|u|\|$$

by 355Ed. But $\{T|u|:T\in A\}$ is upwards-directed and the norm of V is a Fatou norm, so

$$\|T_0u\|\le\sup_{T\in A}\|T|u|\|\le\sup_{T\in A}\|T\|\|u\|.$$

As u is arbitrary, $\|T_0\|\le\sup_{T\in A}\|T\|$. As A is arbitrary, the norm of $\mathsf{L}^\sim(U;V)$ is Fatou. **Q**

371C Theorem Let U be an L-space and V a Dedekind complete Banach lattice with a Fatou norm and the Levi property. Then $\mathsf{B}(U;V)=\mathsf{L}^\sim(U;V)=\mathsf{L}^\times(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property. In particular, $|T|$ is defined and $\||T|\|=\|T\|$ for every $T\in\mathsf{B}(U;V)$.

proof (a) Let $T:U\to V$ be any bounded linear operator. Then $T\in\mathsf{L}^\sim(U;V)$. **P** Take any $u\ge 0$ in U. Set

$$B=\{\sum_{i=0}^n|Tu_i|:u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=u\}\subseteq V^+$$

as in 371A. Then 371A tells us that B is upwards-directed and norm-bounded. Because V has the Levi property, B is bounded above. But just as in part (b) of the proof of 371B, any upper bound of B is also an upper bound of $\{Tv:|v|\le u\}$. As u is arbitrary, $T\in\mathsf{L}^\sim(U;V)$. **Q**

(b) Accordingly $\mathsf{L}^\sim(U;V)=\mathsf{B}(U;V)$. By 371B, this is a Banach lattice with a Fatou norm, and equal to $\mathsf{L}^\times(U;V)$. To see that it also has the Levi property, let $A\subseteq\mathsf{L}^\sim(U;V)$ be any non-empty norm-bounded upwards-directed set. For $u\in U^+$, $\{Tu:T\in A\}$ is non-empty, norm-bounded and upwards-directed in V, so is bounded above in V. By 355Ed, A is bounded above in $\mathsf{L}^\sim(U;V)$.

371D Corollary Let U and V be L-spaces. Then $\mathsf{L}^\sim(U;V)=\mathsf{L}^\times(U;V)=\mathsf{B}(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property.

371E Remarks Note that both these theorems show that $\mathsf{L}^\sim(U;V)$ is a Banach lattice with properties similar to those of V whenever U is an L-space. They can therefore be applied repeatedly, to give facts about $\mathsf{L}^\sim(U_1;\mathsf{L}^\sim(U_2;V))$ where U_1, U_2 are L-spaces and V is a Banach lattice, for instance. I hope that this formula will recall some of those in the theory of bilinear maps and tensor products (see 253Xa-253Xb).

371F The class $\mathcal{T}^{(0)}$ For the sake of applications in the next section, I introduce now a class of operators of great intrinsic interest.

Definition Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be measure algebras. Recall that $M^{1,0}(\mathfrak{A},\bar\mu)$ is the space of those $u\in L^1(\mathfrak{A},\bar\mu)+L^\infty(\mathfrak{A})$ such that $\bar\mu[\![|u|>\alpha]\!]<\infty$ for every $\alpha>0$ (366F-366G, 369P). Write $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ for the set of all linear operators $T:M^{1,0}(\mathfrak{A},\bar\mu)\to M^{1,0}(\mathfrak{B},\bar\nu)$ such that $Tu\in L^1(\mathfrak{B},\bar\nu)$ and $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mathfrak{A},\bar\mu)$, $Tu\in L^\infty(\mathfrak{B})$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})\cap M^{1,0}(\mathfrak{A},\bar\mu)$.

371G Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras.

(a) $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))$. If $T_0:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$ is a linear operator of norm at most 1, and $T_0u\in L^\infty(\mathfrak{B})$ and $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1(\mathfrak{A},\bar\mu)\cap L^\infty(\mathfrak{A})$, then T_0 has a unique extension to a member of $\mathcal{T}^{(0)}$.

(b) If $T\in\mathcal{T}^{(0)}$ then T is order-bounded and $|T|$, taken in

$$\mathsf{L}^\sim(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))=\mathsf{L}^\times(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu)),$$

also belongs to $\mathcal{T}^{(0)}$.

(c) If $T\in\mathcal{T}^{(0)}$ then $\|Tu\|_{1,\infty}\le\|u\|_{1,\infty}$ for every $u\in M^{1,0}(\mathfrak{A},\bar\mu)$.

(d) If $T\in\mathcal{T}^{(0)}$, $p\in[1,\infty[$ and $w\in L^p(\mathfrak{A},\bar\mu)$ then $Tw\in L^p(\mathfrak{B},\bar\nu)$ and $\|Tw\|_p\le\|w\|_p$.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra then $ST\in\mathcal{T}^{(0)}_{\bar\mu,\bar\lambda}$ for every $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ and every $S\in\mathcal{T}^{(0)}_{\bar\nu,\bar\lambda}$.

proof I write $M^{1,0}_{\bar\mu}$, $L^p_{\bar\nu}$ for $M^{1,0}_{\bar\mu}$, $L^p(\mathfrak{B},\bar\nu)$, etc.

(a)(i) If $T\in\mathcal{T}^{(0)}$ and $u\in M^{1,0}_{\bar\mu}$ then there are $v\in L^1_{\bar\mu}$, $w\in L^\infty_{\bar\mu}$ such that $u=v+w$ and $\|v\|_1+\|w\|_\infty=\|u\|_{1,\infty}$ (369Ob); so that

$$\|Tu\|_{1,\infty}\le\|Tv\|_1+\|Tw\|_\infty\le\|v\|_1+\|w\|_\infty\le\|u\|_{1,\infty}.$$

As u is arbitrary, T is in the unit ball of $\mathsf{B}(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$.

(ii) Because the unit balls of $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$ and $\mathsf{B}(L^\infty_{\bar\mu};L^\infty_{\bar\nu})$ are convex, so is $\mathcal{T}^{(0)}$.

(iii) Now suppose that $T_0:L^1_{\bar\mu}\to L^1_{\bar\nu}$ is a linear operator of norm at most 1 such that $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1_{\bar\mu}\cap L^\infty_{\bar\mu}$. By the argument of (i), T_0 is a bounded operator for the $\|\,\|_{1,\infty}$ norms; since $L^1_{\bar\mu}$ is dense in $M^{1,0}_{\bar\mu}$ (369Pc), T_0 has a unique extension to a bounded linear operator $T:M^{1,0}_{\bar\mu}\to M^{1,0}_{\bar\nu}$. Of course $\|Tu\|_1=\|T_0u\|_1\le\|u\|_1$ for every $u\in L^1_{\bar\mu}$.

Now suppose that $u\in L^\infty_{\bar\mu}\cap M^{1,0}_{\bar\mu}$; set $\gamma=\|u\|_\infty$. Let $\epsilon>0$, and set

$$v=(u^+-\epsilon\chi1)^+-(u^--\epsilon\chi1)^+;$$

then $|v|\le|u|$ and $\|u-v\|_\infty\le\epsilon$ and $v\in L^1_{\bar\mu}\cap L^\infty_{\bar\mu}$. Accordingly

$$\|Tu-Tv\|_{1,\infty}\le\|u-v\|_{1,\infty}\le\epsilon,\quad\|Tv\|_\infty=\|T_0v\|_\infty\le\|v\|_\infty\le\gamma.$$

So if we set $w=(|Tu-Tv|-\epsilon\chi1)^+\in L^1_{\bar\nu}$, $\|w\|_1\le\epsilon$; while

$$|Tu| \le |Tv| + w + \epsilon\chi 1 \le (\gamma+\epsilon)\chi 1 + w,$$

so

$$\|(|Tu| - (\gamma+\epsilon)\chi 1)^+\|_1 \le \|w\|_1 \le \epsilon.$$

As ϵ is arbitrary, $|Tu| \le \gamma\chi 1$, that is, $\|Tu\|_\infty \le \|u\|_\infty$. As u is arbitrary, $T \in \mathcal{T}^{(0)}$.

(b) Because $M^{1,0}_{\bar\mu}$ has an order-continuous norm (369Pb), $\mathsf{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\nu}) = \mathsf{L}^\times(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\nu})$ (355Kb). Take any $T \in \mathcal{T}^{(0)}$ and consider $T_0 = T\restriction L^1_{\bar\mu} : L^1_{\bar\mu} \to L^1_{\bar\nu}$. This is an operator of norm at most 1. By 371D, T_0 is order-bounded, and $\||T_0|\| \le 1$, where $|T_0|$ is taken in $\mathsf{L}^\sim(L^1_{\bar\mu}; L^1_{\bar\mu}) = \mathsf{B}(L^1_{\bar\mu}; L^1_{\bar\nu})$. Now if $u \in L^1_{\bar\mu} \cap L^\infty_{\bar\nu}$,

$$||T_0|u| \le |T_0||u| = \sup_{|u'|\le|u|} |T_0u'| \le \|u\|_\infty\chi 1,$$

so $\||T_0|u\|_\infty \le \|u\|_\infty$. By (a), there is a unique $S \in \mathcal{T}^{(0)}$ extending $|T_0|$. Now $Su^+ \ge 0$ for every $u \in L^1_{\bar\mu}$, so $Su^+ \ge 0$ for every $u \in M^{1,0}_{\bar\mu}$ (since the function $u \mapsto (Su^+)^+ - Su^+ : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is continuous and zero on the dense set $L^1_{\bar\mu}$), that is, S is a positive operator; also $S|u| \ge |Tu|$ for every $u \in L^1_{\bar\mu}$, so $Sv \ge S|u| \ge |Tu|$ whenever u, $v \in M^{1,0}_{\bar\mu}$ and $|u| \le v$. This means that $T : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is order-bounded. Because $M^{1,0}_{\bar\nu}$ is Dedekind complete (366Ga), $|T|$ is defined in $\mathsf{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\mu})$.

If $v \ge 0$ in $L^1_{\bar\mu}$, then

$$|T|v = \sup_{|u|\le v} Tu = \sup_{|u|\le v} T_0u = |T_0|v = Sv.$$

Thus $|T|$ agrees with S on $L^1_{\bar\mu}$. Because $M^{1,0}_{\bar\mu}$ is a Banach lattice (or otherwise), $|T|$ is a bounded operator, therefore continuous (2A4Fc), so $|T| = S \in \mathcal{T}^{(0)}$, which is what we needed to know.

(c) We can express u as $v + w$ where $\|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty}$; now $w = u - v \in M^{1,0}_{\bar\mu}$, so we can speak of Tw, and

$$\|Tu\|_{1,\infty} = \|Tv + Tw\|_{1,\infty} \le \|Tv\|_1 + \|Tw\|_\infty \le \|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty},$$

as required.

(d) (This is a modification of 244M.)

(i) Suppose that T, p, w are as described, and that in addition T is positive. The function $t \mapsto |t|^p$ is convex (233Xc), so we can find families $\langle\beta_q\rangle_{q\in\mathbb{Q}}$, $\langle\gamma_q\rangle_{q\in\mathbb{Q}}$ of real numbers such that $|t|^p = \sup_{q\in\mathbb{Q}} \beta_q + \gamma_q(t-q)$ for every $t \in \mathbb{R}$ (233Hb). Then $|u|^p = \sup_{q\in\mathbb{Q}} \beta_q\chi 1 + \gamma_q(u - q\chi 1)$ for every $u \in L^0$. (The easiest way to check this is perhaps to think of L^0 as a quotient of a space of functions, as in 364D; it is also a consequence of 364Xg(iii).) We know that $|w|^p \in L^1_{\bar\mu}$, so we may speak of $T(|w|^p)$; while $w \in M^{1,0}_{\bar\mu}$ (366Ga), so we may speak of Tw.

For any $q \in \mathbb{Q}$, $0^p \ge \beta_q - q\gamma_q$, that is, $q\gamma_q - \beta_q \ge 0$, while $\gamma_q w - |w|^p \le (q\gamma_q - \beta_q)\chi 1$ and $\|(\gamma_q w - |w|^p)^+\|_\infty \le q\gamma_q - \beta_q$. Now this means that

$$\begin{aligned} T(\gamma_q w - |w|^p) &\le T(\gamma_q w - |w|^p)^+ \le \|T(\gamma_q w - |w|^p)^+\|_\infty\chi 1 \\ &\le \|(\gamma_q w - |w|^p)^+\|_\infty\chi 1 \le (q\gamma_q - \beta_q)\chi 1. \end{aligned}$$

Turning this round again,

$$\beta_q\chi 1 + \gamma_q(Tw - q\chi 1) \le T(|w|^p).$$

Taking the supremum over q, $|Tw|^p \le T(|w|^p)$, so that $\int |Tw|^p \le \int |w|^p$ (because $\|Tv\|_1 \le \|v\|_1$ for every $v \in L^1$). Thus $Tw \in L^p$ and $\|Tw\|_p \le \|w\|_p$.

(ii) For a general $T \in \mathcal{T}^{(0)}$, we have $|T| \in \mathcal{T}^{(0)}$, by (b), and $|Tw| \le |T||w|$, so that $\|Tw\|_p \le \||T||w|\|_p \le \|w\|_p$, as required.

(e) This is elementary, because

$$\|STu\|_1 \le \|Tu\|_1 \le \|u\|_1, \quad \|STv\|_\infty \le \|Tu\|_\infty \le \|u\|_\infty$$

whenever $u \in L^1_{\bar\mu}$, $v \in L^\infty_{\bar\mu} \cap M^{1,0}_{\bar\mu}$.

371H Remark In the context of 366H, $T_\pi\restriction M^{1,0}_{\bar\mu} \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, while $P_\pi \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. Thus 366H(a-iv), 366H(b-iii) are special cases of 371Gd.

371X Basic exercises >(a) Let U be an L-space, V a Banach lattice with an order-continuous norm and $T : U \to V$ a bounded linear operator. Let B be the unit ball of U. Show that $|T|[B] \subseteq \overline{T[B]}$.

(b) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}_w(U;V)$ of weakly compact linear operators from U to V (definition: 3A5Kb) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}_w(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and set $U = L^1(\mathfrak{A},\bar\mu)$. Show that $\mathsf{L}^\sim(U;U) = \mathsf{B}(U;U)$ is a Banach lattice with a Fatou norm and the Levi property. Show that its norm is order-continuous iff $\mathfrak{A}$ is finite. (*Hint*: consider operators $u \mapsto u \times \chi a$, where $a \in \mathfrak{A}$.)

>(d) Let U be a Banach lattice, and V a Dedekind complete M-space. Show that $\mathsf{L}^\sim(U;V) = \mathsf{B}(U;V)$ is a Banach lattice with a Fatou norm and the Levi property.

(e) Let U and V be Riesz spaces, of which V is Dedekind complete, and let $T \in \mathsf{L}^\sim(U;V)$. Define $T' \in \mathsf{L}^\sim(V^\sim;U^\sim)$ by writing $T'(h) = hT$ for $h \in V^\sim$. (i) Show that $|T|' \ge |T'|$ in $\mathsf{L}^\sim(V^\sim;U^\sim)$. (ii) Show that $|T|'h = |T'|h$ for every $h \in V^\times$. (*Hint*: show that if $u \in U^+$, $h \in (V^\times)^+$ then $(|T'|h)(u)$ and $h(|T|u)$ are both equal to $\sup\{\sum_{i=0}^n g_i(Tu_i) : |g_i| \le h,\ u_i \ge 0,\ \sum_{i=0}^n u_i = u\}$.)

>(f) Using 371D, but nothing about uniformly integrable sets beyond the definition (354P), show that if U and V are L-spaces, $A \subseteq U$ is uniformly integrable in U, and $T : U \to V$ is a bounded linear operator, then $T[A]$ is uniformly integrable in V.

371Y Further exercises (a) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}(U;V)$ of compact linear operators from U to V (definition: 3A5Ka) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$. (See Krengel 63.)

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, U a Banach space, and $T : L^1(\mathfrak{A},\bar\mu) \to U$ a bounded linear operator. Show that T is a compact linear operator iff $\{\frac{1}{\bar\mu a}T(\chi a) : a \in \mathfrak{A},\ 0 < \bar\mu a < \infty\}$ is relatively compact in U.

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and set $L^1 = L^1(\mathfrak{A},\bar\mu)$. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a stochastically independent sequence of elements of $\mathfrak{A}$ of measure $\frac12$, and define $T : L^1 \to \mathbb{R}^{\mathbb{N}}$ by setting $Tu(n) = \int u - 2\int_{a_n} u$ for each n. Show that $T \in \mathsf{B}(L^1;\boldsymbol{c}_0) \setminus \mathsf{L}^\sim(L^1;\boldsymbol{c}_0)$, where $\boldsymbol{c}_0$ is the Banach lattice of sequences converging to 0. (See 272Yd.)

(d) Regarding T of 371Yc as a map from L^1 to ℓ^∞, show that $|T'| \ne |T|'$ in $\mathsf{L}^\sim((\ell^\infty)^*, L^\infty(\mathfrak{A}))$.

(e) (i) In ℓ^2 define e_i by setting $e_i(i) = 1$, $e_i(j) = 0$ if $j \ne i$. Show that if $T \in \mathsf{L}^\sim(\ell^2;\ell^2)$ then $(|T|e_i|e_j) = |(Te_i|e_j)|$ for all $i, j \in \mathbb{N}$. (ii) Show that for each $n \in \mathbb{N}$ there is an orthogonal $(2^n \times 2^n)$-matrix $\mathbf{A}_n$ such that every coefficient of $\mathbf{A}_n$ has modulus $2^{-n/2}$. (*Hint*: $\mathbf{A}_{n+1} = \frac{1}{\sqrt2}\begin{pmatrix}\mathbf{A}_n & \mathbf{A}_n\\ -\mathbf{A}_n & \mathbf{A}_n\end{pmatrix}$.) (iii) Show that there is a linear isometry $S : \ell^2 \to \ell^2$ such that $|(Se_i|e_j)| = 2^{-n/2}$ if $2^n \le i, j < 2^{n+1}$. (iv) Show that $S \notin \mathsf{L}^\sim(\ell^2;\ell^2)$.

371 Notes and comments The 'Chacon-Krengel theorem', properly speaking (Chacon & Krengel 64), is 371D in the case in which $U = L^1(\mu)$, $V = L^1(\nu)$; of course no new ideas are required in the generalizations here, which I have copied from Fremlin 74a.

Anyone with a training in functional analysis will automatically seek to investigate properties of operators $T : U \to V$ in terms of properties of their adjoints $T' : V^* \to U^*$, as in 371Xe and 371Yd. When U is an L-space, then U^* is a Dedekind complete M-space, and it is easy to see that this forces T' to be order-bounded, for any Banach lattice V (371Xd). But since no important L-space is reflexive, this approach cannot reach 371B-371D without a new idea of some kind. It can however be adapted to the special case in 371Gb (Dunford & Schwartz 57, VIII.6.4).

In fact the results of 371B-371C are characteristic of L-spaces (Fremlin 74b). To see that they fail in the simplest cases in which U is not an L-space and V is not an M-space, see 371Yc-371Ye.

372 The ergodic theorem

I come now to one of the most remarkable topics in measure theory. I cannot do it justice in the space I have allowed for it here, but I can give the basic theorem (372D-372E) and a variety of the corollaries through which it is regularly used (372F-372K), together with brief notes on one of its most famous and characteristic applications (to continued fractions, 372M-372O) and on 'ergodic' and 'mixing' transformations (372P-372R). In the first half of the section (down to 372G) I express the arguments in the abstract language of measure algebras and their associated function spaces, as developed in Chapter 36; the second half, from 372H onwards, contains translations of the results into the language of measure spaces and measurable functions, the more traditional, and more readily applicable, forms.

372A Lemma Let U be a reflexive Banach space and $T : U \to U$ a bounded linear operator of norm at most 1. Then

$$V = \{u + v - Tu : u,\, v \in U,\, Tv = v\}$$

is dense in U.

proof Of course V is a linear subspace of U. **?** Suppose, if possible, that it is not dense. Then there is a non-zero $h \in U^*$ such that $h(v) = 0$ for every $v \in V$ (3A5Ad). Take $u \in U$ such that $h(u) \neq 0$. Set

$$u_n = \frac{1}{n+1} \sum_{i=0}^{n} T^i u$$

for each $n \in \mathbb{N}$, taking T^0 to be the identity operator; because

$$\|T^i u\| \le \|T^i\| \|u\| \le \|T\|^i \|u\| \le \|u\|$$

for each i, $\|u_n\| \le \|u\|$ for every n. Note also that $T^{i+1}u - T^i u \in V$ for every i, so that $h(T^{i+1}u - T^i u) = 0$; accordingly $h(T^i u) = h(u)$ for every i, and $h(u_n) = u$ for every n.

Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Because U is reflexive, $v = \lim_{n\to\mathcal{F}} u_n$ is defined in U for the weak topology on U (3A5Gc). Now $Tv = v$. **P** For each $n \in \mathbb{N}$,

$$Tu_n - u_n = \frac{1}{n+1} \sum_{i=0}^{n} (T^{i+1}u - T^i u) = \frac{1}{n+1}(T^{n+1}u - u)$$

has norm at most $\frac{2}{n+1}\|u\|$. So $\langle Tu_n - u_n \rangle_{n\in\mathbb{N}} \to 0$ for the norm topology U and therefore for the weak topology, and surely $\lim_{n\to\mathcal{F}} Tu_n - u_n = 0$. On the other hand (because T is continuous for the weak topology, 2A5If)

$$Tv = \lim_{n\to\mathcal{F}} Tu_n = \lim_{n\to\mathcal{F}}(Tu_n - u_n) + \lim_{n\to\mathcal{F}} u_n = 0 + v = v,$$

where all the limits are taken for the weak topology. **Q**

But this means that $v \in V$, while

$$h(v) = \lim_{n\to\mathcal{F}} h(u_n) = h(u) \neq 0,$$

contradicting the assumption that $h \in V^\circ$. **X**

372B Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $T : L^1 \to L^1$ a positive linear operator of norm at most 1, where $L^1 = L^1(\mathfrak{A}, \bar{\mu})$. Take any $u \in L^1$ and $m \in \mathbb{N}$, and set

$$a = [\![u > 0]\!] \cup [\![u + Tu > 0]\!] \cup [\![u + Tu + T^2u > 0]\!] \cup \ldots \cup [\![u + Tu + \ldots + T^m u > 0]\!].$$

Then $\int_a u \ge 0$.

proof Set $u_0 = u$, $u_1 = u + Tu, \dots, u_m = u + Tu + \dots + T^m u$, $v = \sup_{i \le m} u_i$, so that $a = [\![v > 0]\!]$. Consider $u + T(v^+)$. We have $T(v^+) \ge Tv \ge Tu_i$ for every $i \le m$ (because T is positive), so that $u + T(v^+) \ge u + Tu_i = u_{i+1}$ for $i < m$, and $u + T(v^+) \ge \sup_{1 \le i \le m} u_i$. Also $u + T(v^+) \ge u$ because $T(v^+) \ge 0$, so $u + T(v^+) \ge v$. Accordingly

$$\int_a u \ge \int_a v - \int_a T(v^+) = \int v^+ - \int_a T(v^+) \ge \|v^+\|_1 - \|Tv^+\|_1 \ge 0$$

because $\|T\| \le 1$.

372C Maximal Ergodic Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T : L^1 \to L^1$ a linear operator, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$, such that $\|Tu\|_1 \le \|u\|_1$ for every $u \in L^1$ and $\|Tu\|_\infty \le \|u\|_\infty$ for every $u \in L^1 \cap L^\infty(\mathfrak{A})$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n \in \mathbb{N}$. Then for any $u \in L^1$, $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in $L^0(\mathfrak{A})$, and $\alpha\bar\mu[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$.

proof (a) To begin with, suppose that T is positive and that $u \ge 0$ in L^1. Note that if $v \in L^1 \cap L^\infty$, then $\|T^i v\|_\infty \le \|v\|_\infty$ for every $i \in \mathbb{N}$, so $\|A_n v\|_\infty \le \|v\|_\infty$ for every n; in particular, $A_n(\chi a) \le \chi 1$ for every n and every a of finite measure.

For $m \in \mathbb{N}$ and $\alpha > 0$, set

$$a_{m\alpha} = [\![\sup_{i \le m} A_i u > \alpha]\!].$$

Then $\alpha\bar\mu a_{m\alpha} \le \|u\|_1$. **P** Set $a = a_{m\alpha}$, $w = u - \alpha\chi a$. Of course $\sup_{i\le m} A_i u$ belongs to L^1, so $\bar\mu a$ is finite and $w \in L^1$. For any $i \le m$,

$$A_i w = A_i u - \alpha A_i(\chi a) \ge A_i u - \alpha\chi 1,$$

so $[\![A_i w > 0]\!] \supseteq [\![A_i u > \alpha]\!]$. Accordingly $a \subseteq b$, where

$$b = \sup_{i\le m} [\![A_i w > 0]\!] = \sup_{i \le m} [\![w + Tw + \dots + T^i w > 0]\!].$$

By 372B, $\int_b w \ge 0$. But this means that

$$\alpha\bar\mu a = \alpha \int_b \chi a = \int_b u - \int_b w \le \int_b u \le \|u\|_1,$$

as claimed. **Q**

It follows that if we set $c_\alpha = \sup_{n\in\mathbb{N}} a_{n\alpha}$, $\bar\mu c_\alpha \le \alpha^{-1}\|u\|_1$ for every $\alpha > 0$ and $\inf_{\alpha>0} c_\alpha = 0$. But this is exactly the criterion in 364Mb for $u^* = \sup_{n\in\mathbb{N}} A_n u$ to be defined in L^0. And $[\![u^* > \alpha]\!] = c_\alpha$, so $\alpha\bar\mu[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$, as required.

(b) Now consider the case of general T, u. In this case T is order-bounded and $\||T|\| \le 1$, where $|T|$ is the modulus of T in $\mathsf{L}^\sim(L^1; L^1) = \mathsf{B}(L^1; L^1)$ (371D). If $w \in L^1 \cap L^\infty$, then

$$||T|w| \le |T||w| = \sup_{|w'| \le |w|} |Tw'| \le \|w\|_\infty \chi 1,$$

so $\||T|w\|_\infty \le \|w\|_\infty$. Thus $|T|$ also satisfies the conditions of the theorem. Setting $B_n = \frac{1}{n+1}\sum_{i=0}^n |T|^i$, $B_n \ge A_n$ in $\mathsf{L}^\sim(L^1; L^1)$ and $B_n|u| \ge A_n u$ for every n. But by (a), $v = \sup_{n\in\mathbb{N}} B_n|u|$ is defined in L^0 and $\alpha\bar\mu[\![v > \alpha]\!] \le \||u|\|_1 = \|u\|_1$ for every $\alpha > 0$. Consequently $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in L^0 and $u^* \le v$, so that $\alpha\bar\mu[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$.

372D We are now ready for a very general form of the Ergodic Theorem. I express it in terms of the space $M^{1,0}$ from 366F and the class $\mathcal{T}^{(0)}$ of operators from 371F. If these formulae are unfamiliar, you may like to glance at the statement of 372E before looking them up.

The Ergodic Theorem: first form Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and set $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar\mu)$, $\mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar\mu,\bar\mu} \subseteq \mathsf{B}(M^{1,0}; M^{1,0})$ as in 371F-371G. Take any $T \in \mathcal{T}^{(0)}$, and set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : M^{1,0} \to M^{1,0}$ for every n. Then for any $u \in M^{1,0}$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent (definition: 367A) and $\| \|_{1,\infty}$-convergent to a member Pu of $M^{1,0}$. The operator $P : M^{1,0} \to M^{1,0}$ is a projection onto the linear subspace $\{u : u \in M^{1,0}, Tu = u\}$, and $P \in \mathcal{T}^{(0)}$.

proof (a) It will be convenient to start with some elementary remarks. First, every A_n belongs to $\mathcal{T}^{(0)}$, by 371Ge and 371Ga. Next, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order-bounded in $L^0 = L^0(\mathfrak{A})$ for any $u \in M^{1,0}$; this is because if

$u = v + w$, where $v \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$ and $w \in L^\infty = L^\infty(\mathfrak{A})$, then $\langle A_n v\rangle_{n\in\mathbb{N}}$ and $\langle A_n(-v)\rangle_{n\in\mathbb{N}}$ are bounded above, by 372C, while $\langle A_n w\rangle_{n\in\mathbb{N}}$ is norm- and order-bounded in L^∞. Accordingly I can uninhibitedly speak of $P^*(u) = \inf_{n\in\mathbb{N}} \sup_{i\ge n} A_i u$ and $P_*(u) = \sup_{n\in\mathbb{N}} \inf_{i\ge n} A_i u$ for any $u \in M^{1,0}$, these both being defined in L^0.

(b) Write V_1 for the set of those $u \in M^{1,0}$ such that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^0; that is, $P^*(u) = P_*(u)$ (367Be). It is easy to see that V is a linear subspace of $M^{1,0}$ (use 367Ca and 367Cd). Also it is closed for $\| \|_{1,\infty}$.

P We know that $|T|$, taken in $\mathrm{L}^{\sim}(M^{1,0}; M^{1,0})$, belongs to $\mathcal{T}^{(0)}$ (371Gb); set $B_n = \frac{1}{n+1}\sum_{i=0}^{n} |T|^i$ for each i.

Suppose that $u_0 \in \overline{V}_1$. Then for any $\epsilon > 0$ there is a $u \in V_1$ such that $\|u_0 - u\|_{1,\infty} \le \epsilon^2$. Write $Pu = P^*(u) = P_*(u)$ for the order*-limit of $\langle A_n u\rangle_{n\in\mathbb{N}}$. Express $u_0 - u$ as $v + w$ where $v \in L^1$, $w \in L^\infty$ and $\|v\|_1 + \|w\|_\infty \le 2\epsilon^2$.

Set $v^* = \sup_{n\in\mathbb{N}} B_n|v|$. Then $\bar{\mu}[\![v^* > \epsilon]\!] \le 2\epsilon$, by 372C. Next, if $w^* = \sup_{n\in\mathbb{N}} B_n|w|$, we surely have $w^* \le 2\epsilon^2\chi 1$. Now

$$|A_n u_0 - A_n u| = |A_n v + A_n w| \le B_n|v| + B_n|w| \le v^* + w^*$$

for every $n \in \mathbb{N}$, that is,

$$A_n u - v^* - w^* \le A_n u_0 \le A_n u + v^* + w^*$$

for every n. Because $\langle A_n u\rangle_{n\in\mathbb{N}}$ order*-converges to Pu,

$$Pu - v^* - w^* \le P_*(u_0) \le P^*(u_0) \le Pu + v^* + w^*,$$

and $P^*(u_0) - P_*(u_0) \le 2(v^* + w^*)$. On the other hand,

$$\bar{\mu}[\![2(v^* + w^*) > 2\epsilon + 4\epsilon^2]\!] \le \bar{\mu}[\![v^* > \epsilon]\!] + \bar{\mu}[\![w^* > 2\epsilon^2]\!] = \bar{\mu}[\![v^* > \epsilon]\!] \le 2\epsilon$$

(using 364Fa for the first inequality). So

$$\bar{\mu}[\![P^*(u_0) - P_*(u_0) > 2\epsilon(1 + 2\epsilon)]\!] \le 2\epsilon.$$

Since ϵ is arbitrary, $\langle A_n u_0\rangle_{n\in\mathbb{N}}$ order*-converges to $P^*(u_0) = P_*(u_0)$, and $u_0 \in V_1$. As u_0 is arbitrary, V_1 is closed. **Q**

(c) Similarly, the set V_2 of those $u \in M^{1,0}$ for which $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent is a linear subspace of $M^{1,0}$, and it also is closed. **P** This is a standard argument. If $u_0 \in \overline{V}_2$ and $\epsilon > 0$, there is a $u \in V_2$ such that $\|u_0 - u\|_{1,\infty} \le \epsilon$. There is an $n \in \mathbb{N}$ such that $\|A_i u - A_j u\|_{1,\infty} \le \epsilon$ for all $i, j \ge n$, and now $\|A_i u_0 - A_j u_0\|_{1,\infty} \le 3\epsilon$ for all $i, j \ge n$, because every A_i has norm at most 1 in $\mathsf{B}(M^{1,0}; M^{1,0})$ (371Gc). As ϵ is arbitrary, $\langle A_i u_0\rangle_{n\in\mathbb{N}}$ is Cauchy; because $M^{1,0}$ is complete, it is convergent, and $u_0 \in V_2$. As u_0 is arbitrary, V_2 is closed. **Q**

(d) Now let V be $\{u+v-Tu : u \in M^{1,0}\cap L^\infty, v \in M^{1,0}, Tv = v\}$. Then $V \subseteq V_1\cap V_2$. **P** If $u \in M^{1,0}\cap L^\infty$, then for any $n \in \mathbb{N}$

$$A_n(u - Tu) = \tfrac{1}{n+1}(u - T^{n+1}u) \to 0$$

for $\| \|_\infty$, and therefore is both order*-convergent and convergent for $\| \|_{1,\infty}$; so $u - Tu \in V_1 \cap V_2$. On the other hand, if $Tv = v$, then of course $A_n v = v$ for every n, so again $v \in V_1 \cap V_2$. **Q**

(e) Consequently $L^2 = L^2(\mathfrak{A}, \bar{\mu}) \subseteq V_1\cap V_2$. **P** $L^2\cap V_1\cap V_2$ is a linear subspace; but also it is closed for the norm topology of L^2, because the identity map from L^2 to $M^{1,0}$ is continuous (369Oe). We know also that $T\restriction L^2$ is an operator of norm at most 1 from L^2 to itself (371Gd). Consequently $W = \{u + v - Tu : u, v \in L^2, Tv = v\}$ is dense in L^2 (372A). On the other hand, given $u \in L^2$ and $\epsilon > 0$, there is a $u' \in L^2\cap L^\infty$ such that $\|u-u'\|_2 \le \epsilon$ (take $u' = (u\wedge\gamma\chi 1)\vee(-\gamma\chi 1)$ for any γ large enough), and now $\|(u-Tu)-(u'-Tu')\|_2 \le 2\epsilon$. Thus $W' = \{u' + v - Tu' : u' \in L^2 \cap L^\infty, v \in L^2, Tv = v\}$ is dense in L^2. But $W' \subseteq V_1 \cap V_2$, by (d) above. Thus $L^2 \cap V_1 \cap V_2$ is dense in L^2, and is therefore the whole of L^2. **Q**

(f) $L^2 \supseteq S(\mathfrak{A}^f)$ is dense in $M^{1,0}$, by 369Pc, so $V_1 = V_2 = M^{1,0}$. This shows that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent and order*-convergent for every $u \in M^{1,0}$. By 367D, the limits are the same; by 367F, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent when regarded as a sequence in $M^{1,0}$. Write Pu for the common value of the limits.

(g) Of course we now have

$$\|Pu\|_\infty \le \sup_{n\in\mathbb{N}} \|A_n u\|_\infty \le \|u\|_\infty$$

for every $u \in L^\infty \cap M^{1,0}$, while

$$\|Pu\|_1 \le \liminf \|A_n u\|_1 \le \|u\|_1$$

for every $u \in L^1$, by Fatou's Lemma. So $P \in \mathcal{T}^{(0)}$. If $u \in M^{1,0}$ and $Tu = u$, then surely $Pu = u$, because $A_n u = u$ for every u. On the other hand, for any $u \in M^{1,0}$, $TPu = Pu$. **P** Because $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent to Pu,

$$\|TPu - Pu\|_{1,\infty} = \lim_{n\to\infty} \|TA_n u - A_n u\|_{1,\infty} = \lim_{n\to\infty} \frac{1}{n+1}\|T^{n+1}u - u\|_{1,\infty} = 0. \text{ **Q**}$$

Thus, writing $U = \{u : Tu = u\}$, $P[M^{1,0}] = U$ and $Pu = u$ for every $u \in U$.

372E The Ergodic Theorem: second form Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and let $T : L^1 \to L^1$, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$, be a linear operator of norm at most 1 such that $Tu \in L^\infty = L^\infty(\mathfrak{A})$ and $\|Tu\|_\infty \le \|u\|_\infty$ whenever $u \in L^1 \cap L^\infty$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : L^1 \to L^1$ for every n. Then for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to an element Pu of L^1. The operator $P : L^1 \to L^1$ is a projection of norm at most 1 onto the linear subspace $\{u : u \in L^1, Tu = u\}$.

proof By 371Ga, there is an extension of T to a member $\tilde T$ of $\mathcal{T}^{(0)}$. So 372D tells us that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to some $Pu \in L^1$ for every $u \in L^1$, and $P : L^1 \to L^1$ is a projection of norm at most 1, because P is the restriction of a projection $\tilde P \in \mathcal{T}^{(0)}$. Also we still have $TPu = Pu$ for every $u \in L^1$, and $Pu = u$ whenever $Tu = u$, so the set of values $P[L^1]$ of P must be exactly $\{u : u \in L^1, Tu = u\}$.

Remark In 372D and 372E I have used the phrase 'order*-convergent' from §367 without always being specific about the partially ordered set in which it is to be interpreted. But, as remarked in 367F, the notion is robust enough for the omission to be immaterial here. Since both $M^{1,0}$ and L^1 are solid linear subspaces of L^0, a sequence in $M^{1,0}$ is order*-convergent to a member of $M^{1,0}$ (when order*-convergence is interpreted in the partially ordered set $M^{1,0}$) iff it is order*-convergent to the same point (when convergence is interpreted in the set L^0); and the same applies to L^1 in place of $M^{1,0}$.

372F Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\pi : \mathfrak{A}^f \to \mathfrak{A}^f$ a measure-preserving ring homomorphism, where $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$. Let $T : M^{1,0} \to M^{1,0}$ be the corresponding Riesz homomorphism, where $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar\mu)$ (366H, in particular part (a-v)). Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in M^{1,0}$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_{1,\infty}$-convergent to some v such that $Tv = v$.

proof By 366H(a-iv), $T \in \mathcal{T}^{(0)}$, as defined in 371F. So the result follows at once from 372D.

372G Corollary Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $T : L^1 \to L^1$ be the corresponding Riesz homomorphism, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent. If we set $Pu = \lim_{n\to\infty} A_n u$ for each u, P is the conditional expectation operator corresponding to the closed subalgebra $\mathfrak{C} = \{a : \pi a = a\}$ of $\mathfrak{A}$.

proof (a) The first part is just a special case of 372F; the point is that because $(\mathfrak{A}, \bar\mu)$ is totally finite, $L^\infty(\mathfrak{A}) \subseteq L^1$, so $M^{1,0}(\mathfrak{A}, \bar\mu) = L^1$. Also (because $\bar\mu 1 = 1$) $\|u\|_\infty \le \|u\|_1$ for every $u \in L^\infty$, so the norm $\| \|_{1,\infty}$ is actually equal to $\| \|_1$.

(b) For the last sentence, recall that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (cf. 333R). By 372D or 372E, P is a projection operator onto the subspace $\{u : Tu = u\}$. Now $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ (365Oc), so $Tu = u$ iff $[\![u > \alpha]\!] \in \mathfrak{C}$ for every $\alpha \in \mathbb{R}$, that is, iff u belongs to the canonical image of $L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ in L^1 (365R). To identify Pu further, observe that if $u \in L^1$, $a \in \mathfrak{C}$ then

$$\int_a Tu = \int_{\pi a} Tu = \int_a u$$

(365Ob). Consequently $\int_a T^i u = \int_a u$ for every $i \in \mathbb{N}$, $\int_a A_n u = \int_a u$ for every $n \in \mathbb{N}$, and $\int_a Pu = \int_a u$ (because Pu is the limit of $\langle A_n u\rangle_{n\in\mathbb{N}}$ for $\| \|_1$). But this is enough to define Pu as the conditional expectation of u on $\mathfrak{C}$ (365R).

372H The Ergodic Theorem is most often expressed in terms of transformations of measure spaces. In the next few corollaries I will formulate such expressions. The translation is straightforward, in view of the following.

Lemma Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar{\mu})$. For $h\in\mathcal{L}^0=\mathcal{L}^0(\mu)$ write $h^\bullet$ for the corresponding member of $L^0=L^0(\mathfrak{A})$ (364Jc). Now let $\phi:X\to X$ be an inverse-measure-preserving function, $\pi:\mathfrak{A}\to\mathfrak{A}$ the corresponding sequentially order-continuous measure-preserving homomorphism defined by setting $\pi E^\bullet=(\phi^{-1}[E])^\bullet$ for $E\in\Sigma$ (324M), and $T:L^0\to L^0$ the Riesz homomorphism defined by setting $T(\chi a)=\chi(\pi a)$ for $a\in\mathfrak{A}$ (364R). Then $Th^\bullet=(h\phi)^\bullet$ for any $h\in\mathcal{L}^0$.

proof Let $\tilde{h}:X\to\mathbb{R}$ be a Σ-measurable function which is equal to h almost everywhere. Because $\phi^{-1}[E]$ is negligible for every negligible set E, $h\phi=\tilde{h}\phi$ a.e., and $\tilde{h}\phi$ is measurable, so $h\phi\in\mathcal{L}^0$. For any $\alpha\in\mathbb{R}$,

$$\begin{aligned}[\![Th^\bullet>\alpha]\!]&=[\![T\tilde{h}^\bullet>\alpha]\!]=\pi[\![\tilde{h}^\bullet>\alpha]\!]=\pi\{x:\tilde{h}(x)>\alpha\}^\bullet\\&=\{x:\tilde{h}(\phi(x))>\alpha\}^\bullet=[\![(\tilde{h}\phi)^\bullet>\alpha]\!]=[\![(h\phi)^\bullet>\alpha]\!].\end{aligned}$$

372I Corollary Let (X,Σ,μ) be a measure space and $\phi:X\to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x\in X$, and $g\phi(x)=g(x)$ for almost every x.

proof Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of (X,Σ,μ), and $\pi:\mathfrak{A}\to\mathfrak{A}$, $T:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ the homomorphisms corresponding to ϕ, as in 372H. Set $u=f^\bullet$ in $L^1(\mathfrak{A},\bar{\mu})$. Then for any $i\in\mathbb{N}$, $T^iu=(f\phi^i)^\bullet$ (372H), so setting $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$, $A_nu=g_n^\bullet$, where $g_n(x)=\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ whenever this is defined. Now we know from 372E or 372F that $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent to some v such that $Tv=v$, so $\langle g_n\rangle_{n\in\mathbb{N}}$ must be convergent almost everywhere (367G), and taking $g=\lim_{n\to\infty}g_n$ where this is defined, $g^\bullet=v$. Accordingly $(g\phi)^\bullet=Tv=v=g^\bullet$ and $g\phi=g$ a.e., as claimed.

372J The following straightforward facts will be useful in the next corollary and elsewhere.

Lemma Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar{\mu})$. Let $\phi:X\to X$ be an inverse-measure-preserving function and $\pi:\mathfrak{A}\to\mathfrak{A}$ the associated homomorphism, as in 372H. Set $\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi c=c\}$, $\mathrm{T}=\{E:E\in\Sigma,\ \phi^{-1}[E]\triangle E\text{ is negligible}\}$ and $\mathrm{T}_0=\{E:E\in\Sigma,\ \phi^{-1}[E]=E\}$. Then T and T_0 are σ-subalgebras of Σ; $\mathrm{T}_0\subseteq\mathrm{T}$, $\mathrm{T}=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{C}\}$, and $\mathfrak{C}=\{E^\bullet:E\in\mathrm{T}_0\}$.

proof It is easy to see that T and T_0 are σ-subalgebras of Σ and that $\mathrm{T}_0\subseteq\mathrm{T}=\{E:E^\bullet\in\mathfrak{C}\}$. So we have only to check that if $c\in\mathfrak{C}$ there is an $E\in\mathrm{T}_0$ such that $E^\bullet=c$. **P** Start with any $F\in\Sigma$ such that $F^\bullet=c$. Now $F\triangle\phi^{-i}[F]$ is negligible for every $i\in\mathbb{N}$, because $(\phi^{-i}[F])^\bullet=\pi^ic=c$. So if we set

$$E=\bigcup_{n\in\mathbb{N}}\bigcap_{i\ge n}\phi^{-i}[F]=\{x:\text{there is an }n\in\mathbb{N}\text{ such that }\phi^i(x)\in F\text{ for every }i\ge n\},$$

$E^\bullet=c$. On the other hand, it is easy to check that $E\in\mathrm{T}_0$. **Q**

372K Corollary Let (X,Σ,μ) be a probability space and $\phi:X\to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x\in X$; $g\phi=g$ a.e., and g is a conditional expectation of f on the σ-algebra $\mathrm{T}=\{E:E\in\Sigma,\ \phi^{-1}[E]\triangle E\text{ is negligible}\}$. If *either* f is Σ-measurable and defined everywhere in X *or* $\phi[E]$ is negligible for every negligible set E, then g is a conditional expectation of f on the σ-algebra $\mathrm{T}_0=\{E:E\in\Sigma,\ \phi^{-1}[E]=E\}$.

proof (a) We know by 372I that g is defined almost everywhere and that $g\phi=g$ a.e. In the language of the proof of 372I, $g^\bullet=v$ is the conditional expectation of $u=f^\bullet$ on the closed subalgebra

$$\mathfrak{C}=\{a:a\in\mathfrak{A},\ \pi a=a\}=\{F^\bullet:F\in\mathrm{T}\}=\{F^\bullet:F\in\mathrm{T}_0\},$$

by 372G and 372J. So v must be expressible as $h^\bullet$ where $h : X \to \mathbb{R}$ is T_0-measurable and is a conditional expectation of f on T_0 (and also on T). Since every set of measure zero belongs to T, $g = h$ $\mu\restriction\mathrm{T}$-a.e., and g is also a conditional expectation of f on T.

(b) Suppose now that f is defined everywhere and Σ-measurable. Here I come to a technical obstruction. The definition of 'conditional expectation' in 233D asks for g to be $\mu\restriction\mathrm{T}_0$-integrable, and since μ-negligible sets do not need to be $\mu\restriction\mathrm{T}_0$-negligible we have some more checking to do, to confirm that $\{x : x \in \operatorname{dom} g,\ g(x) = h(x)\}$ is $\mu\restriction\mathrm{T}_0$-conegligible as well as μ-conegligible.

(i) For $n \in \mathbb{N}$, set $\Sigma_n = \{\phi^{-n}[E] : E \in \Sigma\}$; then Σ_n is a σ-subalgebra of Σ, including T_0. Set $\Sigma_\infty = \bigcap_{n\in\mathbb{N}} \Sigma_n$, still a σ-algebra including T_0. Now any negligible set $E \in \Sigma_\infty$ is $\mu\restriction\mathrm{T}_0$-negligible. **P** For each $n \in \mathbb{N}$ choose $F_n \in \Sigma$ such that $E = \phi^{-n}[F_n]$. Because ϕ is inverse-measure-preserving, every F_n is negligible, so that

$$E^* = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j}[F_n]$$

is negligible. Of course $E = \bigcap_{m\in\mathbb{N}} \phi^{-m}[F_m]$ is included in E^*. Now

$$\phi^{-1}[E^*] = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j-1}[F_n] = \bigcap_{m\ge1} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j}[F_n] = E^*$$

because

$$\bigcup_{n\in\mathbb{N},j\ge1} \phi^{-j}[F_n] \subseteq \bigcup_{n\in\mathbb{N},j\ge0} \phi^{-j}[F_n].$$

So $E^* \in \mathrm{T}_0$ and E is included in a negligible member of T_0, which is what we needed to know. **Q**

(ii) We are assuming that f is Σ-measurable and defined everywhere, so that $g_n = \frac{1}{n+1}\sum_{i=0}^n f\circ\phi^i$ is Σ-measurable and defined everywhere. If we set $g^* = \limsup_{n\to\infty} g_n$, then $g^* : X \to [-\infty,\infty]$ is Σ_∞-measurable. **P** For any $m \in \mathbb{N}$, $f\circ\phi^i$ is Σ_m-measurable for every $i \ge m$, since $\{x : f(\phi^i(x)) > \alpha\} = \phi^{-m}[\{x : f(\phi^{i-m}(x)) > \alpha\}]$ for every α. Accordingly

$$g^* = \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=m}^n f\circ\phi^i$$

is Σ_m-measurable. As m is arbitrary, g^* is Σ_∞-measurable. **Q**

Since h is surely Σ_∞-measurable, and $h = g^*$ μ-a.e., (i) tells us that $h = g^*$ $\mu\restriction\mathrm{T}_0$-a.e. But similarly $h = \liminf_{n\to\infty} g_n$ $\mu\restriction\mathrm{T}_0$-a.e., so we must have $h = g$ $\mu\restriction\mathrm{T}_0$-a.e.; and g, like h, is a conditional expectation of f on T_0.

(c) Finally, suppose that $\phi[E]$ is negligible for every negligible set E. Then every μ-negligible set is $\mu\restriction\mathrm{T}_0$-negligible. **P** If E is μ-negligible, then $\phi[E]$, $\phi^2[E] = \phi[\phi[E]], \dots$ are all negligible, so $E^* = \bigcup_{n\in\mathbb{N}} \phi^n[E]$ is negligible, and there is a measurable negligible set $F \supseteq E^*$. Now $F_* = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \phi^{-n}[F]$ is a negligible set in T_0 including E, so E is $\mu\restriction\mathrm{T}_0$-negligible. **Q** Consequently $g = h$ $\mu\restriction\mathrm{T}_0$-a.e., and in this case also g is a conditional expectation of f on T_0.

372L Remark Parts (b)-(c) of the proof above are dominated by the technical question of the exact definition of 'conditional expectation of f on T_0', and it is natural to be impatient with such details. The kind of example I am concerned about is the following. Let $C \subseteq [0,1]$ be the Cantor set (134G), and $\phi : [0,1] \to [0,1]$ a Borel measurable function such that $\phi[C] = [0,1]$ and $\phi(x) = x$ for $x \in [0,1] \setminus C$. (For instance, we could take ϕ agreeing with the Cantor function on C (134H).) Because C is negligible, ϕ is inverse-measure-preserving for Lebesgue measure μ, and if f is any Lebesgue integrable function then $g(x) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n f(\phi^i(x))$ is defined and equal to $f(x)$ for every $x \in \operatorname{dom} f \setminus C$. But for $x \in C$ we can, by manipulating ϕ, arrange for $g(x)$ to be almost anything; and if f is undefined on C then g will also be undefined on C. On the other hand, C is not $\mu\restriction\mathrm{T}_0$-negligible, because the only member of T_0 including C is $[0,1]$. So we cannot be sure of being able to form $\int g\,d(\mu\restriction\mathrm{T}_0)$.

If instead of Lebesgue measure itself we took its restriction $\mu_{\mathcal{B}}$ to the algebra of Borel subsets of $[0,1]$, then ϕ would still be inverse-measure-preserving for $\mu_{\mathcal{B}}$, but we should now have to worry about the possibility that $f\restriction C$ was non-measurable, so that $g\restriction C$ came out to be non-measurable, even if everywhere defined, and g was not $\mu_{\mathcal{B}}\restriction\mathrm{T}_0$-virtually measurable.

In the statement of 372K I have offered two ways of being sure that the problem does not arise: check that $\phi[E]$ is negligible whenever E is negligible (so that all negligible sets are $\mu\restriction\mathrm{T}_0$-negligible), or check that f is defined everywhere and Σ-measurable. Even if these conditions are not immediately satisfied in a particular application, it may be possible to modify the problem so that they are. For instance, completing the measure will leave ϕ inverse-measure-preserving (343Ac), will not change the integrable functions but will make them all measurable (212F, 212Bc), and may enlarge T_0 enough to make a difference. If our function f is measurable (because the measure is complete, or otherwise) we can extend it to a measurable function defined everywhere (121I) and the corresponding extension of g will be $\mu\restriction\mathrm{T}_0$-integrable. Alternatively, if the difficulty seems to lie in the behaviour of ϕ rather than in the behaviour of f (as in the example above), it may help to modify ϕ on a negligible set.

372M Continued fractions A particularly delightful application of the results above is to a question which belongs as much to number theory as to analysis. It takes a bit of space to describe, but I hope you will agree with me that it is well worth knowing in itself, and that it also illuminates some of the ideas above.

(a) Set $X=[0,1]\setminus\mathbb{Q}$. For $x\in X$, set $\phi(x)=<\frac{1}{x}>$, the fractional part of $\frac{1}{x}$, and $k_1(x)=\frac{1}{x}-\phi(x)$, the integer part of $\frac{1}{x}$; then $\phi(x)\in X$ for each $x\in X$, so we may define $k_n(x)=k_1(\phi^{n-1}(x))$ for every $n\ge1$. The strictly positive integers $k_1(x)$, $k_2(x)$, $k_3(x),\ldots$ are the **continued fraction coefficients** of x. Of course $k_{n+1}(x)=k_n(\phi(x))$ for every $n\ge1$. Now define $\langle p_n(x)\rangle_{n\in\mathbb{N}}$, $\langle q_n(x)\rangle_{n\in\mathbb{N}}$ inductively by setting

$$p_0(x)=0,\quad p_1(x)=1,\quad p_n(x)=p_{n-2}(x)+k_n(x)p_{n-1}(x)\text{ for }n\ge1,$$

$$q_0(x)=1,\quad q_1(x)=k_1(x),\quad q_n(x)=q_{n-2}(x)+k_n(x)q_{n-1}(x)\text{ for }n\ge1.$$

The **continued fraction approximations** to x are the quotients $p_n(x)/q_n(x)$.

(I do not discuss rational x, because for my purposes here these are merely distracting. But if we set $k_1(0)=\infty$, $\phi(0)=0$ then the formulae above produce the conventional values for $k_n(x)$ for rational $x\in[0,1[$. As for the p_n and q_n, use the formulae above until you get to $x=p_n(x)/q_n(x)$, $\phi^n(x)=0$, $k_{n+1}(x)=\infty$, and then set $p_m(x)=p_n(x)$, $q_m(x)=q_n(x)$ for $m\ge n$.)

(b) The point is that the quotients $r_n(x)=p_n(x)/q_n(x)$ are, relatively speaking, good rational approximations to x. (See 372Yf.) We always have $r_{n+1}(x)<x<r_n(x)$ for every odd $n\ge1$ (372Xj). If $x=\pi-3$, then the first few coefficients are

$$k_1=7,\quad k_2=15,\quad k_3=1,$$

$$r_1=\frac{1}{7},\quad r_2=\frac{15}{106},\quad r_3=\frac{16}{113};$$

the first and third of these corresponding to the classical approximations $\pi\simeq\frac{22}{7}$, $\pi\simeq\frac{355}{113}$. Or if we take $x=e-2$, we get

$$k_1=1,\quad k_2=2,\quad k_3=1,\quad k_4=1,\quad k_5=4,\quad k_6=1,$$

$$r_1=1,\quad r_2=\frac{2}{3},\quad r_3=\frac{3}{4},\quad r_4=\frac{5}{7},\quad r_5=\frac{23}{32},\quad r_6=\frac{28}{39};$$

note that the obvious approximations $\frac{17}{24}$, $\frac{86}{120}$ derived from the series for e are not in fact as close as the even terms $\frac{5}{7}$, $\frac{28}{39}$ above, and involve larger numbers.

(c) Now we need a variety of miscellaneous facts about these coefficients, which I list here.

(i) For any $x\in X$, $n\ge1$ we have

$$p_{n-1}(x)q_n(x)-p_n(x)q_{n-1}(x)=(-1)^n,\quad \phi^n(x)=\frac{p_n(x)-xq_n(x)}{xq_{n-1}(x)-p_{n-1}(x)}$$

(induce on n), so

$$x=\frac{p_n(x)+p_{n-1}(x)\phi^n(x)}{q_n(x)+q_{n-1}(x)\phi^n(x)}.$$

(ii) Another easy induction on n shows that for any finite string $\mathbf{m}=(m_1,\dots,m_n)$ of strictly positive integers the set $D_{\mathbf{m}}=\{x:x\in X,\ k_i(x)=m_i$ for $1\le i\le n\}$ is an interval in X on which ϕ^n is monotonic, being strictly increasing if n is even and strictly decreasing if n is odd. (For the inductive step, note just that

$$D_{(m_1,\dots,m_n)}=[\tfrac{1}{m_1+1},\tfrac{1}{m_1}]\cap\phi^{-1}[D_{(m_2,\dots,m_n)}].)$$

(iii) We also need to know that the intervals $D_{\mathbf{m}}$ of (ii) are small; specifically, that if $\mathbf{m}=(m_1,\dots,m_n)$, the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **P** All the coefficients p_i, q_i, for $i\le n$, take constant values p_i^*, q_i^* on $D_{\mathbf{m}}$, since they are determined from the coefficients k_i which are constant on $D_{\mathbf{m}}$ by definition. Now every $x\in D_{\mathbf{m}}$ is of the form $(p_n^*+tp_{n-1}^*)/(q_n^*+tq_{n-1}^*)$ for some $t\in X$ (see (i) above) and therefore lies between p_{n-1}^*/q_{n-1}^* and p_n^*/q_n^*. But the distance between these is

$$\left|\frac{p_n^*q_{n-1}^*-p_{n-1}^*q_n^*}{q_n^*q_{n-1}^*}\right|=\frac{1}{q_n^*q_{n-1}^*},$$

by the first formula in (i). Next, noting that $q_i^*\ge q_{i-1}^*+q_{i-2}^*$ for each $i\ge 2$, we see that $q_i^*q_{i-1}^*\ge 2q_{i-1}^*q_{i-2}^*$ for $i\ge 2$, and therefore that $q_n^*q_{n-1}^*\ge 2^{n-1}$, so that the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **Q**

372N Theorem Set $X=[0,1]\setminus\mathbb{Q}$, and define $\phi:X\to X$ as in 372M. Then for every Lebesgue integrable function f on X,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))=\frac{1}{\ln 2}\int_0^1\frac{f(t)}{1+t}dt$$

for almost every $x\in X$.

proof (a) The integral just written, and the phrase 'almost every', refer of course to Lebesgue measure; but the first step is to introduce another measure, so I had better give a name μ_L to Lebesgue measure on X. Let ν be the indefinite-integral measure on X defined by saying that $\nu E=\frac{1}{\ln 2}\int_E\frac{1}{1+x}\mu_L(dx)$ whenever this is defined. The coefficient $\frac{1}{\ln 2}$ is of course chosen to make $\nu X=1$. Because $\frac{1}{1+x}>0$ for every $x\in X$, $\operatorname{dom}\nu=\operatorname{dom}\mu_L$ and ν has just the same negligible sets as μ_L (234D); I can therefore safely use the terms 'measurable set', 'almost everywhere' and 'negligible' without declaring which measure I have in mind each time.

(b) Now ϕ is inverse-measure-preserving when regarded as a function from (X,ν) to itself. **P** For each $k\ge 1$, set $I_k=\left[\frac{1}{k+1},\frac{1}{k}\right[$. On $X\cap I_k$, $\phi(x)=\frac{1}{x}-k$. Observe that $\phi\restriction I_k:X\cap I_k\to X$ is bijective and differentiable relative to its domain in the sense of §262. Consider, for any measurable $E\subseteq X$,

$$\begin{aligned}\int_E\frac{1}{(y+k)(y+k+1)}\mu_L(dy)&=\int_{I_k\cap\phi^{-1}[E]}\frac{1}{(\phi(x)+k)(\phi(x)+k+1)}|\phi'(x)|\mu_L(dx)\\&=\int_{I_k\cap\phi^{-1}[E]}\frac{x^2}{x+1}\frac{1}{x^2}\mu_L(dx)=\ln 2\cdot\nu(I_k\cap\phi^{-1}[E]),\end{aligned}$$

using 263D (or more primitive results, of course). But

$$\sum_{k=1}^\infty\frac{1}{(y+k)(y+k+1)}=\sum_{k=1}^\infty\frac{1}{y+k}-\frac{1}{y+k+1}=\frac{1}{y+1}$$

for every $y\in[0,1]$, so

$$\nu E=\frac{1}{\ln 2}\sum_{k=1}^\infty\int_E\frac{1}{(y+k)(y+k+1)}\mu_L(dy)=\sum_{k=1}^\infty\nu(I_k\cap\phi^{-1}[E])=\nu\phi^{-1}[E].$$

As E is arbitrary, ν is inverse-measure-preserving. **Q**

(c) The next thing we need to know is that if $E\subseteq X$ and $\phi^{-1}[E]=E$ then E is either negligible or conegligible. **P** I use the sets $D_{\mathbf{m}}$ of 372M(c-ii).

(i) For any string $\mathbf{m}=(m_1,\dots,m_n)$ of strictly positive integers, we have

$$x=\frac{p_n^*+p_{n-1}^*\phi^n(x)}{q_n^*+q_{n-1}^*\phi^n(x)}$$

for every $x\in D_{\mathbf{m}}$, where p_n^*, etc., are defined from $\mathbf{m}$ as in 372M(c-iii). Recall also that ϕ^n is strictly monotonic on $D_{\mathbf{m}}$. So for any interval $I\subseteq[0,1]$ (open, closed or half-open) with endpoints $\alpha<\beta$, $\phi^{-n}[I]\cap D_{\mathbf{m}}$ will be of the form $X\cap J$, where J is an interval with endpoints $(p_n^*+p_{n-1}^*\alpha)/(q_n^*+q_{n-1}^*\alpha)$, $(p_n^*+p_{n-1}^*\beta)/(q_n^*+q_{n-1}^*\beta)$ in some order. This means that we can estimate $\mu_L(\phi^{-n}[I]\cap D_{\mathbf{m}})/\mu_L D_{\mathbf{m}}$, because it is the modulus of

$$\frac{\frac{p_n^*+p_{n-1}^*\alpha}{q_n^*+q_{n-1}^*\alpha}-\frac{p_n^*+p_{n-1}^*\beta}{q_n^*+q_{n-1}^*\beta}}{\frac{p_n^*}{q_n^*}-\frac{p_n^*+p_{n-1}^*}{q_n^*+q_{n-1}^*}}=\frac{(\beta-\alpha)q_n^*(q_n^*+q_{n-1}^*)}{(q_n^*+q_{n-1}^*\alpha)(q_n^*+q_{n-1}^*\beta)}\ge\frac{(\beta-\alpha)q_n^*}{q_n^*+q_{n-1}^*}\ge\frac{1}{2}(\beta-\alpha).$$

Now look at

$$\mathcal{A}=\{E:E\subseteq[0,1]\text{ is Lebesgue measurable, }\mu_L(\phi^{-n}[E]\cap D_{\mathbf{m}})\ge\frac{1}{2}\mu_L E\cdot\mu_L D_{\mathbf{m}}\}.$$

Clearly the union of two disjoint members of $\mathcal{A}$ belongs to $\mathcal{A}$. Because $\mathcal{A}$ contains every subinterval of $[0,1]$ it includes the algebra $\mathcal{E}$ of subsets of $[0,1]$ consisting of finite unions of intervals. Next, the union of any non-decreasing sequence in $\mathcal{A}$ belongs to $\mathcal{A}$, and the intersection of a non-increasing sequence likewise. But this means that $\mathcal{A}$ must include the σ-algebra generated by $\mathcal{E}$ (136G), that is, the Borel σ-algebra. But also, if $E\in\mathcal{A}$ and $H\subseteq[0,1]$ is negligible, then

$$\mu_L(\phi^{-n}[E\triangle H]\cap D_{\mathbf{m}})=\mu_L(\phi^{-n}[E]\cap D_{\mathbf{m}})\ge\frac{1}{2}\mu_L E\cdot\mu_L D_{\mathbf{m}}=\frac{1}{2}\mu_L(E\triangle H)\cdot\mu_L D_{\mathbf{m}}$$

and $E\triangle H\in\mathcal{A}$. And this means that every Lebesgue measurable subset of $[0,1]$ belongs to $\mathcal{A}$ (134Fb).

(ii) ? Now suppose, if possible, that E is a measurable subset of X and that $\phi^{-1}[E]=E$ and E is neither negligible nor conegligible in X. Set $\gamma=\frac{1}{2}\mu_L E>0$. By Lebesgue's density theorem (223B) there is some $x\in X\setminus E$ such that $\lim_{\delta\downarrow0}\psi(\delta)=0$, where $\psi(\delta)=\frac{1}{2\delta}\mu_L(E\cap[x-\delta,x+\delta])$ for $\delta>0$. Take n so large that $\psi(\delta)<\frac{1}{2}\gamma$ whenever $0<\delta\le2^{-n+1}$, and set $m_i=k_i(x)$ for $i\le n$, so that $x\in D_{\mathbf{m}}$. Taking the least δ such that $D_{\mathbf{m}}\subseteq[x-\delta,x+\delta]$, we must have $\delta\le2^{-n+1}$, because the length of $D_{\mathbf{m}}$ is at most 2^{-n+1} (372M(c-iii)), while $\mu_L D_{\mathbf{m}}\ge\delta$, because $D_{\mathbf{m}}$ is an interval. Accordingly

$$\mu_L(E\cap D_{\mathbf{m}})\le\mu_L(E\cap[x-\delta,x+\delta])=2\delta\psi(\delta)<\gamma\delta\le\gamma\mu_L D_{\mathbf{m}}.$$

But we also have

$$\mu_L(E\cap D_{\mathbf{m}})=\mu_L(\phi^{-n}[E]\cap D_{\mathbf{m}})\ge\gamma\mu_L D_{\mathbf{m}},$$

by (i). **X**

This proves the result. **Q**

(d) The final fact we need in preparation is that $\phi[E]$ is negligible for every negligible $E\subseteq X$. This is because ϕ is differentiable relative to its domain (see 263D(ii)).

(e) Now let f be any μ_L-integrable function. Because $\frac{1}{1+x}\le1$ for every x, f is also ν-integrable (235M); consequently, using (b) above and 372K,

$$g(x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(\phi^i(x))$$

is defined for almost every $x\in X$, and is a conditional expectation of f (with respect to the measure ν) on the σ-algebra $\mathrm{T}_0=\{E:E\text{ is measurable, }\phi^{-1}[E]=E\}$. But we have just seen that T_0 consists only of negligible and conegligible sets, so g must be essentially constant; since $\int g\,d\nu=\int f d\nu$, we must have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(\phi^i(x))=\int f d\nu=\frac{1}{\ln 2}\int_0^1\frac{f(t)}{1+t}\mu_L(dt)$$

for almost every x (using 235M to calculate $\int f d\nu$).

372O Corollary For almost every $x \in [0,1] \setminus \mathbb{Q}$,

$$\lim_{n\to\infty} \frac{1}{n}\#(\{i : 1 \le i \le n,\, k_i(x) = k\}) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2))$$

for every $k \ge 1$, where $k_1(x), \dots$ are the continued fraction coefficients of x.

proof In 372N, set $f = \chi(X \cap [\frac{1}{k+1}, \frac{1}{k}])$. Then (for $i \ge 1$) $f(\phi^i(x)) = 1$ if $k_i(x) = k$ and zero otherwise. So

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\#(\{i : 1 \le i \le n,\, k_i(x) = k\}) \\
&\quad= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\phi^i(x)) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x)) \\
&\quad= \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt = \frac{1}{\ln 2}\int_{1/k+1}^{1/k} \frac{1}{1+t}dt \\
&\quad= \frac{1}{\ln 2}(\ln(1+\frac{1}{k}) - \ln(1+\frac{1}{k+1})) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2)),
\end{aligned}$$

for almost every $x \in X$.

372P Mixing and ergodic transformations This seems an appropriate moment for some brief notes on two special types of measure-preserving homomorphism or inverse-measure-preserving function.

Definitions (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) π is **ergodic** if $\{a : \pi a = a\} = \{0, 1\}$, that is, if the fixed-point subalgebra of π is trivial.

(ii) π is **mixing** (sometimes **strongly mixing**) if $\lim_{n\to\infty} \bar\mu(\pi^n a \cap b) = \bar\mu a \cdot \bar\mu b$ for all a, $b \in \mathfrak{A}$.

(b) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function.

(i) ϕ is **ergodic** (also called **metrically transitive**, **indecomposable**) if every measurable set E such that $\phi^{-1}[E] = E$ is either negligible or conegligible.

(ii) ϕ is **mixing** if $\lim_{n\to\infty} \mu(F \cap \phi^{-n}[E]) = \mu E \cdot \mu F$ for all E, $F \in \Sigma$.

(c) Remarks (i) The reason for introducing 'ergodic' homomorphisms in this section is of course 372G/372K; if π in 372G, or ϕ in 372K, is ergodic, then the limit Pu or g must be (essentially) constant, being a conditional expectation on a trivial subalgebra.

(ii) In the definition (b-i) I should perhaps emphasize that we look only at *measurable* sets E. We certainly expect that there will generally be many sets E for which $\phi^{-1}[E] = E$, since any union of orbits of ϕ will have this property.

(iii) Part (c) of the proof of 372N was devoted to showing that the function ϕ there was ergodic; see also 372Xw. For another ergodic transformation see 372Xo. For examples of mixing transformations see 333P, 372Xm, 372Xn, 372Xr, 372Xu, 372Xv.

372Q The following facts are elementary.

Proposition (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) If π is mixing, it is ergodic.

(ii) Let $T : L^0 = L^0(\mathfrak{A}) \to L^0$ be the Riesz homomorphism such that $T(\chi a) = \chi\pi a$ for every $a \in \mathfrak{A}$. Then the following are equiveridical: (α) π is ergodic; (β) the only $u \in L^0$ such that $Tu = u$ are the multiples of $\chi 1$; (γ) for every $u \in L^1 = L^1(\mathfrak{A}, \bar\mu)$, $\langle \frac{1}{n+1}\sum_{i=0}^n T^i u\rangle_{n\in\mathbb{N}}$ order*-converges to $(\int u)\chi 1$.

(b) Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \bar\mu)$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism such that $\pi E^\bullet = (\phi^{-1}[E])^\bullet$ for every $E \in \Sigma$.

(i) ϕ is mixing iff π is, and in this case it is ergodic.

(ii) The following are equiveridical: (α) ϕ is ergodic; (β) π is ergodic; (γ) for every μ-integrable real-valued function f, $\langle\frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))\rangle_{n\in\mathbb{N}}$ converges to $\int f$ for almost every $x \in X$.

proof (a)(i) If π is mixing and $\pi a = a$, then

$$0 = \bar\mu(a \setminus a) = \lim_{n\to\infty} \bar\mu(\pi^n a \setminus a) = \bar\mu a \cdot \bar\mu(1 \setminus a),$$

so one of $\bar\mu a$, $\bar\mu(1 \setminus a)$ must be zero, and $a \in \{0, 1\}$. Thus π is ergodic.

(ii)($\boldsymbol{\alpha}$)$\Rightarrow$($\boldsymbol{\beta}$) $Tu = u$ iff $\pi[\![u > \alpha]\!] = [\![u > \alpha]\!]$ for every α; if π is ergodic, this means that $[\![u > \alpha]\!] \in \{0, 1\}$ for every α, and u must be of the form $\gamma\chi 1$, where $\gamma = \inf\{\alpha : [\![u > \alpha]\!] = 0\}$.

($\boldsymbol{\beta}$)$\Rightarrow$($\boldsymbol{\gamma}$) If (β) is true and $u \in L^1$, then we know from 372G that $\langle\frac{1}{n+1}\sum_{i=0}^{n} T^i u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\,\|_1$-convergent to some v such that $Tv = v$; by (β), v is of the form $\gamma\chi 1$; and

$$\gamma = \int v = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \int T^i u = \int u.$$

($\boldsymbol{\gamma}$)$\Rightarrow$($\boldsymbol{\alpha}$) Assuming (γ), take any $a \in \mathfrak{A}$ such that $\pi a = a$, and consider $u = \chi a$. Then $T^i u = \chi a$ for every i, so

$$\chi a = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} T^i u = (\int u)\chi 1 = \bar\mu a \cdot \chi 1,$$

and a must be either 0 or 1.

(b)(i) Simply translating the definitions, we see that π is mixing iff ϕ is. In this case ϕ is ergodic, as in (a-i).

(ii)($\boldsymbol{\alpha}$)$\Rightarrow$($\boldsymbol{\beta}$) If $\pi a = a$ there is an E such that $\phi^{-1}[E] = E$ and $E^\bullet = a$, by 372L; now $\bar\mu a = \mu E \in \{0, 1\}$, so $a \in \{0, 1\}$.

($\boldsymbol{\beta}$)$\Rightarrow$($\boldsymbol{\gamma}$) Set $u = f^\bullet \in L^1$. In the language of (a), $T^i u = (f\phi^i)^\bullet$ for each i, by 372H, so that

$$(\frac{1}{n+1}\sum_{i=0}^{n} f\phi^i)^\bullet = \frac{1}{n+1}\sum_{i=0}^{n} T^i u$$

is order*-convergent to $(\int u)\chi 1 = (\int f)\chi 1$, and $\frac{1}{n+1}\sum_{i=0}^{n} f\phi^i \to \int f$ a.e.

($\boldsymbol{\gamma}$)$\Rightarrow$($\boldsymbol{\alpha}$) If $\phi^{-1}[E] = E$ then, applying (γ) to $f = \chi E$, we see that $\chi E = \mu E \cdot \chi X$ a.e., so that E is either negligible or conegligible.

372R There is a useful sufficient condition for a homomorphism or function to be mixing.

Proposition (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. If $\bigcap_{n\in\mathbb{N}} \pi^n[\mathfrak{A}] = \{0, 1\}$, then π is mixing.

(b) Let (X, Σ, μ) be a probability space, and $\phi : X \to X$ an inverse-measure-preserving function. Set

$$\mathrm{T} = \{E : \text{for every } n \in \mathbb{N} \text{ there is an } F \in \Sigma \text{ such that } E = \phi^{-n}[F]\}.$$

If every element of T is either negligible or conegligible, ϕ is mixing.

proof (a) Let $T : L^0 = L^0(\mathfrak{A}) \to L^0$ be the Riesz homomorphism associated with π. Take any a, $b \in \mathfrak{A}$ and any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Then $\langle T^n(\chi a)\rangle_{n\in\mathbb{N}}$ is a bounded sequence in the reflexive space $L^2_{\bar\mu} = L^2(\mathfrak{A}, \bar\mu)$, so $v = \lim_{n\to\mathcal{F}} T^n(\chi a)$ is defined for the weak topology of $L^2_{\bar\mu}$. Now for each $n \in \mathbb{N}$ set $\mathfrak{B}_n = \pi^n[\mathfrak{A}]$. This is a closed subalgebra of $\mathfrak{A}$ (314F(a-i)), and contains $\pi^i a$ for every $i \ge n$. So if we identify $L^2(\mathfrak{B}_n, \bar\mu\restriction\mathfrak{B}_n)$ with the corresponding subspace of $L^2_{\bar\mu}$ (366I), it contains $T^i(\chi a)$ for every $i \ge n$; but also it is norm-closed, therefore weakly closed (3A5Ee), so contains v. This means that $[\![v > \alpha]\!]$ must belong to $\mathfrak{B}_n$ for every α and every n. But in this case $[\![v > \alpha]\!] \in \bigcap_{n\in\mathbb{N}} \mathfrak{B}_n = \{0, 1\}$ for every α, and v is of the form $\gamma\chi 1$. Also

$$\gamma = \int v = \lim_{n\to\mathcal{F}} \int T^n(\chi a) = \bar\mu a.$$

So

$$\lim_{n\to\mathcal{F}} \bar\mu(\pi^n a \cap b) = \lim_{n\to\mathcal{F}} \int T^n(\chi a) \times \chi b = \int v \times \chi b = \gamma\bar\mu b = \bar\mu a \cdot \bar\mu b.$$

But this is true of every non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, so we must have $\lim_{n\to\infty}\bar\mu(\pi^n a\cap b)=\bar\mu a\cdot\bar\mu b$ (3A3Lc). As a and b are arbitrary, π is mixing.

(b) The point is that if $a\in\bigcap_{n\in\mathbb{N}}\pi^n[\mathfrak{A}]$, there is an $E\in\mathrm{T}$ such that $E^\bullet=a$. **P** For each $n\in\mathbb{N}$ there is an $a_n\in\mathfrak{A}$ such that $\pi^n a_n=a$; say $a_n=F_n^\bullet$ where $F_n\in\Sigma$. Then $\phi^{-n}[F_n]^\bullet=a$. Set

$$E=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\phi^{-n}[F_n],\quad E_k=\bigcup_{m\ge k}\bigcap_{n\ge m}\phi^{-(n-k)}[F_n]$$

for each k; then $E^\bullet=a$ and

$$\phi^{-k}[E_k]=\bigcup_{m\ge k}\bigcap_{n\ge m}\phi^{-n}[F_n]=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\phi^{-n}[F_n]=E$$

for every k, so $E\in\mathrm{T}$. **Q**

So $\bigcap_{n\in\mathbb{N}}\mathfrak{A}_n=\{0,1\}$ and π and ϕ are mixing.

372X Basic exercises (a) Let U be any reflexive Banach space, and $T:U\to U$ an operator of norm at most 1. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n\in\mathbb{N}$. Show that $Pu=\lim_{n\to\infty}A_nu$ is defined (as a limit for the norm topology) for every $u\in U$, and that $P:U\to U$ is a projection onto $\{u:Tu=u\}$. (*Hint*: show that $\{u:Pu \text{ is defined}\}$ is a closed linear subspace of U containing $Tu-u$ for every $u\in U$.)

(This is a version of the **mean ergodic theorem**.)

>(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$; set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for $n\in\mathbb{N}$. Take any $p\in[1,\infty[$ and $u\in L^p=L^p(\mathfrak{A},\bar\mu)$. Show that $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\,\|_p$-convergent to some $v\in L^p$. (*Hint*: put 372Xa together with 372D.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $P:L^1\to L^1$ be the operator defined as in 365P/366Hb, where $L^1=L^1(\mathfrak{A},\bar\mu)$, so that $\int_a Pu=\int_{\pi a}u$ for $u\in L^1$, $a\in\mathfrak{A}$. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n P^i:L^1\to L^1$ for each i. Show that for any $u\in L^1$, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\,\|_1$-convergent to the conditional expectation of u on the subalgebra $\{a:\pi a=a\}$.

(d) Show that if f is any Lebesgue integrable function on $\mathbb{R}$, and $y\in\mathbb{R}\setminus\{0\}$, then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n f(x+ky)=0$$

for almost every $x\in\mathbb{R}$.

(e) Let (X,Σ,μ) be a measure space and $\phi:X\to X$ an inverse-measure-preserving function. Set $\mathrm{T}=\{E:E\in\Sigma,\ \mu(\phi^{-1}[E]\triangle E)=0\}$, $\mathrm{T}_0=\{E:E\in\Sigma,\ \phi^{-1}[E]=E\}$. (i) Show that $\mathrm{T}=\{E\triangle F:E\in\mathrm{T}_0,\ F\in\Sigma,\ \mu F=0\}$. (ii) Show that a set $A\subseteq X$ is $\mu\restriction\mathrm{T}_0$-negligible iff $\phi^n[A]$ is μ-negligible for every $n\in\mathbb{N}$.

>(f) Let ν be a Radon probability measure on $\mathbb{R}$ such that $\int|t|\nu(dt)$ is finite (cf. 271C-271F). On $X=\mathbb{R}^{\mathbb{N}}$ let λ be the product measure obtained when each factor is given the measure ν. Define $\phi:X\to X$ by setting $\phi(x)(n)=x(n+1)$ for $x\in X$, $n\in\mathbb{N}$. (i) Show that ϕ is inverse-measure-preserving. (*Hint*: 254G. See also 372Xu below.) (iii) Set $\gamma=\int t\nu(dt)$, the expectation of the distribution ν. By considering $\frac{1}{n+1}\sum_{i=0}^n f\circ\phi^i$, where $f(x)=x(0)$ for $x\in X$, show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n x(i)=\gamma$ for λ-almost every $x\in X$.

>(g) Use the Ergodic Theorem to prove Kolmogorov's Strong Law of Large Numbers (273I), as follows. Given a complete probability space (Ω,Σ,μ) and an independent identically distributed sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of measurable functions from Ω to $\mathbb{R}$, set $X=\mathbb{R}^{\mathbb{N}}$ and $F(\omega)=\langle F_n(\omega)\rangle_{n\in\mathbb{N}}$ for $\omega\in\Omega$. Show that if we give each copy of $\mathbb{R}$ the distribution of F_0 then F is inverse-measure-preserving for μ and the product measure λ on X. Now use 372Xf.

(h) Show that the continued fraction coefficients of $\frac{1}{\sqrt{2}}$ are 1, 2, 1, 2, ...

>(i) For $x\in X=[0,1]\setminus\mathbb{Q}$ let $k_1(x),k_2(x),\dots$ be its continued-fraction coefficients. Show that $x\mapsto\langle k_{n+1}(x)-1\rangle_{n\in\mathbb{N}}$ is a bijection between X and $\mathbb{N}^{\mathbb{N}}$ which is a homeomorphism if X is given its usual topology (as a subset of $\mathbb{R}$) and $\mathbb{N}^{\mathbb{N}}$ is given its usual product topology (each copy of $\mathbb{N}$ being given the discrete topology).

(j) For any irrational $x \in [0,1]$ let $k_1(x), k_2(x), \dots$ be its continued-fraction coefficients and $p_n(x)$, $q_n(x)$ the numerators and denominators of its continued-fraction approximations, as described in 372M. Write $r_n(x) = p_n(x)/q_n(x)$. (i) Show that x lies between $r_n(x)$ and $r_{n+1}(x)$ for every $n \in \mathbb{N}$. (ii) Show that $r_{n+1}(x) - r_n(x) = (-1)^n/q_n(x)q_{n+1}(x)$ for every $n \in \mathbb{N}$. (iii) Show that $|x - r_n(x)| \le 1/q_n(x)^2 k_n(x)$ for every $n \ge 1$. (iv) Hence show that for almost every $\gamma \in \mathbb{R}$, the set $\{(p,q) : p \in \mathbb{Z},\, q \ge 1,\, |\gamma - \frac{p}{q}| \le \epsilon/q^2\}$ is infinite for every $\epsilon > 0$.

(k) Let $(\mathfrak{A}, \bar\mu)$ be an atomless probability algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is homogeneous; (ii) there is an ergodic measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$; (iii) there is a mixing measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. (*Hint*: 333P.)

(l) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Show that if $n \ge 1$ then π is mixing iff π^n is mixing. (ii) Show that if $n \ge 1$ and π^n is ergodic then π is ergodic. (iii) Show that if π is an automorphism then it is ergodic, or mixing, iff π^{-1} is.

>(m) Consider the **tent map** $\phi_\alpha(x) = \alpha\min(x, 1-x)$ for $x \in [0,1]$, $\alpha \in [0,2]$. Show that ϕ_2 is inverse-measure-preserving and mixing for Lebesgue measure on $[0,1]$. (*Hint*: show that $\phi_2^{n+1}(x) = \phi_2(<2^n x>)$ for $n \ge 1$, and hence that $\mu(I \cap \phi_2^{-n}[J]) = \mu I \cdot \mu J$ whenever I is of the form $[2^{-n}k, 2^{-n}(k+1)]$ and J is an interval.)

(n) Consider the **quadratic map** $\psi_\beta(x) = \beta x(1-x)$ for $x \in [0,1]$, $\beta \in [0,4]$. Show that ψ_4 is inverse-measure-preserving and mixing for the Radon measure on $[0,1]$ with density function $t \mapsto 1/\pi\sqrt{t(1-t)}$. (*Hint*: show that the transformation $t \mapsto \sin^2\frac{\pi t}{2}$ turns it into the tent map.) Show that for almost every x,

$$\lim_{n\to\infty}\frac{1}{n+1}\#(\{i : i \le n,\ \psi_4^i(x) \le \alpha\}) = \frac{2}{\pi}\arcsin\sqrt{\alpha}$$

for every $\alpha \in [0,1]$.

(o) Let (X, Σ, μ) be Lebesgue measure on $[0,1[$, and fix an irrational number $\alpha \in [0,1[$. (i) Set $\phi(x) = x +_1 \alpha$ for every $x \in [0,1[$, where $x +_1 \alpha$ is whichever of $x+\alpha$, $x+\alpha-1$ belongs to $[0,1[$. Show that ϕ is inverse-measure-preserving. (ii) Show that if $I \subseteq [0,1[$ is an interval then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n \chi I(\phi^i(x)) = \mu I$ for almost every $x \in [0,1[$. (*Hint*: this is Weyl's Equidistribution Theorem (281N).) (iii) Show that ϕ is ergodic. (*Hint*: take the conditional expectation operator P of 372G, and look at $P(\chi I^\bullet)$ for intervals I.) (iv) Show that ϕ^n is ergodic for any $n \in \mathbb{Z} \setminus \{0\}$. (v) Show that ϕ is not mixing.

(p) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a mixing measure-preserving homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the corresponding homomorphism. Let p, $q \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that $\lim_{n\to\infty}\int T^n u \times v = \int u \int v$ whenever $u \in L^p(\mathfrak{A}, \bar\mu)$ and $v \in L^q(\mathfrak{A}, \bar\mu)$. (*Hint*: start with u, $v \in S(\mathfrak{A})$.)

(q) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ a mixing inverse-measure-preserving function. Let p, $q \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that $\lim_{n\to\infty}\int f(\phi^n(x))g(x)dx = \int f \int g$ whenever $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$.

(r) Give $[0,1[$ Lebesgue measure μ, and let $k \ge 2$ be an integer. Define $\phi : [0,1[\to [0,1[$ by setting $\phi(x) = <kx>$, the fractional part of kx. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: if $I = [k^{-n}i, k^{-n}(i+1)[$, $J = [k^{-n}j, k^{-n}(j+1)[$ then $\mu(I \cap \phi^{-m}[J]) = \mu I \cdot \mu J$ for all $m \ge n$.)

(s) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an ergodic inverse-measure-preserving function. Let f be a μ-virtually measurable function defined almost everywhere on X such that $\int f d\mu = \infty$. Show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x)) = \infty$ for almost every $x \in X$. (*Hint*: look at the corresponding limits for $f_m = f \wedge m\chi X$.)

(t) For irrational $x \in [0,1]$, write $k_1(x), k_2(x), \dots$ for the continued-fraction coefficients of x. Show that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n k_i(x) = \infty$ for almost every x. (*Hint*: take ϕ, ν as in 372N, and show that $\int k_1 d\nu = \infty$.)

(u) Let (X,Σ,μ) be any probability space, and let λ be the product measure on $X^{\mathbb{N}}$. Define $\phi:X^{\mathbb{N}}\to X^{\mathbb{N}}$ by setting $\phi(x)(n)=x(n{+}1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ satisfies the conditions of 372R, so is mixing.

(v) Let (X,Σ,μ) be any probability space, and let λ be the product measure on $X^{\mathbb{Z}}$. Define $\phi:X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ by setting $\phi(x)(n)=x(n+1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: show that if C, C' are basic cylinder sets then $\mu(C\cap\phi^{-n}[C'])=\mu C\cdot\mu C'$ for all n large enough.) Show that ϕ does not satisfy the conditions of 372R. (Compare 333P.)

(w) In 372N, let T_1 be the family $\{E:$ for every $n\in\mathbb{N}$ there is a measurable set $F\subseteq X$ such that $\phi^{-n}[F]=E\}$. Show that every member of T_1 is either negligible or conegligible. (*Hint*: the argument of part (c) of the proof of 372N still works.) Hence show that ϕ is mixing for the measure ν.

372Y Further exercises (a) In 372D, show that the null space of the limit operator P is precisely the closure in $M^{1,0}$ of the subspace $\{Tu-u:u\in M^{1,0}\}$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$. Take $p\in\,]1,\infty[$ and $u\in L^p(\mathfrak{A},\bar\mu)$, and set $u^*=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{i=0}^n|T^iu|$. (i) Show that for any $\gamma>0$,

$$\bar\mu[\![u^*>\gamma]\!]\le\frac{2}{\gamma}\int_{[\![|u|>\gamma/2]\!]}|u|.$$

(*Hint*: apply 372C to $(|u|-\frac{1}{2}\gamma\chi1)^+$.) (ii) Show that $\|u^*\|_p\le2(\frac{p}{p-1})^{1/p}\|u\|_p$. (*Hint*: show that $\int_{[\![|u|>\alpha]\!]}|u|=\alpha\bar\mu[\![|u|>\alpha]\!]+\int_\alpha^\infty\bar\mu[\![|u|>\beta]\!]d\beta$; see 365A. Use 366Xa to show that

$$\|u^*\|_p^p\le2p\int_0^\infty\gamma^{p-2}\int_{\gamma/2}^\infty\bar\mu[\![|u|>\beta]\!]d\beta d\gamma+2^p\|u\|_p^p,$$

and reverse the order of integration. Compare 275Yc.) (This is **Wiener's dominated ergodic theorem**.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and T an operator in $\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$. Take $u\in L^1=L^1(\mathfrak{A},\bar\mu)$ such that $h(|u|)\in L^1$, where $h(t)=t\ln t$ for $t\ge1$, 0 for $t\le1$, and $\bar h$ is the corresponding function from $L^0(\mathfrak{A})$ to itself. Set $u^*=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{i=0}^n|T^iu|$. Show that $u^*\in L^1$. (*Hint*: use the method of 372Yb to show that $\int_2^\infty\bar\mu[\![u^*>\gamma]\!]d\gamma\le2\int\bar h(u)$.)

(d) In 372G, suppose that $\mathfrak{A}$ is atomless. Show that there is always an $a\in\mathfrak{A}$ such that $\bar\mu a\le\frac{1}{2}$ and $\inf_{i\le n}\pi^ia\ne0$ for every n, so that (except in trivial cases) $\langle A_n(\chi a)\rangle_{n\in\mathbb{N}}$ will not be $\|\;\|_\infty$-convergent.

(e) Show that an irrational $x\in\,]0,1[$ has an eventually periodic sequence of continued fraction coefficients iff it is a solution of a quadratic equation with integral coefficients.

(f) In the language of 372M-372O and 372Xj, show the following. (i) For any $x\in X$, $n\ge2$, $q_n(x)q_{n-1}(x)\ge2^{n-1}$, $p_n(x)p_{n+1}(x)\ge2^{n-1}$, so that $q_{n+1}(x)p_n(x)\ge2^{n-1}$ and $|1-x/r_n(x)|\le2^{-n+1}$, $|\ln x-\ln r_n(x)|\le2^{-n+2}$. Also $|x-r_n(x)|\ge1/q_n(x)q_{n+2}(x)$. (ii) For any $x\in X$, $n\ge1$, $p_{n+1}(x)=q_n(\phi(x))$ and $q_n(x)\prod_{i=0}^{n-1}r_{n-i}(\phi^i(x))=1$. (iii) For any $x\in X$, $n\ge1$, $|\ln q_n(x)+\sum_{i=0}^{n-1}\ln\phi^i(x)|\le4$. (iv) For almost every $x\in X$,

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(x)=-\frac{1}{\ln2}\int_0^1\frac{\ln t}{1+t}dt=\frac{\pi^2}{12\ln2}.$$

(*Hint*: 225Xi, 282Xo.) (v) For almost every $x\in X$, $\lim_{n\to\infty}\frac{1}{n}\ln|x-r_n(x)|=-\pi^2/6\ln2$. (vi) For almost every $x\in X$, $11^{-n}\le|x-r_n(x)|\le10^{-n}$ and $3^n\le q_n(x)\le4^n$ for all but finitely many n.

(g) In 372M, show that for any measurable set $E\subseteq X$, $\lim_{n\to\infty}\mu_L\phi^{-n}[E]=\nu E$. (*Hint*: recall that ϕ is mixing for ν (372Xw). Hence show that $\lim_{n\to\infty}\int_{\phi^{-n}[E]}g\,d\nu=\nu E\cdot\int g\,d\nu$ for any integrable g. Apply this to a Radon-Nikodým derivative of μ_L with respect to ν.) (I understand that this result is due to Gauss.)

(h) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, with product (X,Λ,λ). Suppose that for each $i\in I$ we are given an inverse-measure-preserving function $\phi_i:X_i\to X_i$. Show that there is a corresponding inverse-measure-preserving function $\phi:X\to X$ given by setting $\phi(x)(i)=\phi_i(x(i))$ for $x\in X$, $i\in I$. Show that if each ϕ_i is mixing so is ϕ.

(i) Give an example of an ergodic measure-preserving automorphism $\phi:[0,1[\,\to[0,1[$ such that ϕ^2 is not ergodic. (*Hint*: set $\phi(x)=\frac{1}{2}(1+\phi_0(2x))$ for $x<\frac{1}{2}$, $x-\frac{1}{2}$ for $x\ge\frac{1}{2}$. See also 387Xg.)

(j) Show that there is an ergodic $\phi:[0,1]\to[0,1]$ such that $(\xi_1,\xi_2)\mapsto(\phi(\xi_1),\phi(\xi_2)):[0,1]^2\to[0,1]^2$ is not ergodic. (*Hint*: 372Xo.)

(k) Let M be an $r\times r$ matrix with integer coefficients and non-zero determinant, where $r\ge 1$. Let $\phi:[0,1[^r\to[0,1[^r$ be the function such that $\phi(x)-Mx\in\mathbb{Z}^r$ for every $x\in[0,1[^r$. Show that ϕ is inverse-measure-preserving for Lebesgue measure on $[0,1[^r$.

372 Notes and comments I have chosen an entirely conventional route to the Ergodic Theorem here, through the Mean Ergodic Theorem (372Xa) or, rather, the fundamental lemma underlying it (372A), and the Maximal Ergodic Theorem (372B-372C). What is not to be found in every presentation is the generality here. I speak of arbitrary $T\in\mathcal{T}^{(0)}$, the operators which are contractions both for $\| \|_1$ and for $\| \|_\infty$, not requiring T to be positive, let alone correspond to a measure-preserving homomorphism. (I do not mention $\mathcal{T}^{(0)}$ in the statement of 372C, but of course it is present in spirit.) The work we have done up to this point puts this extra generality within easy reach, but as the rest of the section shows, it is not needed for the principal examples. Only in 372Xc do I offer an application not associated with a measure-preserving homomorphism or an inverse-measure-preserving function.

The Ergodic Theorem is an 'almost-everywhere pointwise convergence theorem', like the strong law(s) of large numbers and the martingale theorem(s) (§273, §275). Indeed Kolmogorov's form of the strong law can be derived from the Ergodic Theorem (372Xg). There are some very strong family resemblances. For instance, the Maximal Ergodic Theorem corresponds to the most basic of all the martingale inequalities (275D). Consequently we have similar results, obtained by similar methods, concerning the domination of sequences starting from members of L^p (372Yb, 275Yc), though the inequalities are not identical. (Compare also 372Yc with 275Yd.) There are some tantalising reflections of these traits in results surrounding Carleson's theorem on the pointwise convergence of square-integrable Fourier series (see §286 *notes*), but Carleson's theorem seems to be much harder than the others. Other forms of the strong law (273D, 273H) do not appear to fit into quite the same pattern, but I note that here, as with the Ergodic Theorem, we begin with a study of square-integrable functions (see part (e) of the proof of 372D).

After 372D, there is a contraction and concentration in the scope of the results, starting with a simple replacement of $M^{1,0}$ with L^1 (372E). Of course it is almost as easy to prove 372D from 372E as the other way about; I give precedence to 372D only because $M^{1,0}$ is the space naturally associated with the class $\mathcal{T}^{(0)}$ of operators to which these methods apply. Following this I turn to the special family of operators to which the rest of the section is devoted, those associated with measure-preserving homomorphisms (372F), generally on probability spaces (372G). This is the point at which we can begin to identify the limit as a conditional expectation as well as an invariant element.

Next comes the translation into the language of measure spaces and inverse-measure-preserving functions, all perfectly straightforward in view of the lemmas 372H (which was an exercise in §364) and 372J. These turn 372F into 372I and 372G into the main part of 372K.

In 372K-372L I find myself writing at some length about a technical problem. The root of the difficulty is in the definition of 'conditional expectation'. Now it is generally accepted that any pure mathematician has 'Humpty Dumpty's privilege': 'When *I* use a word, it means just what I choose it to mean – neither more nor less'. With any privilege come duties and responsibilities; in this context, the duty to be self-consistent, and the responsibility to try to use terms in ways which will not mystify or mislead the unprepared reader. Having written down a definition of 'conditional expectation' in Volume 2, I must either stick to it, or go back and change it, or very carefully explain exactly what modification I wish to make here. I don't wish to suggest that absolute consistency – in terminology or anything else – is supreme among mathematical virtues. Surely it is better to give local meanings to words, or tolerate ambiguities, than to suppress ideas which cannot be formulated effectively otherwise, and among 'ideas' I wish to include the analogies and resonances which a suitable language can suggest. But I do say that it is always best to be conscious of what one is doing – I go farther: one of the things which mathematics is for, is to raise our consciousness of what our thoughts really are. So I believe it is right to pause occasionally over such questions.

In 372M-372O (see also 372Xj, 372Xt, 372Xw, 372Yf, 372Yg) I make an excursion into number theory. This is a remarkable example of the power of advanced measure theory to give striking results in other branches of mathematics. Everything here is derived from BILLINGSLEY 65, who goes farther than I have space for, and gives references to more. Here let me point to 372Xi; almost accidentally, the construction offers a useful formula for a homeomorphism between two of the most important spaces of descriptive set theory, which will be important to us in Volume 4.

I end the section by introducing two terms, 'ergodic' and 'mixing' transformation, not because I wish to use them for any new ideas (apart from the elementary 372R, these must wait for §§385-386) but because it may help if I immediately classify some of the inverse-measure-preserving functions we have seen (372Xm-372Xo, 372Xr, 372Xt-372Xv). Of course in any application of any ergodic theorem it is of great importance to be able to identify the limits promised by the theorem, and the point about an ergodic transformation is just that our averages converge to constant limits (372Q). Actually proving that a given inverse-measure-preserving function is ergodic is rarely quite trivial (see 372N, 372Xn, 372Xo), though a handful of standard techniques cover a large number of cases, and it is usually obvious when a map is *not* ergodic, so that if an invariant region does not leap to the eye one has a good hope of ergodicity.

I take the opportunity to mention two famous functions from $[0,1]$ to itself, the 'tent' and 'quadratic' maps (372Xm, 372Xn). In the formulae ϕ_α, ψ_β I include redundant parameters; this is because the real importance of these functions lies in the way their behaviour depends, in bewildering complexity, on these parameters. It is only at the extreme values $\alpha = 2$, $\beta = 4$ that the methods of this volume can tell us anything interesting.

373 Decreasing rearrangements

I take a section to discuss operators in the class $\mathcal{T}^{(0)}$ of 371F-371H and §372 and two associated classes $\mathcal{T}$, $\mathcal{T}^\times$ (373A). These turn out to be intimately related to the idea of 'decreasing rearrangement' (373C). In 373D-373F I give elementary properties of decreasing rearrangements; then in 373G-373O I show how they may be used to characterize the set $\{Tu : T \in \mathcal{T}\}$ for a given u. The argument uses a natural topology on the set $\mathcal{T}$ (373K). I conclude with remarks on the possible values of $\int Tu \times v$ for $T \in \mathcal{T}$ (373P-373Q) and identifications between $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ and $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ (373R-373T).

373A Definition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras. Recall that $M^{1,\infty}_{\bar\mu} = L^1(\mathfrak{A}, \bar\mu) + L^\infty(\mathfrak{A})$ is the set of those $u \in L^0(\mathfrak{A})$ such that $(|u| - \alpha\chi 1)^+$ is integrable for some α, its norm $\| \|_{1,\infty}$ being defined by the formulae

$$\begin{aligned}\|u\|_{1,\infty} &= \min\{\|v\|_1 + \|w\|_\infty : v \in L^1,\, w \in L^\infty,\, v + w = u\} \\ &= \min\{\alpha + \int (|u| - \alpha\chi 1)^+ : \alpha \geq 0\}\end{aligned}$$

(369Ob).

(a) $\mathcal{T}_{\bar\mu,\bar\nu}$ will be the space of linear operators $T : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\nu}$ such that $\|Tu\|_1 \leq \|u\|_1$ for every $u \in L^1_{\bar\mu}$ and $\|Tu\|_\infty \leq \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$. (Compare the definition of $\mathcal{T}^{(0)}$ in 371F.)

(b) If $\mathfrak{B}$ is Dedekind complete, so that $M^{1,\infty}_{\bar\mu}$, being a solid linear subspace of the Dedekind complete space $L^0(\mathfrak{B})$, is Dedekind complete, $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ will be $\mathcal{T}_{\bar\mu,\bar\nu} \cap \mathsf{L}^\times(M^{1,\infty}_{\bar\mu}; M^{1,\infty}_{\bar\mu})$.

373B Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras.

(a) $\mathcal{T}=\mathcal{T}_{\bar{\mu},\bar{\nu}}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$.

(b) If $T\in\mathcal{T}$ then $T\restriction M^{1,0}_{\bar{\mu}}$ belongs to $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$, as defined in 371F. So if $T\in\mathcal{T}$, $p\in[1,\infty[$ and $u\in L^p_{\bar{\mu}}$ then $Tu\in L^p_{\bar{\nu}}$ and $\|Tu\|_p\le\|u\|_p$.

(c) If $\mathfrak{B}$ is Dedekind complete and $T\in\mathcal{T}$, then $T\in\mathsf{L}^{\sim}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$ and $T_1\in\mathcal{T}$ whenever $T_1\in\mathsf{L}^{\sim}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$ and $|T_1|\le|T|$; in particular, $|T|\in\mathcal{T}$.

(d) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism, then we have a corresponding operator $T\in\mathcal{T}$ defined by saying that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$. If π is order-continuous, then so is T.

(e) If $(\mathfrak{C},\bar{\lambda})$ is another measure algebra and $T\in\mathcal{T}$, $S\in\mathcal{T}_{\bar{\nu},\bar{\lambda}}$ then $ST\in\mathcal{T}_{\bar{\mu},\bar{\lambda}}$.

proof (a) As 371G, parts (a-i) and (a-ii) of the proof.

(b) If $u\in M^{1,0}_{\bar{\mu}}$ and $\epsilon>0$, then u is expressible as $u'+u''$ where $\|u''\|_\infty\le\epsilon$ and $u'\in L^1_{\bar{\mu}}$. (Set

$$u''=(u^+\wedge\epsilon\chi1)-(u^-\wedge\epsilon\chi1).)$$

So

$$(|Tu|-\epsilon\chi1)^+\le|Tu-Tu''|\in L^1_{\bar{\nu}}.$$

As ϵ is arbitrary, $Tu\in M^{1,0}_{\bar{\nu}}$; as u is arbitrary, $T\restriction M^{1,0}_{\bar{\mu}}\in\mathcal{T}^{(0)}$. Now the rest is a consequence of 371Gd.

(c) Because $M^{1,\infty}_{\bar{\nu}}$ is a solid linear subspace of $L^0(\mathfrak{B})$, which is Dedekind complete because $\mathfrak{B}$ is, $\mathsf{L}^{\sim}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$ is a Riesz space (355Ea).

Take any $u\ge0$ in $M^{1,\infty}_{\bar{\mu}}$. Let $\alpha\ge0$ be such that $(u-\alpha\chi1)^+\in L^1_{\bar{\mu}}$. Because $T\restriction L^1_{\bar{\mu}}$ belongs to $\mathsf{B}(L^1_{\bar{\mu}};L^1_{\bar{\nu}})=\mathsf{L}^{\sim}(L^1_{\bar{\mu}};L^1_{\bar{\nu}})$ (371D), $w_0=\sup\{Tv:v\in L^1_{\bar{\mu}},\ 0\le v\le(u-\alpha\chi1)^+\}$ is defined in $L^1_{\bar{\nu}}$. Now if $v\in M^{1,\infty}_{\bar{\mu}}$ and $0\le v\le u$, we must have

$$Tv=T(v-\alpha\chi1)^++T(v\wedge\alpha\chi1)\le w_0+\alpha\chi1\in M^{1,\infty}_{\bar{\nu}}.$$

Thus $\{Tv:0\le v\le u\}$ is bounded above in $M^{1,\infty}_{\bar{\nu}}$. As u is arbitrary, $T\in\mathsf{L}^{\sim}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$ (355Ba).

Now take T_1 such that $|T_1|\le|T|$ in $\mathsf{L}^{\sim}(M^{1,\infty}_{\bar{\mu}};M^{1,\infty}_{\bar{\nu}})$. 371D also tells us that $\||T\restriction L^1_{\bar{\mu}}|\|=\|T\restriction L^1_{\bar{\mu}}\|$, so that

$$\begin{aligned}\|T_1u\|_1&=\||T_1u|\|_1\le\||T_1||u|\|_1\le\||T||u|\|_1\\&=\|\sup_{|v|\le|u|}Tv\|_1\le\|\sup_{|v|\le|u|}v\|_1=\|u\|_1\end{aligned}$$

for every $u\in L^1_{\bar{\mu}}$ (using one of the formulae in 355Eb for the first equality). At the same time, if $u\in L^\infty(\mathfrak{A})$, then

$$\begin{aligned}|T_1u|&\le|T_1||u|\le|T||u|=\sup_{|v|\le|u|}Tv\\&\le\sup_{|v|\le|u|}\|Tv\|_\infty\chi1\le\sup_{|v|\le|u|}\|v\|_\infty\chi1=\|u\|_\infty\chi1,\end{aligned}$$

so $\|T_1u\|_\infty\le\|u\|_\infty$. Thus $T_1\in\mathcal{T}$.

(d) By 365O and 363F, we have norm-preserving positive linear operators $T_1:L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ and $T_\infty:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ defined by saying that $T_1(\chi a)=\chi(\pi a)$ whenever $\bar{\mu}a<\infty$ and $T_\infty(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$. If $u\in S(\mathfrak{A}^f)=L^1_{\bar{\mu}}\cap S(\mathfrak{A})$ (365F), then $T_1u=T_\infty u$, because both T_1 and T_∞ are linear and they agree on $\{\chi a:\bar{\mu}a<\infty\}$. If $u\ge0$ in $M^{\infty,1}_{\bar{\mu}}=L^1_{\bar{\mu}}\cap L^\infty(\mathfrak{A})$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u=\sup_{n\in\mathbb{N}}u_n$ and

$$\lim_{n\to\infty}\|u-u_n\|_1=\lim_{n\to\infty}\|u-u_n\|_\infty=0$$

(see the proof of 369Od), so that

$$T_1u=\sup_{n\in\mathbb{N}}T_1u_n=\sup_{n\in\mathbb{N}}T_\infty u_n=T_\infty u.$$

Accordingly T_1 and T_∞ agree on $L^1_{\bar{\mu}}\cap L^\infty(\mathfrak{A})$. But this means that if $u\in M^{1,\infty}_{\bar{\mu}}$ is expressed as $v+w=v'+w'$, where v, $v'\in L^1_{\bar{\mu}}$ and w, $w'\in L^\infty(\mathfrak{A})$, we shall have

$$T_1v' + T_\infty w' = T_1v + T_\infty w + T_1(v' - v) - T_\infty(w - w') = T_1v + T_\infty w,$$

because $v' - v = w - w' \in M_{\bar\mu}^{\infty,1}$. Accordingly we have an operator $T : M_{\bar\mu}^{1,\infty} \to M_{\bar\nu}^{1,\infty}$ defined by setting

$$T(v + w) = T_1v + T_\infty w \text{ whenever } v \in L_{\bar\mu}^1,\, w \in L^\infty(\mathfrak{A}).$$

This formula makes it easy to check that T is linear and positive, and it clearly belongs to $\mathcal{T}$.

To see that T is uniquely defined, observe that $T\restriction L_{\bar\mu}^1$ and $T\restriction L^\infty(\mathfrak{A})$ are uniquely defined by the values T takes on $S(\mathfrak{A}^f)$, $S(\mathfrak{A})$ respectively, because these spaces are dense for the appropriate norms.

Now suppose that π is order-continuous. Then T_1 and T_∞ are also order-continuous (365Oa, 363Ff). If $A \subseteq M_{\bar\mu}^{1,\infty}$ is non-empty and downwards-directed and has infimum 0, take $u_0 \in A$ and $\gamma > 0$ such that $(u_0 - \gamma\chi 1)^+ \in L_{\bar\mu}^1$. Set

$$A_1 = \{(u - \gamma\chi 1)^+ : u \in A,\, u \le u_0\}, \quad A_\infty = \{u \wedge \gamma\chi 1 : u \in A\}.$$

Then $A_1 \subseteq L_{\bar\mu}^1$ and $A_\infty \subseteq L^\infty(\mathfrak{A})$ are both downwards-directed and have infimum 0, so $\inf T_1[A_1] = \inf T_\infty[A_\infty] = 0$ in $L^0(\mathfrak{B})$. But this means that $\inf(T_1[A_1] + T_\infty[A_\infty]) = 0$ (351Dc). Now any $w \in T_1[A_1] + T_\infty[A_\infty]$ is expressible as $T(u - \gamma\chi 1)^+ + T(u' \wedge \gamma\chi 1)$ where u, $u' \in A$; because A is downwards-directed, there is a $v \in A$ such that $v \le u \wedge u'$, in which case $Tv \le w$. Accordingly $T[A]$ must also have infimum 0. As A is arbitrary, T is order-continuous.

(e) is obvious, as usual.

373C Decreasing rearrangements The following concept is fundamental to any understanding of the class $\mathcal{T}$. Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write $M_{\bar\mu}^{0,\infty} = M^{0,\infty}(\mathfrak{A}, \bar\mu)$ for the set of those $u \in L^0(\mathfrak{A})$ such that $\bar\mu[\![|u| > \alpha]\!]$ is finite for some $\alpha \in \mathbb{R}$. (See 369N for the ideology of this notation.) It is easy to see that $M_{\bar\mu}^{0,\infty}$ is a solid linear subspace of $L^0(\mathfrak{A})$. Let $(\mathfrak{A}_L, \bar\mu_L)$ be the measure algebra of Lebesgue measure on $[0, \infty[$. For $u \in M_{\bar\mu}^{0,\infty}$ its **decreasing rearrangement** is $u^* \in M_{\bar\mu_L}^{0,\infty}$, defined by setting $u^* = g^\bullet$, where

$$g(t) = \inf\{\alpha : \alpha \ge 0,\, \bar\mu[\![|u| > \alpha]\!] \le t\}$$

for every $t > 0$. (This is always finite because $\inf_{\alpha\in\mathbb{R}} \bar\mu[\![|u| > \alpha]\!] = 0$, by 364A($\beta$) and 321F.)

I will maintain this usage of the symbols $\mathfrak{A}_L$, $\bar\mu_L$, u^* for the rest of this section.

373D Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) For any $u \in M_{\bar\mu}^{0,\infty}$, its decreasing rearrangement u^* may be defined by the formula

$$[\![u^* > \alpha]\!] = [0, \bar\mu[\![|u| > \alpha]\!][^\bullet \text{ for every } \alpha \ge 0,$$

that is,

$$\bar\mu_L[\![u^* > \alpha]\!] = \bar\mu[\![|u| > \alpha]\!] \text{ for every } \alpha \ge 0.$$

(b) If $|u| \le |v|$ in $M_{\bar\mu}^{0,\infty}$, then $u^* \le v^*$; in particular, $|u|^* = u^*$.

(c)(i) If $u = \sum_{i=0}^n \alpha_i\chi a_i$, where $a_0 \supseteq a_1 \supseteq \ldots \supseteq a_n$ and $\alpha_i \ge 0$ for each i, then $u^* = \sum_{i=0}^n \alpha_i\chi\, [0, \bar\mu a_i[^\bullet$.

(ii) If $u = \sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \ldots, a_n$ are disjoint and $|\alpha_0| \ge |\alpha_1| \ge \ldots \ge |\alpha_n|$, then $u^* = \sum_{i=0}^n |\alpha_i|\chi\, [\beta_i, \beta_{i+1}[^\bullet$, where $\beta_i = \sum_{j<i} \bar\mu a_i$ for $i \le n+1$.

(d) If $E \subseteq\,]0, \infty[$ is any Borel set, and $u \in M_{\bar\mu}^0$, then $\bar\mu_L[\![u^* \in E]\!] = \bar\mu[\![|u| \in E]\!]$.

(e) Let $h : [0, \infty[\to [0, \infty[$ be a non-decreasing function such that $h(0) = 0$, and write $\bar h$ for the corresponding functions on $L^0(\mathfrak{A})^+$ and $L^0(\mathfrak{A}_L)^+$ (364I). Then $(\bar h(u))^* = \bar h(u^*)$ whenever $u \ge 0$ in $M_{\bar\mu}^0$. If h is continuous on the left, $(\bar h(u))^* = \bar h(u^*)$ whenever $u \ge 0$ in $M_{\bar\mu}^{0,\infty}$.

(f) If $u \in M_{\bar\mu}^{0,\infty}$ and $\alpha \ge 0$, then

$$(u^* - \alpha\chi 1)^+ = ((|u| - \alpha\chi 1)^+)^*.$$

(g) If $u \in M_{\bar\mu}^{0,\infty}$, then for any $t > 0$

$$\int_0^t u^* = \inf_{\alpha\ge 0} \alpha t + \int (|u| - \alpha\chi 1)^+.$$

(h) If $A \subseteq (M_{\bar\mu}^{0,\infty})^+$ is non-empty and upwards-directed and has supremum $u_0 \in M_{\bar\mu}^{0,\infty}$, then $u_0^* = \sup_{u\in A} u^*$.

proof (a) Set

$$g(t)=\inf\{\alpha:\bar{\mu}[\![|u|>\alpha]\!]\le t\}$$

as in 373C. If $\alpha\ge 0$,

$$g(t)>\alpha\iff\bar{\mu}[\![|u|>\beta]\!]>t\text{ for some }\beta>\alpha\iff\bar{\mu}[\![|u|>\alpha]\!]>t$$

(because $[\![|u|>\alpha]\!]=\sup_{\beta>\alpha}[\![|u|>\beta]\!]$), so

$$[\![u^*>\alpha]\!]=\{t:g(t)>\alpha\}^\bullet=[0,\bar{\mu}[\![|u|>\alpha]\!][^\bullet.$$

Of course this formula defines u^*.

(b) This is obvious, either from the definition in 373C or from (a) just above.

(c)(i) Setting $v=\sum_{i=0}^n\alpha_i\chi\,[0,\bar{\mu}a_i[^\bullet$, we have

$$\begin{aligned}[\![v>\alpha]\!]&=0\text{ if }\sum_{i=0}^n\alpha_i\le\alpha\\&=[0,\bar{\mu}a_j[^\bullet\text{ if }\sum_{i=0}^{j-1}\alpha_i\le\alpha<\sum_{i=0}^{j}\alpha_i\\&=[0,\bar{\mu}a_0[^\bullet\text{ if }0\le\alpha<\alpha_0,\end{aligned}$$

and in all cases is equal to $[0,\bar{\mu}[\![|u|>\alpha]\!][^\bullet$.

(ii) A similar argument applies. (If any a_j has infinite measure, then a_i is irrelevant for $i>j$.)

(d) Fix $\gamma>0$ for the moment, and consider

$$\mathcal{A}=\{E:E\subseteq\,]\gamma,\infty[\text{ is a Borel set, }\bar{\mu}_L[\![u^*\in E]\!]=\bar{\mu}[\![|u|\in E]\!]\},$$

$$\mathcal{I}=\{]\alpha,\infty[\,:\alpha\ge\gamma\}.$$

Then $\mathcal{I}\subseteq\mathcal{A}$ (by (a)), $I\cap J\in\mathcal{I}$ for all I, $J\in\mathcal{I}$, $E\setminus F\in\mathcal{A}$ whenever E, $F\in\mathcal{A}$ and $F\subseteq E$ (because $u\in M^0_{\bar{\mu}}$, so $\bar{\mu}[\![|u|\in E]\!]<\infty$), and $\bigcup_{n\in\mathbb{N}}E_n\in\mathcal{A}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$. So, by the Monotone Class Theorem (136B), $\mathcal{A}$ includes the σ-algebra of subsets of $]\gamma,\infty[$ generated by $\mathcal{I}$; but this must contain $E\cap\,]\gamma,\infty[$ for every Borel set $E\subseteq\mathbb{R}$.

Accordingly, for any Borel set $E\subseteq\,]0,\infty[$,

$$\bar{\mu}_L[\![u^*\in E]\!]=\sup_{n\in\mathbb{N}}\bar{\mu}_L[\![u^*\in E\cap\,]2^{-n},\infty[\,]\!]=\bar{\mu}[\![|u|\in E]\!].$$

(e) For any $\alpha>0$, $E_\alpha=\{t:h(t)>\alpha\}$ is a Borel subset of $]0,\infty[$. If $u\in M^0_{\bar{\mu}}$ then, using (d) above,

$$\bar{\mu}_L[\![\bar{h}(u^*)>\alpha]\!]=\bar{\mu}_L[\![u^*\in E_\alpha]\!]=\bar{\mu}[\![u\in E_\alpha]\!]=\bar{\mu}[\![\bar{h}(u)>\alpha]\!]=\bar{\mu}_L[\![(\bar{h}(u))^*>\alpha]\!].$$

As both $(\bar{h}(u))^*$ and $\bar{h}(u^*)$ are equivalence classes of non-increasing functions, they must be equal.

If h is continuous on the left, then $E_\alpha=\,]\gamma,\infty[$ for some γ, so we no longer need to use (d), and the argument works for any $u\in(M^{0,\infty}_{\bar{\mu}})^+$.

(f) Apply (e) with $h(\beta)=\max(0,\beta-\alpha)$.

(g) Express u^* as $g^\bullet$, where

$$g(s)=\inf\{\alpha:\bar{\mu}[\![|u|>\alpha]\!]\le s\}$$

for every $s>0$. Because g is non-increasing, it is easy to check that, for $t>0$,

$$\int_0^t g=tg(t)+\int_0^\infty\max(0,g(s)-g(t))ds\le\alpha t+\int_0^\infty\max(0,g(s)-\alpha)ds$$

for every $\alpha\ge 0$; so that

$$\int_0^t u^*=\min_{\alpha\ge 0}\alpha t+\int(u^*-\alpha\chi 1)^+.$$

Now

$$\int(u^*-\alpha\chi 1)^+=\int_0^\infty\bar\mu_L[\![(u^*-\alpha\chi 1)^+>\beta]\!]d\beta$$
$$=\int_0^\infty\bar\mu[\![(|u|-\alpha\chi 1)^+>\beta]\!]d\beta=\int(|u|-\alpha\chi 1)^+$$

for every $\alpha\ge 0$, using (f) and 365A, and

$$\int_0^t u^*=\min_{\alpha\ge 0}\alpha t+\int(|u|-\alpha\chi 1)^+.$$

(h)

$$\bar\mu[\![u_0>\alpha]\!]=\bar\mu(\sup_{u\in A}[\![u>\alpha]\!])=\sup_{u\in A}\bar\mu[\![u>\alpha]\!]$$

for any $\alpha>0$, using 364Mb and 321D. So

$$[\![u_0^*>\alpha]\!]=[0,\bar\mu[\![u_0>\alpha]\!][^\bullet=\sup_{u\in A}[0,\bar\mu[\![u>\alpha]\!][^\bullet=[\![\sup_{u\in A}u^*>\alpha]\!]$$

for every α, and $u_0^*=\sup_{u\in A}u^*$.

373E Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Then $\int|u\times v|\le\int u^*\times v^*$ for all u, $v\in M_{\bar\mu}^{0,\infty}$.

proof (a) Consider first the case u, $v\ge 0$ in $S(\mathfrak{A})$. Then we may express u, v as $\sum_{i=0}^m\alpha_i\chi a_i$, $\sum_{j=0}^n\beta_j\chi b_j$ where $a_0\supseteq a_1\supseteq\ldots\supseteq a_m$, $b_0\supseteq\ldots\supseteq b_n$ in $\mathfrak{A}$ and α_i, $\beta_j\ge 0$ for all i, j (361Ec). Now u^*, v^* are given by

$$u^*=\sum_{i=0}^m\alpha_i\chi\,[0,\bar\mu a_i[^\bullet,\quad v^*=\sum_{j=0}^n\beta_j\chi\,[0,\bar\mu b_j[^\bullet$$

(373Dc). So

$$\int u\times v=\sum_{i=0}^m\sum_{j=0}^n\alpha_i\beta_j\bar\mu(a_i\cap b_j)\le\sum_{i=0}^m\sum_{j=0}^n\alpha_i\beta_j\min(\bar\mu a_i,\bar\mu b_j)$$
$$=\sum_{i=0}^m\sum_{j=0}^n\alpha_i\beta_j\mu_L([0,\bar\mu a_i[\cap[0,\bar\mu b_j[)=\int u^*\times v^*.$$

(b) For the general case, we have non-decreasing sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ with suprema $|u|$, $|v|$ respectively (364Kd), so that

$$|u\times v|=|u|\times|v|=\sup_{n\in\mathbb{N}}|u|\times v_n=\sup_{m,n\in\mathbb{N}}u_m\times v_n=\sup_{n\in\mathbb{N}}u_n\times v_n$$

and

$$\int|u\times v|=\int\sup_{n\in\mathbb{N}}u_n\times v_n=\sup_{n\in\mathbb{N}}\int u_n\times v_n\le\sup_{n\in\mathbb{N}}\int u_n^*\times v_n^*\le\int u^*\times v^*,$$

using 373Db.

373F Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and u any member of $M_{\bar\mu}^{0,\infty}$.

(a) For any $p\in[1,\infty]$, $u\in L_{\bar\mu}^p$ iff $u^*\in L_{\bar\mu_L}^p$, and in this case $\|u\|_p=\|u^*\|_p$.

(b)(i) $u\in M_{\bar\mu}^0$ iff $u^*\in M_{\bar\mu_L}^0$;

(ii) $u\in M_{\bar\mu}^{1,\infty}$ iff $u^*\in M_{\bar\mu_L}^{1,\infty}$, and in this case $\|u\|_{1,\infty}=\|u^*\|_{1,\infty}$;

(iii) $u\in M_{\bar\mu}^{1,0}$ iff $u^*\in M_{\bar\mu_L}^{1,0}$;

(iv) $u\in M_{\bar\mu}^{\infty,1}$ iff $u^*\in M_{\bar\mu_L}^{\infty,1}$, and in this case $\|u\|_{\infty,1}=\|u^*\|_{\infty,1}$.

proof (a)(i) Consider first the case $p=1$. In this case

$$\int|u|=\int_0^\infty\bar\mu[\![|u|>\alpha]\!]d\alpha=\int_0^\infty\bar\mu_L[\![u^*>\alpha]\!]d\alpha=\int u^*.$$

(ii) If $1<p<\infty$, then by 373De we have $(|u|^p)^*=(u^*)^p$, so that

$$\|u\|_p^p=\int|u|^p=\int(|u|^p)^*=\int(u^*)^p=\|u^*\|_p^p$$

if either $\|u\|_p$ or $\|u^*\|_p$ is finite. (iii) As for $p=\infty$,

$$\|u\|_\infty \le \gamma \iff [\![|u|>\gamma]\!]=0 \iff [\![u^*>\gamma]\!]=0 \iff \|u^*\|_\infty \le \gamma.$$

(b)(i)

$$\begin{aligned}u\in M^0_{\bar\mu} &\iff \bar\mu[\![|u|>\alpha]\!]<\infty \text{ for every } \alpha>0\\ &\iff \bar\mu_L[\![u^*>\alpha]\!]<\infty \text{ for every } \alpha>0 \iff u^*\in M^0_{\bar\mu_L}.\end{aligned}$$

(ii) For any $\alpha\ge 0$,

$$\int(|u|-\alpha\chi 1)^+=\int(u^*-\alpha\chi 1)^+$$

as in the proof of 373Dg. So $\|u\|_{1,\infty}=\|u^*\|_{1,\infty}$ if either is finite, by the formula in 369Ob.

(iii) This follows from (i) and (ii), because $M^{1,0}=M^0\cap M^{1,\infty}$.

(iv) Allowing ∞ as a value of an integral, we have

$$\begin{aligned}\|u\|_{1,\infty}&=\min\{\alpha+\int(|u|-\alpha\chi 1)^+:\alpha\ge 0\}\\ &=\min\{\alpha+\int(u^*-\alpha\chi 1)^+:\alpha\ge 0\}=\|u^*\|_{1,\infty}\end{aligned}$$

by 369Ob; in particular, $u\in M^{1,\infty}_{\bar\mu}$ iff $u^*\in M^{1,\infty}_{\bar\mu_L}$.

373G Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. If

either $u\in M^{1,\infty}_{\bar\mu}$ and $T\in\mathcal{T}_{\bar\mu,\bar\nu}$

or $u\in M^{1,0}_{\bar\mu}$ and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$,

then $\int_0^t(Tu)^*\le\int_0^t u^*$ for every $t\ge 0$.

proof Set $T_1=T{\restriction}L^1_{\bar\mu}$, so that $\|T_1\|\le 1$ in $\mathrm{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, and $|T_1|$ is defined in $\mathrm{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, also with norm at most 1. If $\alpha\ge 0$, then we can express u as u_1+u_2 where $|u_1|\le(|u|-\alpha\chi 1)^+$ and $|u_2|\le\alpha\chi 1$. (Let $w\in L^\infty(\mathfrak{A})$ be such that $\|w\|_\infty\le 1$, $u=|u|\times w$; set $u_2=w\times(|u|\wedge\alpha\chi 1)$.) So if $\int(|u|-\alpha\chi 1)^+<\infty$,

$$|Tu|\le|Tu_1|+|Tu_2|\le|T_1||u_1|+\alpha\chi 1$$

and

$$\int(|Tu|-\alpha\chi 1)^+\le\int|T_1||u_1|\le\int|u_1|\le\int(|u|-\alpha\chi 1)^+.$$

The formula of 373Dg now tells us that $\int_0^t(Tu)^*\le\int_0^t u^*$ for every t.

373H Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\theta:\mathfrak{A}^f\to\mathbb{R}$ an additive functional, where $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$.

(a) The following are equiveridical:

(α) $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\theta a|=\lim_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|=0$,

(β) there is some $u\in M^{1,0}=M^{1,0}_{\bar\mu}$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and in this case u is uniquely defined.

(b) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable. Then the following are equiveridical:

(α) $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\theta a|=0$, $\quad\limsup_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|<\infty$,

(β) there is some $u\in M^{1,\infty}=M^{1,\infty}_{\bar\mu}$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and again this u is uniquely defined.

proof (a)(i) Assume (α). For a, $c\in\mathfrak{A}^f$, set $\theta_c(a)=\theta(a\cap c)$. Then for each $c\in\mathfrak{A}^f$, there is a unique $u_c\in L^1_{\bar\mu}$ such that $\theta_c a=\int_a u_c$ for every $a\in\mathfrak{A}^f$ (365Eb). Because u_c is unique we must have $u_c=u_d\times\chi c$

whenever $c \subseteq d \in \mathfrak{A}^f$. Next, given $\alpha > 0$, there is a $t_0 \ge 0$ such that $|\theta a| \le \alpha\bar{\mu}a$ whenever $a \in \mathfrak{A}^f$ and $\bar{\mu}a \ge t_0$; so that $\bar{\mu}[\![u_c > \alpha]\!] \le t_0$ for every $c \in \mathfrak{A}^f$, and $e(\alpha) = \sup_{c\in\mathfrak{A}^f} [\![u_c^+ > \alpha]\!]$ is defined in $\mathfrak{A}^f$. Of course $e(\alpha) = [\![u_{e(1)}^+ > \alpha]\!]$ for every $\alpha \ge 1$, so $\inf_{\alpha\in\mathbb{R}} e(\alpha) = 0$, and $v_1 = \sup_{c\in\mathfrak{A}^f} u_c^+$ is defined in $L^0 = L^0(\mathfrak{A})$ (364Mb). Because $[\![v_1 > \alpha]\!] = e(\alpha) \in \mathfrak{A}^f$ for each $\alpha > 0$, $v_1 \in M_{\bar{\mu}}^0$. For any $a \in \mathfrak{A}^f$,

$$v_1 \times \chi a = \sup_{c\in\mathfrak{A}^f} u_c^+ \times \chi a = u_a^+,$$

so $v_1 \in M^{1,0}$ and $\int_a v_1 = \int_a u_a^+$ for every $a \in \mathfrak{A}^f$.

Similarly, $v_2 = \sup_{c\in\mathfrak{A}^f} u_c^-$ is defined in $M^{1,0}$ and $\int_a v_2 = \int_a u_a^-$ for every $a \in \mathfrak{A}^f$. So we can set $u = v_1 - v_2 \in M^{1,0}$ and get

$$\int_a u = \int_a u_a = \theta a$$

for every $a \in \mathfrak{A}^f$. Thus (β) is true.

(ii) Assume (β). If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a(|u| - \epsilon\chi 1)^+ \le \epsilon$ whenever $\bar{\mu}a \le \delta$ (365Ea), so that $|\int_a u| \le \epsilon(1 + \bar{\mu}a)$ whenever $\bar{\mu}a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0} \sup_{\bar{\mu}a\le t} |\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar{\mu}a \le t$, $\frac{1}{t}|\int_a u| \le \epsilon + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty} \tfrac{1}{t} \sup_{\bar{\mu}a\le t} |\int_a u| \le \epsilon.$$

As ϵ is arbitrary, θ satisfies the conditions in (α).

(iii) The uniqueness of u is a consequence of 366Gd.

(b) The argument for (b) uses the same ideas.

(i) Assume (α). Again, for each $c \in \mathfrak{A}^f$, we have $u_c \in L^1$ such that $\theta_c a = \int_a u_c$ for every $a \in \mathfrak{A}^f$; again, set $e(\alpha) = \sup_{c\in\mathfrak{A}^f} [\![u_c^+ > \alpha]\!]$, which is defined because $\mathfrak{A}$ is supposed to be Dedekind complete. This time, there are t_0, $\gamma \ge 0$ such that $|\theta a| \le \gamma\bar{\mu}a$ whenever $a \in \mathfrak{A}^f$ and $\bar{\mu}a \ge t_0$; so that $\bar{\mu}[\![u_c > \gamma]\!] \le t_0$ for every $c \in \mathfrak{A}^f$, and $\bar{\mu}e(\gamma) < \infty$. Accordingly

$$\inf_{\alpha\ge\gamma} e(\alpha) = \inf_{\alpha\ge\gamma} [\![u_{e(\gamma)}^+ > \alpha]\!] = 0,$$

and once more $v_1 = \sup_{c\in\mathfrak{A}^f} u_c^+$ is defined in $L^0 = L^0(\mathfrak{A})$. As before, $v_1 \times \chi a = u_a^+ \in L^1$ for any $a \in \mathfrak{A}^f$, Because $[\![v_1 > \gamma]\!] = e(\gamma) \in \mathfrak{A}^f$, $v_1 \in M^{1,\infty}$. Similarly, $v_2 = \sup_{c\in\mathfrak{A}^f} u_c^-$ is defined in $M^{1,\infty}$, with $v_2\times\chi a = u_a^-$ for every $a \in \mathfrak{A}^f$. So $u = v_1 - v_2 \in M^{1,\infty}$, and

$$\int_a u = \int_a u_a = \theta a$$

for every $a \in \mathfrak{A}^f$.

(ii) Assume (β). Take $\gamma \ge 0$ such that $\beta = \int(|u| - \gamma\chi 1)^+$ is finite. If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a(|u| - \gamma\chi 1)^+ \le \epsilon$ whenever $\bar{\mu}a \le \delta$, so that $|\int_a u| \le \epsilon + \gamma\bar{\mu}a$ whenever $\bar{\mu}a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0} \sup_{\bar{\mu}a\le t} |\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar{\mu}a \le t$, then $\frac{1}{t}|\int_a u| \le \gamma + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty} \tfrac{1}{t} \sup_{\bar{\mu}a\le t} |\int_a u| \le \gamma < \infty,$$

and the function $a \mapsto \int_a u$ satisfies the conditions in (β).

(iii) u is uniquely defined because $u \times \chi a$ must be u_a, as defined in (i), for every $a \in \mathfrak{A}^f$, and $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

373I Lemma Suppose that u, v, $w \in M_{\bar{\mu}_L}^{0,\infty}$ are all equivalence classes of non-negative non-increasing functions. If $\int_0^t u \le \int_0^t v$ for every $t \ge 0$, then $\int u \times w \le \int v \times w$.

proof For $n \in \mathbb{N}$, $i \le 4^n$ set $a_{ni} = [\![w > 2^{-n}i]\!]$; set $w_n = \sum_{i=1}^{4^n} 2^{-n}\chi a_{ni}$. Then each a_{ni} is of the form $[0,t]^\bullet$, so

$$\int u \times w_n = \sum_{i=1}^{4^n} 2^{-n} \int_{a_{ni}} u \le \sum_{i=1}^{4^n} 2^{-n} \int_{a_{ni}} v = \int v \times w_n.$$

Also $\langle w_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum w, so

$$\int u \times w = \sup_{n\in\mathbb{N}} \int u \times w_n \le \sup_{n\in\mathbb{N}} \int v \times w_n = \int v \times w.$$

373J Corollary Suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras and $v\in M^{0,\infty}_{\bar\nu}$. If

either $u\in M^{1,0}_{\bar\mu}$ and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$

or $u\in M^{1,\infty}_{\bar\mu}$ and $T\in\mathcal{T}_{\bar\mu,\bar\nu}$

then $\int|Tu\times v|\le\int u^*\times v^*$.

proof Put 373E, 373G and 373I together.

373K The very weak operator topology of $\mathcal{T}$ Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be two measure algebras. For $u\in M^{1,\infty}_{\bar\mu}$, $w\in M^{\infty,1}_{\bar\nu}$ set

$$\rho_{uw}(S,T)=|\int Su\times w-\int Tu\times w|\text{ for all }S,\,T\in\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\nu}.$$

Then ρ_{uw} is a pseudometric on $\mathcal{T}$. I will call the topology generated by $\{\rho_{uw}:u\in M^{1,\infty}_{\bar\mu},\,w\in M^{\infty,1}_{\bar\nu}\}$ (2A3F) the **very weak operator topology** on $\mathcal{T}$.

373L Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra. Then $\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\nu}$ is compact in its very weak operator topology.

proof Let $\mathcal{F}$ be an ultrafilter on $\mathcal{T}$. If $u\in M^{1,\infty}_{\bar\mu}$, $w\in M^{\infty,1}_{\bar\nu}$ then

$$|\int Tu\times w|\le\int u^*\times w^*<\infty$$

for every $T\in\mathcal{T}$ (373J); $\int u^*\times w^*$ is finite because $u^*\in M^{1,\infty}$ and $w^*\in M^{\infty,1}$ (373F).

In particular, $\{\int Tu\times w:T\in\mathcal{T}\}$ is bounded. Consequently $h_u(w)=\lim_{T\to\mathcal{F}}\int Tu\times w$ is defined in $\mathbb{R}$ (2A3Se). Because $w\mapsto\int Tu\times w$ is additive for every $T\in\mathcal{T}$, so is h_u. Also

$$|h_u(w)|\le\int u^*\times w^*\le\|u^*\|_{1,\infty}\|w^*\|_{\infty,1}=\|u\|_{1,\infty}\|w\|_{\infty,1}$$

for every $w\in M^{\infty,1}_{\bar\nu}$.

$|h_u(\chi b)|\le\int_0^t u^*$ whenever $b\in\mathfrak{B}^f$ and $\bar\nu b\le t$. So

$$\lim_{t\downarrow 0}\sup_{\bar\nu b\le t}|h_u(\chi b)|\le\lim_{t\downarrow 0}\int_0^t u^*=0,$$

$$\limsup_{t\to\infty}\tfrac{1}{t}\sup_{\bar\nu b\le t}|h_u(\chi b)|\le\limsup_{t\to\infty}\tfrac{1}{t}\int_0^t u^*<\infty.$$

Of course $b\mapsto h_u(\chi b)$ is additive, so by 373H there is a unique $Su\in M^{1,\infty}_{\bar\nu}$ such that $h_u(\chi b)=\int_b Su$ for every $b\in\mathfrak{B}^f$. Since both h_u and $w\mapsto\int Su\times w$ are linear and continuous on $M^{\infty,1}_{\bar\nu}$, and $S(\mathfrak{B}^f)$ is dense in $M^{\infty,1}_{\bar\nu}$ (369Od),

$$\int Su\times w=h_u(w)=\lim_{T\to\mathcal{F}}\int Tu\times w$$

for every $w\in M^{\infty,1}_{\bar\nu}$. And this is true for every $u\in M^{1,\infty}_{\bar\mu}$.

For any particular $w\in M^{\infty,1}_{\bar\nu}$, all the maps $u\mapsto\int Tu\times w$ are linear, so $u\mapsto\int Su\times w$ also is; that is, $S:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ is linear.

Now $S\in\mathcal{T}$. $\mathbf{P}$ (α) If $u\in L^1_{\bar\mu}$ and $b,\,c\in\mathfrak{B}^f$, then

$$\begin{aligned}\int_b Su-\int_c Su&=\lim_{T\to\mathcal{F}}\int Tu\times(\chi b-\chi c)\le\sup_{T\in\mathcal{T}}\int Tu\times(\chi b-\chi c)\\&\le\sup_{T\in\mathcal{T}}\|Tu\|_1\|\chi b-\chi c\|_\infty\le\|u\|_1.\end{aligned}$$

But, setting $e=[\![Su>0]\!]$, we have

$$\begin{aligned}\int|Su|&=\int_e Su-\int_{1\setminus e}Su\\&=\sup_{b\in\mathfrak{B}^f,b\subseteq e}\int_b Su+\sup_{c\in\mathfrak{B}^f,c\subseteq 1\setminus e}(-Su)\le\|u\|_1.\end{aligned}$$

(β) If $u \in L^\infty(\mathfrak{A})$, then

$$|\int_b Su| \le \sup_{T\in\mathcal{T}} |\int Tu \times \chi b| \le \sup_{T\in\mathcal{T}} \|Tu\|_\infty \bar\nu b \le \|u\|_\infty \bar\nu b$$

for every $b \in \mathfrak{B}^f$. So $[\![Su > \|u\|_\infty]\!] = [\![-Su > \|u\|_\infty]\!] = 0$ and $\|Su\|_\infty \le \|u\|_\infty$. (Note that both parts of this argument depend on knowing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so that we cannot be troubled by purely infinite elements of $\mathfrak{B}$.) **Q**

Of course we now have $\lim_{T\to\mathcal{F}} \rho_{uw}(T,S) = 0$ for all $u \in M^{1,0}_{\bar\mu}$, $w \in M^{\infty,1}_{\bar\nu}$, so that $S = \lim\mathcal{F}$ in $\mathcal{T}$. As $\mathcal{F}$ is arbitrary, $\mathcal{T}$ is compact (2A3R).

373M Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and u any member of $M^{1,\infty}_{\bar\mu}$. Then $B = \{Tu : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ is compact in $M^{1,\infty}_{\bar\nu}$ for the topology $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$.

proof The point is just that the map $T \mapsto Tu : \mathcal{T}_{\bar\mu,\bar\nu} \to M^{1,0}_{\bar\nu}$ is continuous for the very weak operator topology on $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. So B is a continuous image of a compact set, therefore compact (2A3Nb).

373N Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra and u any member of $M^{1,\infty}_{\bar\mu}$; set $B = \{Tu : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in B, then $\sup_{n\in\mathbb{N}} v_n$ is defined in $M^{1,\infty}_{\bar\nu}$ and belongs to B.

proof By 373M, $\langle v_n\rangle_{n\in\mathbb{N}}$ must have a cluster point $v \in B$ for $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. Now for any $b \in \mathfrak{B}^f$, $\int_b v$ must be a cluster point of $\langle\int_n v_n\rangle_{n\in\mathbb{N}}$, because $w \mapsto \int_b w$ is continuous for $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. But $\langle\int_b v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence, so its only possible cluster point is its supremum; thus $\int_b v = \lim_{n\to\infty}\int_b v_n$. Consequently $v \times \chi b$ must be the supremum of $\{v_n \times \chi b : n \in \mathbb{N}\}$ in L^1. And this is true for every $b \in \mathfrak{B}^f$; as $(\mathfrak{B},\bar\nu)$ is semi-finite, v is the supremum of $\langle v_n\rangle_{n\in\mathbb{N}}$ in $L^0(\mathfrak{B})$ and in $M^{1,\infty}_{\bar\nu}$.

373O Theorem Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be measure algebras and $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$. Then the following are equiveridical:

(i) there is a $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu = v$,

(ii) $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$.

In particular, given $u \in M^{1,\infty}_{\bar\mu}$, there are $S \in \mathcal{T}_{\bar\mu,\bar\mu_L}$, $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ such that $Su = u^*$, $Tu^* = u$.

proof (i)$\Rightarrow$(ii) is Lemma 373G. Accordingly I shall devote the rest of the proof to showing that (ii)$\Rightarrow$(i).

(a) If $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ are measure algebras and $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$, I will say that $v \preccurlyeq u$ if there is a $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu = v$, and that $v \sim u$ if $v \preccurlyeq u$ and $u \preccurlyeq v$. (Properly speaking, I ought to write $(u,\bar\mu) \preccurlyeq (v,\bar\nu)$, because we could in principle have two different measures on the same algebra. But I do not think any confusion is likely to arise in the argument which follows.) By 373Be, $\preccurlyeq$ is transitive and $\sim$ is an equivalence relation. Now we have the following facts.

(b) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and u_1, $u_2 \in M^{1,\infty}_{\bar\mu}$ are such that $|u_1| \le |u_2|$, then $u_1 \preccurlyeq u_2$. **P** There is a $w \in L^\infty(\mathfrak{A})$ such that $u_1 = w \times u_2$ and $\|w\|_\infty \le 1$. Set $Tv = w \times v$ for for $v \in M^{1,\infty}_{\bar\mu}$; then $T \in \mathcal{T}_{\bar\mu,\bar\mu}$ and $Tu_2 = u_1$. **Q** So $u \sim |u|$ for every $u \in M^{1,\infty}_{\bar\mu}$.

(c) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $S(\mathfrak{A})$, then $u \preccurlyeq u^*$. **P** If $u = 0$ this is trivial. Otherwise, express u as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0,\dots,a_n$ are disjoint and non-zero and $\alpha_0 > \alpha_1 \ldots > \alpha_n > 0 \in \mathbb{R}$. If $\bar\mu a_i = \infty$ for any i, take m to be minimal subject to $\bar\mu a_m = \infty$; otherwise, set $m = n$. Then $u^* = \sum_{i=0}^m \alpha_i\chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_0 = 0$, $\beta_j = \sum_{i=0}^{j-1}\bar\mu a_i$ for $1 \le j \le m+1$.

For $i < m$, and for $i = m$ if $\bar\mu a_m < \infty$, define $h_i : M^{1,\infty}_{\bar\mu} \to \mathbb{R}$ by setting

$$h_i(v) = \frac{1}{\bar\mu a_i}\int_{a_i} v$$

for every $v \in M^{1,\infty}_{\bar\mu}$. If $\bar\mu a_m = \infty$, then we need a different idea to define h_m, as follows. Let I be $\{a : a \in \mathfrak{A},\ \bar\mu(a \cap a_m) < \infty\}$. Then I is an ideal of $\mathfrak{A}$ not containing a_m, so there is a Boolean homomorphism $\pi : \mathfrak{A} \to \{0,1\}$ such that $\pi a = 0$ for $a \in I$ and $\pi a_m = 1$ (311D). This induces a corresponding $\|\ \|_\infty$-continuous linear operator $h : L^\infty(\mathfrak{A}) \to L^\infty(\{0,1\}) \cong \mathbb{R}$, as in 363F. Now $h(\chi a) = 0$ whenever $\bar\mu a < \infty$, and accordingly $h(v) = 0$ whenever $v \in M^{\infty,1}_{\bar\mu}$, since $S(\mathfrak{A}^f)$ is dense in $M^{\infty,1}_{\bar\mu}$ for $\| \|_{\infty,1}$ and therefore also for $\| \|_\infty$. But this means that h has a unique extension to a linear functional $h_m : M^{1,\infty}_{\bar\mu} \to \mathbb{R}$ such that $h_m(v) = 0$ for every $v \in L^1_{\bar\mu}$, while $h_m(\chi a_m) = 1$ and $|h(v)| \le \|v\|_\infty$ for every $v \in L^\infty(\mathfrak{A})$.

Having defined h_i for every $i \le m$, define $T : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\mu_L}$ by setting

$$Tv = \sum_{i=0}^m h_i(v)\chi\,[\beta_i, \beta_{i+1}[^\bullet$$

for every $v \in M^{1,\infty}_{\bar\mu}$.

For any $i \le m$, $v \in L^1_{\bar\mu}$,

$$\int_{\beta_i}^{\beta_{i+1}} |Tv| = |h_i(v)|\bar\mu a_i \le \int_{a_i} |v|;$$

summing over i, $\|Tv\|_1 \le \|v\|_1$. Similarly, for any $i \le m$, $v \in L^\infty(\mathfrak{B})$, $|h_i(v)| \le \|v\|_\infty$, so $\|Tv\|_\infty \le \|v\|_\infty$.

Thus $T \in \mathcal{T}_{\bar\mu,\bar\mu_L}$. Since $u^* = Tu$, we conclude that $u^* \preccurlyeq u$, as claimed. **Q**

(d) If $(\mathfrak{A}, \bar\mu)$ is a measure algebra and $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u^* \preccurlyeq u$. **P** Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$. Then $\langle u_n^* \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $M^{1,\infty}_{\bar\mu_L}$ with supremum u^*, by 373Db and 373Dh. Also $u_n^* \preccurlyeq u_n \preccurlyeq u$ for every n, by (b) and (c) of this proof. By 373N, $u^* \preccurlyeq u$. **Q**

(e) If $(\mathfrak{A}, \bar\mu)$ is a measure algebra and $u \ge 0$ in $S(\mathfrak{A})$, then $u \preccurlyeq u^*$. **P** The argument is very similar to that of (c). Again, the result is trivial if $u = 0$; suppose that $u > 0$ and define α_i, a_i, m, β_i as before. This time, set $a'_i = a_i$ for $i < m$, $a'_m = \sup_{m\le j\le n} a_j$, $\tilde u = \sum_{i=0}^m \alpha_i \chi a'_i$; then $u \le \tilde u$ amd $\tilde u^* = u^*$. Set

$$h_i(v) = \frac{1}{\beta_{i+1}-\beta_i} \int_{\beta_i}^{\beta_{i+1}} v$$

if $i \le m$, $\beta_{i+1} < \infty$ (that is, $\bar\mu a_i < \infty$) and $v \in M^{1,\infty}_{\bar\mu_L}$; and if $\bar\mu a_m = \infty$, set

$$h_m(v) = \lim_{k\to\mathcal{F}} \frac{1}{k} \int_0^k v$$

for some non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. As before, we have

$$|h_i(v)|\bar\mu a'_i \le \int_{\beta_i}^{\beta_{i+1}} |v|,$$

whenever $v \in L^1_{\bar\mu_L}$, $i \le m$, while $|h_i(v)| \le \|v\|_\infty$ whenever $v \in L^\infty(\mathfrak{A}_L)$ and $i \le m$. So we can define $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ by setting $Tv = \sum_{i=0}^m h_i(v)\chi a'_i$ for every $v \in M^{1,\infty}_{\bar\mu_L}$, and get

$$u \preccurlyeq \tilde u = Tu^* \preccurlyeq u^*. \ \mathbf{Q}$$

(f) If $(\mathfrak{A}, \bar\mu)$ is a measure algebra and $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u \preccurlyeq u^*$. **P** This time I seek to copy the ideas of (d); there is a new obstacle to circumvent, since $(\mathfrak{A}, \bar\mu)$ might not be localizable. Set

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0,\ \bar\mu[\![u > \alpha]\!] < \infty\}, \quad e = [\![u > \alpha_0]\!].$$

Then $e = \sup_{n\in\mathbb{N}} [\![u > \alpha_0 + 2^{-n}]\!]$ is a countable supremum of elements of finite measure, so the principal ideal $\mathfrak{A}_e$, with its induced measure $\bar\mu_e$, is σ-finite. Now let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$; set $\tilde u = u \times \chi e$ and $\tilde u_n = u_n \times \chi e$, regarded as members of $S(\mathfrak{A}_e)$, for each n. In this case

$$\tilde u_n \preccurlyeq \tilde u_n^* \preccurlyeq u^*$$

for every n. Because $(\mathfrak{A}_e, \bar\mu_e)$ is σ-finite, therefore localizable, 373N tells us that $\tilde u \preccurlyeq u^*$.

Let $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_e}$ be such that $Su^* = \tilde u$. As in part (e), choose a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and set

$$h(v) = \lim_{k\to\mathcal{F}} \frac{1}{k} \int_0^k v$$

for $v \in M^{1,\infty}_{\bar\mu_L}$. Now define $T : M^{1,\infty}_{\bar\mu_L} \to M^{1,\infty}_{\bar\mu}$ by setting

$$Tv = Sv + h(v)\chi(1 \setminus e),$$

here regarding Sv as a member of $M^{1,\infty}_{\bar\mu}$. (I am taking it to be obvious that $M^{1,\infty}_{\bar\mu_e}$ can be identified with $\{w\times\chi e: w\in M^{1,\infty}_{\bar\mu}\}$.) Then it is easy to see that $T\in\mathcal{T}_{\bar\mu_L,\bar\mu}$. Also $u\le Tu^*$, because

$$h(u^*)=\inf\{\alpha:\bar\mu_L[\![u^*>\alpha]\!]<\infty\}=\alpha_0,$$

while $u\times\chi(1\setminus e)\le\alpha_0\chi(1\setminus e)$. So we get $u\preccurlyeq Tu^*\preccurlyeq u^*$. **Q**

(g) Now suppose that u, $v\ge 0$ in $M^{1,\infty}_{\bar\mu_L}$, that $\int_0^t u^*\ge\int_0^t v^*$ for every $t\ge 0$, and that v is of the form $\sum_{i=1}^n\alpha_i\chi a_i$ where $\alpha_1>\ldots>\alpha_n>0$, $a_1,\ldots,a_n\in\mathfrak{A}_L$ are disjoint and $\bar\mu_L a_i<\infty$ for each i. Then $v\preccurlyeq u$. **P** Induce on n. If $n=0$ then $v=0$ and the result is trivial. For the inductive step to $n\ge 1$, if $v^*\le u^*$ we have

$$v\sim v^*\preccurlyeq u^*\sim u,$$

using (b), (d) and (f) above. Otherwise, look at $\phi(t)=\frac{1}{t}\int_0^t u^*$ for $t>0$. We have

$$\phi(t)\ge\frac{1}{t}\int_0^t v^*=\alpha_1$$

for $t\le\beta=\bar\mu a_1$, while $\lim_{t\to\infty}\phi(t)<\alpha_1$, because $(\lim_{t\to\infty}\phi(t))\chi 1\le u^*$ and $v^*\le\alpha_1\chi 1$ and $v^*\not\le u^*$. Becaasue ϕ is continuoue, there is a $\gamma\ge\beta$ such that $\phi(\gamma)=\alpha_1$. Define $T_0\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$T_0w=(\frac{1}{\gamma}\int_0^\gamma w)\chi\,[0,\gamma[^\bullet+(w\times\chi\,[\gamma,\infty[^\bullet)$$

for every $w\in M^{1,\infty}_{\bar\mu_L}$. Then $T_0u^*\preccurlyeq u^*\sim u$, and

$$T_0u^*\times\chi\,[0,\gamma[^\bullet=(\frac{1}{\gamma}\int_0^\gamma u^*)\chi\,[0,\gamma[^\bullet=\alpha_1\chi\,[0,\gamma[^\bullet.$$

We need to know that $\int_0^t T_0u^*\ge\int_0^t v^*$ for every t; this is because

$$\begin{aligned}\int_0^t T_0u^*&=\alpha_1 t\ge\int_0^t v^*\text{ whenever }t\le\gamma,\\&=\int_0^\gamma T_0u^*+\int_\gamma^t T_0u^*=\int_0^t u^*\ge\int_0^t v^*\text{ whenever }t\ge\gamma.\end{aligned}$$

Set

$$u_1=T_0u^*\times\chi\,[\beta,\infty[^\bullet,\quad v_1=v^*\times\chi\,[\beta,\infty[^\bullet.$$

Then u_1^*, v_1^* are just translations of T_0u^*, v^* to the left, so that

$$\int_0^t u_1^*=\int_\beta^{\beta+t}T_0u^*=\int_0^{\beta+t}T_0u^*-\alpha_1\beta\ge\int_0^{\beta+t}v^*-\alpha_1\beta=\int_\beta^{\beta+t}v^*=\int_0^t v_1^*$$

for every $t\ge 0$. Also $v_1=\sum_{i=2}^n\alpha_i\chi\,[\beta_{i-1},\beta_i[^\bullet$ where $\beta_i=\sum_{j=1}^i\bar\mu a_j$ for each j. So by the inductive hypothesis, $v_1\preccurlyeq u_1$.

Let $S\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ be such that $Su_1=v_1$, and define $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$Tw=w\times\chi\,[0,\beta[^\bullet+S(w\times\chi\,[\beta,\infty[^\bullet)\times\chi\,[\beta,\infty[^\bullet$$

for every $w\in M^{1,\infty}_{\bar\mu_L,\bar\mu_L}$. Then $TT_0u^*=v^*$, so $v\sim v^*\preccurlyeq u^*\sim u$, as required. **Q**

(h) We are nearly home. If u, $v\ge 0$ in $M^{1,\infty}_{\bar\mu_L}$ and $\int_0^t v^*\le\int_0^t u^*$ for every $t\ge 0$, then $v\preccurlyeq u$. **P** Let $\langle v_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}_L^f)^+$ with supremum v. Then $v_n^*\le v^*$ for each n, so (g) tells us that $v_n\preccurlyeq u$ for every n. By 373N, for the last time, $v\preccurlyeq u$. **Q**

(i) Finally, suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are arbitrary measure algebras and that $u\in M^{1,\infty}_{\bar\mu}$, $v\in M^{1,\infty}_{\bar\nu}$ are such that $\int_0^t v^*\le\int_0^t u^*$ for every $t\ge 0$. By (b), $v\preccurlyeq|v|$; by (f), $|v|\preccurlyeq|v|^*$; by 373Db, $|v|^*=v^*$; by (h) of this proof, $v^*\preccurlyeq u^*$; by (d), $u^*=|u|^*\preccurlyeq|u|$; and by (b) again, $|u|\preccurlyeq u$.

373P Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a semi-finite measure algebra. Then for any $u\in M^{1,\infty}_{\bar\mu}$ and $v\in M^0_{\bar\nu}$, there is a $T\in\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu\times v=\int u^*\times v^*$.

proof (a) It is convenient to dispose immediately of some elementary questions.

(i) We need only find a $T \in \mathcal{T}$ such that $\int |Tu \times v| \ge \int u^* \times v^*$. **P** Take $v_0 \in L^\infty(\mathfrak{B})$ such that $|Tu \times v| = v_0 \times Tu \times v$ and $\|v_0\|_\infty \le 1$, and set $T_1 w = v_0 \times Tw$ for $w \in M^{1,\infty}_{\bar\mu}$; then $T_1 \in \mathcal{T}$ and

$$\int T_1 u \times v = \int |Tu \times v| \ge \int u^* \times v^* \ge \int T_1 u \times v$$

by 373J. **Q**

(ii) Consequently it will be enough to consider $v \ge 0$, since of course $\int |Tu \times v| = \int |Tu \times |v||$, while $|v|^* = v^*$.

(iii) It will be enough to consider $u = u^*$. **P** If we can find $T \in \mathcal{T}_{\bar\mu_L,\bar\nu}$ such that $\int Tu^* \times v = \int (u^*)^* \times v^*$, then we know from 373O that there is an $S \in \mathcal{T}_{\bar\mu,\bar\mu_L}$ such that $Su = u^*$, so that $TS \in \mathcal{T}$ and

$$\int TSu \times v = \int (u^*)^* \times v^* = \int u^* \times v^*. \text{ **Q**}$$

(iv) It will be enough to consider localizable $(\mathfrak{B}, \bar\nu)$. **P** Assuming that $v \ge 0$, following (ii) above, set $e = [\![v > 0]\!] = \sup_{n\in\mathbb{N}} [\![v > 2^{-n}]\!]$, and let $\bar\nu_e$ be the restriction of $\bar\nu$ to the principal ideal $\mathfrak{B}_e$ generated by e. Then if we write $\tilde v$ for the member of $L^0(\mathfrak{B}_e)$ corresponding to v (so that $[\![\tilde v > \alpha]\!] = [\![v > \alpha]\!]$ for every $\alpha > 0$), $\tilde v^* = v^*$. Also $(\mathfrak{B}_e, \bar\nu_e)$ is σ-finite, therefore localizable. Now if we can find $T \in \mathcal{T}_{\bar\mu,\bar\nu_e}$ such that $\int Tu \times \tilde v = \int u^* \times \tilde v^*$, then ST will belong to $\mathcal{T}_{\bar\mu,\bar\nu}$, where $S : L^0(\mathfrak{B}_e) \to L^0(\mathfrak{B})$ is the canonical embedding defined by the formula

$$[\![Sw > \alpha]\!] = [\![w > \alpha]\!] \text{ if } \alpha \ge 0,$$
$$= [\![w > \alpha]\!] \cup (1 \setminus e) \text{ if } \alpha < 0,$$

and

$$\int STu \times v = \int Tu \times \tilde v = \int u^* \times \tilde v^* = \int u^* \times v^*. \text{ **Q**}$$

(b) So let us suppose henceforth that $\bar\mu = \bar\mu_L$, $u = u^*$ is the equivalence class of a non-increasing non-negative function, $v \ge 0$ and $(\mathfrak{B}, \bar\nu)$ is localizable.

For n, $i \in \mathbb{N}$ set

$$b_{ni} = [\![v > 2^{-n}i]\!], \quad \beta_{ni} = \bar\nu b_{ni}, \quad c_{ni} = b_{ni} \setminus b_{n,i+1}, \quad \gamma_{ni} = \bar\nu c_{ni} = \beta_{ni} - \beta_{n,i+1}$$

(because $\beta_{ni} < \infty$ if $i > 0$; this is really where I use the hypothesis that $v \in M^0$). For $n \in \mathbb{N}$ set

$$K_n = \{i : i \ge 1, \gamma_{ni} > 0\},$$

$$T_n w = \sum_{i \in K_n} \Bigl(\frac{1}{\gamma_{ni}} \int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr) \chi c_{ni}$$

for $w \in M^{1,\infty}_{\bar\mu_L}$; this is defined in $L^0(\mathfrak{B})$ because K_n is countable and $\langle c_{ni}\rangle_{i\in\mathbb{N}}$ is disjoint. Of course $T_n : M^{1,\infty}_{\bar\mu_L} \to L^0(\mathfrak{B})$ is linear. If $w \in L^\infty(\mathfrak{A}_L)$ then

$$\|T_n w\|_\infty = \sup_{i\in K_n} \Bigl|\frac{1}{\gamma_{ni}} \int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \le \|w\|_\infty,$$

and if $w \in L^1_{\bar\mu_L}$ then

$$\|T_n w\|_1 = \sum_{i\in K_n} \Bigl|\frac{1}{\gamma_{ni}} \int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \bar\nu c_{ni} = \sum_{i\in K_n} \Bigl|\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \le \|w\|_1;$$

so $T_n w \in M^{1,\infty}_{\bar\nu}$ for every $w \in M^{1,\infty}_{\bar\mu_L}$, and $T_n \in \mathcal{T}$. It will be helpful to observe that

$$\int_{c_{ni}} T_n w = \int_{\beta_{n,i+1}}^{\beta_{ni}} w$$

whenever $i \ge 1$, since if $i \notin K_n$ then both sides are 0.

Note next that for every n, $i \in \mathbb{N}$,

$$b_{ni} = b_{n+1,2i}, \quad \beta_{ni} = \beta_{n+1,2i}, \quad c_{ni} = c_{n+1,2i} \cup c_{n+1,2i+1}, \quad \gamma_{ni} = \gamma_{n+1,2i} + \gamma_{n+1,2i+1},$$

so that, for $i \ge 1$,

$$\int_{c_{ni}} T_n u = \int_{\beta_{n,i+1}}^{\beta_{ni}} u = \int_{c_{ni}} T_{n+1} u.$$

This means that if T is any cluster point of $\langle T_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ for the very weak operator topology (and such a cluster point exists, by 373L), $\int_{c_{mi}} Tu$ must be a cluster point of $\langle \int_{c_{mi}} T_n u \rangle_{n\in\mathbb{N}}$, and therefore equal to $\int_{c_{mi}} T_m u$, for every $m \in \mathbb{N}$, $i \ge 1$.

Consequently, if $m \in \mathbb{N}$,

$$\int |Tu \times v| \ge \sum_{i=0}^{\infty} \int_{c_{mi}} |Tu| \times v \ge \sum_{i=0}^{\infty} 2^{-m} i \int_{c_{mi}} |Tu|$$

(because $c_{mi} \subseteq [\![v > 2^{-m} i]\!]$)

$$\ge \sum_{i=1}^{\infty} 2^{-m} i |\int_{c_{mi}} Tu| = \sum_{i=1}^{\infty} 2^{-m} i \int_{c_{mi}} T_m u$$

$$= \sum_{i=0}^{\infty} 2^{-m} i \int_{\beta_{m,i+1}}^{\beta_{mi}} u \ge \int u \times (v^* - 2^{-m}\chi 1)^+$$

because

$$[\beta_{m,i+1}, \beta_{mi}]^\bullet \subseteq [\![v^* \le 2^{-m}(i+1)]\!] = [\![(v^* - 2^{-m}\chi 1)^+ \le 2^{-m} i]\!]$$

for each $i \in \mathbb{N}$. But letting $m \to \infty$, we have

$$\int |Tu \times v| \ge \lim_{m\to\infty} \int u \times (v^* - 2^{-m}\chi 1)^+ = \int u \times v^*$$

because $\langle u \times (v^* - 2^{-m}\chi 1)^+ \rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence with supremum $u \times v^*$. In view of the reductions in (a) above, this is enough to complete the proof.

373Q Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, $(\mathfrak{B}, \bar\nu)$ a semi-finite measure algebra, $u \in M^{1,\infty}_{\bar\mu}$ and $v \in M^{0,\infty}_{\bar\nu}$. Then

$$\int u^* \times v^* = \sup\{\int |Tu \times v| : T \in \mathcal{T}_{\bar\mu,\bar\nu}\} = \sup\{\int Tu \times v : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}.$$

proof There is a non-decreasing sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar\nu c_n < \infty$ for every n and $v^* = \sup_{n\in\mathbb{N}} (v \times \chi c_n)^*$. **P** For each rational $q > 0$, we can find a countable non-empty set $B_q \subseteq \mathfrak{B}$ such that

$$b \subseteq [\![|v| > q]\!],\ \bar\nu b < \infty \text{ for every } b \in B_q,$$

$$\sup_{b\in B_q} \bar\nu b = \bar\nu [\![|v| > q]\!]$$

(because $(\mathfrak{B}, \bar\nu)$ is semi-finite). Let $\langle b_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{q\in\mathbb{Q}, q>0} B_q$ and set $c_n = \sup_{i\le n} b_i$, $v_n = v \times \chi c_n$ for each n. Then $\langle |v_n| \rangle_{n\in\mathbb{N}}$ and $\langle v_n^* \rangle_{n\in\mathbb{N}}$ are non-decreasing and $\sup_{n\in\mathbb{N}} v_n^* \le v^*$ in $L^0(\mathfrak{A}_L)$. But in fact $\sup_{n\in\mathbb{N}} v_n^* = v^*$, because

$$\bar\mu_L [\![v^* > q]\!] = \bar\mu [\![|v| > q]\!] = \sup_{n\in\mathbb{N}} \bar\mu [\![v_n > q]\!] = \sup_{n\in\mathbb{N}} \bar\mu_L [\![v_n^* > q]\!] = \bar\mu_L [\![\sup_{n\in\mathbb{N}} v_n^* > q]\!]$$

for every rational $q > 0$, by 373Da. **Q**

For each $n \in \mathbb{N}$ we have a $T_n \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int T_n u \times v_n = \int u^* \times v_n^*$ (373P). Set $S_n w = T_n w \times \chi c_n$ for $n \in \mathbb{N}$, $w \in M^{1,\infty}_{\bar\mu}$; then every S_n belongs to $\mathcal{T}_{\bar\mu,\bar\nu}$, so

$$\sup\{\int Tu \times v : T \in \mathcal{T}_{\bar\mu,\bar\mu}\} \ge \sup_{n\in\mathbb{N}} \int S_n u \times v = \sup_{n\in\mathbb{N}} \int T_n u \times v_n$$

$$= \sup_{n\in\mathbb{N}} \int u^* \times v_n^* = \int u^* \times v^*$$

$$\ge \sup\{\int |Tu \times v| : T \in \mathcal{T}_{\bar\mu,\bar\mu}\} \ge \sup\{\int Tu \times v : T \in \mathcal{T}_{\bar\mu,\bar\mu}\}$$

by 373J, as usual.

373R Order-continuous operators: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, $(\mathfrak{B},\bar{\nu})$ a localizable measure algebra, and $T_0 \in \mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$. Then there is a $T \in \mathcal{T}^{\times} = \mathcal{T}^{\times}_{\bar{\mu},\bar{\nu}}$ extending T_0. If $(\mathfrak{A},\bar{\mu})$ is semi-finite, T is uniquely defined.

proof (a) Suppose first that $T_0 \in \mathcal{T}^{(0)}$ is non-negative, regarded as a member of $\mathsf{L}^{\sim}(M^{1,0}_{\bar{\mu}}; M^{1,0}_{\bar{\nu}})$. In this case T_0 has an extension to an order-continuous positive linear operator $T : M^{1,\infty}_{\bar{\mu}} \to L^0(\mathfrak{B})$ defined by saying that $Tw = \sup\{T_0 u : u \in M^{1,0}_{\bar{\mu}},\ 0 \le u \le w\}$ for every $w \ge 0$ in $M^{1,\infty}_{\bar{\mu}}$. **P** I use 355F. $M^{1,0}_{\bar{\mu}}$ is a solid linear subspace of $M^{1,\infty}_{\bar{\mu}}$. T_0 is order-continuous when its codomain is taken to be $M^{1,0}_{\bar{\nu}}$, as noted in 371Gb, and therefore if its codomain is taken to be $L^0(\mathfrak{B})$, because $M^{1,0}$ is a solid linear subspace in L^0, so the embedding is order-continuous. If $w \ge 0$ in $M^{1,\infty}_{\bar{\mu}}$, let $\gamma \ge 0$ be such that $u_1 = (w - \gamma\chi 1)^+$ is integrable. If $u \in M^{1,0}_{\bar{\mu}}$ and $0 \le u \le w$, then $(u - \gamma\chi 1)^+ \le u_1$, so

$$T_0 u = T_0(u - \gamma\chi 1)^+ + T_0(u \wedge \gamma\chi 1) \le T_0 u_1 + \gamma\chi 1 \in L^0(\mathfrak{B}).$$

Thus $\{T_0 u : u \in M^{1,0}_{\bar{\nu}},\ 0 \le u \le w\}$ is bounded above in $L^0(\mathfrak{B})$, for any $w \ge 0$ in $M^{1,\infty}_{\bar{\mu}}$. $L^0(\mathfrak{B})$ is Dedekind complete, because $(\mathfrak{B},\bar{\nu})$ is localizable, so $\sup\{T_0 u : 0 \le u \le w\}$ is defined in $L^0(\mathfrak{B})$; and this is true for every $w \in (M^{1,\infty}_{\bar{\mu}})^+$. Thus the conditions of 355F are satisfied and we have the result. **Q**

(b) Now suppose that T_0 is any member of $\mathcal{T}^{(0)}$. Then T_0 has an extension to a member of $\mathcal{T}^{\times}$. **P** $|T_0|$, $T_0^+ = \frac{1}{2}(|T_0| + T_0)$ and $T_0^- = \frac{1}{2}(|T_0| - T_0)$, taken in $\mathsf{L}^{\sim}(M^{1,0}_{\bar{\mu}}; M^{1,0}_{\bar{\nu}})$, all belong to $\mathcal{T}^{(0)}$ (371G), so have extensions S, S_1 and S_2 to order-continuous positive linear operators from $M^{1,\infty}_{\bar{\mu}}$ to $L^0(\mathfrak{B})$ as defined in (a). Now for any $w \in L^1_{\bar{\mu}}$,

$$\|Sw\|_1 = \||T_0|w\|_1 \le \|w\|_1,$$

and for any $w \in L^\infty(\mathfrak{A})$,

$$|Sw| \le S|w| = \sup\{|T_0|u : u \in M^{1,0}_{\bar{\mu}},\ 0 \le u \le w\} \le \|w\|_\infty \chi 1,$$

so $\|Sw\|_\infty \le \|w\|_\infty$. Thus $S \in \mathcal{T}$; similarly, S_1 and S_2 can be regarded as operators from $M^{1,\infty}_{\bar{\mu}}$ to $M^{1,\infty}_{\bar{\nu}}$, and as such belong to $\mathcal{T}$. Next, for $w \ge 0$ in $M^{1,\infty}_{\bar{\mu}}$,

$$\begin{aligned} S_1 w + S_2 w &= \sup\{T_0^+ u : u \in M^{1,0}_{\bar{\mu}},\ 0 \le u \le w\} + \sup\{T_0^- u : u \in M^{1,0}_{\bar{\mu}},\ 0 \le u \le w\} \\ &= \sup\{T_0^+ u + T_0^- u : u \in M^{1,0}_{\bar{\mu}},\ 0 \le u \le w\} = Sw. \end{aligned}$$

But this means that

$$S = S_1 + S_2 \ge |S_1 - S_2|$$

and $T = S_1 - S_2 \in \mathcal{T}$, by 373Bc; while of course T extends $T_0^+ - T_0^- = T_0$. Finally, because S_1 and S_2 are order-continuous, $T \in \mathsf{L}^{\times}(M^{1,\infty}_{\bar{\mu}}; M^{1,\infty}_{\bar{\nu}})$, so $T \in \mathcal{T}^{\times}$. **Q**

(c) If $(\mathfrak{A},\bar{\mu})$ is semi-finite, then $M^{1,0}_{\bar{\mu}}$ is order-dense in $M^{1,\infty}_{\bar{\mu}}$ (because it includes $L^1_{\bar{\mu}}$, which is order-dense in $L^0(\mathfrak{A})$); so that the extension T is unique, by 355F(iii).

373S Adjoints in $\mathcal{T}^{(0)}$: Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and T any member of $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$. Then there is a unique operator $T' \in \mathcal{T}^{(0)}_{\bar{\nu},\bar{\mu}}$ such that $\int_a T'(\chi b) = \int_b T(\chi a)$ for every $a \in \mathfrak{A}^f$, $b \in \mathfrak{B}^f$, and now $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar{\mu}}$, $v \in M^{1,0}_{\bar{\nu}}$ are such that $\int u^* \times v^* < \infty$.

proof (a) For each $v \in M^{1,0}_{\bar{\nu}}$ we can define $T'v \in M^{1,0}_{\bar{\mu}}$ by the formula

$$\int_a T'v = \int T(\chi a) \times v$$

for every $a \in \mathfrak{A}^f$. **P** Set $\theta a = \int T(\chi a) \times v$ for each $a \in \mathfrak{A}^f$; because $\int (\chi a)^* \times v^* < \infty$, the integral is defined and finite (373J). Of course $\theta : \mathfrak{A}^f \to \mathbb{R}$ is additive because χ is additive and T, $\times$ and $\int$ are linear. Also

$$\lim_{t\downarrow 0} \sup_{\bar{\mu}a \le t} |\theta a| \le \lim_{t\downarrow 0} \int_0^t v^* = 0,$$

$$\lim_{t\to\infty} \frac{1}{t} \sup_{\bar{\mu}a \le t} |\theta a| \le \lim_{t\to\infty} \tfrac{1}{t} \int_0^t v^* = 0$$

because $v \in M^{1,0}_{\bar\nu}$, so $v^* \in M^{1,0}_{\bar\mu_L}$. By 373Ha, there is a unique $T'v \in M^{1,0}_{\bar\mu}$ such that $\int_a T'v = \theta a$ for every $a \in \mathfrak{A}^f$. **Q**

(b) Because the formula uniquely determines $T'v$, we see that $T' : M^{1,0}_{\bar\nu} \to M^{1,0}_{\bar\mu}$ is linear. Now $T' \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. **P** (i) If $v \in L^1_{\bar\nu}$, then (because $T'v \in M^{1,0}_{\bar\mu}$) $|T'v| = \sup_{a\in\mathfrak{A}^f} |T'v| \times \chi a$, and

$$\begin{aligned}\|T'v\|_1 &= \int |T'v| = \sup_{a\in\mathfrak{A}^f} \int_a |T'v| = \sup_{b,c\in\mathfrak{A}^f} (\int_b T'v - \int_c T'v)\\ &= \sup_{b,c\in\mathfrak{A}^f} \int T(\chi b - \chi c) \times v \le \sup_{b,c\in\mathfrak{A}^f} \int (\chi b - \chi c)^* \times v^*\\ &= \int v^* = \|v\|_1.\end{aligned}$$

(ii) Now suppose that $v \in L^\infty(\mathfrak{B}) \cap M^{1,0}_{\bar\nu}$, and set $\gamma = \|v\|_\infty$. **?** If $a = [\![|T'v| > \gamma]\!] \neq 0$, then $T'v \neq 0$ so $v \neq 0$ and $\gamma > 0$ and $\bar\mu a < \infty$, because $T'v \in M^{1,0}_{\bar\mu}$. Set $b = [\![(T'v)^+ > \gamma]\!]$, $c = [\![(T'v)^- > \gamma]\!]$; then

$$\begin{aligned}\gamma\bar\mu a &< \int_a |T'v| = \int_b T'v - \int_c T'v = \int T(\chi b - \chi c) \times v\\ &\le \gamma\|T(\chi b - \chi c)\|_1 \le \gamma\|\chi b - \chi c\|_1 = \gamma\bar\mu a,\end{aligned}$$

which is impossible. **X** Thus $[\![|T'v| > \gamma]\!] = 0$ and $\|T'v\|_\infty \le \gamma = \|v\|_\infty$.

Putting this together with (i), we see that $T' \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. **Q**

(c) Let $|T|$ be the modulus of T in $\mathrm{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\nu})$, so that $|T| \in \mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$, by 371Gb. If $u \ge 0$ in $M^{1,0}_{\bar\mu}$, $v \ge 0$ in $M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$, let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}^f)^+$ with supremum u. In this case $|T|u = \sup_{n\in\mathbb{N}} |T|u_n$, so $\int |T|u \times v = \sup_{n\in\mathbb{N}} \int |T|u_n \times v$ and

$$|\int Tu \times v - \int Tu_n \times v| \le \int |T|(u - u_n) \times v \to 0$$

as $n \to \infty$, because

$$\int |T|u \times v \le \int u^* \times v^* < \infty.$$

At the same time,

$$|\int u \times T'v - \int u_n \times T'v| \le \int (u - u_n) \times |T'v| \to 0$$

because $\int u \times |T'v| \le \int u^* \times v^* < \infty$. So

$$\int Tu \times v = \lim_{n\to\infty} \int Tu_n \times v = \lim_{n\to\infty} \int u_n \times T'v = \int u \times T'v,$$

the middle equality being valid because each u_n is a linear combination of characteristic functions.

Because T and T' are linear, it follows at once that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

(d) Finally, to see that T' is uniquely defined by the formula in the statement of the theorem, observe that this surely defines $T'(\chi b)$ for every $b \in \mathfrak{B}^f$, by the remarks in (a). Consequently it defines T' on $S(\mathfrak{B}^f)$. Since $S(\mathfrak{B}^f)$ is order-dense in $M^{1,0}_{\bar\nu}$, and any member of $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ must belong to $\mathrm{L}^\times(M^{1,0}_{\bar\nu}; M^{1,0}_{\bar\mu})$ (371Gb), the restriction of T' to $S(\mathfrak{B}^f)$ determines T' (355J).

373T Corollary Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras. Then for any $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ such that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

proof The restriction $T{\restriction}M^{1,0}_{\bar\mu}$ belongs to $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ (373Bb), so there is a unique $S \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ such that $\int u \times Sv = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$ (373S). Now there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ extending S (373R). If $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, $v \ge 0$ in $M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$, then $\int u \times T'v = \int Tu \times v$. **P** If $T \ge 0$, then both are

$$\sup\{\int u_0 \times T'v_0 : u_0 \in M^{1,0}_{\bar\mu},\ v \in M^{1,0}_{\bar\nu},\ 0 \le u_0 \le u,\ 0 \le v_0 \le v\}$$

because both T and T' are (order-)continuous. In general, we can apply the same argument to T^+ and T^-, taken in $\mathrm{L}^{\sim}(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$, since these belong to $\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$, by 373B and 355H, and we shall surely have $T'=(T^+)'-(T^-)'$. **Q** As in 373S, it follows that $\int u\times T'v=\int Tu\times v$ whenever $u\in M^{1,\infty}_{\bar\mu}$, $v\in M^{1,\infty}_{\bar\nu}$ are such that $\int u^*\times v^*<\infty$.

373U Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous measure-preserving Boolean homomorphism. Then the associated map $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$ (373Bd) has an adjoint $P\in\mathcal{T}^{\times}_{\bar\nu,\bar\mu}$ defined by the formula $\int_a P(\chi b)=\bar\nu(b\cap\pi a)$ for $a\in\mathfrak{A}^f$, $b\in\mathfrak{B}^f$.

proof The adjoint $P=T'$ must have the property that

$$\int_a P(\chi b)=\int\chi a\times P(\chi b)=\int T(\chi a)\times\chi b=\int\chi(\pi a)\times\chi b=\bar\nu(\pi a\cap b)$$

for every $a\in\mathfrak{A}^f$, $b\in\mathfrak{B}^f$. To see that this defines P uniquely, let $S\in\mathcal{T}^{\times}_{\bar\nu,\bar\mu}$ be any other operator with the same property. By 373Hb, $S(\chi b)=P(\chi b)$ for every $b\in\mathfrak{B}^f$, so S and P agree on $S(\mathfrak{B}^f)$. Because both P and S are supposed to belong to $\mathrm{L}^{\times}(M^{1,\infty}_{\bar\nu};M^{1,\infty}_{\bar\mu})$, and $S(\mathfrak{B}^f)$ is order-dense in $M^{1,\infty}_{\bar\nu}$, $S=P$, by 355J.

373X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a *ring* homomorphism such that $\bar\nu\pi a\le\bar\mu a$ for every $a\in\mathfrak{A}$. (i) Show that there is a unique $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$, and that T is a Riesz homomorphism. (ii) Show that T is (sequentially) order-continuous iff π is.

>(b) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\mu)$ be measure algebras, and $\phi:\mathbb{R}\to\mathbb{R}$ a convex function such that $\phi(0)\le 0$. Show that if $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ and $T\ge 0$, then $\bar\phi(Tu)\le T(\bar\phi(u))$ whenever $u\in M^{1,\infty}_{\bar\mu}$ is such that $\bar\phi(u)\in M^{1,\infty}_{\bar\mu}$. (*Hint*: 233J, 365Rb.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that if $w\in L^\infty(\mathfrak{A})$ and $\|w\|_\infty\le 1$ then $u\mapsto u\times w:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}^{\times}_{\bar\mu,\bar\mu}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Show that if $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$, $\mathfrak{B}$ respectively, and $\langle T_i\rangle_{i\in I}$ is any family in $\mathcal{T}_{\bar\mu,\bar\nu}$, and *either* I is countable *or* $\mathfrak{B}$ is Dedekind complete, then we have an operator $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu\times\chi b_i=T_i(u\times\chi a_i)\times\chi b_i$ for every $u\in M^{1,\infty}_{\bar\mu,\bar\nu}$, $i\in I$.

>(e) Let I, J be sets and write $\mu=\bar\mu$, $\nu=\bar\nu$ for counting measure on I, J respectively. Show that there is a natural one-to-one correspondence between $\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$ and the set of matrices $\langle a_{ij}\rangle_{i\in I,j\in J}$ such that $\sum_{i\in I}|a_{ij}|\le 1$ for every $j\in J$, $\sum_{j\in J}|a_{ij}|\le 1$ for every $i\in I$.

>(f) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$, and product measure λ on $X\times Y$. Let $h:X\times Y\to\mathbb{R}$ be a measurable function such that $\int|h(x,y)|dx\le 1$ for ν-almost every $y\in Y$ and $\int|h(x,y)|dy\le 1$ for μ-almost every $x\in X$. Show that there is a corresponding $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$ defined by writing $T(f^\bullet)=g^\bullet$ whenever $f\in\mathcal{L}^1(\mu)+\mathcal{L}^\infty(\mu)$ and $g(y)=\int h(x,y)f(x)dx$ for almost every y.

>(g) Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Show that for any μ-integrable function h with $\int|h|d\mu\le 1$ we have a corresponding $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\mu}$ defined by setting $T(f^\bullet)=(h*f)^\bullet$ whenever $g\in\mathcal{L}^1(\mu)+\mathcal{L}^\infty(\mu)$, writing $h*f$ for the convolution of h and f (255E). Explain how this may be regarded as a special case of 373Xf.

>(h) Let (Ω,Σ,μ) be a probability space and $X\in\mathcal{L}^0(\mu)$ a non-negative real-valued random variable on Ω; let ν_X be its distribution (271C). Write $u=X^\bullet\in L^0(\mu)\cong L^0_{\bar\mu}$, where $(\mathfrak{A},\bar\mu)$ is the measure algebra of (Ω,Σ,μ). Show that $\bar\mu_L[\![u^*>\alpha]\!]=\Pr(X>\alpha)$ for every α, so that each of u^*, ν_X is uniquely determined by the other.

(i) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism; let $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ be the corresponding operator (373Bd). Show that $(Tu)^*=u^*$ for every $u\in M^{1,\infty}_{\bar\mu}$.

(j) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and A a subset of $L^1_{\bar\mu}$. Show that the following are equiveridical: (i) A is uniformly integrable; (ii) $\{u^* : u \in A\}$ is uniformly integrable in $L^1_{\bar\mu_L}$; (iii) $\lim_{t\downarrow 0}\sup_{u\in A}\int_0^t u^* = 0$.

(l) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $A \subseteq (M^0_{\bar\mu})^+$ a non-empty downwards-directed set. Show that $(\inf A)^* = \inf_{u\in A} u^*$ in $L^0(\mathfrak{A}_L)$.

(k) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that $\|u\|_{1,\infty} = \int_0^1 u^*$ for every $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$.

(m) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and ϕ a Young's function (369Xc). Write $U_{\phi,\bar\mu} \subseteq L^0(\mathfrak{A})$, $U_{\phi,\bar\nu} \subseteq L^0(\mathfrak{B})$ for the corresponding Orlicz spaces. (i) Show that if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ and $u \in U_{\phi,\bar\mu}$, then $Tu \in U_{\phi,\bar\nu}$ and $\|Tu\|_\phi \le \|u\|_\phi$. (ii) Show that $u \in U_{\phi,\bar\mu}$ iff $u^* \in U_{\phi,\bar\mu_L}$, and in this case $\|u\|_\phi = \|u^*\|_\phi$.

>(n) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a totally finite measure algebra. Show that if $A \subseteq L^1_{\bar\mu}$ is uniformly integrable, then $\{Tu : u \in A,\ T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ is uniformly integrable in $L^1_{\bar\nu}$.

(o) (i) Give examples of u, $v \in L^1(\mathfrak{A}_L)$ such that $(u+v)^* \not\le u^* + v^*$. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is any measure algebra and u, $v \in M^{0,\infty}_{\bar\mu}$, then $\int_0^t (u+v)^* \le \int_0^t u^* + v^*$ for every $t \ge 0$.

(p) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be two measure algebras. For $u \in M^{1,0}_{\bar\mu}$, $w \in M^{\infty,1}_{\bar\nu}$ set

$$\rho_{uw}(S,T) = |\int (Su - Tu) \times w| \text{ for all } S,\ T \in \mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}.$$

The topology generated by the pseudometrics ρ_{uw} is the **very weak operator topology** on $\mathcal{T}^{(0)}$. Show that $\mathcal{T}^{(0)}$ is compact in this topology.

(q) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras and let $u \in M^{1,0}_{\bar\mu}$. (i) Show that $B = \{Tu : T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}\}$ is compact for the topology $\mathfrak{T}_s(M^{1,0}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. (ii) Show that any non-decreasing sequence in B has a supremum in $L^0(\mathfrak{B})$ which belongs to B.

(r) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$. Show that the following are equiveridical: (i) there is a $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ such that $Tu = v$; (ii) $\int_0^t u^* \le \int_0^t v^*$ for every $t \ge 0$.

(s) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Suppose that u_1, $u_2 \in M^{1,\infty}_{\bar\mu}$ and $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int_0^t v^* \le \int_0^t (u_1+u_2)^*$ for every $t \ge 0$. Show that there are v_1, $v_2 \in M^{1,\infty}_{\bar\nu}$ such that $v_1 + v_2 = v$ and $\int_0^t v_i^* \le \int_0^t u_i^*$ for both i, every $t \ge 0$.

>(t) Set $g(t) = t/(t+1)$ for $t \ge 0$, and set $v = g^\bullet$, $u = \chi[0,1]^\bullet \in L^\infty(\mathfrak{A}_L)$. Show that $\int u^* \times v^* = 1 > \int Tu \times v$ for every $T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$.

(u) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and for $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ define $T' \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ as in 373S. Show that $T'' = T$.

(v) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and give $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ their very weak operator topologies (373Xp). Show that the map $T \mapsto T' : \mathcal{T}^{(0)}_{\bar\mu,\bar\nu} \to \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ is an isomorphism for the convex, order and topological structures of the two spaces. (By the 'convex structure' of a convex set C in a linear space I mean the operation $(x,y,t) \mapsto tx + (1-t)y : C \times C \times [0,1] \to C$.)

373Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $u = f^\bullet$ and $v = g^\bullet$ in $L^0(\mathfrak{A})$, where $f(t) = t$, $g(t) = 1 - 2|t - \frac{1}{2}|$ for $t \in [0,1]$. Show that $u^* = v^*$, but that there is no measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $T_\pi v = u$, writing $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ for the operator induced by π, as in 364R. (*Hint*: show that $\{[\![v > \alpha]\!] : \alpha \in \mathbb{R}\}$ does not τ-generate $\mathfrak{A}$.)

(b) Let $(\mathfrak{A},\bar\mu)$ be a totally finite homogeneous measure algebra of uncountable Maharam type. Let u, $v\in(M^{1,\infty}_{\bar\mu})^+$ be such that $u^*=v^*$. Show that there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T_\pi u=v$.

(c) Let u, $v\in M^{1,\infty}_{\bar\mu_L}$ be such that $u=u^*$, $v=v^*$ and $\int_0^t v\le\int_0^t u$ for every $t\ge 0$. Show that there is a non-negative $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Tu=v$ and $\int_0^t Tw\le\int_0^t w$ for every $w\ge 0$ in $M^{1,\infty}$. Show that any such T must belong to $\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $u\in M^{1,\infty}_{\bar\mu}$. (i) Suppose that $w\in S(\mathfrak{B}^f)$. Show directly (without quoting the result of 373O, but possibly using some of the ideas of the proof) that for every $\gamma<\int u^*\times w^*$ there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu\times w\ge\gamma$. (ii) Suppose that $(\mathfrak{B},\bar\nu)$ is localizable and that $v\in M^{1,\infty}_{\bar\nu}\setminus\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. Show that there is a $w\in S(\mathfrak{B}^f)$ such that $\int v\times w>\sup\{\int Tu\times w:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. (*Hint*: use 373M and the Hahn-Banach theorem in the following form: if U is a linear space with the topology $\mathfrak{T}_s(U,V)$ defined by a linear subspace V of $\mathrm{L}(U;\mathbb{R})$, $C\subseteq U$ is a non-empty closed convex set, and $v\in U\setminus C$, then there is an $f\in V$ such that $f(v)>\sup_{u\in C}f(u)$.) (iii) Hence prove 373O for localizable $(\mathfrak{B},\bar\nu)$. (iv) Now prove 373O for general $(\mathfrak{B},\bar\nu)$.

(e) (i) Define $v\in L^\infty(\mathfrak{A}_L)$ as in 373Xt. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tv=v^*$. (ii) Set $h(t)=1+\max(0,\frac{\sin t}{t})$ for $t>0$, $w=h^\bullet\in L^\infty(\mathfrak{A}_L)$. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tw^*=w$.

(f) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that $\mathcal{T}_{\bar\mu,\bar\mu_L}=\mathcal{T}^\times_{\bar\mu,\bar\mu_L}$ can be identified, as convex ordered space, with $\mathcal{T}^\times_{\bar\mu_L,\bar\mu}$, and that this is a proper subset of $\mathcal{T}_{\bar\mu_L,\bar\mu}$.

(g) Show that the adjoint operation of 373T is not as a rule continuous for the very weak operator topologies of $\mathcal{T}^\times_{\bar\mu,\bar\nu}$, $\mathcal{T}^\times_{\bar\nu,\bar\mu}$.

373 Notes and comments 373A-373B are just alternative expressions of concepts already treated in 371F-371H. My use of the simpler formula $\mathcal{T}_{\bar\mu,\bar\nu}$ symbolizes my view that $\mathcal{T}$, rather than $\mathcal{T}^{(0)}$ or $\mathcal{T}^\times$, is the most natural vehicle for these ideas; I used $\mathcal{T}^{(0)}$ in §§371-372 only because that made it possible to give theorems which applied to all measure algebras, without demanding localizability (compare 371Gb with 373Bc).

The obvious examples of operators in $\mathcal{T}$ are those derived from measure-preserving Boolean homomorphisms, as in 373Bd, and their adjoints (373U). Note that the latter include conditional expectation operators. In return, we find that operators in $\mathcal{T}$ share some of the characteristic properties of the operators derived from Boolean homomorphisms (373Bb, 373Xb, 373Xm). Other examples are multiplication operators (373Xc), operators obtained by piecing others together (373Xd) and kernel operators of the type described in 373Xe-373Xf, including convolution operators (373Xg). (For a general theory of kernel operators, see §376 below.)

Most of the section is devoted to the relationships between the classes $\mathcal{T}$ of operators and the 'decreasing rearrangements' of 373C. If you like, the decreasing rearrangement u^* of u describes the 'distribution' of $|u|$ (373Xh); but for $u\notin M^0$ it loses some information (373Xt, 373Ye). It is important to be conscious that even when $u\in L^0(\mathfrak{A}_L)$, u^* is not necessarily obtained by 'rearranging' the elements of the algebra $\mathfrak{A}_L$ by a measure-preserving automorphism (which would, of course, correspond to an automorphism of the measure space $([0,\infty[\,,\mu_L)$, by 344C). I will treat 'rearrangements' of this narrower type in the next section; for the moment, see 373Ya. Apart from this, the basic properties of decreasing rearrangements are straightforward enough (373D-373F). The only obscure area concerns the relationship between $(u+v)^*$ and u^*, v^* (see 373Xo).

In 373G I embark on results involving both decreasing rearrangements and operators in $\mathcal{T}$, leading to the characterization of the sets $\{Tu:T\in\mathcal{T}\}$ in 373O. In one direction this is easy, and is the content of 373G. In the other direction it depends on a deeper analysis, and the easiest method seems to be through studying the 'very weak operator topology' on $\mathcal{T}$ (373K-373L), even though this is an effective tool only when one of the algebras involved is localizable (373L). A functional analyst is likely to feel that the method is both natural and illuminating; but from the point of view of a measure theorist it is not perfectly satisfactory, because it is essentially non-constructive. While it tells us that there are operators $T\in\mathcal{T}$ acting in the required ways, it gives only the vaguest of hints concerning what they actually look like.

Of course the very weak operator topology is interesting in its own right; and see also 373Xp-373Xq.

The proof of 373O can be thought of as consisting of three steps. Given that $\int_0^t v^* \le \int_0^t u^*$ for every t, then I set out to show that v is expressible as T_1v^* (parts (c)-(d) of the proof), that v^* is expressible as T_2u^* (part (g)) and that u^* is expressible as T_3u (parts (e)-(f)), each T_i belonging to an appropriate $\mathcal{T}$. In all three steps the general case follows easily from the case of u, $v \in S(\mathfrak{A})$, $S(\mathfrak{B})$. If we are willing to use a more sophisticated version of the Hahn-Banach theorem than those given in 3A5A and 363R, there is an alternative route (373Yd). I note that the central step above, from u^* to v^*, can be performed with an order-continuous T_2 (373Yc), but that in general neither of the other steps can (373Ye), so that we cannot use $\mathcal{T}^\times$ in place of $\mathcal{T}$ here.

A companion result to 373O, in that it also shows that $\{Tu : T \in \mathcal{T}\}$ is large enough to reach natural bounds, is 373P; given u and v, we can find T such that $\int Tu \times v$ is as large as possible. In this form the result is valid only for $v \in M^{(0)}$ (373Xt). But if we do not demand that the supremum should be attained, we can deal with other v (373Q).

We already know that every operator in $\mathcal{T}^{(0)}$ is a difference of order-continuous operators, just because $M^{1,0}$ has an order-continuous norm (371Gb). It is therefore not surprising that members of $\mathcal{T}^{(0)}$ can be extended to members of $\mathcal{T}^\times$, at least when the codomain $M^{1,\infty}_{\bar\nu}$ is Dedekind complete (373R). It is also very natural to look for a correspondence between $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathcal{T}_{\bar\nu,\bar\mu}$, because if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ we shall surely have an adjoint operator $(T{\restriction}L^1_{\bar\mu})'$ from $(L^1_{\bar\nu})^*$ to $(L^1_{\bar\mu})^*$, and we can hope that this will correspond to some member of $\mathcal{T}_{\bar\nu,\bar\mu}$. But when we come to the details, the normed-space properties of a general member of $\mathcal{T}$ are not enough (373Yf), and we need some kind of order-continuity. For members of $\mathcal{T}^{(0)}$ this is automatically present (373S), and now the canonical isomorphism between $\mathcal{T}^{(0)}$ and $\mathcal{T}^\times$ gives us an isomorphism between $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ and $\mathcal{T}^\times_{\bar\nu,\bar\mu}$ when $\bar\mu$ and $\bar\nu$ are localizable (373T).

374 Rearrangement-invariant spaces

As is to be expected, many of the most important function spaces of analysis are symmetric in various ways; in particular, they share the symmetries of the underlying measure algebras. The natural expression of this is to say that they are 'rearrangement-invariant' (374E). In fact it turns out that in many cases they have the stronger property of '$\mathcal{T}$-invariance' (374A). In this section I give a brief account of the most important properties of these two kinds of invariance. In particular, $\mathcal{T}$-invariance is related to a kind of transfer mechanism, enabling us to associate function spaces on different measure algebras (374B-374D). As for rearrangement-invariance, the salient fact is that on the most important measure algebras many rearrangement-invariant spaces are $\mathcal{T}$-invariant (374K, 374M).

374A $\mathcal{T}$-invariance: Definitions Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Recall that I write

$$M^{1,\infty}_{\bar\mu} = L^1_{\bar\mu} + L^\infty(\mathfrak{A}) \subseteq L^0(\mathfrak{A}),$$

$$M^{\infty,1}_{\bar\mu} = L^1_{\bar\mu} \cap L^\infty(\mathfrak{A}),$$

$$M^{0,\infty}_{\bar\mu} = \{u : u \in L^0(\mathfrak{A}),\ \inf_{\alpha>0} \bar\mu[\![|u| > \alpha]\!] < \infty\},$$

(369N, 373C).

(a) I will say that a subset A of $M^{1,\infty}_{\bar\mu}$ is **$\mathcal{T}$-invariant** if $Tu \in A$ whenever $u \in A$ and $T \in \mathcal{T} = \mathcal{T}_{\bar\mu,\bar\mu}$ (definition: 373Aa).

(b) I will say that an extended Fatou norm τ on L^0 is **$\mathcal{T}$-invariant** or **fully symmetric** if $\tau(Tu) \le \tau(u)$ whenever $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}$.

(c) As in §373, I will write $(\mathfrak{A}_L, \bar\mu_L)$ for the measure algebra of Lebesgue measure on $[0, \infty[$, and $u^* \in M^{0,\infty}_{\bar\mu_L}$ for the decreasing rearrangement of any u belonging to any $M^{0,\infty}_{\bar\mu}$ (373C).

374B The first step is to show that the associate of a $\mathcal{T}$-invariant norm is $\mathcal{T}$-invariant.

Theorem Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Let L^τ be the Banach lattice defined from τ (369G), and τ' the associate extended Fatou norm (369H-369I). Then

(i) $M^{\infty,1}_{\bar{\mu}} \subseteq L^\tau \subseteq M^{1,\infty}_{\bar{\mu}}$;

(ii) τ' is also $\mathcal{T}$-invariant, and $\int u^*\times v^* \le \tau(u)\tau'(v)$ for all u, $v\in M^{0,\infty}_{\bar{\mu}}$.

proof (a) I check first that $L^\tau\subseteq M^{0,\infty}_{\bar{\mu}}$. **P** Take any $u\in L^0(\mathfrak{A})\setminus M^{0,\infty}_{\bar{\mu}}$. There is surely some $w>0$ in L^τ, and we can suppose that $w=\chi a$ for some a of finite measure. Now, for any $n\in\mathbb{N}$,

$$(|u|\wedge n\chi 1)^* = n\chi 1\ge nw^*$$

in $L^0(\mathfrak{A}_L)$, because $\bar{\mu}[\![|u|>n]\!]=\infty$. So there is a $T\in\mathcal{T}_{\bar{\mu},\bar{\mu}}$ such that $T(|u|\wedge n\chi 1)=nw$, by 373O, and

$$\tau(u)\ge\tau(|u|\wedge n\chi 1)\ge\tau(T(|u|\wedge n\chi 1))=\tau(nw)=n\tau(w).$$

As n is arbitrary, $\tau(u)=\infty$. As u is arbitrary, $L^\tau\subseteq M^{0,\infty}_{\bar{\mu}}$. **Q**

(b) Next, $\int u^*\times v^*\le\tau(u)\tau'(v)$ for every u, $v\in M^{0,\infty}_{\bar{\mu}}$. **P** If $u\in M^{1,\infty}_{\bar{\mu}}$, then

$$\int u^*\times v^* = \sup\{\int|Tu\times v| : T\in\mathcal{T}_{\bar{\mu},\bar{\mu}}\}$$

(373Q)

$$\le\sup\{\tau(Tu)\tau'(v):T\in\mathcal{T}_{\bar{\mu},\bar{\mu}}\}=\tau(u)\tau'(v).$$

Generally, setting $u_n=|u|\wedge n\chi 1$, $\langle u_n^*\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u^* (373Db, 373Dh), so

$$\int u^*\times v^*=\sup_{n\in\mathbb{N}}\int u_n^*\times v^*\le\sup_{n\in\mathbb{N}}\tau(u_n)\tau'(v)=\tau(u)\tau'(v). \text{ } \mathbf{Q}$$

(c) Consequently, $L^\tau\subseteq M^{1,\infty}_{\bar{\mu}}$. **P** If $\mathfrak{A}=\{0\}$, this is trivial. Otherwise, take $u\in L^\tau$. There is surely some non-zero a such that $\tau'(\chi a)<\infty$; now, setting $v=\chi a$,

$$\int_0^{\bar{\mu}a}u^*=\int u^*\times v^*\le\tau(u)\tau'(v)<\infty$$

by (b) above. But this means that $u^*\in M^{1,\infty}_{\bar{\mu}}$, so that $u\in M^{1,\infty}_{\bar{\mu}}$ (373F(b-ii)). **Q**

(d) Next, τ' is $\mathcal{T}$-invariant. **P** Suppose that $v\in M^{0,\infty}_{\bar{\mu}}$, $T\in\mathcal{T}_{\bar{\mu},\bar{\mu}}$, $u\in L^0(\mathfrak{A})$ and $\tau(u)\le 1$. Then $u\in M^{1,\infty}_{\bar{\mu}}$, by (c), so

$$\int|u\times Tv|\le\int u^*\times v^*\le\tau(u)\tau'(v)\le\tau'(v),$$

using 373J for the first inequality. Taking the supremum over u, we see that $\tau'(Tv)\le\tau'(v)$; as T and v are arbitrary, τ' is $\mathcal{T}$-invariant. **Q**

(e) Finally, putting (d) and (c) together, $L^{\tau'}\subseteq M^{1,\infty}_{\bar{\mu}}$, so that $L^\tau\supseteq M^{\infty,1}_{\bar{\mu}}$, using 369J and 369O.

374C For any $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ there are corresponding norms on $L^0(\mathfrak{A})$ for any semi-finite measure algebra, as follows.

Theorem Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and $(\mathfrak{A},\bar{\mu})$ a semi-finite measure algebra.

(a) There is a $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ defined by setting

$$\tau(u)=\theta(u^*) \text{ if } u\in M^{0,\infty}_{\bar{\mu}},$$
$$=\infty \text{ if } u\in L^0(\mathfrak{A})\setminus M^{0,\infty}_{\bar{\mu}}.$$

(b) Writing θ', τ' for the associates of θ and τ, we now have

$$\tau'(v) = \theta'(v^*) \text{ if } v \in M^{0,\infty}_{\bar\mu},$$
$$= \infty \text{ if } v \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}.$$

(c) If θ is an order-continuous norm on the Banach lattice L^θ, then τ is an order-continuous norm on L^τ.

proof (a)(i) The argument seems to run better if I use a different formula to define τ: set

$$\tau(u) = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu},\, w \in L^0(\mathfrak{A}_L),\, \theta'(w) \le 1\}$$

for $u \in L^0(\mathfrak{A})$. (By 374B, $w \in M^{1,\infty}_{\bar\mu}$ whenever $\theta'(w) \le 1$, so there is no difficulty in defining Tw.) Now $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$. **P** ($\alpha$) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then $w \in M^{1,\infty}_{\bar\mu_L}$, by 374B(i), so there is an $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Sw = w^*$ (373O). Accordingly $\theta'(w^*) \le \theta'(w)$ (because θ' is $\mathcal{T}$-invariant, by 374B); now

$$\int |u \times Tw| \le \int u^* \times w^* \le \theta(u^*)\theta'(w^*) \le \theta(u^*)\theta'(w) \le \theta(u^*);$$

as w is arbitrary, $\tau(u) \le \theta(u^*)$. (β) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then

$$\int |u^* \times w| \le \int (u^*)^* \times w^*$$

(373E)

$$= \int u^* \times w^* = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}\}$$

(373Q)

$$\le \tau(u).$$

But because θ is the associate of θ' (369I(ii)), this means that $\theta(u^*) \le \tau(u)$. **Q**

(ii) Now τ is an extended Fatou norm on $L^0(\mathfrak{A})$. **P** Of the conditions in 369F, (i)-(iv) are true just because $\tau(u) = \sup_{v\in B} \int |u \times v|$ for some set $B \subseteq L^0$. As for (v) and (vi), observe that if $u \in M^{\infty,1}_{\bar\mu}$ then $u^* \in M^{\infty,1}_{\bar\mu_L}$ (373F(b-iv)), so that $\tau(u) = \theta(u^*) < \infty$, by 374B(i), while also

$$u \ne 0 \Longrightarrow u^* \ne 0 \Longrightarrow \tau(u) = \theta(u^*) > 0.$$

As $M^{\infty,1}_{\bar\mu}$ is order-dense in $L^0(\mathfrak{A})$ (this is where I use the hypothesis that $(\mathfrak{A}, \bar\mu)$ is semi-finite), 369F(v)-(vi) are satisfied, and τ is an extended Fatou norm. **Q**

(iii) τ is $\mathcal{T}$-invariant. **P** Take $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$. There are $S_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ and $S_1 \in \mathcal{T}_{\bar\mu,\bar\mu_L}$ such that $S_0u^* = u$, $S_1Tu = (Tu)^*$ (373O); now $S_1TS_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ (373Be), so

$$\tau(Tu) = \theta((Tu)^*) = \theta(S_1TS_0u^*) \le \theta(u^*) = \tau(u)$$

because θ is $\mathcal{T}$-invariant. **Q**

(iv) We can now return to the definition of τ. I have already remarked that $\tau(u) = \theta(u^*)$ if $u \in M^{0,\infty}_{\bar\mu}$. For other u, we must have $\tau(u) = \infty$ just because τ is a $\mathcal{T}$-invariant extended Fatou norm (374B(i)). So the definitions in the statement of the theorem and (i) above coincide.

(b) We surely have $\tau'(v) = \infty$ if $v \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}$, by 374B, because τ', like τ, is a $\mathcal{T}$-invariant extended Fatou norm. So take $v \in M^{0,\infty}_{\bar\mu}$.

(i) If $u \in L^0(\mathfrak{A})$ and $\tau(u) \le 1$, then

$$\int |v \times u| \le \int v^* \times u^* \le \theta'(v^*)\theta(u^*) = \theta'(v^*)\tau(u) \le \theta'(v^*);$$

as u is arbitrary, $\tau'(v) \le \theta'(v^*)$.

(ii) If $w \in L^0(\mathfrak{A}_L)$ and $\theta(w) \le 1$, then

$$\int |v^* \times w| \le \int v^* \times w^* = \sup\{\int |v \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu}\}$$

(373Q)

$$\le \sup\{\tau'(v)\tau(Tw) : T \in \mathcal{T}_{\bar\mu_L,\bar\mu}\} = \sup\{\tau'(v)\theta((Tw)^*) : T \in \mathcal{T}_{\bar\mu_L,\bar\mu}\}$$
$$\le \sup\{\tau'(v)\theta(STw) : T \in \mathcal{T}_{\bar\mu_L,\bar\mu},\ S \in \mathcal{T}_{\bar\mu,\bar\mu_L}\}$$

(because, given T, we can find an S such that $STw = (Tw)^*$, by 373O)

$$\le \sup\{\tau'(v)\theta(Tw) : T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}\} \le \tau'(v).$$

As w is arbitrary, $\theta'(v^*) \le \tau'(v)$ and the two are equal. This completes the proof of (b).

(c)(i) The first step is to note that $L^\tau \subseteq M^0_{\bar\mu}$. **P?** Suppose that $u \in L^\tau \setminus M^0_{\bar\mu}$, that is, that $\bar\mu[\![|u| > \alpha]\!] = \infty$ for some $\alpha > 0$. Then $u^* \ge \alpha\chi 1$ in $L^0(\mathfrak{A}_L)$, so $L^\infty(\mathfrak{A}_L) \subseteq L^\theta$. For each $n \in \mathbb{N}$, set $v_n = \chi\,[n,\infty[^\bullet$. Then $v_n^* = v_0$, so we can find a $T_n \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $T_n v_n = v_0$ (373O), and $\theta(v_n) \ge \theta(v_0)$ for every n. But as $\langle v_n\rangle_{n\in\mathbb{N}}$ is a decreasing sequence with infimum 0, this means that θ is not an order-continuous norm. **XQ**

(ii) Now suppose that $A \subseteq L^\tau$ is non-empty and downwards-directed and has infimum 0. Then $\inf_{u\in A}\bar\mu[\![u > \alpha]\!] = 0$ for every $\alpha > 0$ (put 364Nb and 321F together). But this means that $B = \{u^* : u \in A\}$ must have infimum 0; since B is surely downwards-directed, $\inf_{v\in B}\theta(v) = 0$, that is, $\inf_{u\in A}\tau(u) = 0$. As A is arbitrary, τ is an order-continuous norm.

374D What is more, every $\mathcal{T}$-invariant extended Fatou norm can be represented in this way.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Then there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A}_L)$ such that $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$.

proof I use the method of 374C. If $\mathfrak{A} = \{0\}$ the result is trivial; assume that $\mathfrak{A} \ne \{0\}$.

(a) Set

$$\theta(w) = \sup\{\textstyle\int |w \times Tv| : T \in \mathcal{T}_{\bar\mu,\bar\mu_L},\ v \in L^0(\mathfrak{A}),\ \tau'(v) \le 1\}$$

for $w \in L^0(\mathfrak{A}_L)$. Note that

$$\theta(w) = \sup\{\textstyle\int w^* \times v^* : v \in L^0(\mathfrak{A}),\ \tau'(v) \le 1\}$$

for every $w \in M^{0,\infty}_{\bar\mu_L}$, by 373J and 373Q again.

θ is an extended Fatou norm on $L^0(\mathfrak{A}_L)$. **P** As in 374C, the conditions 369F(i)-(iv) are elementary. If $w > 0$ in $L^0(\mathfrak{A}_L)$, take any $v \in L^0(\mathfrak{A})$ such that $0 < \tau'(v) \le 1$; then $w^* \times v^* \ne 0$ so $\theta(w) \ge \int w^* \times v^* > 0$. So 369F(v) is satisfied. As for 369F(vi), if $w > 0$ in $L^0(\mathfrak{A}_L)$, take a non-zero $a \in \mathfrak{A}$ of finite measure such that $\alpha = \tau(\chi a) < \infty$. Let $\beta > 0$, $b \in \mathfrak{A}_L$ be such that $0 < \bar\mu_L b \le \bar\mu a$ and $\beta\chi b \le w$; then

$$\theta(\chi b) = \sup_{\tau'(v)\le 1}\textstyle\int(\chi b)^* \times v^* \le \sup_{\tau'(v)\le 1}\int(\chi a)^* \times v^* \le \tau(\chi a) < \infty$$

by 374B(ii). So $\theta(\beta\chi b) < \infty$ and 369F(vi) is satisfied. Thus θ is an extended Fatou norm. **Q**

(b) θ is $\mathcal{T}$-invariant. **P** If $T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ and $w \in M^{1,\infty}_{\bar\mu_L}$, then

$$\theta(Tw) = \sup_{\tau'(v)\le 1}\textstyle\int(Tw)^* \times v^* \le \sup_{\tau'(v)\le 1}\int w^* \times v^* = \theta(w)$$

by 373G and 373I. **Q**

(c) $\theta(u^*) = \tau(u)$ for every $u \in M^{0,\infty}_{\bar\mu}$. **P** We have

$$\tau(u) = \sup_{\tau'(v)\le 1}\textstyle\int |u \times v| \le \sup_{\tau'(v)\le 1}\int u^* \times v^* \le \tau(u),$$

using 369I, 373E and 374B. So

$$\theta(u^*) = \sup_{\tau'(v)\le 1}\textstyle\int u^* \times v^* = \tau(u)$$

by the remark in (a) above. **Q**

374E I turn now to rearrangement-invariance. Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that a subset A of $L^0 = L^0(\mathfrak{A})$ is **rearrangement-invariant** if $T_\pi u \in A$ whenever $u \in A$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean automorphism, writing $T_\pi : L^0 \to L^0$ for the isomorphism corresponding to π (364R).

(b) I will say that an extended Fatou norm τ on L^0 is **rearrangement-invariant** if $\tau(T_\pi u) = \tau(u)$ whenever $u \in L^0$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism.

374F Remarks (a) If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a sequentially order-continuous measure-preserving Boolean homomorphism, then $T_\pi\restriction M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}_{\bar\mu,\bar\mu}$; this is obvious from the definition of $M^{1,\infty} = L^1 + L^\infty$ and the basic properties of T_π (364R). Accordingly, any $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ must be rearrangement-invariant, since (by 374B) we shall have $\tau(u) = \tau(T_\pi(u)) = \infty$ when $u \notin M^{1,\infty}_{\bar\mu}$. Similarly, any $\mathcal{T}$-invariant subset of $M^{1,\infty}_{\bar\mu}$ will be rearrangement-invariant.

(b) I seek to describe cases in which rearrangement-invariance implies $\mathcal{T}$-invariance. This happens only for certain measure algebras; in order to shorten the statements of the main theorems I introduce a special phrase.

374G Definition I say that a measure algebra $(\mathfrak{A},\bar\mu)$ is **quasi-homogeneous** if for any non-zero a, $b \in \mathfrak{A}$ there is a measure-preserving Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi a \cap b \ne 0$.

374H Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous;

(ii) *either* $\mathfrak{A}$ is purely atomic and every atom of $\mathfrak{A}$ has the same measure *or* there is a $\kappa \ge \omega$ such that the principal ideal $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a \in \mathfrak{A}$ of non-zero finite measure.

proof (i)$\Rightarrow$(ii) Suppose that $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous.

(α) Suppose that $\mathfrak{A}$ has an atom a. In this case, for any $b \in \mathfrak{A} \setminus \{0\}$ there is an automorphism π of $(\mathfrak{A},\bar\mu)$ such that $\pi a \cap b \ne 0$; now πa must be an atom, so $\pi a = \pi a \cap b$ and πa is an atom included in b. As b is arbitrary, $\mathfrak{A}$ is purely atomic; moreover, if b is an atom, then it must be equal to πa and therefore of the same measure as a, so all atoms of $\mathfrak{A}$ have the same measure.

(β) Now suppose that $\mathfrak{A}$ is atomless. In this case, if $a \in \mathfrak{A}$ has finite non-zero measure, $\mathfrak{A}_a$ is homogeneous. **P?** Otherwise, there are non-zero b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ are homogeneous and of different Maharam types, by Maharam's theorem (332B, 332H). But now there is supposed to be an automorphism π such that $\pi b \cap c \ne 0$, in which case $\mathfrak{A}_b$, $\mathfrak{A}_{\pi b}$, $\mathfrak{A}_{\pi b\cap c}$ and $\mathfrak{A}_c$ must all have the same Maharam type. **XQ**

Consequently, if a, $b \in \mathfrak{A}$ are both of non-zero finite measure, the Maharam types of $\mathfrak{A}_a$, $\mathfrak{A}_{a\cup b}$ and $\mathfrak{A}_b$ must all be the same infinite cardinal κ.

(ii)$\Rightarrow$(i) Assume (ii), and take a, $b \in \mathfrak{A}\setminus\{0\}$. If $a \cap b \ne 0$ we can take π to be the identity automorphism and stop. So let us suppose that $a \cap b = 0$.

(α) If $\mathfrak{A}$ is purely atomic and every atom has the same measure, then there are atoms $a_0 \subseteq a$, $b_0 \subseteq b$. Set

$$\begin{aligned}\pi c &= c \text{ if } c \supseteq a_0 \cup b_0 \text{ or } c \cap (a_0 \cup b_0) = 0,\\ &= c \mathbin{\triangle} (a_0 \cup b_0) \text{ otherwise.}\end{aligned}$$

Then it is easy to check that π is a measure-preserving automorphism of $\mathfrak{A}$ such that $\pi a_0 = b_0$, so that $\pi a \cap b \ne 0$.

(β) If $\mathfrak{A}_c$ is Maharam-type-homogeneous with the same infinite Maharam type κ for every non-zero c of finite measure, set $\gamma = \min(1,\bar\mu a,\bar\mu b) > 0$. Because $\mathfrak{A}$ is atomless, there are $a_0 \subseteq a$, $b_0 \subseteq b$ with $\bar\mu a_0 = \bar\mu b_0 = \gamma$ (331C). Now $\mathfrak{A}_{a_0}$, $\mathfrak{A}_{b_0}$ are homogeneous with the same Maharam type and the same magnitude, so by Maharam's theorem (331I) there is a measure-preserving isomorphism $\pi_0 : \mathfrak{A}_{a_0} \to \mathfrak{A}_{b_0}$. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi c = (c \setminus (a_0 \cup b_0)) \cup \pi_0(c \cap a_0) \cup \pi_0^{-1}(c \cap b_0);$$

then it is easy to see that π is a measure-preserving automorphism of $\mathfrak{A}$ and that $\pi a \cap b \neq 0$.

Remark We shall return to these ideas in Chapter 38. In particular, the construction of π from π_0 in the last part of the proof will be of great importance; in the language of 381G, $\pi = (\overleftarrow{a_0 \,{}_{\pi_0}\, b_0})$.

374I Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra. Then

(a) whenever a, $b \in \mathfrak{A}$ have the same finite measure, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$ are isomorphic as measure algebras;

(b) there is a subgroup Γ of the additive group $\mathbb{R}$ such that (α) $\bar{\mu}a \in \Gamma$ whenever $a \in \mathfrak{A}$ and $\bar{\mu}a < \infty$ (β) whenever $a \in \mathfrak{A}$, $\gamma \in \Gamma$ and $0 \le \gamma \le \bar{\mu}a$ then there is a $c \subseteq a$ such that $\bar{\mu}c = \gamma$.

proof If $\mathfrak{A}$ is purely atomic, with all its atoms of measure γ_0, set $\Gamma = \gamma_0\mathbb{Z}$, and the results are elementary. If $\mathfrak{A}$ is atomless, set $\Gamma = \mathbb{R}$; then (a) is a consequence of Maharam's theorem, and (b) is a consequence of 331C, already used in the proof of 374H.

374J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra and u, $v \in M^{0,\infty}_{\bar{\mu}}$. Let Aut be the group of measure-preserving automorphisms of $\mathfrak{A}$. Then

$$\int u^* \times v^* = \sup_{\pi \in \mathrm{Aut}} \int |u \times T_\pi v|,$$

where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

proof (a) Suppose first that u, v are non-negative and belong to $S(\mathfrak{A}^f)$, where $\mathfrak{A}^f$ is the ring $\{a : \bar{\mu}a < \infty\}$, as usual. Then they can be expressed as $u = \sum_{i=0}^m \alpha_i \chi a_i$, $v = \sum_{j=0}^n \beta_j \chi b_j$ where $\alpha_0 \ge \ldots \alpha_m \ge 0$, $\beta_0 \ge \ldots \ge \beta_n \ge 0$, $a_0, \ldots, a_m$ are disjoint and of finite measure, and $b_0, \ldots, b_n$ are disjoint and of finite measure. Extending each list by a final term having a coefficient of 0, if need be, we may suppose that $\sup_{i \le m} a_i = \sup_{j \le n} b_j$.

Let $(t_0, \ldots, t_s)$ enumerate in ascending order the set

$$\{0\} \cup \{\textstyle\sum_{i=0}^k \bar{\mu}a_i : k \le m\} \cup \{\textstyle\sum_{j=0}^k \bar{\mu}b_j : k \le n\}.$$

Then every t_r belongs to the subgroup Γ of 374Ib, and $t_s = \sum_{i=0}^m \bar{\mu}a_i = \sum_{j=0}^n \bar{\mu}b_j$. For $1 \le r \le s$ let $k(r)$, $l(r)$ be minimal subject to the requirements $t_r \le \sum_{i=0}^{k(r)} \bar{\mu}a_i$, $t_r \le \sum_{j=0}^{l(r)} \bar{\mu}b_j$. Then $\bar{\mu}a_i = \sum_{k(r)=i} t_r - t_{r-1}$, so (using 374Ib) we can find a disjoint family $\langle c_r \rangle_{1 \le r \le s}$ such that $c_r \subseteq a_{k(r)}$ and $\bar{\mu}c_r = t_r - t_{r-1}$ for each r. Similarly, there is a disjoint family $\langle d_r \rangle_{1 \le r \le s}$ such that $d_r \subseteq b_{l(r)}$ and $\bar{\mu}d_r = t_r - t_{r-1}$ for each r. Now the principal ideals $\mathfrak{A}_{c_r}$, $\mathfrak{A}_{d_r}$ are isomorphic for every r, by 374Ia; let $\pi_r : \mathfrak{A}_{d_r} \to \mathfrak{A}_{c_r}$ be measure-preserving isomorphisms. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi a = (a \setminus \sup_{1 \le r \le s} d_r) \cup \sup_{1 \le r \le s} \pi_r(a \cap d_r);$$

because

$$\sup_{r \le s} c_r = \sup_{i \le m} a_i = \sup_{j \le n} b_j = \sup_{r \le s} d_r,$$

$\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism.

Now

$$u = \sum_{r=1}^s \alpha_{k(r)} \chi c_r, \quad v = \sum_{r=1}^s \beta_{l(r)} \chi d_r,$$

$$u^* = \sum_{r=1}^s \alpha_{k(r)} \chi\, [t_{r-1}, t_r[^\bullet, \quad v^* = \sum_{r=1}^s \beta_{l(r)} \chi\, [t_{r-1}, t_r[^\bullet,$$

so

$$\int u \times T_\pi v = \sum_{r=1}^s \alpha_{k(r)} \beta_{l(r)} \bar{\mu}c_r = \sum_{r=1}^s \alpha_{k(r)} \beta_{l(r)} (t_r - t_{r-1}) = \int u^* \times v^*.$$

(b) Now take any u_0, $v_0 \in M^{0,\infty}_{\bar{\mu}}$. Set

$$A = \{u : u \in S(\mathfrak{A}^f),\, 0 \le u \le |u_0|\}, \quad B = \{v : v \in S(\mathfrak{A}^f),\, 0 \le v \le |v_0|\}.$$

Then A is an upwards-directed set with supremum $|u_0|$, because $(\mathfrak{A},\bar{\mu})$ is semi-finite, so $\{u^*:u\in A\}$ is an upwards-directed set with supremum $|u_0|^*=u_0^*$ (373Db, 373Dh). Similarly $\{v^*:v\in B\}$ is upwards-directed and has supremum v_0^*, so $\{u^*\times v^*:u\in A,\,v\in B\}$ is upwards-directed and has supremum $u_0^*\times v_0^*$.

Consequently, if $\gamma<\int u_0^*\times v_0^*$, there are $u\in A$, $v\in B$ such that $\gamma\le\int u^*\times v^*$. Now, by (a), there is a $\pi\in\mathrm{Aut}$ such that

$$\gamma\le\int u\times T_\pi v\le\int|u_0|\times T_\pi|v_0|=\int|u_0\times T_\pi v_0|$$

because T_π is a Riesz homomorphism. As γ is arbitrary,

$$\int u_0^*\times v_0^*\le\sup_{\pi\in\mathrm{Aut}}\int|u_0\times T_\pi v_0|.$$

But the reverse inequality is immediate from 373J.

374K Theorem Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra, and τ a rearrangement -invariant extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Then τ is $\mathcal{T}$-invariant.

proof Write τ' for the associate of τ. Then 374J tells us that for any u, $v\in M^{0,\infty}_{\bar{\mu}}$,

$$\int u^*\times v^*=\sup_{\pi\in\mathrm{Aut}}\int|T_\pi u\times v|\le\sup_{\pi\in\mathrm{Aut}}\tau(T_\pi u)\tau'(v)=\tau(u)\tau'(v),$$

writing u^*, v^* for the decreasing rearrangements of u and v, and Aut for the group of measure-preserving automorphisms of $(\mathfrak{A},\bar{\mu})$. But now, if $u\in M^{1,\infty}_{\bar{\mu}}$ and $T\in\mathcal{T}_{\bar{\mu},\bar{\mu}}$,

$$\tau(Tu)=\sup\{\int|Tu\times v|:\tau'(v)\le1\}$$

(by 369I)

$$\le\sup\{\int u^*\times v^*:\tau'(v)\le1\}$$

(by 373J)

$$\le\tau(u).$$

As T, u are arbitrary, τ is $\mathcal{T}$-invariant.

374L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra. Suppose that u, $v\in(M^{0,\infty}_{\bar{\mu}})^+$ are such that $\int u^*\times v^*=\infty$. Then there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\int u\times T_\pi v=\infty$.

proof I take three cases separately.

(a) Suppose that $\mathfrak{A}$ is purely atomic. Then it is surely infinite, since otherwise $\int u^*\times v^*$ could not be infinite. Let γ be the common measure of its atoms. For each $n\in\mathbb{N}$, set

$$\alpha_n=\inf\{\alpha:\alpha\ge0,\,\bar{\mu}[\![u>\alpha]\!]\le2^n\gamma\}.$$

Then $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is non-increasing, and

$$\bar{\mu}[\![u>\alpha_n]\!]\le2^n\gamma\le\bar{\mu}[\![u\ge\alpha_n]\!].$$

We can therefore choose a sequence $\langle\tilde{a}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ inductively so that

$$[\![u>\alpha_n]\!]\subseteq\tilde{a}_n\subseteq[\![u\ge\alpha_n]\!],\quad\bar{\mu}\tilde{a}_n=2^n\gamma,\quad\tilde{a}_n\subseteq\tilde{a}_{n+1},$$

for each n. Now if $\alpha_{n+1}\le\alpha<\alpha_n$, $2^n\gamma<\bar{\mu}[\![u>\alpha]\!]\le2^{n+1}\gamma$. So if we set

$$\tilde{u}=\|u\|_\infty\chi\,[0,\gamma[^\bullet\vee\sup_{n\in\mathbb{N}}\alpha_n\chi\,[2^n\gamma,2^{n+1}\gamma[^\bullet,$$

then $u^*\le\tilde{u}$ in $L^\infty(\mathfrak{A}_L)$. Set $a_n=\tilde{a}_{n+1}\setminus\tilde{a}_n$ for each n; then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and

$$\bar{\mu}a_n=2^n\gamma,\quad a_n\subseteq[\![u\ge\alpha_{n+1}]\!]$$

for each n.

Similarly, we can find a non-increasing sequence $\langle\beta_n\rangle_{n\in\mathbb{N}}$ in $[0,\infty[$ and a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that

$$\bar{\mu}b_n = 2^n\gamma, \quad b_n \subseteq [\![v \ge \beta_{n+1}]\!]$$

for each n, while

$$v^* \le \tilde{v} = \|v\|_\infty\chi\,[0,\gamma[^\bullet \vee \sup_{n\in\mathbb{N}}\alpha_n\chi\,[2^n\gamma, 2^{n+1}\gamma[^\bullet$$

in $L^\infty(\mathfrak{A}_L)$.

Now we are supposing that $\int u^*\times v^* = \infty$, so we must have

$$\infty = \int \tilde{u}\times\tilde{v} = \gamma\|u\|_\infty\|v\|_\infty + \sum_{n=0}^\infty 2^n\gamma\alpha_n\beta_n.$$

Because $2^{n+1}\alpha_{n+1}\beta_{n+1} \le 2\cdot 2^n\alpha_n\beta_n$ for each n, we must have

$$\sum_{n=0}^\infty 2^{2n+1}\alpha_{2n+1}\beta_{2n+1} = \infty.$$

At this point, recall that we are dealing with a purely atomic algebra in which every atom has measure γ. Let A_n, B_n be the sets of atoms included in a_n, b_n for each n, and $A = \bigcup_{n\in\mathbb{N}} A_n \cup B_n$. Then $\#(A_n) = \#(B_n) = 2^n$ for each n. We therefore have a bijection $\phi : A \to A$ such that $\phi[B_{2n}] = A_{2n}$ for every n. (The point is that $A\setminus\bigcup_{n\in\mathbb{N}} A_{2n}$ and $A\setminus\bigcup_{n\in\mathbb{N}} B_{2n}$ are both countably infinite.) Define $\pi : \mathfrak{A}\to\mathfrak{A}$ by setting

$$\pi c = (c\setminus \sup A) \cup \sup_{a\in A, a\subseteq c}\phi a.$$

Then π is well-defined (because A is countable), and it is easy to check that it is a measure-preserving Boolean automorphism (because it is just a permutation of the atoms); and $\pi b_{2n} = a_{2n}$ for every n. Consequently

$$\int u\times T_\pi v \ge \sum_{n=0}^\infty \alpha_{2n+1}\beta_{2n+1}\bar{\mu}a_{2n} = \sum_{n=0}^\infty 2^{2n}\gamma\alpha_{2n+1}\beta_{2n+1} = \infty.$$

So we have found a suitable automorphism.

(b) Next, consider the case in which $(\mathfrak{A},\bar{\mu})$ is atomless and of finite magnitude γ. Of course $\gamma > 0$. For each $n\in\mathbb{N}$ set

$$\alpha_n = \inf\{\alpha : \alpha\ge 0,\ \bar{\mu}[\![u>\alpha]\!] \le 2^{-n}\gamma\}.$$

Then

$$u^* \le \sup_{n\in\mathbb{N}}\alpha_{n+1}\chi\,[2^{-n-1}\gamma, 2^{-n}\gamma[^\bullet.$$

Also

$$\bar{\mu}[\![u>\alpha_n]\!] \le 2^{-n}\gamma \le \bar{\mu}[\![u\ge\alpha_n]\!]$$

for each n, so we can choose inductively a decreasing sequence $\langle\tilde{a}_n\rangle_{n\in\mathbb{N}}$ such that

$$[\![u>\alpha_n]\!] \subseteq \tilde{a}_n \subseteq [\![u\ge\alpha_n]\!]$$

and $\bar{\mu}\tilde{a}_n = 2^{-n}\gamma$ for each n. Set $a_n = \tilde{a}_n\setminus\tilde{a}_{n+1}$; then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\bar{\mu}a_n = 2^{-n-1}\gamma$, $a_n\subseteq[\![u\ge\alpha_n]\!]$ for each n.

In the same way, we can find $\langle\beta_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ such that $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, $\bar{\mu}b_n = 2^{-n-1}\gamma$ and $b_n\subseteq[\![v\ge\beta_n]\!]$ for each n, and $v^*\le\sup_{n\in\mathbb{N}}\beta_{n+1}\chi\,[2^{-n-1}\gamma, 2^{-n}\gamma[^\bullet$.

Now all the principal ideals $\mathfrak{A}_{a_n}$, $\mathfrak{A}_{b_n}$ are homogeneous and of the same Maharam type, so there are measure-preserving isomorphisms $\pi_n : \mathfrak{A}_{b_n}\to\mathfrak{A}_{a_n}$. Define $\pi:\mathfrak{A}\to\mathfrak{A}$ by setting $\pi c = \sup_{n\in\mathbb{N}}\pi_n(c\cap a_n)$; then π is a measure-preserving automorphism of $\mathfrak{A}$, and $\pi b_n = a_n$ for each n. Since $u\times\chi a_n \ge \alpha_n\chi a_n$, $v\times\chi b_n\ge\beta_n\chi b_n$ for each n,

$$\int u\times T_\pi v \ge \sum_{n=0}^\infty 2^{-n-1}\gamma\alpha_n\beta_n;$$

but on the other hand,

$$\int u^*\times v^* \le \sum_{n=0}^\infty 2^{-n-1}\gamma\alpha_{n+1}\beta_{n+1} \le 2\int u\times T_\pi v.$$

So $\int u\times T_\pi v = \infty$.

(c) Thirdly, consider the case in which $\mathfrak{A}$ is atomless and not totally finite; take κ to be the common Maharam type of all the principal ideals $\mathfrak{A}_a$ where $0<\bar{\mu}a<\infty$. In this case, set

$$\alpha_n = \inf\{\alpha : \bar\mu[\![u > \alpha]\!] \le 2^n\}, \quad \beta_n = \inf\{\alpha : \bar\mu[\![v > \alpha]\!] \le 2^n\}$$

for each $n \in \mathbb{Z}$. This time

$$u^* \le \sup_{n\in\mathbb{Z}} \alpha_n \chi\,[2^n, 2^{n+1}[^\bullet, \quad v^* \le \sup_{n\in\mathbb{Z}} \beta_n \chi\,[2^n, 2^{n+1}[^\bullet.$$

There are disjoint families $\langle a_n\rangle_{n\in\mathbb{Z}}$, $\langle b_n\rangle_{n\in\mathbb{Z}}$ such that $\bar\mu a_n = \bar\mu b_n = 2^n$ for each n and

$$u \ge \sup_{n\in\mathbb{Z}} \alpha_{n+1}\chi a_n, \quad v \ge \sup_{n\in\mathbb{Z}} \beta_{n+1}\chi b_n.$$

(This time, start by fixing $\tilde a_0$ such that $\bar\mu\tilde a_0 = 1$ and $[\![u > \alpha_0]\!] \subseteq \tilde a_0 \subseteq [\![u \ge \alpha_0]\!]$, and choose $\tilde a_{n+1} \supseteq \tilde a_n$ for $n \ge 0$, $\tilde a_{n-1} \subseteq \tilde a_n$ for $n \le 0$.)

Set $d^* = \sup_{n\in\mathbb{Z}} a_n \cup \sup_{n\in\mathbb{Z}} b_n$. Then

$$d_1 = d^* \setminus \sup_{n\in\mathbb{Z}} a_{2n}, \quad d_2 = d^* \setminus \sup_{n\in\mathbb{Z}} b_{2n}$$

both have magnitude ω and Maharam type κ. So there is a measure-preserving isomorphism $\tilde\pi : \mathfrak{A}_{d_2} \to \mathfrak{A}_{d_1}$ (332J). At the same time, for each $n \in \mathbb{Z}$ there is a measure-preserving isomorphism $\pi_n : \mathfrak{A}_{b_{2n}} \to \mathfrak{A}_{a_{2n}}$. So once again we can assemble these to form a measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$, defined by the formula

$$\pi c = (c \setminus d^*) \cup \tilde\pi(c \cap d_2) \cup \sup_{n\in\mathbb{Z}} \pi_n(c \cap b_{2n}).$$

Just as in (a) and (b) above,

$$\int u \times T_\pi v \ge \sum_{n=-\infty}^{\infty} 2^{2n}\alpha_{2n+1}\beta_{2n+1},$$

while

$$\int u^* \times v^* \le \sum_{n=-\infty}^{\infty} 2^n \alpha_n \beta_n$$

is infinite. Because

$$2^{2n+2}\alpha_{2n+2}\beta_{2n+2} + 2^{2n+1}\alpha_{2n+1}\beta_{2n+1} \le 6 \cdot 2^{2n}\alpha_{2n+1}\beta_{2n+1}$$

for every n,

$$\int u \times T_\pi v \ge \frac{1}{6}\int u^* \times v^* = \infty.$$

Thus we have a suitable π in any of the cases allowed by 374H.

374M Proposition Let $(\mathfrak{A}, \bar\mu)$ be a quasi-homogeneous localizable measure algebra, and $U \subseteq L^0 = L^0(\mathfrak{A})$ a solid linear subspace which, regarded as a Riesz space, is perfect. If U is rearrangement-invariant and $M^{\infty,1}_{\bar\mu} \subseteq U \subseteq M^{1,\infty}_{\bar\mu}$, then U is $\mathcal{T}$-invariant.

proof Set $V = \{v : u \times v \in L^1$ for every $u \in U\}$, so that V is a solid linear subspace of L^0 which can be identified with $U^\times$ (369C), and U becomes $\{u : u \times v \in L^1$ for every $v \in V\}$; note that $M^{\infty,1}_{\bar\mu} \subseteq V \subseteq M^{1,\infty}_{\bar\mu}$ (using 369Q).

If $u \in U^+$, $v \in V^+$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, then $T_\pi u \in U$, so $\int v \times T_\pi u < \infty$; by 374L, $\int u^* \times v^*$ is finite. But this means that if $u \in U$, $v \in V$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$,

$$\int |Tu \times v| \le \int u^* \times v^* < \infty.$$

As v is arbitrary, $Tu \in U$; as T and u are arbitrary, U is $\mathcal{T}$-invariant.

374X Basic exercises >(a) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq M^{1,\infty}_{\bar\mu}$ a $\mathcal{T}$-invariant set. (i) Show that A is solid. (ii) Show that if A is a linear subspace and not $\{0\}$, then it includes $M^{\infty,1}_{\bar\mu}$. (iii) Show that if $u \in A$, $v \in M^{0,\infty}_{\bar\mu}$ and $\int_0^t v^* \le \int_0^t u^*$ for every $t > 0$, then $v \in A$. (iv) Show that if $(\mathfrak{B}, \bar\nu)$ is any other measure algebra, then $B = \{Tu : u \in A, T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ and $C = \{v : v \in M^{1,\infty}_{\bar\nu}, Tv \in A$ for every $T \in \mathcal{T}_{\bar\nu,\bar\mu}\}$ are $\mathcal{T}$-invariant subsets of $M^{1,\infty}_{\bar\nu}$, and that $B \subseteq C$. Give two examples in which $B \subset C$. Show that if $(\mathfrak{A}, \bar\mu) = (\mathfrak{A}_L, \bar\mu_L)$ then $B = C$.

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that the extended Fatou norm $\| \|_p$ on $L^0(\mathfrak{A})$ is $\mathcal{T}$-invariant for every $p \in [1, \infty]$. (*Hint*: 371Gd.)

(c) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, and ϕ a Young's function (369Xc). Let τ_ϕ, $\tilde{\tau}_\phi$ be the corresponding Orlicz norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$. Show that $\tilde{\tau}_\phi(Tu) \le \tau_\phi(u)$ for every $u \in L^0(\mathfrak{A})$, $T \in \mathcal{T}_{\bar{\mu},\bar{\nu}}$. (*Hint*: 369Xn, 373Xm.) In particular, τ_ϕ is $\mathcal{T}$-invariant.

(d) Show that if $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra and τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$, then the Banach lattice L^τ defined from τ is $\mathcal{T}$-invariant.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on L^0 which is an order-continuous norm on L^τ. Show that $L^\tau \subseteq M^{1,0}_{\bar{\mu}}$.

(f) Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ and $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ two semi-finite measure algebras. Let τ_1, τ_2 be the extended Fatou norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ defined from θ by the method of 374C. Show that $\tau_2(Tu) \le \tau_1(u)$ whenever $u \in M^{1,\infty}_{\bar{\mu}}$ and $T \in \mathcal{T}_{\bar{\mu},\bar{\nu}}$.

>(g) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$, and set $\tau(u) = \sup_{0<\bar{\mu}a<\infty} \frac{1}{\sqrt{\bar{\mu}a}} \int_a |u|$ for $u \in L^0(\mathfrak{A})$. Show that τ is a $\mathcal{T}$-invariant extended Fatou norm. Find examples of $(\mathfrak{A},\bar{\mu})$ for which τ is, and is not, order-continuous on L^τ.

(h) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. (i) Show that there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{B})$ defined by setting $\theta(v) = \sup\{\tau(Tv) : T \in \mathcal{T}_{\bar{\nu},\bar{\mu}}\}$ for $v \in M^{1,\infty}_{\bar{\nu}}$. (ii) Show that $\theta'(w) = \sup\{\tau'(Tw) : T \in \mathcal{T}_{\bar{\nu},\bar{\mu}}\}$ for every $w \in M^{1,\infty}_{\bar{\nu}}$. (iii) Show that when $(\mathfrak{A},\bar{\mu}) = (\mathfrak{A}_L,\bar{\mu}_L)$ then $\theta(v) = \tau(v^*)$ for every $v \in M^{0,\infty}_{\bar{\nu}}$. (iv) Show that when $(\mathfrak{B},\bar{\nu}) = (\mathfrak{A}_L,\bar{\mu}_L)$ then $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar{\mu}}$.

(i) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Suppose that L^τ is a $\mathcal{T}$-invariant subset of L^0. Show that there is a $\mathcal{T}$-invariant extended Fatou norm $\tilde{\tau}$ which is equivalent to τ in the sense that, for some $M > 0$, $\tilde{\tau}(u) \le M\tau(u) \le M^2\tilde{\tau}(u)$ for every $u \in L^0$. (*Hint*: show first that $\int u^* \times v^* < \infty$ for every $u \in L^\tau$ and $v \in L^{\tau'}$, then that $\sup_{\tau(u)\le 1,\tau'(v)\le 1} \int u^* \times v^* < \infty$.)

(j) Suppose that τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and that $0 < w = w^* \in M^{1,\infty}_{\bar{\mu}_L}$. Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. Show that the function $u \mapsto \tau(w \times u^*)$ extends to a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A})$. (*Hint*: $\tau(w \times u^*) = \sup\{\tau(w \times Tu) : T \in \mathcal{T}_{\bar{\mu},\bar{\mu}_L}\}$ for $u \in M^{1,\infty}_{\bar{\mu}_L}$.) (When $\tau = \| \|_p$ these norms are called **Lorentz norms**; see Lindenstrauss & Tzafriri 79, p. 121.)

(k) Let $(\mathfrak{A},\bar{\mu})$ be $\mathcal{P}\mathbb{N}$ with counting measure. Identify $L^0(\mathfrak{A})$ with $\mathbb{R}^{\mathbb{N}}$. Let U be $\{u : u \in \mathbb{R}^{\mathbb{N}},\ \{n : u(n) \ne 0\}$ is finite$\}$. Show that U is a perfect Riesz space, and is rearrangement-invariant but not $\mathcal{T}$-invariant.

(l) Let $(\mathfrak{A},\bar{\mu})$ be an atomless quasi-homogeneous localizable measure algebra, and $U \subseteq L^0(\mathfrak{A})$ a rearrangement-invariant solid linear subspace which is a perfect Riesz space. Show that $U \subseteq M^{1,\infty}_{\bar{\mu}}$ and that U is $\mathcal{T}$-invariant. (*Hint*: assume $U \ne \{0\}$. Show that (i) $\chi a \in U$ whenever $\bar{\mu}a < \infty$ (ii) $V = \{v : v \times u \in L^1\ \forall\ u \in U\}$ is rearrangement-invariant (iii) U, $V \subseteq M^{1,\infty}$.)

374Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $U \subseteq M^{1,\infty}_{\bar{\mu}}$ a non-zero $\mathcal{T}$-invariant Riesz subspace which, regarded as a Riesz space, is perfect. (i) Show that U includes $M^{\infty,1}_{\bar{\mu}}$. (ii) Show that its dual $\{v : v \in L^0,\ v \times u \in L^1\ \forall\ u \in U\}$ (which in this exercise I will denote by $U^\times$) is also $\mathcal{T}$-invariant, and is $\{v : v \in M^{0,\infty}_{\bar{\mu}},\ \int u^* \times v^* < \infty\ \forall\ u \in U\}$. (iii) Show that for any localizable measure algebra $(\mathfrak{B},\nu)$ the set $V = \{v : v \in M^{1,\infty}_{\bar{\nu}},\ Tv \in U\ \forall\ T \in \mathcal{T}_{\bar{\nu},\bar{\mu}}\}$ is a perfect Riesz subspace of $L^0(\mathfrak{B})$, and that $V^\times = \{v : v \in M^{1,\infty}_{\bar{\nu}},\ Tv \in U^\times\ \forall\ T \in \mathcal{T}_{\bar{\nu},\bar{\mu}}\}$. (iv) Show that if, in (i)-(iii), $(\mathfrak{A},\bar{\mu}) = (\mathfrak{A}_L,\bar{\mu}_L)$, then $V = \{v : v \in M^{0,\infty},\ v^* \in U\}$. (v) Show that if, in (iii), $(\mathfrak{B},\bar{\nu}) = (\mathfrak{A}_L,\bar{\mu}_L)$, then $U = \{u : u \in M^{0,\infty},\ u^* \in V\}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and suppose that $1 \le q \le p < \infty$. Let $w_{pq} \in L^0(\mathfrak{A}_L)$ be the equivalence class of the function $t \mapsto t^{(q-p)/p}$. (i) Show that for any $u \in L^0(\mathfrak{A})$,

$$\int w_{pq} \times (u^*)^q = p \int_0^\infty t^{q-1} (\bar{\mu}[\![|u| > t]\!])^{q/p} dt.$$

(ii) Show that we have an extended Fatou norm $\| \|_{p,q}$ on $L^0(\mathfrak{A})$ defined by setting

$$\|u\|_{p,q} = \bigl(p\int_0^\infty t^{q-1}(\bar\mu[\![|u|>t]\!])^{q/p}dt\bigr)^{1/q}$$

for every $u \in L^0(\mathfrak{A})$. (*Hint*: use 374Xj with $w = w_{pq}^{1/q}$, $\| \| = \| \|_q$.) (iii) Show that if $(\mathfrak{B},\bar\nu)$ is another measure algebra and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$, then $\|Tu\|_{p,q} \le \|u\|_{p,q}$ for every $u \in M^{1,\infty}_{\bar\mu}$. (iv) Show that $\| \|_{p,q}$ is an order-continuous norm on $L^{\| \|_{p,q}}$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a homogeneous measure algebra of uncountable Maharam type, and u, $v \ge 0$ in $M^0_{\bar\mu}$ such that $u^* = v^*$. Show that there is a measure-preserving automorphism π of $\mathfrak{A}$ such that $T_\pi u = v$, where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

(d) In $L^0(\mathfrak{A}_L)$ let u be the equivalence class of the function $f(t) = te^{-t}$. Show that there is no Boolean automorphism π of $\mathfrak{A}_L$ such that $T_\pi u = u^*$. (*Hint*: show that $\mathfrak{A}_L$ is τ-generated by $\{[\![u^* > \alpha]\!] : \alpha > 0\}$.)

(e) Let $(\mathfrak{A},\bar\mu)$ be a quasi-homogeneous semi-finite measure algebra and $C \subseteq L^0(\mathfrak{A})$ a solid convex order-closed rearrangement-invariant set. Show that $C \cap M^{1,\infty}_{\bar\mu}$ is $\mathcal{T}$-invariant.

374 Notes and comments I gave this section the title ‘rearrangement-invariant spaces’ because it looks good on the Contents page, and it follows what has been common practice since Luxemburg 67b; but actually I think that it's $\mathcal{T}$-invariance which matters, and that rearrangement-invariant spaces are significant largely because the important ones are $\mathcal{T}$-invariant. The particular quality of $\mathcal{T}$-invariance which I have tried to bring out here is its transferability from one measure algebra (or measure space, of course) to another. This is what I take at a relatively leisurely pace in 374B-374D and 374Xf, and then encapsulate in 374Xh and 374Ya. The special place of the Lebesgue algebra $(\mathfrak{A}_L,\bar\mu_L)$ arises from its being more or less the simplest algebra over which every $\mathcal{T}$-invariant set can be described; see 374Xa.

I don't think this section is particularly easy, and (as in §373) there are rather a lot of unattractive names in it; but once one has achieved a reasonable familiarity with the concepts, the techniques used can be seen to amount to half a dozen ideas – non-trivial ideas, to be sure – from §§369 and 373. From §369 I take concepts of duality: the symmetric relationship between a perfect Riesz space $U \subseteq L^0$ and the representation of its dual (369C-369D), and the notion of associate extended Fatou norms (369H-369K). From §373 I take the idea of ‘decreasing rearrangement’ and theorems guaranteeing the existence of useful members of $\mathcal{T}_{\bar\mu,\bar\nu}$ (373O-373Q). The results of the present section all depend on repeated use of these facts, assembled in a variety of patterns.

There is one new method here, but an easy one: the construction of measure-preserving automorphisms by joining isomorphisms together, as in the proofs of 374H and 374J. I shall return to this idea, in greater generality and more systematically investigated, in §381. I hope that the special cases here will give no difficulty.

While $\mathcal{T}$-invariance is a similar phenomenon for both extended Fatou norms and perfect Riesz spaces (see 374Xh, 374Ya), the former seem easier to deal with. The essential difference is I think in 374B(i); with a $\mathcal{T}$-invariant extended Fatou norm, we are necessarily confined to $M^{1,\infty}$, the natural domain of the methods used here. For perfect Riesz spaces we have examples like $\mathbb{R}^{\mathbb{N}} \cong L^0(\mathcal{P}\mathbb{N})$ and its dual, the space of eventually-zero sequences (374Xk); these are rearrangement-invariant but not $\mathcal{T}$-invariant, as I have defined it. This problem does not arise over atomless algebras (374Xl).

I think it is obvious that for algebras which are not quasi-homogeneous (374G) rearrangement-invariance is going to be of limited interest; there will be regions between which there is no communication by means of measure-preserving automorphisms, and the best we can hope for is a discussion of quasi-homogeneous components, if they exist, corresponding to the partition of unity used in the proof of 332J. There is a special difficulty concerning rearrangement-invariance in $L^0(\mathfrak{A}_L)$: two elements can have the same decreasing rearrangement without being rearrangements of each other in the strict sense (373Ya, 374Yd). The phenomenon of 373Ya is specific to algebras of countable Maharam type (374Yc). You will see that some of the labour of 374L is because we have to make room for the pieces to move in. 374J is easier just because in that context we can settle for a supremum, rather than an actual infinity, so the rearrangement needed (part (a) of the proof) can be based on a region of finite measure.

375 Kwapien's theorem

In §368 and the first part of §369 I examined maps from various types of Riesz space into L^0 spaces. There are equally striking results about maps out of L^0 spaces. I start with some relatively elementary facts about positive linear operators from L^0 spaces to Archimedean Riesz spaces in general (375A-375D), and then turn to a remarkable analysis, due essentially to S.Kwapien, of the positive linear operators from a general L^0 space to the L^0 space of a semi-finite measure algebra (375I), with a couple of simple corollaries.

375A Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and W an Archimedean Riesz space. If $T : L^0(\mathfrak{A}) \to W$ is a positive linear operator, it is sequentially order-continuous.

proof (a) The first step is to observe that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is any non-increasing sequence in $L^0 = L^0(\mathfrak{A})$ with infimum 0, and $\epsilon > 0$, then $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above in L^0. **P** For $k \in \mathbb{N}$ set $a_k = \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$; set $a = \inf_{k\in\mathbb{N}} a_k$. **?** Suppose, if possible, that $a \neq 0$. Because $u_n \le u_0$, $n(u_n - \epsilon u_0) \le nu_0$ for every n and

$$a \subseteq a_0 \subseteq [\![u_0 > 0]\!] = [\![\epsilon u_0 > 0]\!] = \sup_{n\in\mathbb{N}} [\![\epsilon u_0 - u_n > 0]\!].$$

So there is some $m \in \mathbb{N}$ such that $a' = a \cap [\![\epsilon u_0 - u_m > 0]\!] \neq 0$. Now, for any $n \ge m$, any $k \in \mathbb{N}$,

$$a' \cap [\![n(u_n - \epsilon u_0) > k]\!] \subseteq [\![\epsilon u_0 - u_m > 0]\!] \cap [\![u_m - \epsilon u_0 > 0]\!] = 0.$$

But $a' \subseteq \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$, so in fact

$$a' \subseteq \sup_{n\le m} [\![n(u_n - \epsilon u_0) > k]\!] = [\![v > k]\!],$$

where $v = \sup_{n\le m} n(u_n - \epsilon u_0)$. And this means that $\inf_{k\in\mathbb{N}} [\![v > k]\!] \supseteq a' \neq 0$, which is impossible. **X** Accordingly $a = 0$; by 364Ma, $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above. **Q**

(b) Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^0 with infimum 0, and that $w \in W$ is a lower bound for $\{Tu_n : n \in \mathbb{N}\}$. Take any $\epsilon > 0$. By (a), $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ has an upper bound v in L^0. Because T is positive,

$$w \le Tu_n = T(u_n - \epsilon u_0) + T(\epsilon u_0) \le T(\frac{1}{n}v) + T(\epsilon u_0) = \frac{1}{n}Tv + \epsilon Tu_0$$

for every $n \ge 1$. Because W is Archimedean, $w \le \epsilon Tu_0$. But this is true for every $\epsilon > 0$, so (again because W is Archimedean) $w \le 0$. As w is arbitrary, $\inf_{n\in\mathbb{N}} Tu_n = 0$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous (351Gb).

375B Proposition Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra. Then $L^0(\mathfrak{A})^\times = \{0\}$.

proof ? Suppose, if possible, that $h : L^0(\mathfrak{A}) \to \mathbb{R}$ is a non-zero order-continuous positive linear functional. Then there is a $u > 0$ in L^0 such that $h(v) > 0$ whenever $0 < v \le u$ (356H). Because $\mathfrak{A}$ is atomless, there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq [\![u > 0]\!]$ for each n, so that $u_n = u \times \chi a_n > 0$, while $u_m \wedge u_n = 0$ if $m \neq n$. Now however

$$v = \sup_{n\in\mathbb{N}} n(h(u_n))^{-1}u_n$$

is defined in L^0, by 368K, and $h(v) \ge n$ for every n, which is impossible. **X**

375C Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, W an Archimedean Riesz space, and $T : L^0(\mathfrak{A}) \to W$ an order-continuous Riesz homomorphism. Then $V = T[L^0(\mathfrak{A})]$ is an order-closed Riesz subspace of W.

proof The kernel U of T is a band in $L^0 = L^0(\mathfrak{A})$ (352Oe), and must be a projection band (353I), because L^0 is Dedekind complete (364O). Since $U + U^\perp = L^0$, $T[U] + T[U^\perp] = V$, that is, $T[U^\perp] = V$; since $U \cap U^\perp = \{0\}$, T is an isomorphism between $U^\perp$ and V. Now suppose that $A \subseteq V$ is upwards-directed and has a least upper bound $w \in W$. Then $B = \{u : u \in U^\perp, Tu \in A\}$ is upwards-directed and $T[B] = A$. The point is that B is bounded above in L^0. **P?** If not, then $\{u^+ : u \in B\}$ cannot be bounded above, so there is a $u_0 > 0$ in L^0 such that $nu_0 = \sup_{u\in B} nu_0 \wedge u^+$ for every $n \in \mathbb{N}$ (368A). Since $B \subseteq U^\perp$, $u_0 \in U^\perp$ and $Tu_0 > 0$. But now, because T is an order-continuous Riesz homomorphism,

$$nTu_0 = \sup_{u\in B} T(nu_0 \wedge u^+) = \sup_{v\in A} nTu_0 \wedge v^+ \le w^+$$

for every $n \in \mathbb{N}$, which is impossible. **XQ**

Set $u^* = \sup B$; then $Tu^* = \sup A = w$ and $w \in V$. As A is arbitrary, V is order-closed.

375D Corollary Let W be an Archimedean Riesz space and V an order-dense Riesz subspace which is isomorphic to $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$. Then $V = W$.

proof Apply 375C to an isomorphism $T : L^0(\mathfrak{A}) \to V$ to see that V is order-closed in W.

375E I come now to the deepest result of this section, concerning positive linear operators from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$ where $\mathfrak{B}$ is a measure algebra. I approach through a couple of lemmas which are striking enough in their own right.

The following temporary definition will be useful.

Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. I will say that a function $\phi : \mathfrak{A} \to \mathfrak{B}$ is a **σ-subhomomorphism** if

$\phi(a \cup a') = \phi(a) \cup \phi(a')$ for all a, $a' \in \mathfrak{A}$,

$\inf_{n\in\mathbb{N}} \phi(a_n) = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

Now we have the following easy facts.

375F Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism.

(a) $\phi(0) = 0$, $\phi(a) \subseteq \phi(a')$ whenever $a \subseteq a'$, and $\phi(a) \setminus \phi(a') \subseteq \phi(a \setminus a')$ for every a, $a' \in \mathfrak{A}$.

(b) If $\bar\mu$, $\bar\nu$ are measures such that $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are totally finite measure algebras, then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\bar\nu\phi(a) \le \epsilon$ whenever $\bar\mu a \le \delta$.

proof (a) This is elementary. Set every $a_n = 0$ in the second clause of the definition 375E to see that $\phi(0) = 0$. The other two parts are immediate consequences of the first clause.

(b) (Compare 232B, 327Bb.) **?** Suppose, if possible, otherwise. Then for every $n \in \mathbb{N}$ there is an $a_n \in \mathfrak{A}$ such that $\bar\mu a_n \le 2^{-n}$ and $\bar\nu\phi(a_n) \ge \epsilon$. Set $c_n = \sup_{i\ge n} a_i$ for each n; then $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 (since $\bar\mu c_n \le 2^{-n+1}$ for each n), but $\bar\nu\phi(c_n) \ge \epsilon$ for every n, so $\inf_{n\in\mathbb{N}} \phi c_n$ cannot be 0. **X**

375G Lemma Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and an $m \in \mathbb{N}$ such that $b \cap \inf_{j\le m} \phi(a_j) = 0$ whenever $a_0, \dots, a_m \in \mathfrak{A}$ are disjoint.

proof (a) Suppose first that $\mathfrak{A}$ is atomless and that $\bar\mu 1 = 1$.

Set $\epsilon = \frac{1}{5}\bar\nu b_0$ and let $m \ge 1$ be such that $\bar\nu\phi(a) \le \epsilon$ whenever $\bar\mu a \le \frac{1}{m}$. We need to know that $(1 - \frac{1}{m})^m \le \frac{1}{2}$; this is because (if $m \ge 2$) $\ln m - \ln(m - 1) \ge \frac{1}{m}$, so $m\ln(1 - \frac{1}{m}) \le -1 \le -\ln 2$.

Set

$$C = \{\inf_{j\le m} \phi(a_j) : a_0, \dots, a_m \in \mathfrak{A} \text{ are disjoint}\}.$$

? Suppose, if possible, that $b_0 \subseteq \sup C$. Then there are $c_0, \dots, c_k \in C$ such that $\bar\nu(b_0 \cap \sup_{i\le k} c_i) \ge 4\epsilon$. For each $i \le k$ choose disjoint $a_{i0}, \dots, a_{im} \in \mathfrak{A}$ such that $c_i = \inf_{j\le m} \phi(a_{ij})$. Let D be the set of atoms of the finite subalgebra of $\mathfrak{A}$ generated by $\{a_{ij} : i \le k, j \le m\}$, so that D is a finite partition of unity in $\mathfrak{A}$, and every a_{ij} is the join of the members of D it includes. Set $p = \#(D)$, and for each $d \in D$ take a maximal disjoint set $E_d \subseteq \{e : e \subseteq d, \bar\mu e = \frac{1}{pm}\}$, so that $\bar\mu(d \setminus \sup E_d) < \frac{1}{pm}$; set

$$d^* = 1 \setminus \sup(\bigcup_{d\in D} E_d) = \sup_{d\in D}(d \setminus \sup E_d),$$

so that $\bar\mu d^*$ is a multiple of $\frac{1}{pm}$ and is less than $\frac{1}{m}$. Let E^* be a disjoint set of elements of measure $\frac{1}{pm}$ with union d^*, and take $E = E^* \cup \bigcup_{d\in D} E_d$, so that E is a partition of unity in $\mathfrak{A}$, $\bar\mu e = \frac{1}{pm}$ for every $e \in E$, and $a_{ij} \setminus d^*$ is the join of the members of E it includes for every $i \le k$, $j \le m$.

Set

$$\mathcal{K} = \{K : K \subseteq E, \#(K) = p\}, \quad M = \#(\mathcal{K}) = \frac{(mp)!}{p!(mp-p)!}.$$

For every $K\in\mathcal{K}$, $\bar\mu(\sup K)=\frac{1}{m}$ so $\bar\nu\phi(\sup K)\le\epsilon$. So if we set

$$v=\sum_{K\in\mathcal{K}}\chi\phi(\sup K),$$

$\int v\le\epsilon M$. On the other hand,

$$\bar\nu(b_0\cap\sup_{i\le k}c_i)\ge 4\epsilon,\quad \bar\nu\phi(d^*)\le\epsilon,$$

so $\bar\nu b_1\ge 3\epsilon$, where

$$b_1=b_0\cap\sup_{i\le k}c_i\setminus\phi(d^*).$$

Accordingly $\int v\le\frac{1}{3}M\bar\nu b_1$ and

$$b_2=b_1\cap[\![v<\tfrac{1}{2}M]\!]$$

is non-zero.

Because $b_2\subseteq b_1$, there is an $i\le k$ such that $b_2\cap c_i\ne 0$. Now

$$b_2\cap c_i\subseteq c_i\setminus\phi(d^*)=\inf_{j\le m}\phi(a_{ij})\setminus\phi(d^*)\subseteq\ \inf_{j\le m}\phi(a_{ij}\setminus d^*).$$

But every $a_{ij}\setminus d^*$ is the join of the members of E it includes, so

$$\begin{aligned}b_2\cap c_i&\subseteq\inf_{j\le m}\phi(a_{ij}\setminus d^*)\subseteq\inf_{j\le m}\phi(\sup\{e:e\in E,\ e\subseteq a_{ij}\})\\&=\inf_{j\le m}\sup\{\phi(e):e\in E,\ e\subseteq a_{ij}\}\\&=\sup\{\inf_{j\le m}\phi(e_j):e_0,\dots,e_m\in E\text{ and }e_j\subseteq a_{ij}\text{ for every }j\}.\end{aligned}$$

So there are $e_0,\dots,e_m\in E$ such that $e_j\subseteq a_{ij}$ for each j and $b_3=b_2\cap\inf_{j\le m}\phi(e_j)\ne 0$. Because $a_{i0},\dots,a_{im}$ are disjoint, $e_0,\dots,e_m$ are distinct; set $J=\{e_0,\dots,e_m\}$. Then whenever $K\in\mathcal{K}$ and $K\cap J\ne\emptyset$, $b_3\subseteq\phi(\sup K)$.

So let us calculate the size of $\mathcal{K}_1=\{K:K\in\mathcal{K},\ K\cap J\ne\emptyset\}$. This is

$$\begin{aligned}M-\tfrac{(mp-m-1)!}{p!(mp-p-m-1)!}&=M\bigl(1-\tfrac{(mp-p)(mp-p-1)\dots(mp-p-m)}{mp(mp-1)\dots(mp-m)}\bigr)\\&\ge M\bigl(1-(\tfrac{mp-p}{mp})^{m+1}\bigr)\ge\frac{1}{2}M.\end{aligned}$$

But this means that $b_3\subseteq[\![v\ge\frac{1}{2}M]\!]$, while also $b_3\subseteq[\![v<\frac{1}{2}M]\!]$; which is surely impossible. **X**

Accordingly $b_0\not\subseteq\sup C$, and we can take $b=b_0\setminus\sup C$.

(b) Now for the general case. Let A be the set of atoms of $\mathfrak{A}$, and set $d=1\setminus\sup A$. Then $\mathfrak{A}_d$ is atomless, so there are a non-zero $b_1\subseteq b_0$ and an $n\in\mathbb{N}$ such that $b_1\cap\inf_{j\le n}\phi(a_j)=0$ whenever $a_0,\dots,a_n\in\mathfrak{A}_d$ are disjoint. **P** If $\bar\mu d>0$ this follows from (a), if we apply it to $\phi\restriction\mathfrak{A}_d$ and $(\bar\mu d)^{-1}\bar\mu\restriction\mathfrak{A}_d$. If $\bar\mu d=0$ then we can just take $b_1=b_0$, $n=0$. **Q**

Let $\delta>0$ be such that $\bar\nu\phi(a)<\bar\nu b_1$ whenever $\bar\mu a\le\delta$. Let $A_1\subseteq A$ be a finite set such that $\bar\mu(\sup A_1)\ge\bar\mu(\sup A)-\delta$, and set $r=\#(A)$, $d^*=\sup(A\setminus A_1)$. Then $\bar\mu d^*\le\delta$ so $b=b_1\setminus\phi(d^*)\ne 0$. Try $m=n+r$. If $a_0,\dots,a_m$ are disjoint, then at most r of them can meet $\sup A_1$, so (re-ordering if necessary) we can suppose that $a_0,\dots,a_n$ are disjoint from $\sup A_1$, in which case $a_j\setminus d^*\subseteq d$ for each $j\le m$. But in this case (because $b\cap\phi(d^*)=0$)

$$b\cap\inf_{j\le m}\phi(a_j)\subseteq b\cap\inf_{j\le n}\phi(a_j)=b\cap\inf_{j\le n}\phi(a_j\cap d)=0$$

by the choice of n and b_1.

Thus in the general case also we can find appropriate b and m.

375H Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras and $\phi:\mathfrak{A}\to\mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0\in\mathfrak{B}$ there are a non-zero $b\subseteq b_0$ and a finite partition of unity $C\subseteq\mathfrak{A}$ such that $a\mapsto b\cap\phi(a\cap c)$ is a ring homomorphism for every $c\in C$.

proof By 375G, we can find b_1, m such that $0\ne b_1\subseteq b_0$ and $b_1\cap\inf_{j\le m}\phi(a_j)=0$ whenever $a_0,\dots,a_m\in\mathfrak{A}$ are disjoint. Do this with the smallest possible m. If $m=0$ then $b_1\cap\phi(1)=0$, so we can take

$b = b_1$, $C = \{1\}$. Otherwise, because m is minimal, there must be disjoint $c_1, \dots , c_m \in \mathfrak{A}$ such that $b = b_1 \cap \inf_{1\le j\le m} \phi(c_j) \ne 0$. Set $c_0 = 1 \setminus \sup_{1\le j\le m} c_j$, $C = \{c_0, c_1, \dots , c_m\}$; then C is a partition of unity in $\mathfrak{A}$. Set $\pi_j(a) = b \cap \phi(a \cap c_j)$ for each $a \in \mathfrak{A}$, $j \le m$. Then we always have $\pi_j(a \cup a') = \pi_j(a) \cup \pi_j(a')$ for all a, $a' \in \mathfrak{A}$, because ϕ is a subhomomorphism.

To see that every π_j is a ring homomorphism, we need only check that $\pi_j(a \cap a') = 0$ whenever $a \cap a' = 0$. (Compare 312H(iv).) In the case $j = 0$, we actually have $\pi_0(a) = 0$ for every a, because $b \cap \phi(c_0) = b_1 \cap \inf_{0\le j\le m} \phi(c_j) = 0$ by the choice of b_1 and m. When $1 \le j \le m$, if $a \cap a' = 0$, then

$$\pi_j(a) \cap \pi_j(a') = b_1 \cap \inf_{1\le i\le m, i\ne j} \phi(c_j) \cap \phi(a) \cap \phi(a')$$

is again 0, because a, a', $c_1, \dots , c_{j-1}, c_{j+1}, \dots , c_m$ are disjoint. So we have a suitable pair b, C.

375I Theorem Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra and $(\mathfrak{B}, \bar\nu)$ a semi-finite measure algebra. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Then we can find B, $\langle A_b\rangle_{b\in B}$ such that B is a partition of unity in $\mathfrak{B}$, each A_b is a finite partition of unity in $\mathfrak{A}$, and $u \mapsto T(u \times \chi a) \times \chi b$ is a Riesz homomorphism for every $b \in B$, $a \in A_b$.

proof (a) Write B^* for the set of potential members of B; that is, the set of those $b \in \mathfrak{B}$ such that there is a finite partition of unity $A \subseteq \mathfrak{A}$ such that T_{ab} is a Riesz homomorphism for every $a \in A$, writing $T_{ab}(u) = T(u \times \chi a) \times \chi b$. If I can show that B^* is order-dense in $\mathfrak{B}$, this will suffice, since there will then be a partition of unity $B \subseteq B^*$.

(b) So let b_0 be any non-zero member of $\mathfrak{B}$; I seek a non-zero member of B^* included in b_0. Of course there is a non-zero $b_1 \subseteq b_0$ with $\bar\nu b_1 < \infty$. Let $\gamma > 0$ be such that $b_2 = b_1 \cap [\![T(\chi 1) \le \gamma]\!]$ is non-zero. Define $\mu : \mathfrak{A} \to [0, \infty[$ by setting $\mu a = \int_{b_2} T(\chi a)$ for every $a \in \mathfrak{A}$. Then μ is countably additive, because χ, T and $\int$ are all additive and sequentially order-continuous (using 375A). Set $\mathcal{N} = \{a : \mu a = 0\}$; then $\mathcal{N}$ is a σ-ideal of $\mathfrak{A}$, and $(\mathfrak{C}, \bar\mu)$ is a totally finite measure algebra, where $\mathfrak{C} = \mathfrak{A}/\mathcal{N}$ and $\bar\mu a^\bullet = \mu a$ for every $a \in \mathfrak{A}$ (just as in 321H).

(c) We have a function ϕ from $\mathfrak{C}$ to the principal ideal $\mathfrak{B}_{b_2}$ defined by saying that $\phi a^\bullet = b_2 \cap [\![T(\chi a) > 0]\!]$ for every $a \in \mathfrak{A}$. **P** If a_1, $a_2 \in \mathfrak{A}$ are such that $a_1^\bullet = a_2^\bullet$ in $\mathfrak{C}$, this means that $a_1 \triangle a_2 \in \mathcal{N}$; now

$$\begin{aligned}[\![T(\chi a_1) > 0]\!] \triangle [\![T(\chi a_2) > 0]\!] &\subseteq [\![|T(\chi a_1) - T(\chi a_2)| > 0]\!]\\ &\subseteq [\![T(|\chi a_1 - \chi a_2|) > 0]\!] = [\![T\chi(a_1 \triangle a_2) > 0]\!]\end{aligned}$$

is disjoint from b_2 because $\int_{b_2} T\chi(a_1 \triangle a_2) = 0$. Accordingly $b_2 \cap [\![T(\chi a_1) > 0]\!] = b_2 \cap [\![T(\chi a_2) > 0]\!]$ and we can take this common value for $\phi(a_1^\bullet) = \phi(a_2^\bullet)$. **Q**

(d) Now ϕ is a σ-subhomomorphism. **P** (i) For any a_1, $a_2 \in \mathfrak{A}$ we have

$$[\![T\chi(a_1 \cup a_2) > 0]\!] = [\![T(\chi a_1) > 0]\!] \cup [\![T(\chi a_2) > 0]\!]$$

because

$$T(\chi a_1) \vee T(\chi a_2) \le T\chi(a_1 \cup a_2) \le T(\chi a_1) + T(\chi a_2).$$

So $\phi(c_1 \cup c_2) = \phi(c_1) \cup \phi(c_2)$ for all c_1, $c_2 \in \mathfrak{C}$. (ii) If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$ with infimum 0, choose $a_n \in \mathfrak{A}$ such that $a_n^\bullet = c_n$ for each n, and set $\tilde a_n = \inf_{i\le n} a_i \setminus \inf_{i\in\mathbb{N}} a_i$ for each n; then $\tilde a_n^\bullet = c_n$ so $\phi(c_n) = [\![T(\chi\tilde a_n) > 0]\!]$ for each n, while $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}} \tilde a_n = 0$. **?** Suppose, if possible, that $b' = \inf_{n\in\mathbb{N}} \phi(c_n) \ne 0$; set $\epsilon = \frac{1}{2}\bar\nu b'$. Then $\bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > 0]\!]) \ge 2\epsilon$ for every $n \in \mathbb{N}$. For each n, take $\alpha_n > 0$ such that $\bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > \alpha_n]\!]) \ge \epsilon$. Then $u = \sup_{n\in\mathbb{N}} n\alpha_n^{-1}\chi\tilde a_n$ is defined in $L^0(\mathfrak{A})$ (because $\sup_{n\in\mathbb{N}} [\![n\alpha_n^{-1}\chi\tilde a_n > k]\!] \subseteq \tilde a_m$ if $k \ge \max_{i\le m} i\alpha_i^{-1}$, so $\inf_{k\in\mathbb{N}} \sup_{n\in\mathbb{N}} [\![n\alpha_n^{-1}\chi\tilde a_n > k]\!] = 0$). But now

$$\bar\nu(b_2 \cap [\![Tu > n]\!]) \ge \bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > \alpha_n]\!]) \ge \epsilon$$

for every n, so $\inf_{n\in\mathbb{N}} [\![Tu > n]\!] \ne 0$, which is impossible. **X** Thus $\inf_{n\in\mathbb{N}} \phi(c_n) = 0$; as $\langle c_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ϕ is a σ-subhomomorphism. **Q**

(e) By 375H, there are a non-zero $b \in \mathfrak{B}_{b_2}$ and a finite partition of unity $C \subseteq \mathfrak{C}$ such that $d \mapsto b \cap \phi(d \cap c)$ is a ring homomorphism for every $c \in C$. There is a partition of unity $A \subseteq \mathfrak{A}$, of the same size as C, such that $C = \{a^\bullet : a \in A\}$. Now T_{ab} is a Riesz homomorphism for every $a \in A$. **P** It is surely a positive linear operator. If u_1, $u_2 \in L^0(\mathfrak{A})$ and $u_1 \wedge u_2 = 0$, set $e_i = [\![u_i > 0]\!]$ for each i, so that $e_1 \cap e_2 = 0$. Observe that $u_i = \sup_{n\in\mathbb{N}} u_i \wedge n\chi e_i$, so that

$$[\![T_{ab}u_i > 0]\!] = \sup_{n\in\mathbb{N}} [\![T_{ab}(u_i \wedge n\chi e_i) > 0]\!] \subseteq [\![T_{ab}(\chi e_i) > 0]\!] = b \cap [\![T\chi(e_i \cap a) > 0]\!]$$

for both i (of course T_{ab}, like T, is sequentially order-continuous). But this means that

$$\begin{aligned}[\![T_{ab}u_1 > 0]\!] \cap [\![T_{ab}u_2 > 0]\!] &\subseteq b \cap [\![T\chi(e_1 \cap a) > 0]\!] \cap [\![T\chi(e_2 \cap a) > 0]\!] \\ &= b \cap \phi(e_1^\bullet \cap a^\bullet) \cap \phi(e_2^\bullet \cap a^\bullet) = 0\end{aligned}$$

because $a^\bullet \in C$, so $d \mapsto \phi(d \cap a^\bullet)$ is a ring homomorphism, while $e_1^\bullet \cap e_2^\bullet = 0$. So $T_{ab}u_1 \wedge T_{ab}u_2 = 0$. As u_1 and u_2 are arbitrary, T_{ab} is a Riesz homomorphism (352G(iv)). **Q**

(f) Thus $b \in B^*$. As b_0 is arbitrary, B^* is order-dense, and we're home.

375J Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Dedekind complete Riesz space such that $U^\times$ separates the points of U. If $T : L^0(\mathfrak{A}) \to U$ is a positive linear operator, there is a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of Riesz homomorphisms from $L^0(\mathfrak{A})$ to U such that $T = \sum_{n=0}^\infty T_n$, in the sense that $Tu = \sup_{n\in\mathbb{N}} \sum_{i=0}^n T_i u$ for every $u \geq 0$ in $L^0(\mathfrak{A})$.

proof By 369A, U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{B})$ for some localizable measure algebra $(\mathfrak{B}, \bar\nu)$; being Dedekind complete, it is solid in $L^0(\mathfrak{B})$ (353K). Regard T as an operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, and take B, $\langle A_b\rangle_{b\in B}$ as in 375I. Note that $L^0(\mathfrak{B})$ can be identified with $\prod_{b\in B} L^0(\mathfrak{B}_b)$ (364S, 322K). For each $b \in B$ let $f_b : A_b \to \mathbb{N}$ be an injection. If $b \in B$ and $n \in f_b[A_b]$, set $T_{nb}(u) = \chi b \times T(u \times \chi a)$; otherwise set $T_{nb} = 0$. Then $T_{nb} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B}_b)$ is a Riesz homomorphism; because A_b is a finite partition of unity, $\sum_{n=0}^\infty T_{nb}u = \chi b \times Tu$ for every $u \in L^0(\mathfrak{A})$. But this means that if we set $T_n u = \langle T_{nb}u\rangle_{b\in B}$,

$$T_n : L^0(\mathfrak{A}) \to \textstyle\prod_{b\in B} L^0(\mathfrak{B}_b) \cong L^0(\mathfrak{B})$$

is a Riesz homomorphism for each n; and $T = \sum_{n=0}^\infty T_n$. Of course every T_n is an operator from $L^0(\mathfrak{A})$ to U because $|T_n u| \leq T|u| \in U$ for every $u \in L^0(\mathfrak{A})$.

375K Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, $(\mathfrak{B}, \bar\nu)$ is a semi-finite measure algebra, and there is any non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, then there is a non-trivial sequentially order-continuous ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) If $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are homogeneous probability algebras and $\tau(\mathfrak{A}) > \tau(\mathfrak{B})$, then $\mathsf{L}^\sim(L^0(\mathfrak{A}); L^0(\mathfrak{B})) = \{0\}$.

proof (a) It is probably quickest to look at the proof of 375I: starting from a non-zero positive linear operator $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, we move to a non-zero σ-subhomomorphism $\phi : \mathfrak{A}/\mathcal{N} \to \mathfrak{B}$ and thence to a non-zero ring homomorphism from $\mathfrak{A}/\mathcal{N}$ to $\mathfrak{B}$, corresponding to a non-zero ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, which is sequentially order-continuous because it is dominated by ϕ. Alternatively, quoting 375I, we have a non-zero Riesz homomorphism $T_1 : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, and it is easy to check that $a \mapsto [\![T(\chi a) > 0]\!]$ is a non-zero sequentially order-continuous ring homomorphism.

(b) Use (a) and 331J.

375X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W an Archimedean Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be a positive linear operator. Show that T is order-continuous iff $T\chi : \mathfrak{A} \to W$ is order-continuous.

(b) Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra and W a Banach lattice. Show that the only order-continuous positive linear operator from $L^0(\mathfrak{A})$ to W is the zero operator.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W an Archimedean Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be an order-continuous Riesz homomorphism such that $T[L^0(\mathfrak{A})]$ is order-dense in W. Show that T is surjective.

(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism as defined in 375E. Show that ϕ is sequentially order-continuous.

>(e) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $[0, 1]$ and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. (i) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{G})$ to $L^0(\mathfrak{A})$. (*Hint*: suppose $T : L^0(\mathfrak{G}) \to L^0(\mathfrak{A})$ were such an operator. Reduce to the case $T(\chi 1) \le \chi 1$. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ enumerate an order-dense subset of $\mathfrak{G}$ (316Yn). For each $n \in \mathbb{N}$ take non-zero $b'_n \subseteq b_n$ such that $\int T(\chi b'_n) \le 2^{-n-2} \int T(\chi 1)$ and consider $T\chi(\sup_{n\in\mathbb{N}} b'_n)$.) (ii) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{G})$. (*Hint*: suppose $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{G})$ were such an operator. For each $n \in \mathbb{N}$ choose $a_n \in \mathfrak{A}$, $\alpha_n > 0$ such that $\bar\mu a_n \le 2^{-n}$ and if $b_n \subseteq [\![T(\chi 1) > 0]\!]$ then $b_n \cap [\![T(\chi a_n) > \alpha_n]\!] \ne 0$. Consider Tu where $u = \sum_{n=0}^\infty n\alpha_n^{-1}\chi a_n$.)

(f) In 375J, show that for any $u \in L^0(\mathfrak{A})$

$$\inf_{n\in\mathbb{N}} \sup_{m\ge n} [\![|Tu - \textstyle\sum_{i=0}^m T_i u| > 0]\!] = 0.$$

>(g) Prove directly, without quoting 375E-375K, that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra then every positive linear functional from $L^0(\mathfrak{A})$ to $\mathbb{R}$ is a finite sum of Riesz homomorphisms.

375Y Further exercises (a) Show that the following are equiveridical: (i) there is a purely atomic probability space (X, Σ, μ) such that $\Sigma = \mathcal{P}X$ and $\mu\{x\} = 0$ for every $x \in X$; (ii) there are a set X and a Riesz homomorphism $f : \mathbb{R}^X \to \mathbb{R}$ which is not order-continuous; (iii) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a positive linear operator $f : L^0(\mathfrak{A}) \to \mathbb{R}$ which is not order-continuous; (iv) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous; (v) there are a Dedekind complete Riesz space U and a sequentially order-continuous Riesz homomorphism $f : U \to \mathbb{R}$ which is not order-continuous; *(vi) there are an atomless Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous. (Compare 363S.)

(b) Give an example of an atomless Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that $L^0(\mathfrak{A})^\sim \ne \{0\}$.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras of which $\mathfrak{B}$ is weakly σ-distributive. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Show that $a \mapsto [\![T(\chi a) > 0]\!] : \mathfrak{A} \to \mathfrak{B}$ is a σ-subhomomorphism.

(d) Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra and U a Riesz space such that $U^\times$ separates the points of U. Suppose that $T : L^0(\mathfrak{A}) \to U$ is an order-continuous positive linear operator. Show that $T[L^0(\mathfrak{A})]$ is order-closed.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras, and $\phi : \mathfrak{A} \to \mathfrak{B}$ an σ-subhomomorphism such that $\phi 1_{\mathfrak{A}} = 1_{\mathfrak{B}}$. Show that there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi a \subseteq \phi a$ for every $a \in \mathfrak{A}$.

(f) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. Give an example of a non-zero positive linear operator $T : L^0 \to L^0$ such that there is no non-zero Riesz homomorphism $S : L^0 \to L^0$ with $S \le T$.

375Z Problem Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. If $T : L^0 \to L^0$ is a positive linear operator, must $T[L^0]$ be order-closed?

375 Notes and comments Both this section, and the earlier work on linear operators into L^0 spaces, can be regarded as describing different aspects of a single fact: L^0 spaces are very large. The most explicit statements of this principle are 368E and 375D: every Archimedean Riesz space can be embedded into a Dedekind complete L^0 space, but no such L^0 space can be properly embedded as an order-dense Riesz subspace of any other Archimedean Riesz space. Consequently there are many maps into L^0 spaces (368B). But by the same token there are few maps out of them (375B, 375Kb), and those which do exist have a variety of special properties (375A, 375I).

The original version of Kwapien's theorem (KWAPIEN 73) was the special case of 375I in which $\mathfrak{A}$ is the Lebesgue measure algebra. The ideas of the proof here are mostly taken from KALTON, PECK & ROBERTS 84. I have based my account on the concept of 'subhomomorphism' (375E); this seems to be an effective tool when $\mathfrak{B}$ is weakly (σ,∞)-distributive (375Yc), but less useful in other cases. The case $\mathfrak{B}=\{0,1\}$, $L^0(\mathfrak{B})\cong\mathbb{R}$ is not entirely trivial and is worth working through on its own (375Xg).

I mention 375C for the sake of its corollary 375D, but have to admit that I am not sure it is the best possible result. For the cases of principal interest to a measure theorist, there is something a good deal stronger (375Yd). Further questions concern possible relaxations of the hypotheses of 375B. One has a straightforward resolution (375Yb); others can be reduced to the Banach-Ulam problem by the techniques of 363S (375Ya).

376 Kernel operators

The theory of linear integral equations is in large part the theory of operators T defined from formulae of the type

$$(Tf)(y)=\int k(x,y)f(x)dx$$

for some function k of two variables. I make no attempt to study the general theory here. However, the concepts developed in this book make it easy to discuss certain aspects of such operators defined between the 'function spaces' of measure theory, meaning spaces of equivalence classes of functions, and indeed allow us to do some of the work in the abstract theory of Riesz spaces, omitting all formal mention of measures (376D, 376H, 376P). I give a very brief account of two theorems characterizing kernel operators in the abstract (376E, 376H), with corollaries to show the form these theorems can take in the ordinary language of integral kernels (376J, 376N). To give an idea of the kind of results we can hope for in this area, I go a bit farther with operators with domain L^1 (376Mb, 376P, 376S).

I take the opportunity to spell out versions of results from §253 in the language of this volume (376B-376C).

376A Kernel operators To give an idea of where this section is going, I will try to describe the central idea in a relatively concrete special case. Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces; you can take them both to be $[0,1]$ with Lebesgue measure if you like. Let λ be the product measure on $X\times Y$. If $k\in\mathcal{L}^1(\lambda)$, then $\int k(x,y)dx$ is defined for almost every y, by Fubini's theorem; so if $f\in\mathcal{L}^\infty(\mu)$ then $g(y)=\int k(x,y)f(x)dx$ is defined for almost every y. Also

$$\int g(y)dy=\int k(x,y)f(x)dxdy$$

is defined, because $(x,y)\mapsto k(x,y)f(x)$ is λ-virtually measurable, defined λ-a.e. and is dominated by a multiple of the integrable function k. Thus k defines a function from $\mathcal{L}^\infty(\mu)$ to $\mathcal{L}^1(\nu)$. Changing f on a set of measure 0 will not change g, so we can think of this as an operator from $L^\infty(\mu)$ to $\mathcal{L}^1(\nu)$; and of course we can move immediately to the equivalence class of g in $L^1(\nu)$, so getting an operator T_k from $L^\infty(\mu)$ to $L^1(\nu)$. This operator is plainly linear; also it is easy to check that $\pm T_k\le T_{|k|}$, so that $T_k\in\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$, and that $\|T_k\|\le\int|k|$. Moreover, changing k on a λ-negligible set does not change T_k, so that in fact we can speak of T_w for any $w\in L^1(\lambda)$.

I think it is obvious, even before investigating them, that operators representable in this way will be important. We can immediately ask what their properties will be and whether there is any straightforward way of recognising them. We can look at the properties of the map $w\mapsto T_w:L^1(\lambda)\to\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$. And we can ask what happens when $L^\infty(\mu)$ and $L^1(\nu)$ are replaced by other function spaces, defined by extended Fatou norms or otherwise. Theorems 376E and 376H answer questions of this kind.

It turns out that the formula $g(y)=\int k(x,y)f(x)dx$ gives rise to a variety of technical problems, and it is much easier to characterize Tu in terms of its action on the dual. In the language of the special case above, if $h\in\mathcal{L}^\infty(\nu)$, then we shall have

$$\int k(x,y)f(x)h(y)d(x,y)=\int g(y)h(y)dy;$$

since $g^\bullet \in L^1(\nu)$ is entirely determined by the integrals $\int g(y)h(y)dy$ as h runs over $\mathcal{L}^\infty(\nu)$, we can define the operator T in terms of the functional $(f,h) \mapsto \int k(x,y)f(x)h(y)d(x,y)$. This enables us to extend the results from the case of σ-finite spaces to general strictly localizable spaces; perhaps more to the point in the present context, it gives them natural expressions in terms of function spaces defined from measure algebras rather than measure spaces, as in 376E.

Before going farther along this road, however, I give a couple of results relating the theorems of §253 to the methods of this volume.

376B The canonical map $L^0 \times L^0 \to L^0$: Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product (325E). Then we have a bilinear map $(u,v) \mapsto u \otimes v : L^0(\mathfrak{A}) \times L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ with the following properties.

(a) For any $u \in L^0(\mathfrak{A})$, $v \in L^0(\mathfrak{B})$, $\alpha \in \mathbb{R}$,

$$[\![u \otimes \chi 1_{\mathfrak{B}} > \alpha]\!] = [\![u > \alpha]\!] \otimes 1_{\mathfrak{B}}, \quad [\![\chi 1_{\mathfrak{A}} \otimes v > \alpha]\!] = 1_{\mathfrak{A}} \otimes [\![v > \alpha]\!]$$

where for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ I write $a \otimes b$ for the corresponding member of $\mathfrak{A} \otimes \mathfrak{B}$ (315M), identified with a subalgebra of $\mathfrak{C}$ (325Dc).

(b)(i) For any $u \in L^0(\mathfrak{A})^+$, the map $v \mapsto u \otimes v : L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(ii) For any $v \in L^0(\mathfrak{B})^+$, the map $u \mapsto u \otimes v : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(c) In particular, $|u \otimes v| = |u| \otimes |v|$ for all $u \in L^0(\mathfrak{A})$, $v \in L^0(\mathfrak{B})$.

(d) For any $u \in L^0(\mathfrak{A})^+$ and $v \in L^0(\mathfrak{B})^+$, $[\![u \otimes v > 0]\!] = [\![u > 0]\!] \otimes [\![v > 0]\!]$.

proof The canonical maps $a \mapsto a \otimes 1_{\mathfrak{B}}$, $b \mapsto 1_{\mathfrak{A}} \otimes b$ from $\mathfrak{A}$, $\mathfrak{B}$ to $\mathfrak{C}$ are order-continuous Boolean homomorphisms (325Da), so induce order-continuous multiplicative Riesz homomorphisms from $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ to $L^0(\mathfrak{C})$ (364R); write $\tilde u$, $\tilde v$ for the images of $u \in L^0(\mathfrak{A})$, $v \in L^0(\mathfrak{B})$. Observe that $|\tilde u| = |u|^\sim$, $|\tilde v| = |v|^\sim$ and $(\chi 1_{\mathfrak{A}})^\sim = (\chi 1_{\mathfrak{B}})^\sim = \chi 1_{\mathfrak{C}}$. Now set $u \otimes v = \tilde u \times \tilde v$. The properties listed in (a)-(c) are just a matter of putting the definition in 364Ra together with the fact that $L^0(\mathfrak{C})$ is an f-algebra (364E). As for $[\![u \otimes v > 0]\!] = [\![\tilde u \times \tilde v > 0]\!]$, this is (for non-negative u, v) just

$$[\![\tilde u > 0]\!] \cap [\![\tilde v > 0]\!] = ([\![u > 0]\!] \otimes 1_{\mathfrak{B}}) \cap (1_{\mathfrak{A}} \otimes [\![v > 0]\!]) = [\![u > 0]\!] \otimes [\![v > 0]\!].$$

376C For L^1 spaces we have a similar result, with additions corresponding to the Banach lattice structures of the three spaces.

Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$.

(a) If $u \in L^1_{\bar\mu} = L^1(\mathfrak{A},\bar\mu)$ and $v \in L^1_{\bar\nu} = L^1(\mathfrak{B},\bar\nu)$ then $u \otimes v \in L^1_{\bar\lambda} = L^1(\mathfrak{C},\bar\lambda)$ and

$$\int u \otimes v = \int u \int v, \quad \|u \otimes v\|_1 = \|u\|_1 \|v\|_1.$$

(b) Let W be a Banach space and $\phi : L^1_{\bar\mu} \times L^1_{\bar\nu} \to W$ a bounded bilinear map. Then there is a unique bounded linear operator $T : L^1_{\bar\lambda} \to W$ such that $T(u \otimes v) = \phi(u,v)$ for all $u \in L^1_{\bar\mu}$ and $v \in L^1_{\bar\nu}$, and $\|T\| = \|\phi\|$.

(c) Suppose, in (b), that W is a Banach lattice. Then

(i) T is positive iff $\phi(u,v) \geq 0$ for all u, $v \geq 0$;

(ii) T is a Riesz homomorphism iff $u \mapsto \phi(u,v_0) : L^1_{\bar\mu} \to W$, $v \mapsto \phi(u_0,v) : L^1_{\bar\nu} \to W$ are Riesz homomorphisms for all $v_0 \geq 0$ in $L^1_{\bar\nu}$ and $u_0 \geq 0$ in $L^1_{\bar\mu}$.

proof (a) I refer to the proof of 325D. Let (X,Σ,μ) and (Y,T,ν) be the Stone spaces of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ (321K), so that $(\mathfrak{C},\bar\lambda)$ can be identified with the measure algebra of the c.l.d. product measure λ on $X \times Y$ (part (a) of the proof of 325D), and $L^1_{\bar\mu}$, $L^1_{\bar\nu}$, $L^1_{\bar\lambda}$ can be identified with $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$ (365B). Now if $f \in \mathcal{L}^0(\mu)$ and $g \in \mathcal{L}^0(\nu)$ then $f \otimes g \in \mathcal{L}^0(\lambda)$ (253Cb), and it is easy to check that $(f \otimes g)^\bullet \in L^0(\bar\lambda)$ corresponds to $f^\bullet \otimes g^\bullet$ as defined in 376B. (Look first at the cases in which one of f, g is a constant function with value 1.) By 253E, we have a canonical map $(f^\bullet, g^\bullet) \mapsto (f \otimes g)^\bullet$ from $L^1(\mu) \times L^1(\nu)$ to $L^1(\lambda)$, with $\int f \otimes g = \int f \int g$ (253D); so that if $u \in L^1_{\bar\mu}$ and $v \in L^1_{\bar\nu}$ we must have $u \otimes v \in L^1_{\bar\lambda}$, with $\int u \otimes v = \int u \int v$. As in 253E, it follows that $\|u \otimes v\|_1 = \|u\|_1 \|v\|_1$.

(b) In view of the situation described in (a) above, this is now just a translation of the same result about $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$, which is Theorem 253F.

(c) Identifying the algebraic free product $\mathfrak{A}\otimes\mathfrak{B}$ with its canonical image in $\mathfrak{C}$ (325Dc), I write $(\mathfrak{A}\otimes\mathfrak{B})^f$ for $\{c : c\in\mathfrak{A}\otimes\mathfrak{B},\ \bar\lambda c<\infty\}$, so that $(\mathfrak{A}\otimes\mathfrak{B})^f$ is a subring of $\mathfrak{C}$. Recall that any member of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\le n} a_i\otimes b_i$ where $a_0,\dots,a_n$ are disjoint (315Na); evidently this will belong to $(\mathfrak{A}\otimes\mathfrak{B})^f$ iff $\bar\mu a_i\cdot\bar\nu b_i$ is finite for every i.

The next fact to lift from previous theorems is in part (e) of the proof of 253F: the linear span M of $\{\chi(a\otimes b) : a\in\mathfrak{A}^f,\ b\in\mathfrak{B}^f\}$ is norm-dense in $L^1_{\bar\lambda}$. Of course M can also be regarded as the linear span of $\{\chi c : c\in(\mathfrak{A}\otimes\mathfrak{B})^f\}$, or $S(\mathfrak{A}\otimes\mathfrak{B})^f$. (Strictly speaking, this last remark relies on 361J; the identity map from $(\mathfrak{A}\otimes\mathfrak{B})^f$ to $\mathfrak{C}$ induces an injective Riesz homomorphism from $S(\mathfrak{A}\otimes\mathfrak{B})^f$ into $S(\mathfrak{C})\subseteq L^0(\mathfrak{C})$. To see that $\chi c\in M$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$, we need to know that c can be expressed as a disjoint union of members of $\mathfrak{A}\otimes\mathfrak{B}$, as noted above.)

(i) If T is positive then of course $\phi(u,v)=T(u\otimes v)\ge 0$ whenever $u,\ v\ge 0$, since $u\otimes v\ge 0$. On the other hand, if ϕ is non-negative on $U^+\times V^+$, then, in particular, $T\chi(a\otimes b)=\phi(\chi a,\chi b)\ge 0$ whenever $\bar\mu a\cdot\bar\nu b<\infty$. Consequently $T(\chi c)\ge 0$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $Tw\ge 0$ whenever $w\ge 0$ in $M\cong S(\mathfrak{A}\otimes\mathfrak{B})^f$, as in 361Ga.

Now this means that $T|w|\ge 0$ whenever $w\in M$. But as M is norm-dense in $L^1_{\bar\lambda}$, $w\mapsto T|w|$ is continuous and W^+ is closed, it follows that $T|w|\ge 0$ for every $w\in L^1_{\bar\lambda}$, that is, that T is positive.

(ii) If T is a Riesz homomorphism then of course $u\mapsto\phi(u,v_0)=T(u\otimes v_0)$, $v\mapsto\phi(u_0,v)=T(u_0\otimes v)$ are Riesz homomorphisms for $v_0,\ u_0\ge 0$. On the other hand, if all these maps are Riesz homomorphisms, then, in particular,

$$\begin{aligned}T\chi(a\otimes b)\wedge T\chi(a'\otimes b')&=\phi(\chi a,\chi b)\wedge\phi(\chi a',\chi b')\\&\le\phi(\chi a,\chi b+\chi b')\wedge\phi(\chi a',\chi b+\chi b')\\&=\phi(\chi a\wedge\chi a',\chi b+\chi b')=0\end{aligned}$$

whenever $a,\ a'\in\mathfrak{A}^f$, $b,\ b'\in\mathfrak{B}^f$ and $a\cap a'=0$. Similarly, $T\chi(a\otimes b)\wedge T\chi(a'\otimes b')=0$ if $b\cap b'=0$. But this means that $T\chi c\wedge T\chi c'=0$ whenever $c,\ c'\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $c\cap c'=0$. **P** Express c, c' as $\sup_{i\le m} a_i\otimes b_i$, $\sup_{j\le n} a'_j\otimes b'_j$ where a_i, a'_j, b_i, b'_j all have finite measure. Now if $i\le m$, $j\le n$, $(a_i\cap a'_j)\otimes(b_i\cap b'_j)=(a_i\otimes b_i)\cap(a'_j\otimes b'_j)=0$, so one of $a_i\cap a'_j$, $b_i\cap b'_j$ must be zero, and in either case $T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0$. Accordingly

$$\begin{aligned}T\chi c\wedge T\chi c'&\le\Bigl(\sum_{i=0}^m T\chi(a_i\otimes b_i)\Bigr)\wedge\Bigl(\sum_{j=0}^n T\chi(a'_j\times b'_j)\Bigr)\\&\le\sum_{i\le m,j\le n} T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0,\end{aligned}$$

using 352Fa for the second inequality. **Q**

This implies that $T\restriction M$ must be a Riesz homomorphism (361Gc), that is, $T|w|=|Tw|$ for all $w\in M$. Again because M is dense in $L^1_{\bar\lambda}$, $T|w|=|Tw|$ for every $w\in L^1_{\bar\lambda}$, and T is a Riesz homomorphism.

376D Abstract integral operators: Definition The following concept will be used repeatedly in the theorems below; it is perhaps worth giving it a name. Let U be a Riesz space and V a Dedekind complete Riesz space, so that $\mathsf{L}^\times(U;V)$ is a Dedekind complete Riesz space (355H). If $f\in U^\times$ and $v\in V$ write $P_{fv}u=f(u)v$ for each $u\in U$; then $P_{fv}\in\mathsf{L}^\times(U;V)$. **P** If $f\ge 0$ in $U^\times$ and $v\ge 0$ in $V^\times$ then P_{fv} is a positive linear operator from U to V which is order-continuous because if $A\subseteq U$ is non-empty, downwards-directed and has infimum 0, then (as V is Archimedean)

$$\inf_{u\in A}P_{fv}(u)=\inf_{u\in A}f(u)v=0.$$

Of course $(f,g)\mapsto P_{fg}$ is bilinear, so $P_{fv}\in\mathsf{L}^\times(U;V)$ for every $f\in U^\times$, $v\in V$. **Q** Now I call a linear operator from U to V an **abstract integral operator** if it is in the band of $\mathsf{L}^\times(U;V)$ generated by $\{P_{fv} : f\in U^\times,\ v\in V\}$.

The first result describes these operators when U, V are expressed as subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ for measure algebras $\mathfrak{A}$, $\mathfrak{B}$ and V is perfect.

376E Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$, and $U\subseteq L^0(\mathfrak{A})$, $V\subseteq L^0(\mathfrak{B})$ order-dense Riesz subspaces. Write W for the set of those $w\in L^0(\mathfrak{C})$ such that $w\times(u\otimes v)$ is integrable for every $u\in U$, $v\in V$. Then we have an operator $w\mapsto T_w:W\to\mathsf{L}^\times(U;V^\times)$ defined by setting

$$T_w(u)(v)=\int w\times(u\otimes v)$$

for every $w\in W$, $u\in U$ and $v\in V$. The map $w\mapsto T_w$ is a Riesz space isomorphism between W and the band of abstract integral operators in $\mathsf{L}^\times(U;V^\times)$.

proof (a) The first thing to check is that the formula offered does define a member $T_w(u)$ of $V^\times$ for any $w\in W$, $u\in U$. **P** Of course $T_w(u)$ is a linear operator because $\int$ is linear and $\otimes$ and $\times$ are bilinear. It belongs to $V^\sim$ because, writing $g(v)=\int|w|\times(|u|\otimes v)$, g is a positive linear operator and $|T_w(u)(v)|\le g(|v|)$ for every v. (I am here using 376Bc to see that $|w\times(u\otimes v)|=|w|\times(|u|\otimes|v|)$.) Also $g\in V^\times$ because $v\mapsto|u|\otimes v$, $w'\mapsto|w|\times w'$ and $\int$ are all order-continuous; so $T_w(u)$ also belongs to $V^\times$. **Q**

(b) Next, for any given $w\in W$, the map $T_w:U\to V^\times$ is linear (again because $\otimes$ and $\times$ are bilinear). It is helpful to note that W is a solid linear subspace of $L^0(\mathfrak{C})$. Now if $w\ge0$ in W, then $T_w\in\mathsf{L}^\times(U;V^\times)$. **P** If u, $v\ge0$ then $u\otimes v\ge0$, $w\times(u\otimes v)\ge0$ and $T_w(u)(v)\ge0$; as v is arbitrary, $T_w(u)\ge0$ whenever $u\ge0$; as u is arbitrary, T_w is positive. If $A\subseteq U$ is non-empty, downwards-directed and has infimum 0, then $T_w[A]$ is downwards-directed, and for any $v\in V^+$

$$(\inf T_w[A])(v)=\inf_{u\in A}T_w(u)(v)=\inf_{u\in A}\int w\times(u\otimes v)=0$$

because $u\mapsto u\otimes v$ is order-continuous. So $\inf T_w[A]=0$; as A is arbitrary, T_w is order-continuous. **Q**

For general $w\in W$, we now have $T_w=T_{w^+}-T_{w^-}\in L^\times(U;V^\times)$.

(c) Ths shows that $w\mapsto T_w$ is a map from W to $L^\times(U;V^\times)$. Running through the formulae once again, it is linear, positive and order-continuous; this last because, given a non-empty downwards-directed $C\subseteq W$ with infimum 0, then for any $u\in U^+$, $v\in V^+$

$$(\inf_{w\in C}T_w)(u)(v)\le\inf_{w\in C}\int w\times(u\otimes v)=0$$

(because $\int$ and $\times$ are order-continuous); as v is arbitrary, $(\inf_{w\in C}T_w)(u)=0$; as u is arbitrary, $\inf_{w\in C}T_w=0$.

(d) All this is easy, being nothing but a string of applications of the elementary properties of $\otimes$, $\times$ and $\int$. But I think a new idea is needed for the next fact: the map $w\mapsto T_w:W\to\mathsf{L}^\times(U;V^\times)$ is a Riesz homomorphism. **P** Write $\mathfrak{D}$ for the set of those $d\in\mathfrak{C}$ such that $T_w\wedge T_{w'}=0$ whenever w, $w'\in W^+$, $[\![w>0]\!]\subseteq d$ and $[\![w'>0]\!]\subseteq 1_{\mathfrak{C}}\setminus d$. (i) If d_1, $d_2\in\mathfrak{D}$, w, $w'\in W^+$, $[\![w>0]\!]\subseteq d_1\cup d_2$ and $[\![w'>0]\!]\cap(d_1\cup d_2)=0$, then set $w_1=w\times\chi d_1$, $w_2=w-w_1$. In this case

$$[\![w_1>0]\!]\subseteq d_1,\quad [\![w_2>0]\!]\subseteq d_2,$$

so

$$T_{w_1}\wedge T_{w'}=T_{w_2}\wedge T_{w'}=0,\quad T_w\wedge T_{w'}\le(T_{w_1}\wedge T_{w'})+(T_{w_2}\wedge T_{w'})=0.$$

As w, w' are arbitrary, $d_1\cup d_2\in\mathfrak{D}$. Thus $\mathfrak{D}$ is closed under $\cup$. (ii) The symmetry of the definition of $\mathfrak{D}$ means that $1_{\mathfrak{C}}\setminus d\in\mathfrak{D}$ whenever $d\in\mathfrak{D}$. (iii) Of course $0\in\mathfrak{D}$, just because $T_w=0$ if $w\in W^+$ and $[\![w>0]\!]=0$; so $\mathfrak{D}$ is a subalgebra of $\mathfrak{C}$. (iv) If $D\subseteq\mathfrak{D}$ is non-empty and upwards-directed, with supremum c in $\mathfrak{C}$, and if w, $w'\in W^+$ are such that $[\![w>0]\!]\subseteq c$, $[\![w'>0]\!]\cap c=0$, then consider $\{w\times\chi d:d\in D\}$. This is upwards-directed, with supremum w; so $T_w=\sup_{d\in D}T_{w\times\chi d}$, because the map $q\mapsto T_q$ is order-continuous. Also $T_{w\times\chi d}\wedge T_{w'}=0$ for every $d\in D$, so $T_w\wedge T_{w'}=0$. As w, w' are arbitrary, $c\in\mathfrak{D}$; as D is arbitrary, $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$. (v) If $a\in\mathfrak{A}$ and w, $w'\in W^+$ are such that $[\![w>0]\!]\subseteq a\otimes1_{\mathfrak{B}}$, $[\![w'>0]\!]\cap(a\otimes1_{\mathfrak{B}})=0$, then any $u\in U^+$ is expressible as u_1+u_2 where $u_1=u\times\chi a$, $u_2=u\times\chi(1_{\mathfrak{A}}\setminus a)$. Now

$$T_w(u_2)(v)=\int w\times(u_2\otimes v)=\int w\times\chi(a\otimes1_{\mathfrak{B}})\times(u\otimes v)\times\chi((1_{\mathfrak{A}}\setminus a)\otimes1_{\mathfrak{B}})=0$$

for every $v \in V$, so $T_w(u_2) = 0$. Similarly, $T_{w'}(u_1) = 0$. But this means that

$$(T_w \wedge T_{w'})(u) \le T_w(u_2) + T_{w'}(u_1) = 0.$$

As u is arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $a \otimes 1_{\mathfrak{B}} \in \mathfrak{D}$. (vi) Now suppose that $b \in \mathfrak{B}$ and that w, $w' \in W^+$ are such that $[\![w > 0]\!] \subseteq 1_{\mathfrak{A}} \otimes b$, $[\![w' > 0]\!] \cap (1_{\mathfrak{A}} \otimes b) = 0$. If $u \in U^+$, $v \in V^+$ then

$$(T_w \wedge T_{w'})(u)(v) \le \int w \times (u \otimes (v \times \chi(1_{\mathfrak{B}} \setminus b))) + \int w' \times (u \otimes (v \times \chi b)) = 0.$$

As u, v are arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $1_{\mathfrak{A}} \otimes b \in \mathfrak{D}$. (vii) This means that $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$ including $\mathfrak{A} \otimes \mathfrak{B}$, and is therefore the whole of $\mathfrak{C}$ (325D(c-ii)). (viii) Now take any w, $w' \in W$ such that $w \wedge w' = 0$, and consider $c = [\![w > 0]\!]$. Then $[\![w' > 0]\!] \subseteq 1_{\mathfrak{C}} \setminus c$ and $c \in \mathfrak{D}$, so $T_w \wedge T_{w'} = 0$. This is what we need to be sure that $w \mapsto T_w$ is a Riesz homomorphism (352G). **Q**

(e) The map $w \mapsto T_w$ is injective. **P** (i) If $w > 0$ in W, then consider

$$A = \{a : a \in \mathfrak{A}, \exists\, u \in U, \chi a \le u\}, \quad B = \{b : b \in \mathfrak{B}, \exists\, v \in V, \chi b \le v\}.$$

Because U and V are order-dense in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, A and B are order-dense in $\mathfrak{A}$ and $\mathfrak{B}$. Also both are upwards-directed. So $\sup_{a \in A, b \in B} a \otimes b = 1_{\mathfrak{C}}$ and $0 < \int w = \sup_{a \in A, b \in B} \int_{a \otimes b} w$. Take $a \in A$, $b \in B$ such that $\int_{a \otimes b} w > 0$; then there are $u \in U$, $v \in V$ such that $\chi a \le u$, $\chi b \le v$, so that

$$T_w(u)(v) \ge \int_{a \otimes b} w > 0$$

and $T_w > 0$. (ii) For general non-zero $w \in W$, we now have $|T_w| = T_{|w|} > 0$ so $T_w \ne 0$. **Q**

Thus $w \mapsto T_w$ is an order-continuous injective Riesz homomorphism.

(f) Write $\tilde{W}$ for $\{T_w : w \in W\}$, so that $\tilde{W}$ is a Riesz subspace of $\mathsf{L}^\times(U; V^\times)$ isomorphic to W, and $\widehat{W}$ for the band it generates in $\mathsf{L}^\times(U; V^\times)$. Then $\tilde{W}$ is order-dense in $\widehat{W}$. **P** Suppose that $S > 0$ in $\widehat{W} = \tilde{W}^{\perp\perp}$ (353Ba). Then $S \notin \tilde{W}^\perp$, so there is a $w \in W$ such that $S \wedge T_w > 0$. Set $w_1 = w \wedge \chi 1_{\mathfrak{C}}$. Then $w = \sup_{n \in \mathbb{N}} w \wedge n w_1$, so $T_w = \sup_{n \in \mathbb{N}} T_w \wedge n T_{w_1}$ and $R = S \wedge T_{w_1} > 0$.

Set $U_1 = U \cap L^1(\mathfrak{A}, \bar{\mu})$. Because U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$, U_1 is an order-dense Riesz subspace of $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$, therefore also norm-dense. Similarly $V_1 = V \cap L^1(\mathfrak{B}, \bar{\nu})$ is a norm-dense Riesz subspace of $L^1_{\bar{\nu}} = L^1(\mathfrak{B}, \bar{\nu})$. Define $\phi_0 : U_1 \times V_1 \to \mathbb{R}$ by setting $\phi_0(u, v) = R(u)(v)$ for $u \in U_1$, $v \in V_1$. Then ϕ_0 is bilinear, and

$$\begin{aligned} |\phi_0(u, v)| &= |R(u)(v)| \le |R(u)|(|v|) \le R(|u|)(|v|) \le T_{w_1}(|u|)(|v|) \\ &= \int w_1 \times (|u| \otimes |v|) \le \int |u| \otimes |v| = \|u\|_1 \|v\|_1 \end{aligned}$$

for all $u \in U_1$, $v \in V_1$, because $0 \le R \le T_{w_1}$ in $\mathsf{L}^\times(U; V^\times)$. Because U_1, V_1 are norm-dense in $L^1_{\bar{\mu}}$, $L^1_{\bar{\nu}}$ respectively, ϕ_0 has a unique extension to a continuous bilinear map $\phi : L^1_{\bar{\mu}} \times L^1_{\bar{\nu}} \to \mathbb{R}$. (To reduce this to standard results on linear operators, think of R as a function from U_1 to V_1^*; since every member of V_1^* has a unique extension to a member of $(L^1_{\bar{\nu}})^*$, we get a corresponding function $R_1 : U_1 \to (L^1_{\bar{\nu}})^*$ which is continuous and linear, so has a unique extension to a continuous linear operator $R_2 : L^1_{\bar{\mu}} \to (L^1_{\bar{\nu}})^*$, and we set $\phi(u, v) = R_2(u)(v)$.)

By 376C, there is a unique $h \in (L^1_{\bar{\lambda}})^* = L^1(\mathfrak{C}, \bar{\lambda})^*$ such that $h(u \otimes v) = \phi(u, v)$ for every $u \in L^1_{\bar{\mu}}$, $v \in L^1_{\bar{\nu}}$. Because $(\mathfrak{C}, \bar{\lambda})$ is localizable, this h corresponds to a $w' \in L^\infty(\mathfrak{C})$ (365Jc), and

$$\int w' \times (u \otimes v) = h(u \otimes v) = \phi_0(u, v) = R(u)(v)$$

for every $u \in U_1$, $v \in V_1$.

Because U_1 is norm-dense in $L^1_{\bar{\mu}}$, U_1^+ is dense in $(L^1_{\bar{\mu}})^+$, and similarly V_1^+ is dense in $(L^1_{\bar{\nu}})^+$, so $U_1^+ \times V_1^+$ is dense in $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$; now ϕ_0 is non-negative on $U_1^+ \times V_1^+$, so ϕ (being continuous) is non-negative on $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$. By 376Cc, $h \ge 0$ in $(L^1_{\bar{\lambda}})^*$ and $w' \ge 0$ in $L^\infty(\mathfrak{C})$. In the same way, because $\phi_0(u, v) \le T_w(u)(v)$ for $u \in U_1^+$ and $v \in V_1^+$, $w' \le w_1 \le w$ in $L^0(\mathfrak{C})$, so $w' \in W$. We have

$$T_{w'}(u)(v) = \int w' \times (u \otimes v) = R(u)(v)$$

for all $u \in U_1$, $v \in V_1$. If $u \in U_1^+$, then $T_{w'}(u)$ and $R(u)$ are both order-continuous, so must be identical, since V_1 is order-dense in V. This means that $T_{w'}$ and R agree on U_1. But as both are themselves order-continuous linear operators, and U_1 is order-dense in U, they must be equal.

Thus $0 < T_{w'} \le S$ in $\mathrm{L}^\times(U;V^\times)$. As S is arbitrary, $\tilde W$ is quasi-order-dense in $\widehat W$, therefore order-dense (353A). **Q**

(g) Because $w \mapsto T_w : W \mapsto \tilde W$ is an injective Riesz homomorphism, we have an inverse map $Q : \tilde W \to L^0(\mathfrak C)$, setting $Q(T_w) = w$; this is a Riesz homomorphism, and it is order-continuous because W is solid in $L^0(\mathfrak C)$, so that the embedding $W \subseteq L^0(\mathfrak C)$ is order-continuous. By 368B, Q has an extension to an order-continuous Riesz homomorphism $\tilde Q : \widehat W \to L^0(\mathfrak C)$. Because $Q(S) > 0$ whenever $S > 0$ in $\tilde W$, $\tilde Q(S) > 0$ whenever $S > 0$ in $\widehat W$, so $\tilde Q$ is injective. Now $\tilde Q(S) \in W$ for every $S \in \widehat W$. **P** It is enough to look at non-negative S. In this case, $\tilde Q(S)$ must be $\sup\{\tilde Q(T_w) : w \in W, T_w \le S\} = \sup C$, where $C = \{w : T_w \le S\} \subseteq W$. Take $u \in U^+$, $v \in V^+$. Then $\{w \times (u \otimes v) : w \in C\}$ is upwards-directed, because C is, and

$$\sup_{w\in C} \int w \times (u \otimes v) = \sup_{w \in C} T_w(u)(v) \le S(u)(v) < \infty.$$

So $\tilde Q(S) \times (u \otimes v) = \sup_{w\in C} w \times (u \otimes v)$ belongs to $L^1_{\tilde\lambda}$ (365Df). As u and v are arbitrary, $\tilde Q(S) \in W$. **Q**

(h) Of course this means that $\tilde W = \widehat W$ and $\tilde Q = Q$, that is, that $w \mapsto T_w : W \mapsto \widehat W$ is a Riesz space isomorphism.

(i) I have still to check on the identification of $\widehat W$ as the band Z of abstract integral operators in $\mathrm{L}^\times(U;V^\times)$. Write $P_{fg}(u) = f(u)g$ for $f \in U^\times$, $g \in V^\times$ and $u \in U$.

Set

$$U^\# = \{u : u \in L^0(\mathfrak A),\ u \times u' \in L^1_{\bar\mu} \text{ for every } u' \in U\},$$

$$V^\# = \{v : v \in L^0(\mathfrak B),\ v \times v' \in L^1_{\bar\nu} \text{ for every } v' \in V\}.$$

From 369C we know that if we set $f_u(u') = \int u \times u'$ for $u \in U^\#$ and $u' \in U$, then $f_u \in U^\times$ for every $u \in U^\#$, and $u \mapsto f_u$ is an isomorphism between $U^\#$ and an order-dense Riesz subspace of $U^\times$. Similarly, setting $g_v(v') = \int v \times v'$ for $v \in V^\#$ and $v' \in V$, $v \mapsto g_v$ is an isomorphism between $V^\#$ and an order-dense Riesz subspace of $V^\times$.

If $u \in U^\#$, $v \in V^\#$ then

$$\int (u \otimes v) \times (u' \otimes v') = \int (u \times u') \otimes (v \times v') = (\int u \times u')(\int v \times v') = f_u(u')g_v(v')$$

for every $u' \in U$, $v' \in V$, so $u \otimes v \in W$ and $T_{u\otimes v} = P_{f_u g_v}$.

Now take $f \in (U^\times)^+$ and $g \in (V^\times)^+$. Set $A = \{u : u \in U^\#, u \ge 0, f_u \le f\}$ and $B = \{v : v \in V^\#, v \ge 0, g_v \le g\}$. These are upwards-directed, so $C = \{u \otimes v : u \in A, v \in B\}$ is upwards-directed in $L^0(\mathfrak C)$. Because $\{f_u : u \in U^\#\}$ is order-dense in $U^\times$, $f = \sup_{u\in A} f_u$; by 355Ed, $f(u') = \sup_{u\in A} f_u(u')$ for every $u' \in U^+$. Similarly, $g(v') = \sup_{v\in B} f_v(v')$ for every $v' \in V^+$.

? Suppose, if possible, that C is not bounded above in $L^0(\mathfrak C)$. Because $\mathfrak C$ and $L^0(\mathfrak C)$ are Dedekind complete,

$$c = \inf_{n\in\mathbb N} \sup_{u\in A, v\in B} [\![u \otimes v \ge n]\!]$$

must be non-zero (364Ma). Because U and V are order-dense in $L^0(\mathfrak A)$, $L^0(\mathfrak B)$ respectively,

$$1_{\mathfrak A} = \sup\{[\![u' > 0]\!] : u' \in U\}, \quad 1_{\mathfrak B} = \sup\{[\![v' > 0]\!] : v' \in V\},$$

and there are $u' \in U^+$, $v' \in V^+$ such that $c \cap [\![u' > 0]\!] \otimes [\![v' > 0]\!] \ne 0$, so that $\int_c u' \otimes v' > 0$. But now, for any $n \in \mathbb N$,

$$\begin{aligned}
f(u')g(v') &\ge \sup_{u\in A,v\in B} f_u(u')g_v(v') \\
&= \sup_{u\in A,v\in B} \int (u\otimes v)\times(u'\otimes v') \\
&\ge \sup_{u\in A,v\in B} \int ((u\otimes v)\wedge n\chi c)\times(u'\otimes v') \\
&= \int \sup_{u\in A,v\in B} ((u\otimes v)\wedge n\chi c)\times(u'\otimes v')
\end{aligned}$$

(because $w\mapsto\int w\times(u'\otimes v')$ is order-continuous)

$$= \int (n\chi c)\times(u'\otimes v') = n\int_c u'\otimes v',$$

which is impossible. **X**

Thus C is bounded above in $L^0(\mathfrak{C})$, and has a supremum $w\in L^0(\mathfrak{C})$. If $u'\in U^+$, $v'\in V^+$ then

$$\begin{aligned}
\int w\times(u'\otimes v') &= \sup_{u\in A,v\in B} \int (u\otimes v)\times(u'\otimes v') \\
&= \sup_{u\in A,v\in B} f_u(u')g_v(v') = f(u')g(v') = P_{fg}(u')(v').
\end{aligned}$$

Thus $w\in W$ and

$$P_{fg} = T_w \in \tilde{W} \subseteq \widehat{W}.$$

And this is true for any non-negative $f\in U^\times$, $g\in V^\times$. Of course it follows that $P_{fg}\in\widehat{W}$ for every $f\in U^\times$, $g\in V^\times$; as $\widehat{W}$ is a band, it must include Z.

(j) Finally, $\widehat{W}\subseteq Z$. **P** Since $Z=Z^{\perp\perp}$, it is enough to show that $\widehat{W}\cap Z^\perp=\{0\}$. Take any $T>0$ in $\widehat{W}$. There are $u_0'\in U^+$, $v_0'\in V^+$ such that $T(u_0')(v_0')>0$. So there is a $v\in V^\#$ such that $0\le g_v\le T(u_0')$ and $g_v(v_0')>0$, that is, $\int v\times v_0'>0$. Because V is order-dense in $L^0(\mathfrak{B})$, there is a $v_1'\in V$ such that $0<v_1'\le v_0'\times\chi[\![v>0]\!]$, so that

$$0<\int v\times v_1' = g_v(v_1') \le T(u_0')(v_1')$$

and $[\![v_1'>0]\!]\subseteq[\![v>0]\!]$.

Now consider the functional $u'\mapsto h(u')=T(u')(v_1'):U\to\mathbb{R}$. This belongs to $(U^\times)^+$ and $h(u_0')>0$, so there is a $u\in U^\#$ such that $0\le f_u\le h$ and $f_u(u_0')>0$. This time, $\int u\times u_0'>0$ so (because U is order-dense in $L^0(\mathfrak{A})$) there is a $u_1'\in U$ such that $h(u_1')>0$ and $[\![u_1'>0]\!]\subseteq[\![u>0]\!]$.

We can express T as T_w where $w\in W^+$. In this case, we have

$$\int w\times(u_1'\otimes v_1') = T(u_1')(v_1') = h(u_1')>0,$$

so

$$\begin{aligned}
0\ne[\![w>0]\!]\cap[\![u_1'\otimes v_1'>0]\!] &= [\![w>0]\!]\cap([\![u_1'>0]\!]\otimes[\![v_1'>0]\!]) \\
&\subseteq [\![w>0]\!]\cap([\![u>0]\!]\otimes[\![v>0]\!]) = [\![w>0]\!]\cap[\![u\otimes v>0]\!],
\end{aligned}$$

and $w\wedge(u\otimes v)>0$, so

$$T_w\wedge P_{f_u g_v} = T_w\wedge T_{u\otimes v} = T_{w\wedge(u\otimes v)}>0.$$

Thus $T\notin Z^\perp$. Accordingly $\widehat{W}\cap Z^\perp=\{0\}$ and $\widehat{W}\subseteq Z^{\perp\perp}=Z$. **Q**

Since we already know that $Z\subseteq\widehat{W}$, this completes the proof.

376F Corollary Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be localizable measure spaces, with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$. Let $U\subseteq L^0(\mathfrak{A})$, $V\subseteq L^0(\mathfrak{B})$ be perfect order-dense solid linear subspaces, and $T:U\to V$ a linear operator. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $w\in L^0(\mathfrak{C})$ such that $\int w\times(u\otimes v')$ is defined and equal to $\int Tu\times v'$ whenever $u\in U$ and $v'\in L^0(\mathfrak{B})$ is such that $v'\times v$ is integrable for every $v\in V$.

proof Setting $V^{\#}=\{v':v'\in L^0(\mathfrak{B}),\,v\times v'\in L^1$ for every $v\in V\}$, we know that we can identify $V^{\#}$ with $V^{\times}$ and V with $(V^{\#})^{\times}$ (369C). So the equivalence of (i) and (ii) is just 376E applied to $V^{\#}$ in place of V.

376G Lemma Let U be a Riesz space, V an Archimedean Riesz space, $T:U\to V$ a linear operator, $f\in(U^{\sim})^+$ and $e\in V^+$. Suppose that $0\le Tu\le f(u)e$ for every $u\in U^+$. Then if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty}g(u_n)=0$ whenever $g\in U^{\sim}$ and $|g|\le f$, $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V (definition: 367A).

proof Let V_e be the solid linear subspace of V generated by e; then $Tu\in V_e$ for every $u\in U$. We can identify V_e with an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, with e corresponding to χX (353M). For $x\in X$, set $g_x(u)=(Tu)(x)$ for every $u\in U$; then $0\le g_x(u)\le f(u)$ for $u\ge 0$, so $|g_x|\le f$ and $\lim_{n\to\infty}(Tu_n)(x)=0$. As x is arbitrary, $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$, by 367L, and therefore in V_e, because V_e is order-dense in $C(X)$ (367F). But V_e, regarded as a subspace of V, is solid, so 367F also tells us that $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

376H Theorem Let U be a Riesz space and V a weakly (σ,∞)-distributive Dedekind complete Riesz space (definition: 368N). Suppose that $T\in\mathsf{L}^{\times}(U;V)$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^{\times}$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V;

(iii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U and $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^{\times}$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

proof For $f\in U^{\times}$, $v\in V$ and $u\in U$ set $P_{fv}(u)=f(u)v$. Write $Z\subseteq\mathsf{L}^{\times}(U;V)$ for the space of abstract integral operators.

(a)(i)⇒(iii) Suppose that $T\in Z^+$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^{\times}$. Note that $\{P_{fv}:f\in U^{\times+},\,v\in V^+\}$ is upwards-directed, so that $T=\sup\{T\wedge P_{fv}:f\in U^{\times+},\,v\in V^+\}$ (352Va).

Take $u^*\in U^+$ such that $|u_n|\le u^*$ for every n, and set $w=\inf_{n\in\mathbb{N}}\sup_{m\ge n}Tu_m$, which is defined because $|Tu_n|\le Tu^*$ for every n. Now $w\le(T-P_{fv})^+(u^*)$ for every $f\in U^{\times+}$, $v\in V^+$. **P** Setting $T_1=T\wedge P_{fv}$, $w_0=(T-P_{fv})^+(u^*)$ we have

$$Tu_n-T_1u_n\le|T-T_1|(u^*)=(T-P_{fv})^+(u^*)=w_0$$

for every $n\in\mathbb{N}$, so $Tu_n\le w_0+T_1u_n$. On the other hand, $0\le T_1u\le f(u)v$ for every $u\in U^+$, so by 376G we must have $\inf_{n\in\mathbb{N}}\sup_{m\ge n}T_1u_m=0$. Accordingly

$$w\le w_0+\inf_{n\in\mathbb{N}}\sup_{m\ge n}T_1u_m=w_0.\ \mathbf{Q}$$

But as $\inf\{(T-P_{fv})^+:f\in U^{\times+},\,v\in V^+\}=0$, $w\le 0$. Similarly (or applying the same argument to $\langle -u_n\rangle_{n\in\mathbb{N}}$), $\sup_{n\in\mathbb{N}}\inf_{n\in\mathbb{N}}Tu_n\ge 0$ and $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to zero.

For general $T\in Z$, this shows that $\langle T^+u_n\rangle_{n\in\mathbb{N}}$ and $\langle T^-u_n\rangle_{n\in\mathbb{N}}$ both order*-converge to 0, so $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367C. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (iii) is satisfied.

(b)(iii)⇒(ii) is trivial.

(c)(ii)⇒(i) **?** Now suppose, if possible, that (ii) is satisfied, but that $T\notin Z$. Because $\mathsf{L}^{\times}(U;V)$ is Dedekind complete (355H), Z is a projection band (353I), so T is expressible as T_1+T_2 where $T_1\in Z$, $T_2\in Z^{\perp}$ and $T_2\ne 0$. At least one of T_2^+, T_2^- is non-zero; replacing T by $-T$ if need be, we may suppose that $T_2^+>0$.

Because T_2^+, like T, belongs to $\mathrm{L}^\times(U;V)$, its kernel U_0 is a band in U, which cannot be the whole of U, and there is a $u_0 > 0$ in $U_0^\perp$. In this case $T_2^+ u_0 > 0$; because $T_2^+ \wedge (T_2^- + |T_1|) = 0$, there is a $u_1 \in [0, u_0]$ such that $T_2^+(u_0 - u_1) + (T_2^- + |T_1|)(u_1) \not\geq T_2^+ u_0$, so that

$$Tu_1 \geq T_2 u_1 - |T_1|(u_1) \not\leq 0$$

and $Tu_1 \neq 0$. Now this means that the sequence $(Tu_1, Tu_1, \dots)$ is not order*-convergent to zero, so there must be some $f \in U^\times$ such that $(f(u_1), f(u_1), \dots)$ does not converge to 0, that is, $f(u_1) \neq 0$; replacing f by $|f|$ if necessary, we may suppose that $f \geq 0$ and that $f(u_1) > 0$.

By 356H, there is a u_2 such that $0 < u_2 \leq u_1$ and $g(u_2) = 0$ whenever $g \in U^\times$ and $g \wedge f = 0$. Because $0 < u_2 \leq u_0$, $u_2 \in U_0^\perp$ and $v_0 = T_2^+ u_2 > 0$. Consider $P_{fv_0} \in Z$. Because $T_2 \in Z^\perp$, $T_2^+ \wedge P_{fv_0} = 0$; set $S = P_{fv_0} + T_2^-$, so that $T_2^+ \wedge S = 0$. Then

$$\inf_{u \in [0,u_2]} T_2^+(u_2 - u) + Su = 0, \quad \sup_{u \in [0,u_2]} T_2^+ u - Su = v_0$$

(use 355Ec for the first equality, and then subtract both sides from v_0). Now $Su \geq f(u)v_0$ for every $u \geq 0$, so that for any $\epsilon > 0$

$$\sup_{u \in [0,u_2], f(u) \geq \epsilon} T_2^+ u - Su \leq (1 - \epsilon)v_0$$

and accordingly

$$\sup_{u \in [0,u_2], f(u) \leq \epsilon} T_2^+ u = v_0,$$

since the join of these two suprema is surely at least v_0, while the second is at most v_0. Note also that

$$v_0 = \sup_{u \in [0,u_2], f(u) \leq \epsilon} T_2^+ u = \sup_{0 \leq u' \leq u \leq u_2, f(u) \leq \epsilon} T_2 u' = \sup_{0 \leq u' \leq u_2, f(u') \leq \epsilon} T_2 u'.$$

For $k \in \mathbb{N}$ set $A_k = \{u : 0 \leq u \leq u_2,\ f(u) \leq 2^{-k}\}$. We know that

$$B_k = \{\sup_{u \in I} T_2 u : I \subseteq A_k \text{ is finite}\}$$

is an upwards-directed set with supremum v_0 for each k. Because V is weakly (σ, ∞)-distributive, we can find a sequence $\langle v_k' \rangle_{k \in \mathbb{N}}$ such that $v_k' \in B_k$ for every k and $v_1 = \inf_{k \in \mathbb{N}} v_k' > 0$. For each k let $I_k \subseteq A_k$ be a finite set such that $v_k' = \sup_{u \in I_k} T_2 u$.

Because each I_k is finite, we can build a sequence $\langle u_n' \rangle_{n \in \mathbb{N}}$ in $[0, u_2]$ enumerating each in turn, so that $\lim_{n \to \infty} f(u_n') = 0$ (since $f(u) \leq 2^{-k}$ if $u \in I_k$) while $\sup_{m \geq n} T_2 u_m' \geq v_1$ for every n (since $\{u_m' : m \geq n\}$ always includes some I_k). Now $\langle T_2 u_n' \rangle_{n \in \mathbb{N}}$ does not order*-converge to 0.

However, $\lim_{n \to \infty} g(u_n') = 0$ for every $g \in U^\times$. **P** Express $|g|$ as $g_1 + g_2$ where g_1 belongs to the band of $U^\times$ generated by f and $g_2 \wedge f = 0$ (353Hc). Then $g_2(u_n') = g_2(u_2) = 0$ for every n, by the choice of u_2. Also $g_1 = \sup_{n \in \mathbb{N}} g_1 \wedge nf$ (352Vb); so, given $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that $(g_1 - mf)^+(u_2) \leq \epsilon$ and $(g_1 - mf)^+(u_n') \leq \epsilon$ for every $n \in \mathbb{N}$. But this means that

$$|g(u_n')| \leq |g|(u_n') \leq \epsilon + mf(u_n')$$

for every n, and $\limsup_{n \to \infty} |g(u_n')| \leq \epsilon$; as ϵ is arbitrary, $\lim_{n \to \infty} g(u_n') = 0$. **Q**

Now, however, part (a) of this proof tells us that $\langle T_1 u_n' \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0, because $T_1 \in Z$, while $\langle Tu_n' \rangle_{n \in \mathbb{N}}$ is order*-convergent to 0, by hypothesis; so $\langle T_2 u_n' \rangle_{n \in \mathbb{N}} = \langle Tu_n' - T_1 u_n' \rangle_{n \in \mathbb{N}}$ order*-converges to 0. **X**

This contradiction shows that every operator satisfying the condition (ii) must be in Z.

376I The following elementary remark will be useful for the next corollary and also for Theorem 376S.

Lemma Let (X, Σ, μ) be a σ-finite measure space and U an order-dense solid linear subspace of $L^0(\mu)$. Then there is a non-decreasing sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of measurable subsets of X, with union X, such that $\chi X_n^\bullet \in U$ for every $n \in \mathbb{N}$.

proof Write $\mathfrak{A}$ for the measure algebra of μ, so that $L^0(\mu)$ can be identified with $L^0(\mathfrak{A})$ (364Jc). $A = \{a : a \in \mathfrak{A} \setminus \{0\},\ \chi a \in U\}$ is order-dense in $\mathfrak{A}$, so includes a partition of unity $\langle a_i \rangle_{i \in I}$. Because μ is σ-finite, $\mathfrak{A}$ is ccc (322G) and I is countable, so we can take I to be a subset of $\mathbb{N}$. Choose $E_i \in \Sigma$ such that $E_i^\bullet = a_i$ for $i \in I$; set $E = X \setminus \bigcup_{i \in I} E_i$, $X_n = E \cup \bigcup_{i \in I, i \leq n} E_i$ for $n \in \mathbb{N}$.

376J Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$. Let $U\subseteq L^0(\mu)$, $V\subseteq L^0(\nu)$ be perfect order-dense solid linear subspaces, and $T:U\to V$ a linear operator. Write $\mathcal{U}=\{f:f\in\mathcal{L}^0(\mu),\ f^\bullet\in U\}$, $\mathcal{V}^\#=\{h:h\in\mathcal{L}^0(\nu),\ h^\bullet\times v\in L^1$ for every $v\in V\}$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $k\in\mathcal{L}^0(\lambda)$ such that

(α) $\int|k(x,y)f(x)h(y)|d(x,y)<\infty$ for every $f\in\mathcal{U}$, $h\in\mathcal{V}^\#$,

(β) if $f\in\mathcal{U}$ and we set $g(y)=\int k(x,y)f(x)dx$ wherever this is defined, then $g\in\mathcal{L}^0(\nu)$ and $Tf^\bullet=g^\bullet$;

(iii) $T\in\mathsf{L}^\sim(U;V)$ and whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty}h(u_n)=0$ for every $h\in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

Remark I write '$d(x,y)$' above to indicate integration with respect to the product measure λ. Recall that in the terminology of §251, λ can be taken to be either the 'primitive' or 'c.l.d.' product measure (251K).

proof The idea is of course to identify $L^0(\mu)$ and $L^0(\nu)$ with $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, where $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are the measure algebras of μ and ν, so that their localizable measure algebra free product can be identified with the measure algebra of λ (325Eb), while $V^\#=\{h^\bullet:h\in\mathcal{V}^\#\}$ can be identified with $V^\times$.

(a)(i)$\Rightarrow$(ii) By 376F, there is a $w\in L^0(\lambda)$ such that $\int w\times(u\otimes v')$ is defined and equal to $\int Tu\times v'$ for every $u\in U$, $v'\in V^\#$. Express w as $k^\bullet$ where $k\in\mathcal{L}^0(\lambda)$. If $f\in\mathcal{U}$ and $h\in\mathcal{V}^\#$ then $\int|k(x,y)f(x)h(y)|d(x,y)=\int|w\times(f^\bullet\otimes h^\bullet|$ is finite, so (ii-α) is satisfied.

Now take any $f\in\mathcal{U}$, and set $g(y)=\int k(x,y)f(x)dx$ whenever this is defined in $\mathbb{R}$. Write $\mathcal{F}$ for the set of those $F\in\mathrm{T}$ such that $\chi F\in\mathcal{V}^\#$. Then for any $F\in\mathcal{F}$, g is defined almost everywhere on F and $g\restriction F$ is ν-virtually measurable. **P** $\int k(x,y)f(x)\chi F(y)d(x,y)$ is defined in $\mathbb{R}$, so by Fubini's theorem (252B, 252C) $g_F(y)=\int k(x,y)f(x)\chi F(y)dx$ is defined for almost every y, and is ν-virtually measurable; now $g\restriction F=g_F\restriction F$. **Q** Next, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ with union Y, by 376I, because V is perfect and order-dense, so $V^\#$ must also be order-dense in $L^0(\nu)$.

For each $n\in\mathbb{N}$, there is a measurable set $F'_n\subseteq F_n\cap\operatorname{dom}g$ such that $g\restriction F_n$ is measurable and $F_n\setminus F'_n$ is negligible. Setting $G=\bigcup_{n\in\mathbb{N}}F'_n$, G is conegligible and $g\restriction G$ is measurable, so $g\in\mathcal{L}^0(\nu)$.

If $\tilde g\in L^0(\nu)$ represents $Tu\in L^0(\nu)$, then for any $F\in\mathcal{F}$

$$\int_F\tilde g=\int Tu\times(\chi F)^\bullet=\int_F g.$$

In particular, this is true whenever $F\in\mathrm{T}$ and $F\subseteq F_n$. So g and $\tilde g$ agree almost everywhere on F_n, for each n, and $g=\tilde g$ a.e. Thus g also represents Tu, as required in (ii-β).

(b)(ii)$\Rightarrow$(i) Set $w=k^\bullet$ in $L^0(\lambda)$. If $f\in\mathcal{U}$ and $h\in\mathcal{V}^\#$ the hypothesis (α) tells us that $(x,y)\mapsto k(x,y)f(x)h(y)$ is integrable (because it surely belongs to $\mathcal{L}^0(\lambda)$). By Fubini's theorem,

$$\int k(x,y)f(x)h(y)d(x,y)=\int g(y)h(y)dy$$

where $g(y)=\int k(x,y)f(x)dx$ for almost every y, so that $Tf^\bullet=g^\bullet$, by (β). But this means that, setting $u=f^\bullet$, $v'=h^\bullet$,

$$\int w\times(u\otimes v')=\int Tu\times v';$$

and this is true for every $u\in U$, $v'\in V^\#$.

Thus T satisfies the condition 376F(ii), and is an abstract integral operator.

(b)(i)$\Rightarrow$(iii) Because V is weakly (σ,∞)-distributive (368S), this is covered by 376H(i)$\Rightarrow$(iii).

(c)(iii)$\Rightarrow$(i) Suppose that T satisfies (iii). The point is that T^+ is order-continuous. **P?** Otherwise, let $A\subseteq U$ be a non-empty downwards-directed set, with infimum 0, such that $v_0=\inf_{u\in A}T^+(u)>0$. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering X, and set $a_n=X_n^\bullet$ for each n. For each n, $\inf_{u\in A}[\![u>2^{-n}]\!]=0$, so we can find $\tilde u_n\in A$ such that $\bar\mu(a_n\cap[\![\tilde u_n>2^{-n}]\!])\le 2^{-n}$. Set $u_n=\inf_{i\le n}\tilde u_i$ for each n; then $\langle u_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0; also, $[0,u_n]$ meets A for each n, so that $v_0\le\sup\{Tu:0\le u\le u_n\}$ for each n. Because V is weakly (σ,∞)-distributive, we can find a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite sets such that $I_n\subseteq[0,u_n]$ for each n and $v_1=\inf_{n\in\mathbb{N}}\sup_{u\in I_n}(Tu)^+>0$. Enumerating $\bigcup_{n\in\mathbb{N}}I_n$ as $\langle u'_n\rangle_{n\in\mathbb{N}}$, as in part (d) of the proof of 376H, we see that $\langle u'_n\rangle_{n\in\mathbb{N}}$ is order-bounded and $\lim_{n\to\infty} f(u'_n) = 0$ for every $f \in U^\times$ (indeed, $\langle u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in U), while $\langle Tu'_n\rangle_{n\in\mathbb{N}} \not\to 0$ in V. **XQ**

Similarly, T^- is order-continuous, so $T \in \mathsf{L}^\times(U;V)$. Accordingly T is an abstract integral operator by condition (ii) of 376H.

376K As an application of the ideas above, I give a result due to N.Dunford (376N) which was one of the inspirations underlying the theory. Following the method of Zaanen 83, I begin with a couple of elementary lemmas.

Lemma Let U and V be Riesz spaces. Then there is a Riesz space isomorphism $T \mapsto T' : \mathsf{L}^\times(U;V^\times) \to \mathsf{L}^\times(V;U^\times)$ defined by the formula

$$(T'v)(u) = (Tu)(v) \text{ for every } u \in U,\ v \in V.$$

If for $f \in U^\times$, $g \in V^\times$ we write $P_{fg}(u) = f(u)g$ for every $u \in U$, then $P_{fg} \in \mathsf{L}^\times(U;V^\times)$ and $P'_{fg} = P_{gf}$ in $\mathsf{L}^\times(V;U^\times)$. Consequently T is an abstract integral operator iff T' is.

proof All the ideas involved have already appeared. For positive $T \in \mathsf{L}^\times(U;V^\times)$ the functional $(u,v) \mapsto (Tu)(v)$ is bilinear and order-continuous in each variable separately; so (just as in the first part of the proof of 376E) corresponds to a $T' \in \mathsf{L}^\times(V;U^\times)$. The map $T \mapsto T' : \mathsf{L}^\times(U;V^\times)^+ \to \mathsf{L}^\times(V;U^\times)^+$ is evidently an additive, order-preserving bijection, so extends to an isomorphism between $\mathsf{L}^\times(U;V^\times)$ and $\mathsf{L}^\times(V;U^\times)$ given by the same formula. I remarked in part (i) of the proof of 376E that every P_{fg} belongs to $\mathsf{L}^\times(U;V^\times)$, and the identification $P'_{fg} = P_{gf}$ is just a matter of checking the formulae. Of course it follows at once that the bands of abstract integral operators must also be matched by the map $T \mapsto T'$.

376L Lemma Let U be a Riesz space with an order-continuous norm. If $w \in U^+$ there is a $g \in U^\times$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le w$ and $g(u) \le \delta$.

proof (a) As remarked in 356D, $U^* = U^\sim = U^\times$. Set

$$A = \{v : v \in U \text{ and there is an } f \in (U^\times)^+ \text{ such that } f(u) > 0 \text{ whenever } 0 < u \le |v|\}.$$

Then $v' \in A$ whenever $|v'| \le |v| \in A$ and $v + v' \in A$ for all v, $v' \in A$ (if $f(u) > 0$ whenever $0 < u \le |v|$ and $f'(u) > 0$ whenever $0 < u \le |v'|$, then $(f + f')(u) > 0$ whenever $0 < u \le |v + v'|$); moreover, if $v_0 > 0$ in U, there is a $v \in A$ such that $0 < v \le v_0$. **P** Because $U^\times = U^*$ separates the points of U, there is a $g > 0$ in $U^\times$ such that $g(v_0) > 0$; now by 356H there is a $v \in]0, v_0]$ such that g is strictly positive on $]0, v]$, so that $v \in A$. **Q** But this means that A is an order-dense solid linear subspace of U.

(b) In fact $w \in A$. **P** $w = \sup B$, where $B = A \cap [0, w]$. Because B is upwards-directed, $w \in \overline{B}$ (354Ea), and there is a sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ in B converging to w for the norm. For each n, choose $f_n \in (U^\times)^+$ such that $f_n(u) > 0$ whenever $0 < u \le u'_n$. Set

$$f = \sum_{n=0}^\infty \frac{1}{2^n(1+\|f_n\|)} f_n$$

in $U^* = U^\times$. Then whenever $0 < u \le w$ there is some $n \in \mathbb{N}$ such that $u \wedge u'_n > 0$, so that $f_n(u) > 0$ and $f(u) > 0$. So f witnesses that $w \in A$. **Q**

(c) Take $g \in (U^\times)^+$ such that $g(u) > 0$ whenever $0 < u \le w$. This g serves. **P?** Otherwise, there is some $\epsilon > 0$ such that for every $n \in \mathbb{N}$ we can find a $u_n \in [0, w]$ with $g(u_n) \le 2^{-n}$ and $\|u_n\| \ge \epsilon$. Set $v_n = \sup_{i\ge n} u_i$; then $0 \le v_n \le w$, $g(v_n) \le 2^{-n+1}$ and $\|v_n\| \ge \epsilon$ for every $n \in \mathbb{N}$. But $\langle v_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so $v = \inf_{n\in\mathbb{N}} v_n$ must be non-zero, while $0 \le v \le w$ and $g(v) = 0$; which is impossible. **XQ**

Thus we have found an appropriate g.

376M Theorem (a) Let U be a Banach lattice with an order-continuous norm and V a Dedekind complete M-space. Then every bounded linear operator from U to V is an abstract integral operator.

(b) Let U be an L-space and V a Banach lattice with order-continuous norm. Then every bounded linear operator from U to $V^\times$ is an abstract integral operator.

proof (a) By 355Kb and 355C, $\mathsf{L}^\times(U;V)=\mathsf{L}^\sim(U;V)\subseteq\mathsf{B}(U;V)$; but since norm-bounded sets in V are also order-bounded, $\{Tu:|u|\le u_0\}$ is bounded above in V for every $T\in\mathsf{B}(U;V)$ and $u_0\in U^+$, and $\mathsf{B}(U;V)=\mathsf{L}^\times(U;V)$.

I repeat ideas from the proof of 376H. (I cannot quote 376H directly as I am not assuming that V is weakly (σ,∞)-distributive.) **?** Suppose, if possible, that $\mathsf{B}(U;V)$ is not the band Z of abstract integral operators. In this case there is a $T>0$ in $Z^\perp$. Take $u_1\ge 0$ such that $Tu_1\ne 0$. Let $f\ge 0$ in $U^\times$ be such that for every $\epsilon>0$ there is a $\delta>0$ such that $\|u\|\le\epsilon$ whenever $0\le u\le u_1$ and $f(u)\le\delta$ (376L). Set $v_0=Tu_1$. Then, just as in part (d) of the proof of 376H,

$$\sup_{u\in[0,u_1],f(u)\le\delta}Tu=v_0$$

for every $\delta>0$. But there is a $\delta>0$ such that $\|T\|\|u\|\le\frac{1}{2}\|v_0\|$ whenever $0\le u\le u_1$ and $f(u)\le\delta$; in which case $\|\sup_{u\in[0,u_1],f(u)\le\delta}Tu\|\le\frac{1}{2}\|v_0\|$, which is impossible. **X**

Thus $Z=\mathsf{B}(U;V)$, as required.

(b) Because V has an order-continuous norm, $V^*=V^\times=V^\sim$; and the norm of V^* is a Fatou norm with the Levi property (356Da). So $\mathsf{B}(U;V^*)=\mathsf{L}^\times(U;V^\times)$, by 371C. By 376K, this is canonically isomorphic to $\mathsf{L}^\times(V;U^\times)$. Now $U^\times=U^*$ is an M-space (356Pb). By (a), every member of $\mathsf{L}^\times(V;U^\times)$ is an abstract integral operator; but the isomorphism between $\mathsf{L}^\times(V;U^\times)$ and $\mathsf{L}^\times(U;V^\times)$ matches the abstract integral operators in each space (376K), so every member of $\mathsf{B}(U;V^*)$ is also an abstract integral operator, as claimed.

376N Corollary: Dunford's theorem Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T:L^1(\mu)\to L^p(\nu)$ a bounded linear operator, where $1<p\le\infty$. Then there is a measurable function $k:X\times Y\to\mathbb{R}$ such that $Tf^\bullet=g_f^\bullet$, where $g_f(y)=\int k(x,y)f(x)dx$ almost everywhere, for every $f\in\mathcal{L}^1(\mu)$.

proof Set $q=\frac{p}{p-1}$ if p is finite, 1 if $p=\infty$. We can identify $L^p(\nu)$ with $V^\times$, where $V=L^q(\nu)\cong L^p(\nu)^\times$ (366Dc, 365Jc) has an order-continuous norm because $1\le q<\infty$. By 376Mb, T is an abstract integral operator. By 376F/376J, T is represented by a kernel, as claimed.

376O Under the right conditions, weakly compact operators are abstract integral operators.

Lemma Let U be a Riesz space, and W a solid linear subspace of $U^\sim$. If $C\subseteq U$ is relatively compact for the weak topology $\mathfrak{T}_s(U,W)$ (3A5E), then for every $g\in W^+$ and $\epsilon>0$ there is a $u^*\in U^+$ such that $g(|u|-u^*)^+\le\epsilon$ for every $u\in C$.

proof Let W_g be the solid linear subspace of W generated by g. Then W_g is an Archimedean Riesz space with order unit, so $W_g^\times$ is a band in the L-space $W_g^*=W_g^\sim$ (356Na), and is therefore an L-space in its own right (354O). For $u\in U$, $h\in W_g^\times$ set $(Tu)(h)=h(u)$; then T is an order-continuous Riesz homomomorphism from U to $W_g^\times$ (356F).

Now W_g is perfect. **P** I use 356K. W_g is Dedekind complete because it is a solid linear subspace of the Dedekind complete space $U^\sim$. $W_g^\times$ separates the points of W because $T[U]$ does. If $A\subseteq W_g$ is upwards-directed and $\sup_{h\in A}\phi(h)$ is finite for every $\phi\in W_g^\times$, then A acts on $W_g^\times$ as a set of bounded linear functionals which, by the Uniform Boundedness Theorem (3A5Ha), is uniformly bounded; that is, there is some $M\ge 0$ such that $\sup_{h\in A}|\phi(h)|\le M\|\phi\|$ for every $\phi\in W_g^\times$. Because g is the standard order unit of W_g, we have $\|\phi\|=|\phi|(g)$ and $|\phi(h)|\le M|\phi|(g)$ for every $\phi\in W_g^\times$, $h\in A$. In particular,

$$h(u)\le|h(u)|=|(Tu)(h)|\le M|Tu|(g)=MTu(g)=Mg(u)$$

for every $h\in A$, $u\in U^+$. But this means that $h\le Mg$ for every $h\in A$ and A is bounded above in W_g. Thus all the conditions of 356K are satisfied and W_g is perfect. **Q**

Accordingly T is continuous for the topologies $\mathfrak{T}_s(U,W)$ and $\mathfrak{T}_s(W_g^\times,W_g^{\times\times})$, because every element ϕ of $W_g^{\times\times}$ corresponds to a member of $W_g\subseteq W$, so 3A5Ec applies.

Now we are supposing that C is relatively compact for $\mathfrak{T}_s(U,W)$, that is, is included in some compact set C'; accordingly $T[C']$ is compact and $T[C]$ is relatively compact for $\mathfrak{T}_s(W_g^\times,W_g^{\times\times})$. Since $W_g^\times$ is an L-space, $T[C]$ is uniformly integrable (356Q); consequently (ignoring the trivial case $C=\emptyset$) there are $\phi_0,\dots,\phi_n\in T[C]$ such that $\|(|\phi|-\sup_{i\le n}|\phi_i|)^+\|\le\epsilon$ for every $\phi\in T[C]$ (354Rb), so that $(|\phi|-\sup_{i\le n}|\phi_i|)^+(g)\le\epsilon$ for every $\phi\in T[C]$.

Translating this back into terms of C itself, and recalling that T is a Riesz homomorphism, we see that there are $u_0,\dots,u_n\in C$ such that $g(|u|-\sup_{i\le n}|u_i|)^+\le\epsilon$ for every $u\in C$. Setting $u^*=\sup_{i\le n}|u_i|$ we have the result.

376P Theorem Let U be an L-space and V a perfect Riesz space. If $T:U\to V$ is a linear operator such that $\{Tu:u\in U,\ \|u\|\le 1\}$ is relatively compact for the weak topology $\mathfrak{T}_s(V,V^\times)$, then T is an abstract integral operator.

proof (a) For any $g\ge 0$ in $V^\times$, $M_g=\sup_{\|u\|\le 1}g(|Tu|)$ is finite. **P** By 376O, there is a $v^*\in V^+$ such that $g(|Tu|-v^*)^+\le 1$ whenever $\|u\|\le 1$; now $M_g\le g(v^*)+1$. **Q** Considering $\|u\|^{-1}u$, we see that $g(|Tu|)\le M_g\|u\|$ for every $u\in U$.

Next, we find that $T\in\mathsf{L}^\sim(U;V)$. **P** Take $u\in U^+$. Set

$$B=\{\sum_{i=0}^n|Tu_i|:u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=u\}\subseteq V^+.$$

Then B is upwards-directed. (Cf. 371A.) If $g\ge 0$ in $V^\times$,

$$\begin{aligned}\sup_{v\in B}g(v)&=\sup\{\sum_{i=0}^n g(|Tu_i|):\sum_{i=0}^n u_i=u\}\\&\le\sup\{\sum_{i=0}^n M_g\|u_i\|:\sum_{i=0}^n u_i=u\}=M_g\|u\|\end{aligned}$$

is finite. By 356K, B is bounded above in V; and of course any upper bound for B is also an upper bound for $\{Tu':0\le u'\le u\}$. As u is arbitrary, T is order-bounded. **Q**

Because U is a Banach lattice with an order-continuous norm, $T\in\mathsf{L}^\times(U;V)$ (355Kb).

(b) Since we can identify $\mathsf{L}^\times(U;V)$ with $\mathsf{L}^\times(U;V^{\times\times})$, we have an adjoint operator $T'\in\mathsf{L}^\times(V^\times;U^\times)$, as in 376K. Now if $g\ge 0$ in $V^\times$ and $\langle g_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[0,g]$ such that $\lim_{n\to\infty}g_n(v)=0$ for every $v\in V$, $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $U^\times$. **P** For any $\epsilon>0$, there is a $v^*\in V^+$ such that $g(|Tu|-v^*)^+\le\epsilon$ whenever $\|u\|\le 1$; consequently

$$\begin{aligned}\|T'g_n\|&=\sup_{\|u\|\le 1}(T'g_n)(u)=\sup_{\|u\|\le 1}g_n(Tu)\\&\le g_n(v^*)+\sup_{\|u\|\le 1}g_n(|Tu|-v^*)^+\\&\le g_n(v^*)+\sup_{\|u\|\le 1}g(|Tu|-v^*)^+\le g_n(v^*)+\epsilon\end{aligned}$$

for every $n\in\mathbb{N}$. As $\lim_{n\to\infty}g_n(v^*)=0$, $\limsup_{n\to\infty}\|T'g_n\|\le\epsilon$; as ϵ is arbitrary, $\langle\|T'g_n\|\rangle_{n\in\mathbb{N}}\to 0$. But as $U^\times$ is an M-space (356Pb), it follows that $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. **Q**

By 368Pc, V is weakly (σ,∞)-distributive. By 376H, T' is an abstract integral operator, so T also is, by 376K.

376Q Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T:L^1(\mu)\to L^1(\nu)$ a weakly compact bounded linear operator. Then there is a function $k:X\times Y\to\mathbb{R}$ such that $Tf^\bullet=g_f^\bullet$, where $g_f(y)=\int k(x,y)f(x)dx$ almost everywhere, for every $f\in\mathcal{L}^1(\mu)$.

proof This follows from 376P and 376J, just as in 376N.

376R So far I have mentioned actual kernel functions $k(x,y)$ only as a way of giving slightly more concrete form to the abstract kernels of 376E. But of course they can provide new structures and insights. I give one result as an example. The following lemma is useful.

Lemma Let (X,Σ,μ) be a measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Suppose that k is a λ-integrable real-valued function. Then for any $\epsilon>0$ there is a finite partition $E_0,\dots,E_n$ of X into measurable sets such that $\|k-k_1\|_1\le\epsilon$, where

$$k_1(x,y) = \frac{1}{\mu E_i}\int_{E_i} k(t,y)dt \text{ whenever } x \in E_i,\ 0 < \mu E_i < \infty$$

$$\text{and the integral is defined in } \mathbb{R},$$

$$= 0 \text{ in all other cases.}$$

proof Once again I refer to the proof of 253F: there are sets $H_0,\dots,H_r$ of finite measure in X, sets $F_0,\dots,F_r$ of finite measure in Y, and $\alpha_0,\dots,\alpha_r$ such that $\|k-k_2\|_1 \le \frac{1}{2}\epsilon$, where $k_2 = \sum_{j=0}^{r}\alpha_i\chi(H_j\times F_j)$. Let $E_0,\dots,E_n$ be the partition of X generated by $\{H_i : i \le r\}$. Then for any $i \le n$, $\int_{E_i\times Y}|k-k_1|$ is defined and is at most $2\int_{E_i\times Y}|k-k_2|$. **P** If $\mu E_i = 0$, this is trivial, as both are zero. If $\mu E_i = \infty$, then again the result is elementary, since both k_1 and k_2 are zero on $E_i\times Y$. So let us suppose that $0 < \mu E_i < \infty$. In this case $\int_{E_i} k(t,y)dt$ must be defined for almost every y, by Fubini's theorem. So k_1 is defined almost everywhere on $E_i\times Y$, and

$$\int_{E_i\times Y}|k-k_1| = \int_Y\int_{E_i}|k(x,y)-k_1(x,y)|dxdy.$$

Now take some fixed $y \in Y$ such that

$$\beta = \frac{1}{\mu E_i}\int_{E_i} k(t,y)dt$$

is defined. Then $\beta = k_1(x,y)$ for every $x \in E_i$. For every $x \in E_i$, we must have $k_2(x,y) = \alpha$ where $\alpha = \sum\{\alpha_j : E_i \subseteq H_j,\ y \in F_j\}$. But in this case, because $\int_{E_i} k(x,y) - \beta\,dx = 0$, we have

$$\int_{E_i}\max(0,k(x,y)-\beta)dx = \int_{E_i}\max(0,\beta-k(x,y))dx = \frac{1}{2}\int_{E_i}|k(x,y)-k_1(x,y)|dx.$$

If $\beta \ge \alpha$,

$$\int_{E_i}\max(0,k(x,y)-\beta)dx \le \int_{E_i}\max(0,k(x,y)-\alpha)dx \le \int_{E_i}|k(x,y)-k_2(x,y)|dx;$$

if $\beta \le \alpha$,

$$\int_{E_i}\max(0,\beta-k(x,y))dx \le \int_{E_i}\max(0,\alpha-k(x,y))dx \le \int_{E_i}|k(x,y)-k_2(x,y)|dx;$$

in either case,

$$\frac{1}{2}\int_{E_i}|k(x,y)-k_1(x,y)|dx \le \int_{E_i}|k(x,y)-k_2(x,y)|dx.$$

This is true for almost every y, so integrating with respect to y we get the result. **Q**

Now, summing over i, we get

$$\int|k-k_1| \le 2\int|k-k_2| \le \epsilon,$$

as required.

376S Theorem Let (X,Σ,μ) be a complete locally determined measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Let τ be an extended Fatou norm on $L^0(\nu)$ and write $\mathcal{L}^{\tau'}$ for $\{g : g \in \mathcal{L}^0(\nu),\ \tau'(g^\bullet) < \infty\}$, where τ' is the associate extended Fatou norm of τ (369H-369I). Suppose that $k \in \mathcal{L}^0(\lambda)$ is such that $k\times(f\otimes g)$ is integrable whenever $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$. Then we have a corresponding linear operator $T : L^1(\mu) \to L^\tau$ defined by saying that $\int(Tf^\bullet)\times g^\bullet = \int k\times(f\otimes g)$ whenever $f \in \mathcal{L}^1(\mu)$, $g \in \mathcal{L}^{\tau'}$.

For $x \in X$ set $k_x(y) = k(x,y)$ whenever this is defined. Then $k_x \in \mathcal{L}^0(\nu)$ for almost every x; set $v_x = k_x^\bullet \in L^0(\nu)$ for such x. In this case $x \mapsto \tau(v_x)$ is measurable and defined and finite almost everywhere, and $\|T\| = \operatorname{ess\,sup}_x \tau(v_x)$.

Remarks The discussion of extended Fatou norms in §369 regarded them as functionals on spaces of the form $L^0(\mathfrak{A})$. I trust that no-one will be offended if I now speak of an extended Fatou norm on $L^0(\nu)$, with the associated function spaces L^τ, $L^{\tau'} \subseteq L^0$, taking for granted the identification in 364Jc.

Recall that $(f\otimes g)(x,y) = f(x)g(y)$ for $x \in \operatorname{dom} f$, $y \in \operatorname{dom} g$ (253B).

By 'ess $\sup_x \tau(v_x)$' I mean

$$\inf\{M : M \ge 0,\ \{x : v_x \text{ is defined and } \tau(v_x) \le M\} \text{ is conegligible}\}$$

(see 243D).

proof (a) To see that the formula $(f,g) \mapsto \int k \times (f \otimes g)$ gives rise to an operator in $\mathrm{L}^\times(U;(L^{\tau'})^\times)$, it is perhaps quickest to repeat the argument of parts (a) and (b) of the proof of 376E. (We are not quite in a position to quote 376E, as stated, because the localizable measure algebra free product there might be strictly larger than the measure algebra of λ; see 325B.) The first step, of course, is to note that changing f or g on a negligible set does not affect the integral $\int k \times (f \otimes g)$, so that we have a bilinear functional on $L^1 \times L^{\tau'}$; and the other essential element is the fact that the maps $f^\bullet \mapsto (f \otimes \chi Y)^\bullet$, $g^\bullet \mapsto (\chi X \otimes g)^\bullet$ are order-continuous (put 325A and 364Rc together).

By 369K, we can identify $(L^{\tau'})^\times$ with L^τ, so that T becomes an operator in $\mathrm{L}^\times(U;L^\tau)$. Note that it must be norm-bounded (355C).

(b) By 376I, there is a non-decreasing sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of measurable sets in Y, covering Y, such that $\chi Y_n \in \mathcal{L}^{\tau'}$ for every n. Set $X_0 = \{x : x \in X,\ k_x \in \mathcal{L}^0(\nu)\}$. Then X_0 is conegligible in X. **P** Let $E \in \Sigma$ be any set of finite measure. Then for any $n \in \mathbb{N}$, $k \times (\chi E \otimes \chi Y_n)$ is integrable, that is, $\int_{E\times Y_n} k$ is defined and finite; so by Fubini's theorem $\int_{Y_n} k_x$ is defined and finite for almost every $x \in E$. Consequently, for almost every $x \in E$, $k_x \times \chi Y_n \in \mathcal{L}^0(\nu)$ for every $n \in \mathbb{N}$, that is, $k_x \in \mathcal{L}^0(\nu)$, that is, $x \in X_0$.

Thus $E \setminus X_0$ is negligible for every set E of finite measure. Because μ is complete and locally determined, X_0 is conegligible. **Q**

This means that v_x and $\tau(v_x)$ are defined for almost every x.

(c) $\tau(v_x) \le \|T\|$ for almost every x. **P** Take any $E \in \Sigma$ of finite measure, and $n \in \mathbb{N}$. Then $k\times\chi(E\times Y_n)$ is integrable. For each $r \in \mathbb{N}$, there is a finite partition $E_{r0},\dots,E_{r,m(r)}$ of E into measurable sets such that $\int_{E\times Y_n} |k - k^{(r)}| \le 2^{-r}$, where

$$\begin{aligned} k^{(r)}(x,y) &= \frac{1}{\mu E_{ri}} \int_{E_{ri}} k(t,y)dt \text{ whenever } y \in Y_n,\ x \in E_{ri},\ \mu E_{ri} > 0 \\ &\qquad\qquad \text{and the integral is defined in } \mathbb{R}, \\ &= 0 \text{ otherwise} \end{aligned}$$

(376R). Now $k^{(r)}$ is also integrable over $E \times Y_n$, so $k_x^{(r)} \in \mathcal{L}^0(\nu)$ for almost every $x \in E$, writing $k_x^{(r)}(y) = k^{(r)}(x,y)$, and we can speak of $v_x^{(r)} = (k_x^{(r)})^\bullet$ for almost every x. Note that $k_x^{(r)} = k_{x'}^{(r)}$ whenever x, x' belong to the same E_{ri}.

If $\mu E_{ri} > 0$, then $v_x^{(r)}$ must be defined for every $x \in E_{ri}$. If $v' \in L^{\tau'}$ is represented by $g \in \mathcal{L}^{\tau'}$ then

$$\begin{aligned} \int k \times (\chi E_{ri} \otimes (g \times \chi Y_n)) &= \int_{E_{ri}\times Y_n} k(t,y)g(y)d(t,y) \\ &= \mu E_{ri} \int k^{(r)}(x,y)g(y)dy = \mu E_{ri} \int v_x^{(r)} \times v' \end{aligned}$$

for any $x \in E_{ri}$. But this means that

$$\mu E_{ri} \int v_x^{(r)} \times v' = \int T(\chi E_{ri}^\bullet) \times v' \times \chi Y_n^\bullet$$

for every $v' \in L^{\tau'}$, so

$$v_x^{(r)} = \frac{1}{\mu E_{ri}} T(\chi E_{ri}^\bullet) \times \chi Y_n^\bullet, \qquad \tau(v_x^{(r)}) \le \frac{1}{\mu E_{ri}} \|T\| \|\chi E_{ri}^\bullet\|_1 = \|T\|$$

for every $x \in E_{ri}$. This is true whenever $\mu E_{ri} > 0$, so in fact $\tau(v_x^{(r)}) \le \|T\|$ for almost every $x \in E$.

Because $\sum_{r\in\mathbb{N}} \int_{E\times Y_n} |k - k^{(r)}| < \infty$, we must have $k(x,y) = \lim_{r\to\infty} k^{(r)}(x,y)$ for almost every $(x,y) \in E \times Y_n$. Consequently, for almost every $x \in E$, $k(x,y) = \lim_{r\to\infty} k^{(r)}(x,y)$ for almost every $y \in Y_n$, that is,

$\langle v_x^{(r)}\rangle_{r\in\mathbb{N}}$ order*-converges to $v_x \times \chi Y_n^\bullet$ (in $L^0(\nu)$) for almost every $x \in E$. But this means that, for almost every $x \in E$,

$$\tau(v_x \times \chi Y_n^\bullet) \le \liminf_{r\to\infty} \tau(v_x^{(r)}) \le \|T\|$$

(369Mc). Now

$$\tau(v_x) = \lim_{n\to\infty} \tau(v_x \times \chi Y_n^\bullet) \le \|T\|$$

for almost every $x \in E$.

As in (b), this implies (since E is arbitrary) that $\tau(v_x) \le \|T\|$ for almost every $x \in X$. **Q**

(d) I now show that $x \mapsto \tau(v_x)$ is measurable. **P** Take $\gamma \in [0,\infty[$ and set $A = \{x : x \in X_0,\, \tau(v_x) \le \gamma\}$. Suppose that $\mu E < \infty$. Let G be a measurable envelope of $A \cap E$ (132Ed). Set $\tilde{k}(x,y) = k(x,y)$ when $x \in G$ and $(x,y) \in \operatorname{dom} k$, 0 otherwise. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$, then

$$\int \tilde{k}(x,y)f(x)g(y)d(x,y) = \int_{G\times Y} k(x,y)f(x)g(y)d(x,y) = \int_G f(x)\int_Y k(x,y)g(y)dydx$$

is defined.

For $x \in X_0$, set $h(x) = \int |\tilde{k}(x,y)g(y)|dy$. Then h is finite almost everywhere and measurable. For $x \in A \cap E$,

$$\int |\tilde{k}(x,y)g(y)|dy = \int |v_x \times g^\bullet| \le \gamma\tau'(g^\bullet).$$

So the measurable set $G' = \{x : h(x) \le \gamma\tau'(g^\bullet)\}$ includes $A \cap E$, and $\mu(G \setminus G') = 0$. Consequently

$$|\int \tilde{k}(x,y)f(x)g(y)d(x,y)| \le \int_G |f(x)|h(x)dx \le \gamma\|f\|_1\tau'(g^\bullet),$$

and this is true for every $f \in \mathcal{L}^1(\mu)$, $g \in \mathcal{L}^{\tau'}$.

Now we have an operator $\tilde{T} : L^1(\mu) \to L^\tau$ defined by the formula

$$\int(\tilde{T}f^\bullet) \times g^\bullet = \int \tilde{k} \times (f \otimes g) \text{ when } f \in \mathcal{L}^1(\nu) \text{ and } g \in \mathcal{L}^{\tau'},$$

and the formula just above tells us that $|\int \tilde{T}u \times v'| \le \gamma\|u\|_1\tau'(v')$ for every $u \in L^1(\nu)$, $v' \in L^{\tau'}$; that is, $\tau(\tilde{T}u) \le \gamma\|u\|_1$ for every $u \in L^1(\mu)$; that is, $\|\tilde{T}\| \le \gamma$. But now (c) tells us that $\tau(\tilde{v}_x) \le \gamma$ for almost every $x \in X$, where $\tilde{v}_x$ is the equivalence class of $y \mapsto \tilde{k}(x,y)$, that is, $\tilde{v}_x = v_x$ for $x \in G \cap X_0$, 0 for $x \in X \setminus G$. So $\tau(v_x) \le \gamma$ for almost every $x \in G$, and $G \setminus A$ is negligible. But this means that $A \cap E$ is measurable. As E is arbitrary, A is measurable; as γ is arbitrary, $x \mapsto \tau(v_x)$ is measurable. **Q**

(e) Finally, the ideas in (d) show that $\|T\| \le \operatorname{ess\,sup}_x \tau(v_x)$. **P** Set $M = \operatorname{ess\,sup}_x \tau(v_x)$. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$, then

$$\int |k(x,y)f(x)g(y)|d(x,y) \le \int |f(x)|\tau(v_x)\tau'(g^\bullet)dx \le M\|f\|_1\tau'(g^\bullet);$$

as g is arbitrary, $\tau(Tf^\bullet) \le M\|f\|_1$; as f is arbitrary, $\|T\| \le M$. **Q**

376X Basic exercises >(a) Let μ be Lebesgue measure on $\mathbb{R}$. Let h be a μ-integrable real-valued function with $\|h\|_1 \le 1$, and set $k(x,y) = h(y-x)$ whenever this is defined. Show that if f is in either $\mathcal{L}^1(\mu)$ or $\mathcal{L}^\infty(\mu)$ then $g(y) = \int k(x,y)f(x)dx$ is defined for almost every $y \in \mathbb{R}$, and that this formula gives rise to an operator $T \in \mathcal{T}^\times_{\bar\mu,\bar\mu}$ as defined in 373A. (*Hint*: 255H.)

(b) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$, and take $p \in [1,\infty]$. Show that if $u \in L^p(\mathfrak{A},\bar\mu)$ and $v \in L^p(\mathfrak{B},\bar\nu)$ then $u \otimes v \in L^p(\mathfrak{C},\bar\lambda)$ and $\|u \otimes v\|_p = \|u\|_p\|v\|_p$.

>(c) Let U, V, W be Riesz spaces, of which V and W are Dedekind complete, and suppose that $T \in \mathrm{L}^\times(U;V)$ and $S \in \mathrm{L}^\times(V;W)$. Show that if either S or T is an abstract integral operator, so is ST.

(d) Let h be a Lebesgue integrable function on $\mathbb{R}$, and f a square-integrable function. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions such that (α) $|f_n| \le f$ for every n (β) $\lim_{n\to\infty}\int_E f_n = 0$ for every measurable set E of finite measure. Show that $\lim_{n\to\infty}(h * f_n)(y) = 0$ for almost every $y \in \mathbb{R}$, where $h * f_n$ is the convolution of h and f_n. (*Hint*: 376Xa, 376F.)

(e) Let U and V be Riesz spaces, of which V is Dedekind complete. Suppose that $W \subseteq U^\sim$ is a solid linear subspace, and that T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u \mapsto f(u)v$, where $f \in W$ and $v \in V$. Show that whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in W$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

(f) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace such that $U^\times$ separates the points of U. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty} f(|u_n|) = 0$ for every $f \in U^\times$; (ii) $\langle u_n\rangle_{n\in\mathbb{N}} \to 0$ for the topology of convergence in measure on L^0. (*Hint*: by 367T, condition (ii) is intrinsic to U, so we can replace $(\mathfrak{A}, \bar\mu)$ by a localizable algebra and use the representation in 369D.)

(g) Let U be a Banach lattice with an order-continuous norm, and V a weakly (σ,∞)-distributive Riesz space. Show that for $T \in \mathsf{L}^\sim(U;V)$ the following are equiveridical: (i) T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u \mapsto f(u)v$ where $f \in U^\sim$, $v \in V$; (ii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ which is norm-convergent to 0; (iii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U which is weakly convergent to 0.

(h) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, with product measure λ on $X \times Y$, and measure algebras $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$. Suppose that $k \in \mathcal{L}^0(\lambda)$. Show that the following are equiveridical: (i)(α) if $f \in \mathcal{L}^1(\mu)$ then $g_f(y) = \int k(x,y)f(x)dx$ is defined for almost every y and $g_f \in \mathcal{L}^1(\nu)$ (β) there is an operator $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ defined by setting $Tf^\bullet = g_f^\bullet$ for every $f \in \mathcal{L}^1(\mu)$; (ii) $\int |k(x,y)|dy \le 1$ for almost every $x \in X$, $\int |k(x,y)|dx \le 1$ for almost every $y \in Y$.

>(i) Let μ be Lebesgue measure on $\mathbb{R}$. Give an example of a measurable function $k : [0,1]^2 \to \mathbb{R}$ such that, for any $f \in \mathcal{L}^2(\mu)$, $g_f(y) = \int k(x,y)f(x)dx$ is defined for every y and $\|g_f\|_2 = \|f\|_2$, but k is not integrable, so the linear isometry on $L^2 = L^2(\mu)$ defined by k does not belong to $\mathsf{L}^\sim(L^2;L^2)$. (*Hint*: 371Ye.)

(j) Let (X, Σ, μ) be a σ-finite measure space and (Y, T, ν) a complete locally determined measure space. Let $U \subseteq L^0(\mu)$, $V \subseteq L^0(\nu)$ be solid linear subspaces, of which V is order-dense; write $V^\# = \{v : v \in L^0(\nu),\ v \times v' \text{ is integrable for every } v' \in V\}$, $\mathcal{U} = \{f : f \in \mathcal{L}^0(\nu),\ f^\bullet \in U\}$, $\mathcal{V} = \{g : g \in \mathcal{L}^0(\nu),\ g^\bullet \in V\}$, $\mathcal{V}^\# = \{h : h \in \mathcal{L}^0(\nu),\ h^\bullet \in V^\#\}$. Let λ be the c.l.d. product measure on $X \times Y$, and $k \in \mathcal{L}^0(\lambda)$ a function such that $k \times (f \otimes g)$ is integrable for every $f \in \mathcal{U}$, $g \in \mathcal{V}$. (i) Show that for any $f \in \mathcal{U}$, $h_f(y) = \int k(x,y)f(x)dx$ is defined for almost every $y \in Y$, and that $h_f \in \mathcal{V}^\#$. (ii) Show that we have a map $T \in \mathsf{L}^\times(U;V^\#)$ defined *either* by writing $Tf^\bullet = h_f^\bullet$ for every $f \in \mathcal{U}$ *or* by writing $\int (Tf^\bullet) \times g^\bullet = \int k \times (f \otimes g)$ for every $f \in \mathcal{U}$, $g \in \mathcal{V}$.

(k) Let (X, Σ, μ), (Y, T, ν) and (Z, Λ, λ) be σ-finite measure spaces, and U, V, W perfect order-dense solid linear subspaces of $L^0(\mu)$, $L^0(\nu)$ and $L^0(\lambda)$ respectively. Suppose that $T : U \to V$ and $S : V \to W$ are abstract integral operators corresponding to kernels $k_1 \in \mathcal{L}^0(\mu \times \nu)$, $k_2 \in \mathcal{L}^0(\nu \times \lambda)$, writing $\mu \times \nu$ for the (c.l.d. or primitive) product measure on $X \times Y$. Show that $ST : U \to W$ is represented by the kernel $k \in \mathcal{L}^0(\mu \times \lambda)$ defined by setting $k(x,z) = \int k_1(x,y)k_2(y,z)dy$ whenever this integral is defined.

(l) Let U be a perfect Riesz space. Show that a set $C \subseteq U$ is relatively compact for $\mathfrak{T}_s(U, U^\times)$ iff for every $g \in (U^\times)^+$, $\epsilon > 0$ there is a $u^* \in U$ such that $g(|u| - u^*)^+ \le \epsilon$ for every $u \in C$. (*Hint*: 376O and the proof of 356Q.)

>(m) Let μ be Lebesgue measure on $[0,1]$, and ν counting measure on $[0,1]$. Set $k(x,y) = 1$ if $x = y$, 0 otherwise. Show that 376S fails in this context (with, e.g., $\tau = \| \|_\infty$).

(n) Suppose, in 376Xj, that $U = L^\tau$ for some extended Fatou norm on $L^0(\mu)$ and that $V = L^1(\nu)$, so that $V^\# = L^\infty(\nu)$. Set $k_y(x) = k(x,y)$ whenever this is defined, $u_y = k_y^\bullet$ whenever $k_y \in \mathcal{L}^0(\nu)$. Show that $u_y \in L^\tau$ for almost every $y \in Y$, and that the norm of T in $\mathsf{B}(L^\tau;L^\infty)$ is $\operatorname{ess\,sup}_y \tau'(u_y)$. (*Hint*: do the case of totally finite Y first.)

376Y Further exercises (a) Let U, V and W be linear spaces (over any field F) and $\phi : U \times V \to W$ a bilinear map. Let W_0 be the linear subspace of W generated by $\phi[U \times V]$. Show that the following are equiveridical: (i) for every linear space Z over F and every bilinear $\psi : U \times V \to Z$, there is a (unique) linear operator $T : W_0 \to Z$ such that $T\phi = \psi$ (ii) whenever $u_0, \dots , u_n \in U$ are linearly independent and $v_0, \dots , v_n \in V$ are non-zero, $\sum_{i=0}^{n} \phi(u_i, v_i) \neq 0$ (iii) whenever $u_0, \dots , u_n \in U$ are non-zero and $v_0, \dots , v_n \in V$ are linearly independent, $\sum_{i=0}^{n} \phi(u_i, v_i) \neq 0$ (iv) for any Hamel bases $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of U and V, $\langle\phi(u_i, v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0 (v) for some pair $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of Hamel bases of U and V, $\langle\phi(u_i, v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0.

(b) Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras, and $(\mathfrak{C}, \bar\lambda)$ their localizable measure algebra free product. Show that $\otimes : L^0(\mathfrak{A}) \times L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ satisfies the equivalent conditions of 376Ya.

(c) Let (X, Σ, μ) and (Y, T, ν) be semi-finite measure spaces and λ the c.l.d. product measure on $X \times Y$. Show that the map $(f, g) \mapsto f \otimes g : \mathcal{L}^0(\mu) \times \mathcal{L}^0(\nu) \to \mathcal{L}^0(\lambda)$ induces a map $(u, v) \mapsto u \otimes v : L^0(\mu) \times L^0(\nu) \to L^0(\lambda)$ possessing all the properties described in 376B and 376Ya.

(d) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of $\{0, 1\}^{\omega_1}$ with its usual measure, and $\langle b_\xi\rangle_{\xi<\omega_1}$ the canonical independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{A}$. Set $U = L^2(\mathfrak{A}, \bar\mu)$ and $V = \{v : v \in \mathbb{R}^{\omega_1}, \{\xi : v(\xi) \neq 0\}$ is countable$\}$. Define $T : U \to V$ by setting $Tu(\xi) = 2\int_{b_\xi} u - \int u$ for $\xi < \omega_1$, $u \in U$. Show that (i) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$ (ii) $T \notin \mathrm{L}^\sim(U; V)$.

(e) Let U be a Riesz space with the countable sup property (definition: 241Yd) such that $U^\times$ separates the points of U, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty} f(v\wedge|u_n|) = 0$ for every $f \in U^\times$, $v \in U^+$; (ii) every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which is order*-convergent to 0.

(f) Let U be an Archimedean Riesz space and $\mathfrak{A}$ a weakly (σ, ∞)-distributive Dedekind complete Boolean algebra. Suppose that $T : U \to L^0 = L^0(\mathfrak{A})$ is a linear operator such that $\langle|Tu_n|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded and order*-convergent to 0 in U. Show that $T \in \mathrm{L}^\sim_c(U; L^0)$, so that if U has the countable sup property then $T \in \mathrm{L}^\times(U; L^0)$.

(g) Suppose that (Y, T, ν) is a probability space in which $\mathrm{T} = \mathcal{P}Y$, $\nu\{y\} = 0$ for every $y \in Y$. (See 363S.) Take $X = Y$ and let μ be counting measure on X; let λ be the c.l.d. product measure on $X \times Y$, and set $k(x, y) = 1$ if $x = y$, 0 otherwise. Show that we have an operator $T : L^\infty(\mu) \to L^\infty(\nu)$ defined by setting $Tf = g^\bullet$ whenever $f \in L^\infty(\mu) \cong \ell^\infty(X)$ and $g(y) = \int k(x, y)f(x)dx = f(y)$ for every $y \in Y$. Show that T does not belong to $\mathrm{L}^\times(L^\infty(\mu); L^\infty(\nu))$ and in particular does not satisfy the condition (ii) of 376J.

(h) Give an example of an abstract integral operator $T : \ell^2 \to L^1(\mu)$, where μ is Lebesgue measure on $[0, 1]$, such that $\langle Te_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent in $L^1(\mu)$, where $\langle e_n\rangle_{n\in\mathbb{N}}$ is the standard orthonormal sequence in ℓ^2.

(i) Let U be an L-space and write $\mathfrak{G}$ for the regular open algebra of $\mathbb{R}$. Show that any bounded linear operator from U to $L^\infty(\mathfrak{G})$ is an abstract integral operator.

(j) Let U be an L-space and V a Banach lattice with an order-continuous norm. Let $T \in \mathrm{L}^\sim(U; V)$. Show that the following are equiveridical: (i) T is an abstract integral operator; (ii) $T[C]$ is norm-compact in V whenever C is weakly compact in U. (*Hint*: start with the case in which C is order-bounded, and remember that it is weakly sequentially compact.)

(k) Let (X, Σ, μ) be a semi-finite measure space. Show that 376S is valid for (X, Σ, μ) iff (X, Σ, μ) has locally determined negligible sets (213I).

(l) Let (X, Σ, μ) be a complete locally determined measure space and (Y, T, ν), (Z, Λ, λ) two σ-finite measure spaces. Suppose that τ, θ are extended Fatou norms on $L^0(\nu)$, $L^0(\lambda)$ respectively, and that $T : L^1(\mu) \to L^\tau$ is an abstract integral operator, with corresponding kernel $k \in \mathcal{L}^0(\mu\times\nu)$, while $S \in \mathrm{L}^\times(L^\tau; L^\theta)$, so that $ST : L^1(\mu) \to L^\theta$ is an abstract integral operator (376Xc); let $\tilde k \in \mathcal{L}^0(\mu \times \lambda)$ be the corresponding kernel. For $x \in X$ set $v_x = k^\bullet_x$ when this is defined in L^τ, as in 376S, and similarly take $w_x = \tilde k^\bullet_x \in L^\theta$. Show that $Sv_x = w_x$ for almost every $x \in X$.

(m) Set $k(m,n)=1/\pi(n-m+\frac{1}{2})$ for m, $n\in\mathbb{Z}$. (i) Show that $\sum_{n=-\infty}^{\infty}k(m,n)^2=1$ for every $m\in\mathbb{Z}$. (*Hint*: 282Xo.) (ii) Show that $\sum_{n=-\infty}^{\infty}k(m,n)k(m',n)=0$ for all distinct m, $m'\in\mathbb{Z}$. (*Hint*: look at $k(m,n)-k(m',n)$.) (iii) Show that there is a norm-preserving linear operator T from $\ell^2=\ell^2(\mathbb{Z})$ to itself given by setting $(Tu)(n)=\sum_{m=-\infty}^{\infty}k(m,n)u(m)$ for every n. (iv) Show that T^2 is the identity operator on ℓ^2. (v) Show that $T\notin\mathsf{L}^{\sim}(\ell^2;\ell^2)$. (*Hint*: consider $\sum_{m,n=-\infty}^{\infty}|k(m,n)|x(m)x(n)$ where $x(n)=1/\sqrt{|n|}\ln|n|$ for $|n|\ge 2$.) (T is a form of the **Hilbert transform.**)

(n) (i) Show that there is a compact linear operator from ℓ^2 to itself which is not in $\mathsf{L}^{\sim}(\ell^2;\ell^2)$. (*Hint*: start from the operator S of 371Ye.) (ii) Show that the identity operator on ℓ^2 is an abstract integral operator.

376 Notes and comments I leave 376Yb to the exercises because I do not rely on it for any of the work here, but of course it is an essential aspect of the map $\otimes:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ I discuss in this section. The conditions in 376Ya are characterizations of the 'tensor product' of two linear spaces, a construction of great importance in abstract linear algebra (and, indeed, in modern applied linear algebra; it is by no means trivial even in the finite-dimensional case). In particular, note that conditions (ii), (iii) of 376Ya apply to arbitrary subspaces of U and V if they apply to U and V themselves.

The principal ideas used in 376B-376C have already been set out in §§253 and 325. Here I do little more than list the references. I remark however that it is quite striking that $L^1(\mathfrak{C},\bar\lambda)$ should have no fewer than three universal mapping theorems attached to it (376Ca, 376C(b-i) and 376C(b-ii)).

The real work of this section begins in 376E. As usual, much of the proof is taken up with relatively straightforward verifications, as in parts (a) and (b), while part (i) is just a manoeuvre to show that it doesn't matter if $\mathfrak{A}$ and $\mathfrak{B}$ aren't Dedekind complete, because $\mathfrak{C}$ is. But I think that parts (d), (f) and (j) have ideas in them. In particular, part (f) is a kind of application of the Radon-Nikodým theorem (through the identification of $L^1(\mathfrak{C},\bar\lambda)^*$ with $L^\infty(\mathfrak{C})$).

I have split 376E from 376H because the former demands the language of measure algebras, while the latter can be put into the language of pure Riesz space theory. Asking for a weakly (σ,∞)-distributive space V in 376H is a way of applying the ideas to $V=L^0$ as well as to Banach function spaces. (When $V=L^0$, indeed, variations on the hypotheses are possible, using 376Yf.) But it is a reminder of one of the directions in which it is often possible to find generalizations of ideas beginning in measure theory.

The condition '$\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^\times$' (376H(ii)) seems natural in this context, and gives marginally greater generality than some alternatives (because it does the right thing when $U^\times$ does not separate the points of U), but it is not the only way of expressing the idea; see 376Xf and 376Ye. Note that the conditions (ii) and (iii) of 376H are significantly different. In 376H(iii) we could easily have $|u_n|=u^*$ for every n; for instance, if $u_n=2\chi a_n-\chi 1$ for some stochastically independent sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of elements of measure $\frac{1}{2}$ in a probability algebra (272Yd).

If you have studied compact linear operators between Banach spaces (definition: 3A5Ka), you will have encountered the condition '$Tu_n\to 0$ strongly whenever $u_n\to 0$ weakly'. The conditions in 376H and 376J are of this type. If a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in a Riesz space U is order-bounded and order*-convergent to 0, then $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^\times$ (367Xf). Visibly this latter condition is associated with weak convergence, and order*-convergence is (in Banach lattices) closely related to norm convergence (367D-367E). In the context of 376H, an abstract integral operator is one which transforms convergent sequences of a weak type into convergent sequences of a stronger type. The relationship between the classes of (weakly) compact operators and abstract integral operators is interesting, but outside the scope of this book; I leave you with 376P-376Q and 376Y, and a pair of elementary examples to guard against extravagant conjecture (376Yn).

376O belongs to an extensive general theory of weak compactness in perfect Riesz spaces, based on adaptations of the concept of 'uniform integrability'. I give the next step in 376Xl. For more information see Fremlin 74a, chap. 8.

Note that 376Mb and 376P overlap when $V^\times$ in 376Mb is reflexive – for instance, when V is an L^p space for some $p\in\,]1,\infty[$ – since then every bounded linear operator from L^1 to $V^\times$ must be weakly compact. I give 376Yi as a hint that there may be more to be said about the case in which the codomain is not perfect, indeed not weakly (σ,∞)-distributive. For more information on the representation of operators see Dunford & Schwartz 57, particularly Table VI in the notes to Chapter VI.

As soon as we leave formulations in terms of the spaces $L^0(\mathfrak{A})$ and their subspaces, and return to the original conception of a kernel operator in terms of integrating functions against sections of a kernel, we are necessarily involved in the pathology of Fubini's theorem for general measure spaces. In general, the repeated integrals $\int\int k(x,y)dxdy$, $\int\int k(x,y)dydx$ need not be equal, and something has to give (376Xm). Of course this particular worry disappears if the spaces are σ-finite, as in 376J. In 376S I take the trouble to offer a more general condition, fairly near to the best possible result (376Yk), mostly as a reminder that the techniques developed in Volume 2 do enable us sometimes to go beyond the σ-finite case. Note that this is one of the many contexts in which anything we can prove about probability spaces will be true of all σ-finite spaces; but that we cannot make the next step, to all strictly localizable spaces.

376S verges on the theory of integration of vector-valued functions, which I don't wish to enter here; but it also seems to have a natural place in the context of this chapter. It is of course a special property of L^1 spaces. The formula $\|T_k\| = \operatorname{ess\,sup}_x \tau(k_x^\bullet)$ shows that $\|T_{|k|}\| = \|T_k\|$; now we know fron 376E that $T_{|k|} = |T_k|$, so we get a special case of the Chacon-Krengel theorem (371D). Reversing the roles of X and Y, we find ourselves with an operator from L^τ to L^∞ (376Xn), which is the other standard context in which $\|T\| = \||T|\|$ (371Xd). I include two exercises on L^2 spaces (376Xi, 376Ym) designed to emphasize the fact that $\mathsf{B}(U;V)$ is included in $\mathsf{L}^\sim(U;V)$ only in very special cases.

The history of the theory here is even more confusing than that of mathematics in general, because so many of the ideas were developed in national schools in very imperfect contact with each other. My own account gives no hint of how this material arose; I ought in particular to note that 376N is one of the oldest results, coming (essentially) from Dunford 36. For further references, see Zaanen 83, chap. 13.

Chapter 38

Automorphism groups

As with any mathematical structure, every measure algebra has an associated symmetry group, the group of all measure-preserving automorphisms. In this chapter I set out to describe some of the remarkable features of these groups. I begin with a section on automorphism groups of general Dedekind complete Boolean algebras (§381), before continuing with applications of the ideas developed there to measure algebras (§382); the principal results of these sections concern the expression of a general automorphism as a product of involutions (381N, 382D) and a description of the normal subgroups of automorphism groups of homogeneous algebras (381S, 382H). I continue with a discussion of circumstances under which these automorphism groups determine the underlying algebras and/or have few outer automorphisms (§383).

One of the outstanding open problems of the subject is the 'isomorphism problem', the classification of automorphisms of measure algebras up to conjugacy in the automorphism group. I offer a section on 'entropy', the most important numerical invariant enabling us to distinguish some non-conjugate automorphisms (§384). For Bernoulli shifts on the Lebesgue measure algebra (384Q-384S), the isomorphism problem is solved by Ornstein's theorem; I present a complete proof of this theorem in §§385-386. Finally, in §387, I give Dye's theorem, describing the full subgroups generated by single automorphisms of measure algebras of countable Maharam type.

381 Automorphism groups of Boolean algebras

My aim in this chapter is to describe the automorphism groups of measure algebras, but as usual I prefer to begin with results which can be expressed in the language of general Boolean algebras, even though on this occasion it is necessary to introduce some special terminology. I will however restrict myself to the ideas which seem necessary for the theorems in the next two sections. I begin with some elementary general results on the construction of automorphisms by piecing together fragments of others (381B-381C), a discussion of the concept of an element supporting an automorphism (381D-381F, 381L), an introduction to a 'cycle notation' for certain automorphisms (381G-381I), and a version of Frolík's theorem on the structure of automorphisms (381J).

The principal theorems are 381N, giving a sufficient condition for every automorphism to be a product of involutions, and 381S, describing the normal subgroups of certain groups of automorphisms. Both depend on Dedekind completeness of the Boolean algebra and on the concept of 'full' subgroup of $\operatorname{Aut}\mathfrak{A}$ (381M); the latter also requires the group to have 'many involutions' (381P-381R). Both concepts are chosen with a view to the next section, where the results will be applied to groups of measure-preserving automorphisms.

381A The group $\operatorname{Aut}\mathfrak{A}$ For any Boolean algebra $\mathfrak{A}$, I write $\operatorname{Aut}\mathfrak{A}$ for the set of automorphisms of $\mathfrak{A}$, that is, the set of bijective Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{A}$. This is a group, being a subgroup of the group of all bijections from $\mathfrak{A}$ to itself (use 312G). Note that every member of $\operatorname{Aut}\mathfrak{A}$ is order-continuous; this is because it must be an isomorphism of the order structure of $\mathfrak{A}$, and is also a consequence of 313P(a-ii).

381B Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is finite
or that I is countable and $\mathfrak{A}$ is Dedekind σ-complete
or that $\mathfrak{A}$ is Dedekind complete.
Suppose that for each $i\in I$ we have an isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ between the corresponding principal ideals. The there is a unique $\pi\in\operatorname{Aut}\mathfrak{A}$ such that $\pi c=\pi_i c$ whenever $i\in I$ and $c\subseteq a_i$.

proof By 315F, we may identify $\mathfrak{A}$ with each of the products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

381C Corollary Let $\mathfrak{A}$ be a homogeneous Boolean algebra, and A, B two partitions of unity in $\mathfrak{A}$, neither containing 0. Let $\theta : A \to B$ be a bijection. Suppose

either that A, B are finite

or that A, B are countable and $\mathfrak{A}$ is Dedekind σ-complete

or that $\mathfrak{A}$ is Dedekind complete.

Then there is an automorphism of $\mathfrak{A}$ extending θ.

proof For every $a \in A$, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{\theta a}$ are isomorphic to the whole algebra $\mathfrak{A}$, and therefore to each other; let $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_{\theta a}$ be an isomorphism. Now apply 381B.

381D Elements supporting an automorphism The following concept will be extremely useful. If $\mathfrak{A}$ is a Boolean algebra, $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$ I will say that π is **supported** by a, or a **supports** π, if $\pi d = d$ for every $d \subseteq 1 \setminus a$. We have the following elementary facts.

381E Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\pi \in \operatorname{Aut}\mathfrak{A}$ is supported by $a \in \mathfrak{A}$ then $\pi a = a$ and $\pi d \subseteq a$ for every $d \subseteq a$.

(b) If π, $\phi \in \operatorname{Aut}\mathfrak{A}$ are both supported by $a \in \mathfrak{A}$ so are π^{-1} and $\pi\phi$.

(c) If $\pi \in \operatorname{Aut}\mathfrak{A}$ is supported by $a \in \mathfrak{A}$ and $\phi \in \operatorname{Aut}\mathfrak{A}$ then $\phi\pi\phi^{-1}$ is supported by ϕa.

(d) If π, $\phi \in \operatorname{Aut}\mathfrak{A}$ are supported by a, b respectively and $a \cap b = 0$, then $\pi\phi = \phi\pi$.

(e) Let $\pi \in \operatorname{Aut}\mathfrak{A}$ and let A be the set of $a \in \mathfrak{A}$ supporting π. Then A is non-empty and closed under arbitrary infima; also $b \in A$ whenever $b \supseteq a \in A$. In particular, if π is supported by both a and b then it is supported by $a \cap b$.

(f) For any $\pi \in \operatorname{Aut}\mathfrak{A}$, $a \in \mathfrak{A}$, a supports π iff $d \subseteq a$ whenever $d \cap \pi d = 0$.

(g) If $\pi \in \operatorname{Aut}\mathfrak{A}$ is supported by $a \in \mathfrak{A}$, and ϕ, $\psi \in \operatorname{Aut}\mathfrak{A}$ are such that $\phi d = \psi d$ for every $d \subseteq a$, then $\phi\pi\phi^{-1} = \psi\pi\psi^{-1}$.

proof (a) $\pi(1 \setminus a) = 1 \setminus a$, so $\pi a = a$, and if $d \subseteq a$ then $\pi d \subseteq \pi a = a$.

(b) If $d \cap a = 0$ then $\pi d = d = \phi d$ so $\pi^{-1}d = d = \pi\phi d$.

(c) If $d \cap \phi a = 0$ then $\phi^{-1}d \cap a = 0$ so $\pi\phi^{-1}d = \phi^{-1}d$ and $\phi\pi\phi^{-1}d = d$.

(d) If $d \subseteq a$, then $\phi d = d$ so $\pi\phi d = \pi d$. Also $\pi d \subseteq \pi a = a$, so $\phi\pi d = \pi d$; thus $\phi\pi d = \pi\phi d$. Similarly $\phi\pi d = \pi\phi d$ if $d \subseteq b$. Finally, if $d \subseteq 1 \setminus (a \cup b)$, then $\pi\phi d = \phi\pi d = d$. By the 'uniqueness' assertion of 381B, or otherwise, $\pi\phi = \phi\pi$.

(e) Of course $1 \in A$, because $\pi 0 = 0$; and it is also obvious that if $b \supseteq a \in A$ then $b \in A$. If $B \subseteq A$ is non-empty and $c = \inf B$ is defined in $\mathfrak{A}$, then for any $d \subseteq 1 \setminus c$ we have

$$d = d \setminus c = \sup_{b \in B} d \setminus b,$$

so (because π is order-continuous)

$$\pi d = \sup_{b \in B} \pi(d \setminus b) = \sup_{b \in B} d \setminus b = d.$$

So c supports π.

(f)(i) If a supports π and $d \cap \pi d = 0$, then

$$d \setminus a = \pi(d \setminus a) \cap d \subseteq \pi d \cap d = 0,$$

so $d \subseteq a$.

(ii) If a does not support π, there is a $c \subseteq 1 \setminus a$ such that $\pi c \neq c$. So one of $c \setminus \pi c$, $\pi c \setminus c$ is non-zero. If $c \setminus \pi c \neq 0$, take this for d; then $d \not\subseteq a$ and $\pi d \cap d \subseteq \pi c \setminus \pi c = 0$. Otherwise, take $d = \pi^{-1}(\pi c \setminus c)$; then $0 \neq d \subseteq c$, so $d \not\subseteq a$, while

$$d \cap \pi d = (c \setminus \pi^{-1}c) \cap (\pi c \setminus c) = 0.$$

(g) For $d \subseteq a$, $\psi^{-1}\phi d = \psi^{-1}\psi d = d$, so $\psi^{-1}\phi$ is supported by $1 \setminus a$. By (d), $\pi\psi^{-1}\phi = \psi^{-1}\phi\pi$, so

$$\phi\pi\phi^{-1} = \psi\psi^{-1}\phi\pi\phi^{-1} = \psi\pi\psi^{-1}\phi\phi^{-1} = \psi\pi\psi^{-1}.$$

381F Corollary Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If $e \in \mathfrak{A}$ is such that there is any $b \subseteq e$ for which $\pi b \neq b$, then there is a non-zero $a \subseteq e$ such that $a \cap \pi a = 0$.

proof Because $1 \setminus e$ does not support π, 381Ef tells us that there is a $d \not\subseteq 1 \setminus e$ such that $d \cap \pi d = 0$; now set $a = d \cap e$.

381G Cyclic automorphisms: Definition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that a, b are disjoint members of $\mathfrak{A}$ and that $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = b$. I will write $(\overleftarrow{a \,{}_{\pi}\, b})$ for the member ψ of $\operatorname{Aut}\mathfrak{A}$ defined by setting

$$\begin{aligned}\psi d &= \pi d \text{ if } d \subseteq a,\\ &= \pi^{-1} d \text{ if } d \subseteq b,\\ &= d \text{ if } d \subseteq 1 \setminus (a \cup b).\end{aligned}$$

Observe that in this case (if $a \neq 0$) ψ is an involution, that is, has order 2 in the group $\operatorname{Aut}\mathfrak{A}$; involutions of this type I will call **exchanging involutions**.

(b) More generally, if $a_1, \ldots, a_n$ are disjoint elements of $\mathfrak{A}$ and $\pi_i \in \operatorname{Aut}\mathfrak{A}$ are such that $\pi_i a_i = a_{i+1}$ for each $i < n$, then I will write

$$(\overleftarrow{a_1 \,{}_{\pi_1}\, a_2 \,{}_{\pi_2}\, \ldots \,{}_{\pi_{n-1}}\, a_n})$$

for that $\psi \in \operatorname{Aut}\mathfrak{A}$ such that

$$\begin{aligned}\psi d &= \pi_i d \text{ if } 1 \leq i < n,\ d \subseteq a_i,\\ &= \pi_1^{-1}\pi_2^{-1} \ldots \pi_{n-1}^{-1} d \text{ if } d \subseteq a_n,\\ &= d \text{ if } d \subseteq 1 \setminus \sup_{i \leq n} a_i.\end{aligned}$$

(c) It will occasionally be convenient to use the same notation when each π_i is a Boolean isomorphism between the principal ideals $\mathfrak{A}_{a_i}$ and $\mathfrak{A}_{a_{i+1}}$, rather than an automorphism of the whole algebra $\mathfrak{A}$.

Remark The point of this notation is that we can expect to use the standard techniques for manipulating cycles that are (I suppose) familiar to you from elementary group theory; the principal change is that we have to keep track of the subscripted automorphisms ${}_{\pi}$. The following results are typical.

381H Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\psi = (\overleftarrow{a \,{}_{\pi}\, b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then

$$\psi = (\overleftarrow{a \,{}_{\psi}\, b}) = (\overleftarrow{b \,{}_{\psi}\, a}) = (\overleftarrow{b \,{}_{\pi^{-1}}\, a})$$

is supported by $a \cup b$.

(b) If $\pi = (\overleftarrow{a \,{}_{\pi}\, b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then for any $\phi \in \operatorname{Aut}\mathfrak{A}$,

$$\phi\pi\phi^{-1} = (\overleftarrow{\phi a \,{}_{\phi\pi\phi^{-1}}\, \phi b})$$

is another exchanging involution.

(c) If $\pi = (\overleftarrow{a \,{}_{\pi}\, b})$ and $\phi = (\overleftarrow{c \,{}_{\phi}\, d})$ are exchanging involutions, and a, b, c, d are all disjoint, then π and ϕ commute, and $\psi = \pi\phi = \phi\pi$ is also an exchanging involution, being $(\overleftarrow{a \cup c \,{}_{\psi}\, b \cup d})$.

proof (a) Check the action of ψ on the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$, $\mathfrak{A}_{1 \setminus (a \cup b)}$.

(b) $\phi a \cap \phi b = \phi(a \cap b) = 0$ and

$$\phi\pi\phi^{-1}\phi a = \phi\pi a = \phi b,$$

so $\psi = (\overleftarrow{\phi a \,{}_{\phi\pi\phi^{-1}}\, \phi b})$ is well-defined. Now check the action of ψ on the principal ideals $\mathfrak{A}_{\phi a}$, $\mathfrak{A}_{\phi b}$, $\mathfrak{A}_{1 \setminus \phi(a \cup b)}$.

(c) Check the action of ψ on each of the principal ideals $\mathfrak{A}_a, \ldots, \mathfrak{A}_e$, where $e = 1 \setminus (a \cup b \cup c \cup d)$.

381I Remark I must emphasize that while, after a little practice, calculations of this kind become easy and safe, they are absolutely dependent on all the cycles present involving only members of one list of disjoint elements of $\mathfrak{A}$. If, for instance, a, b, c are disjoint, then

$$(\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{b\,{}_{\phi}\,c})=(\overleftarrow{a\,{}_{\pi}\,b\,{}_{\phi}\,c}).$$

But if $a\cap c\ne 0$ then there is no expression for the product in this language. Secondly, of course, we must be scrupulous in checking, at every use of the notation $(\overleftarrow{a_1\,{}_{\pi_1}\,\ldots\,a_n})$, that $a_1,\ldots,a_n$ are disjoint and that $\pi_i a_i=a_{i+1}$ for $i<n$. Thirdly, a significant problem can arise if the automorphisms involved don't match. Consider for instance the product

$$\psi=(\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{a\,{}_{\phi}\,b}).$$

Then we have $\psi d=\pi^{-1}\phi d$ if $d\subseteq a$, $\pi\phi^{-1}d$ if $d\subseteq b$; ψ is not necessarily expressible as a product of 'disjoint' cycles. Clearly there are indefinitely complex variations possible on this theme. A possible formal expression of a sufficient condition to avoid these difficulties is the following. Restrict yourself to calculations involving a fixed list $a_1,\ldots,a_n$ of disjoint elements of $\mathfrak{A}$ for which you can describe a family of isomorphisms $\phi_{ij}:\mathfrak{A}_{a_i}\to\mathfrak{A}_{a_j}$ such that ϕ_{ii} is always the identity on $\mathfrak{A}_{a_i}$, $\phi_{jk}\phi_{ij}=\phi_{ik}$ for all i, j, k, and whenever $a_i\,{}_{\pi}\,a_j$ appears in a cycle of the calculation, then π agrees with ϕ_{ij} on $\mathfrak{A}_{a_i}$. Of course this would be intolerably unwieldy if it were really necessary to exhibit all the ϕ_{ij} every time. I believe however that it is usually easy enough to form a mental picture of the actions of the isomorphisms involved sufficiently clear to offer confidence that such ϕ_{ij} are indeed present; and in cases of doubt, then *after* performing the formal operations it is always straightforward to check that the calculations are valid, by looking at the actions of the automorphisms on each relevant principal ideal.

381J The following facts may help to put the concept of 'exchanging involution' in its proper place.

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and π any automorphism of $\mathfrak{A}$. Then there is a partition of unity (a',a'',b',b'',c,e) of $\mathfrak{A}$ such that

$$\pi a'=b',\ \pi a''=b'',\ \pi b''=c,\ \pi(b'\cup c)=a'\cup a'',\ \pi d=d\text{ for every }d\subseteq e.$$

proof (a) Let P be the set $\{p:p\in\mathfrak{A},\,p\cap\pi p=0\}$. Then P has a maximal element. **P** Of course $P\ne\emptyset$, as $0\in P$. If $Q\subseteq P$ is non-empty and upwards-directed, set $p=\sup Q$, which is defined because $\mathfrak{A}$ is Dedekind complete; then $\pi p=\sup\pi[Q]$ (since π, being an automorphism, is surely order-continuous). If q_1, $q_2\in Q$, there is a $q\in Q$ such that $q_1\cup q_2\subseteq q$, so $q_1\cap\pi q_2\subseteq q\cap\pi q=0$. By 313Bc, $\sup Q\cap\sup\pi[Q]=0$, that is, $p\cap\pi p=0$. This means that $p\in P$ and is an upper bound for Q in P. As Q is arbitrary, Zorn's Lemma tells us that P has a maximal element. **Q**

(b) Let a be a maximal element of P and set $b=\pi a$, $c=\pi b\setminus a$, $e=1\setminus(a\cup b\cup c)$.

(i) ? Suppose, if possible, that there is some $d\subseteq e$ such that $\pi d\ne d$. By 381F, there is a non-zero $d\subseteq e$ such that $d\cap\pi d=0$. Consider $\tilde a=a\cup\pi^{-1}d$; then $a\subset\tilde a$. But

$$\tilde a\cap\pi\tilde a=(a\cap b)\cup(a\cap d)\cup((b\cup d)\cap\pi^{-1}d)=0$$

because

$$\pi((b\cup d)\cap\pi^{-1}d)\subseteq d\cap(a\cup c\cup\pi d)=0.$$

So $\tilde a$ is a member of P strictly greater than a, which is impossible. **X**

Thus $\pi d=d$ for every $d\subseteq e$; in particular, $\pi e=e$.

(ii) Because $a\cap b=0$, $b\cap c\subseteq\pi(a\cap b)=0$, and (a,b,c,e) is a partition of unity. Because $\pi(a\cup b\cup e)$ includes $b\cup c\cup e$, $\pi c\subseteq 1\setminus(b\cup c\cup e)=a$.

(c) Set $b''=\pi^{-1}c\subseteq b$, $b'=b\setminus b''$, $a'=\pi^{-1}b'$, $a''=\pi^{-1}b''$. Then everything required in the statement of the lemma has been covered, with the possible exception of '$\pi(b'\cup c)=a'\cup a''$'; but as $b'\cup c=1\setminus(a\cup b''\cup e)$,

$$\pi(b'\cup c)=1\setminus(\pi a\cup\pi b''\cup\pi e)=1\setminus(b\cup c\cup e)=a=a'\cup a''.$$

So the proof is complete.

381K Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Then every involution in $\operatorname{Aut}\mathfrak{A}$ is an exchanging involution in the sense of 381G.

proof Let π be an involution and let a', a'', b', b'', c, e be a partition of unity as in 381J. Then

$$c = \pi^2 a'' = a'', \quad c \cap a'' = 0,$$

so $a'' = b'' = c = 0$ and $\pi d = d$ whenever $d \subseteq 1 \setminus (a' \cup \pi a') = e$, while $a' \cap \pi a' = 0$.

381L The support of an automorphism For Dedekind complete Boolean algebras $\mathfrak{A}$, the following concept is very useful. For any $\pi \in \operatorname{Aut}\mathfrak{A}$, set $\operatorname{supp}\pi = \inf\{a : \pi \text{ is supported by } a\}$, the **support** of π. Note that $\operatorname{supp}\pi$ supports π (381Ee); it is the smallest element of $\mathfrak{A}$ supporting π. By 381Ef, $\operatorname{supp}\pi$ is also $\sup\{d : d \cap \pi d = 0\}$. Observe that in the language of 381J $\operatorname{supp}\pi$ is just $a' \cup a'' \cup b' \cup b'' \cup c = 1 \setminus e$.

Note that $\operatorname{supp}\pi^{-1} = \operatorname{supp}\pi$, and that $\operatorname{supp}(\pi\pi') \subseteq \operatorname{supp}\pi \cup \operatorname{supp}\pi'$ for all π, $\pi' \in \operatorname{Aut}\mathfrak{A}$, by 381Ed. Moreover, if π, $\phi \in \operatorname{Aut}\mathfrak{A}$, then

$$\operatorname{supp}(\phi\pi\phi^{-1}) = \phi(\operatorname{supp}\pi).$$

P By 381Ec, $\phi(\operatorname{supp}\pi)$ supports $\phi\pi\phi^{-1}$, so $\operatorname{supp}(\phi\pi\phi^{-1}) \subseteq \phi(\operatorname{supp}\pi)$; but also

$$\operatorname{supp}\pi \subseteq \phi^{-1}(\operatorname{supp}(\phi\pi\phi^{-1})),$$

so $\phi(\operatorname{supp}\pi) \subseteq \operatorname{supp}(\phi\pi\phi^{-1})$. **Q**

381M In order to apply the argument of the next theorem simultaneously to the groups $\operatorname{Aut}\mathfrak{A}$ here and to the groups $\operatorname{Aut}_\mu\mathfrak{A}$ of the next section, I introduce the following terminology.

Definition If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **full** if whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = \pi_i a_i$ whenever $i \in I$ and $a \subseteq a_i$, then $\pi \in G$.

381N Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$. Then every member of G is expressible as the product of at most eight involutions belonging to G.

proof (a) For each $n \in \mathbb{N}$, let $\mathcal{I}_n$ be the set of elements of G expressible as the product of n or fewer involutions belonging to G (so that $\mathcal{I}_0$ consists of the identity ι alone). It will be helpful to note that $\pi^{-1} \in \mathcal{I}_n$, $\phi\pi\phi^{-1} \in \mathcal{I}_n$ whenever $\pi \in \mathcal{I}_n$ and $\phi \in G$. In particular, for instance, $\pi\psi\pi^{-1}\psi^{-1} = (\pi\psi\pi^{-1})\psi^{-1} \in \mathcal{I}_2$ whenever $\psi \in \mathcal{I}_1$ and $\pi \in G$.

Note that if $a \in \mathfrak{A}$, $\pi \in G$ and $\pi a \cap a = 0$, then the involution $(\overleftarrow{a\,{}_\pi\,\pi a})$ belongs to G, because it is made up of the actions of π, π^{-1} and ι on $\mathfrak{A}_a$, $\mathfrak{A}_{\pi a}$ and $\mathfrak{A}_{1\setminus(a\cup\pi a)}$ respectively, and G is full.

(b) If $\pi \in G$ there are $a \in \mathfrak{A}$ and involutions ϕ_1, $\phi_2 \in G$ such that $\pi a \cap a = 0$ and $\phi_1\phi_2\pi$ is supported by a. **P** Take a', a'', b', b'', c, e from 381J, and set $a = b' \cup c$. Then certainly $\pi a = a' \cup a''$ is disjoint from a. Set

$$\phi_1 = (\overleftarrow{b''\,{}_\pi\,c}), \quad \phi_2 = (\overleftarrow{a' \cup a''\,{}_\pi\,b' \cup b''}).$$

Then both ϕ_1, ϕ_2 are involutions in G. Now trace the action of $\phi_1\phi_2\pi$ on each of the principal ideals $\mathfrak{A}_{a'\cup a''}$, $\mathfrak{A}_{b''}$, $\mathfrak{A}_e$; we get

$$\phi_1\phi_2\pi d = \iota\pi^{-1}\pi d = d \text{ if } d \subseteq a' \cup a'',$$

$$\phi_1\phi_2\pi d = \pi^{-1}\iota\pi d = d \text{ if } d \subseteq b'',$$

$$\phi_1\phi_2\pi d = \iota\iota d = d \text{ if } d \subseteq e.$$

(I write the identity operator ι into these formulae in the hope of suggesting how the three functions act in each case.) So $\operatorname{supp}(\phi_1\phi_2\pi) \subseteq b' \cup c = a$. **Q**

(c) Suppose that π, $\psi \in G$, $a \in \mathfrak{A}$, π is supported by a and $\psi a \cap a = 0$. Then there are ϕ_1, ϕ_2, ϕ_3, $\psi' \in G$ and $\tilde{b} \subseteq \psi a$ such that (i) ϕ_1, ϕ_2, ϕ_3 are all involutions supported by $a \cup \tilde{b}$ (ii) $\psi'\tilde{b} \subseteq \psi a \setminus \tilde{b}$ (iii) $\phi_1\phi_2\phi_3\pi$ is supported by $\tilde{b}$. **P** Take a', a'', b', b'', c, e from 381J; then

$$a' \cup a'' \cup b' \cup b'' \cup c = \operatorname{supp} \pi \subseteq a.$$

Set

$$b = b' \cup c, \quad \tilde{b}' = \psi b', \quad \tilde{c} = \psi c, \quad \tilde{b} = \psi b = \tilde{b}' \cup \tilde{c} \subseteq \psi a.$$

Note that $b \cap \tilde{b} \subseteq a \cap \psi a = 0$. Set $\psi' = \psi\pi\psi^{-1}$, so that

$$\psi'\tilde{b} = \psi\pi b = \psi(a' \cup a'')$$

is disjoint from $\psi b = \tilde{b}$.

Set

$$\pi_1 = (\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{a''\,{}_{\pi}\,b''})\pi,$$

so that (just as in (b) above) π_1 is supported by $b' \cup c = b$. Set

$$\phi_1 = (\overleftarrow{b\,{}_{\psi\pi_1}\,\tilde{b}}) \in \mathcal{I}_1, \quad \pi_2 = \phi_1(\overleftarrow{b\,{}_{\psi}\,\tilde{b}})\pi_1.$$

Examining the action of π_2 on $\mathfrak{A}_b$, we see that

$$\pi_2 d = \phi_1\psi\pi_1 d = \pi_1^{-1}\psi^{-1}\psi\pi_1 d = d \text{ if } d \subseteq b,$$

so that $\operatorname{supp} \pi_2 \cap b = 0$; but as

$$\operatorname{supp} \pi_2 \subseteq \operatorname{supp} \phi_1 \cup \operatorname{supp}(\overleftarrow{b\,{}_{\psi}\,\tilde{b}}) \cup \operatorname{supp} \pi_1 = b \cup \tilde{b},$$

$\operatorname{supp} \pi_2 \subseteq \tilde{b}$.

Now we have an alternative expression for π_2, as follows:

$$\begin{aligned}
\pi_2 &= \phi_1(\overleftarrow{b\,{}_{\psi}\,\tilde{b}})\pi_1 \\
&= \phi_1(\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{c\,{}_{\psi}\,\tilde{c}})(\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{a''\,{}_{\pi}\,b''})\pi \\
&= \phi_1(\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{c\,{}_{\psi}\,\tilde{c}})(\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a''\,{}_{\pi}\,b''})\pi
\end{aligned}$$

(because $(\overleftarrow{a'\,{}_{\pi}\,b'})$ commutes with $(\overleftarrow{b''\,{}_{\pi}\,c})$ and $(\overleftarrow{c\,{}_{\psi}\,\tilde{c}})$)

$$= \phi_1(\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{c\,{}_{\pi^{-1}}\,b''\,{}_{\pi^{-1}}\,a''\,{}_{\psi\pi^2}\,\tilde{c}})\pi$$

(of course this calculation depends on the fact that c, b'', a'', $\tilde{c}$ are all disjoint)

$$\begin{aligned}
&= \phi_1(\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a''\,{}_{\psi\pi^2}\,\tilde{c}})(\overleftarrow{b''\,{}_{\psi\pi}\,\tilde{c}})\pi \\
&= \phi_1(\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a''\,{}_{\psi\pi^2}\,\tilde{c}})(\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{b''\,{}_{\psi\pi}\,\tilde{c}})\pi
\end{aligned}$$

(because $(\overleftarrow{a'\,{}_{\pi}\,b'})$ commutes with $(\overleftarrow{b''\,{}_{\pi}\,c})$ and $(\overleftarrow{a''\,{}_{\psi\pi^2}\,\tilde{c}})$)

$$= \phi_1\phi_2\phi_3\pi$$

where

$$\phi_2 = (\overleftarrow{b'\,{}_{\psi}\,\tilde{b}'})(\overleftarrow{b''\,{}_{\pi}\,c})(\overleftarrow{a''\,{}_{\psi\pi^2}\,\tilde{c}}),$$

$$\phi_3 = (\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{b''\,{}_{\psi\pi}\,\tilde{c}})$$

are involutions in G, each being a product of disjoint involutions. Now

$$\operatorname{supp} \phi_1 \subseteq b \cup \tilde{b},$$

$$\operatorname{supp} \phi_2 \subseteq b \cup \tilde{b}' \cup b'' \cup c \cup a'' \cup \tilde{c},$$

$$\operatorname{supp} \phi_3 \subseteq a' \cup b' \cup b'' \cup \tilde{c}$$

are all included in $a \cup \tilde{b}$, and we already know that

$$\operatorname{supp}(\phi_1\phi_2\phi_3\pi) = \operatorname{supp} \pi_2 \subseteq \tilde{b}. \quad \mathbf{Q}$$

(d) We are ready for the assault. Take any $\pi \in G$. Choose $\langle\pi_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$, $\langle\phi_{1n}\rangle_{n\in\mathbb{N}}$, $\langle\phi_{2n}\rangle_{n\in\mathbb{N}}$, $\langle\phi_{3n}\rangle_{n\in\mathbb{N}}$, $\langle\psi_n\rangle_{n\in\mathbb{N}}$ as follows. Start by defining a_0, ϕ_1, ϕ_2 from π as in (b), so that ϕ_1, $\phi_2 \in \mathcal{I}_1$, $\pi a_0 \cap a_0 = 0$ and $\phi_1\phi_2\pi$ is supported by a_0. Set $\pi_0 = \phi_1\phi_2\pi$, $\psi_0 = \pi$ and $b_0 = \pi a_0$. For the inductive step, given ψ_n and π_n in G, $a_n \supseteq \operatorname{supp}\pi_n$ and $b_n = \psi_n a_n$ disjoint from a_n, use (c) to find ϕ_{1n}, ϕ_{2n}, $\phi_{3n} \in \mathcal{I}_1$, $a_{n+1} \subseteq b_n$ and $\psi_{n+1} \in G$ such that $\psi_{n+1}a_{n+1} \subseteq b_n \setminus a_{n+1}$, $\pi_{n+1} = \phi_{1n}\phi_{2n}\phi_{3n}\pi_n$ is supported by a_{n+1}, and all the ϕ_{in} are supported by $a_n \cup a_{n+1}$; set $b_{n+1} = \psi_{n+1}a_{n+1}$, and continue.

Evidently $\langle b_n\rangle_{n\in\mathbb{N}}$ is now non-increasing, while $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint; consequently each of the sequences

$$\langle a_{2k} \cup a_{2k+1}\rangle_{k\in\mathbb{N}}, \quad \langle a_{2k+1} \cup a_{2k+2}\rangle_{k\in\mathbb{N}}$$

is disjoint.

(e) For $r = 0$, $r = 1$ define $\theta_r \in G$ by setting

$$\begin{aligned}\theta_r d &= \pi_{2k+r}\pi_{2k+r+1}^{-1}d \text{ if } k \in \mathbb{N},\ d \subseteq a_{2k+r} \cup a_{2k+r+1},\\ &= d \text{ if } d \cap \sup_{n\ge r} a_n = 0.\end{aligned}$$

(The definition is valid because $\operatorname{supp}\pi_n \subseteq a_n$ for every n, and produces a member of G because G is full.) Now $\theta_r \in \mathcal{I}_3$.

P For each $n \in \mathbb{N}$,

$$\pi_n\pi_{n+1}^{-1} = \phi_{3n}^{-1}\phi_{2n}^{-1}\phi_{1n}^{-1},$$

and all these automorphisms are supported by $a_n \cup a_{n+1}$. So if we set

$$\begin{aligned}\tilde\phi_{ir} d &= \phi_{i,2k+r}d \text{ if } d \subseteq a_{2k+r} \cup a_{2k+r+1},\\ &= d \text{ if } d \cap \sup_{n\ge r} a_n = 0,\end{aligned}$$

each $\tilde\phi_{ir}$ will be in $\mathcal{I}_1$, and

$$\theta_r = \tilde\phi_{3r}^{-1}\tilde\phi_{2r}^{-1}\tilde\phi_{1r}^{-1}$$

belongs to $\mathcal{I}_3$. **Q**

(f) Consequently $\theta_1\theta_0 \in \mathcal{I}_6$. But $\theta_1\theta_0 = \pi_0$. **P** If $d \subseteq a_0$, then

$$\theta_0 d = \pi_0 d \subseteq a_0, \quad \theta_1\theta_0 d = \pi_0 d.$$

If $d \subseteq a_{2k+1}$, where $k \in \mathbb{N}$, then

$$\theta_0 d = \pi_{2k+1}^{-1}d \subseteq a_{2k+1}, \quad \theta_1\theta_0 d = \pi_{2k+1}\pi_{2k+1}^{-1}d = d = \pi_0 d.$$

If $d \subseteq a_{2k+2}$, then

$$\theta_0 d = \pi_{2k+2}d \subseteq a_{2k+2}, \quad \theta_1\theta_0 d = \pi_{2k+2}^{-1}\pi_{2k+2}d = d = \pi_0 d;$$

and finally, if $d \cap \sup_{n\in\mathbb{N}} a_n = 0$,

$$\theta_0 d = \theta_1 d = \theta_1\theta_0 d = \pi_0 d = d.$$

Putting these together, $\pi_0 = \theta_1\theta_0$. **Q**

(g) Finally,

$$\pi = \phi_2^{-1}\phi_1^{-1}\pi_0 \in \mathcal{I}_8.$$

As π is arbitrary, the theorem is proved.

381O Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$. Then any $\pi \in G$ other than the identity is expressible as a product of finitely many involutions in G with supports included in $\operatorname{supp}\pi$.

proof In fact the construction in the proof of 381N achieves this almost automatically. Alternatively, setting $a = \operatorname{supp}\pi$, let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a, and set

$$G_a = \{\phi{\restriction}\mathfrak{A}_a : \phi \in G,\ \operatorname{supp}\phi \subseteq a\}.$$

It is easy to check that G_a is a full subgroup of $\operatorname{Aut}\mathfrak{A}_a$, so that $\pi{\restriction}\mathfrak{A}_a$ is expressible as a product of involutions belonging to G_a. These are all of the form $\phi_i{\restriction}\mathfrak{A}_a$ where $\phi_i \in G$ and $\operatorname{supp}\phi_i \subseteq a$, so that every ϕ_i is also an involution and π is the product of the ϕ_i.

381P Definition Let $\mathfrak{A}$ be a Boolean algebra, and G a subgroup of the automorphism group $\operatorname{Aut}\mathfrak{A}$. I will say that G **has many involutions** if for every non-zero $a \in \mathfrak{A}$ there is an involution $\pi \in G$ which is supported by a.

381Q Lemma Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has many involutions, and in fact every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution.

proof If $a \in \mathfrak{A}\setminus\{0\}$, then there is a b such that $0 \ne b \subset a$. By 381C there is a $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi b = a \setminus b$; now $(\overleftarrow{b \,{}_{\pi}\, a\setminus b})$ is an involution with support a.

381R Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then every non-zero element of $\mathfrak{A}$ is the support of an involution belonging to G.

proof By the definition 381P,

$$C = \{\operatorname{supp}\pi : \pi \in G \text{ is an involution}\}$$

is order-dense in $\mathfrak{A}$. So if $a \in \mathfrak{A}\setminus\{0\}$ there is a disjoint $B \subseteq C$ such that $\sup B = a$ (313K). For each $b \in B$ let $\pi_b \in G$ be an involution with support B. Define $\pi \in G$ by setting $\pi d = \pi_b d$ for $d \subseteq b \in B$, $\pi d = d$ if $d \cap a = 0$; then $\pi \in G$ is an involution with support a.

381S Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then a subset H of G is a normal subgroup of G iff it is of the form

$$\{\pi : \pi \in G,\ \operatorname{supp}\pi \in I\}$$

for some ideal $I \lhd \mathfrak{A}$ which is G**-invariant**, that is, such that $\pi a \in I$ for every $a \in I$, $\pi \in G$.

proof (a) I deal with the easy implication first. Let $I \lhd \mathfrak{A}$ be a G-invariant ideal and set $H = \{\pi : \pi \in G,\ \operatorname{supp}\pi \in I\}$. Because the support of the identity automorphism ι is $0 \in I$, $\iota \in H$. If ϕ, $\psi \in H$ and $\pi \in G$, then

$$\operatorname{supp}(\phi\psi) \subseteq \operatorname{supp}\phi \cup \operatorname{supp}\psi \in I,$$

$$\operatorname{supp}(\psi^{-1}) = \operatorname{supp}\psi \in I,$$

$$\operatorname{supp}(\pi\psi\pi^{-1}) = \pi(\operatorname{supp}\psi) \in I$$

and $\phi\psi$, ψ^{-1}, $\pi\psi\pi^{-1}$ all belong to H; so $H \lhd G$.

(b) For the rest of the proof, therefore, I suppose that H is a normal subgroup of G and seek to express it in the given form. We can in fact describe the ideal I immediately, as follows. Set

$$J = \{a : a \in \mathfrak{A},\ \pi \in H \text{ whenever } \pi \in G \text{ is an involution, } \operatorname{supp}\pi \subseteq a\};$$

then $0 \in J$ and $a \in J$ whenever $a \subseteq b \in J$. Also $\pi a \in J$ whenever $a \in J$, $\pi \in G$. **P** If $\phi \in G$ is an involution, $\operatorname{supp}\phi \subseteq \pi a$ then $\phi_1 = \pi^{-1}\phi\pi$ is an involution in G and

$$\operatorname{supp}\phi_1 = \pi^{-1}(\operatorname{supp}\phi) \subseteq a,$$

so $\phi_1 \in H$ and $\phi = \pi\phi_1\pi^{-1} \in H$. As ϕ is arbitrary, $\pi a \in J$. **Q**

I do not know how to prove directly that J is an ideal, so let us set

$$I = \{a_1 \cup a_2 \cup \ldots \cup a_n : a_1, \ldots, a_n \in J\};$$

then $I \lhd \mathfrak{A}$, and $\pi a \in I$ for every $a \in I$, $\pi \in G$.

(c) If $a \in \mathfrak{A}$, $\psi \in H$ and $a \cap \psi a = 0$ then $a \in J$. **P** If $a = 0$, this is trivial. Otherwise, let $\pi \in G$ be an involution with $\operatorname{supp} \pi \subseteq a$; say $\pi = (\overleftarrow{b\,_{\pi}\,c})$ where $b \cup c \subseteq a$. By 381R there is an involution $\pi_1 \in G$ such that $\operatorname{supp} \pi_1 = b$; say $\pi_1 = (\overleftarrow{b'\,_{\pi_1}\,b''})$ where $b' \cup b'' = b$. Set

$$c' = \pi b', \quad c'' = \pi b'' = c \setminus c',$$

$$\pi_2 = \pi_1 \pi \pi_1 \pi^{-1} = (\overleftarrow{b'\,_{\pi_1}\,b''})(\overleftarrow{c'\,_{\pi\pi_1\pi^{-1}}\,c''}), \quad \pi_3 = (\overleftarrow{b'\,_{\pi}\,c'}),$$

$$\phi = \pi_2^{-1} \psi \pi_2 \psi^{-1} \in H,$$

$$\bar{\pi} = \pi_3^{-1} \phi \pi_3 \phi^{-1} = \pi_3^{-1} \pi_2^{-1} \psi \pi_2 \psi^{-1} \pi_3 \psi \pi_2^{-1} \psi^{-1} \pi_2 \in H.$$

Now

$$\operatorname{supp}(\psi \pi_2 \psi^{-1}) = \psi(\operatorname{supp} \pi_2) = \psi(b \cup c) \subseteq \psi a$$

is disjoint from

$$\operatorname{supp} \pi_3 = b' \cup c' \subseteq a,$$

so π_3 commutes with $\psi \pi_2 \psi^{-1}$, and

$$\begin{aligned} \bar{\pi} &= \pi_3^{-1} \pi_2^{-1} \pi_3 \psi \pi_2 \psi^{-1} \psi \pi_2^{-1} \psi^{-1} \pi_2 \\ &= \pi_3^{-1} \pi_2^{-1} \pi_3 \pi_2 \\ &= (\overleftarrow{b'\,_{\pi}\,c'})(\overleftarrow{b'\,_{\pi_1}\,b''})(\overleftarrow{c'\,_{\pi\pi_1\pi^{-1}}\,c''})(\overleftarrow{b'\,_{\pi}\,c'})(\overleftarrow{b'\,_{\pi_1}\,b''})(\overleftarrow{c'\,_{\pi\pi_1\pi^{-1}}\,c''}) \\ &= (\overleftarrow{b'\,_{\pi}\,c'})(\overleftarrow{b''\,_{\pi}\,c''}) \\ &= \pi. \end{aligned}$$

So $\pi \in H$. As π is arbitrary, $a \in J$. **Q**

(d) If $\pi = (\overleftarrow{a\,_{\pi}\,b})$ is an involution in G and $a \in J$, then $\pi \in H$. **P** By 381R again, there is an involution $\psi \in G$ such that $\operatorname{supp} \psi = a$; because $a \in J$, $\psi \in H$. Express ψ as $(\overleftarrow{a'\,_{\psi}\,a''})$ where $a' \cup a'' = a$. Set $b' = \pi a'$, $b'' = \pi a''$, so that $\pi = (\overleftarrow{a'\,_{\pi}\,b'})(\overleftarrow{a''\,_{\pi}\,b''})$, and

$$\psi_1 = \psi \pi \psi \pi^{-1} = (\overleftarrow{a'\,_{\psi}\,a''})(\overleftarrow{b'\,_{\pi\psi\pi^{-1}}\,b''}) \in H.$$

As $\psi_1(a' \cup b') = a'' \cup b''$ is disjoint from $a' \cup b'$, $a' \cup b' \in J$, by (c), and $\pi_1 = (\overleftarrow{a'\,_{\pi}\,b'}) \in H$; similarly, $a'' \cup b'' \in J$, so $\pi_2 = (\overleftarrow{a''\,_{\pi}\,b''}) \in H$ and $\pi = \pi_1 \pi_2$ belongs to H. **Q**

(e) If $\pi \in G$ is an involution and $\operatorname{supp} \pi \in I$, then $\pi \in H$. **P** Express π as $(\overleftarrow{a\,_{\pi}\,b})$. Let $a_1, \dots, a_n \in J$ be such that $a \cup b \subseteq a_1 \cup \dots \cup a_n$. Set

$$c_j = a \cap a_j \setminus \sup_{i<j} a_i, \quad b_j = \pi c_j, \quad \pi_j = (\overleftarrow{c_j\,_{\pi}\,b_j})$$

for $1 \le j \le n$; then every c_j belongs to J, so every π_j belongs to H (by (d)) and $\pi = \pi_1 \dots \pi_n \in H$. **Q**

(f) If $\pi \in G$ and $\operatorname{supp} \pi \in I$ then $\pi \in H$. **P** By 381O, π is a product of involutions in G all with supports included in $\operatorname{supp} \pi$; by (e), they all belong to H, so π also does. **Q**

(g) We are nearly home. So far we know that I is a G-invariant ideal and that $\pi \in H$ whenever $\pi \in G$, $\operatorname{supp} \pi \in I$. On the other hand, $\operatorname{supp} \pi \in I$ for every $\pi \in H$. **P** Take a', a'', b', b'', c from Frolík's theorem (381J). Then

$$a' \cap \pi a' = b' \cap \pi b' = \dots = c \cap \pi c = 0,$$

so $a', \dots, c$ all belong to J, by (c), and $\operatorname{supp} \pi = a' \cup \dots \cup c$ belongs to I. **Q**

So H is precisely the set of members of G with supports in I, as required.

381T Corollary Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra. Then $\operatorname{Aut} \mathfrak{A}$ is simple.

proof If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, let H be a normal subgroup of $\operatorname{Aut}\mathfrak{A}$. Then by 381S and 381Q there is an invariant ideal I of $\mathfrak{A}$ such that $H = \{\pi : \operatorname{supp}\pi \in I\}$. But if H is non-trivial so is I; say $a \in I \setminus \{0\}$. If $a = 1$ then certainly $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$. Otherwise, there is a $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi a = 1 \setminus a$ (as in 381C), so $1 \setminus a \in I$, and again $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$.

Remark I ought to remark that in fact $\operatorname{Aut}\mathfrak{A}$ is simple for any homogeneous Dedekind σ-complete Boolean algebra; see Štěpánek & Rubin 89, Theorem 5.9b.

381X Basic exercises >(a) Let X be a set and Σ an algebra of subsets of X containing all singleton sets. Show that $\operatorname{Aut}\Sigma$ can be identified with the group of bijections $\phi : X \to X$ such that $\phi[E] \in \Sigma$, $\phi^{-1}[E] \in \Sigma$ for every $E \in \Sigma$.

>(b) Let $\mathfrak{A}$ be a Boolean algebra and G any subgroup of $\operatorname{Aut}\mathfrak{A}$. Let H be the set of those $\pi \in \operatorname{Aut}\mathfrak{A}$ such that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $\phi \in G$ such that $\pi c = \phi c$ for every $c \subseteq b$. Show that H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, the smallest full subgroup of $\mathfrak{A}$ including G.

>(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G any subgroup of $\mathfrak{A}$. Show that an element π of $\operatorname{Aut}\mathfrak{A}$ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G iff there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ and a family $\langle \pi_i\rangle_{i\in I}$ in G such that $\pi a = \pi_i a$ whenever $i \in I$ and $a \subseteq a_i$.

(d) For a Boolean algebra $\mathfrak{A}$, let us say that a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **countably full** if whenever $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = \pi_i a_i$ whenever $i \in I$ and $a \subseteq a_i$, then $\pi \in G$. Show that if $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$, then every member of G is expressible as a product of at most eight involutions belonging to G.

>(e) Let X be any set. Show that any automorphism of the Boolean algebra $\mathcal{P}X$ is expressible as a product of at most two involutions.

(f) Recall that in any group G, a **commutator** in G is an element of the form $ghg^{-1}h^{-1}$ where g, $h \in G$. Show that if $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions then every involution in G is a commutator in G, so that every element of G is expressible as a product of finitely many commutators.

(g) Give an example of a Dedekind complete Boolean algebra $\mathfrak{A}$ such that not every member of $\operatorname{Aut}\mathfrak{A}$ is a product of commutators in $\operatorname{Aut}\mathfrak{A}$.

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ has many involutions. Show that if $H \lhd \operatorname{Aut}\mathfrak{A}$ then every member of H is expressible as the product of at most eight involutions belonging to H.

(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that the partially ordered set $\mathcal{H}$ of normal subgroups of G is a distributive lattice, that is, $H \cap K_1K_2 = (H \cap K_1)(H \cap K_2)$, $H(K_1 \cap K_2) = HK_1 \cap HK_2$ for all H, K_1, $K_2 \in \mathcal{H}$.

(j) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that if H is a the normal subgroup of G generated by a finite subset of G, then it is the normal subgroup generated by a single involution.

(k) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show (i) that there is an involution $\pi \in G$ such that every member of G is expressible as a product of conjugates of π in G (ii) any proper normal subgroup of G is included in a maximal proper normal subgroup of G.

(l) Let G be any group. Show that if π, $\phi \in G$ are involutions then $\pi\phi$ is conjugate to its inverse.

381Y Further exercises (a) Find a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with an automorphism which cannot be expressed either as a product of finitely many involutions in $\operatorname{Aut}\mathfrak{A}$, nor as a product of finitely many commutators in $\operatorname{Aut}\mathfrak{A}$. (This seems to require a certain amount of ingenuity.)

(b) Let X be a set and Σ a countably generated σ-subalgebra of subsets of X. (i) Show that if $f : X \to X$ is a bijection such that $\Sigma = \{E : E \subseteq X, f^{-1}[E] \in \Sigma\}$, then there are disjoint E', E'', F', F'', $G \in \Sigma$ such that $f^{-1}[E'] = F'$, $f^{-1}[E''] = F''$, $f^{-1}[F''] = G$, $f^{-1}[F' \cup G] = E' \cup E''$, and $f(x) = x$ for every $x \in X \setminus (E' \cup E'' \cup F' \cup F'' \cup G$. (*Hint*: there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $E_n \cap f^{-1}[E_n] = \emptyset$ for every n and $\bigcup_{n\in\mathbb{N}} E_n = \{x : f(x) \neq x\}$.) (In particular, any member of $\operatorname{Aut}\Sigma$ has a support.) (ii) Show that every involution in $\operatorname{Aut}\Sigma$ is an exchanging involution.

(c) Let X be a set and Σ a countably generated σ-subalgebra of subsets of X. Show that any member of $\operatorname{Aut}\Sigma$ is expressible as a product of at most eight involutions in $\operatorname{Aut}\Sigma$.

(d) Let $\mathfrak{A}$ be a homogeneous Boolean algebra which is isomorphic to the simple power $\mathfrak{A}^{\mathbb{N}}$. (For instance, $\mathfrak{A}$ could be the measure algebra of Lebesgue measure on $\mathbb{R}$.) Show that any automorphism of $\mathfrak{A}$ is the product of at most five exchanging involutions. (Cf. Štěpánek & Rubin 89, Corollary 5.9a(ii).)

381Z Problem In 381N, is 'eight' best possible?

381 Notes and comments The ideas above are adapted from Štěpánek & Rubin 89 and Fathi 78. Lemma 381J is a form of what is sometimes called 'Frolík's theorem', following Frolík 68.

The two main results 381N and 381S, as written out above, both involve careful algebra. It seems to me that we can distinguish two essential methods. (i) There are arguments involving *finitely* many automorphisms, carefully pieced together from descriptions of their actions on different parts of the algebra, as in (a)-(c) of the proof of 381N, and the whole of the proof of 381S; similar ideas can be used in 381Xf. It is in these that I believe that the 'cycle notation' introduced in 381G-381I can be of value. Generally the hope is that we can use intuitions derived from the theory of permutation groups (that is, the case $\mathfrak{A} = \mathcal{P}X$) to guide us. (ii) There is the argument in 381N involving a *sequence* of automorphisms, designed to express an automorphism π_0 supported by an element a_0 as the product of an automorphism θ_0 supported by $\sup_{k\in\mathbb{N}} a_k$ with an automorphism θ_1 supported by $\sup_{k\ge1} a_k$, so chosen that the actions of the θ_r on $\sup_{k\ge1} a_k$ cancel out and we are left with π_0 as the residue. (For an account of the origins of this idea see Štěpánek & Rubin 89.) Since we know of no automorphisms except those which can be derived from the original automorphism π, the method has to be to some extent constructive. The idea is that each π_{n+1} is not exactly a copy on a_{n+1} of the preceding π_n, but a modification of it by involutions. At each stage of the induction we have to mention an auxiliary element ψ_n of G in order to be sure that there will be room (in $b_n = \psi_n a_n$) for the next step, safely disjoint from the preceding a_k. When we come to build the mutually cancelling pair θ_0, θ_1 we find that they incorporate the modifiers ϕ_{1n}, ϕ_{2n}, ϕ_{3n}, which can be assembled into the involutions $\tilde{\phi}_{ir}$ in part (f) of the proof.

I note that the assumption of 'Dedekind completeness' (as opposed to Dedekind σ-completeness) in 381N is used only in parts (b) and (c) of the proof, when applying Frolík's theorem. Consequently we have a slight generalization possible (381Xd); but we do need the full hypothesis for the theorem as stated (381Ya). There is however a very important special case, when $\mathfrak{A}$ is a countably generated σ-algebra, for which we have a version of Frolík's theorem available for different reasons (381Yb), and can get a corresponding theorem to match 381N (381Yc).

A natural question arising from 381T is: does *every* homogeneous Boolean algebra have a simple automorphism group? This leads into deep water. As remarked after 381T, every homogeneous Dedekind σ-complete algebra has a simple automophism group. Using the continuum hypothesis, it is possible to construct a homogeneous Boolean algebra which does not have a simple automorphism group; but as far as I am aware no such construction is known which does not rely on some special axiom outside ordinary set theory. See Štěpánek & Rubin 89, §5.

In 381Z I ask whether the number 'eight' appearing in 381N is actually best possible. The argument is complex enough to make it seem that there may be room for improvement – see 381Xe and 381Yd. Ornstein

& Shields 73[1] present examples of automorphisms in the full subgroup $G = \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ of measure-preserving automorphisms of the Lebesgue probability algebra which are not conjugate (in G) to their inverses, and therefore cannot be expressible as the product of two involutions in G.

382 Automorphism groups of measure algebras

I turn now to the group of measure-preserving automorphisms of a measure algebra, seeking to apply the results of the last section. The principal theorems are 382D, which is a straightforward special case of 381N, and 382I, corresponding to 381T. I give another example of the use of 381S to describe the normal subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (382J).

382A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will write $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ for the set of all measure-preserving automorphisms of $\mathfrak{A}$. This is a group, being a subgroup of the group $\mathrm{Aut}\,\mathfrak{A}$ of all Boolean automorphisms of $\mathfrak{A}$.

382B Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is countable
or that $(\mathfrak{A},\bar\mu)$ is localizable.
Suppose that for each $i\in I$ we have a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi c=\pi_i c$ whenever $i\in I$ and $c\subseteq a_i$.

proof (Compare 381B.) By 322K, we may identify $\mathfrak{A}$ with each of the simple products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

382C Corollary If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra, then, in the language of 381M, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a full subgroup of $\mathrm{Aut}\,\mathfrak{A}$.

382D Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then every measure-preserving automorphism of $\mathfrak{A}$ is expressible as the product of at most eight measure-preserving involutions.

proof This is immediate from 382C and 381N.

382E Lemma If $(\mathfrak{A},\bar\mu)$ is a homogeneous semi-finite measure algebra, it is σ-finite, therefore localizable.

proof If $\mathfrak{A}=\{0\}$, this is trivial. Otherwise there is an $a\in\mathfrak{A}$ such that $0<\bar\mu a<\infty$. The principal ideal $\mathfrak{A}_a$ is ccc (322G), so $\mathfrak{A}$ also is, and $(\mathfrak{A},\bar\mu)$ must be σ-finite (by 322G again).

382F Lemma Let $(\mathfrak{A},\bar\mu)$ be a homogeneous semi-finite measure algebra.

(a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ with $\bar\mu a_i=\bar\mu b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, then whenever $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ with $\bar\mu a_i=\bar\mu b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

proof (a) By 382E, $(\mathfrak{A},\bar\mu)$ is σ-finite, therefore localizable. For each $i\in I$, the principal ideals $\mathfrak{A}_{a_i}$, $\mathfrak{A}_{b_i}$ are homogeneous, of the same measure and the same Maharam type (being $\tau(\mathfrak{A})$ if $a_i\ne0$, 0 if $a_i=0$). Because they are ccc, they are of the same magnitude, as defined in 332G, and there is a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ (332J). By 382B there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi d=\pi_i d$ for every $i\in I$, $d\subseteq a_i$; and this π serves.

(b) Set $a^*=1\setminus\sup_{i\in I}a_i$, $b^*=1\setminus\sup_{i\in I}b_i$. We must have

$$\bar\mu a^*=\bar\mu1-\sum_{i\in I}\bar\mu a_i=\bar\mu1-\sum_{i\in I}\bar\mu b_i=\bar\mu b^*,$$

so adding a^*, b^* to the families we obtain partitions of unity to which we can apply the result of (a).

[1] I am indebted to G.Hjorth for the reference.

382G Lemma (a) If $(\mathfrak{A},\bar{\mu})$ is an atomless semi-finite measure algebra, then $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ have many involutions.

(b) If $(\mathfrak{A},\bar{\mu})$ is an atomless localizable measure algebra, then every element of $\mathfrak{A}$ is the support of some involution in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

proof (a) If $a\in\mathfrak{A}\setminus\{0\}$, then by 332A there is a non-zero $b\subseteq a$, of finite measure, such that the principal ideal $\mathfrak{A}_b$ is (Maharam-type-)homogeneous. Now because $\mathfrak{A}$ is atomless, there is a $c\subseteq b$ such that $\bar{\mu}c=\frac{1}{2}\bar{\mu}b$ (331C), so that $\mathfrak{A}_c$ and $\mathfrak{A}_{b\setminus c}$ are isomorphic measure algebras. If $\theta:\mathfrak{A}_c\to\mathfrak{A}_{b\setminus c}$ is any measure-preserving isomorphism, then $\pi=(\overleftarrow{c\,{}_\theta\,b\setminus c})$ is an involution in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ (and therefore in $\operatorname{Aut}\mathfrak{A}$) supported by a.

(b) Use 382C, (a) and 381R.

382H Corollary Let $(\mathfrak{A},\bar{\mu})$ be an atomless localizable measure algebra. Then

(a) the lattice of normal subgroups of $\operatorname{Aut}\mathfrak{A}$ is isomorphic to the lattice of $\operatorname{Aut}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$;

(b) the lattice of normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is isomorphic to the lattice of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$.

proof Use 381S. Taking G to be either $\operatorname{Aut}\mathfrak{A}$ or $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, and $\mathcal{I}$ to be the family of G-invariant ideals in $\mathfrak{A}$, we have a map $I\mapsto H_I=\{\pi:\pi\in G,\ \operatorname{supp}\pi\in I\}$ from $\mathcal{I}$ to the family $\mathcal{H}$ of normal subgroups of G. Of course this map is order-preserving; 381S tells us that it is surjective; and 382Gb tells us that it is injective and its inverse is order-preserving, since if $a\in I\setminus J$ there is a $\pi\in G$ with $\operatorname{supp}\pi=a$, so that $\pi\in H_I\setminus H_J$. Thus we have an order-isomorphism between $\mathcal{H}$ and $\mathcal{I}$.

382I Normal subgroups of $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ 381S provides the machinery for a full description of the normal subgroups of $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ when $(\mathfrak{A},\bar{\mu})$ is an atomless localizable measure algebra, as we know that they correspond exactly to the invariant ideals of $\mathfrak{A}$. The general case is complicated. But the following are simple enough.

Theorem Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous semi-finite measure algebra.

(a) $\operatorname{Aut}\mathfrak{A}$ is simple.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is simple.

(c) If $(\mathfrak{A},\bar{\mu})$ is not totally finite, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has exactly one non-trivial proper normal subgroup.

proof (a) This is a special case of 381T.

(b)-(c) The point is that the only possible $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **P** If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. Let $I\lhd\mathfrak{A}$ be an invariant ideal.

(i) If $I\not\subseteq\mathfrak{A}^f$, take $a\in I$ with $\bar{\mu}a=\infty$. By 382E, $\mathfrak{A}$ is σ-finite, so a has the same magnitude ω as 1. By 332I, there is a partition of unity $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with $\bar{\mu}e_n=1$ for every n; setting $b=\sup_{n\in\mathbb{N}}e_{2n}$, $b'=1\setminus b$, we see that both b and b' are of infinite measure. Similarly we can divide a into c, c', both of infinite measure. Now by 332J the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_{b'}$, $\mathfrak{A}_c$, $\mathfrak{A}_{1\setminus c}$ are all isomorphic as measure algebras, so that there are automorphisms π, $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that

$$\pi c=b,\quad \phi c=b'.$$

But this means that both b and b' belong to I, so that $1=b\cup b'\in I$ and $I=\mathfrak{A}$.

(ii) If $I\subseteq\mathfrak{A}^f$ and $I\neq\{0\}$, take any non-zero $a\in I$. If b is any member of $\mathfrak{A}$, then (because $\mathfrak{A}$ is atomless) b can be partitioned into $b_0,\dots,b_n$, all of measure at most $\bar{\mu}a$. Then for each i there is a $b_i'\subseteq a$ such that $\bar{\mu}b_i'=\bar{\mu}b_i$; since this common measure is finite, $\bar{\mu}(1\setminus b_i')=\bar{\mu}(1\setminus b_i)$. By 332J and 382Fa, there is a $\pi_i\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi_ib_i'=b_i$, so that b_i belongs to I. Accordingly $b\in I$. As b is arbitrary, $I=\mathfrak{A}^f$.

Thus the only invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **Q**

By 382Hb we therefore have either one, two or three normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, according to whether $\bar{\mu}1$ is zero, finite and not zero, or infinite.

Remark For the Lebesgue probability algebra, (b) is due to Fathi 78. The extension to algebras of uncountable Maharam type is from Choksi & Prasad 82.

382J The language of §352 offers a way of describing another case.

Proposition Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$, and let K be $\{\kappa : e_\kappa \ne 0\}$. Let $\mathcal{H}$ be the lattice of normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. Then

(i) if K is finite, $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$;

(ii) if K is infinite, then $\mathcal{H}$ is isomorphic, as partially ordered set, to the lattice of solid linear subspaces of ℓ^∞.

proof (a) Let $\mathcal{I}$ be the family of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$, so that $\mathcal{H}\cong\mathcal{I}$, by 382Hb. For a, $b\in\mathfrak{A}$, say that $a\preccurlyeq b$ if there is some $k\in\mathbb{N}$ such that $\bar{\mu}(a\cap e_\kappa)\le k\bar{\mu}(b\cap e_\kappa)$ for every $\kappa\in K$. Then an ideal I of $\mathfrak{A}$ is $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant iff $a\in I$ whenever $a\preccurlyeq b\in I$. **P** (α) Suppose that I is $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ invariant and that $b\in I$, $\bar{\mu}(a\cap e_\kappa)\le k\bar{\mu}(b\cap e_\kappa)$ for every $\kappa\in K$. Then for each κ we can find $a_{\kappa1},\dots,a_{\kappa k}$ such that $a\cap e_\kappa=\sup_{i\le k}a_{\kappa i}$ and $\bar{\mu}a_{\kappa i}\le\bar{\mu}(b\cap e_\kappa)$ for every i. Now there are measure-preserving automorphisms $\pi_{\kappa i}$ of the principal ideal $\mathfrak{A}_{e_\kappa}$ such that $\pi_{\kappa i}a_{\kappa i}\subseteq b$. Setting $\pi_i d=\sup_{\kappa\in K}\pi_{\kappa i}(d\cap e_\kappa)$ for every $d\in\mathfrak{A}$, and $a_i=\sup_{\kappa\in K}a_{\kappa i}$, we have $\pi_i\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\pi_i a_i\subseteq b$, so $a_i\in I$ for each i; also $a=\sup_{i\le k}a_i$, so $a\in I$. (β) On the other hand, if $a\in\mathfrak{A}$ and $\pi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, then

$$\bar{\mu}(\pi a\cap e_\kappa)=\bar{\mu}\pi(a\cap e_\kappa)=\bar{\mu}(a\cap e_\kappa)$$

for every $\kappa\in K$, because $\pi e_\kappa=e_\kappa$, so that $\pi a\preccurlyeq a$. So if I satisfies the condition, $\pi[I]\subseteq I$ for every $\pi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $I\in\mathcal{I}$. **Q**

(b) Consequently, for $I\in\mathcal{I}$ and $\kappa\in K$, $e_\kappa\in I$ iff there is some $a\in I$ such that $a\cap a_\kappa\ne0$, since in this case $e_\kappa\preccurlyeq a$. (This is where I use the hypothesis that $(\mathfrak{A},\bar{\mu})$ is totally finite.) It follows that if K is finite, any $I\in\mathcal{I}$ is the principal ideal generated by $\sup\{e_\kappa : e_\kappa\in I\}$. Conversely, of course, all such ideals are $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant. Thus $\mathcal{I}$ is in a natural order-preserving correspondence with $\mathcal{P}K$, and $\mathcal{H}\cong\mathcal{P}K$.

(c) Now suppose that K is infinite; enumerate it as $\langle\kappa_n\rangle_{n\in\mathbb{N}}$. Define $\theta:\mathfrak{A}\to\ell^\infty$ by setting $\theta a=\langle\bar{\mu}(a\cap e_{\kappa_n})/\bar{\mu}(e_{\kappa_n})\rangle_{n\in\mathbb{N}}$ for $a\in\mathfrak{A}$; so that

$$a\preccurlyeq b \text{ iff there is some } k \text{ such that } \theta a\le k\theta b,$$

$$\theta a\le\theta(a\cup b)\le\theta a+\theta b\le 2\theta(a\cup b)$$

for all a, $b\in\mathfrak{A}$, while $\theta(1_{\mathfrak{A}})$ is the standard order unit $\mathbf{1}$ of ℓ^∞. Let $\mathcal{U}$ be the family of solid linear subspaces of ℓ^∞ and define functions $I\mapsto V_I:\mathcal{I}\to\mathcal{U}$, $U\mapsto J_U:\mathcal{U}\to\mathcal{I}$ by saying

$$V_I=\{f : f\in\ell^\infty,\ |f|\le k\theta a \text{ for some } a\in I,\ k\in\mathbb{N}\},$$

$$J_U=\{a : a\in\mathfrak{A},\ \theta a\in U\}.$$

The properties of θ just listed ensure that $V_I\in\mathcal{U}$, $J_U\in\mathcal{I}$ for every $I\in\mathcal{I}$, $U\in\mathcal{U}$. Of course both $I\mapsto V_I$ and $U\mapsto J_U$ are order-preserving. If $I\in\mathcal{I}$, then

$$J_{V_I}=\{a : \exists\, b\in I,\ a\preccurlyeq b\}=I,$$

Finally, $V_{J_U}=U$ for every $U\in\mathcal{U}$. **P**

$$V_{J_U}=\{f : \exists\, a\in\mathfrak{A},\ k\in\mathbb{N},\ |f|\le k\theta a\in U\}\subseteq U$$

because U is a solid linear subspace. But also, given $g\in U$, there is an $a\in\mathfrak{A}$ such that $\bar{\mu}(a\cap e_{\kappa_n})=\min(1,|g(n)|)\bar{\mu}(e_{\kappa_n})$ for every n (because $\mathfrak{A}$ is atomless); in which case

$$\theta a\le|g|\le\max(1,\|g\|_\infty)\theta a$$

so $a\in J_U$ and $g\in V_{J_U}$. Thus $U=V_{J_U}$. **Q** So the functions $I\mapsto V_I$ and $U\mapsto J_U$ are the two halves of an order-isomorphism between $\mathcal{I}$ and $\mathcal{U}$, and $\mathcal{H}\cong\mathcal{I}\cong\mathcal{U}$, as claimed.

382X Basic exercises >(a) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$. (i) Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is a simple group iff *either* there is just one infinite cardinal κ such that $e_\kappa\ne0$, that e_κ has finite measure and all the atoms of $\mathfrak{A}$ (if any) have different measures *or* $\mathfrak{A}$ is purely atomic and there is just one pair of atoms of the same measure *or* $\mathfrak{A}$ is purely atomic and all its atoms have different measures. (ii) Show that $\operatorname{Aut}\mathfrak{A}$ is a simple group iff *either*

$(\mathfrak{A}, \bar{\mu})$ is σ-finite and there is just one infinite cardinal κ such that $e_\kappa \neq 0$ and $\mathfrak{A}$ has at most one atom *or* $\mathfrak{A}$ is purely atomic and has at most two atoms.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. (i) Show that $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\operatorname{Aut}_{\bar{\nu}_\kappa} \mathfrak{B}_\kappa$ where κ is an infinite cardinal and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ is the measure algebra of the usual measure on $\{0,1\}^\kappa$. (ii) Show that $\operatorname{Aut} \mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\operatorname{Aut} \mathfrak{B}_\kappa$.

(c) Show that if $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra of magnitude greater than $\mathfrak{c}$, its automorphism group $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is not simple.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless localizable measure algebra. For each infinite cardinal κ write e_κ for the Maharam-type-κ component of $\mathfrak{A}$. For π, $\psi \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ show that π belongs to the normal subgroup of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ generated by ψ iff there is a $k \in \mathbb{N}$ such that

$$\operatorname{mag}(e_\kappa \cap \operatorname{supp} \pi) \leq k \operatorname{mag}(e_\kappa \cap \operatorname{supp} \psi) \text{ for every infinite cardinal } \kappa,$$

writing $\operatorname{mag} a$ for the magnitude of a, and setting $k\zeta = \zeta$ if $k > 0$ and ζ is an infinite cardinal.

>(e) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. For $n \in \mathbb{N}$ set $e_n = [-n, n]^\bullet \in \mathfrak{A}$. Let $G \leq \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ be the group consisting of measure-preserving automorphisms π such that $\operatorname{supp} \pi \subseteq e_n$ for some n. Show that G is simple. (*Hint*: show that G is the union of an increasing sequence of simple subgroups.)

(f) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless totally finite measure algebra. Let $\mathcal{H}$ be the lattice of normal subgroups of $\operatorname{Aut} \mathfrak{A}$. Show that $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$ for some countable set K.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless localizable measure algebra which is not σ-finite, and suppose that $\tau(\mathfrak{A}_a) = \tau(\mathfrak{A}_b)$ whenever a, $b \in \mathfrak{A}$ and $0 < \bar{\mu} a \leq \bar{\mu} b < \infty$. Let κ be the magnitude of $\mathfrak{A}$. (i) Show that the lattice $\mathcal{H}$ of normal subgroups of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is well-ordered, with least member $\{\iota\}$, next member $\{\pi : \bar{\mu}(\pi a) < \infty$ whenever $\bar{\mu} a < \infty\}$, and one member H_ζ for each infinite cardinal ζ less than or equal to κ, setting

$$H_\zeta = \{\pi : \pi \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A},\ \operatorname{mag}(\pi a) \leq \zeta \text{ whenever } \operatorname{mag} a \leq \zeta\},$$

where $\operatorname{mag} a$ is the magnitude of a. (ii) Show that the lattice $\mathcal{H}'$ of normal subgroups of $\operatorname{Aut} \mathfrak{A}$ is well-ordered, with least member $\{\iota\}$ and one member H'_ζ for each infinite cardinal ζ less than or equal to κ, setting

$$H'_\zeta = \{\pi : \pi \in \operatorname{Aut} \mathfrak{A},\ \operatorname{mag}(\pi a) \leq \zeta \text{ whenever } \operatorname{mag} a \leq \zeta\}.$$

382Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless totally finite measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$, $\operatorname{Aut} \mathfrak{A}$ have the same (cardinal) number of normal subgroups.

(b) Let X be a set. Show that $\operatorname{Aut} \mathcal{P}X$ has one normal subgroup if $\#(X) \leq 1$, two if $\#(X) = 2$, three if $\#(X) = 3$ or $5 \leq \#(X) \leq \omega$, four if $\#(X) = 4$ or $\#(X) = \omega_1$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving homomorphism. Take any $a \in \mathfrak{A}$ and set $c = \sup_{n \geq 1} \pi^n a$. (i) Show that $a \subseteq c$ and that $c = \sup_{m \geq n} \pi^m a$ for every $n \in \mathbb{N}$. (ii) Set $a_n = \pi^n a \setminus \sup_{1 \leq i < n} \pi^i a$ for $n \geq 1$. Show that $\sum_{n=1}^\infty n \bar{\mu}(a \cap a_n) = \bar{\mu} c$. (*Hint*: For $j < k$ set $a_{jk} = \pi^k a \cap \pi^j a \setminus \sup_{j < i < k} \pi^i a = \pi^j(a \cap a_{k-j})$. Show that, for any n, $\langle a_{jk} \rangle_{j \leq n < k}$ is disjoint and has union $\sup_{i \leq n} \pi^i a$.)

(d) Let (X, Σ, μ) be a totally finite measure space and $f : X \to X$ an inverse-measure-preserving function. Take $E \in \Sigma$ and set $F = \{x : \exists\, n \geq 1,\ f^n(x) \in E\}$. (i) Show that $E \setminus F$ is negligible. (ii) For $x \in E \cap F$ set $k_x = \min\{n : n \geq 1,\ f^n(x) \in E\}$. Show that $\int_E k_x \mu(dx) = \mu F$. (This is a simple form of the **Recurrence Theorem**.)

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that there are a measure algebra $(\mathfrak{B}, \bar{\nu})$, a measure-preserving automorphism $\phi : \mathfrak{B} \to \mathfrak{B}$

and a closed subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ such that $\phi[\mathfrak{C}] \subseteq \mathfrak{C}$ and $(\mathfrak{C}, \bar\nu\restriction\mathfrak{C}, \phi\restriction\mathfrak{C})$ is isomorphic to $(\mathfrak{A}, \bar\mu, \pi)$. (*Hint*: Take $\mathfrak{B}_1 \subseteq \mathfrak{A}^{\mathbb{N}}$ to be the subalgebra consisting of sequences $\mathbf{a} = (\alpha_0, \alpha_1, \dots)$ such that $\sum_{n=0}^{\infty} \bar\mu(\alpha_{n+1} \mathbin{\triangle} \pi\alpha_n) < \infty$. For $\mathbf{a} \in \mathfrak{B}_1$ set $\bar\nu_1\mathbf{a} = \lim_{n\to\infty} \bar\mu\alpha_n$, $\phi_1\mathbf{a} = (\alpha_1, \alpha_2, \dots)$. Let $\mathcal{I}$ be the ideal $\{\mathbf{a} : \bar\nu_1\mathbf{a} = 0\}$ and let $\mathfrak{B}$ be the quotient $\mathfrak{B}_1/\mathcal{I}$; define $\phi : \mathfrak{B} \to \mathfrak{B}$ by setting $\phi\mathbf{a}^\bullet = (\phi_1\mathbf{a})^\bullet$. For $a \in \mathfrak{A}$ let $a^* \in \mathfrak{B}$ be the equivalence class of the sequence $\langle \pi^n a\rangle_{n\in\mathbb{N}}$; set $\mathfrak{C} = \{a^* : a \in \mathfrak{A}\}$.) (This is an abstract version of a construction known as the 'natural extension' of an inverse-measure-preserving function; see Petersen 83, 1.3G.)

(f) Let (X, Σ, μ) be a measure space in which Σ is countably generated as σ-algebra, and write $\operatorname{Aut}_\mu \Sigma$ for the group of automorphisms $\phi : \Sigma \to \Sigma$ such that $\mu\phi(E) = \mu E$ for every $E \in \Sigma$. Show that every member of $\operatorname{Aut}_\mu \Sigma$ is expressible as a product of at most eight involutions belonging to $\operatorname{Aut}_\mu \Sigma$. (*Hint*: 381Yc.)

382 Notes and comments This section is short because there are no substantial new techniques to be developed. 382D is simply a matter of checking that the hypotheses of 381N are satisfied (and these hypotheses were of course chosen with 382D in mind), and 382I is similarly direct from 381S. 382I-382J, 382Xd and 382Xg are variations on a theme. In a general Boolean algebra $\mathfrak{A}$ with a group G of automorphisms, we have a transitive, reflexive relation $\preccurlyeq_G$ defined by saying that $a \preccurlyeq_G b$ if there are $\pi_1, \dots, \pi_k \in G$ such that $a \subseteq \sup_{i\le k} \pi_i b$; the point about localizable measure algebras is that the functions 'Maharam type' and 'magnitude' enable us to describe this relation when $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$, and the essence of 381S is that in that context π belongs to the normal subgroup of G generated by ψ iff $\operatorname{supp}\pi \preccurlyeq_G \operatorname{supp}\psi$.

Most of the work of this chapter is focused on atomless measure algebras. There are various extra complications which appear if we allow atoms. The most striking are in the next section; here I mention only 382Xa and 382Yb.

383 Outer automorphisms

Continuing with the investigation of the abstract group-theoretic nature of the automorphism groups $\operatorname{Aut}\mathfrak{A}$, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, I devote a section to some remarkable results concerning isomorphisms between them. Under any of a variety of conditions, any isomorphism between two groups $\operatorname{Aut}\mathfrak{A}$, $\operatorname{Aut}\mathfrak{B}$ must correspond to an isomorphism between the underlying Boolean algebras (383E, 383F, 383J, 383M); consequently $\operatorname{Aut}\mathfrak{A}$ has few, or no, outer automorphisms (383G, 383K, 383O). I organise the section around a single general result (383D).

383A Lemma Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ which has many involutions (definition: 381P). Then for every non-zero $a \in \mathfrak{A}$ there is an automorphism $\psi \in G$, of order 4, which is supported by a.

proof Let $\pi \in G$ be an involution supported by a. Let $b \subseteq a$ be such that $\pi b \ne b$. Then at least one of $b \setminus \pi b$, $\pi b \setminus b = \pi(b \setminus \pi b)$ is non-zero, so in fact both are. Let ϕ be an involution supported by $b \setminus \pi b$. Then $\pi\phi\pi = \pi\phi\pi^{-1}$ is an involution supported by $\pi b \setminus b$, so commutes with ϕ, and the product $\phi\pi\phi\pi$ is an involution. But this means that $\psi = \phi\pi$ has order 4, and of course it is supported by a because ϕ and π both are.

383B A note on supports Since in this section we shall be looking at more than one automorphism group at a time, I shall need to call on the following elementary extension of a fact in 381Ec. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. If $\pi \in \operatorname{Aut}\mathfrak{A}$ is supported by $a \in \mathfrak{A}$, then $\theta\pi\theta^{-1} \in \operatorname{Aut}\mathfrak{B}$ is supported by θa. (Use the same argument as in 381Ec.) Accordingly, if a is the support of π then θa will be the support of $\theta\pi\theta^{-1}$, as in 381L.

383C Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Boolean algebras, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. If θ_1, $\theta_2 : \mathfrak{A} \to \mathfrak{B}$ are distinct isomorphisms, then there is a $\phi \in G$ such that $\theta_1\phi\theta_1^{-1} \ne \theta_2\phi\theta_2^{-1}$.

proof Because $\theta_1 \ne \theta_2$, $\theta = \theta_2^{-1}\theta_1$ is not the identity automorphism on $\mathfrak{A}$, and there is some non-zero $a \in \mathfrak{A}$ such that $\theta a \cap a = 0$. Let $\pi \in G$ be an involution supported by a; then $\theta\pi\theta^{-1}$ is supported by θa, so cannot be equal to π, and $\theta_1\pi\theta_1^{-1} \ne \theta_2\pi\theta_2^{-1}$.

383D Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and G and H subgroups of $\operatorname{Aut}\mathfrak{A}$, $\operatorname{Aut}\mathfrak{B}$ respectively, both having many involutions. Let $q : G \to H$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$.

proof (a) The first half of the proof is devoted to setting up some structures in the group G. Let $\pi \in G$ be any involution. Set

$$C_\pi = \{\phi : \phi \in G,\, \phi\pi = \pi\phi\},$$

the centralizer of π in G;

$$U_\pi = \{\phi : \phi \in C_\pi,\, \phi = \phi^{-1},\, \phi\psi\phi\psi^{-1} = \psi\phi\psi^{-1}\phi \text{ for every } \psi \in C_\pi\},$$

the set of involutions in C_π commuting with all their conjugates in C_π, together with the identity,

$$V_\pi = \{\phi : \phi \in G,\, \phi\psi = \psi\phi \text{ for every } \psi \in U_\phi\},$$

the centralizer of U_π in G,

$$S_\pi = \{\phi^2 : \phi \in V_\pi\},$$

$$W_\pi = \{\phi : \phi \in G,\, \phi\psi = \psi\phi \text{ for every } \psi \in S_\pi\},$$

the centralizer of S_π in G.

(b) The point of this list is to provide a purely group-theoretic construction corresponding to the support of π in $\mathfrak{A}$. In the next few paragraphs of the proof (down to (f)), I set out to describe the objects just introduced in terms of their action on $\mathfrak{A}$. First, note that π is an exchanging involution (381K); express it as $(\overleftarrow{a'\,{}_\pi\,a''})$, so that the support of π is $a_\pi = a' \cup a''$.

(c) I start with two elementary properties of C_π:

(i) $\phi(a_\pi) = a_\pi$ for every $\phi \in C_\pi$. **P** As remarked in 381L, the support of $\pi = \phi\pi\phi^{-1}$ is $\phi(a_\pi)$, so this must be a_π. **Q**

(ii) If $\phi \in C_\pi$ and ϕ is not supported by a_π, there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $d \cap \phi d = 0$, by 381Ef.

(d) Now for the properties of U_π:

(i) If $\phi \in U_\pi$, then ϕ is supported by a_π.

P(α)**?** Suppose first that there is a $d \subseteq 1 \setminus a_\pi$ such that $d \cap (\phi d \cup \phi^2 d) = 0$. Let $\psi \in G$ be an involution supported by d. Then $\operatorname{supp}\psi \cap \operatorname{supp}\pi = 0$, so $\psi \in C_\pi$. There is a $c \subseteq d$ such that $\psi c \neq c$, so

$$\psi\phi\psi^{-1}\phi c = \psi\phi^2 c = \phi^2 c,$$

because $d \cap (\phi c \cup \phi^2 c) = 0$, while

$$\phi\psi\phi\psi^{-1}c = \phi^2\psi^{-1}c,$$

because $d \cap \phi\psi^{-1}c = 0$; but this means that $\psi\phi\psi^{-1}\phi c \neq \phi\psi\phi\psi^{-1}c$, so ϕ and $\psi\phi\psi^{-1}$ do not commute, and $\phi \notin U_\pi$. **X**

(β)**?** Suppose that ϕ^2 is not supported by a_π. Then, as remarked in (b), there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi^2 d \cap d = 0$. Now $d \not\subseteq \phi^2 d$, so $d \not\subseteq \phi d$; set $d' = d \setminus \phi d$. Then $d' \cap \phi d' = d' \cap \phi^2 d' = 0$ and $0 \neq d' \subseteq 1 \setminus a_\pi$; but this is impossible, by (α). **X**

(γ) Thus $\phi^2 d = d$ for every $d \subseteq 1 \setminus a_\pi$. **?** Suppose, if possible, that ϕ is not supported by a_π. Then there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi d \cap d = 0$. By 383A, there is a $\psi \in G$, of order 4, supported by d. Because $d \cap a_\pi = 0$, $\psi \in C_\pi$. Because $\psi \neq \psi^{-1}$, there is a $c \subseteq d$ such that $\psi c \neq \psi^{-1}c$; but now $\phi c \cap d = \phi\psi^{-1}c \cap d = 0$, so

$$\psi\phi\psi^{-1}\phi c = \psi\phi^2 c = \psi c \neq \psi^{-1}c = \phi^2\psi^{-1}c = \phi\psi\phi\psi^{-1}c,$$

and ϕ does not commute with its conjugate $\psi\phi\psi^{-1}$, contradicting the assumption that $\phi \in U_\pi$. **X**

So ϕ is supported by a_π, as claimed. **Q**

(ii) If $u \in \mathfrak{A}$ and $\pi u = u$, then $\pi_u \in U_\pi$, where

$$\pi_u d = \pi d \text{ if } d \subseteq u, \qquad \pi_u d = d \text{ if } d \cap u = 0,$$

that is, $\pi_u = (\overleftarrow{a' \cap u \,{}_\pi\, a'' \cap u})$. **P** For any $\psi \in \operatorname{Aut}\mathfrak{A}$,

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a' \cap u) \,{}_{\psi\pi\psi^{-1}}\, \psi(a'' \cap u)})$$

(381Hb). (α) Accordingly

$$\pi\pi_u\pi^{-1} = (\overleftarrow{a'' \cap u \,{}_\pi\, a' \cap u}) = \pi_u$$

and $\pi_u \in C_\pi$. (β) If $\psi \in C_\pi$, then

$$\pi = \psi\pi\psi^{-1} = (\overleftarrow{\psi a' \,{}_{\psi\pi\psi^{-1}}\, \psi a''}) = (\overleftarrow{\psi a' \,{}_\pi\, \psi a''}).$$

So

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a' \cap u) \,{}_{\psi\pi\psi^{-1}}\, \psi(a'' \cap u)}) = (\overleftarrow{\psi a' \cap \psi u \,{}_\pi\, \psi a'' \cap \psi u}) = \pi_{\psi u}.$$

Now if $\pi v = v$ then $\pi_u\pi_v = \pi_{u\triangle v} = \pi_v\pi_u$; in particular, $\pi_{\psi u}\pi_u = \pi_u\pi_{\psi u}$. As ψ is arbitrary, $\pi_u \in U_\pi$. **Q**

In particular, of course, $\pi = \pi_1$ belongs to U_π.

(e) The two parts of (d) lead directly to the properties we need of V_π.

(i) $V_\pi \subseteq C_\pi$, because $\pi \in U_\pi$. Consequently $\phi a_\pi = a_\pi$ for every $\phi \in V_\pi$.

(ii) If $\phi \in V_\pi$ then $\phi d \subseteq d \cup \pi d$ for every $d \subseteq a_\pi$. **P?** Suppose, if possible, otherwise. Set $u_0 = d \cup \pi d$, so that $\pi u_0 = u_0$, and $u = \phi u_0 \setminus u_0 \ne 0$; also $u \subseteq \phi a_\pi = a_\pi$. Since $\pi\phi u_0 = \phi\pi u_0 = \phi u_0$, $\pi u = u$. Set $v = u \cap a'$, so that $u = v \cup \pi v$ and $v \ne \pi v$. Because $u \cap \phi v \subseteq \phi(u_0 \cap u) = 0$,

$$\pi_u\phi v = \phi v \ne \phi\pi v = \phi\pi_u v,$$

which is impossible. **XQ**

(iii) It follows that $\phi^2 d = d$ whenever $\phi \in V_\pi$ and $d \subseteq a_\pi$. **P** Let e be the support of ϕ. Recall that $e = \sup\{c : c \cap \phi c = 0\}$ (381L), so that $d \cap e = \sup\{c : c \subseteq d,\ c \cap \phi c = 0\}$. Now if $c \subseteq a_\pi$ and $c \cap \phi c = 0$, we know that $\phi c \subseteq c \cup \pi c$, so in fact $\phi c \subseteq \pi c$. This shows that $\phi(d \cap e) \subseteq \pi(d \cap e)$. Also, because $\pi\phi = \phi\pi$, by (i), we have

$$\phi^2(d \cap e) \subseteq \phi\pi(d \cap e) = \pi\phi(d \cap e) \subseteq \pi^2(d \cap e) = d \cap e.$$

Of course $\phi^2(d \setminus e) = d \setminus e$, so $\phi^2 d \subseteq d$. This is true for every $d \subseteq a_\pi$. But as also $\phi^2 a_\pi = \phi a_\pi = a_\pi$, $\phi^2 d = d$ for every $d \subseteq a_\pi$. **Q**

(iv) The final thing we need to know about V_π is that $\phi \in V_\pi$ whenever $\phi \in G$ and $\operatorname{supp}\phi \cap a_\pi = 0$; this is immediate from (d-i) above.

(f) From (e-iii), we see that if $\phi \in S_\pi$ then $\operatorname{supp}\phi \cap a_\pi = 0$. But we also see from (e-iv) that if $0 \ne c \subseteq 1 \setminus a_\pi$ there is an involution in S_π supported by c; for there is a member ψ of G, of order 4, supported by c, and now $\psi \in V_\pi$ so $\psi^2 \in S_\pi$, while ψ^2 is an involution.

(g) Consequently, W_π is just the set of members of G supported by a_π. **P** (i) If $\operatorname{supp}\phi \subseteq a_\pi$ and $\psi \in S_\pi$, then $\operatorname{supp}\psi \cap a_\pi = 0$, as noted in (e), so $\phi\psi = \psi\phi$; as ψ is arbitrary, $\phi \in W_\pi$. (ii) If $\operatorname{supp}\phi \not\subseteq a_\pi$, then take a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi d \cap d = 0$. Let $\psi \in S_\pi$ be an involution supported by d; then if $c \subseteq d$ is such that $\psi c \ne c$,

$$\phi\psi c \ne \phi c = \psi\phi c,$$

and $\phi\psi \ne \psi\phi$ so $\phi \notin W_\pi$. **Q**

(h) We can now return to consider the isomorphism $q : G \to H$. If $\pi \in G$ is an involution, then $q(\pi) \in H$ is an involution, and it is easy to check that

$$q[C_\pi] = C_{q(\pi)},$$

$$q[U_\pi] = U_{q(\pi)},$$

$$q[V_\pi] = V_{q(\pi)},$$

$$q[S_\pi] = S_{q(\pi)},$$

$$q[W_\pi] = W_{q(\pi)},$$

defining $C_{q(\pi)}, \dots , W_{q(\pi)} \subseteq H$ as in (a) above. So we see that, for any $\phi \in G$,

$$\operatorname{supp}\phi \subseteq \operatorname{supp}\pi \iff \operatorname{supp}q(\phi) \subseteq \operatorname{supp}q(\pi).$$

(i) Define $\theta : \mathfrak{A} \to \mathfrak{B}$ by writing

$$\theta a = \sup\{\operatorname{supp}q(\pi) : \pi \in G \text{ is an involution and } \operatorname{supp}\pi \subseteq a\}$$

for every $a \in \mathfrak{A}$. Evidently θ is order-preserving. Now if $a \in \mathfrak{A}$, $\pi \in G$ is an involution and $\operatorname{supp}\pi \not\subseteq a$, $\operatorname{supp}q(\pi) \not\subseteq \theta a$. **P** There is a $\phi \in G$, of order 4, supported by $\operatorname{supp}\pi \setminus a$. Now ϕ^2 is an involution supported by $\operatorname{supp}\pi$, so $\operatorname{supp}q(\phi^2) \subseteq \operatorname{supp}q(\pi)$. On the other hand, if $\pi' \in G$ is an involution supported by a, then $\phi \in V_{\pi'}$ and $\phi^2 \in S_{\pi'}$, so $q(\phi^2) \in S_{q(\pi')}$ and $\operatorname{supp}q(\phi^2) \cap \operatorname{supp}q(\pi') = 0$. As π' is arbitrary, $\operatorname{supp}q(\phi^2) \cap \theta a = 0$; so

$$\operatorname{supp}q(\pi) \setminus \theta a \supseteq \operatorname{supp}q(\phi^2) \neq 0. \quad \mathbf{Q}$$

(j) In the same way, we can define $\theta^* : \mathfrak{B} \to \mathfrak{A}$ by setting

$$\theta^* b = \sup\{\operatorname{supp}q^{-1}(\pi) : \pi \in H \text{ is an involution and } \operatorname{supp}\pi \subseteq b\}$$

for every $b \in \mathfrak{B}$. Now $\theta^*\theta a = a$ for every $a \in \mathfrak{A}$. **P** (α) If $0 \neq u \subseteq a$, there is an involution $\pi \in G$ supported by u. Now $q(\pi)$ is an involution in H supported by θa, so

$$u \cap \theta^*\theta a \supseteq u \cap \operatorname{supp}q^{-1}q(\pi) = \operatorname{supp}\pi \neq 0.$$

As u is arbitrary, $a \subseteq \theta^*\theta a$. (β) If $\pi \in H$ is an involution supported by θa, then $\phi = q^{-1}(\pi)$ is an involution in G with $\operatorname{supp}q(\phi) = \operatorname{supp}\pi \subseteq \theta a$, so $\operatorname{supp}\phi \subseteq a$, by (i) above; as π is arbitrary, $\theta^*\theta a \subseteq a$. **Q**

Similarly, $\theta\theta^* b = b$ for every $b \in \mathfrak{B}$. But this means that θ and θ^* are the two halves of an order-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. By 312L, both are Boolean homomorphisms.

(k) If $\pi \in G$ is an involution, then $\theta(\operatorname{supp}\pi) = \operatorname{supp}q(\pi)$. **P** By the definition of θ, $\operatorname{supp}q(\pi) \subseteq \theta(\operatorname{supp}\pi)$. On the other hand,

$$\operatorname{supp}q(\pi) = \theta\theta^*(\operatorname{supp}q(\pi)) \supseteq \theta(\operatorname{supp}q^{-1}q(\pi)) = \theta(\operatorname{supp}\pi). \quad \mathbf{Q}$$

Similarly, if $\pi \in H$ is an involution, $\theta^{-1}(\operatorname{supp}\pi) = \theta^*(\operatorname{supp}\pi) = \operatorname{supp}q^{-1}(\pi)$.

(l) We are nearly home. Let us confirm that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$. **P?** Otherwise, $\psi = q(\phi)^{-1}\theta\phi\theta^{-1}$ is not the identity automorphism on $\mathfrak{B}$, and there is a non-zero $b \in \mathfrak{B}$ such that $\psi b \cap b = 0$, that is, $\theta\phi\theta^{-1}b \cap q(\phi)b = 0$. Let $\pi \in H$ be an involution supported by b. Then $q^{-1}(\pi)$ is supported by $\theta^{-1}b$, by (j), so $\phi\theta^{-1}b$ supports $\phi q^{-1}(\pi)\phi^{-1}$ and $\theta\phi\theta^{-1}b$ supports $q(\phi q^{-1}(\pi)\phi^{-1}) = q(\phi)\pi q(\phi)^{-1}$. On the other hand, $q(\phi)b$ also supports $q(\phi)\pi q(\phi)^{-1}$, which is not the identity automorphism; so these two elements of $\mathfrak{B}$ cannot be disjoint. **XQ**

(m) Finally, θ is unique by 383C.

Remark The ideas of the proof here are taken from Eigen 82.

383E The rest of this section may be regarded as a series of corollaries of this theorem. But I think it will be apparent that they are very substantial results.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless homogeneous Boolean algebras, and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then every $\phi \in \operatorname{Aut}\mathfrak{A}$ has a unique extension to a Boolean homomorphism $\hat\phi : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ (314Tb). Because the extension is unique, we must have $(\phi\psi)\hat{}= \hat\phi\hat\psi$ for all ϕ, $\psi \in \operatorname{Aut}\mathfrak{A}$; consequently, $\hat\phi$ and $\widehat{\phi^{-1}}$ are inverses of each other, and $\hat\phi \in \operatorname{Aut}\widehat{\mathfrak{A}}$ for each $\phi \in \operatorname{Aut}\mathfrak{A}$; moreover, $\phi \mapsto \hat\phi$ is a group homomorphism. Of course it is injective, so we have a subgroup $G = \{\hat\phi : \phi \in \operatorname{Aut}\mathfrak{A}\}$ of $\operatorname{Aut}\widehat{\mathfrak{A}}$ which is isomorphic to $\operatorname{Aut}\mathfrak{A}$. Clearly

$$G = \{\phi : \phi \in \operatorname{Aut}\widehat{\mathfrak{A}},\ \phi u \in \mathfrak{A} \text{ for every } u \in \mathfrak{A}\}.$$

If $a \in \widehat{\mathfrak{A}}$ is non-zero, then there is a non-zero $u \subseteq a$ belonging to $\mathfrak{A}$. Because $\mathfrak{A}$ is atomless and homogeneous, there is an involution $\pi \in \mathrm{Aut}\, A$ supported by u (381Q); now $\hat{\pi} \in G$ is an involution supported by a. As a is arbitrary, G has many involutions.

Similarly, writing $\widehat{\mathfrak{B}}$ for the Dedekind completion of $\mathfrak{B}$, we have a subgroup $H = \{\hat{\psi} : \psi \in \mathrm{Aut}\,\mathfrak{B}\}$ of $\mathrm{Aut}\,\widehat{\mathfrak{B}}$ isomorphic to $\mathrm{Aut}\,\mathfrak{B}$, and with many involutions. Let $\hat{q} : G \to H$ be the corresponding isomorphism, so that $\hat{q}(\hat{\phi}) = \widehat{q(\phi)}$ for every $\phi \in \mathrm{Aut}\,\mathfrak{A}$.

By 383D, there is a Boolean isomorphism $\hat{\theta} : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{B}}$ such that $\hat{q}(\phi) = \hat{\theta}\phi\hat{\theta}^{-1}$ for every $\phi \in G$.

(b) If $u \in \mathfrak{A}$, then $\hat{\theta}u \in \mathfrak{B}$. **P** It is enough to consider the case $u \notin \{0, 1\}$, since surely $\hat{\theta}0 = 0$, $\hat{\theta}1 = 1$. Take any $w \in \mathfrak{B}$ which is neither 0 nor 1; then there is an involution in $\mathrm{Aut}\,\mathfrak{B}$ with support w (381Q again); the corresponding member π of H is still an involution with support w. Its image $\hat{q}^{-1}(\pi)$ in G is an involution with support $a = \hat{\theta}^{-1}w \in \widehat{\mathfrak{A}}$; of course $0 \neq a \neq 1$. Take non-zero u_1, $u_3 \in \mathfrak{A}$ such that $u_1 \subset a$ and $u_3 \subseteq 1 \setminus a$; set $u_2 = 1 \setminus (u_1 \cup u_3)$. Because $\mathfrak{A}$ is homogeneous, there are ϕ, $\psi \in G$ such that $\phi u_1 = u$, $\psi u_1 = u_1$, $\psi u_2 = u_3$; set $\phi_2 = \phi\psi$. Then we have

$$u = \phi u_1 \subseteq \phi(\mathrm{supp}\,\hat{q}^{-1}(\pi)) = \mathrm{supp}(\phi\hat{q}^{-1}(\pi)\phi^{-1}) \subseteq \phi(u_1 \cup u_2) = u \cup \phi u_2,$$

$$u = \phi_2 u_1 \subseteq \phi_2(\mathrm{supp}\,\hat{q}^{-1}(\pi)) = \mathrm{supp}(\phi_2\hat{q}^{-1}(\pi)\phi_2^{-1}) \subseteq u \cup \phi_2 u_2 = u \cup \phi u_3,$$

so

$$\phi(\mathrm{supp}\,\hat{q}^{-1}(\pi)) \cap \phi_2(\mathrm{supp}\,\hat{q}^{-1}(\pi)) = u,$$

and

$$\begin{aligned}
\hat{\theta}u &= \hat{\theta}(\phi(\mathrm{supp}\,\hat{q}^{-1}(\pi))) \cap \hat{\theta}(\phi_2(\mathrm{supp}\,\hat{q}^{-1}(\pi))) \\
&= \hat{\theta}(\mathrm{supp}\,\phi\hat{q}^{-1}(\pi)\phi^{-1}) \cap \hat{\theta}(\mathrm{supp}\,\phi_2\hat{q}^{-1}(\pi)\phi_2^{-1}) \\
&= \hat{\theta}(\mathrm{supp}\,\hat{q}^{-1}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1})) \cap \hat{\theta}(\mathrm{supp}\,\hat{q}^{-1}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1})) \\
&= \hat{\theta}\hat{\theta}^{-1}(\mathrm{supp}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1})) \cap \hat{\theta}\hat{\theta}^{-1}(\mathrm{supp}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1})) \\
&= \mathrm{supp}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1}) \cap \mathrm{supp}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1}) \\
&= \hat{q}(\phi)(\mathrm{supp}\,\pi) \cap \hat{q}(\phi_2)(\mathrm{supp}\,\pi) = \hat{q}(\phi)w \cap \hat{q}(\phi_2)w \in \mathfrak{B}
\end{aligned}$$

because both $\hat{q}(\phi)$ and $\hat{q}(\phi_2)$ belong to H. **Q**

Similarly, $\hat{\theta}^{-1}v \in \mathfrak{A}$ for every $v \in \mathfrak{B}$, and $\theta = \hat{\theta}{\restriction}\mathfrak{A}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

We now have

$$q(\phi) = \hat{q}(\hat{\phi}){\restriction}\mathfrak{B} = (\hat{\theta}\hat{\phi}\hat{\theta}^{-1}){\restriction}\mathfrak{B} = \theta\phi\theta^{-1}$$

for every $\phi \in \mathrm{Aut}\,\mathfrak{A}$. Finally, θ is unique by 383C, as before.

383F Corollary If $\mathfrak{A}$ and $\mathfrak{B}$ are atomless homogeneous Boolean algebras with isomorphic automorphism groups, they are isomorphic as Boolean algebras.

Remark Of course a one-element Boolean algebra $\{0\}$ and a two-element Boolean algebra $\{0, 1\}$ have isomorphic automorphism groups without being isomorphic.

383G Corollary If $\mathfrak{A}$ is a homogeneous Boolean algebra, then $\mathrm{Aut}\,\mathfrak{A}$ has no outer automorphisms.

proof If $\mathfrak{A} = \{0, 1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless, so if q is any automorphism of $\mathrm{Aut}\,\mathfrak{A}$, there is a Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \mathrm{Aut}\,\mathfrak{A}$, and q is an inner automorphism.

383H Definitions Complementary to the notion of 'many involutions' is the following concept.

(a) A Boolean algebra $\mathfrak{A}$ is **rigid** if the only automorphism of $\mathfrak{A}$ is the identity automorphism.

(b) A Boolean algebra $\mathfrak{A}$ is **nowhere rigid** if no non-trivial principal ideal of $\mathfrak{A}$ is rigid.

383I Lemma Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is nowhere rigid;

(ii) for every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, not the identity, supported by a;

(iii) for every $a \in \mathfrak{A} \setminus \{0\}$ there are distinct non-zero b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ they generate are isomorphic;

(iv) the automorphism group $\operatorname{Aut}\mathfrak{A}$ has many involutions.

proof (a)(ii)⇒(i) If $a \in \mathfrak{A} \setminus \{0\}$, let $\phi \in \operatorname{Aut}\mathfrak{A}$ be a non-trivial automophism supported by a; then $\phi\restriction\mathfrak{A}_a$ is a non-trivial automorphism of the principal ideal $\mathfrak{A}_a$, so $\mathfrak{A}_a$ is not rigid.

(b)(i)⇒(iii) There is a non-trivial automorphism ψ of $\mathfrak{A}_a$; now if $b \in \mathfrak{A}_a$ is such that $\psi b = c \neq b$, $\mathfrak{A}_b$ is isomorphic to $\phi[\mathfrak{A}_b] = \mathfrak{A}_c$.

(c)(iii)⇒(iv) Take any non-zero $a \in \mathfrak{A}$. By (iii), there are distinct b, $c \subseteq a$ such that $\mathfrak{A}_b$, $\mathfrak{A}_c$ are isomorphic. At least one of $b \setminus c$, $c \setminus b$ is non-zero; suppose the former. Let $\psi : \mathfrak{A}_b \to \mathfrak{A}_c$ be an isomorphism, and set $d = b \setminus c$, $d' = \psi(b \setminus c)$; then $d' \subseteq c$, so $d' \cap d = 0$, and $\phi = (\overleftarrow{d \,{}_{\psi}\, d'})$ is an involution supported by a.

(d)(iv)⇒(ii) is trivial.

383J Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Dedekind complete Boolean algebras and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof Put 383I(i)⇒(iv) and 383D together.

383K Corollary Let $\mathfrak{A}$ be a nowhere rigid Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

383L Examples I note the following examples of nowhere rigid algebras.

(a) An atomless homogeneous Boolean algebra is nowhere rigid.
(b) Any principal ideal of a nowhere rigid Boolean algebra is nowhere rigid.
(c) A simple product of nowhere rigid Boolean algebras is nowhere rigid.
(d) Any atomless semi-finite measure algebra is nowhere rigid.
(e) A free product of nowhere rigid Boolean algebras is nowhere rigid.
(f) The Dedekind completion of a nowhere rigid Boolean algebra is nowhere rigid.

Indeed, the difficulty is to find an atomless Boolean algebra which is not nowhere rigid; for a variety of constructions of rigid algebras, see Bekkali & Bonnet 89.

383M Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be atomless localizable measure algebras, and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ the corresponding groups of measure-preserving automorphisms. Let $q : \operatorname{Aut}_{\bar\mu}\mathfrak{A} \to \operatorname{Aut}_{\bar\nu}\mathfrak{B}$ be an isomorphism. Then there is a Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

proof The point is just that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has many involutions. **P** Let $a \in \mathfrak{A} \setminus \{0\}$. Then there is a non-zero $b \subseteq a$ such that the principal ideal $\mathfrak{A}_b$ is Maharam-type-homogeneous. Take $c \subseteq b$, $d \subseteq b \setminus c$ such that $\bar\mu c = \bar\mu d = \min(1, \frac{1}{2}\bar\mu b)$ (331C). The principal ideals $\mathfrak{A}_c$, $\mathfrak{A}_d$ are now isomorphic as measure algebras (331I); let $\psi : \mathfrak{A}_c \to \mathfrak{A}_d$ be a measure-preserving isomorphism. Then $(\overleftarrow{c \,{}_{\psi}\, d}) \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is an involution supported by a. **Q**

Similarly, $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ has many involutions, and the result follows at once from 383D.

383N To make proper use of the last theorem we need the following result.

Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ (332G) and for each $\gamma \in]0, \infty[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Then the following are equiveridical:

(i) for every $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar\nu}\mathfrak{B}$;

(ii)(α) for every infinite cardinal κ there is an $\alpha_\kappa > 0$ such that $\bar\nu(\theta a) = \alpha_\kappa\bar\mu a$ for every $a \subseteq e_\kappa$ (β) for every $\gamma \in \,]0,\infty[$ there is an $\alpha_\gamma > 0$ such that $\bar\nu(\theta a) = \alpha_\gamma\bar\mu a$ for every $a \in A_\gamma$.

proof (a)(i)⇒(ii)(α) Let κ be an infinite cardinal. The point is that if a, $a' \subseteq e_\kappa$ and $\bar\mu a = \bar\mu a' < \infty$ then $\bar\nu(\theta a) = \bar\nu(\theta a')$. **P** The principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{a'}$ are isomorphic as measure algebras; moreover, by 332J, the principal ideals $\mathfrak{A}_{e_\kappa\setminus a}$, $\mathfrak{A}_{e_\kappa\setminus a'}$ are isomorphic. We therefore have a $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ such that $\phi a = a'$. Consequently $\psi\theta a = \theta a'$, where $\psi = \theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar\nu}\mathfrak{B}$, and $\bar\nu(\theta a) = \bar\nu(\theta a')$. **Q**

If $e_\kappa = 0$ we can take $\alpha_\kappa = 1$. Otherwise fix on some $c_0 \subseteq e_\kappa$ such that $0 < \bar\mu c_0 < \infty$; take $b \subseteq \theta c_0$ such that $0 < \bar\nu b < \infty$, and set $c = \theta^{-1}b$, $\alpha_\kappa = \bar\nu b/\bar\mu c$. Then we shall have $\bar\nu(\theta a) = \bar\nu(\theta c) = \alpha_\kappa\bar\mu a$ whenever $a \subseteq e_\kappa$ and $\bar\mu a = \bar\mu c$. But we can find for any $n \ge 1$ a partition $c_{n1}, \dots, c_{nn}$ of c into elements of measure $\frac{1}{n}\bar\mu c$; since $\bar\nu(\theta c_{ni}) = \bar\nu(\theta c_{nj})$ for all i, $j \le n$, we must have $\bar\nu(\theta c_{ni}) = \frac{1}{n}\bar\nu(\theta c) = \alpha_\kappa\bar\mu c_{ni}$ for all i. So if $a \subseteq e_\kappa$ and $\bar\mu a = \frac{1}{n}\bar\mu c$, $\bar\nu(\theta a) = \bar\nu(\theta c_{n1}) = \alpha_\kappa\bar\mu a$. Now suppose that $a \subseteq e_\kappa$ and $\bar\mu a = \frac{k}{n}\bar\mu c$ for some k, $n \ge 1$; then a can be partitioned into k sets of measure $\frac{1}{n}\bar\mu c$, so in this case also $\bar\nu(\theta a) = \alpha_\kappa\bar\mu a$. Finally, for any $a \subseteq e_\kappa$, set

$$D = \{d : d \subseteq a,\ \bar\mu d \text{ is a rational multiple of } \bar\mu c\},$$

and let $D' \subseteq D$ be a maximal upwards-directed set. Then $\sup D' = a$, so $\theta[D']$ is an upwards-directed set with supremum θa, and

$$\bar\nu(\theta a) = \sup_{d\in D'}\bar\nu(\theta d) = \sup_{d\in D'}\alpha_\kappa\bar\mu d = \alpha_\kappa\bar\mu a.$$

(β) Let $\gamma \in \,]0,\infty[$. If $A_\gamma = \emptyset$ take $\alpha_\gamma = 1$. Otherwise, fix on any $c \in A_\gamma$ and set $\alpha_\gamma = \bar\nu(\theta c)/\gamma$. If $a \in A_\gamma$ then there is a $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ exchanging the atoms a, c, so that $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar\nu}\mathfrak{B}$ exchanges the atoms θa, θc, and

$$\bar\nu(\theta a) = \bar\nu(\theta c) = \alpha_\gamma\bar\mu a.$$

(b)(ii)⇒(i) Now suppose that the conditions (α) and (β) are satisfied, that $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and that $a \in \mathfrak{A}$. For each infinite cardinal κ, we have $\phi e_\kappa = e_\kappa$, so

$$\bar\nu(\theta\phi(e_\kappa \cap a)) = \alpha_\kappa\bar\mu(\phi(e_\kappa \cap a)) = \alpha_\kappa\bar\mu(e_\kappa \cap a) = \bar\nu(\theta(e_\kappa \cap a)).$$

Similarly, if we write $a_\gamma = \sup A_\gamma$, then for each $\gamma \in \,]0,\infty[$ we have $\phi[A_\gamma] = A_\gamma$, $\phi a_\gamma = a_\gamma$, and for $c \subseteq a_\gamma$ we have

$$\bar\mu c = \gamma\#(\{e : e \in A_\gamma,\ e \subseteq c\});$$

so

$$\begin{aligned}\bar\nu(\theta\phi(a_\gamma \cap a)) &= \alpha_\gamma\gamma\#(\{e : e \in A_\gamma,\ e \subseteq \phi a\})\\ &= \alpha_\gamma\gamma\#(\{e : e \in A_\gamma,\ e \subseteq a\})\\ &= \sum_{e\in A_\gamma, e\subseteq a}\bar\nu(\theta e) = \bar\nu(\theta(a_\gamma \cap a)).\end{aligned}$$

Putting these together,

$$\begin{aligned}\bar\nu(\theta\phi a) &= \sum_{\kappa \text{ is an infinite cardinal}}\bar\nu(\theta\phi(e_\kappa \cap a)) + \sum_{\gamma\in]0,\infty[}\bar\nu(\theta\phi(a_\gamma \cap a))\\ &= \sum_{\kappa \text{ is an infinite cardinal}}\bar\nu(\theta(e_\kappa \cap a)) + \sum_{\gamma\in]0,\infty[}\bar\nu(\theta(a_\gamma \cap a))\\ &= \bar\nu(\theta a).\end{aligned}$$

But this means that

$$\bar\nu(\theta\phi\theta^{-1}b) = \bar\nu(\theta\theta^{-1}b) = \bar\nu b$$

for every $b \in \mathfrak{B}$, and $\theta\phi\theta^{-1}$ is measure-preserving, as required by (i).

383O Corollary If $(\mathfrak{A},\bar\mu)$ is an atomless totally finite measure algebra, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has no outer automorphisms.

proof Let $q:\operatorname{Aut}_{\bar\mu}\mathfrak{A}\to\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ be any automorphism. By 383M, there is a corresponding $\theta\in\operatorname{Aut}\mathfrak{A}$ such that $q(\phi)=\theta\phi\theta^{-1}$ for every $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. By 383N, there is for each infinite cardinal κ an $\alpha_\kappa>0$ such that $\bar\mu(\theta a)=\alpha_\kappa\bar\mu a$ whenever $a\subseteq e_\kappa$, the Maharam-type-κ component of $\mathfrak{A}$. But since $\theta e_\kappa=e_\kappa$ and $\bar\mu e_\kappa<\infty$ for every κ, we must have $\alpha_\kappa=1$ whenever $e_\kappa\ne0$; as $\mathfrak{A}$ is atomless,

$$\begin{aligned}\bar\mu(\theta a)&=\sum_{\kappa\text{ is an infinite cardinal}}\bar\mu(\theta(a\cap e_\kappa))\\&=\sum_{\kappa\text{ is an infinite cardinal}}\alpha_\kappa\bar\mu(a\cap e_\kappa)\\&=\sum_{\kappa\text{ is an infinite cardinal}}\bar\mu(a\cap e_\kappa)=\bar\mu a\end{aligned}$$

for every $a\in\mathfrak{A}$. Thus $\theta\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and q is an inner automorphism.

383P The results above are satisfying and complete in their own terms, but leave open a number of obvious questions concerning whether some of the hypotheses can be relaxed. Atoms can produce a variety of complications (see 383Ya-383Yd below). To show that we really do need to assume that our algebras are Dedekind complete or localizable, I offer the following.

Example (a) There are an atomless localizable measure algebra $(\mathfrak{A},\bar\mu)$ and an atomless semi-finite measure algebra $(\mathfrak{B},\bar\nu)$ such that $\operatorname{Aut}\mathfrak{A}\cong\operatorname{Aut}\mathfrak{B}$, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}\cong\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ but $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

proof Let $(\mathfrak{A}_0,\bar\mu_0)$ be an atomless homogeneous probability algebra; for instance, the measure algebra of Lebesgue measure on the unit interval. Let $\mathfrak{A}$ be the simple product Boolean algebra $\mathfrak{A}_0^{\omega_1}$, and $\bar\mu$ the corresponding measure (322K); then $(\mathfrak{A},\bar\mu)$ is an atomless localizable measure algebra. In $\mathfrak{A}$ let I be the set

$$\{a:a\in\mathfrak{A}\text{ and the principal ideal }\mathfrak{A}_a\text{ is ccc}\};$$

then I is an ideal of $\mathfrak{A}$, the σ-ideal generated by the elements of finite measure (cf. 322G). Set

$$\mathfrak{B}=\{a:a\in\mathfrak{A},\text{ either }a\in I\text{ or }1\setminus a\in I\}.$$

Then $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, so if we set $\bar\nu=\bar\mu\restriction\mathfrak{B}$ then $(\mathfrak{B},\bar\nu)$ is a measure algebra in its own right.

The definition of I makes it plain that it is invariant under all Boolean automorphisms of $\mathfrak{A}$; so $\mathfrak{B}$ is also invariant under all automorphisms, and we have a homomorphism $\phi\mapsto q(\phi)=\phi\restriction\mathfrak{B}:\operatorname{Aut}\mathfrak{A}\to\operatorname{Aut}\mathfrak{B}$. On the other hand, because $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{A}$ is Dedekind complete, every automorphism of $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{A}$ (see part (a) of the proof of 383E). So q is actually an isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$. Moreover, still because $\mathfrak{B}$ is order-dense, $q(\phi)$ is measure-preserving iff ϕ is measure-preserving, so $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is isomorphic to $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$. But of course there is no Boolean isomorphism, let alone a measure algebra isomorphism, between $\mathfrak{A}$ and $\mathfrak{B}$, because $\mathfrak{A}$ is Dedekind complete while $\mathfrak{B}$ is not.

Remark Thus the hypothesis 'Dedekind complete' in 383D and 383J (and 'localizable' in 383M), and the hypothesis 'homogeneous' in 383E-383F, are essential.

(b) There is an atomless semi-finite measure algebra $(\mathfrak{C},\bar\lambda)$ such that $\operatorname{Aut}\mathfrak{C}$ has an outer automorphism.

proof In fact we can take $\mathfrak{C}$ to be the simple product of $\mathfrak{A}$ and $\mathfrak{B}$ above. I claim that the isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ gives rise to an outer automorphism of $\operatorname{Aut}\mathfrak{C}$; this seems very natural, but I think there is a fair bit to check, so I take the argument in easy stages.

(i) We may identify the Dedekind completion of $\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}$ with $\mathfrak{A}\times\mathfrak{A}$. For $\phi\in\operatorname{Aut}\mathfrak{C}$, we have a corresponding $\hat\phi\in\operatorname{Aut}(\mathfrak{A}\times\mathfrak{A})$. Now $\mathfrak{B}\times\mathfrak{A}$ is invariant under $\hat\phi$. **P** Consider first $\phi(0,1)=(a_1,b_1)\in\mathfrak{C}$. The corresponding principal ideal $\mathfrak{C}_{(a_1,b_1)}\cong\mathfrak{A}_{a_1}\times\mathfrak{B}_{b_1}$ of $\mathfrak{C}$ must be isomorphic to the principal ideal $\mathfrak{C}_{(0,1)}\cong\mathfrak{B}$; so that if $(a,b)\in\mathfrak{C}$ and $(a,b)\subseteq(a_1,b_1)$, then just one of the principal ideals $\mathfrak{C}_{(a,b)}\cong\mathfrak{A}_a\times\mathfrak{B}_b$, $\mathfrak{C}_{(a_1\setminus a,b_1\setminus b)}\cong\mathfrak{A}_{a_1\setminus a}\times\mathfrak{B}_{b_1\setminus b}$ is ccc. But this can only happen if $\mathfrak{A}_{a_1}$ is ccc and $\mathfrak{B}_{b_1}$ is not; that is, if a_1 and $1\setminus b_1$ belong to I. Consequently $\hat\phi(0,a)\subseteq(a_1,b_1)$ belongs to $\mathfrak{B}\times\mathfrak{A}$ for every $a\in\mathfrak{A}$. We also find that

$$\phi(1,0)=(1,1)\setminus\phi(0,1)=(1\setminus a_1,1\setminus b_1)\in\mathfrak{B}\times\mathfrak{A}.$$

Now if $b \in I$, then

$$\mathfrak{C}_{\phi(b,0)} \cong \mathfrak{C}_{(b,0)} \cong \mathfrak{A}_b$$

is ccc and

$$\phi(b,0) \in I \times I \subseteq \mathfrak{B} \times \mathfrak{A};$$

while

$$\phi(1 \setminus b, 0) = (1 \setminus a_1, 1 \setminus b_1) \setminus \phi(b,0) \in \mathfrak{B} \times \mathfrak{A}.$$

This shows that $\phi(b,0) \in \mathfrak{B} \times \mathfrak{A}$ for every $b \in \mathfrak{B}$. So

$$\hat{\phi}(b,a) = \hat{\phi}(b,0) \cup \hat{\phi}(0,a) \in \mathfrak{B} \times \mathfrak{A}$$

for every $b \in \mathfrak{B}$, $a \in \mathfrak{A}$. **Q**

(ii) Let $\theta : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ be the involution defined by setting $\theta(a,b) = (b,a)$ for all a, $b \in \mathfrak{A}$. Take $\phi \in \operatorname{Aut}\mathfrak{C}$ and consider $\psi = \theta\hat{\phi}\theta^{-1} \in \operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$. If $c = (a,b) \in \mathfrak{C}$, then $\theta^{-1}c = (b,a) \in \mathfrak{B} \times \mathfrak{A}$, so $\hat{\phi}\theta^{-1}c \in \mathfrak{B} \times \mathfrak{A}$, by (i), and $\psi c \in \mathfrak{A} \times \mathfrak{B} = \mathfrak{C}$. This shows that $\psi{\restriction}\mathfrak{C}$ is a homomorphism from $\mathfrak{C}$ to itself. Of course $\psi^{-1} = \theta\hat{\phi}^{-1}\theta^{-1}$ has the same property. So we have a map $q : \operatorname{Aut}\mathfrak{C} \to \operatorname{Aut}\mathfrak{C}$ given by setting

$$q(\phi) = \theta\hat{\phi}\theta^{-1}{\restriction}\mathfrak{C}$$

for $\phi \in \operatorname{Aut}\mathfrak{C}$. Evidently q is an automorphism.

(iii) ? Suppose, if possible, that q were an inner automorphism. Let $\chi \in \operatorname{Aut}\mathfrak{C}$ be such that $q(\phi) = \chi\phi\chi^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{C}$. Then

$$\hat{\chi}\hat{\phi}\hat{\chi}^{-1} = \widehat{q(\phi)} = \theta\hat{\phi}\theta^{-1}$$

for every $\phi \in \operatorname{Aut}\mathfrak{C}$. Since $G = \{\hat{\phi} : \phi \in \operatorname{Aut}\mathfrak{C}\}$ is a subgroup of $\operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$ with many involutions, the 'uniqueness' assertion of 383D tells us that $\hat{\chi} = \theta$. But

$$\theta[\mathfrak{C}] = \mathfrak{B} \times \mathfrak{A} \neq \mathfrak{C} = \chi[\mathfrak{C}] = \hat{\chi}[\mathfrak{C}],$$

so this cannot be. **X**

Thus q is the required outer automorphism of $\operatorname{Aut}\mathfrak{C}$.

Remark Thus the hypothesis 'homogeneous' in 383E, and the hypothesis 'Dedekind complete' in 383J, are necessary.

383Q Example Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Then $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has an outer automorphism. **P** Set $f(x) = 2x$ for $x \in \mathbb{R}$. Then $E \mapsto f^{-1}[E] = \frac{1}{2}E$ is a Boolean automorphism of the domain Σ of μ, and $\mu(\frac{1}{2}E) = \frac{1}{2}\mu E$ for every $E \in \Sigma$ (263A, or otherwise). So we have a corresponding $\theta \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\theta E^\bullet = (\frac{1}{2}E)^\bullet$ for every $E \in \Sigma$, and $\bar{\mu}(\theta a) = \frac{1}{2}\bar{\mu}a$ for every $a \in \mathfrak{A}$. By 383N, we have an automorphism q of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ defined by setting $q(\phi) = \theta\phi\theta^{-1}$ for every measure-preserving automorphism ϕ. But q is now an outer automorphism of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, because (by 383D) the only possible automorphism of $\mathfrak{A}$ corresponding to q is θ, and θ is not measure-preserving. **Q**

Thus the hypothesis 'totally finite' in 383O cannot be omitted.

383X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is nowhere rigid; (ii) for every $a \in \mathfrak{A} \setminus \{0\}$ and $n \in \mathbb{N}$ there are disjoint non-zero $b_0, \dots, b_n \subseteq a$ such that the principal ideals $\mathfrak{A}_{b_i}$ they generate are all isomorphic; (iii) for every $a \in \mathfrak{A} \setminus \{0\}$ and $n \geq 1$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, of order n, supported by a.

(b) Let $\mathfrak{A}$ be a nowhere rigid Boolean algebra. Show that its Dedekind completion is nowhere rigid.

(c) Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra and $\mathfrak{B}$ a nowhere rigid Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ is isomorphic to $\operatorname{Aut}\mathfrak{B}$. Show that there is an invariant order-dense subalgebra of $\mathfrak{B}$ which is isomorphic to $\mathfrak{A}$.

(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Boolean algebras. Show that if $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ are isomorphic, then the Dedekind completions $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{B}}$ are isomorphic.

(e) Find two non-isomorphic atomless totally finite measure algebras $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ such that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ are isomorphic. (This is easy.)

(f) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras and $\theta:\mathfrak{A}\to\mathfrak{B}$ a Boolean isomorphism. Show that the following are equiveridical: (i) for every $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\theta\phi\theta^{-1}\in\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$; (ii)($\alpha$) for every infinite cardinal κ there is an $\alpha_\kappa>0$ such that $\bar{\nu}(\theta a)=\alpha_\kappa\bar{\mu}a$ whenever $a\in\mathfrak{A}$ and the principal ideal $\mathfrak{A}_a$ is Maharam-type-homogeneous with Maharam type κ; (β) for every $\gamma\in\,]0,\infty[$ there is an $\alpha_\gamma>0$ such that $\bar{\nu}(\theta a)=\alpha_\gamma\bar{\mu}a$ whenever $a\in\mathfrak{A}$ is an atom of measure γ.

(g) Let $q:\operatorname{Aut}\mathfrak{C}\to\operatorname{Aut}\mathfrak{C}$ be the automorphism of 383Pb. Show that $q(\phi)$ is measure-preserving whenever ϕ is measure-preserving, so that $q\restriction\operatorname{Aut}_{\bar{\lambda}}\mathfrak{C}$ is an outer automorphism of $\operatorname{Aut}_{\bar{\lambda}}\mathfrak{C}$.

383Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be localizable measure algebras such that $\operatorname{Aut}\mathfrak{A}\cong\operatorname{Aut}\mathfrak{B}$. Show that *either* $\mathfrak{A}\cong\mathfrak{B}$ *or* one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom and the other is atomless.

(b) Let $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ be localizable measure algebras such that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}\cong\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$. Show that *either* $(\mathfrak{A},\bar{\mu})\cong(\mathfrak{B},\bar{\nu})$ *or* there is some $\gamma\in\,]0,\infty[$ such that one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom of measure γ and the other has none *or* there are γ, $\gamma'\in\,]0,\infty[$ such that the number of atoms of $\mathfrak{A}$ of measure γ is equal to the number of atoms of $\mathfrak{B}$ of measure γ', but not to the number of atoms of $\mathfrak{A}$ of measure γ'.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Show that there is an outer automorphism of $\operatorname{Aut}\mathfrak{A}$ iff $\mathfrak{A}$ has exactly six atoms.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ and for each $\gamma\in\,]0,\infty[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Show that there is an outer automorphism of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ iff

either there is an infinite cardinal κ such that $\bar{\mu}e_\kappa=\infty$
or there are distinct γ, $\delta\in\,]0,\infty[$ such that $\#(A_\gamma)=\#(A_\delta)\ge 2$
or there is a $\gamma\in\,]0,\infty[$ such that $\#(A_\gamma)=6$
or there are γ, $\delta\in\,]0,\infty[$ such that $\#(A_\gamma)=2<\#(A_\delta)<\omega$.

383 Notes and comments Let me recapitulate the results above. If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras, any isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if *either* $\mathfrak{A}$ and $\mathfrak{B}$ are atomless and homogeneous (383E) *or* they are nowhere rigid and Dedekind complete (383J). If $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are atomless localizable measure algebras, then any automorphism between $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ (383M) which if $\bar{\mu}=\bar{\nu}$ is totally finite will be measure-preserving (383O).

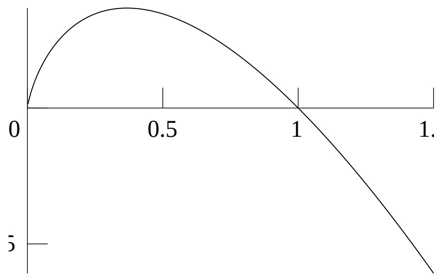
These results may appear a little less surprising if I remark that the elementary Boolean algebras $\mathcal{P}X$ give rise to some of the same phenomena. The automorphism group of $\mathcal{P}X$ can be identified with the group S_X of all permutations of X, and this has no outer automorphisms unless X has just six elements. Some of the ideas of the fundamental theorem 383D can be traced through in the purely atomic case also, though of course there are significant changes to be made, and some serious complications arise, of which the most striking surround the remarkable fact that S_6 *does* have an outer automorphism (Burnside 11, §162; Rotman 84, Theorem 7.8). I have not attempted to incorporate these in the main results. For localizable measure algebras, where the only rigid parts are atoms, the complications are superable, and I think I have listed them all (383Ya-383Yd).

384 Entropy

Perhaps the most glaring problem associated with the theory of measure-preserving homomorphisms and automorphisms is the fact that we have no generally effective method of determining when two homomorphisms are the same, in the sense that two structures $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are isomorphic, where $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}$, $\phi:\mathfrak{B}\to\mathfrak{B}$ are Boolean homomorphisms. Of course the first part of the problem is to decide whether $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are isomorphic; but this is solved (at least for localizable algebras) by Maharam's theorem (see 332J). The difficulty lies in the homomorphisms. Even when we know that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are both isomorphic to the Lebesgue measure algebra, the extraordinary variety of constructions of homomorphisms – corresponding in part to the variety of measure spaces with such measure algebras, each with its own natural inverse-measure-preserving functions – means that the question of which are isomorphic to each other is continually being raised. In this section I give the most elementary ideas associated with the concept of 'entropy', up to the Kolmogorov-Sinaĭ theorem (384P). This is an invariant which can be attached to any measure-preserving homomorphism on a probability algebra, and therefore provides a useful method for distinguishing non-isomorphic homomorphisms.

The main work of the section deals with homomorphisms on measure algebras, but as many of the most important ones arise from inverse-measure-preserving functions on measure spaces I comment on the extra problems arising in the isomorphism problem for such functions (384T-384V). I should remark that some of the lemmas will be repeated in stronger forms in the next section.

384A Notation Throughout this section, I will use the letter q to denote the function from $[0,\infty[$ to $\mathbb{R}$ defined by saying that $q(t)=-t\ln t=t\ln\frac{1}{t}$ if $t>0$, $q(0)=0$.



The function q

We shall need the following straightforward facts concerning q.

(a) q is continuous on $[0,\infty[$ and differentiable on $]0,\infty[$; $q'(t)=-1-\ln t$ and $q''(t)=-\frac{1}{t}$ for $t>0$. Because q'' is negative, q is concave, that is, $-q$ is convex. q has a unique maximum at $(\frac{1}{e},\frac{1}{e})$.

(b) If $s\ge 0$, $t>0$ then $q'(s+t)\le q'(t)$; consequently

$$q(s+t)=q(s)+\int_0^t q'(s+\tau)d\tau\le q(s)+q(t)$$

for s, $t\ge 0$. It follows that $q(\sum_{i=0}^n s_i)\le\sum_{i=0}^n q(s_i)$ for all $s_0,\dots,s_n\ge 0$ and (because q is continuous) $q(\sum_{i=0}^\infty s_i)\le\sum_{i=0}^\infty q(s_i)$ for every non-negative summable series $\langle s_i\rangle_{i\in\mathbb{N}}$.

(c) If s, $t\ge 0$ then $q(st)=sq(t)+tq(s)$; more generally, if $n\ge 1$ and $s_i\ge 0$ for $i\le n$ then

$$q(\prod_{i=0}^n s_i)=\sum_{j=0}^n q(s_j)\prod_{i\ne j}s_i.$$

(d) The function $t\mapsto q(t)+q(1-t)$ has a unique maximum at $(\frac{1}{2},\ln 2)$. $(\frac{d}{dt}(q(t)+q(1-t))=\ln\frac{1-t}{t}.)$ It follows that for every $\epsilon>0$ there is a $\delta>0$ such that $|t-\frac{1}{2}|\le\epsilon$ whenever $q(t)+q(1-t)\ge\ln 2-\delta$.

(e) If $0\le t\le\frac{1}{2}$, then $q(1-t)\le q(t)$. **P** Set $f(t)=q(t)-q(1-t)$. Then

$$f''(t)=-\frac{1}{t}+\frac{1}{1-t}=\frac{2t-1}{t(1-t)}\le 0$$

for $0<t\le\frac{1}{2}$, while $f(0)=f(\frac{1}{2})=0$, so $f(t)\ge 0$ for $0\le t\le\frac{1}{2}$. **Q**

(f) (i) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, I will write $\bar{q}$ for the function from $L^0(\mathfrak{A})^+$ to $L^0(\mathfrak{A})$ defined from q (364I). Note that because $0 \le q(t) \le 1$ for $t \in [0,1]$, $0 \le \bar{q}(u) \le \chi 1$ if $0 \le u \le \chi 1$.

(ii) By (b), $\bar{q}(u+v) \le \bar{q}(u) + \bar{q}(v)$ for all u, $v \ge 0$ in $L^0(\mathfrak{A})$. (Represent $\mathfrak{A}$ as the measure algebra of a measure space, so that $\bar{q}(f^\bullet) = (qf)^\bullet$, as in 364Jb.)

(iii) Similarly, if u, $v \in L^0(\mathfrak{A})^+$, then $\bar{q}(u \times v) = u \times \bar{q}(v) + v \times \bar{q}(u)$.

384B Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{A}, \bar{\mu})$ the corresponding conditional expectation operator (365R). Then $\int \bar{q}(u) \le q(\int u)$ and $P(\bar{q}(u)) \le \bar{q}(Pu)$ for every $u \in L^\infty(\mathfrak{A})^+$.

proof Apply the remarks in 365Rb to $-q$. ($\bar{q}(u) \in L^\infty \subseteq L^1$ for every $u \in (L^\infty)^+$ because q is bounded on every bounded interval in $[0, \infty[$.)

384C Definition Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. If A is a partition of unity in $\mathfrak{A}$, its **entropy** is $H(A) = \sum_{a \in A} q(\bar{\mu}a)$, where q is the function defined in 384A.

Remarks (a) In the definition of 'partition of unity' (311Gc) I allowed 0 to belong to the family. In the present context this is a mild irritant, and when convenient I shall remove 0 from the partitions of unity considered here (as in 384F below). But because $q(0) = 0$, it makes no difference; $H(A) = H(A \setminus \{0\})$ whenever A is a partition of unity. So if you wish you can read 'partition of unity' in this section to mean 'partition of unity not containing 0', if you are willing to make an occasional amendment in a formula. In important cases, in fact, A is of the form $\{a_i : i \in I\}$ or $\{a_i : i \in I\} \setminus \{0\}$, where $\langle a_i \rangle_{i \in I}$ is an indexed partition of unity, with $a_i \cap a_j = 0$ for $i \ne j$, but no restriction in the number of i with $a_i = 0$; in this case, we still have $H(A) = \sum_{i \in I} q(\bar{\mu}a_i)$.

(b) Many authors prefer to use $\log_2$ in place of ln. This makes sense in terms of one of the intuitive approaches to entropy as the 'information' associated with a partition. See Petersen 83, §5.1.

384D Definition Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Let $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{A}, \bar{\mu})$ be the conditional expectation operator associated with $\mathfrak{B}$. Then the **conditional entropy of** A **on** $\mathfrak{B}$ is

$$H(A|\mathfrak{B}) = \textstyle\sum_{a \in A} \int \bar{q}(P\chi a),$$

where $\bar{q}$ is defined as in 384A.

384E Elementary remarks (a) In the formula

$$\textstyle\sum_{a \in A} \int \bar{q}(P\chi a),$$

we have $0 \le P(\chi a) \le \chi 1$ for every a, so $\bar{q}(P\chi a) \ge 0$ and every term in the sum is non-negative; accordingly $H(A|\mathfrak{B})$ is well-defined in $[0, \infty]$.

(b) $H(A) = H(A|\{0,1\})$, since if $\mathfrak{B} = \{0,1\}$ then $P(\chi a) = \bar{\mu}a\chi 1$, so that $\int \bar{q}(P\chi a) = q(\bar{\mu}a)$. If $A \subseteq \mathfrak{B}$, $H(A|\mathfrak{B}) = 0$, since $P(\chi a) = \chi a$, $\bar{q}(P\chi a) = 0$ for every a.

384F Definition If $\mathfrak{A}$ is a Boolean algebra and A, $B \subseteq \mathfrak{A}$ are partitions of unity, I write $A \vee B$ for the partition of unity $\{a \cap b : a \in A, b \in B\} \setminus \{0\}$. (See 384Xq.)

384G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra. Let $A \subseteq \mathfrak{A}$ be a partition of unity.

(a) If B is another partition of unity in $\mathfrak{A}$, then

$$H(A|\mathfrak{B}) \le H(A \vee B|\mathfrak{B}) \le H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(b) If $\mathfrak{B}$ is purely atomic and D is the set of its atoms, then $H(A \vee D) = H(D) + H(A|\mathfrak{B})$.

(c) If $\mathfrak{C} \subseteq \mathfrak{B}$ is another closed subalgebra of $\mathfrak{A}$, then $H(A|\mathfrak{C}) \ge H(A|\mathfrak{B})$. In particular, $H(A) \ge H(A|\mathfrak{B})$.

(d) Suppose that $\langle \mathfrak{B}_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{B} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n}$. If $H(A) < \infty$ then

$$H(A|\mathfrak{B}) = \lim_{n\to\infty} H(A|\mathfrak{B}_n).$$

In particular, if $A \subseteq \mathfrak{B}$ then $\lim_{n\to\infty} H(A|\mathfrak{B}_n) = 0$.

proof Write P for the conditional expectation operator on $L^1(\mathfrak{A}, \bar{\mu})$ associated with $\mathfrak{B}$.

(a)(i) If B is infinite, enumerate it as $\langle b_j\rangle_{j\in\mathbb{N}}$; if it is finite, enumerate it as $\langle b_j\rangle_{j\le n}$ and set $b_j = 0$ for $j > n$. For any $a \in A$,

$$\chi a = \sum_{j=0}^{\infty} \chi(a \cap b_j), \quad P(\chi a) = \sum_{j=0}^{\infty} P\chi(a \cap b_j),$$

$$\begin{aligned}\bar{q}(P\chi a) &= \lim_{n\to\infty} \bar{q}(\sum_{j=0}^{n} P\chi(a \cap b_j))\\ &\le \lim_{n\to\infty} \sum_{j=0}^{n} \bar{q}(P\chi(a \cap b_j)) = \sum_{j=0}^{\infty} \bar{q}(P\chi(a \cap b_j))\end{aligned}$$

where all the infinite sums are to be regarded as order*-limits of the corresponding finite sums (see §367), and the middle inequality is a consequence of 384Af(ii). Accordingly

$$\begin{aligned}H(A \vee B|\mathfrak{B}) &= \sum_{a\in A, b\in B, a\cap b\ne 0} \int \bar{q}(P\chi(a \cap b))\\ &= \sum_{a\in A} \sum_{j=0}^{\infty} \int \bar{q}(P\chi(a \cap b_j)) \ge \sum_{a\in A} \int \bar{q}(P\chi a_i) = H(A|\mathfrak{B}).\end{aligned}$$

(ii) Suppose for the moment that A and B are both finite. For $a \in \mathfrak{A}$ set $u_a = P(\chi a)$. If $a, b \in \mathfrak{A}$ we have $0 \le u_{a\cap b} \le u_b$ in $L^0(\mathfrak{B})$, so we may choose $v_{ab} \in L^0(\mathfrak{B})$ such that $0 \le v_{ab} \le \chi 1$ and $u_{a\cap b} = v_{ab} \times u_b$.

For any $b \in B$, $\sum_{a\in A} u_{a\cap b} = u_b$ (because $\sum_{a\in A} \chi(a \cap b) = \chi b$), so $u_b \times \sum_{a\in A} v_{ab} = u_b$. Since $[\![|\bar{q}(u_b)| > 0]\!] \subseteq [\![u_b > 0]\!]$, $\bar{q}(u_b) \times \sum_{a\in A} v_{ab} = \bar{q}(u_b)$.

For any $a \in A$,

$$\bar{q}(u_a) = \bar{q}(\sum_{b\in B} u_{a\cap b}) = \bar{q}(\sum_{b\in B} u_b \times v_{ab}) = \bar{q}(P(\sum_{b\in B} \chi b \times v_{ab}))$$

(because $v_{ab} \in L^0(\mathfrak{B})$ for every b, so $P(\chi b \times v_{ab}) = P(\chi b) \times v_{ab}$)

$$\ge P(\bar{q}(\sum_{b\in B} \chi b \times v_{ab}))$$

(384B)

$$= P(\sum_{b\in B} \chi b \times \bar{q}(v_{ab}))$$

(because B is disjoint)

$$= \sum_{b\in B} u_b \times \bar{q}(v_{ab})$$

(because $\bar{q}(v_{ab}) \in L^0(\mathfrak{B})$ for every b).

Putting these together,

$$\begin{aligned}H(A \vee B|\mathfrak{B}) &= \sum_{a\in A, b\in B} \int \bar{q}(u_{a\cap b}) = \sum_{a\in A, b\in B} \int \bar{q}(u_b \times v_{ab})\\ &= \sum_{a\in A, b\in B} \int u_b \times \bar{q}(v_{ab}) + \sum_{a\in A, b\in B} \int v_{ab} \times \bar{q}(u_b)\\ &\le \sum_{a\in A} \int \bar{q}(u_a) + \sum_{b\in B} \int \bar{q}(u_b) = H(A|\mathfrak{B}) + H(B|\mathfrak{B}).\end{aligned}$$

(iii) For general partitions of unity A and B, take any finite set $C \subseteq A \vee B$. Then $C \subseteq \{a \cap b : a \in A_0,\, b \in B_0\}$ where $A_0 \subseteq A$ and $B_0 \subseteq B$ are finite. Set

$$A' = A_0 \cup \{1 \setminus \sup A_0\}, \quad B' = B_0 \cup \{1 \setminus \sup B_0\},$$

so that A' and B' are finite partitions of unity and $C \subseteq A' \vee B'$. Now

$$\sum_{c \in C} \int \bar{q}(P\chi c) \le \sum_{c \in A' \vee B'} \int \bar{q}(P\chi c) = H(A' \vee B'|\mathfrak{B}) \le H(A'|\mathfrak{B}) + H(B'|\mathfrak{B})$$

(by (ii))

$$\le H(A' \vee A|\mathfrak{B}) + H(B' \vee B|\mathfrak{B})$$

(by (i))

$$= H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

As C is arbitrary,

$$H(A \vee B|\mathfrak{B}) = \sum_{c \in A \vee B} \int \bar{q}(P\chi c) \le H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(b) It follows from 384Ab that $\sum_{d \in D} q(\bar{\mu}(a \cap d)) \ge q(\bar{\mu}a)$ for any $a \in A$.
Now, because $\mathfrak{B}$ is purely atomic and D is its set of atoms,

$$P(\chi a) = \sum_{d \in D} \frac{\bar{\mu}(a \cap d)}{\bar{\mu}d} \chi d, \quad \bar{q}(P(\chi a)) = \sum_{d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \chi d$$

for every $a \in A$,

$$H(A|\mathfrak{B}) = \sum_{a \in A, d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \bar{\mu}d.$$

Putting these together,

$$\begin{aligned} H(A \vee D) &= \sum_{a \in A, d \in D} q(\bar{\mu}(a \cap d)) = \sum_{a \in A, d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \bar{\mu}d + \frac{\bar{\mu}(a \cap d)}{\bar{\mu}d} q(\bar{\mu}d) \\ &= H(A|\mathfrak{B}) + \sum_{d \in D} q(\bar{\mu}d) = H(A|\mathfrak{B}) + H(D). \end{aligned}$$

(c) Write $P_{\mathfrak{C}}$ for the conditional expectation operator corresponding to $\mathfrak{C}$. If $a \in \mathfrak{A}$,

$$\bar{q}(P_{\mathfrak{C}}\chi a) = \bar{q}(P_{\mathfrak{C}}P\chi a) \ge P_{\mathfrak{C}}\bar{q}(P\chi a)$$

by 384B. So

$$H(A|\mathfrak{C}) = \sum_{a \in A} \int \bar{q}(P_{\mathfrak{C}}\chi a) \ge \sum_{a \in A} \int P_{\mathfrak{C}}\bar{q}(P\chi a) = \sum_{a \in A} \int \bar{q}(P\chi a) = H(A|\mathfrak{B}).$$

Taking $\mathfrak{C} = \{0, 1\}$, we get $H(A) \ge H(A|\mathfrak{B})$.

(d) Let P_n be the conditional expectation operator corresponding to $\mathfrak{B}_n$, for each n. Fix $a \in A$. Then $P(\chi a)$ is the order*-limit of $\langle P_n(\chi a)\rangle_{n \in \mathbb{N}}$, by Lévy's martingale theorem (367Kb). Consequently (because q is continuous) $\langle \bar{q}(P_n\chi a)\rangle_{n \in \mathbb{N}}$ is order*-convergent to $\bar{q}(P\chi a)$ for every $a \in A$ (367I). Also, because $0 \le P_n\chi a \le \chi 1$ for every n, $0 \le \bar{q}(P_n\chi a) \le \frac{1}{e}\chi 1$ for every n. By the Dominated Convergence Theorem (367J), $\lim_{n \to \infty} \int \bar{q}(P_n\chi a) = \int \bar{q}(P\chi a)$.
By 384B, we also have

$$0 \le \int \bar{q}(P_n\chi a) \le q(\int P_n(\chi a)) = q(\int \chi a) = q(\bar{\mu}a)$$

for every $a \in A$ and $n \in \mathbb{N}$; since also

$$0 \le \int \bar{q}(P\chi a) = q(\bar{\mu}a),$$

we have $|\int \bar{q}(P_n\chi a) - \int \bar{q}(P\chi a)| \le q(\bar{\mu}a)$ for every $a \in A$, $n \in \mathbb{N}$.

Now we are supposing that $H(A)$ is finite. Given $\epsilon > 0$, we can find a finite set $I \subseteq A$ such that $\sum_{a\in A\setminus I} q(\bar\mu a) \le \epsilon$, and an $n_0 \in \mathbb{N}$ such that

$$\sum_{a\in I} |\int \bar q(P_n\chi a) - \int \bar q(P\chi a)| \le \epsilon$$

for every $n \ge n_0$; in which case

$$\sum_{a\in A\setminus I} |\int \bar q(P_n\chi a) - \int \bar q(P\chi a)| \le \sum_{a\in A\setminus I} q(\bar\mu a) \le \epsilon$$

and $|H(A|\mathfrak{B}_n) - H(A|\mathfrak{B})| \le 2\epsilon$ for every $n \ge n_0$. As ϵ is arbitrary, $H(A|\mathfrak{B}) = \lim_{n\to\infty} H(A|\mathfrak{B}_n)$.

384H Corollary Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Then $H(A) \le H(A \vee B) \le H(A) + H(B)$.

proof Take $\mathfrak{B} = \{0, 1\}$ in 384Ga.

384I Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. If $A \subseteq \mathfrak{A}$ is a partition of unity, then $H(\pi[A]) = H(A)$.

proof $\sum_{a\in A} q(\bar\mu\pi a) = \sum_{a\in A} q(\bar\mu a)$.

384J Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Let A be the set of its atoms. Then the following are equiveridical:

(i) *either* $\mathfrak{A}$ is not purely atomic *or* $\mathfrak{A}$ is purely atomic and $H(A) = \infty$;

(ii) there is a partition of unity $B \subseteq \mathfrak{A}$ such that $H(B) = \infty$;

(iii) for every $\gamma \in \mathbb{R}$ there is a finite partition of unity $C \subseteq \mathfrak{A}$ such that $H(C) \ge \gamma$.

proof (i)$\Rightarrow$(ii) We need examine only the case in which $\mathfrak{A}$ is not purely atomic. Let $a \in \mathfrak{A}$ be a non-zero element such that the principal ideal $\mathfrak{A}_a$ is atomless. By 331C we can choose inductively a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq a$ and $\bar\mu a_n = 2^{-n-1}\bar\mu a$. Now, for each $n \in \mathbb{N}$, choose a disjoint set B_n such that

$$\#(B_n) = 2^{2^n}, \quad b \subseteq a_n \text{ and } \bar\mu b = 2^{-2^n}\bar\mu a_n \text{ for each } b \in B_n.$$

Set

$$B = \bigcup_{n\in\mathbb{N}} B_n \cup \{1 \setminus a\}.$$

Then B is a partition of unity in $\mathfrak{A}$ and

$$\begin{aligned} H(B) &\ge \sum_{n=0}^{\infty} \sum_{b\in B_n} q(\bar\mu B_n) = \sum_{n=0}^{\infty} 2^{2^n} q\Big(\frac{\bar\mu a}{2^{n+1+2^n}}\Big) \\ &= \sum_{n=0}^{\infty} \frac{\bar\mu a}{2^{n+1}} \ln\Big(\frac{2^{n+1+2^n}}{\bar\mu a}\Big) \ge \sum_{n=0}^{\infty} \frac{\bar\mu a}{2^{n+1}} 2^n \ln 2 = \infty. \end{aligned}$$

(ii)$\Rightarrow$(iii) Enumerate B as $\langle b_i\rangle_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$, $C_n = \{b_i : i \le n\} \cup \{1 \setminus \sup_{i\le n} b_i\}$ is a finite partition of unity, and

$$\lim_{n\to\infty} H(C_n) \ge \lim_{n\to\infty} \sum_{i=0}^{n} q(\bar\mu b_i) = H(B) = \infty.$$

(iii)$\Rightarrow$(i) We need only consider the case in which $\mathfrak{A}$ is purely atomic and A is its set of atoms. In this case, $A \vee C = A$ for every partition of unity $C \subseteq \mathfrak{A}$, so $H(C) \le H(A)$ for every C (384H), and $H(A)$ must be infinite.

384K Definition Let $\mathfrak{A}$ be a Boolean algebra. If $\pi : \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, $A \subseteq \mathfrak{A}$ is a partition of unity and $n \ge 1$, write $D_n(A, \pi)$ for the partition of unity generated by $\{\pi^i a : a \in A,\ 0 \le i < n\}$, that is, $\{\inf_{i<n} \pi^i a_i : a_i \in A \text{ for every } i < n\} \setminus \{0\}$. Observe that $D_1(A, \pi) = A \setminus \{0\}$ and

$$D_{n+1}(A, \pi) = D_n(A, \pi) \vee \pi^n[A] = A \vee \pi[D_n(A, \pi)]$$

for every $n \ge 1$.

384L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $A\subseteq\mathfrak{A}$ be a partition of unity. Then $\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))=\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))$ is defined in $[0,\infty]$.

proof (a) Set $\alpha_0=0$, $\alpha_n=H(D_n(A,\pi))$ for $n\ge1$. Then $\alpha_{m+n}\le\alpha_m+\alpha_n$ for all m, $n\ge0$. **P** If m, $n\ge1$, $D_{m+n}(A,\pi)=D_m(A,\pi)\vee\pi^m[D_n(A,\pi)]$. So 384Ga tells us that

$$H(D_{m+n}(A,\pi))\le H(D_m(A,\pi))+H(\pi^m[D_n(A,\pi)])=H(D_m(A,\pi))+H(D_n(A,\pi))$$

because π is measure-preserving. **Q**

(b) If $\alpha_1=\infty$ then of course $H(D_n(A,\pi))\ge H(A)=\infty$ for every n, by 384H, so $\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))=\infty=\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))$. Otherwise, $\alpha_n\le n\alpha_1$ is finite for every n. Set $\alpha=\inf_{n\ge1}\frac{1}{n}\alpha_n$. If $\epsilon>0$ there is an $m\ge1$ such that $\frac{1}{m}\alpha_m\le\alpha+\epsilon$. Set $M=\max_{j<m}\alpha_j$. Now, for any $n\ge m$, there are $k\ge1$, $j<m$ such that $n=km+j$, so that

$$\alpha_n\le k\alpha_m+\alpha_j,\quad \frac{1}{n}\alpha_n\le\frac{k}{n}\alpha_m+\frac{M}{n}\le\frac{1}{m}\alpha_m+\frac{M}{n}.$$

Accordingly $\limsup_{n\to\infty}\frac{1}{n}\alpha_n\le\alpha+\epsilon$. As ϵ is arbitrary,

$$\alpha\le\liminf_{n\to\infty}\frac{1}{n}\alpha_n\le\limsup_{n\to\infty}\frac{1}{n}\alpha_n\le\alpha$$

and $\lim_{n\to\infty}\frac{1}{n}\alpha_n=\alpha$ is defined in $[0,\infty]$.

Remark See also 384Yb and 385Nc below.

384M Definition Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. For any partition of unity $A\subseteq\mathfrak{A}$, set

$$h(\pi,A)=\inf_{n\ge1}\tfrac{1}{n}H(D_n(A,\pi))=\lim_{n\to\infty}\tfrac{1}{n}H(D_n(A,\pi))$$

(384L). Now the **entropy** of π is

$$h(\pi)=\sup\{h(\pi,A):A\subseteq\mathfrak{A}\text{ is a finite partition of unity}\}.$$

Remarks (a) We always have

$$h(\pi,A)\le H(D_1(A,\pi))=H(A).$$

(b) Observe that if π is the identity automorphism then $D_n(A,\pi)=A\setminus\{0\}$ for every A and n, so that $h(\pi)=0$.

384N Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Let $\pi:\mathfrak{A}\to\mathfrak{A}$ be a measure-preserving Boolean homomorphism. Then $h(\pi,A)\le h(\pi,B)+H(A|\mathfrak{B})$, where $\mathfrak{B}$ is the closed subalgebra of $\mathfrak{A}$ generated by B.

proof We may suppose that $0\notin B$, since removing 0 from B changes neither $D_n(B,\pi)$ nor $\mathfrak{B}$. For each $n\in\mathbb{N}$, set $A_n=\pi^n[A]$, $B_n=\pi^n[B]$. Let $\mathfrak{B}_n=\pi^n[\mathfrak{B}]$ be the closed subalgebra of $\mathfrak{A}$ generated by B_n, and $\mathfrak{B}_n^*$ the closed subalgebra of $\mathfrak{A}$ generated by $D_n(B,\pi)$. Then $H(A_n|\mathfrak{B}_n)=H(A|\mathfrak{B})$ for each n. **P** The point is that, because $\mathfrak{B}$ is purely atomic and B is its set of atoms,

$$H(A|\mathfrak{B})=\sum_{a\in A,b\in B}q(\tfrac{\bar{\mu}(a\cap b)}{\bar{\mu}b})\bar{\mu}b$$

as in the proof of 384Gb. Similarly,

$$H(A_n|\mathfrak{B}_n)=\sum_{a\in A,b\in B}q(\tfrac{\bar{\mu}(\pi^na\cap\pi^nb)}{\bar{\mu}(\pi^nb)})\bar{\mu}(\pi^nb)=H(A|\mathfrak{B}).\ \textbf{Q}$$

Accordingly, for any $n\ge1$,

$$H(D_n(A,\pi)|\mathfrak{B}_n^*) \le \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}_n^*)$$

(by 384Ga)

$$\le \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}_i)$$

(by 384Gc)

$$= nH(A|\mathfrak{B}).$$

Now

$$h(\pi,A) = \lim_{n\to\infty} \frac{1}{n} H(D_n(A,\pi)) \le \limsup_{n\to\infty} \frac{1}{n} H(D_n(A,\pi) \vee D_n(B,\pi))$$

(384Ga)

$$\le \limsup_{n\to\infty} \frac{1}{n} H(D_n(B,\pi)) + \frac{1}{n} H(D_n(A,\pi)|\mathfrak{B}_n^*)$$

(384Gb)

$$\le h(\pi,B) + H(A|\mathfrak{B}).$$

Remark Compare 385Nd below.

384O Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism, and $A \subseteq \mathfrak{A}$ a partition of unity such that $H(A) < \infty$. Then $h(\pi,A) \le h(\pi)$.

proof If A is finite, this is immediate from the definition of $h(\pi)$; so suppose that A is infinite. Enumerate A as $\langle a_i\rangle_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $a_0,\dots,a_n$; then $\mathfrak{B}_n$ has atoms $B_n = \{a_0,\dots,a_n,b_n\}$ where $b_n = \sup_{i>n} a_i$. Now

$$H(A|\mathfrak{B}_n) = \textstyle\sum_{i=n+1}^{\infty} q(\frac{\alpha_i}{\beta_n})\beta_n$$

where $\alpha_i = \bar\mu a_i$, $\beta_n = \bar\mu b_n$, because if P_n is the conditional expectation associated with $\mathfrak{B}_n$ then $P_n(\chi a_i) = \chi a_i$ if $i \le n$, $P_n(\chi a_i) = \frac{\alpha_i}{\beta_n}\chi b_n$ if $i > n$. But

$$\sum_{i=n+1}^{\infty} \beta_n q(\frac{\alpha_i}{\beta_n}) = \sum_{i=n+1}^{\infty} q(\alpha_i) - \frac{\alpha_i}{\beta_n} q(\beta_n)$$

$$= \sum_{i=n+1}^{\infty} q(\alpha_i) - q(\beta_n) \to 0 \text{ as } n\to\infty$$

because $\sum_{i=0}^{\infty} q(\alpha_i) < \infty$ and $\lim_{n\to\infty}\beta_n = 0$. So by 384N and 384Gd we get

$$h(\pi,A) \le h(\pi,B_n) + H(A|\mathfrak{B}_n) \le h(\pi) + H(A|\mathfrak{B}_n) \to h(\pi)$$

as $n\to\infty$, and $h(\pi,A) \le h(\pi)$.

Remark Compare 385Nb below.

384P Theorem (Kolmogorov 58, Sinaĭ 59) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) Suppose that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi,A)$.

(ii) Suppose that π is an automorphism, and that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{Z}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi,A)$.

proof I take the two arguments together. In both cases, by 384O, we have $h(\pi, A) \le h(\pi)$, so I have to show that if $B \subseteq \mathfrak{A}$ is any finite partition of unity, then $h(\pi, B) \le h(\pi, A)$. For (i), let A_n be the partition of unity generated by $\bigcup_{0\le j<n} \pi^j[A]$; for (ii), let A_n be the partition of unity generated by $\bigcup_{-n\le j<n} \pi^j[A]$. Then $h(\pi, A_n) = h(\pi, A)$ for every n. **P** In case (i), we have $D_m(A_n, \pi) = D_{m+n}(A, \pi)$ for every m, so that

$$\lim_{m\to\infty} \frac{1}{m} H(D_m(A_n, \pi)) = \lim_{m\to\infty} \frac{1}{m} H(D_{m+n}(A, \pi))$$

$$= \lim_{m\to\infty} \frac{1}{m} H(D_m(A, \pi)).$$

In case (ii), we have $D_m(A_n, \pi) = \pi^{-n}[D_{m+2n}(A, \pi)]$ for every m, so that

$$\lim_{m\to\infty} \frac{1}{m} H(D_m(A_n, \pi)) = \lim_{m\to\infty} \frac{1}{m} H(D_{m+2n}(A, \pi))$$

$$= \lim_{m\to\infty} \frac{1}{m} H(D_m(A, \pi)). \textbf{ Q}$$

Let $\mathfrak{A}_n$ be the purely atomic closed subalgebra of $\mathfrak{A}$ generated by A_n; our hypothesis is that the closed subalgebra generated by $\bigcup_{n\in\mathbb{N}} A_n$ is $\mathfrak{A}$ itself, that is, that $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ is dense. But this means that $\lim_{n\to\infty} H(B|\mathfrak{A}_n) = 0$ (384Gd). Since

$$h(\pi, B) \le h(\pi, A_n) + H(B|\mathfrak{A}_n) = h(\pi, A) + H(B|\mathfrak{A}_n)$$

for every n (384N), we have the result.

384Q Bernoulli shifts Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) π is a **one-sided Bernoulli shift** if there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent (that is, $\bar{\mu}(\inf_{j\le k} \pi^j a_j) = \prod_{j=0}^k \bar{\mu} a_j$ for all $a_0, \dots, a_k \in \mathfrak{A}_0$; see 325L) (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

(b) π is a **two-sided Bernoulli shift** if it is an automorphism and there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

It is important to be aware that a Bernoulli shift can have many, and (in the case of a two-sided shift) very different, root algebras; this is the subject of §386 below.

384R Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Bernoulli shift, either one- or two-sided, with root algebra $\mathfrak{A}_0$.

(i) If $\mathfrak{A}_0$ is purely atomic, then $h(\pi) = H(A)$, where A is the set of atoms of $\mathfrak{A}_0$.

(ii) If $\mathfrak{A}_0$ is not purely atomic, then $h(\pi) = \infty$.

proof (a) The point is that for any partition of unity $C \subseteq \mathfrak{A}_0$, $h(\pi, C) = H(C)$. **P** For any $n \ge 1$, $D_n(C, \pi)$ is the partition of unity consisting of elements of the form $\inf_{j<n} \pi^j c_j$, where $c_0, \dots, c_{n-1} \in C$. So

$$H(D_n(C, \pi)) = \sum_{c_0,\dots,c_{n-1}\in C} q(\bar{\mu}(\inf_{j<n} \pi^j c_j)) = \sum_{c_0,\dots,c_{n-1}\in C} q(\prod_{j=0}^{n-1} \bar{\mu} c_j))$$

$$= \sum_{c_0,\dots,c_{n-1}\in C} \sum_{j=0}^{n-1} q(\bar{\mu} c_j) \prod_{i\ne j} \bar{\mu} c_i$$

(384Ac)

$$= \sum_{j=0}^{n-1} \sum_{c\in C} q(\bar{\mu} c) = nH(C).$$

So

$$h(\pi, C) = \lim_{n\to\infty} \frac{1}{n} H(D_n(C, \pi)) = H(C). \text{ } \mathbf{Q}$$

(b) If $\mathfrak{A}_0$ is purely atomic and $H(A) < \infty$, the result can now be read off from 384P, because the closed subalgebra of $\mathfrak{A}$ generated by A is $\mathfrak{A}_0$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[A]$ or $\bigcup_{k\in\mathbb{Z}} \pi^k[A]$ is $\mathfrak{A}$; so $h(\pi) = h(\pi, A) = H(A)$.

(c) Otherwise, 384J tells us that there are finite partitions of unity $C \subseteq \mathfrak{A}_0$ such that $H(C)$ is arbitrarily large. Since $h(\pi) \ge h(\pi, C) = H(C)$ for any such C, by (a) and the definition of $h(\pi)$, $h(\pi)$ must be infinite, as claimed.

384S Remarks (a) The standard construction of a Bernoulli shift is from a product space, as follows. If (X, Σ, μ_0) is any probability space, write μ for the product measure on $X^{\mathbb{N}}$; let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ the set of equivalence classes of sets of the form $\{x : x(0) \in E\}$ where $E \in \Sigma$, so that $(\mathfrak{A}_0, \bar\mu{\restriction}\mathfrak{A}_0)$ can be identified with the measure algebra of μ_0. We have an inverse-measure-preserving function $f : X^{\mathbb{N}} \to X^{\mathbb{N}}$ defined by setting

$$f(x)(n) = x(n+1) \text{ for every } x \in X^{\mathbb{N}},\ n \in \mathbb{N},$$

and f induces, as usual, a measure-preserving homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. Now π is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$. $\mathbf{P}$ (i) If $a_0, \dots, a_k \in \mathfrak{A}_0$, express each a_j as $\{x : x(0) \in E_j\}^\bullet$, where $E_j \in \Sigma$. Now

$$\pi^j a_j = (f^{-j}\{x : x(0) \in E_j\})^\bullet = \{x : x(j) \in E_j\}^\bullet$$

for each j, so

$$\bar\mu(\inf_{j\le k} \pi^j a_j) = \mu(\bigcap_{j\le k}\{x : x(j) \in E_j\}) = \prod_{j=0}^k \mu_0 E_j = \prod_{j=0}^k \bar\mu a_j.$$

Thus $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent. (ii) The closed subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[\mathfrak{A}_0]$ must contain $\{x : x(k) \in E\}^\bullet$ for every $k \in \mathbb{N}$, $E \in \Sigma$, so must contain $W^\bullet$ for every W in the σ-algebra generated by sets of the form $\{x : x(k) \in E\}$; but every set measured by μ is equivalent to such a set W. So $\mathfrak{A}' = \mathfrak{A}$. $\mathbf{Q}$

(b) The same method gives us two-sided Bernoulli shifts. Again let (X, Σ, μ_0) be a probability space, and this time write μ for the product measure on $X^{\mathbb{Z}}$; again let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ the set of equivalence classes of sets of the form $\{x : x(0) \in E\}$ where $E \in \Sigma$, so that $(\mathfrak{A}_0, \bar\mu{\restriction}\mathfrak{A}_0)$ can once more be identified with the measure algebra of μ_0. This time, we have a measure space automorphism $f : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ defined by setting

$$f(x)(n) = x(n+1) \text{ for every } x \in X^{\mathbb{Z}},\ n \in \mathbb{Z},$$

and f induces a measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. The arguments used above show that π is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

(c) I remarked above that a Bernoulli shift will normally have many root algebras. But it is important to know that, up to isomorphism, any root algebra is associated with just one Bernoulli shift of each type.

$\mathbf{P}$**(i)** Given a probability algebra $(\mathfrak{A}_0, \bar\mu_0)$ then we can identify it with the measure algebra of a probability space (X, Σ, μ_0) (321J), and now the constructions of (a) and (b) provide Bernoulli shifts with root algebras isomorphic to $(\mathfrak{A}_0, \bar\mu_0)$.

(ii) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be probability algebras with one-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0 : \mathfrak{A}_0 \to \mathfrak{B}_0$ is a measure-preserving isomorphism. Then $(\mathfrak{A}, \bar\mu)$ can be identified with the probability algebra free product of $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ (325L), while $(\mathfrak{B}, \bar\nu)$ can be identified with the probability algebra free product of $\langle \pi^k[\mathfrak{B}_0]\rangle_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, $\phi^k\theta_0(\pi^k)^{-1}$ is a measure-preserving isomorphism between $\pi^k[\mathfrak{A}_0]$ and $\phi^k[\mathfrak{B}_0]$. Assembling these, we have a measure-preserving isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $\theta a = \phi^k\theta_0(\pi^k)^{-1}a$ whenever $k \in \mathbb{N}$ and $a \in \pi^k[\mathfrak{A}_0]$, that is, $\theta\pi^k a = \phi^k\theta_0 a$ for every $a \in \mathfrak{A}_0$, $k \in \mathbb{N}$. Of course θ extends θ_0.

If we set

$$\mathfrak{C} = \{a : a \in \mathfrak{A},\, \theta\pi a = \phi\theta a\},$$

then $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. If $a \in \mathfrak{A}_0$ and $k \in \mathbb{N}$, then

$$\theta\pi(\pi^k a) = \theta\pi^{k+1} a = \phi^{k+1}\theta_0 a = \phi(\phi^k\theta_0 a) = \phi\theta(\pi^k a),$$

so $\pi^k a \in \mathfrak{C}$. Thus $\phi^k[\mathfrak{A}_0] \subseteq \mathfrak{C}$ for every $k \in \mathbb{N}$, and $\mathfrak{C} = \mathfrak{A}$.

This means that $\theta : \mathfrak{A} \to \mathfrak{B}$ is such that $\phi = \theta\pi\theta^{-1}$; θ is an isomorphism between the structures $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ extending the isomorphism θ_0 from $\mathfrak{A}_0$ to $\mathfrak{B}_0$.

(iii) Now suppose that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are probability algebras with two-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0 : \mathfrak{A}_0 \to \mathfrak{B}_0$ is a measure-preserving isomorphism. Repeating (ii) word for word, but changing each $\mathbb{N}$ into $\mathbb{Z}$, we find that θ_0 has an extension to a measure-preserving isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $\theta\pi = \phi\theta$, so that once more the structures $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are isomorphic. **Q**

(d) The classic problem to which the theory of this section was directed was the following: suppose we have two two-sided Bernoulli shifts π and ϕ, one based on a root algebra with two atoms of measure $\frac{1}{2}$ and the other on a root algebra with three atoms of measure $\frac{1}{3}$; are they isomorphic? The Kolmogorov-Sinaĭ theorem tells us that they are not, because $h(\pi) = \ln 2$ and $h(\phi) = \ln 3$ are different. The question of which Bernoulli shifts *are* isomorphic is addressed, and (for countably-generated algebras) solved, in §386 below.

(e) We shall need to know that any Bernoulli shift (either one- or two-sided) is ergodic. In fact, it is mixing. **P** Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Bernoulli shift with root algebra $\mathfrak{A}_0$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{A}_0]$ (if π is two-sided). If b, $c \in \mathfrak{B}$, there is some $n \in \mathbb{N}$ such that both belong to the algebra $\mathfrak{B}_n$ generated by $\bigcup_{j\le n} \pi^j[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{|j|\le n} \pi^j[\mathfrak{A}_0]$ (if π is two-sided). If now $k > 2n$, $\pi^k b$ belongs to the algebra generated by $\bigcup_{j>n} \pi^j[\mathfrak{A}_0]$. But this is independent of $\mathfrak{B}_n$ (cf. 325Xf, 272K), so

$$\bar{\mu}(c \cap \pi^k b) = \bar{\mu}c \cdot \bar{\mu}(\pi^k b) = \bar{\mu}c \cdot \bar{\mu}b.$$

And this is true for every $k \ge n$. Generally, if b, $c \in \mathfrak{A}$ and $\epsilon > 0$, there are b', $c' \in \mathfrak{B}$ such that $\bar{\mu}(b \triangle b') \le \epsilon$, $\bar{\mu}(c \triangle c') \le \epsilon$, so that

$$\begin{aligned}
\limsup_{k\to\infty} |\bar{\mu}(c \cap \pi^k b) - \bar{\mu}c \cdot \bar{\mu}b| &\le \limsup_{k\to\infty} |\bar{\mu}(c' \cap \pi^k b') - \bar{\mu}c' \cdot \bar{\mu}b'| \\
&\qquad + \bar{\mu}(c \triangle c') + \bar{\mu}(\pi^k b \triangle \pi^k b') + |\bar{\mu}c \cdot \bar{\mu}b - \bar{\mu}c' \cdot \bar{\mu}b'| \\
&\le 0 + \epsilon + \epsilon + |\bar{\mu}c - \bar{\mu}c'| + |\bar{\mu}b - \bar{\mu}b'| \\
&\le 4\epsilon.
\end{aligned}$$

As ϵ, b, and c are arbitrary, π is mixing. By 372Qa, it is ergodic. **Q**

(f) The following elementary remark will be useful. If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a closed subalgebra such that $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent, then $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent. **P** If $J \subseteq \mathbb{Z}$ is finite and $\langle a_j\rangle_{j\in J}$ is a family in $\mathfrak{A}_0$, take $n \in \mathbb{N}$ such that $-n \le j$ for every $j \in J$; then

$$\bar{\mu}(\inf_{j\in J} \pi^j a_j) = \bar{\mu}(\inf_{j\in J} \pi^{n+j} a_j) = \prod_{j\in J} \bar{\mu}a_j. \text{ **Q**}$$

384T Isomorphic homomorphisms (a) In this section I have spoken of 'isomorphic homomorphisms' without offering a formal definition. I hope that my intention was indeed obvious, and that the next sentence will merely confirm what you have already assumed. If $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ are measure algebras, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ are functions, then I say that $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ are isomorphic if there is a measure-preserving isomorphism $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\pi_2 = \phi\pi_1\phi^{-1}$. In this context, using Maharam's theorem or otherwise, we can expect to be able to decide whether $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ are or are not isomorphic; and if they are, we have a good hope of being able to describe a measure-preserving isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$. In this case, of course, $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ will be isomorphic to $(\mathfrak{A}_1, \bar{\mu}_1, \pi_2')$ where $\pi_2' = \theta^{-1}\pi_2\theta$. So now we have to decide whether $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ is isomorphic to $(\mathfrak{A}_1, \bar{\mu}_1, \pi_2')$; and when π_1, π_2

are measure-preserving Boolean automorphisms, this is just the question of whether π_1, π_2' are conjugate in the group $\mathrm{Aut}_{\bar\mu_1}(\mathfrak{A}_1)$ of measure-preserving automorphisms of $\mathfrak{A}_1$. Thus the isomorphism problem, as stated here, is very close to the classical group-theoretic problem of identifying the conjugacy classes in $\mathrm{Aut}_{\bar\mu}(\mathfrak{A})$ for a measure algebra $(\mathfrak{A},\bar\mu)$. But we also want to look at measure-preserving homomorphisms which are not automorphisms, so there would be something left even if the conjugacy problem were solved. (In effect, we are studying conjugacy in the semigroup of all measure-preserving Boolean homomorphisms, not just in its group of invertible elements.)

The point of the calculation of the entropy of a homomorphism is that it is an invariant under this kind of isomorphism; so that if π_1, π_2 have different entropies then $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ cannot be isomorphic. Of course the properties of being 'ergodic' or 'mixing' (see 372P) are also invariant.

(b) All the main work of this section has been done in terms of measure algebras; part of my purpose in this volume has been to insist that this is often the right way to proceed, and to establish a language which makes the arguments smooth and natural. But of course a large proportion of the most important homomorphisms arise in the context of measure spaces, and I take a moment to discuss such applications. Suppose that we have two quadruples (X_1,Σ_1,μ_1,f_1), (X_2,Σ_2,μ_2,f_2) where, for each i, (X_i,Σ_i,μ_i) is a measure space and $f_i:X_i\to X_i$ is an inverse-measure-preserving function. Then we have associated structures $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$, $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ where $(\mathfrak{A}_i,\bar\mu_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is the measure-preserving homomorphism defined by the usual formula $\pi_iE^\bullet=f_i^{-1}[E]^\bullet$. Now we can call (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) isomorphic if there is a measure space isomorphism $g:X_1\to X_2$ such that $f_2=gf_1g^{-1}$. In this case $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are isomorphic under the obvious isomorphism $\phi(E^\bullet)=g[E]^\bullet$ for every $E\in\Sigma_1$.

It is not the case that if the $(\mathfrak{A}_i,\bar\mu_i,\pi_i)$ are isomorphic, then the (X_i,Σ_i,μ_i,f_i) are; in fact we do not even need to have an isomorphism of the measure spaces (for instance, one could be Lebesgue measure, and the other the Stone space of the Lebesgue measure algebra). Even when $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are actually identical, f_1 and f_2 need not be isomorphic. There are two examples in §343 of a probability space (X,Σ,μ) with a measure space automorphism $f:X\to X$ such that $f(x)\ne x$ for every $x\in X$ but the corresponding automorphism on the measure algebra is the identity (343I, 343J); writing f_0 for the identity map from X to itself, (X,Σ,μ,f_0) and (X,Σ,μ,f) are non-isomorphic but give rise to the same $(\mathfrak{A},\bar\mu,\pi)$.

(c) Even with Lebesgue measure, we can have a problem in a formal sense. Take (X,Σ,μ) to be $[0,1]$ with Lebesgue measure, and set $f(0)=1$, $f(1)=0$, $f(x)=x$ for $x\in\,]0,1[$; then f is not isomorphic to the identity function on X, but induces the identity automorphism on the measure algebra. But in this case we can sort things out just by discarding the negligible set $\{0,1\}$, and for Lebesgue measure such a procedure is effective in a wide variety of situations. To formalize it I offer the following definition.

384U Definition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be measure spaces, and $f_1:X_1\to X_1$, $f_2:X_2\to X_2$ two inverse-measure-preserving functions. I will say that the structures (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are **almost isomorphic** if there are conegligible sets $X_i'\subseteq X_i$ such that $f_i[X_i']\subseteq X_i'$ for both i and the structures $(X_i',\Sigma_i',\mu_i',f_i')$ are isomorphic in the sense of 384Tb, where Σ_i' is the algebra of relatively measurable subsets of X_i', μ_i' is the subspace measure on X_i' and $f_i'=f_i\restriction X_i'$.

384V I leave the elementary properties of this notion to the exercises (384Xn-384Xp), but I spell out the result for which the definition is devised. I phrase it in the language of §§342-343; if the terms are not immediately familiar, start by imagining that both (X_i,Σ_i,μ_i) are measurable subspaces of $\mathbb{R}$ endowed with some Radon measure (342J, 343H), or indeed that both are $[0,1]$ with Lebesgue measure.

Proposition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be perfect, complete, strictly localizable and countably separated measure spaces, and $(\mathfrak{A}_1,\bar\mu_1)$, $(\mathfrak{A}_2,\bar\mu_2)$ their measure algebras. Suppose that $f_1:X_1\to X_1$, $f_2:X_2\to X_2$ are inverse-measure-preserving functions and that $\pi_1:\mathfrak{A}_1\to\mathfrak{A}_1$, $\pi_2:\mathfrak{A}_2\to\mathfrak{A}_2$ are the measure-preserving Boolean homomorphisms they induce. If $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are isomorphic, then (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic.

proof Because $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$ are isomorphic, we surely have $\mu_1X_1=\mu_2X_2$. If both are zero, we can take $X_1'=X_2'=\emptyset$ and stop; so let us suppose that $\mu_1X_1>0$. Let $\phi:\mathfrak{A}_1\to\mathfrak{A}_2$ be a measure-preserving automorphism such that $\pi_2=\phi\pi_1\phi^{-1}$. Because both μ_1 and μ_2 are complete and strictly localizable and compact (343K), there are inverse-measure-preserving functions $g_1 : X_1 \to X_2$, $g_2 : X_2 \to X_1$ representing ϕ^{-1}, ϕ respectively (343B). Now $g_1g_2 : X_2 \to X_2$, $g_2g_1 : X_1 \to X_1$, $f_2g_1 : X_1 \to X_2$ and $g_1f_1 : X_1 \to X_2$ represent, respectively, the identity automorphism on $\mathfrak{A}_2$, the identity automorphism on $\mathfrak{A}_1$, the homomorphism $\phi^{-1}\pi_2 = \pi_1\phi^{-1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ and the homomorphism $\pi_1\phi^{-1}$ again. Next, because both μ_1 and μ_2 are countably separated, the sets $E_1 = \{x : g_2g_1(x) = x\}$, $H = \{x : f_2g_1(x) = g_1f_1(x)\}$ and $E_2 = \{y : g_1g_2(y) = y\}$ are all conegligible (343F). As in part (b) of the proof of 344I, $g_1{\restriction}E_1$ and $g_2{\restriction}E_2$ are the two halves of a bijection, a measure space isomorphism if E_1 and E_2 are given their subspace measures. Set $G_0 = E_1 \cap H$, and for $n \in \mathbb{N}$ set $G_{n+1} = G_n \cap f_1^{-1}[G_n]$. Then every G_n is conegligible, so $X_1' = \bigcap_{n\in\mathbb{N}} G_n$ is conegligible. Because X_1' is a conegligible subset of E_1, $h = g_1{\restriction}X_1'$ is a measure space isomorphism between X_1' and $X_2' = g_1[X_1']$, which is conegligible in X_2. Because $f_1[G_{n+1}] \subseteq G_n$ for each n, $f_1[X_1'] \subseteq X_1'$. Because $X_1' \subseteq H$, $g_1f_1(x) = f_2g_1(x)$ for every $x \in X_1'$. Next, if $y \in X_2'$, $g_2(y) \in X_1'$, so

$$f_2(y) = f_2g_1g_2(y) = g_1f_1g_2(y) \in g_1[f_1[X_1']] \subseteq g_1[X_1'] = X_2'.$$

Accordingly we have $f_2' = hf_1'h^{-1}$, where $f_i' = f_i{\restriction}X_i'$ for both i.

Thus h is an isomorphism between (X_1', f_1') and (X_2', f_2'), and $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are almost isomorphic.

384X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $A \subseteq \mathfrak{A}$ a partition of unity. Show that if $\#(A) = n$ then $H(A) \le \ln n$.

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$, enumerated as $\langle a_n\rangle_{n\in\mathbb{N}}$. Set $a_n^* = \sup_{i>n} a_i$, $A_n = \{a_0, \dots, a_n, a_n^*\}$ for each n. Show that $H(A_n|\mathfrak{B}) \le H(A_{n+1}|\mathfrak{B})$ for every n, and that $H(A|\mathfrak{B}) = \lim_{n\to\infty} H(A_n|\mathfrak{B})$.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B}) = 0$ iff $A \subseteq \mathfrak{B}$.

(d) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B}) = H(A)$ iff $\bar\mu(a \cap b) = \bar\mu a \cdot \bar\mu b$ for every $a \in A$, $b \in \mathfrak{B}$. (*Hint*: for 'only if', start with the case $\mathfrak{B} = \{0, b, 1 \setminus b, 1\}$ and use 384Gc.)

(e) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Show that $H(A \vee B) = H(A) + H(B)$ iff $\bar\mu(a \cap b) = \bar\mu a \cdot \bar\mu b$ for all $a \in A$, $b \in B$. Show that $H(A \vee B) = H(A)$ iff every member of A is included in some member of B, that is, iff $A = A \vee B$.

(f) Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar\lambda)$ (325K). Suppose that $\pi_i : \mathfrak{A}_i \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$, and that $\pi : \mathfrak{C} \to \mathfrak{C}$ is the measure-preserving Boolean homomorphism they induce. Show that $h(\pi) = \sum_{i\in I} h(\pi_i)$. (*Hint*: use 384Gb and 384Gd to show that $h(\pi)$ is the supremum of $h(\pi, A)$ as A runs over the finite partitions of unity in $\bigotimes_{i\in I} \mathfrak{A}_i$. Use this to reduce to the case $I = \{0, 1\}$. Now show that if $A_i \subseteq \mathfrak{A}_i$ is a finite partition of unity for each i, and $A = \{a_0 \otimes a_1 : a_0 \in A_0,\ a_1 \in A_1\}$, then $H(A) = H(A_0) + H(A_1)$, so that $h(\pi, A) = h(\pi_0, A_0) + h(\pi_1, A_1)$.)

>(g) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. Show that $h(\pi^{-1}) = h(\pi)$.

(h) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi^k) = kh(\pi)$ for any $k \in \mathbb{N}$. (*Hint*: if $A \subseteq \mathfrak{A}$ is a partition of unity, $h(\pi^k, A) \le h(\pi^k, D_k(A, \pi)) = kh(\pi, A)$.)

>(i) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Suppose there is a partition of unity $A \subseteq \mathfrak{A}$ such that (α) $\bar\mu(a \cap \pi b) = \bar\mu a \cdot \bar\mu b$ for every $a \in A$, $b \in \mathfrak{A}$ (β) $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}} \pi^n[A]$. Show that π is a one-sided Bernoulli shift, and that $h(\pi) = H(A)$. (ii) Suppose that π is a one-sided Bernoulli shift of finite entropy. Show that there is a partition of unity satisfying (α) and (β).

>(j) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1[$. Fix an integer $k\ge 2$, and define $f:[0,1[\,\to[0,1[$ by setting $f(x)=<kx>$, the fractional part of kx, for every $x\in[0,1[$; let $\pi:\mathfrak{A}\to\mathfrak{B}$ be the corresponding homomorphism. (Cf. 372Xr.) Show that π is a one-sided Bernoulli shift and that $h(\pi)=\ln k$. (*Hint*: in 384Xi, set $A=\{a_0,\dots,a_{k-1}\}$ where $a_i=\bigl[\frac{i}{k},\frac{i+1}{k}\bigr[^\bullet$ for $i<k$.)

>(k) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $f(x)=2\min(x,1-x)$ for $x\in[0,1]$ (see 372Xm). Show that the corresponding homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is a one-sided Bernoulli shift and that $h(\pi)=\ln 2$. (*Hint*: in 384Xi, set $A=\{a,1\setminus a\}$ where $a=[0,\frac{1}{2}]^\bullet$.)

(l) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift. Show that π^{-1} is a two-sided Bernoulli shift and that there is a measure-preserving involution $\phi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi^{-1}=\phi\pi\phi^{-1}$, so that π is a product of two involutions in $\operatorname{Aut}_{\bar{\mu}}(\mathfrak{A})$.

(m) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar{\lambda})$ their probability algebra free product. Suppose that for each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$, and that $\pi:\mathfrak{C}\to\mathfrak{C}$ is the measure-preserving homomorphism induced by $\langle\pi_i\rangle_{i\in I}$ (325Xd). (i) Show that if every π_i is a one-sided Bernoulli shift so is π. (ii) Show that if every π_i is a two-sided Boolean shift so is π.

(n) Show that the relation 'almost isomorphic to' (384U) is an equivalence relation.

(o) Show that the concept of 'almost isomorphism' described in 384U is not changed if we amend the definition to require that the subspaces X_1', X_2' should be measurable.

(p) Show that if (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic quadruples as described in 384U, then $(\mathfrak{A}_1,\bar{\mu}_1,\pi_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2,\pi_2)$ are isomorphic, where for each i $(\mathfrak{A}_i,\bar{\mu}_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is the measure-preserving Boolean homomorphism derived from $f_i:X_i\to X_i$.

(q) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and write $\mathcal{A}$ for the set of partitions of unity in $\mathfrak{A}$ not containing 0, ordered by saying that $A\le B$ if every member of B is included in some member of A. (i) Show that $\mathcal{A}$ is a Dedekind complete lattice, and can be identified with the lattice of purely atomic closed subalgebras of $\mathfrak{A}$. Show that for A, $B\in\mathcal{A}$, $A\vee B$, as defined in 384F, is $\sup\{A,B\}$ in $\mathcal{A}$. (ii) Show that $H(A\vee B)+H(A\wedge B)\le H(A)+H(B)$ for all A, $B\in\mathcal{A}$, where $\vee$, $\wedge$ are the lattice operations on $\mathcal{A}$. (iii) Set $\mathcal{A}_1=\{A:A\in\mathcal{A},\,H(A)<\infty\}$. For A, $B\in\mathcal{A}_1$ set $\rho(A,B)=2H(A\vee B)-H(A)-H(B)$. Show that ρ is a metric on $\mathcal{A}_1$ (the **entropy metric**). (iv) Show that if $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism, then $|h(\pi,A)-h(\pi,B)|\le\rho(A,B)$ for all A, $B\in\mathcal{A}_1$. (iv) Show that the lattice operations $\vee$, $\wedge$ are ρ-continuous on $\mathcal{A}_1$. (v) Show that $H:\mathcal{A}_1\to[0,\infty[$ is order-continuous. (vi) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$, then $A\mapsto H(A|\mathfrak{B})$ is order-continuous on $\mathcal{A}_1$.

384Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and write $\mathfrak{P}$ for the lattice of closed subalgebras of $\mathfrak{A}$. Show that if A is any partition of unity in $\mathfrak{A}$ of finite entropy, then the order-preserving function $\mathfrak{B}\mapsto -H(A|\mathfrak{B}):\mathfrak{P}\to\,]-\infty,0]$ is order-continuous.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, A a partition of unity in $\mathfrak{A}$ of finite entropy, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi,A)=\lim_{n\to\infty}H(A|\mathfrak{B}_n)$, where $\mathfrak{B}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{1\le i\le n}\pi^i[A]$. (*Hint*: use 384Gb to show that $H(A|\mathfrak{B}_n)=H(D_{n+1}(A,\pi))-H(D_n(A,\pi))$ and observe that $\lim_{n\to\infty}H(A|\mathfrak{B}_n)$ is defined.)

(c) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that there is a partition of unity A of finite entropy such that the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\ge 1}\pi^i[A]$ is $\mathfrak{A}$. Show that $h(\pi)=0$. (*Hint*: use 384Yb and 384Pa.)

(d) Let μ be Lebesgue measure on $[0,1[$, and take any $\alpha\in\,]0,1[$. Let $f:[0,1[\,\to[0,1[$ be the measure space automorphism defined by saying that $f(x)$ is to be one of $x+\alpha$, $x+\alpha-1$. Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of $([0,1[,\mu)$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ the measure-preserving automorphism corresponding to f. Show that $h(\pi)=0$. (*Hint*: if $\alpha\in\mathbb{Q}$, use 384Xh; otherwise use 384Yc with $A=\{a,1\setminus a\}$ where $a=\bigl[0,\frac{1}{2}\bigr[^\bullet$.)

(e) Set $X = [0,1] \setminus \mathbb{Q}$, let ν be the measure on X defined by setting $\nu E = \frac{1}{\ln 2}\int_E \frac{1}{1+x}dx$ for every Lebesgue measurable set $E \subseteq X$, and for $x \in X$ let $f(x)$ be the fractional part $<\frac{1}{x}>$ of $\frac{1}{x}$. Recall that f is inverse-measure-preserving for ν (372N). Let $(\mathfrak{A}, \bar{\nu})$ be the measure algebra of (X, ν) and $\pi : \mathfrak{A} \to \mathfrak{A}$ the homomorphism corresponding to f. Show that $h(\pi) = \pi^2/6\ln 2$. (*Hint*: use the Kolmogorov-Sinaǐ theorem and 372Yf(v).)

(f) Consider the triplets $([0,1[\,, \mu_1, f_1)$ and $([0,1], \mu_2, f_2)$ where μ_1, μ_2 are Lebesgue measure on $[0,1[$, $[0,1]$ respectively, $f_1(x) = <2x>$ for each $x \in [0,1[$, and $f_2(x) = 2\min(x, 1-x)$ for each $x \in [0,1]$. Show that these structures are almost isomorphic in the sense of 384U, and give a formula for an isomorphism.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\mathcal{A}_1$ the set of partitions of unity of finite entropy not containing 0, as in 384Xq. Show that $\mathcal{A}_1$ is complete under the entropy metric. (*Hint*: show that if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}_1$ and $\sup_{n\in\mathbb{N}} H(A_n) < \infty$, then the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} A_n$ is purely atomic.)

384 Notes and comments In preparing this section I have been heavily influenced by Petersen 83. I have taken almost the shortest possible route to Theorem 384P, the original application of the theory, ignoring both the many extensions of these ideas and their intuitive underpinning in the concept of the quantity of 'information' carried by a partition. For both of these I refer you to Petersen 83. The techniques described there are I think sufficiently powerful to make possible the calculation of the entropy of any of the measure-preserving homomorphisms which have yet appeared in this treatise.

Of course the idea of entropy of a partition, or of a homomorphism, can be translated into the language of probability spaces and inverse-measure-preserving functions; if (X, Σ, μ) is a probability space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, then partitions of unity in $\mathfrak{A}$ correspond (subject to decisions on how to treat negligible sets) to countable partitions of X into measurable sets, and an inverse-measure-preserving function from X to itself gives rise to a measure-preserving homomorphism $\pi_f : \mathfrak{A} \to \mathfrak{A}$; so we can define the entropy of f to be $h(\pi_f)$. The whole point of the language I have sought to develop in this volume is that we can do this when and if we choose; in particular, we are not limited to those homomorphisms which are representable by inverse-measure-preserving functions. But of course a large proportion of the most important examples do arise in this way (see 384Xj, 384Xk). The same two examples are instructive from another point of view: the case $k = 2$ of 384Xj is (almost) isomorphic to the tent map of 384Xk. The similarity is obvious, but exhibiting an actual isomorphism is I think another matter (384Yf).

I must say 'almost' isomorphic here because the doubling map on $[0,1[$ is everywhere two-to-one, while the tent map is not, so they cannot be isomorphic in any exact sense. This is the problem grappled with in 384T-384V. In some moods I would say that a dislike of such contortions is a sign of civilized taste. Certainly it is part of my motivation for working with measure algebras whenever possible. But I have to say also that new ideas in this topic arise more often than not from actual measure spaces, and that it is absolutely necessary to be able to operate in the more concrete context.

385 More about entropy

In preparation for the next two sections, I present a number of basic facts concerning measure-preserving homomorphisms and entropy. Compared with the work to follow, they are mostly fairly elementary, but the Halmos-Rokhlin-Kakutani lemma (385E) and the Shannon-McMillan-Breiman theorem (385G), in their full strengths, go farther than one might expect.

385A Periodic and aperiodic parts If X is a set and $f : X \to X$ is a bijection, then the orbits $\Omega_x = \{f^n(x) : n \in \mathbb{Z}\}$ of f are described by their cardinalities, and X has a natural decomposition into the sets $X_i = \{x : \#(\Omega_x) = i\}$ for $1 \le i \le \omega$. Corresponding to this is a partition of unity of a Dedekind complete Boolean algebra with an automorphism, as follows.

Definition If $\mathfrak{A}$ is a Boolean algebra, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **periodic**, with **period** $n \ge 2$, if $\mathfrak{A} \ne \{0\}$, π^n is the identity operator and whenever $b \in \mathfrak{A} \setminus \{0\}$ and $1 \le i < n$ there is a $c \subseteq b$ such that

$\pi^i c \neq c$. π is **periodic with period** 1 iff it is the identity operator. π is **aperiodic** if for every non-zero $b \in \mathfrak{A}$, $n \geq 1$ there is a $c \subseteq b$ such that $\pi^n c \neq c$. I remark immediately that if π is aperiodic, so is π^n for every $n \geq 1$. Note that if $\mathfrak{A} = \{0\}$ then the trivial automorphism of $\mathfrak{A}$ is counted both as aperiodic and as periodic with period 1.

385B Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism which is periodic with period $n \geq 2$. Then π is a Boolean automorphism and there is an $a \in \mathfrak{A}$ such that $(a, \pi a, \pi^2 a, \dots , \pi^{n-1} a)$ is a partition of unity in $\mathfrak{A}$; that is (in the language of 381G) π is of the form $(\overleftarrow{a_1 \,{}_\pi\, a_2 \,{}_\pi\, \dots \,{}_\pi\, a_n})$ where $(a_1, \dots , a_n)$ is a partition of unity in $\mathfrak{A}$.

proof Because $\pi^n : \mathfrak{A} \to \mathfrak{A}$ is injective and surjective, so is π, and π is a Boolean automorphism, therefore order-continuous. Set $D = \{d : d \in \mathfrak{A},\ \pi^i d \cap d = 0 \text{ whenever } 1 \leq i < n\}$. Then the supremum of any upwards-directed subset of D is defined in $\mathfrak{A}$ (because $\mathfrak{A}$ is Dedekind complete) and belongs to D (because π^i is order-continuous for every i – use 313Bc); so D has a maximal element a, by Zorn's lemma. **?** If $(a, \pi a, \dots , \pi^{n-1} a)$ is not a partition of unity in $\mathfrak{A}$, set $b_0 = 1 \setminus (a \cup \pi a \cup \dots \cup \pi^{n-1} a)$. Then

$$\pi b_0 = 1 \setminus (\pi a \cup \dots \cup \pi^{n-1} a \cup a) = b_0.$$

Now we can choose $b_1, \dots , b_{n-1}$ such that $0 \neq b_i \subseteq b_{i-1}$ and $\pi^i b_i \cap b_i = 0$ for $1 \leq i < n$. **P** Given that $b_{i-1} \neq 0$, where $1 \leq i < n$, then (by the definition of 'periodic' homomorphism) there is a $c \subseteq b_{i-1}$ such that $\pi^i c \neq c$. If $c \setminus \pi^i c \neq 0$, take $b_i = c \setminus \pi^i c$. Otherwise, $\pi^i c \setminus c \neq 0$ so $c \setminus \pi^{-i} c \neq 0$ and we can take $b_i = c \setminus \pi^{-i} c$. **Q** At the end of this induction, set $d = b_{n-1}$; then $d \cap \pi^i d = 0$ for $1 \leq i < n$, so $d \in D$; also $d \cap \pi^i a = \pi^i d \cap a = 0$ for $1 \leq i < n$ (because $\pi^i d \subseteq \pi^i b_0 = b_0$ for every i), so $a \cup d \in D$, which is supposed to be impossible. **X**

Thus $(a, \pi a, \dots , \pi^{n-1} a)$ is a partition of unity, as required.

385C Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an injective order-continuous Boolean homomorphism. Then there is a partition of unity $\langle c_i \rangle_{1 \leq i \leq \omega}$ in $\mathfrak{A}$ such that $\pi c_i = c_i$ for every i and $\pi \restriction \mathfrak{A}_{c_n}$ is periodic with period n whenever n is finite and $c_n \neq 0$, while $\pi \restriction \mathfrak{A}_{c_\omega}$ is aperiodic.

Remark As usual, I write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

proof For each $n \geq 1$, set

$$B_n = \{b : b \in \mathfrak{A},\ \pi^n c = c \text{ for every } c \subseteq b\}, \quad b_n = \sup B_n.$$

Because π is a Boolean homomorphism, B_n is an ideal of $\mathfrak{A}$; because π is order-continuous, $b_n \in B_n$, that is, B_n is the principal ideal $\mathfrak{A}_{b_n}$. Also $\pi b_n = b_n$. **P** If $b \in B_n$ and $c \subseteq \pi b$, then $\pi^{n-1} c \subseteq \pi^n b = b$, so

$$\pi^{n-1}(\pi^n c) = \pi^n(\pi^{n-1} c) = \pi^{n-1} c.$$

But π^{n-1}, like π, is injective, so $\pi^n c = c$. As c is arbitrary, $\pi b \in B_n$. As π is order-continuous,

$$\pi b_n = \sup_{b \in B_n} \pi b \subseteq b_n.$$

But this means that $\pi^{i+1} b_n \subseteq \pi^i b_n$ for every i, so

$$b_n = \pi^n b_n \subseteq \pi b_n \subseteq b_n$$

and $\pi b_n = b_n$. **Q**

Set

$$c_n = b_n \setminus \sup_{1 \leq i < n} b_i \text{ for } n \in \mathbb{N}, \quad c_\omega = 1 \setminus \sup_{n \geq 1} b_n = 1 \setminus \sup_{n \geq 1} c_n.$$

Then $\langle c_i \rangle_{1 \leq i \leq \omega}$ serves. **P** Because $\pi b_n = b_n$ for every n, $\pi c_i = c_i$ for every $i \leq \omega$. So $\pi c \subseteq \pi c_i = c_i$ for every $c \subseteq c_i$, and $\pi \restriction \mathfrak{A}_{c_i}$ is a Boolean homomorphism from $\mathfrak{A}_{c_i}$ to itself, for every $i \leq \omega$. If $1 \leq i < \omega$ and $c \subseteq c_i$, then $c \subseteq b_i$ so $\pi^i c = c$; accordingly $(\pi \restriction \mathfrak{A}_{c_i})^i = \pi \restriction \mathfrak{A}_{c_i}$. If $1 \leq j < i \leq \omega$ and $c \subseteq c_i$ is non-zero, then $c \not\subseteq b_j$, so there is a non-zero $d \subseteq c$ such that $\pi^j d \neq d$. This shows both that $\pi \restriction \mathfrak{A}_{c_i}$ is periodic with period i if $1 \leq i < \omega$ and that $\pi \restriction \mathfrak{A}_{c_\omega}$ is aperiodic. So we have an appropriate partition $\langle c_i \rangle_{i \leq \omega}$. **Q**

385D The following elementary fact is essential.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Then

$$\pi(\sup_{n\in\mathbb{N}}\pi^n a)=\sup_{n\in\mathbb{N}}\pi^n a,\quad a\subseteq\sup_{n\ge1}\pi^n a$$

for every $a\in\mathfrak{A}$.

proof Set $c=\sup_{n\in\mathbb{N}}\pi^n a$. Then $\pi c=\sup_{n\ge1}\pi^n$, because π is order-continuous (324Kb), so $\pi c\subseteq c$; as π is measure-preserving and $\bar{\mu}c<\infty$, $\pi c=c$. Now

$$a\subseteq c=\pi c=\sup_{n\in\mathbb{N}}\pi^{n+1}a=\sup_{n\ge1}\pi^n a.$$

Remark See 382Yc-382Yd.

385E The Halmos-Rokhlin-Kakutani lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Set $\mathfrak{C}=\{c:\pi c=c\}$. Then $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$, and the following are equiveridical:

(i) π is aperiodic;

(ii) $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ (definition: 331A);

(iii) whenever $n\ge1$ and $0\le\gamma<\frac{1}{n}$ there is an $a\in\mathfrak{A}$ such that a, πa, $\pi^2 a,\dots,\pi^{n-1}a$ are disjoint and $\bar{\mu}(a\cap c)=\gamma\bar{\mu}c$ for every $c\in\mathfrak{C}$;

(iv) whenever $n\ge1$, $0\le\gamma<\frac{1}{n}$ and $B\subseteq\mathfrak{A}$ is finite, there is an $a\in\mathfrak{A}$ such that a, πa, $\pi^2 a,\dots,\pi^{n-1}a$ are disjoint and $\bar{\mu}(a\cap b)=\gamma\bar{\mu}b$ for every $b\in B$.

proof Of course $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ because π is a Boolean homomorphism, and is (order-)closed because π is (order-)continuous (324Kb).

(i)⇒(ii) ? Suppose, if possible, that π is aperiodic, but there is a non-zero $a\in\mathfrak{A}$ such that $\mathfrak{A}_a=\{a\cap c:c\in\mathfrak{C}\}$. Let $n\ge1$ be such that $b=a\cap\pi^n a\ne0$ (385D). If $d\subseteq b$ there is some $c\in\mathfrak{C}$ such that $d=a\cap c$ and

$$\pi^n d=\pi^n a\cap c\supseteq b\cap c\supseteq d;$$

because π is measure-preserving, $\pi^n d=d$. But this means that π is not aperiodic. **X**

(ii)⇒(iii) Set $\delta=\frac{1}{n}(\frac{1}{n}-\gamma)>0$. By 331B, there is a $d\in\mathfrak{A}$ such that $\bar{\mu}(c\cap d)=\delta\bar{\mu}c$ for every $c\in\mathfrak{C}$. Set $d_k=\pi^k d\setminus\sup_{i<k}\pi^i d$ for $k\in\mathbb{N}$. Note that

$$d_{j+k}=\pi^{j+k}d\setminus\sup_{i<j+k}\pi^i d\subseteq\pi^{j+k}d\setminus\sup_{i<k}\pi^{j+i}d=\pi^j d_k$$

whenever j, $k\in\mathbb{N}$. Next, $\pi^i d_j\cap d_k\subseteq\sup_{m\le i}d_m$ for any i, j, $k\in\mathbb{N}$ such that $i+j\ne k$. **P** (α) If $k\le i$ this is obvious. (β) If $i<k<i+j$ then

$$\pi^i d_j\cap d_k\subseteq\pi^i d_j\cap\pi^i d_{k-i}=\pi^i(d_j\cap d_{k-i})=0.$$

(γ) If $i+j<k$, then

$$\pi^i d_j\cap d_k\subseteq\pi^{i+j}d\cap d_k=0.\ \mathbf{Q}$$

Setting $c^*=\sup_{i\in\mathbb{N}}d_i=\sup_{i\in\mathbb{N}}\pi^i d$, we have $\pi c^*=c^*$, by 385D, so that $c^*\in\mathfrak{C}$ and $\bar{\mu}(d\setminus c^*)=\delta\bar{\mu}(1\setminus c^*)$; but as $d\subseteq c^*$, $c^*=1$.

Set $a^*=\sup_{m\in\mathbb{N}}d_{mn}$ (the mn here is a product, not a double subscript!), $d^*=\sup_{i<n}d_i=\sup_{i<n}\pi^i d$. Then

$$\bar{\mu}(c\cap d^*)\le\sum_{i=0}^{n-1}\bar{\mu}(c\cap\pi^i d)=\sum_{i=0}^{n-1}\bar{\mu}\pi^i(c\cap d)=n\bar{\mu}(c\cap d)=n\delta\bar{\mu}c$$

for every $c\in\mathfrak{C}$. Next, $\pi^i d_{mn}\supseteq d_{mn+i}$ for all m and i, so

$$\sup_{i<n}\pi^i a^*=\sup_{i\in\mathbb{N}}d_i=1.$$

Consequently

$$\bar{\mu}c\le\sum_{i=0}^{n-1}\bar{\mu}(c\cap\pi^i a^*)=n\bar{\mu}(c\cap a^*),$$

$$\bar{\mu}(c\cap a^*\setminus d^*)\ge\bar{\mu}(c\cap a^*)-\bar{\mu}(c\cap d^*)\ge(\tfrac{1}{n}-n\delta)\bar{\mu}c=\gamma\bar{\mu}c$$

for every $c\in\mathfrak{C}$.

By 331B again (applied to the principal ideal of $\mathfrak{A}$ generated by $a^*\setminus d^*$) there is an $a\subseteq a^*\setminus d^*$ such that $\bar\mu(a\cap c)=\gamma\bar\mu c$ for every $c\in\mathfrak{C}$. For $0<i<n$,

$$\pi^ia^*\cap a^*=\sup_{k,l\in\mathbb{N}}\pi^id_{kn}\cap d_{ln}\subseteq\sup_{m\le i}d_m\subseteq d^*,$$

so $\pi^ia\cap a=0$; accordingly a, $\pi a,\dots,\pi^{n-1}a$ are all disjoint and (iii) is satisfied.

(iii)⇒(iv) Note that $\mathfrak{A}$ is certainly atomless, since for every $k\ge 1$ we can find a $c\in\mathfrak{A}$ such that $c,\pi c,\dots,\pi^{k-1}c$ are disjoint and $\bar\mu c=\frac{\bar\mu 1}{k+1}$, so that we have a partition of unity consisting of sets of measure $\frac{\bar\mu 1}{k+1}$. Let B' be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by B, and $m=\#(B')$. Let $\delta>0$ and r, $k\in\mathbb{N}$ be such that

$$3\delta\le(1-n\gamma)\bar\mu b\text{ for every }b\in B',\quad m(\bar\mu 1)^2<r\delta^2,\quad k\delta\ge\bar\mu 1.$$

By (iii), there is a $c\in\mathfrak{A}$ such that $c,\pi c,\dots,\pi^{nr(k+1)-1}c$ are disjoint and $\bar\mu(\sup_{i<nr(k+1)}\pi^ic)=1-\delta$. For $j<r$, set $e_j=\sup_{n(k+1)j\le i<n(k+1)(j+1)}\pi^ic$, $d_j=\sup_{i<k}\pi^{n(k+1)j+ni}c$. Observe that $d_j,\pi d_j,\dots,\pi^{n-1}d_j$ are disjoint, and that $\pi^id_j\subseteq e_j$ for $i<2n$. Set $e=\sup_{j<r}e_j=\sup_{i<nr(k+1)}\pi^ic$, so that $\bar\mu e=1-\delta$.

Suppose we choose $d\in\mathfrak{A}$ by the following random process. Take $s(0),\dots,s(r-1)$ independently in $\{0,\dots,n-1\}$, so that $\Pr(s(j)=l)=\frac{1}{n}$ for each $l<n$, and set $d=\sup_{j<r}\pi^{s(j)}d_j$. Because we certainly have $\pi^i\pi^{s(j)}d_j\subseteq e_j$ whenever $i<n$, $d,\pi d,\dots,\pi^{n-1}d$ will be disjoint. Now for any $b\in\mathfrak{A}$,

$$\Pr\bigl(\bar\mu(d\cap b)\le\frac{1}{n}(\bar\mu b-3\delta)\bigr)<\frac{1}{m}.$$

P We can express the random variable $\bar\mu(d\cap b)$ as $X=\sum_{j=0}^{r-1}X_j$, where $X_j=\bar\mu(\pi^{s(j)}d_j\cap b)$. Then the X_j are independent random variables. For each j, X_j takes values between 0 and $k\bar\mu c\le\frac{\bar\mu 1}{nr}$, and has expectation $\frac{1}{n}\bar\mu(e_j'\cap b)$, where

$$e_j'=\sup_{i<n}\pi^id_j=\sup_{n(k+1)j\le i<n(k+1)j+nk}\pi^ic.$$

So X has expectation $\frac{1}{n}\bar\mu(e'\cap b)$ where $e'=\sup_{j<r}e_j'$. Now

$$e_j\setminus e_j'=\sup_{n(k+1)j+nk\le i<n(k+1)(j+1)}\pi^ic$$

has measure $n\bar\mu c\le\frac{n\bar\mu 1}{nr(k+1)}$ for each j, so $\bar\mu(e\setminus e')\le\frac{\bar\mu 1}{k+1}$ and $\bar\mu(1\setminus e')\le 2\delta$; thus $\mathbb{E}(X)\ge\frac{1}{n}(\bar\mu b-2\delta)$, while

$$\operatorname{Var}(X)=\sum_{j=0}^{r-1}\operatorname{Var}(X_j)\le r\bigl(\frac{\bar\mu 1}{nr}\bigr)^2=\frac{(\bar\mu 1)^2}{n^2r}.$$

But this means that

$$\frac{(\bar\mu 1)^2}{n^2r}\ge\bigl(\frac{\delta}{n}\bigr)^2\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr),$$

and

$$\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr)\le\frac{(\bar\mu 1)^2}{r\delta^2}<\frac{1}{m}$$

by the choice of r. **Q**

This is true for every $b\in B'$, while $\#(B')=m$. There must therefore be some choice of $s(0),\dots,s(r-1)$ such that, taking $d^*=\sup_{j<r}\pi^{s(j)}d_j$,

$$\bar\mu(d^*\cap b)\ge\frac{1}{n}(\bar\mu b-3\delta)\ge\gamma\bar\mu b$$

for every $b\in B'$, while $d^*,\pi d^*,\dots,\pi^{n-1}d^*$ are disjoint. Because $\mathfrak{A}$ is atomless, there is a $d\subseteq d^*$ such that $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B'$. Since every member of B is a disjoint union of members of B', $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B$.

(iv)⇒(i) If $a\in\mathfrak{A}\setminus\{0,1\}$ and $n\ge 1$ then (iv) tells us that there is a $b\in\mathfrak{A}$ such that $b,\pi b,\dots,\pi^nb$ are all disjoint and $\bar\mu(1\setminus\sup_{i\le n}\pi^ib)<\bar\mu a$. Now there must be some $i<n$ such that $d=\pi^ib\cap a\ne 0$, in which case

$$d\cap\pi^nd\subseteq\pi^ib\cap\pi^{i+n}b=\pi^i(b\cap\pi^nb)=0,$$

and $\pi^n d \ne d$. As n and a are arbitrary, π is aperiodic.

385F Corollary An ergodic measure-preserving Boolean homomorphism on an atomless totally finite measure algebra is aperiodic.

proof This is (ii)⇒(i) of 385E in the case $\mathfrak{C} = \{0, 1\}$.

385G I turn now to a celebrated result which is a kind of strong law of large numbers.

The Shannon-McMillan-Breiman theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $A \subseteq \mathfrak{A}$ a partition of unity of finite entropy. For each $n \ge 1$, set

$$w_n = \frac{1}{n} \sum_{d \in D_n(A,\pi)} \ln(\frac{1}{\bar{\mu}d})\chi d,$$

where $D_n(A, \pi)$ is the partition of unity generated by $\{\pi^i a : a \in A,\, i < n\}$, as in 384K. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is norm-convergent in $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ to w say; moreover, $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to w (definition: 367A). If $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the Riesz homomorphism defined by π, so that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$ (364R), then $Tw = w$.

proof (Petersen 83) We may suppose that $0 \notin A$.

(a) For each $n \in \mathbb{N}$, let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{\pi^i a : a \in A,\, 1 \le i \le n\}$, B_n the set of its atoms, and P_n the corresponding conditional expectation operator on $L^1 = L^1(\mathfrak{A})$ (365R). Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and P the corresponding conditional expectation operator. Observe that $B_n = \pi[D_n(A, \pi)]$ and that, in the language of 384F, $D_{n+1}(A, \pi) = A \vee B_n$. Let $\mathfrak{C}$ be the fixed-point algebra $\{c : c \in \mathfrak{A},\, \pi c = c\}$ and Q the associated conditional expectation. Set $L^0 = L^0(\mathfrak{A})$, and let $\bar{\ln}$ be the function from $\{v : [\![v > 0]\!] = 1\}$ to L^0 corresponding to $\ln :]0, \infty[\to \mathbb{R}$ (364I).

(b) It will save a moment later if I note an elementary fact here: if $v \in L^1$, then $\langle \frac{1}{n} T^n v\rangle_{n\ge1}$ is order*-convergent and $\| \|_1$-convergent to 0. **P** We know from the ergodic theorem (372G) that $\langle \tilde{v}_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to Qv, where $\tilde{v}_n = \frac{1}{n+1} \sum_{i=0}^{n} T^i v$. Now $\frac{1}{n} T^n v = \frac{n+1}{n} \tilde{v}_n - \tilde{v}_{n-1}$ is order*-convergent and $\| \|_1$-convergent to $Qv - Qv = 0$ (using 367C for 'order*-convergent'). **Q**

(c) Set

$$v_n = \sum_{a\in A} P_n(\chi a) \times \chi a = \sum_{a\in A, b\in B_n} \frac{\bar{\mu}(a\cap b)}{\bar{\mu}b} \chi(a \cap b).$$

By Lévy's martingale theorem (275I, 367Kb),

$$\langle v_n \times \chi a\rangle_{n\in\mathbb{N}} = \langle P_n(\chi a) \times \chi a\rangle_{n\in\mathbb{N}}$$

is order*-convergent to $P(\chi a) \times \chi a$ for every $a \in A$; consequently $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $v = \sum_{a\in A} P(\chi a) \times \chi a$. It follows that $\langle \bar{\ln} v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln} v$. **P** The point is that, for any $a \in A$, $n \in \mathbb{N}$, $a \subseteq [\![P_n(\chi a) > 0]\!]$, so that $[\![v_n > 0]\!] = 1$ for every n, and $\bar{\ln} v_n$ is defined. Similarly, $\bar{\ln} v$ is defined, and $\langle \bar{\ln} v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln} v$ by 367I. **Q** As $0 \le v_n \le \chi 1$ for every n, $\langle v_n\rangle_{n\in\mathbb{N}} \to v$ for $\| \|_1$, by the Dominated Convergence Theorem (367J).

Next, $\langle \bar{\ln} v_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^1. **P** Of course $\bar{\ln} v_n \le 0$ for every n, because $P_n(\chi a) \le P_n(\chi 1) \le \chi 1$ for each a, so $v_n \le \chi 1$. To see that $\{\bar{\ln} v_n : n \in \mathbb{N}\}$ is bounded below in L^1, we use an idea from the fundamental martingale inequality 275D. Set $v_* = \inf_{n\in\mathbb{N}} v_n$. For $\alpha > 0$, $a \in A$ and $n \in \mathbb{N}$ set

$$b_{an}(\alpha) = [\![P_n(\chi a) < \alpha]\!] \cap \inf_{i<n} [\![P_i(\chi a) \ge \alpha]\!],$$

so that

$$[\![v_* < \alpha]\!] = \sup_{a\in A, n\in\mathbb{N}} a \cap b_{an}(\alpha).$$

Now $b_{an}(\alpha) \in \mathfrak{B}_n$, so

$$\bar{\mu}(a \cap b_{an}(\alpha)) = \int_{b_{an}(\alpha)} \chi a = \int_{b_{an}(\alpha)} P_n(\chi a) \le \alpha\bar{\mu}(b_{an}(\alpha)),$$

and

$$\bar\mu(a\cap[\![v_*<\alpha]\!])\le\min(\bar\mu a,\sum_{n=0}^\infty\bar\mu(a\cap b_{an}(\alpha)))$$
$$\le\min(\bar\mu a,\alpha\sum_{n=0}^\infty\bar\mu b_{an}(\alpha))\le\min(\bar\mu a,\alpha).$$

Letting $\alpha\downarrow 0$, $\bar\mu(a\cap[\![v_*=0]\!])=0$ for every $a\in A$, so $[\![v_*>0]\!]=1$, and $\bar{\ln}\,v_*$ is defined. Moreover,

$$\bar\mu(a\cap[\![-\bar{\ln}\,v_*>-\ln\alpha]\!])=\bar\mu(a\cap[\![v_*<\alpha]\!])\le\min(\bar\mu a,\alpha)$$

for every $a\in A$, $\alpha>0$; that is,

$$\bar\mu(a\cap[\![-\bar{\ln}\,v_*>\beta]\!])\le\min(\bar\mu a,e^{-\beta})$$

for every $a\in A$, $\beta\in\mathbb{R}$. Accordingly

$$\int(-\bar{\ln}\,v_*)=\int_0^\infty\bar\mu[\![-\bar{\ln}\,v_*>\beta]\!]d\beta=\sum_{a\in A}\int_0^\infty\bar\mu(a\cap[\![-\bar{\ln}\,v_*>\beta]\!])d\beta$$
$$\le\sum_{a\in A}\int_0^\infty\min(\bar\mu a,e^{-\beta})d\beta$$
$$=\sum_{a\in A}\Bigl(\int_0^{\ln(1/\bar\mu a)}\bar\mu a\,d\beta+\int_{\ln(1/\bar\mu a)}^\infty e^{-\beta}d\beta\Bigr)$$
$$=\sum_{a\in A}\bigl(\ln(\frac{1}{\bar\mu a})\bar\mu a+e^{\ln\bar\mu a}\bigr)$$
$$=\sum_{a\in A}\ln(\frac{1}{\bar\mu a})\bar\mu a+\sum_{a\in A}\bar\mu a=H(A)+1<\infty$$

because A has finite entropy. But this means that $\bar{\ln}\,v_*$ belongs to L^1, and of course it is a lower bound for $\{\bar{\ln}\,v_n:n\in\mathbb{N}\}$. **Q**

By 367J again, $\bar{\ln}\,v\in L^1$ and $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}\to\bar{\ln}\,v$ for $\|\,\|_1$.

(d) Fix $n\in\mathbb{N}$ for the moment. For each $d\in D_{n+1}(A,\pi)$ let d' be the unique element of B_n such that $d\subseteq d'$. Then

$$(n+1)w_{n+1}=\sum_{d\in D_{n+1}(A,\pi)}\ln(\frac{1}{\bar\mu d'})\chi d-\sum_{d\in D_{n+1}(A,\pi)}\ln(\frac{\bar\mu d}{\bar\mu d'})\chi d$$
$$=\sum_{b\in B_n}\ln(\frac{1}{\bar\mu b})\chi b-\sum_{a\in A,b\in B_n,a\cap b\ne0}\ln(\frac{\bar\mu(a\cap b)}{\bar\mu b})\chi(a\cap b)$$
$$=\sum_{d\in D_n(A,\pi)}\ln(\frac{1}{\bar\mu(\pi d)})\chi(\pi d)-\bar{\ln}\,v_n$$
$$=T(nw_n)-\bar{\ln}\,v_n.$$

Inducing on n, starting from

$$w_1=\textstyle\sum_{a\in A}\ln(\frac{1}{\bar\mu a})\chi a=-\bar{\ln}\,v_0,$$

we get

$$nw_n=\textstyle\sum_{i=0}^{n-1}T^i(-\bar{\ln}\,v_{n-i-1}),\quad w_n=\frac{1}{n}\sum_{i=0}^{n-1}T^i(-\bar{\ln}\,v_{n-i-1})$$

for every $n\ge1$.

(e) Set $w'_n=\frac{1}{n}\sum_{i=0}^{n-1}T^i(-\bar{\ln}\,v)$ for $n\ge1$. By the Ergodic Theorem, $\langle w'_n\rangle_{n\ge1}$ is order*-convergent and $\|\,\|_1$-convergent to $w=Q(-\bar{\ln}\,v)$, and $Tw=w$. To estimate $w_n-w'_n$, set $u^*_n=\sup_{k\ge n}|\bar{\ln}\,v_k-\bar{\ln}\,v|$ for each $n\in\mathbb{N}$. Then $\langle u_n^*\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence, $u_0^*\in L^1$ (by (c) above), and $\inf_{n\in\mathbb{N}}u_n^*=0$ because $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln}\,v$. Now, whenever $n>m\in\mathbb{N}$,

$$\begin{aligned}|w_n-w_n'|&\le\frac{1}{n}\sum_{i=0}^{n-1}T^i|\bar{\ln}\,v-\bar{\ln}\,v_{n-i-1}|\\&=\frac{1}{n}\Bigl(\sum_{i=0}^{n-m-1}T^i|\bar{\ln}\,v-\bar{\ln}\,v_{n-i-1}|+\sum_{i=n-m}^{n-1}T^i|\bar{\ln}\,v-\bar{\ln}\,v_{n-i-1}|\Bigr)\\&\le\frac{1}{n}\Bigl(\sum_{i=0}^{n-m-1}T^iu_m^*+\sum_{j=0}^{m-1}T^{n-1-j}|\bar{\ln}\,v-\bar{\ln}\,v_j|\Bigr)\\&\le\frac{1}{n-m}\Bigl(\sum_{i=0}^{n-m-1}T^iu_m^*+\sum_{j=0}^{m-1}T^{n-1-j}u_0^*\Bigr)\\&=\frac{1}{n-m}\sum_{i=0}^{n-m-1}T^iu_m^*+\frac{1}{n-m}T^{n-m}\sum_{j=0}^{m-1}T^{m-1-j}u_0^*\\&\le\frac{1}{n-m}\sum_{i=0}^{n-m-1}T^iu_m^*+\frac{1}{n-m}T^{n-m}\tilde{u}_m,\end{aligned}$$

setting $\tilde{u}_m=\sum_{j=0}^{m-1}T^{m-1-j}u_0^*$.

Holding m fixed and letting $n\to\infty$, we know that

$$\frac{1}{n-m}\sum_{i=0}^{n-m-1}T^iu_m^*$$

is order*-convergent and $\|\;\|_1$-convergent to Qu_m^*. As for the other term, $\frac{1}{n-m}T^{n-m}\tilde{u}_m$ is order*-convergent and $\|\;\|_1$-convergent to 0, by (b). What this means is that

$$\limsup_{n\to\infty}|w_n-w_n'|\le Qu_m^*,$$

$$\limsup_{n\to\infty}\|w_n-w_n'\|_1\le\|Qu_m^*\|_1$$

for every $m\in\mathbb{N}$. Since $\langle Qu_m^*\rangle_{m\in\mathbb{N}}$ is surely a non-decreasing sequence with infimum 0,

$$\limsup_{n\to\infty}|w_n-w_n'|=0,\quad\limsup_{n\to\infty}\|w_n-w_n'\|_1=0.$$

Since w_n' is order*-convergent and $\|\;\|_1$-convergent to w, so is w_n.

385H Corollary If, in 385G, π is ergodic, then $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\;\|_1$-convergent to $h(\pi,A)\chi1$.

proof Because the limit w in 385G has $Tw=w$, it must be of the form $\gamma\chi1$, because π is ergodic. Now $\gamma=\int w$ must be

$$\lim_{n\to\infty}\int w_n=\lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)}\ln(\frac{1}{\bar{\mu}d})\bar{\mu}d=\lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)}q(\bar{\mu}d)$$

(where q is the function of 384A)

$$=\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))=h(\pi,A).$$

385I Definition Set $p(t)=t\ln t$ for $t>0$, $p(0)=0$; for any Dedekind σ-complete Boolean algebra $\mathfrak{A}$, let $\bar{p}:L^0(\mathfrak{A})^+\to L^0(\mathfrak{A})$ be the corresponding function, as in 364I. (Thus $p=-q$ where q is the function of 384A.)

385J Lemma (Csiszár 67, Kullback 67) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and u a member of $L^1(\mathfrak{A},\bar\mu)^+$ such that $\int u=1$. Then

$$(\int|u-\chi 1|)^2\le 2\int\bar p(u).$$

proof Set $a=[\![u<1]\!]$, $\alpha=\bar\mu a$, $\beta=\int_a u$, $b=1\setminus a$. Then $\bar\mu b=1-\alpha$ and $\int_b u=1-\beta$. Surely $\beta\le\alpha<1$. If $\alpha=0$ then $u=\chi 1$ and the result is trivial; so let us suppose that $0<\alpha<1$. Because the function p is convex,

$$\int_a\bar p(u)\ge\bar\mu a\cdot p(\frac{1}{\bar\mu a}\int_a u)=\alpha p(\frac{\beta}{\alpha})=p(\beta)-\beta\ln\alpha,$$

(using 233Ib/365Rb for the first inequality), and similarly

$$\int_b\bar p(u)\ge p(1-\beta)-(1-\beta)\ln(1-\alpha).$$

Also

$$\int|u-\chi 1|=\int_a(\chi 1-u)+\int_b(u-\chi 1)=\alpha-\beta+(1-\beta)-(1-\alpha)=2(\alpha-\beta),$$

so

$$\begin{aligned}\int\bar p(u)-\frac{1}{2}(\int|u-\chi 1|)^2&\ge p(\beta)-\beta\ln\alpha+p(1-\beta)-(1-\beta)\ln(1-\alpha)-2(\alpha-\beta)^2\\&=\phi(\beta)\end{aligned}$$

say. Now ϕ is continuous on $[0,1]$ and arbitrarily often differentiable on $]0,1[$,

$$\phi(\alpha)=0,$$

$$\phi'(t)=\ln t-\ln\alpha-\ln(1-t)+\ln(1-\alpha)+4(\alpha-t)\text{ for }t\in\,]0,1[,$$

$$\phi'(\alpha)=0,$$

$$\phi''(t)=\frac{1}{t}+\frac{1}{1-t}-4\ge 0\text{ for }t\in\,]0,1[.$$

So $\phi(t)\ge 0$ for $t\in[0,1]$ and, in particular, $\phi(\beta)\ge 0$; but this means that

$$\int\bar p(u)-\frac{1}{2}(\int|u-\chi 1|)^2\ge 0,$$

that is, $(\int|u-\chi 1|)^2\le 2\int\bar p(u)$, as claimed.

385K Corollary Whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and A, B are partitions of unity of finite entropy,

$$\sum_{a\in A,b\in B}|\bar\mu(a\cap b)-\bar\mu a\cdot\bar\mu b|\le\sqrt{2(H(A)+H(B)-H(A\vee B))}.$$

proof Replacing A, B by $A\setminus\{0\}$ and $B\setminus\{0\}$ if necessary, we may suppose that neither A nor B contains $\{0\}$. Let $(\mathfrak{C},\bar\lambda)$ be the probability algebra free product of $(\mathfrak{A},\bar\mu)$ with itself (325E, 325K). Set

$$u=\sum_{a\in A,b\in B}\frac{\bar\mu(a\cap b)}{\bar\mu a\cdot\bar\mu b}\chi(a\otimes b)\in L^0(\mathfrak{C});$$

then u is non-negative and integrable and $\int u=\sum_{a\in A,b\in B}\bar\mu(a\cap b)=1$. Now

$$\begin{aligned}\int \bar{p}(u) &= \sum_{a\in A, b\in B} \bar{\mu}(a\cap b)\ln\frac{\bar{\mu}(a\cap b)}{\bar{\mu}a\cdot\bar{\mu}b} \\ &= -H(A\vee B) - \sum_{a\in A, b\in B}\bar{\mu}(a\cap b)\ln\bar{\mu}a - \sum_{a\in A, b\in B}\bar{\mu}(a\cap b)\ln\bar{\mu}b \\ &= -H(A\vee B) - \sum_{a\in A}\bar{\mu}a\ln\bar{\mu}a - \sum_{b\in B}\bar{\mu}b\ln\bar{\mu}b \\ &= H(A)+H(B)-H(A\vee B).\end{aligned}$$

On the other hand,

$$\int |u-\chi 1| = \sum_{a\in A, b\in B}\bar{\mu}a\cdot\bar{\mu}b|\tfrac{\bar{\mu}(a\cap b)}{\bar{\mu}a\cdot\bar{\mu}b} - 1| = \sum_{a\in A, b\in B}|\bar{\mu}(a\cap b) - \bar{\mu}a\cdot\bar{\mu}b|.$$

So what we are seeking to prove is that

$$\int |u-\chi 1| \le \sqrt{2\int \bar{p}(u)},$$

which is 385J.

385L The next six lemmas are notes on some more or less elementary facts which will be used at various points in the next section. The first two are nearly trivial.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Then

$$\bar{\mu}(\sup_{i\in I} a_i\cap b_i) = 1 - \frac{1}{2}\sum_{i\in I}\bar{\mu}(a_i\triangle b_i).$$

proof

$$\begin{aligned}\bar{\mu}(\sup_{i\in I} a_i\cap b_i) = \sum_{i\in I}\bar{\mu}(a_i\cap b_i) &= \sum_{i\in I}\frac{1}{2}(\bar{\mu}a_i + \bar{\mu}b_i - \bar{\mu}(a_i\triangle b_i)) \\ &= 1 - \frac{1}{2}\sum_{i\in I}\bar{\mu}(a_i\triangle b_i).\end{aligned}$$

385M Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, $\langle B_k\rangle_{k\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0\in B_0$, and $\langle c_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$ such that $c_i\in\overline{\bigcup_{k\in\mathbb{N}} B_k}$ for every $i\in I$. Then

$$\lim_{k\to\infty}\sup_{i\in I}\rho(c_i, B_k) = 0,$$

writing $\rho(c,B) = \inf_{b\in B}\bar{\mu}(c\triangle b)$ for $c\in\mathfrak{A}$ and non-empty $B\subseteq\mathfrak{A}$, as in 3A4I.

proof Let $\epsilon>0$. Then $J=\{j : j\in I,\ \bar{\mu}c_j\ge\epsilon\}$ is finite. For each $j\in J$, $\lim_{k\to\infty}\rho(c_i,B_k)=0$, by 3A4I, while

$$\rho(c_i,B_k)\le\bar{\mu}(c_i\triangle 0) = \bar{\mu}c_i\le\epsilon$$

for every $i\in I\setminus J$. So

$$\limsup_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)\le\max(\epsilon,\limsup_{k\to\infty}\sup_{i\in J}\rho(c_i,B_k)) = \epsilon.$$

As ϵ is arbitrary, we have the result.

385N Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let A, B and C be partitions of unity in $\mathfrak{A}$.

(a) $H(A\vee B\vee C)+H(C)\le H(B\vee C)+H(A\vee C)$.

(b) $h(\pi,A)\le h(\pi,A\vee B)\le h(\pi,A)+h(\pi,B)\le h(\pi,A)+H(B)$.

(c) If $H(A)<\infty$,

$$h(\pi, A) = \inf_{n\in\mathbb{N}} H(D_{n+1}(A,\pi)) - H(D_n(A,\pi))$$
$$= \lim_{n\to\infty} H(D_{n+1}(A,\pi)) - H(D_n(A,\pi)).$$

(d) If $H(A) < \infty$ and $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] \subseteq \mathfrak{B}$, then $h(\pi, A) \le h(\pi\restriction\mathfrak{B}) + H(A|\mathfrak{B})$.

proof (a) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by C, so that $\mathfrak{C}$ is purely atomic and C is the set of its atoms. Then

$$H(A\vee B\vee C) + H(C) = H(A\vee B|\mathfrak{C}) + 2H(C)$$
$$\le H(A|\mathfrak{C}) + H(B|\mathfrak{C}) + 2H(C) = H(A\vee C) + H(B\vee C)$$

by 384Gb and 384Ga.

(b) We need only observe that $D_n(A\vee B,\pi) = D_n(A,\pi)\vee D_n(B,\pi)$ for every $n\in\mathbb{N}$, being the partition of unity generated by $\{\pi^i a : i < n,\, a\in A\}\cup\{\pi^i b : i < n,\, b\in B\}$. Consequently

$$h(\pi, A) = \lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi)) \le \lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi)\vee D_n(B,\pi))$$
$$= \lim_{n\to\infty}\frac{1}{n}H(D_n(A\vee B,\pi)) = h(\pi, A\vee B)$$
$$\le \lim_{n\to\infty}\frac{1}{n}(H(D_n(A,\pi) + H(D_n(B,\pi))) = h(\pi, A) + h(\pi, B)$$
$$\le h(\pi, A) + H(B)$$

as remarked in 384M.

(c) Set $\gamma_n = H(D_{n+1}(A,\pi)) - H(D_n(A,\pi))$ for each $n\in\mathbb{N}$. By 384H, $\gamma_n \ge 0$. From (a) we see that

$$\gamma_{n+1} = H(A\vee\pi[D_{n+1}(A,\pi)]) - H(A\vee\pi[D_n(A,\pi)])$$
$$\le H(\pi[D_{n+1}(A,\pi)]) - H(\pi[D_n(A,\pi]) = \gamma_n$$

for every $n\in\mathbb{N}$. So $\lim_{n\to\infty}\gamma_n = \inf_{n\in\mathbb{N}}\gamma_n$; write γ for the common value. Now

$$h(\pi, A) = \lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi)) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\gamma_i = \gamma$$

(273Ca).

(d) Let $P : L^1_{\bar\mu} \to L^1_{\bar\mu}$ be the conditional expectation operator corresponding to $\mathfrak{B}$. Let $\langle b_k\rangle_{k\in\mathbb{N}}$ be a sequence running over $\{[\![P(\chi a) > q]\!] : a\in A,\, q\in\mathbb{Q}\}$, so that $b_k\in\mathfrak{B}$ for every k, and for each $k\in\mathbb{N}$ let $\mathfrak{B}_k\subseteq\mathfrak{B}$ be the subalgebra generated by $\{b_i : i\le k\}$; let P_k be the conditional expectation operator corresponding to $\mathfrak{B}_k$. Writing $\mathfrak{B}_\infty\subseteq\mathfrak{B}$ for $\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$, and P_∞ for the corresponding conditional expectation operator, then $P(\chi a)\in L^0(\mathfrak{B}_\infty)$, so $P_\infty(\chi a) = P(\chi a)$, for every $a\in A$. So

$$H(A|\mathfrak{B}) = \sum_{a\in A}\int q(P\chi a) = H(A|\mathfrak{B}_\infty) = \lim_{k\to\infty} H(A|\mathfrak{B}_k),$$

by 384Gd.

For each k, let B_k be the set of atoms of $\mathfrak{B}_k$. Then

$$h(\pi, A) \le h(\pi, B_k) + H(A|\mathfrak{B}_k) \le h(\pi\restriction\mathfrak{B}) + H(A|\mathfrak{B}_k)$$

by 384N and the definition of $h(\pi\restriction\mathfrak{B})$. So

$$h(\pi, A) \le h(\pi\restriction\mathfrak{B}) + \lim_{k\to\infty} H(A|\mathfrak{B}_k) = h(\pi\restriction\mathfrak{B}) + H(A|\mathfrak{B}).$$

385O Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra.

(a) There is a function $h:\mathfrak{A}\to\mathfrak{B}$ such that $\bar\mu(a\triangle h(a))=\rho(a,\mathfrak{B})$ for every $a\in\mathfrak{A}$ and $h(a)\cap h(a')=0$ whenever $a\cap a'=0$.

(b) If A is a partition of unity in $\mathfrak{A}$, then $H(A|\mathfrak{B})\le\sum_{a\in A}q(\rho(a,\mathfrak{B}))$, where q is the function of 384A.

(c) If $\mathfrak{B}$ is atomless and $\langle a_i\rangle_{i\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$, then there is a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar\mu b_i=\bar\mu a_i$ and $\bar\mu(b_i\triangle a_i)\le 2\rho(a_i,\mathfrak{B})$ for every $i\in\mathbb{N}$.

proof (a) Let $P:L^1_{\bar\mu}\to L^1_{\bar\mu}$ be the conditional expectation operator associated with $\mathfrak{B}$. For any $c\in\mathfrak{B}$,

$$\begin{aligned}\int|P(\chi a)-\chi c|&=\int_{1\setminus c}P(\chi a)+\bar\mu c-\int_c P(\chi a)=\int_{1\setminus c}\chi a+\bar\mu c-\int_c\chi a\\&=\bar\mu(a\setminus c)+\bar\mu c-\bar\mu(a\cap c)=\bar\mu(a\triangle c).\end{aligned}$$

If $a\in\mathfrak{A}$ set $h(a)=[\![P(\chi a)>\frac12]\!]$. Then $|P(\chi a)-\chi h(a)|\le|P(\chi a)-\chi c|$ for any $c\in\mathfrak{B}$, so

$$\begin{aligned}\rho(a,\mathfrak{B})&=\inf_{c\in\mathfrak{B}}\bar\mu(a\triangle c)=\inf_{c\in\mathfrak{B}}\int|P(\chi a)-\chi c|\\&=\int|P(\chi a)-\chi h(a)|=\bar\mu(a\triangle h(a)).\end{aligned}$$

If $a\cap a'=0$, then

$$P(\chi a)+P(\chi a')=P\chi(a\cup a')\le\chi 1,$$

so

$$h(a)\cap h(a')=[\![P(\chi a)>\tfrac12]\!]\cap[\![P(\chi a')>\tfrac12]\!]\subseteq[\![P(\chi a)+P(\chi a')>1]\!]=0,$$

by 364D(b-i).

(b) By 384Ae, $q(1-t)\le q(t)$ whenever $0\le t\le\frac12$. Consequently $q(t)\le q(\min(t,1-t))$ for every $t\in[0,1]$, and $\bar q(u)\le\bar q(u\wedge(\chi1-u))$ whenever $u\in L^0(\mathfrak{A})$ and $0\le u\le\chi 1$. Fix $a\in A$ for the moment. We have

$$\bar q(P(\chi a))\le\bar q(P(\chi a)\wedge(\chi1-P(\chi a))=\bar q(|P(\chi a)-\chi h(a)|).$$

Consequently

$$\int\bar q(P\chi a)\le\int\bar q(|P(\chi a)-\chi h(a)|)\le q\Bigl(\int|P(\chi a)-\chi h(a)|\Bigr)$$

(because q is concave)

$$=q(\rho(a,\mathfrak{B})).$$

Summing over a,

$$H(A|\mathfrak{B})=\sum\nolimits_{a\in A}\int\bar q(P\chi a)\le\sum\nolimits_{a\in A}q(\rho(a,\mathfrak{B})).$$

(c) Set $b'_i=h(a_i)$ for each $i\in\mathbb{N}$. Then $\langle b'_i\rangle_{i\in\mathbb{N}}$ is disjoint. Next, for each $i\in\mathbb{N}$, take $b''_i\in\mathfrak{B}$ such that $b''_i\subseteq b'_i$ and $\bar\mu b''_i=\min(\bar\mu a_i,\bar\mu b''_i)$; then $\langle b''_i\rangle_{i\in\mathbb{N}}$ is disjoint and $\bar\mu b''_i\le\bar\mu a_i$ for every i. We can therefore find a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ such that $b_i\supseteq b''_i$ and $\bar\mu b_i=\bar\mu a_i$ for every i. (Use 331C to choose $\langle d_i\rangle_{i\in\mathbb{N}}$ inductively so that $d_i\subseteq 1\setminus(\sup_{j<i}d_j\cup\sup_{j\in\mathbb{N}}b''_j)$ and $\bar\mu d_i=\bar\mu a_i-\bar\mu b''_i$ for each i, and set $b_i=b''_i\cup d_i$.)

Take any $i\in\mathbb{N}$. If $\bar\mu b'_i>\bar\mu a_i$, then

$$\begin{aligned}\bar\mu(a_i\triangle b_i)&=\bar\mu(a_i\triangle b''_i)\le\bar\mu(a_i\triangle b'_i)+\bar\mu(b'_i\triangle b''_i)\\&=\bar\mu(a_i\triangle b'_i)+\bar\mu b'_i-\bar\mu a_i\le 2\bar\mu(a_i\triangle b'_i)=2\rho(a_i,\mathfrak{B}).\end{aligned}$$

If $\bar\mu b'_i\le\bar\mu a_i$, then

$$\begin{aligned}\bar\mu(a_i\triangle b_i)&\le\bar\mu(a_i\triangle b'_i)+\bar\mu(b'_i\triangle b_i)\\&=\bar\mu(a_i\triangle b'_i)+\bar\mu a_i-\bar\mu b'_i\le 2\bar\mu(a_i\triangle b'_i)=2\rho(a_i,\mathfrak{B}).\end{aligned}$$

385P Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Suppose that $B\subseteq\mathfrak{A}$. For $k\in\mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b: b\in B,\ |j|\le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b: b\in B,\ j\in\mathbb{Z}\}$. Then

(a) $\mathfrak{B}$ is the topological closure $\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$.

(b) $\pi[\mathfrak{B}]=\mathfrak{B}$.

(c) If $\mathfrak{C}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}]=\mathfrak{C}$, and $a\in\mathfrak{B}_k$, then

$$\rho(a,\mathfrak{C})\le(2k+1)\sum_{b\in B}\rho(b,\mathfrak{C}).$$

proof (a) Because $\langle\mathfrak{B}_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, $\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k$ is a subalgebra of $\mathfrak{A}$, so its closure also is (323J), and must be $\mathfrak{B}$.

(b) Of course $\pi^{-1}[\mathfrak{B}_{k+1}]$ is a closed subalgebra of $\mathfrak{A}$ containing $\pi^j b$ whenever $|j|\le k$ and $b\in B$, so includes $\mathfrak{B}_k$; thus $\pi[\mathfrak{B}_k]\subseteq\mathfrak{B}_{k+1}\subseteq\mathfrak{B}$ for every k, and

$$\pi[\mathfrak{B}]=\pi[\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}]\subseteq\overline{\bigcup_{k\in\mathbb{N}}\pi[\mathfrak{B}_k]}\subseteq\overline{\mathfrak{B}}\subseteq\mathfrak{B}$$

because π is continuous (324Kb). Similarly, $\pi^{-1}[\mathfrak{B}]\subseteq\mathfrak{B}$ and $\pi[\mathfrak{B}]=\mathfrak{B}$.

(c) For each $b\in B$, choose $c_b\in\mathfrak{C}$ such that $\bar{\mu}(b\triangle c_b)=\rho(c_b,\mathfrak{C})$ (385Oa). Set

$$e=\sup_{|j|\le k}\sup_{b\in B}\pi^j(b\triangle c_b);$$

then

$$\bar{\mu}e\le(2k+1)\sum_{b\in B}\bar{\mu}(b\triangle c_b)=(2k+1)\sum_{b\in B}\rho(b,\mathfrak{C}).$$

Now

$$\mathfrak{B}'=\{d:d\in\mathfrak{A},\exists\,c\in\mathfrak{C}\text{ such that }d\setminus e=c\setminus e\}$$

is a subalgebra of $\mathfrak{A}$. By 314Fa, applied to the order-continuous homomorphism $c\mapsto c\setminus e:\mathfrak{C}\to\mathfrak{A}_{1\setminus e}$, $\{c\setminus e: c\in\mathfrak{C}\}$ is an order-closed subalgebra of the principal ideal $\mathfrak{A}_{1\setminus e}$; by 313Id, applied to the order-continuous function $d\mapsto d\setminus e:\mathfrak{A}\to\mathfrak{A}_{1\setminus e}$, $\mathfrak{B}'$ is order-closed. If $b\in B$ and $|j|\le k$, then $\pi^j b\triangle\pi^j c_b\subseteq e$, so $\pi^j b\in\mathfrak{B}'$; accordingly $\mathfrak{B}'\supseteq\mathfrak{B}_k$. Now $a\in\mathfrak{B}_k$, so there is a $c\in\mathfrak{C}$ such that $a\triangle c\subseteq e$, and

$$\rho(a,\mathfrak{C})\le\bar{\mu}(a\triangle c)\le\bar{\mu}e\le(2k+1)\sum_{b\in B}\rho(b,\mathfrak{C}),$$

as claimed.

385Q Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and suppose *either* that $\mathfrak{A}$ is not purely atomic *or* that it is purely atomic and $H(D_0)=\infty$, where D_0 is the set of atoms of $\mathfrak{A}$. Then whenever $A\subseteq\mathfrak{A}$ is a partition of unity and $H(A)\le\gamma\le\infty$, there is a partition of unity B, refining A, such that $H(B)=\gamma$.

proof (a) By 384J, there is a partition of unity D_1 such that $H(D_1)=\infty$. Set $D=D_1\vee A$; then we still have $H(D)=\infty$. Enumerate D as $\langle d_i\rangle_{i\in\mathbb{N}}$. Choose $\langle B_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $B_0=A$. Given that B_k is a partition of unity, then if $H(B_k\vee\{d_k,1\setminus d_k\})\le\gamma$, set $B_{k+1}=B_k\vee\{d_k,1\setminus d_k\}$; otherwise set $B_{k+1}=B_k$.

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}B_k$. Note that, for each $d\in D$,

$$\{c:c\in\mathfrak{A},\ d\subseteq c\text{ or }d\cap c=0\}$$

is a closed subalgebra of $\mathfrak{A}$ including every B_k, so includes $\mathfrak{B}$. If $b\in\mathfrak{B}\setminus\{0\}$, there is surely some $d\in D$ such that $b\cap d\ne0$, so $b\supseteq d$; thus $\mathfrak{B}$ must be purely atomic. Let B be the set of atoms of $\mathfrak{B}$. Because $A=B_0\subseteq\mathfrak{B}$, B refines A.

(b) $H(B)\le\gamma$. **P** For each $k\in\mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by B_k, so that $\mathfrak{B}=\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$. Suppose that $b_0,\dots,b_n$ are distinct members of B. Then for each $k\in\mathbb{N}$ we can find disjoint $b_{0k},\dots,b_{nk}\in\mathfrak{B}_k$ such that $\bar{\mu}(b_{ik}\triangle b_i)\le\rho(b_i,\mathfrak{B}_k)$ for every $i\le n$ (385Oa). Accordingly $\bar{\mu}b_i=\lim_{k\to\infty}\bar{\mu}b_{ik}$ for each i, and

$$\sum_{i=0}^n q(\bar{\mu}b_i)=\lim_{k\to\infty}\sum_{i=0}^n q(\bar{\mu}b_{ik})\le\sup_{k\in\mathbb{N}}H(B_k)\le\gamma.$$

As $b_0,\dots,b_n$ are arbitrary, $H(B)\le\gamma$. **Q**

(c) $H(B) \ge \gamma$. **P?** Suppose otherwise. We know that

$$\lim_{k\to\infty} H(\{d_k, 1 \setminus d_k\}) = \lim_{k\to\infty} q(\bar{\mu}d_k) + q(1 - \bar{\mu}d_k) = 0.$$

Let $m \in \mathbb{N}$ be such that $H(B) + H(\{d_k, 1 \setminus d_k\}) \le \gamma$ for every $k \ge m$. Because B refines B_k, we must have

$$H(B_k \vee \{d_k, 1 \setminus d_k\}) \le H(B_k) + H(\{d_k, 1 \setminus d_k\}) \le \gamma,$$

so that $B_{k+1} = B_k \vee \{d_k, 1 \setminus d_k\}$ for every $k \ge m$. But this means that $d_k \in B$ for every $k \ge m$, so that

$$\gamma > H(B) \ge \sum_{k=m}^{\infty} q(\bar{\mu}d_k) = \infty,$$

which is impossible. **XQ**

Thus B has the required properties.

385X Basic exercises >(a) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism; set $\mathfrak{C} = \{c : \pi c = c\}$. Show that π is periodic, with period $n \ge 1$, iff $\pi \restriction \mathfrak{A}_c$ has order n in the group $\operatorname{Aut} \mathfrak{A}_c$ whenever $c \in \mathfrak{C} \setminus \{0\}$. Show that π is aperiodic iff $\pi \restriction \mathfrak{A}_c$ has infinite order in the group $\operatorname{Aut} \mathfrak{A}_c$ whenever $c \in \mathfrak{C} \setminus \{0\}$.

(b) In 385C, show that the family $\langle c_i \rangle_{1 \le i \le \omega}$ is uniquely determined.

>(c) Show that for a Bernoulli shift π, the Shannon-McMillan-Breiman theorem is a special case of the Ergodic Theorem. (*Hint*: $w_n = \frac{1}{n} \sum_{i=0}^{n-1} T^i w_1$.)

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\langle B_k \rangle_{k\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0 \in B_0$, and $\langle c_i \rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Show that

$$\lim_{k\to\infty} \sum_{i\in I} \rho(c_i, B_k) = \sum_{i\in I} \rho(c_i, B)$$

where $B = \overline{\bigcup_{k\in\mathbb{N}} B_k}$.

>(e) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism and A a partition of unity in $\mathfrak{A}$. Show that $h(\pi, D_n(A, \pi)) = h(\pi, A) = h(\pi, \pi[A])$ for any $n \ge 1$.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that $B \subseteq \mathfrak{A}$. For $k \in \mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \in \mathbb{N}\}$. Show that

$$\mathfrak{B} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{B}_k}, \quad \pi[\mathfrak{B}] \subseteq \mathfrak{B},$$

and that if $\mathfrak{C}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}] \subseteq \mathfrak{C}$, and $a \in \mathfrak{B}_k$, then $\rho(a, \mathfrak{C}) \le (k+1) \sum_{b\in B} \rho(b, \mathfrak{C})$.

385Y Further exercises (a) Give an example to show that the word 'injective' in the statement of 385C is essential.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving Boolean homomorphism. Set $\mathfrak{C} = \{c : \pi c = c\}$. Show that whenever $n \ge 1$, $0 \le \gamma < \frac{1}{n}$ and $B \subseteq \mathfrak{A}$ is finite, there is an $a \in \mathfrak{A}$ such that $a, \pi a, \pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap b \cap c) = \gamma \bar{\mu}(b \cap c)$ for every $b \in B$, $c \in \mathfrak{C}$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{P}$ be the set of all closed subalgebras of $\mathfrak{A}$ which are invariant under π, ordered by inclusion. Show that $\mathfrak{B} \mapsto h(\pi \restriction \mathfrak{B}) : \mathfrak{P} \to [0, \infty]$ is order-preserving and order-continuous on the left, in the sense that if $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and upwards-directed then $h(\pi \restriction \sup \mathfrak{Q}) = \sup_{\mathfrak{B}\in\mathfrak{Q}} h(\pi \restriction \mathfrak{B})$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism of finite entropy. Let $\mathfrak{P}$ be the set of all π-invariant closed subalgebras of $\mathfrak{A}$. Show that $\mathfrak{B} \mapsto h(\pi \restriction \mathfrak{B}) : \mathfrak{P} \to [0, \infty[$ is order-continuous. (*Hint*: if $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and downwards-directed, then for any partition of unity $A \subseteq \mathfrak{A}$, $H(A | \inf \mathfrak{Q}) = \sup_{\mathfrak{B}\in\mathfrak{Q}} H(A | \mathfrak{B})$.)

385 Notes and comments I have taken the trouble to give sharp forms of the Halmos-Rokhlin-Kakutani lemma (385E) and the Cziszár-Kullback inequality (385J); while it is possible to get through the principal results of the next two sections with rather less, the formulae become better focused if we have the exact expressions available. Of course one can always go farther still (385Yb). Ornstein's theorem in §386 (though not Sinaǐ's, as stated there) can be deduced from the special case of the Shannon-McMillan-Breiman theorem (385G) in which the homomorphism π is a Bernoulli shift, which can be deduced from the Ergodic Theorem (385Xc).

Lemma 385D is the starting point of the theory of 'recurrence'; the next steps are in 382Yc-382Yd and 387E-387F.

386 Ornstein's theorem

I come now to the most important of the handful of theorems known which enable us to describe automorphisms of measure algebras up to isomorphism: two two-sided Bernoulli shifts (on algebras of countable Maharam type) of the same entropy are isomorphic (386I, 386K). This is hard work. It requires both delicate ϵ-δ analysis and substantial skill with the manipulation of measure-preserving homomorphisms. The proof is based on two difficult lemmas (386C and 386F), and includes Sinaǐ's theorem (386E, 386L), describing the Bernoulli shifts which arise as factors of a given ergodic automorphism.

386A The following definitions offer a language in which to express the ideas of this section.

Definitions Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) A **Bernoulli partition** for π is a partition of unity $\langle a_i\rangle_{i\in I}$ such that

$$\bar\mu(\inf_{j\le k}\pi^j a_{i(j)})=\prod_{j=0}^k\bar\mu a_{i(j)}$$

whenever $i(0),\dots,i(k)\in I$.

(b) If π is an automorphism, a Bernoulli partition $\langle a_i\rangle_{i\in I}$ for π is **(two-sidedly) generating** if the closed subalgebra generated by $\{\pi^j a_i: i\in I,\ j\in\mathbb{Z}\}$ is $\mathfrak{A}$ itself.

(c) A **factor** of $(\mathfrak{A},\bar\mu,\pi)$ is a triple $(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\pi\restriction\mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]=\mathfrak{B}$.

386B Remarks Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism and $\langle a_i\rangle_{i\in I}$ a Bernoulli partition for π.

(a) $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent, where $\mathfrak{A}_0$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_i: i\in I\}$. **P** Suppose that $c_j\in\pi^j[\mathfrak{A}_0]$ for $j\le k$. Then each $\pi^{-j}c_j\in\mathfrak{A}_0$ is expressible as $\sup_{i\in I_j}a_i$ for some $I_j\subseteq I$. Now

$$\begin{aligned}\bar\mu(\inf_{j\le k}c_j)&=\bar\mu(\sup_{i_0\in I_0,\dots,i_k\in I_k}\inf_{j\le k}\pi^j a_{i_j})\\&=\sum_{i_0\in I_0,\dots,i_k\in I_k}\bar\mu(\inf_{j\le k}\pi^j a_{i_j})=\sum_{i_0\in I_0,\dots,i_k\in I_k}\prod_{j=0}^k\bar\mu a_{i_j}\\&=\prod_{j=0}^k\sum_{i\in I_j}\bar\mu a_i=\prod_{j=0}^k\bar\mu(\sup_{i\in I_j}a_i)=\prod_{j=0}^k\bar\mu c_j.\end{aligned}$$

As $c_0,\dots,c_k$ are arbitrary, $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent. **Q**

(b) If π is an automorphism, then $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent, by 384Sf.

(c) Setting $A=\{a_i: i\in I\}\setminus\{0\}$, we have $h(\pi,A)=H(A)$, as in part (a) of the proof of 384R, so $h(\pi)\ge H(A)$.

(d) If $H(A)>0$, then $\mathfrak{A}$ is atomless. **P** As A contains at least two elements of non-zero measure, $\gamma=\max_{a\in A}\bar{\mu}a<1$. Because $\langle a_i\rangle_{i\in I}$ is a Bernoulli partition, every member of $D_k(A,\pi)$ has measure at most γ^k, for any $k\in\mathbb{N}$. Thus any atom of $\mathfrak{A}$ could have measure at most $\inf_{k\in\mathbb{N}}\gamma^k=0$. **Q**

(e) If $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$, then $h(\pi\restriction\mathfrak{B})\le h(\pi)$, just because $h(\pi\restriction\mathfrak{B})$ is calculated from the action of π on a smaller set of partitions. If $\mathfrak{C}^+$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a_i:i\in I,\,j\in\mathbb{N}\}$, then $\pi[\mathfrak{C}^+]\subseteq\mathfrak{C}^+$ (compare 385Pb), and $\pi\restriction\mathfrak{C}^+$ is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$ and entropy $H(A)$, so that

$$H(A)=h(\pi\restriction\mathfrak{C}^+)$$

by the Kolmogorov-Sinaǐ theorem (384P, 384R).

(f) If π is an automorphism, and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a_i:i\in I,\,j\in\mathbb{Z}\}$, then $\pi[\mathfrak{C}]=\mathfrak{C}$ (385Pb) and $\pi\restriction\mathfrak{C}$ is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

(g) Thus every Bernoulli partition for π gives rise to a factor of $(\mathfrak{A},\bar{\mu},\pi)$ which is a one-sided Bernoulli shift, and if π is an automorphism we can extend this to the corresponding two-sided Bernoulli shift. If π has a generating Bernoulli partition then it is itself a Bernoulli shift.

(h) Now suppose that $(\mathfrak{B},\bar{\nu})$ is another probability algebra, $\phi:\mathfrak{B}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism, and $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition for ϕ such that $\bar{\nu}b_i=\bar{\mu}a_i$ for every i. We have a unique measure-preserving Boolean homomorphism $\theta^+:\mathfrak{C}^+\to\mathfrak{B}$ such that $\theta^+(\pi^j a_i)=\phi^j b_i$ for every $i\in I$, $j\in\mathbb{N}$. (Apply 324P.) Now $\theta^+\pi=\phi\theta^+$. (The set $\{a:\theta^+\pi a=\phi\theta^+a\}$ is a closed subalgebra of $\mathfrak{C}^+$ containing every $\pi^j a_i$.)

(i) If, in (g) above, π and ϕ are both automorphisms, then the same arguments show that we have a unique measure-preserving Boolean homomorphism $\theta:\mathfrak{C}\to\mathfrak{B}$ such that $\theta a_i=b_i$ for every $i\in I$ and $\theta\pi=\phi\theta$.

386C Lemma Let $(\mathfrak{A},\bar{\mu})$ be an atomless probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^\infty\gamma_i=1,\quad \sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

where q is the function of 384A. Then for any $\epsilon>0$ we can find a partition $\langle a'_i\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{A}$ such that

(i) $\{i:a'_i\ne0\}$ is finite,

(ii) $\sum_{i=0}^\infty|\gamma_i-\bar{\mu}a'_i|\le\epsilon$,

(iii) $\sum_{i=0}^\infty\bar{\mu}(a'_i\mathbin{\triangle}a_i)\le\epsilon+6\sqrt{\sum_{i=0}^\infty|\bar{\mu}a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$

where $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$,

(iv) $H(A')\le h(\pi,A')+\epsilon$

where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$.

proof (a) Of course $h(\pi,A)\le H(A)$, by 384Ma, so the square root $\sqrt{2(H(A)-h(\pi,A))}$ gives no difficulty. Set $\beta=\sqrt{\sum_{i=0}^\infty|\bar{\mu}a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$, $\delta=\min(\frac14,\frac1{24}\epsilon)$.

There is a sequence $\langle\bar{\gamma}_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\{i:\bar{\gamma}_i>0\}$ is finite, $\sum_{i=0}^\infty\bar{\gamma}_i=1$, $\sum_{i=0}^\infty|\bar{\gamma}_i-\gamma_i|\le2\delta^2$ and $\sum_{i=0}^\infty q(\bar{\gamma}_i)\le h(\pi)$. **P** Take $k\in\mathbb{N}$ such that $\sum_{i=k}^\infty\gamma_i\le\delta^2$, and set $\bar{\gamma}_i=\gamma_i$ for $i<k$, $\bar{\gamma}_k=\sum_{i=k}^\infty\gamma_i$ and $\bar{\gamma}_i=0$ for $i>k$; then $q(\bar{\gamma}_k)\le\sum_{i=k}^\infty q(\gamma_i)$ (384Ab), so

$$\sum_{i=0}^\infty q(\bar{\gamma}_i)\le\sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

while

$$\sum_{i=0}^\infty|\bar{\gamma}_i-\gamma_i|\le\bar{\gamma}_k+\sum_{i=k}^\infty\gamma_i\le2\delta^2.\text{ **Q**}$$

Because $\sum_{i=0}^\infty q(\bar{\gamma}_i)$ is finite, there is a partition of unity C in $\mathfrak{A}$, of finite entropy, such that $\sum_{i=0}^\infty q(\bar{\gamma}_i)\le h(\pi,C)+3\delta$; replacing C by $C\vee A$ if need be (note that $C\vee A$ still has finite entropy, by 384H), we may suppose that C refines A.

There is a sequence $\langle\gamma'_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{i=0}^{\infty}\gamma'_i=1$, $\{i:\gamma'_i>0\}$ is finite, $\sum_{i=0}^{\infty}|\gamma'_i-\gamma_i|\le 4\delta^2$ and

$$\sum_{i=0}^{\infty}q(\gamma'_i)=h(\pi,C)+3\delta.$$

P Take $k\in\mathbb{N}$ such that $\bar\gamma_i=0$ for $i>k$. Take $r\ge 1$ such that $\delta^2\ln(\frac{r}{\delta^2})\ge h(\pi,C)+3\delta$ and set

$$\begin{aligned}\tilde\gamma_i&=(1-\delta^2)\bar\gamma_i \text{ for } i\le k,\\ &=\frac{1}{r}\delta^2 \text{ for } k+1\le i\le k+r,\\ &=0 \text{ for } i>k+r.\end{aligned}$$

Then

$$\sum_{i=0}^{\infty}|\tilde\gamma_i-\bar\gamma_i|=2\delta^2,\quad \sum_{i=0}^{\infty}|\tilde\gamma_i-\gamma_i|\le 4\delta^2,$$

$$\sum_{i=0}^{k+r}q(\bar\gamma_i)\le h(\pi,C)+3\delta\le\delta^2\ln(\tfrac{r}{\delta^2})=rq(\tfrac{\delta^2}{r})\le\sum_{i=0}^{k+r}q(\tilde\gamma_i).$$

Now the function

$$\alpha\mapsto\sum_{i=0}^{k+r}q(\alpha\bar\gamma_i+(1-\alpha)\tilde\gamma_i):[0,1]\to\mathbb{R}$$

is continuous, so there is some $\alpha\in[0,1]$ such that

$$\sum_{i=0}^{k+r}q(\alpha\bar\gamma_i+(1-\alpha)\tilde\gamma_i)=h(\pi,C)+3\delta,$$

and we can set $\gamma'_i=\alpha\bar\gamma_i+(1-\alpha)\tilde\gamma_i$ for every i; of course

$$\sum_{i=0}^{\infty}|\gamma'_i-\gamma_i|\le\alpha\sum_{i=0}^{\infty}|\bar\gamma_i-\gamma_i|+(1-\alpha)\sum_{i=0}^{\infty}|\tilde\gamma_i-\gamma_i|\le 4\delta^2. \textbf{ Q}$$

Set $M=\{i:\gamma'_i\ne 0\}$, so that M is finite.

(b) Let $\eta\in\,]0,\delta]$ be so small that

(i) $|q(s)-q(t)|\le\frac{\delta}{1+\#(M)}$ whenever s, $t\in[0,1]$ and $|s-t|\le 3\eta$,

(ii) $\sum_{c\in C}q(\min(\bar\mu c,2\eta))\le\delta$,

(iii) $\eta\le\frac{1}{6}$.

(Actually, (iii) is a consequence of (i). For (ii) we must of course rely on the fact that $\sum_{c\in C}q(\bar\mu c)$ is finite.)

Let ν be the probability measure on M defined by saying that $\nu\{i\}=\gamma'_i$ for every $i\in M$, and λ the product measure on $M^{\mathbb{N}}$. Define $X_{ij}:M^{\mathbb{N}}\to\{0,1\}$, for $i\in M$ and $j\in\mathbb{N}$, and $Y_j:M^{\mathbb{N}}\to\mathbb{R}$, for $j\in\mathbb{N}$, by setting

$$\begin{aligned}X_{ij}(\omega)&=1 \text{ if } \omega(j)=i,\\ &=0 \text{ otherwise},\\ Y_j(\omega)&=\ln(\gamma'_{\omega(j)}) \text{ for every} \omega\in M^{\mathbb{N}}.\end{aligned}$$

Then, for each $i\in M$, $\langle X_{ij}\rangle_{j\in\mathbb{N}}$ is an independent sequence of random variables, all with expectation γ'_i, and $\langle Y_j\rangle_{j\in\mathbb{N}}$ is also an independent sequence of random variables, all with expectation

$$\sum_{i\in M}\gamma'_i\ln\gamma'_i=-\sum_{i=0}^{\infty}q(\gamma'_i)=-h(\pi,C)-3\delta.$$

Let $n\ge 1$ be so large that

(iv) $\bar\mu[\![w_n-h(\pi,C)\chi 1\ge\delta]\!]<\eta$, where

$$w_n=\frac{1}{n}\sum_{d\in D_n(C,\pi)}\ln(\frac{1}{\bar\mu d})\chi d;$$

(v)

$$\Pr\bigl(\sum_{i\in M}|\frac{1}{n}\sum_{j=0}^{n-1}X_{ij}-\gamma'_i|\le\eta\bigr)\ge 1-\delta,$$

$$\Pr\bigl(|\frac{1}{n}\sum_{j=0}^{n-1}Y_j+h(\pi,C)+3\delta|\le\delta\bigr)\ge 1-\delta;$$

(vi) $e^{n\delta}\ge 2,\quad \frac{1}{n+1}\le\eta,\quad q(\frac{1}{n+1})+q(\frac{n}{n+1})\le\delta;$

these will be true for all sufficiently large n, using the Shannon-McMillan-Breiman theorem (385H) for (iv) and the strong law of large numbers (in any of the forms 273D, 273H or 273I) for (v).

(c) There is a family $\langle b_{ji}\rangle_{j<n,i\in M}$ such that

(α) for each $j<n$, $\langle b_{ji}\rangle_{i\in M}$ is a partition of unity in $\mathfrak{A}$,

(β) $\bar\mu(\inf_{j<n}b_{j,i(j)})=\prod_{j=0}^{n-1}\gamma'_{i(j)}$ for every $i(0),\dots,i(n-1)\in M$,

(γ) $\sum_{i\in M}\bar\mu(b_{ji}\cap\pi^j a_i)\ge 1-\beta^2-4\delta^2$ for every $j<n$.

P Construct $\langle b_{ji}\rangle_{i\in M}$ for $j=n-1$, $n-2,\dots,0$, as follows. Given b_{ji}, for $k<j<n$, such that

$$\bar\mu(\inf_{j\le k}\pi^j a_{i(j)}\cap\inf_{k<j<n}b_{j,i(j)})=\bar\mu(\inf_{j\le k}\pi^j a_{i(j)})\cdot\prod_{j=k+1}^{n-1}\gamma'_{i(j)}$$

for every $i(0),\dots,i(n-1)\in M$ (of course this hypothesis is trivial for $k=n-1$), let B_k be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by $\{b_{ji}:i\in M,\ k<j<n\}$. Then $\bar\mu(b\cap d)=\bar\mu b\cdot\bar\mu d$ for every $b\in B_k$, $d\in D_{k+1}(A,\pi)$.

Now

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\gamma'_i\bar\mu c|\\
&\le\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c|+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma'_i|\sum_{c\in D_k(A,\pi)}\bar\mu c\\
&\le\sum_{i=0}^{\infty}|\gamma_i-\gamma'_i|+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c|\\
&\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sqrt{2\bigl(H(\pi^k[A])+H(D_k(A,\pi))-H(D_{k+1}(A,\pi))\bigr)}
\end{aligned}$$

(by 385K, because $D_{k+1}(A,\pi)=\pi^k[A]\wedge D_k(A,\pi)$)

$$\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}$$

(because $h(\pi,A)\le H(D_{k+1}(A,\pi))-H(D_k(A,\pi))$, by 385Nc)

$$=\beta^2+4\delta^2.$$

Choose a partition of unity $\langle b_{ki}\rangle_{i\in M}$ such that, for each $c\in D_k(A,\pi)$, $b\in B_k$, $i\in M$,

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu(b\cap c),$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\ge\gamma'_i\bar\mu(b\cap c)\text{ then }b_{ki}\cap b\cap c\subseteq\pi^k a_i,$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\le\gamma'_i\bar\mu(b\cap c)\text{ then }\pi^k a_i\cap b\cap c\subseteq b_{ki}.$$

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Note that in these formulae we always have

$$\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c,\quad \bar\mu(\pi^k a_i\cap b\cap c)=\bar\mu(\pi^k a_i\cap c)\cdot\bar\mu b.$$

Consequently

$$\begin{aligned}\sum_{i\in M}\bar\mu(\pi^k a_i\cap b_{ki}) &= \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i\in M}\bar\mu(b\cap c\cap(\pi^k a_i\cap b_{ki}))\\ &= \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\min(\bar\mu(b\cap c\cap\pi^k a_i),\gamma'_i\bar\mu(b\cap c))\\ &\ge \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu(b\cap c\cap\pi^k a_i)-|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|\\ &= 1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|\\ &= 1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu b\cdot|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|\\ &= 1-\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|\ge 1-\beta^2-4\delta^2.\end{aligned}$$

Also we have

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu b\cdot\bar\mu c$$

for every $b\in B_k$, $c\in D_k(A,\pi)$ and $i\in M$, so the (downwards) induction proceeds. **Q**

(d) Let B be the set of atoms of the algebra generated by $\{b_{ji}: j<n,\,i\in M\}$. For $b\in B$, $d\in D_n(C,\pi)$ set

$$I_{bd}=\{j: j<n,\,\exists\, i\in M,\, b\subseteq b_{ji},\, d\subseteq\pi^j a_i\}.$$

Then, for any $j<n$,

$$\sup\{b\cap d: b\in B,\, d\in D_n(C,\pi),\, j\in I_{bd}\}=\sup_{i\in M} b_{ji}\cap\pi^j a_i,$$

because C refines A, so every $\pi^j a_i$ is a supremum of members of $D_n(C,\pi)$. Accordingly

$$\sum_{b\in B,d\in D_n(C,\pi)}\#(I_{bd})\bar\mu(b\cap d)=\sum_{j=0}^{n-1}\sum_{i\in M}\bar\mu(b_{ji}\cap\pi^j a_i)\ge n(1-\beta^2-4\delta^2).$$

Set

$$e_0=\sup\{b\cap d: b\in B,\, d\in D_n(C,\pi),\,\#(I_{bd})\ge n(1-\beta-4\delta)\};$$

then $\bar\mu e_0\ge 1-\beta-\delta$.

(e) Let $B'\subseteq B$ be the set of those $b\in B$ such that

$$\bar\mu b\le e^{-n(h(\pi,C)+2\delta)},\qquad \textstyle\sum_{i\in M}|\gamma'_i-\frac{1}{n}\#(\{j:j<n,\, b\subseteq b_{ji}\})|\le\eta.$$

Then $\bar\mu(\sup B')\ge 1-2\delta$. **P** Set

$$\begin{aligned}B'_1&=\{b: b\in B,\,\bar\mu b\le e^{-n(h(\pi,C)+2\delta)}\}\\ &=\{b: b\in B,\, h(\pi,C)+2\delta+\frac{1}{n}\ln(\bar\mu b)\le 0\}\\ &=\{\inf_{j<n} b_{i,i(j)}: i(0),\dots,i(n-1)\in M,\, h(\pi,C)+2\delta+\frac{1}{n}\sum_{j=0}^{n-1}\ln\gamma'_{i(j)}\le 0\}.\end{aligned}$$

Then

$$\begin{aligned}\bar\mu(\sup B_1') &= \Pr(h(\pi,C)+2\delta+\frac{1}{n}\sum_{j=0}^{n-1}Y_j\le 0)\\ &\ge \Pr(|h(\pi,C)+3\delta+\frac{1}{n}\sum_{j=0}^{n-1}Y_j|\le\delta)\ge 1-\delta\end{aligned}$$

by the choice of n. On the other hand, setting

$$\begin{aligned}B_2' &= \{b:b\in B,\ \sum_{i\in M}|\gamma_i'-\frac{1}{n}\#(\{j:j<n,\ b\subseteq b_{ji}\})|\le\eta\}\\ &= \{\inf_{j<n} b_{i,i(j)}:i(0),\dots,i(n-1)\in M,\ \sum_{i\in M}|\gamma_i'-\frac{1}{n}\#(\{j:i(j)=i\})|\le\eta\},\end{aligned}$$

we have

$$\bar\mu(\sup B_2')=\Pr(\textstyle\sum_{i\in M}|\gamma_i'-\frac{1}{n}\sum_{j=0}^{n-1}X_{ij}|\le\eta)\ge 1-\delta$$

by the other half of clause (b-v). Since $B'=B_1'\cap B_2'$, $\bar\mu(\sup B)\ge 1-2\delta$. **Q**

Let D_0' be the set of those $d\in D_n(C,\pi)$ such that

$$\frac{1}{n}\ln(\frac{1}{\bar\mu d})\le h(\pi,C)+\delta,\quad\text{i.e.,}\quad \bar\mu d\ge e^{-n(h(\pi,C)+\delta)};$$

by (b-iv), $\bar\mu(\sup D_0')>1-\eta$. Let $D'\subseteq D_0'$ be a finite set such that $\bar\mu(\sup D')\ge 1-\eta$. If $d\in D'$, $b\in B'$ then

$$\bar\mu d\ge e^{-n(h(\pi,C)+\delta)}\ge e^{n\delta}\bar\mu b\ge 2\bar\mu b.$$

Since $\bar\mu(\sup D')\le 1\le 2\bar\mu(\sup B')$ (remember that $\delta\le\frac{1}{4}$, $\#(D')\le\#(B')$.

Set $e_1=e_0\cap\sup B'$, so that $\bar\mu e_1\ge 1-\beta-3\delta$, and

$$D''=\{d:d\in D',\ \bar\mu(d\cap e_1)\ge\frac{1}{2}\bar\mu d\};$$

then

$$\bar\mu(\sup(D'\setminus D''))\le 2\bar\mu(1\setminus e_1)\le 2\beta+6\delta,$$

so

$$\bar\mu(\sup D'')\ge 1-2\beta-6\delta-\eta\ge 1-2\beta-7\delta.$$

(f) If $d_1,\dots,d_k\in D''$ are distinct,

$$\bar\mu(\sup_{1\le i\le k}d_i\cap e_1)\ge\frac{k}{2}\inf_{i\le k}\bar\mu d_i\ge k\sup_{b\in B'}\bar\mu b,$$

and

$$\#(\{b:b\in B',\ b\cap e_0\cap\sup_{1\le i\le k}d_i\}\ne 0)\ge k.$$

By the Marriage Lemma (3A1K), there is an injective function $f_0:D''\to B'$ such that $d\cap f_0(d)\cap e_0\ne 0$ for every $d\in D''$. Because $\#(D')\le\#(B')$, we can extend f_0 to an injective function $f:D'\to B'$.

(g) By the Halmos-Rokhlin-Kakutani lemma, in the strong form 385E(iv), there is an $a\in\mathfrak{A}$ such that $a,\pi^{-1}a,\dots,\pi^{-n+1}a$ are disjoint and $\bar\mu(a\cap d)=\frac{1}{n+1}\bar\mu d$ for every $d\in D'\cup\{1\}$. Set $e=\sup\{\pi^{-j}(a\cap d):j<n,\ d\in D'\}$. Because $\langle\pi^{-j}(a\cap d)\rangle_{j<n,d\in D'}$ is disjoint,

$$\bar\mu e=\textstyle\sum_{j=0}^{n-1}\sum_{d\in D'}\bar\mu(a\cap d)=\frac{n}{n+1}\sum_{d\in D'}\bar\mu d\ge(1-\eta)^2\ge 1-2\eta.$$

(h) For $i\in M$, set

$$a_i'=\sup\{\pi^{-j}(a\cap d):j<n,\ d\in D',\ f(d)\subseteq b_{ji}\}.$$

Then the a'_i are disjoint. **P** Suppose that i, $i' \in M$ are distinct. If j, $j' < n$ and d, $d' \in D'$ and $f(d) \subseteq b_{ji}$, $f(d') \subseteq b_{j'i'}$, then either $j \neq j'$ or $j = j'$. In the former case,

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j}a \cap \pi^{-j'}a = 0.$$

In the latter case, $b_{ji} \cap b_{j'i'} = 0$, so $f(d) \neq f(d')$ and $d \neq d'$ and

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j}(d \cap d') = 0. \text{ **Q**}$$

Observe that

$$\sup_{i \in M} a'_i = \sup_{j<n, d \in D'} \pi^{-j}(a \cap d) = e$$

because if $j < n$ and $d \in D'$ there must be some $i \in M$ such that $f(d) \subseteq b_{ji}$. Take any $m \in \mathbb{N} \setminus M$ and set $a'_m = 1 \setminus e$, $a'_i = 0$ for $i \in \mathbb{N} \setminus (M \cup \{m\})$; then $\langle a'_i \rangle_{i \in \mathbb{N}}$ is a partition of unity. Now

$$\begin{aligned}
\sum_{i \in M} |\bar\mu a'_i - \gamma'_i| &\le \sum_{i \in M} \gamma'_i |1 - n\bar\mu(a \cap \sup D')| + \sum_{i \in M} |\bar\mu a'_i - n\gamma'_i \bar\mu(a \cap \sup D')| \\
&\le 1 - \tfrac{n}{n+1}\bar\mu(\sup D') \\
&\quad + \sum_{i \in M} |\sum_{j=0}^{n-1} \sum_{\substack{d \in D' \\ f(d) \subseteq b_{ji}}} \bar\mu(\pi^{-j}(a \cap d)) - n\gamma'_i \sum_{d \in D'} \bar\mu(a \cap d)| \\
&\le 1 - (1-\eta)^2 \\
&\quad + \sum_{d \in D'} \sum_{i \in M} |\bar\mu(a \cap d) \cdot \#(\{j : j < n,\, f(d) \subseteq b_{ji}\}) - n\gamma'_i \bar\mu(a \cap d)| \\
&\le 1 - (1-\eta)^2 + \sum_{d \in D'} \bar\mu(a \cap d) n\eta
\end{aligned}$$

(see the definition of B' in (d) above)

$$\le 2\eta + n\eta\bar\mu a \le 3\eta.$$

So

$$\begin{aligned}
\sum_{i=0}^{\infty} |\bar\mu a'_i - \gamma_i| &\le \bar\mu a'_m + \sum_{i \in M} |\bar\mu a'_i - \gamma'_i| + \sum_{i=0}^{\infty} |\gamma'_i - \gamma_i| \\
&\le 2\eta + 3\eta + 4\delta^2 \le 6\delta \le \epsilon.
\end{aligned}$$

We shall later want to know that $|\bar\mu a'_i - \gamma'_i| \le 3\eta$ for every i; for $i \in M$ this is covered by the formulae above, for $i = m$ it is true because $\bar\mu a'_m = 1 - \bar\mu e \le 2\eta$ (see (g)), and for other i it is trivial.

(i) The next step is to show that $\sum_{i=0}^{\infty} \bar\mu(a'_i \cap a_i) \ge 1 - 3\beta - 12\delta$. **P** It is enough to consider the case in which $3\beta + 12\delta < 1$. We know that

$$\begin{aligned}
\sup_{i \in \mathbb{N}} a'_i \cap a_i &\supseteq \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D', \\
&\qquad\qquad \exists\, i \in M \text{ such that } f(d) \subseteq b_{ji} \text{ and } d \subseteq \pi^j a_i\} \\
&= \sup\{\pi^{-j}(a \cap d) : d \in D',\, j \in I_{f(d),d}\}
\end{aligned}$$

(see (d) for the definition of I_{bd}) has measure at least $\sum_{d \in D'} \#(I_{f(d),d})\bar\mu(a \cap d)$.

For any $d \in D''$, we arranged that $d \cap f(d) \cap e_0 \neq 0$. This means that there must be some $b \in B$ and $d' \in D_n(C, \pi)$ such that $d \cap f(d) \cap b \cap f(d') \neq 0$ and $\#(I_{bd'}) \ge n(1 - \beta - 4\delta)$; of course $b = f(d)$ and $d' = d$, so that $\#(I_{f(d),d})$ must be at least $n(1 - \beta - 4\delta)$. Accordingly

$$\begin{aligned}
\sum_{i=0}^{\infty} \bar\mu(a'_i \cap a_i) &\ge \sum_{d \in D''} n(1 - \beta - 4\delta)\bar\mu(a \cap d) = n(1 - \beta - 4\delta)\tfrac{1}{n+1}\bar\mu(\sup D'') \\
&\ge (1-\eta)(1 - \beta - 4\delta)(1 - 2\beta - 7\delta) \ge 1 - 3\beta - 12\delta. \text{ **Q**}
\end{aligned}$$

But this means that

$$\sum_{i=0}^{\infty}\bar\mu(a'_i\triangle a_i)=2(1-\sum_{i=0}^{\infty}\bar\mu(a'_i\cap a_i))\le 6\beta+24\delta\le\epsilon+6\beta$$

(using 385L for the equality).

(j) Finally, we need to estimate $H(A')$ and $h(\pi,A')$, where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$. For the former, we have $H(A')\le h(\pi,C)+4\delta$. **P** $|\bar\mu a'_i-\gamma'_i|\le 3\eta$ for every i, by (h) above. So by (b-i),

$$H(A')=\sum_{i\in M\cup\{m\}}q(\bar\mu a'_i)\le\delta+\sum_{i=0}^{\infty}q(\gamma'_i)=h(\pi,C)+4\delta.\ \mathbf{Q}$$

(k) Consider the partition of unity

$$A''=A'\vee\{a,1\setminus a\}.$$

Let $\mathfrak{D}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^jc:j\in\mathbb{Z},\,c\in A''\}$.

(i) $a\cap d\in\mathfrak{D}$ for every $d\in D'$. **P** Of course $a\cap e\in\mathfrak{D}$, because $1\setminus e=a'_m$. If $d'\in D'$ and $d'\ne d$, then (because f is injective) $f(d)\ne f(d')$; there must therefore be some $k<n$ and distinct $i,\,i'\in M$ such that $f(d)\subseteq b_{ki}$ and $f(d')\subseteq b_{ki'}$. But this means that $\pi^{-k}(a\cap d)\subseteq a'_i$ and $\pi^{-k}(a\cap d')\subseteq a'_{i'}$, so that $a\cap d\subseteq\pi^ka'_i$ and $a\cap d'\cap\pi^ka'_i=0$.

What this means is that if we set

$$\tilde d=a\cap e\cap\inf\{\pi^ka'_i:k<n,\,i\in M,\,a\cap d\subseteq\pi^ka'_i\},$$

we get a member of $\mathfrak{D}$ (because every $a'_i\in\mathfrak{D}$, and $\pi[\mathfrak{D}]=\mathfrak{D}$) including $a\cap d$ and disjoint from $a\cap d'$ whenever $d'\in D'$ and $d'\ne d$. But as $a\cap\pi^{-j}a=0$ if $0<j<n$, $a\cap e$ must be $\sup\{a\cap d':d'\in D'\}$, and $a\cap d=\tilde d$ belongs to $\mathfrak{D}$. **Q**

(ii) Consequently $c\cap e\in\mathfrak{D}$ for every $c\in C$. **P** We have

$$\begin{aligned}c\cap e&=\sup\{c\cap\pi^{-j}(a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(\pi^jc\cap a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(a\cap d):j<n,\,d\in D',\,d\subseteq\pi^jc\}\end{aligned}$$

(because if $d\in D'$ and $j<n$ then either $d\subseteq\pi^jc$ or $d\cap\pi^jc=0$)

$$\in\mathfrak{D}$$

because $a\cap d\in\mathfrak{D}$ for every $d\in D'$ and $\pi^{-1}[\mathfrak{D}]=\mathfrak{D}$. **Q**

(iii) It follows that $h(\pi,A'')\ge h(\pi,C)-\delta$. **P** For any $c\in C$,

$$\rho(c,\mathfrak{D})\le\bar\mu(c\triangle(c\cap e))=\bar\mu(c\setminus e)\le\min(\bar\mu c,2\eta)\le\frac{1}{3}.$$

So

$$h(\pi,C)\le h(\pi\restriction\mathfrak{D})+H(C|\mathfrak{D})$$

(385Nd, because $\pi[\mathfrak{D}]=\mathfrak{D}$)

$$\le h(\pi,A'')+\sum_{c\in C}q(\rho(c,\mathfrak{D}))$$

(by the Kolmogorov-Sinaĭ theorem (384P) and 385Ob)

$$\le h(\pi,A'')+\sum_{c\in C}q(\min(\bar\mu c,2\eta))$$

(because q is monotonic on $[0,\frac{1}{3}]$)

$$\le h(\pi,A'')+\delta$$

by the choice of η. **Q**

(iv) Finally, $h(\pi,A')\ge h(\pi,C)-2\delta$. **P** Using 385Nb,

$$\begin{aligned}h(\pi,C)-\delta\le h(\pi,A'')&\le h(\pi,A')+H(\{a,1\setminus a\})\\&=h(\pi,A')+q(\bar\mu a)+q(1-\bar\mu a)\\&=h(\pi,A')+q(\tfrac{1}{n+1})+q(\tfrac{n}{n+1})\le h(\pi,A')+\delta\end{aligned}$$

by the choice of n. **Q**

(l) Putting these together,

$$H(A')\le h(\pi,C)+4\delta\le h(\pi,A')+6\delta\le h(\pi,A')+\epsilon,$$

and the proof is complete.

386D Corollary Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^\infty\gamma_i=1,\quad\sum_{i=0}^\infty q(\gamma_i)\le h(\pi).$$

Then for any $\epsilon>0$ we can find a Bernoulli partition $\langle a_i^*\rangle_{i\in\mathbb{N}}$ for π such that $\bar\mu a_i^*=\gamma_i$ for every $i\in\mathbb{N}$ and

$$\sum_{i=0}^\infty\bar\mu(a_i^*\triangle a_i)\le\epsilon+6\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}},$$

writing $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$.

proof (a) Set $\beta=\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$. Let $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that

$$\sum_{n=0}^\infty\epsilon_n+6\sqrt{\epsilon_n+\sqrt{2\epsilon_n}}\le\epsilon.$$

Using 386C, we can choose inductively, for $n\in\mathbb{N}$, partitions of unity $\langle a_{ni}\rangle_{i\in\mathbb{N}}$ such that, for each $n\in\mathbb{N}$,

$$\sum_{i=0}^\infty|\gamma_i-\bar\mu a_{ni}|\le\epsilon_n,$$

$$H(A_n)\le h(\pi,A_n)+\epsilon_n<\infty$$

(writing $A_n=\{a_{ni}:i\in\mathbb{N}\}\setminus\{0\}$),

$$\sum_{i=0}^\infty\bar\mu(a_{n+1,i}\triangle a_{ni})\le\epsilon_{n+1}+6\sqrt{\epsilon_n+\sqrt{2\epsilon_n}},$$

while

$$\sum_{i=0}^\infty\bar\mu(a_{0i}\triangle a_i)\le\epsilon_0+6\beta.$$

On completing the induction, we see that

$$\sum_{n=0}^\infty\sum_{i=0}^\infty\bar\mu(a_{n+1,i}\triangle a_{ni})\le\sum_{n=1}^\infty\epsilon_n+\sum_{n=0}^\infty 6\sqrt{\epsilon_n+\sqrt{2\epsilon_n}}<\infty.$$

In particular, given $i\in\mathbb{N}$, $\sum_{n=0}^\infty\bar\mu(a_{n+1,i}\triangle a_{ni})$ is finite, so $\langle a_{ni}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space $\mathfrak{A}$ (323Gc), and has a limit a_i^*, with

$$\bar\mu a_i^*=\lim_{n\to\infty}\bar\mu a_{ni}=\gamma_i$$

(323C). If $i\ne j$,

$$a_i^*\cap a_j^*=\lim_{n\to\infty}a_{ni}\cap a_{nj}=0$$

(using 323B), so $\langle a_i^*\rangle_{i\in\mathbb{N}}$ is disjoint; since

$$\sum_{i=0}^\infty\bar\mu a_i^*=\sum_{i=0}^\infty\gamma_i=1,$$

$\langle a_i^*\rangle_{i\in\mathbb{N}}$ is a partition of unity. We also have

$$\sum_{i=0}^{\infty}\bar\mu(a_i^*\triangle a_i)\le\sum_{i=0}^{\infty}\bar\mu(a_{0i}\triangle a_i)+\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\bar\mu(a_{n+1,i}\triangle a_{ni})$$

$$\le\epsilon_0+6\beta+\sum_{n=1}^{\infty}\epsilon_n+\sum_{n=0}^{\infty}6\sqrt{\epsilon_n+\sqrt{2\epsilon_n}}\le\epsilon+6\beta.$$

(b) Now take any $i(0),\dots,i(k)\in\mathbb{N}$. For each $j<k$, $n\in\mathbb{N}$,

$$H(\pi^j[A_n])+H(D_j(A_n,\pi))-H(D_{j+1}(A_n,\pi))\le H(A_n)-h(\pi,A_n)\le\epsilon_n$$

(using 385Nc). But this means that

$$\sum_{d\in D_j(A_n,\pi)}\sum_{i=0}^{\infty}|\bar\mu(d\cap\pi^j a_{ni})-\bar\mu d\cdot\bar\mu a_{ni}|\le\sqrt{2\epsilon_n},$$

by 385K. *A fortiori*,

$$|\bar\mu(d\cap\pi^j a_{ni})-\bar\mu d\cdot\bar\mu a_{ni}|\le\sqrt{2\epsilon_n}$$

for every $d\in D_j(A_n,\pi)$, $i\in\mathbb{N}$. Inducing on r, we see that

$$|\bar\mu(\inf_{j\le r}\pi^j a_{n,i(j)})-\prod_{j=0}^{r}\bar\mu a_{n,i(j)}|\le r\sqrt{2\epsilon_n}\to 0$$

as $n\to\infty$, for any $r\le k$. Because $\bar\mu$, $\cap$ and π are all continuous (323C, 323B, 324Kb),

$$\bar\mu(\inf_{j\le k}\pi^j a^*_{i(j)})=\lim_{n\to\infty}\bar\mu(\inf_{j\le k}\pi^j a_{n,i(j)})$$

$$=\lim_{n\to\infty}\prod_{j\le k}\bar\mu a_{n,i(j)}=\prod_{j\le k}\gamma_{i(j)}.$$

As $i(0),\dots,i(k)$ are arbitrary, $\langle a_i^*\rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π.

386E Sinaĭ's theorem (atomic case) (Sinaĭ 62) Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{i=0}^{\infty}\gamma_i=1$ and $\sum_{i=0}^{\infty}q(\gamma_i)\le h(\pi)$. Then there is a Bernoulli partition $\langle a_i^*\rangle_{i\in\mathbb{N}}$ for π such that $\bar\mu a_i^*=\gamma_i$ for every $i\in\mathbb{N}$.

proof Apply 386D from any starting point, e.g., $a_0=1$, $a_i=0$ for $i>0$.

386F Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$, $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras of $\mathfrak{A}$ generated by $\{\pi^j b_i:i\in\mathbb{N},\,j\in\mathbb{Z}\}$ and $\{\pi^j c_i:i\in\mathbb{N},\,j\in\mathbb{Z}\}$. Suppose that $\mathfrak{C}\subseteq\mathfrak{B}$. Then for any $\epsilon>0$ we can find a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that

(i) $d_i\in\mathfrak{C}$ for every $i\in\mathbb{N}$,

(ii) $\bar\mu d_i=\bar\mu b_i$ for every $i\in\mathbb{N}$,

(iii) $\bar\mu(\phi c_i\triangle c_i)\le\epsilon$ for every $i\in\mathbb{N}$,

where $\phi:\mathfrak{B}\to\mathfrak{C}$ is the measure-preserving Boolean homomorphism such that $\phi b_i=d_i$ for every i and $\pi\phi=\phi\pi$ (386Bi).

proof (a) Set $B=\{b_i:i\in\mathbb{N}\}\setminus\{0\}$, $C=\{c_i:i\in\mathbb{N}\}\setminus\{0\}$. If only one c_i is non-zero, then $H(C)=0$, so $H(B)=0$ and $\mathfrak{B}=\{0,1\}$, in which case $\mathfrak{B}=\mathfrak{C}$ and we take $d_i=b_i$ and stop. Otherwise, $\mathfrak{C}$ is atomless (386Bd).

For $k\in\mathbb{N}$, let $\mathfrak{B}_k\subseteq\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i:i\le k,\,|j|\le k\}$. Because $\mathfrak{C}\subseteq\mathfrak{B}$, there is an $m\in\mathbb{N}$ such that

$$\rho(c_i,\mathfrak{B}_m)\le\tfrac14\epsilon\text{ for every }i\in\mathbb{N}$$

(385M). Let η, $\xi>0$ be such that

$$\eta+6\sqrt[4]{2\eta}\le\frac{\epsilon}{4(2m+1)},\quad \xi\le\min(\frac{\epsilon}{4},\frac{1}{6}),\quad \sum_{i=0}^{\infty}q(\min(2\xi,\bar\mu c_i))\le\eta.$$

(The last is achievable because $\sum_{i=0}^{\infty}q(\bar\mu c_i)$ is finite.) Let $r\ge m$ be such that

$$\rho(c_i,\mathfrak{B}_r)\le\xi \text{ for every } i\in\mathbb{N}.$$

Let $n\ge r$ be such that

$$\frac{2r+1}{2n+2}\le\xi,\quad \bar\mu c_i\le\xi \text{ for every } i>n.$$

(b) Let $\langle b_i'\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{C}$ such that $\bar\mu b_i'=\bar\mu b_i$ for every $i\in\mathbb{N}$. Let U be the set of atoms of the subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i: i\le n,\ |j|\le n\}\cup\{\pi^j c_i: i\le n,\ |j|\le n\}$, and V the set of atoms of the subalgebra of $\mathfrak{C}$ generated by $\{\pi^j b_i': i\le n,\ |j|\le n\}\cup\{\pi^j c_i: i\le n,\ |j|\le n\}$. For each $v\in V$, choose a disjoint family $\langle d_{vu}\rangle_{u\in U}$ in $\mathfrak{C}$ such that $\sup_{u\in U}d_{vu}=v$ and $\bar\mu d_{vu}=\bar\mu(v\cap u)$ for every $u\in U$. By 385E(iv), there is an $a\in\mathfrak{C}$ such that $a,\pi a,\dots,\pi^{2n}a$ are disjoint and $\bar\mu(a\cap d_{vu})=\frac{1}{2n+2}\bar\mu(d_{vu})$ for every $u\in U$ and $v\in V$. ($\pi\restriction\mathfrak{C}$ is a Bernoulli shift, therefore ergodic, by 384Se, therefore aperiodic, by 385F.) Set $e=\sup_{|j|\le n}\pi^j a$, $\tilde e=\sup_{|j|\le n-r}\pi^j a$; then

$$\bar\mu\tilde e=(2(n-r)+1)\bar\mu a=1-\frac{2r+1}{2n+2}.$$

Let $\mathfrak{C}_{\tilde e}$ be the principal ideal of $\mathfrak{C}$ generated by $\tilde e$.

(c) The family $\langle\pi^{-j}(a\cap d_{vu})\rangle_{|j|\le n,u\in U,v\in V}$ is disjoint. **P** All we have to note is that the families $\langle d_{vu}\rangle_{u\in U,v\in V}$ and

$$\langle\pi^{-j}a\rangle_{|j|\le n}=\langle\pi^{-n}(\pi^{n+j}a)\rangle_{|j|\le n}$$

are disjoint. **Q** Consequently, if we set

$$\hat b_i=\sup_{|j|\le n}\sup_{v\in V}\sup_{u\in U,u\subseteq\pi^j b_i}\pi^{-j}(a\cap d_{vu})\in\mathfrak{C}$$

for $i\in\mathbb{N}$, $\langle\hat b_i\rangle_{i\in\mathbb{N}}$ is disjoint, since a given triple (j,u,v) can contribute to at most one $\hat b_i$.

Of course $\hat b_i\subseteq\sup_{|j|\le n}\pi^{-j}a_i=e$ for every i. If $i\le n$, we also have $\bar\mu\hat b_i=\bar\mu e\cdot\bar\mu b_i$. **P** For $|j|\le n$, $\pi^j b_i$ is a union of members of U, so

$$\bar\mu\hat b_i=\sum_{j=-n}^{n}\sum_{v\in V}\sum_{u\in U,u\subseteq\pi^j b_i}\bar\mu(\pi^{-j}(a\cap d_{vu}))$$

(because $\langle\pi^{-j}(a\cap d_{vu})\rangle_{|j|\le n,u\in U,v\in V}$ is disjoint)

$$=\sum_{j=-n}^{n}\sum_{v\in V}\sum_{u\in U,u\subseteq\pi^j b_i}\bar\mu(a\cap d_{vu})=\frac{1}{2n+2}\sum_{j=-n}^{n}\sum_{v\in V}\sum_{u\in U,u\subseteq\pi^j b_i}\bar\mu d_{vu}$$

(by the choice of a)

$$=\frac{1}{2n+2}\sum_{j=-n}^{n}\sum_{v\in V}\sum_{u\in U,u\subseteq\pi^j b_i}\bar\mu(v\cap u)$$

(by the choice of d_{vu})

$$=\frac{1}{2n+2}\sum_{j=-n}^{n}\sum_{u\in U,u\subseteq\pi^j b_i}\bar\mu u=\frac{1}{2n+2}\sum_{j=-n}^{n}\bar\mu(\pi^j b_i)$$

(because $\pi^j b_i$ is a disjoint union of members of U when $i\le n$, $|j|\le n$)

$$=\frac{2n+1}{2n+2}\bar\mu b_i=\bar\mu e\cdot\bar\mu b_i.\ \mathbf{Q}$$

Again because $\mathfrak{C}$ is atomless, we can choose a partition of unity $\langle b_i^*\rangle_{i\in\mathbb{N}}\in\mathfrak{C}$ such that $\bar\mu b_i^*=\bar\mu b_i$ for every i, while $b_i^*\supseteq\hat b_i$ and $b_i^*\cap e=\hat b_i$ for $i\le n$.

(d) Let $\mathfrak{E}$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i : i \le n,\ |j| \le r\} \cup \{\pi^j c_i : i \le n,\ |j| \le r\}$. Define $\theta : \mathfrak{E} \to \mathfrak{C}_{\tilde{e}}$ by setting

$$\theta b = \sup_{|j|\le n-r} \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^j b} \pi^{-j}(a \cap d_{vu})$$

for $b \in \mathfrak{E}$.

(i) θ is a Boolean homomorphism. **P** The point is that if $|j| \le n-r$ and $b \in \mathfrak{E}$, then $\pi^j b$ belongs to the algebra generated by $\{\pi^k b_i : i \le n,\ |k| \le n\} \cup \{\pi^k c_i : i \le n,\ |k| \le n\}$, so is a union of members of U. Since each map

$$b \mapsto \pi^{-j}(a \cap d_{vu}) \text{ if } u \subseteq \pi^j b, \quad 0 \text{ otherwise}$$

is a Boolean homomorphism from $\mathfrak{E}$ to the principal ideal generated by $\pi^{-j}(a \cap d_{vu})$, and

$$\langle \pi^{-j}(a \cap d_{vu})\rangle_{|j|\le n-r, u\in U, v\in V}$$

is a partition of unity in $\mathfrak{C}_{\tilde{e}}$, θ also is a Boolean homomorphism. **Q**

(ii) $\bar\mu(\theta b) \le \bar\mu b$ for every $b \in \mathfrak{E}$. **P** (Compare (c) above.)

$$\begin{aligned}\bar\mu(\theta b) &= \sum_{j=-n+r}^{n-r} \sum_{v\in V} \sum_{u\in U, u\subseteq \pi^j b} \bar\mu\pi^{-j}(a \cap d_{vu}) \\ &= \frac{1}{2n+2} \sum_{j=-n+r}^{n-r} \sum_{v\in V} \sum_{u\in U, u\subseteq \pi^j b} \bar\mu(v \cap u) = \frac{2n-2r+1}{2n+2}\bar\mu b \le \bar\mu b. \ \mathbf{Q}\end{aligned}$$

(iii) $\theta(\pi^k b_i) = \tilde{e} \cap \pi^k b_i^*$ for $i \le n$, $|k| \le r$. **P** Of course $\pi^k b_i \in \mathfrak{E}$. If $|j| \le n-r$, then $|j+k| \le n$, so

$$\begin{aligned}\pi^{-j} a \cap \theta(\pi^k b_i) &= \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^{j+k} b_i} \pi^{-j}(a \cap d_{vu}) \\ &= \pi^k\bigl(\sup_{v\in V} \sup_{u\in U, u\subseteq \pi^{j+k} b_i} \pi^{-j-k}(a \cap d_{vu})\bigr) \\ &= \pi^k(\pi^{-j-k} a \cap \hat{b}_i) = \pi^{-j} a \cap \pi^k(e \cap b_i^*) = \pi^{-j} a \cap \pi^k b_i^*\end{aligned}$$

because $\pi^{-j} a \subseteq \pi^k e$. Taking the supremum of these pieces we have

$$\theta(\pi^k b_i) = \sup_{|j|\le n-r} \pi^{-j} a \cap \theta(\pi^k b_i) = \sup_{|j|\le n-r} \pi^{-j} a \cap \pi^k b_i^* = \tilde{e} \cap \pi^k b_i^*. \ \mathbf{Q}$$

It follows that

$$\theta(\pi^k(1 \setminus \sup_{i\le l} b_i)) = \tilde{e} \cap \pi^k(1 \setminus \sup_{i\le l} b_i^*))$$

if $l \le n$, $|k| \le r$.

(iv) Finally, $\theta c_i = c_i \cap \tilde{e}$ for every $i \le n$. **P** If $|j| \le n-r$, $v \in V$ then either $v \subseteq \pi^j c_i$ or $v \cap \pi^j c_i = 0$. In the former case,

$$d_{vu} = v \cap u = 0 \text{ whenever } u \in U \text{ and } u \not\subseteq \pi^j c_i,$$

so that

$$v = \sup_{u\in U} d_{vu} = \sup_{u\in U, u\subseteq \pi^j c_i} d_{vu};$$

in the latter case, $d_{vu} = v \cap u = 0$ whenever $u \subseteq \pi^j c_i$. So we have

$$v \cap \pi^j c_i = \sup_{u\in U, u\subseteq \pi^j c_i} d_{vu}$$

for every $v \in V$, and

$$\begin{aligned}\theta c_i &= \sup_{|j|\le n-r}\sup_{v\in V}\sup_{u\in U,u\subseteq\pi^j c_i}\pi^{-j}(a\cap d_{vu})\\ &= \sup_{|j|\le n-r}\pi^{-j}(a\cap\sup_{v\in V}\sup_{u\in U,u\subseteq\pi^j c_i}d_{vu})\\ &= \sup_{|j|\le n-r}\pi^{-j}(a\cap\sup_{v\in V}(v\cap\pi^j c_i))\\ &= \sup_{|j|\le n-r}\pi^{-j}(a\cap\pi^j c_i) = c_i\cap\sup_{|j|\le n-r}\pi^{-j}a = c_i\cap\tilde e. \ \mathbf{Q}\end{aligned}$$

(e) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i^* : i\in\mathbb{N},\ |j|\in\mathbb{Z}\}$. Then for every $b\in\mathfrak{B}_r$ there is a $b^*\in\mathfrak{B}^*$ such that $\theta b = b^*\cap\tilde e$. **P** The set of b for which this is true is a subalgebra of $\mathfrak{A}$ containing $\pi^k b_i$ for $i\le r$ and $|k|\le r$, by (d-iii). **Q** It follows that

$$\rho(c_i,\mathfrak{B}^*)\le 2\xi \text{ for } i\in\mathbb{N}.$$

P If $i>n$ this is trivial, because $\bar\mu c_i\le\xi$, by the choice of n. Otherwise, $c_i\in\mathfrak{C}$. Take $b\in\mathfrak{B}_r$ such that $\bar\mu(b\mathbin{\triangle}c_i)=\rho(c_i,\mathfrak{B}_r)\le\xi$ (385Oa). Let $b^*\in\mathfrak{B}^*$ be such that $\theta b=b^*\cap\tilde e$. Then

$$\begin{aligned}\rho(c_i,\mathfrak{B}_k^*) &\le \bar\mu(c_i\mathbin{\triangle}b^*)\le 1-\bar\mu\tilde e+\bar\mu(\tilde e\cap(c_i\mathbin{\triangle}b^*))\\ &= \tfrac{2r+1}{2n+2}+\bar\mu((\tilde e\cap c_i)\mathbin{\triangle}\theta b)=\tfrac{2r+1}{2n+2}+\bar\mu(\theta c_i\mathbin{\triangle}\theta b)\end{aligned}$$

(by (d-iv))

$$= \tfrac{2r+1}{2n+2}+\bar\mu(\theta(c_i\mathbin{\triangle}b))\le\tfrac{2r+1}{2n+2}+\bar\mu(c_i\mathbin{\triangle}b)$$

(by (d-ii))

$$\le 2\xi$$

by the choice of n. **Q**

(f) Set $B^*=\{b_i^*: i\in\mathbb{N}\}\setminus\{0\}$. Then $H(B^*)=h(\pi,C)\le h(\pi,B^*)+\eta$. **P**

$$H(B^*)=H(B)=H(C)$$

(because $\bar\mu b_i^*=\bar\mu b_i$ for every i, and we supposed from the beginning that $H(C)=H(B)$)

$$=h(\pi,C)$$

(because C is a Bernoulli partition, see 386Bc)

$$\le h(\pi\restriction\mathfrak{B}_r^*)+H(C|\mathfrak{B}_r^*)$$

(385Nd)

$$\le h(\pi\restriction\mathfrak{B}^*)+\sum_{i=0}^{\infty}q(\rho(c_i,\mathfrak{B}_r^*))$$

(by the definition of $h(\pi\restriction\mathfrak{B}^*)$, and 385Ob)

$$\le h(\pi,B^*)+\sum_{i=0}^{\infty}q(\min(2\xi,\bar\mu c_i))$$

(by the Kolmogorov-Sinaĭ theorem, 384P(ii), and (e) above, recalling that $\xi\le\frac{1}{6}$, so that q is monotonic on $[0,2\xi]$)

$$\le h(\pi,B^*)+\eta$$

by the choice of ξ. **Q**

(g) By 386D, applied to $\pi{\restriction}\mathfrak{C}$ and the partition $\langle b_i^*\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{C}$ and the sequence $\langle\gamma_i\rangle_{i\in\mathbb{N}} = \langle\bar\mu b_i^*\rangle_{i\in\mathbb{N}}$, we have a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{C}$ such that $\bar\mu d_i = \bar\mu b_i^* = \bar\mu b_i$ for every $i\in\mathbb{N}$ and

$$\textstyle\sum_{i=0}^{\infty}\bar\mu(d_i\triangle b_i^*) \le \eta + 6\sqrt[4]{2\eta} \le \frac{\epsilon}{4(2m+1)}.$$

Let $\mathfrak{D}\subseteq\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i\in\mathbb{N},\ j\in\mathbb{Z}\}$. Then $(\mathfrak{B},\pi{\restriction}\mathfrak{B},\langle b_i\rangle_{i\in\mathbb{N}})$ is isomorphic to $(\mathfrak{D},\pi{\restriction}\mathfrak{D},\langle d_i\rangle_{i\in\mathbb{N}})$, with an isomorphism $\phi:\mathfrak{B}\to\mathfrak{D}$ such that $\phi\pi=\pi\phi$ and $\phi b_i = d_i$ for every $i\in\mathbb{N}$ (386Bi).

(h) Set

$$e^* = \tilde e\setminus \sup_{|j|\le m, i\in\mathbb{N}}\pi^j(d_i\triangle b_i^*).$$

Then $\phi(\pi^j b_i)\cap e^* = \theta(\pi^j b_i)\cap e^*$ whenever $i\le m$ and $|j|\le m$. **P**

$$\begin{aligned}\phi(\pi^j b_i)\cap e^* &= \pi^j(\phi b_i)\cap e^* = \pi^j d_i\cap e^*\\ &= \pi^j b_i^*\cap e^* = \pi^j b_i^*\cap\tilde e\cap e^* = \theta(\pi^j b_i)\cap e^*\end{aligned}$$

by (d-iii), because i, $|j|\le r\le n$. **Q** Since $b\mapsto\phi b\cap e^* : \mathfrak{A}\to\mathfrak{A}_{e^*}$, $b\mapsto\theta b\cap e^* : \mathfrak{C}\to\mathfrak{A}_{e^*}$ are Boolean homomorphisms, $\phi b\cap e^* = \theta b\cap e^*$ for every $b\in\mathfrak{B}_m$.

Now $\bar\mu(c_i\triangle\phi c_i)\le\epsilon$ for every $i\in\mathbb{N}$. **P** If $i>n$ then of course

$$\bar\mu(\phi c_i\triangle c_i)\le 2\bar\mu c_i\le 2\xi\le\epsilon.$$

If $i\le n$, then (by the choice of m) there is a $b\in\mathfrak{B}_m$ such that $\bar\mu(c_i,b)\le\frac14\epsilon$. So

$$\begin{aligned}\phi c_i\triangle c_i &\subseteq (\phi c_i\triangle\phi b)\cup(\phi b\triangle\theta b)\cup(\theta b\triangle\theta c_i)\cup(\theta c_i\triangle c_i)\\ &\subseteq \phi(c_i\triangle b)\cup(1\setminus e^*)\cup\theta(b\triangle c_i)\end{aligned}$$

(using the definition of e^* and (d-iv)) has measure at most

$$\bar\mu(c_i\triangle b)+\bar\mu(1\setminus e^*)+\bar\mu(b\triangle c_i)$$

(by (d-ii), since b and c both belong to $\mathfrak{C}$)

$$\begin{aligned}&\le 2\bar\mu(c_i\triangle b)+\bar\mu(1\setminus\tilde e)+(2m+1)\sum_{i=0}^{\infty}\bar\mu(d_i\triangle b_i^*)\\ &\le\frac{\epsilon}{2}+\frac{2r+1}{2n+2}+\frac{\epsilon}{4}\le\epsilon,\end{aligned}$$

as required. **Q**

386G Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$, $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i\in\mathbb{N},\ j\in\mathbb{Z}\}$ and $\{\pi^j c_i : i\in\mathbb{N},\ j\in\mathbb{Z}\}$. Suppose that $\mathfrak{C}\subseteq\mathfrak{B}$. Then for any $\epsilon>0$ we can find a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that

(i) $\bar\mu d_i=\bar\mu c_i$ for every $i\in\mathbb{N}$,

(ii) $\bar\mu(d_i\triangle c_i)\le\epsilon$ for every $i\in\mathbb{N}$,

(iii) writing $\mathfrak{D}$ for the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i\in\mathbb{N},\ j\in\mathbb{Z}\}$, $\rho(b_i,\mathfrak{D})\le\epsilon$ for every $i\in\mathbb{N}$.

proof (a) By 386F, there is a Bernoulli partition $\langle b_i^*\rangle_{i\in\mathbb{N}}$ for π such that $b_i^*\in\mathfrak{C}$ for every $i\in\mathbb{N}$, $\bar\mu b_i^*=\bar\mu b_i$ for every $i\in\mathbb{N}$, and $\bar\mu(\phi c_i\triangle c_i)\le\frac14\epsilon$ for every $i\in\mathbb{N}$, where $\phi:\mathfrak{B}\to\mathfrak{C}$ is the measure-preserving Boolean homomorphism such that $\phi b_i = b_i^*$ for every i and $\pi\phi=\phi\pi$. Note that this implies that $\pi^{-1}\phi=\phi\pi^{-1}$, and generally that $\pi^j\phi=\phi\pi^{-j}$ for every $j\in\mathbb{Z}$; so $\phi[\mathfrak{B}]\subseteq\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\phi\pi^j b_i : i\in\mathbb{N},\ j\in\mathbb{Z}\}=\{\pi^j b_i^* : i\in\mathbb{N},\ j\in\mathbb{Z}\}$ (324L), and is invariant under the action of π and π^{-1}.

Let $m\in\mathbb{N}$ be such that

$$\rho(c_i,\mathfrak{B}_m)\le\tfrac14\epsilon\text{ for every }i\in\mathbb{N},$$

where $\mathfrak{B}_m$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \in \mathbb{N}, |j| \le m\}$ (385M). Let $\eta \in]0,\pi]$ be such that

$$(2m+1)\sum_{i=0}^{\infty} \min(\eta, 2\bar\mu b_i) \le \tfrac{1}{4}\epsilon.$$

By 386F again, applied to $\pi\restriction\mathfrak{C}$, there is a Bernoulli partition $\langle c_i^*\rangle_{i\in\mathbb{N}}$ for π such that $c_i^* \in \phi[\mathfrak{B}]$, $\bar\mu c_i^* = \bar\mu c_i$ and $\bar\mu(\psi b_i^* \triangle b_i^*) \le \eta$ for every $i \in \mathbb{N}$, where $\psi : \mathfrak{C} \to \mathfrak{C}$ is the measure-preserving Boolean homomorphism such that $\psi c_i = c_i^*$ for every $i \in \mathbb{N}$ and $\psi\pi = \pi\psi$. Once again, $\psi[\mathfrak{C}]$ will be the closed subalgebra of $\mathfrak{A}$ generated by $\{\psi\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\} = \{\pi^j c_i^* : i \in \mathbb{N}, j \in \mathbb{Z}\}$; because every c_i^* belongs to $\phi[\mathfrak{B}]$, $\psi[\mathfrak{C}] \subseteq \phi[\mathfrak{B}]$.

(b) Now $\bar\mu(c_i^* \triangle \phi c_i) \le \epsilon$ for every $i \in \mathbb{N}$. **P** There is a $b \in \mathfrak{B}_m$ such that $\bar\mu(c_i \triangle b) \le \frac{1}{4}\epsilon$. We know that $\phi[\mathfrak{B}_m]$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\phi\pi^j b_i : i \in \mathbb{N}, |j| \le m\} = \{\pi^j b_i^* : i \in \mathbb{N}, |j| \le m\}$, and contains ϕb. Because

$$\psi(\phi b) \triangle \phi b \subseteq \sup_{i\in\mathbb{N},|j|\le m} \psi(\pi^j b_i^*) \triangle \pi^j b_i^* = \sup_{|j|\le m} \pi^j(\sup_{i\in\mathbb{N}} \psi b_i^* \triangle b_i^*),$$

we have

$$\begin{aligned}\bar\mu(\psi\phi b \triangle \phi b) &\le (2m+1)\sum_{i=0}^{\infty} \bar\mu(\psi b_i^* \triangle b_i^*)\\ &\le (2m+1)\sum_{i=0}^{\infty} \min(\eta, 2\bar\mu b_i) \le \frac{1}{4}\epsilon.\end{aligned}$$

But this means that

$$\begin{aligned}\bar\mu(c_i^* \triangle \phi c_i) &= \bar\mu(\psi c_i \triangle \phi c_i) \le \bar\mu(\psi c_i \triangle \psi\phi b) + \bar\mu(\psi\phi b \triangle \phi b) + \bar\mu(\phi b \triangle \phi c_i)\\ &\le \bar\mu(c_i \triangle \phi b) + \frac{\epsilon}{4} + \bar\mu(b \triangle c_i) \le \bar\mu(c_i \triangle \phi c_i) + \bar\mu(\phi c_i \triangle \phi b) + \frac{\epsilon}{2}\\ &\le \frac{\epsilon}{4} + \bar\mu(c_i \triangle b) + \frac{\epsilon}{2} \le \epsilon. \textbf{ Q}\end{aligned}$$

(c) Set $d_i = \phi^{-1}c_i^*$ for each i; this is well-defined because ϕ is injective and $c_i^* \in \phi[\mathfrak{B}]$. Write $\mathfrak{D}$ for the closed subalgebra of $\mathfrak{A}$ or of $\mathfrak{B}$ generated by $\{\pi^j d_i : i \in \mathbb{N}, j \in \mathbb{Z}\} = \{\phi^{-1}\psi\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$; then $\mathfrak{D} = \phi^{-1}[\psi[\mathfrak{C}]]$, by 324L again, because $\phi^{-1} : \phi[\mathfrak{B}] \to \mathfrak{B}$ is a measure-preserving homomorphism. Then $\bar\mu d_i = \bar\mu c_i^* = \bar\mu c_i$ for every $i \in \mathbb{N}$, and $\langle d_i\rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π. **P** If $i(0),\dots,i(n) \in \mathbb{N}$, then

$$\begin{aligned}\bar\mu(\inf_{j\le n} \pi^j d_{i(j)}) &= \bar\mu(\inf_{j\le n} \pi^j \phi^{-1} c_{i(j)}^*) = \bar\mu(\phi(\inf_{j\le n} \pi^j \phi^{-1} c_{i(j)}^*))\\ &= \bar\mu(\inf_{j\le n} \pi^j c_{i(j)}^*)) = \prod_{j\le n} \bar\mu c_{i(j)}^* = \prod_{j\le n} \bar\mu d_{i(j)}. \textbf{ Q}\end{aligned}$$

Next,

$$\bar\mu(c_i \triangle d_i) = \bar\mu(\phi c_i \triangle \phi d_i) = \bar\mu(\phi c_i \triangle c_i^*) \le \epsilon$$

for every i, by (b). Finally, if $i \in \mathbb{N}$, then ψb_i^* belongs to $\psi[\mathfrak{C}]$, while $\mathfrak{D} = \phi^{-1}[\psi[\mathfrak{C}]]$, so

$$\rho(b_i, \mathfrak{D}) = \rho(\phi b_i, \psi[\mathfrak{C}]) \le \bar\mu(\phi b_i \triangle \psi b_i^*) = \bar\mu(b_i^* \triangle \psi b_i^*) \le \eta \le \epsilon.$$

This completes the proof.

386H Lemma Let $(\mathfrak{A}, \bar\mu)$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$, $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that

(i) $\bar\mu d_i = \bar\mu c_i$ for every $i \in \mathbb{N}$,

(ii) $\bar\mu(d_i \triangle c_i) \le \epsilon$ for every $i \in \mathbb{N}$,

(iii) the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ is $\mathfrak{B}$.

proof (a) To begin with (down to the end of (c) below) suppose that $\mathfrak{A} = \mathfrak{B}$. Choose $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$, $\langle \delta_n \rangle_{n \in \mathbb{N}}$, $\langle r_n \rangle_{n \in \mathbb{N}}$ and $\langle \langle d_{ni} \rangle_{i \in \mathbb{N}} \rangle_{n \in \mathbb{N}}$ inductively, as follows. Start with $d_{0i} = c_i$ for every i, $r_0 = 0$. Given that $\langle d_{ni} \rangle_{i \in \mathbb{N}}$ is a Bernoulli partition with $\bar{\mu} d_{ni} = \bar{\mu} c_i$ for every i, take $\epsilon_n > 0$ such that

$$(2r_m + 1)\epsilon_n \le 2^{-n} \text{ for every } m \le n,$$

$\delta_n > 0$ such that

$$\delta_n \le 2^{-n-1}\epsilon, \quad \sum_{i=0}^{\infty} \min(\delta_n, 2\bar{\mu} c_i) \le \epsilon_n,$$

and use 386G to find a Bernoulli partition $\langle d_{n+1,i} \rangle_{i \in \mathbb{N}}$ for π such that

$$\bar{\mu} d_{n+1,i} = \bar{\mu} c_i, \quad \bar{\mu}(d_{n+1,i} \triangle d_{ni}) \le \delta_n, \quad \rho(b_i, \mathfrak{D}^{(n+1)}) \le 2^{-n-1}$$

for every $i \in \mathbb{N}$, where $\mathfrak{D}^{(n+1)}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_{n+1,i} : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Let r_{n+1} be such that

$$\rho(b_i, \mathfrak{D}^{(n+1)}_{r_{n+1}}) \le 2^{-n}$$

for every $i \in \mathbb{N}$, where $\mathfrak{D}^{(n+1)}_{r_{n+1}}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_{n+1,i} : i \in \mathbb{N}, |j| \le r_{n+1}\}$. Continue.

(b) For any $i \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \bar{\mu}(d_{n+1,i} \triangle d_{ni}) \le \sum_{n=0}^{\infty} \delta_n \le \epsilon,$$

so $\langle d_{ni} \rangle_{n \in \mathbb{N}}$ has a limit d_i in $\mathfrak{A}$. Of course

$$\bar{\mu}(c_i \triangle d_i) \le \sum_{n=0}^{\infty} \bar{\mu}(d_{n+1,i} \triangle d_{ni}) \le \epsilon$$

for every i. We must have

$$\bar{\mu} d_i = \lim_{n \to \infty} \bar{\mu} d_{ni} = \bar{\mu} c_i$$

for each i, and if $i \ne j$ then

$$d_i \cap d_j = \lim_{n \to \infty} d_{ni} \cap d_{nj} = 0;$$

since $\sum_{i=0}^{\infty} \bar{\mu} c_i = 1$, $\langle d_i \rangle_{i \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. For any $i(0), \dots, i(k)$ in $\mathbb{N}$,

$$\bar{\mu}(\inf_{j \le k} \pi^j d_{i(j)}) = \lim_{n \to \infty} \bar{\mu}(\inf_{j \le k} \pi^j d_{n,i(j)}) = \lim_{n \to \infty} \prod_{j=0}^{k} \bar{\mu} d_{n,i(j)} = \prod_{j=0}^{k} \bar{\mu} d_{i(j)},$$

so $\langle d_i \rangle_{i \in \mathbb{N}}$ is a Bernoulli partition.

(c) Let $\mathfrak{D}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Then $b_j \in \mathfrak{D}$ for every $j \in \mathbb{N}$. **P** Fix $m \in \mathbb{N}$. Then $\rho(b_j, \mathfrak{D}^{(m+1)}_{r_{m+1}}) \le 2^{-m}$, so there is a $b \in \mathfrak{D}^{(m+1)}_{r_{m+1}}$ such that $\bar{\mu}(b_j \triangle b) \le 2^{-m}$. Now

$$\begin{aligned} \sum_{i=0}^{\infty} \rho(d_{m+1,i}, \mathfrak{D}) &\le \sum_{i=0}^{\infty} \bar{\mu}(d_{m+1,i} \triangle d_i) \le \sum_{i=0}^{\infty} \sum_{k=m+1}^{\infty} \bar{\mu}(d_{k+1,i} \triangle d_{ki}) \\ &\le \sum_{k=m+1}^{\infty} \sum_{i=0}^{\infty} \min(2\bar{\mu} c_i, \delta_k) \le \sum_{k=m+1}^{\infty} \epsilon_k. \end{aligned}$$

So

$$\begin{aligned} \rho(b, \mathfrak{D}) &\le (2r_{m+1} + 1) \sum_{i=0}^{\infty} \rho(d_{m+1,i}, \mathfrak{D}) \\ &\le \sum_{k=m+1}^{\infty} (2r_{m+1} + 1)\epsilon_k \le \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m}, \end{aligned}$$

and

$$\rho(b_j, \mathfrak{D}) \le \bar{\mu}(b_j \triangle b) + \rho(b, \mathfrak{D}) \le 2^{-m} + 2^{-m} = 2 \cdot 2^{-m}.$$

As m is arbitrary, $\rho(b_j, \mathfrak{D}) = 0$ and $b_j \in \mathfrak{D}$. **Q**

(d) This completes the proof if $\mathfrak{A} = \mathfrak{B}$. For the general case, apply the arguments above to $(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B}, \pi \restriction \mathfrak{B})$.

386I Ornstein's theorem (finite entropy case) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, and $\pi:\mathfrak{A}\to\mathfrak{A}$, $\phi:\mathfrak{B}\to\mathfrak{B}$ two-sided Bernoulli shifts of the same finite entropy. Then $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are isomorphic.

proof (a) Let $\langle a_i\rangle_{i\in\mathbb{N}}$, $\langle b_i\rangle_{i\in\mathbb{N}}$ be (two-sided) generating Bernoulli partitions in $\mathfrak{A}$, $\mathfrak{B}$ respectively. By the Kolmogorov-Sinaĭ theorem, $\langle a_i\rangle_{i\in\mathbb{N}}$ and $\langle b_i\rangle_{i\in\mathbb{N}}$ both have entropy equal to $h(\pi)=h(\phi)$. If this entropy is zero, then $\mathfrak{A}$ and $\mathfrak{B}$ are both $\{0,1\}$, and the result is trivial; so let us assume that $h(\pi)>0$, so that $\mathfrak{A}$ is atomless (386Bd).

(b) By Sinaĭ's theorem (386E), there is a Bernoulli partition $\langle c_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}c_i=\bar{\nu}b_i$ for every $i\in\mathbb{N}$. By 386H, there is a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}d_i=\bar{\mu}c_i$ for every i and the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i:i\in\mathbb{N},\,j\in\mathbb{Z}\}$ is $\mathfrak{A}$. But now $(\mathfrak{A},\bar{\mu},\pi,\langle d_i\rangle_{i\in\mathbb{N}})$ is isomorphic to $(\mathfrak{B},\bar{\nu},\phi,\langle b_i\rangle_{i\in\mathbb{N}})$, so $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are isomorphic.

386J Using the same methods, we can extend the last result to the case of Bernoulli shifts of infinite entropy. The first step uses the ideas of 386C, as follows.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving automorphism. Suppose that $\langle a_i\rangle_{i\in I}$ is a finite Bernoulli partition for π, with $\#(I)=r$ and $\bar{\mu}a_i=1/r$ for every $i\in I$, and that $h(\pi)\ge\ln 2r$. Then for any $\epsilon>0$ there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I,j\in\{0,1\}}$ for π such that

$$\bar{\mu}(a_i\triangle(b_{i0}\cup b_{i1}))\le\epsilon,\quad \bar{\mu}b_{i0}=\bar{\mu}b_{i1}=\frac{1}{2r}$$

for every $i\in I$.

proof (a) Let $\delta>0$ be such that

$$\delta+6\sqrt{4\delta}\le\epsilon.$$

Let $\eta>0$ be such that

$$\eta<\ln 2,\quad \sqrt{8\eta}\le\delta,$$

and

$$|t-\tfrac{1}{2}|\le\delta \text{ whenever } t\in[0,1] \text{ and } q(t)+q(1-t)\ge\ln 2-4\eta$$

(384Ad). We have

$$H(A)=rq(\frac{1}{r})=\ln r,$$

and $\bar{\mu}d=r^{-n}$ whenever $n\in\mathbb{N}$, $d\in D_n(A,\pi)$.

Note that $\mathfrak{A}$ is atomless. **P?** If $a\in\mathfrak{A}$ is an atom, then (because π is ergodic) $\sup_{j\in\mathbb{N}}\pi^j a=1$, and $\mathfrak{A}$ is purely atomic, with finitely many atoms all of the same size as a; but this means that $H(C)\le\ln(\frac{1}{\bar{\mu}a})$ for every partition of unity $C\subseteq\mathfrak{A}$, so that

$$h(\pi,C)=\lim_{n\to\infty}\frac{1}{n}H(D_n(C,\pi),\pi)\le\lim_{n\to\infty}\frac{1}{n}\ln(\frac{1}{\bar{\mu}a})=0$$

for every partition of unity C, and

$$0=h(\pi)\ge\ln 2r\ge\ln 2. \textbf{ XQ}$$

(b) There is a finite partition of unity $C\subseteq\mathfrak{A}$ such that

$$h(\pi,C)=\ln 2r-\eta,$$

and C refines $A=\{a_i:i\in I\}\setminus\{0\}$. **P** Because $h(\pi)\ge\ln 2r$, there is a finite partition of unity C' such that $h(\pi,C')\ge\ln 2r-\eta$; replacing C' by $C'\vee A$ if need be, we may suppose that C' refines A; take such a C' of minimal size. Because $H(C')\ge h(\pi,C')>H(A)$, there must be distinct c_0, $c_1\in C'$ included in the same member of A. Because $\mathfrak{A}$ is atomless, the principal ideal generated by c_1 has a closed subalgebra isomorphic, as measure algebra, to the measure algebra of Lebesgue measure on $[0,1]$, up to a scalar multiple of the measure; and in particular there is a family $\langle d_t\rangle_{t\in[0,1]}$ such that $d_s\subseteq d_t$ whenever $s\le t$, $d_1=c_1$ and $\bar{\mu}d_t=t\bar{\mu}c_1$ for every $t\in[0,1]$. Let D_t be the partition of unity

$$(C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup d_t, c_1 \setminus d_t\}$$

for each $t \in [0,1]$. Then

$$h(\pi, D_1) = h(\pi, (C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup c_1\}) < \ln 2r - \eta,$$

by the minimality of $\#(C')$, while

$$h(\pi, D_0) = h(\pi, C') \geq \ln 2r - \eta.$$

Using 384N, we also have, for any s, $t \in [0,1]$ such that $|s-t| \leq \frac{1}{e}$,

$$h(\pi, D_s) - h(\pi, D_t) \leq H(D_s|\mathfrak{D}_t)$$

(where $\mathfrak{D}_t$ is the closed subalgebra generated by D_t)

$$\leq q(\rho(c_0 \cup d_s, \mathfrak{D}_t)) + q(\rho(c_1 \setminus d_s, \mathfrak{D}_t))$$

(by 385Ob, because $D_s \setminus \mathfrak{D}_t \subseteq \{c_0 \cup d_s, c_1 \setminus d_s\}$)

$$\leq q(\bar{\mu}((c_0 \cup d_s) \triangle (c_0 \cup d_t))) + q(\bar{\mu}((c_1 \setminus d_s) \triangle (c_1 \setminus d_t)))$$
$$= 2q(\bar{\mu}(d_s \triangle d_t)) = 2q(|s-t|\bar{\mu}c_1)$$

because q is monotonic on $[0, |s-t|\bar{\mu}c_1]$. But this means that $t \mapsto h(\pi, D_t)$ is continuous and there must be some t such that $h(\pi, D_t) = \ln 2r - \eta$; take $C = D_t$. **Q**

(c) Let $\xi > 0$ be such that

$$\xi \leq \eta, \quad \xi \leq \frac{1}{6}, \quad q(2\xi) + q(1-2\xi) \leq \eta,$$

$$\textstyle\sum_{c \in C} q(\min(2\xi, \bar{\mu}c)) \leq \eta.$$

Let $n \in \mathbb{N}$ be such that

$$\tfrac{1}{n+1} \leq \xi, \quad q(\tfrac{1}{n+1}) + q(\tfrac{n}{n+1}) \leq \eta,$$

$$\bar{\mu}[\![w_n - h(\pi, C)\chi 1 \geq \eta]\!] \leq \xi,$$

where

$$w_n = \tfrac{1}{n} \textstyle\sum_{d \in D_n(C,\pi)} \ln(\frac{1}{\bar{\mu}d})\chi d.$$

(The Shannon-McMillan-Breiman theorem, 385H, assures us that any sufficiently large n has these properties.)

(d) Let D be the set of those $d \in D_n(C, \pi)$ such that

$$\bar{\mu}d \geq (2r)^{-n}, \quad \text{i.e.,} \quad \frac{1}{n}\ln(\frac{1}{\bar{\mu}d}) \leq \ln 2r.$$

Then $\bar{\mu}(\sup D) \geq 1 - \xi$, by the choice of n, because $h(\pi, C) = \ln 2r - \eta$. Note that every member of D is included in some member of $D_n(A, \pi)$, because C refines A. If $b \in D_n(A, \pi)$, then $\bar{\mu}b = r^{-n}$, so $\#(\{d : d \in D, d \subseteq b\}) \leq 2^n$; we can therefore find a function $f : D \to \{0,1\}^n$ such that f is injective on $\{d : d \in D, d \subseteq b\}$ for every $b \in D_n(A, \pi)$.

(e) By 385E(iv), as usual, there is an $a \in \mathfrak{A}$ such that $a, \pi^{-1}a, \dots, \pi^{-n+1}a$ are disjoint and $\bar{\mu}(a \cap d) = \frac{1}{n+1}\bar{\mu}d$ for every $d \in D_n(C, \pi)$. Set

$$e = \sup_{d \in D, j < n} \pi^{-j}(a \cap d);$$

then

$$\bar{\mu}e = \textstyle\sum_{j=0}^{n-1} \sum_{d \in D} \bar{\mu}(a \cap d) = \frac{n}{n+1}\bar{\mu}(\sup D) \geq (1-\xi)^2 \geq 1 - 2\xi.$$

(f) Set

$$c^* = \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D,\, f(d)(j) = 1\}.$$

(I am identifying members of $\{0,1\}^n$ with functions from $\{0,\dots,n-1\}$ to $\{0,1\}$.) Set

$$A^* = A \vee \{c^*, 1 \setminus c^*\}, \quad A' = A^* \vee \{a, 1 \setminus a\} \vee \{e, 1 \setminus e\},$$

and let $\mathfrak{A}'$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a : a \in A',\, j \in \mathbb{Z}\}$. Then $a \cap d \in \mathfrak{A}'$ for every $d \in D$. **P** Set

$$\tilde{d} = \mathrm{upr}(a \cap d, \mathfrak{A}') = \inf\{c : c \in \mathfrak{A}',\, c \supseteq a \cap d\} \in \mathfrak{A}'.$$

Let b be the element of $D_n(A,\pi)$ including d. Because a, b, $e \in \mathfrak{A}'$,

$$\tilde{d} \subseteq a \cap b \cap e = \sup_{d' \in D} a \cap b \cap d' = \sup\{a \cap d' : d' \in D,\, d' \subseteq b\}.$$

Now if $d' \in D$, $d' \subseteq b$ and $d' \neq d$, then $f(d') \neq f(d)$. Let j be such that $f(d')(j) \neq f(d)(j)$; then $\pi^{-j}(a \cap d)$ is included in one of c^*, $1 \setminus c^*$ and $\pi^{-j}(a \cap d')$ in the other. This means that one of $\pi^j c^*$, $1 \setminus \pi^j c^*$ is a member of $\mathfrak{A}'$ including $a \cap d$ and disjoint from $a \cap d'$, so that $\tilde{d} \cap d' = 0$. Thus $\tilde{d}$ must be actually equal to $a \cap d$, and $a \cap d \in \mathfrak{A}'$. **Q**

Next, $c \cap e \in \mathfrak{A}'$ for every $c \in C$. **P** $\langle \pi^{-j}(a \cap d)\rangle_{j<n,\, d\in D}$ is a disjoint family in $\mathfrak{A}'$ with supremum e. But whenever $d \in D$, $j < n$ we must have $d \subseteq \pi^j c'$ for some $c' \in C$, so either $d \subseteq \pi^j c$ or $d \cap \pi^j c = 0$; thus $\pi^{-j}(a \cap d)$ must be either included in c or disjoint from it. Accordingly

$$c \cap e = \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D,\, d \subseteq \pi^j c\} \in \mathfrak{A}'. \text{ **Q**}$$

Consequently $h(\pi, A') \geq \ln 2r - 2\eta$. **P** For any $c \in C$,

$$\rho(c, \mathfrak{A}') \leq \bar{\mu}(c \triangle (c \cap e)) = \bar{\mu}(c \setminus e) \leq \min(\bar{\mu}c, 2\xi) \leq \tfrac{1}{3},$$

so

$$\ln 2r - \eta = h(\pi, C) \leq h(\pi \restriction \mathfrak{A}') + H(C|\mathfrak{A}')$$

(385Nd)

$$\leq h(\pi, A') + \sum_{c \in C} q(\rho(c, \mathfrak{A}'))$$

(by the Kolmogorov-Sinaĭ theorem and 385Ob)

$$\leq h(\pi, A') + \sum_{c \in C} q(\min(\bar{\mu}c, 2\xi)) \leq h(\pi, A') + \eta$$

by the choice of ξ. **Q**

Finally, $h(\pi, A^*) \geq \ln 2r - 4\eta$. **P**

$$\ln 2r - 2\eta \leq h(\pi, A') \leq h(\pi, A^*) + H(\{a, 1 \setminus a\}) + H(\{e, 1 \setminus e\})$$

(applying 385Nb twice)

$$= h(\pi, A^*) + q(\bar{\mu}a) + q(1 - \bar{\mu}a) + q(\bar{\mu}e) + q(1 - \bar{\mu}e)$$

$$\leq h(\pi, A^*) + q(\tfrac{1}{n}) + q(\tfrac{n}{n+1}) + q(2\xi) + q(1 - 2\xi)$$

$$\leq h(\pi, A^*) + \eta + \eta = h(\pi, A^*) + 2\eta. \text{ **Q**}$$

(g) It follows that $H(\{c^*, 1 \setminus c^*\}) \geq \ln 2 - 4\eta$. **P** We have

$$\ln 2r - 4\eta \leq h(\pi, A^*) \leq H(A^*) \leq H(A) + H(\{c^*, 1 \setminus c^*\}) = \ln r + H(\{c^*, 1 \setminus c^*\}),$$

so $H(\{c^*, 1 \setminus c^*\}) \geq \ln 2 - 4\eta$. **Q** Thus $q(\bar{\mu}c^*) + q(1 - \bar{\mu}c^*) \geq \ln 2 - 4\eta$; by the choice of η, $|\bar{\mu}c^* - \frac{1}{2}| \leq \delta$.

Next,

$$\textstyle\sum_{i \in I} |\bar{\mu}(a_i \cap c^*) - \frac{1}{2r}| + |\bar{\mu}(a_i \setminus c^*) - \frac{1}{2r}| \leq 3\delta.$$

P By 385K,

$$\begin{aligned}\sum_{i\in I}|\bar\mu(a_i\cap c^*)-\frac{1}{r}\bar\mu c^*|+|\bar\mu(a_i\setminus c^*)-\frac{1}{r}\bar\mu(1\setminus c^*)|&\\ \le\sqrt{2(H(A)+H(\{c^*,1\setminus c^*\})-H(A^*))}&\\ \le\sqrt{2(\ln r+\ln 2-\ln 2r+4\eta)}=\sqrt{8\eta}\le\delta.&\end{aligned}$$

So

$$\begin{aligned}\sum_{i\in I}|\bar\mu(a_i\cap c^*)-\frac{1}{2r}|+|\bar\mu(a_i\setminus c^*)-\frac{1}{2r}|&\\ \le\sum_{i\in I}\big(|\bar\mu(a_i\cap c^*)-\frac{1}{r}\bar\mu c^*|+\frac{1}{r}|\bar\mu c^*-\frac{1}{2}|&\\ +|\bar\mu(a_i\setminus c^*)-\frac{1}{r}\bar\mu(1\setminus c^*)|+\frac{1}{r}|\bar\mu(1\setminus c^*)-\frac{1}{2}|\big)&\\ \le\delta+|\bar\mu c^*-\frac{1}{2}|+|\bar\mu(1\setminus c^*)-\frac{1}{2}|\le 3\delta.\ \mathbf{Q}&\end{aligned}$$

(h) Now apply 386D to the partition of unity A^*, indexed as $\langle a^*_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $a^*_{i1}=a_i\cap c^*$ and $a^*_{i0}=a_i\setminus c^*$, and $\langle\gamma_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $\gamma_{ij}=\frac{1}{2r}$ for all i, j. We have

$$\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|\le 3\delta$$

by (g), while

$$H(A^*)-h(\pi,A^*)\le\ln r+\ln 2-\ln 2r+4\eta=4\eta,$$

so

$$\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|+\sqrt{2(H(A^*)-h(\pi,A^*))}\le 3\delta+\sqrt{8\eta}\le 4\delta.$$

Also

$$\sum_{i\in I,j\in\{0,1\}}q(\gamma_{ij})=\ln 2r\le h(\pi).$$

So 386D tells us that there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I,j\in\{0,1\}}$ for π such that $\bar\mu b^*_{ij}=\frac{1}{2r}$ for all i, j and

$$\sum_{i\in I,j\in\{0,1\}}\bar\mu(b_{ij}\triangle a^*_{ij})\le\delta+6\sqrt{4\delta}\le\epsilon.$$

Now of course

$$\begin{aligned}\sum_{i\in I}\bar\mu(a_i\triangle(b_{i0}\cup b_{i1}))&\le\sum_{i\in I}\bar\mu((a_i\cap c^*)\triangle b_{i1})+\bar\mu((a_i\setminus c^*)\triangle b_{i0})\\&=\sum_{i\in I,j\in\{0,1\}}\bar\mu(a^*_{ij}\triangle b_{ij})\le\epsilon,\end{aligned}$$

as required.

386K Ornstein's theorem (infinite entropy case) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift of infinite entropy. Then $(\mathfrak{A},\bar\mu,\pi)$ is isomorphic to $(\mathfrak{B},\bar\nu,\phi)$, where $(\mathfrak{B},\bar\nu)$ is the measure algebra of the usual measure on $[0,1]^{\mathbb{Z}}$, and ϕ is the standard Bernoulli shift on $\mathfrak{B}$ (384Sb).

proof (a) We have to find a root algebra $\mathfrak{C}$ for π which is isomorphic to the measure algebra of the usual measure on $[0,1]$. The materials we have to start with are a root algebra $\mathfrak{A}_0$ of $\mathfrak{A}$ such that *either* $\mathfrak{A}_0$ is not purely atomic *or* $H(A_0)=\infty$, where A_0 is the set of atoms of $\mathfrak{A}_0$.

Because $\mathfrak{A}$ has countable Maharam type, there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_0$ such that $\{d_n:n\in\mathbb{N}\}$ is dense in the metric of $\mathfrak{A}_0$.

(b) There is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of partitions of unity in $\mathfrak{A}_0$ such that C_{n+1} refines C_n, $H(C_n)=n\ln 2$ and d_n is a union of members of C_{n+1} for every n. **P** We have

$$\infty=\sup\{H(C):C\subseteq\mathfrak{A}_0 \text{ is a partition of unity}\}$$

(384J). Choose the C_n inductively, as follows. Start with $C_0=\{0,1\}$. Given C_n with $H(C_n)=n\ln 2$, set $C'_n=C_n\vee\{d_n,1\setminus d_n\}$; then

$$H(C'_n)\le H(C_n)+H(\{d_n,1\setminus d_n\})\le(n+1)\ln 2.$$

By 385Q, there is a partition of unity C_{n+1}, refining C'_n, such that $H(C_{n+1})=(n+1)\ln 2$. Continue. **Q**

(c) For each $n\in\mathbb{N}$, let $\mathfrak{C}_n$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a:a\in C_n,\ j\in\mathbb{Z}\}$. Then $\langle\mathfrak{C}_n\rangle_{n\in\mathbb{N}}$ is increasing, and $d_n\in\mathfrak{C}_{n+1}$, $\pi[\mathfrak{C}_n]=\mathfrak{C}_n$ and

$$h(\pi\restriction\mathfrak{C}_n)=h(\pi,C_n)=H(C_n)=n\ln 2$$

for every n.

Choose inductively, for each $n\in\mathbb{N}$, $\epsilon_n>0$, $r_n\in\mathbb{N}$ and a Bernoulli partition $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ in $\mathfrak{C}_n$, as follows. Start with $b_{n\emptyset}=1$. (See 3A1H for the notation I am using here.) Given that $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ is a Bernoulli partition for π which generates $\mathfrak{C}_n$, in the sense that $\mathfrak{C}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_{n\sigma}:\sigma\in\{0,1\}^n,\ j\in\mathbb{Z}\}$, and $\bar\mu b_{n\sigma}=2^{-n}$ for every σ, take $\epsilon_n>0$ such that

$$(2r_m+1)\epsilon_n\le 2^{-n} \text{ for every } m<n.$$

We know that

$$h(\pi\restriction\mathfrak{C}_{n+1})=(n+1)\ln 2=\ln(2\cdot 2^n).$$

So we can apply 386J to $(\mathfrak{C}_{n+1},\pi\restriction\mathfrak{C}_{n+1})$ to see that there is a Bernoulli partition $\langle b'_{n\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for π such that

$$b'_{n\tau}\in\mathfrak{C}_{n+1},\quad \bar\mu b'_{n\tau}=2^{-n-1}$$

for every $\tau\in\{0,1\}^{n+1}$,

$$\bar\mu(b_{n\sigma}\triangle(b'_{n,\sigma^\frown 0}\cup b'_{n,\sigma^\frown 1}))\le 2^{-n}\epsilon_n$$

for every $\sigma\in\{0,1\}^n$. By 386H (with $\mathfrak{B}=\mathfrak{C}=\mathfrak{C}_{n+1}$), there is a Bernoulli partition $\langle b_{n+1,\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for $\pi\restriction\mathfrak{C}_{n+1}$ such that the closed subalgebra generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\ j\in\mathbb{Z}\}$ is $\mathfrak{C}_{n+1}$, $\bar\mu b_{n+1,\tau}=2^{-n-1}$ for every $\tau\in\{0,1\}^{n+1}$, and

$$\sum_{\tau\in\{0,1\}^{n+1}}\bar\mu(b_{n+1,\tau}\triangle b'_{n\tau})\le\epsilon_n.$$

For each $k\in\mathbb{N}$, let $\mathfrak{B}_k^{(n+1)}$ be the closed subalgebra of $\mathfrak{C}_{n+1}$ generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\ |j|\le k\}$. Since $d_m\in\mathfrak{C}_{m+1}\subseteq\mathfrak{C}_{n+1}$ for every $m\le n$, there is an $r_n\in\mathbb{N}$ such that

$$\rho(d_m,\mathfrak{B}_{r_n}^{(n+1)})\le 2^{-n} \text{ for every } m\le n.$$

Continue.

(d) Fix $m\le n\in\mathbb{N}$ for the moment. For $\sigma\in\{0,1\}^m$, set

$$b_{n\sigma}=\sup\{b_{n\tau}:\tau\in\{0,1\}^n,\ \tau \text{ extends } \sigma\}.$$

(If $n=m$, then of course σ is the unique member of $\{0,1\}^m$ extending itself, so this formula is safe.) Then

$$\bar\mu b_{n\sigma}=2^{-n}\#(\{\tau:\tau\in\{0,1\}^n,\ \tau \text{ extends } \sigma\})=2^{-n}2^{n-m}=2^{-m}.$$

Next, if σ, $\sigma'\in\{0,1\}^m$ are distinct, there is no member of $\{0,1\}^n$ extending both, so $b_{n\sigma}\cap b_{n\sigma'}=0$; thus $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity. If $\sigma(0),\dots,\sigma(k)\in\{0,1\}^m$, then

$$\begin{aligned}\bar\mu(\inf_{j\le k}\pi^j b_{m,\sigma(j)}) &= \bar\mu(\sup_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\inf_{j\le k}\pi^j b_{n,\tau(j)})\\ &= \sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\bar\mu(\inf_{j\le k}\pi^j b_{n,\tau(j)})\\ &= \sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}(2^{-n})^{k+1}\\ &= (2^{n-m})^{k+1}(2^{-n})^{k+1} = (2^{-m})^{k+1} = \prod_{j=0}^k \bar\mu b_{n,\sigma(j)},\end{aligned}$$

so $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition.

(e) If $m \le n \in \mathbb{N}$, then

$$\textstyle\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma}) \le 2\epsilon_n$$

whenever $m \le n \in \mathbb{N}$. **P** We have

$$\begin{aligned}\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma}) &\le \sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle b_{n+1,\tau})\\ &= \sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle(b_{n+1,\tau^\frown 0}\cup b_{n+1,\tau^\frown 1}))\\ &\le \sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle(b'_{n,\tau^\frown 0}\cup b'_{n,\tau^\frown 1})) + \sum_{\upsilon\in\{0,1\}^{n+1}}\bar\mu(b'_{n\upsilon}\triangle b_{n+1,\upsilon})\\ &\le \sum_{\tau\in\{0,1\}^n}2^{-n}\epsilon_n + \epsilon_n = 2\epsilon_n. \quad \mathbf{Q}\end{aligned}$$

(f) In particular, for any $m \in \mathbb{N}$ and $\sigma \in \{0,1\}^m$,

$$\textstyle\sum_{n=m}^\infty \bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma}) \le \sum_{n=m}^\infty 2\epsilon_n < \infty.$$

So we can define $b_\sigma = \lim_{n\to\infty} b_{n\sigma}$ in $\mathfrak{A}$. We have

$$\bar\mu b_\sigma = \lim_{n\to\infty}\bar\mu b_{n\sigma} = 2^{-m};$$

and if σ, $\sigma' \in \{0,1\}^m$ are distinct, then

$$b_\sigma \cap b_{\sigma'} = \lim_{n\to\infty} b_{n\sigma}\cap b_{n\sigma'} = 0,$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity in $\mathfrak{A}$. If $\sigma(0),\dots,\sigma(k) \in \{0,1\}^m$, then

$$\begin{aligned}\bar\mu(\inf_{j\le k}\pi^j b_{\sigma(j)}) &= \lim_{n\to\infty}\bar\mu(\inf_{j\le k}\pi^j b_{n,\sigma(j)})\\ &= \lim_{n\to\infty}\prod_{j=0}^k \bar\mu b_{n,\sigma(j)} = \prod_{j=0}^k \bar\mu b_{\sigma(j)},\end{aligned}$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition for π. If $\sigma \in \{0,1\}^m$, then

$$b_{\sigma^\frown 0}\cup b_{\sigma^\frown 1} = \lim_{n\to\infty} b_{n,\sigma^\frown 0}\cup b_{n,\sigma^\frown 1} = \lim_{n\to\infty} b_{n,\sigma} = b_\sigma.$$

(g) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{m\in\mathbb{N}}\{b_\sigma : \sigma \in \{0,1\}^m\}$. Then $\mathfrak{C}$ is atomless and countably generated, so $(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. Now $\bar\mu(\inf_{j\le k}\pi^j e_j) = \prod_{j=0}^k \bar\mu e_j$ for all $e_0,\dots,e_k \in \mathfrak{C}$. **P** Let $\epsilon > 0$. For $m \in \mathbb{N}$, let $\mathfrak{C}_m$ be the subalgebra of $\mathfrak{C}$

generated by $\{b_\sigma : \sigma \in \{0,1\}^m\}$. $\langle \mathfrak{E}_m \rangle_{m\in\mathbb{N}}$ is non-decreasing, so $\overline{\bigcup_{m\in\mathbb{N}} \mathfrak{E}_m}$ is a closed subalgebra of $\mathfrak{A}$, and must be $\mathfrak{E}$. Now the function

$$(a_0, \dots , a_k) \to \bar\mu(\inf_{j\le k} \pi^j a_j) - \prod_{j=0}^k \bar\mu a_j : \mathfrak{A}^{k+1} \to \mathbb{R}$$

is continuous and zero on $\mathfrak{E}_m^{k+1}$ for every m, by 386Bb, so is zero on $\mathfrak{E}^{k+1}$, and in particular is zero at $(e_0, \dots , e_k)$, as required. **Q**

By 384Sf, $\langle \pi^j[\mathfrak{E}] \rangle_{j\in\mathbb{Z}}$ is independent.

(h) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_\sigma : \sigma \in \bigcup_{m\in\mathbb{N}}\{0,1\}^m,\ j \in \mathbb{Z}\}$; then $\mathfrak{E} \subseteq \mathfrak{B}^*$, so $\mathfrak{B}^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{j\in\mathbb{Z}} \pi^j[\mathfrak{E}]$. It follows from (e) that, for any $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{\sigma\in\{0,1\}^m} \rho(b_{m\sigma}, \mathfrak{B}^*) &\le \sum_{\sigma\in\{0,1\}^m} \bar\mu(b_{m\sigma}, b_\sigma)\\ &\le \sum_{\sigma\in\{0,1\}^m} \sum_{n=m}^\infty \bar\mu(b_{n\sigma}, b_{n+1,\sigma}) \le 2\sum_{n=m}^\infty \epsilon_n.\end{aligned}$$

So if $b \in \mathfrak{B}_{r_m}^{(m+1)}$,

$$\begin{aligned}\rho(b, \mathfrak{B}^*) &\le (2r_m + 1) \sum_{\sigma\in\{0,1\}^{m+1}} \rho(b_{m+1,\sigma}, \mathfrak{B}^*)\\ &\le 2(2r_m + 1) \sum_{n=m+1}^\infty \epsilon_n \le 2 \sum_{n=m+1}^\infty 2^{-n} = 2^{-m+1}.\end{aligned}$$

It follows that, whenever $m \le n$ in $\mathbb{N}$,

$$\rho(d_m, \mathfrak{B}^*) \le \rho(d_m, \mathfrak{B}_{r_n}^{(n+1)}) + 2^{-n+1} \le 2^{-n} + 2^{-n+1}$$

by the choice of r_n. Letting $n \to \infty$, we see that $\rho(d_m, \mathfrak{B}^*) = 0$, that is, $d_m \in \mathfrak{B}^*$, for every $m \in \mathbb{N}$. But this means that $\mathfrak{A}_0 \subseteq \mathfrak{B}^*$, by the choice of $\langle d_m \rangle_{m\in\mathbb{N}}$. Accordingly $\pi^j[\mathfrak{A}_0] \subseteq \mathfrak{B}^*$ for every j and $\mathfrak{B}^*$ must be the whole of $\mathfrak{A}$.

(i) Thus π is a two-sided Bernoulli shift with root algebra $\mathfrak{E}$; by 384Sc, $(\mathfrak{A}, \bar\mu, \pi)$ is isomorphic to $(\mathfrak{B}, \bar\nu, \phi)$.

386L Corollary: Sinaĭ's theorem (general case) Let $(\mathfrak{A}, \bar\mu)$ be an atomless probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. Let $(\mathfrak{B}, \bar\nu)$ be a probability algebra of countable Maharam type, and $\phi : \mathfrak{B} \to \mathfrak{B}$ a one- or two-sided Bernoulli shift with $h(\phi) \le h(\pi)$. Then $(\mathfrak{B}, \bar\nu, \phi)$ is isomorphic to a factor of $(\mathfrak{A}, \bar\mu, \pi)$.

proof (a) To begin with (down to the end of (b)) suppose that ϕ is two-sided. Let $\mathfrak{B}_0$ be a root algebra for ϕ. If $\mathfrak{B}_0$ is purely atomic, then there is a generating Bernoulli partition $\langle b_i \rangle_{i\in\mathbb{N}}$ for ϕ of entropy $h(\phi)$. By 386E, there is a Bernoulli partition $\langle c_i \rangle_{i\in\mathbb{N}}$ for π such that $\bar\mu c_i = \bar\nu b_i$ for every i. Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Now $(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C}, \pi{\restriction}\mathfrak{C})$ is a factor of $(\mathfrak{A}, \bar\mu, \pi)$ isomorphic to $(\mathfrak{B}, \bar\nu, \phi)$.

(b) If $\mathfrak{B}_0$ is not purely atomic, then there is still a partition of unity $\langle b_i \rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}_0$ of infinite entropy. Again, let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$, where $\langle c_i \rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π such that $\bar\mu c_i = \bar\nu b_i$ for every i. Now $\pi{\restriction}\mathfrak{C}$ is a Bernoulli shift of infinite entropy and $\mathfrak{C}$ has countable Maharam type, so 386K tells us that there is a closed subalgebra $\mathfrak{C}_0 \subseteq \mathfrak{C}$ such that $\langle \pi^k[\mathfrak{C}_0] \rangle_{k\in\mathbb{N}}$ is independent and $(\mathfrak{C}_0, \bar\mu{\restriction}\mathfrak{C}_0)$ is isomorphic to the measure algebra of Lebesgue measure on $[0, 1]$. But $(\mathfrak{B}_0, \bar\nu{\restriction}\mathfrak{B}_0)$ is a probability algebra of countable Maharam type, so is isomorphic to a closed subalgebra $\mathfrak{C}_1$ of $\mathfrak{C}_0$ (332N). Of course $\langle \pi^k[\mathfrak{C}_1] \rangle_{k\in\mathbb{N}}$ is independent, so if we take $\mathfrak{C}_1^*$ to be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{C}_1]$, $\pi{\restriction}\mathfrak{C}_1^*$ will be a two-sided Bernoulli shift isomorphic to ϕ (384Sf).

(c) If ϕ is a one-sided Bernoulli shift, then 384Sa and 384Sc show that $(\mathfrak{B}, \bar\nu, \phi)$ can be represented in terms of a product measure on a space $X^{\mathbb{N}}$ and the standard shift operator on $X^{\mathbb{N}}$. Now this extends naturally to the standard two-sided Bernoulli shift represented by the product measure on $X^{\mathbb{Z}}$, as described in 384Sb; so that $(\mathfrak{B}, \bar\nu, \phi)$ becomes represented as a factor of $(\mathfrak{B}', \bar\nu', \phi')$ where ϕ' is a two-sided Bernoulli shift with the same entropy as ϕ (since the entropy is determined by the root algebra, by 384R). By (a)-(b), $(\mathfrak{B}', \bar{\nu}', \phi')$ is isomorphic to a factor of $(\mathfrak{A}, \bar{\mu}, \pi)$, so $(\mathfrak{B}, \bar{\nu}, \phi)$ also is.

Remark Thus $(\mathfrak{A}, \bar{\mu}, \pi)$ has factors which are Bernoulli shifts based on root algebras of all countably-generated types permitted by the entropy of π.

386X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a one- or two-sided Bernoulli shift. Show that π^n is a Bernoulli shift for any $n \geq 1$. (*Hint*: if $\mathfrak{A}_0$ is a root algebra for π, the closed subalgebra generated by $\bigcup_{j<n} \pi^j[\mathfrak{A}_0]$ is a root algebra for π^n.)

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and $\pi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ a measure-preserving automorphism such that $\pi[\mathfrak{B}] = \mathfrak{B}$. Show that if π is ergodic or mixing, so is $\pi \restriction \mathfrak{B}$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra of countable Maharam type, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a two-sided Bernoulli shift. Show that for any $n \geq 1$ there is a Bernoulli shift $\phi : \mathfrak{A} \to \mathfrak{A}$ such that $\phi^n = \pi$. (*Hint*: construct a Bernoulli shift ψ such that $h(\psi) = \frac{1}{n} h(\pi)$, and use 384Xh and Ornstein's theorem to show that π is isomorphic to ψ^n.)

(d) Let $\langle \alpha_i \rangle_{i \in \mathbb{N}}$, $\langle \beta_i \rangle_{i \in \mathbb{N}}$ be non-negative real sequences such that $\sum_{i=0}^{\infty} \alpha_i = \sum_{i=0}^{\infty} \beta_i = 1$ and $\sum_{i=0}^{\infty} q(\alpha_i) = \sum_{i=0}^{\infty} q(\beta_i)$. Let μ_0, ν_0 be the measures on $\mathbb{N}$ defined by the formulae

$$\mu_0 E = \sum_{i \in E} \alpha_i, \quad \nu_0 E = \sum_{i \in E} \beta_i$$

for $E \subseteq \mathbb{N}$. Set $X = \mathbb{N}^{\mathbb{Z}}$ and let μ, ν be the product measures on X derived from μ_0 and ν_0. Show that there is a bijection $f : X \to X$ such that ν is precisely the image measure μf^{-1} and f is translation-invariant, that is, $f(x\theta) = f(x)\theta$ for every $x \in X$, where $\theta(n) = n + 1$ for every $n \in \mathbb{Z}$.

386Y Further exercises (a) Suppose that $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are probability algebras with one-sided Bernoulli shifts, and that they are isomorphic. Show that they have isomorphic root algebras. (*Hint*: apply the results of §333 to $(\mathfrak{A}, \bar{\mu}, \pi[\mathfrak{A}])$.)

386 Notes and comments The arguments here are expanded from Smorodinsky 71 and Ornstein 74. I have sought the most direct path to 386I and 386K; of course there is a great deal more to be said (386Xc is a hint), and, in particular, extensions of the methods here provide powerful theorems enabling us to show that automorphisms are Bernoulli shifts. (See Ornstein 74.)

387 Dye's theorem

I have repeatedly said that any satisfactory classification theorem for automorphisms of measure algebras remains elusive. There is however a classification, at least for the Lebesgue measure algebra, of the 'orbit structures' corresponding to measure-preserving automorphisms; in fact, they are defined by the fixed-point subalgebras, which I described in §333. We have to work hard for this result, but the ideas are instructive.

387A Full subgroups Recall the definition of 'full' subgroup (381M): if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then a subgroup G of its automorphism group is full if whenever $\langle a_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$, and $\phi \in \mathrm{Aut}\,\mathfrak{A}$ is such that for each $i \in I$ there is a $\pi_i \in G$ such that $\phi b = \pi_i b$ for every $b \subseteq a_i$, then $\phi \in G$. I take the opportunity to note two facts: (i) if $\mathfrak{A}$ is a Dedekind complete Boolean algebra and $c \in \mathfrak{A}$, then $\{\pi : \pi \in \mathrm{Aut}\,\mathfrak{A}, \pi c = c\}$ is full (ii) if $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra, then the set of measure-preserving automorphisms of $\mathfrak{A}$ is a full subgroup of $\mathrm{Aut}\,\mathfrak{A}$ (382C).

Evidently the intersection of any family of full subgroups of $\mathrm{Aut}\,\mathfrak{A}$ is full, so we can speak of the full subgroup G_π generated by a given automorphism π. From the facts cited above (or otherwise) we see that (i)$'$ if $\pi \in \mathrm{Aut}\,\mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra $\{c : \pi c = c\}$, then $\phi c = c$ for every $\phi \in G_\pi$ and $c \in \mathfrak{C}$ (ii)$'$ if $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra and $\pi \in \mathrm{Aut}\,\mathfrak{A}$ is measure-preserving, then every member of G_π is measure-preserving.

Now the subgroups G_π are easy to describe, as follows.

387B Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and π, ϕ Boolean automorphisms of $\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{Z}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$;

(iii) there is a partition of unity $\langle a_n \rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$.

proof (i)⇒(ii) Let G be the set of those automorphisms satisfying (ii). Then G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$. **P** ($\alpha$) Suppose that ψ_1, $\psi_2 \in G$ and $a \in \mathfrak{A}\setminus\{0\}$. Then there are a non-zero $b_1 \subseteq a$ and an $m \in \mathbb{Z}$ such that $\psi_2 c = \pi^m c$ for every $c \subseteq b_1$. Next, there are a non-zero $b_2 \subseteq \psi_2 b_1$ and an $n \in \mathbb{Z}$ such that $\psi_1 c = \pi^n c$ for every $c \subseteq b_2$. Set $b = \psi_2^{-1} b_2$, so that $0 \ne b \subseteq a$ and

$$\psi_1\psi_2 c = \pi^n \psi_2 c = \pi^n \pi^m c = \pi^{m+n} c$$

for every $c \subseteq b$. As a is arbitrary, $\psi_1\psi_2 \in G$. (β) Suppose that $\psi \in G$ and $a \in \mathfrak{A}\setminus\{0\}$. Then there are a non-zero $b_1 \subseteq \psi^{-1}a$ and an $n \in \mathbb{Z}$ such that $\psi c = \pi^n c$ for every $c \subseteq b_1$. Set $b = \psi b_1$, so that $0 \ne b \subseteq a$; then for any $c \subseteq b$,

$$\pi^{-n} c = \pi^{-n}\psi\psi^{-1} c = \psi^{-1} c.$$

As a is arbitrary, $\psi^{-1} \in G$. (γ) Of course $\pi \in G$, so G is a subgroup of $\operatorname{Aut}\mathfrak{A}$. ($\delta$) Now suppose that $\langle a_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ and that $\psi \in \operatorname{Aut}\mathfrak{A}$ is such that for every $i \in I$ there is a $\psi_i \in G$ such that $\psi c = \psi_i c$ for every $c \subseteq a_i$. If $a \in \mathfrak{A}\setminus\{0\}$, then there is an $i \in I$ such that $a \cap a_i \ne 0$, and there are a non-zero $b \subseteq a \cap a_i$ and an $n \in \mathbb{Z}$ such that $\psi c = \psi_i c = \pi^n c$ for every $c \subseteq b$. As a is arbitrary, $\psi \in G$. As $\langle a_i \rangle_{i\in I}$ is arbitrary, G is full. **Q**

Since G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$ containing π, it includes the full subgroup generated by π and, in particular, $\phi \in G$.

(ii)⇒(iii) For $n \in \mathbb{Z}$, let B_n be the set of those $b \in \mathfrak{A}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$. Set $b_n = \sup B_n$ for each n; then if $c \subseteq b_n$,

$$\phi c = \phi(\sup_{b\in B_n} b \cap c) = \sup_{b\in B_n} \phi(b\cap c) = \sup_{b\in B_n} \pi^n(b\cap c) = \pi^n c.$$

Set

$$a_n = b_n \setminus \sup_{0\le i<n} b_i \text{ if } n \in \mathbb{N},$$
$$= b_n \setminus \sup_{i>n} b_i \text{ if } n \in \mathbb{Z}\setminus\mathbb{N};$$

then $\langle a_n \rangle_{n\in\mathbb{Z}}$ is disjoint,

$$\sup_{n\in\mathbb{Z}} a_n = \sup_{n\in\mathbb{Z}} b_n = \sup(\bigcup_{n\in\mathbb{Z}} B_n) = 1,$$

and $\phi c = \pi^n c$ for every $c \subseteq a_n$, $n \in \mathbb{Z}$; so (iii) is true.

(iii)⇒(i) is trivial.

387C Orbit structures I said that this section was directed to a classification of 'orbit structures', without saying what these might be. In fact what I will do is to classify the full subgroups generated by measure-preserving automorphisms of the Lebesgue measure algebra. One aspect of the relation with 'orbits' is the following.

Proposition Let (X,Σ,μ) be a localizable countably separated measure space (definition: 343D), with measure algebra $(\mathfrak{A},\bar\mu)$. Suppose that f and g are measure space automorphisms from X to itself, inducing measure-preserving automorphisms π, ϕ of $\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for almost every $x \in X$, there is an $n \in \mathbb{Z}$ such that $g(x) = f^n(x)$;

(iii) for almost every $x \in X$, $\{g^n(x) : n \in \mathbb{Z}\} \subseteq \{f^n(x) : n \in \mathbb{Z}\}$.

proof (i)⇒(ii) Let $\langle H_k \rangle_{k\in\mathbb{N}}$ be a sequence in Σ which separates the points of X; we may suppose that $H_0 = X$. By 387B, there is a partition of unity $\langle a_n \rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c = \pi^n c$ for every $c \subseteq a_n$, $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ let $E_n \in \Sigma$ be such that $E_n^\bullet = a_n$; then $Y_0 = \bigcup_{n\in\mathbb{Z}} E_n$ is conegligible. The transformation f^n induces π^n, so for any $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ the set

$$F_{nk} = \{x : f^n(x) \in E_n \cap H_k,\ g(x) \notin E_n \cap H_k\}$$
$$\cup \{x : g(x) \in E_n \cap H_k,\ f^n(x) \notin E_n \cap H_k\}$$

is negligible, and $Y = g^{-1}[Y_0] \setminus \bigcup_{n\in\mathbb{Z},k\in\mathbb{N}} F_{nk}$ is conegligible. Now, for any $x \in Y$, there is some n such that $g(x) \in E_n$, so that $f(x) \in E_n$ and $\{k : g(x) \in H_k\} = \{k : f^n(x) \in H_k\}$ and $g(x) = f^n(x)$. As Y is conegligible, (ii) is satisfied.

(ii)⇒(iii) For $x \in X$, set $\Omega_x = \{f^n(x) : n \in \mathbb{Z}\}$; we are supposing that $A_0 = \{x : g(x) \notin \Omega_x\}$ is negligible. Set $A = \bigcup_{n\in\mathbb{Z}} g^{-n}[A_0]$, so that A is negligible and $g^n(x) \in X \setminus A$ for every $x \in X \setminus A$, $n \in \mathbb{Z}$.

Suppose that $x \in X \setminus A$ and $n \in \mathbb{N}$. Then $g^n(x) \in \Omega_x$. **P** Induce on n. Of course $g^0(x) = x \in \Omega_x$. For the inductive step to $n+1$, $g^n(x) \in \Omega_x \setminus A_0$, so there is a $k \in \mathbb{Z}$ such that $g^n(x) = f^k(x)$. At the same time, there is an $i \in \mathbb{Z}$ such that $g(g^n(x)) = f^i(g^n(x))$, so that $g^{n+1}(x) = f^{i+k}(x) \in \Omega_x$. Thus the induction continues. **Q**

Consequently $g^{-n}(x) \in \Omega_x$ whenever $x \in X \setminus A$ and $n \in \mathbb{N}$. **P** Since $g^{-n}(x) \in X \setminus A$, there is a $k \in \mathbb{Z}$ such that $x = g^n g^{-n}(x) = f^k g^{-n}(x)$ and $g^{-n}(x) = f^{-k}(x) \in \Omega_x$. **Q**

Thus $\{g^n(x) : n \in \mathbb{Z}\} \subseteq \Omega_x$ for every x in the conegligible set $X \setminus A$.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Set

$$E_n = \{x : g(x) = f^n(x)\} = X \setminus \bigcup_{k\in\mathbb{N}}(g^{-1}H_k \triangle f^{-n}[H_k]),$$

for $n \in \mathbb{Z}$. Then (ii) tells us that $\bigcup_{n\in\mathbb{Z}} E_n$ is conegligible, so $\bigcup_{n\in\mathbb{Z}} g[E_n]$ is conegligible. But also each E_n is measurable, so $g[E_n]$ also is, and we can set $a_n = g[E_n]^\bullet$. Now for $y \in g[E_n]$, $y = f^n(g^{-1}(y))$, that is, $g^{-1}(y) = f^{-n}(y)$; so $\phi a = \pi^n a$ for every $a \subseteq a_n$. Since $\sup_{n\in\mathbb{Z}} a_n = 1$ in $\mathfrak{A}$, ϕ belongs to the full subgroup generated by π.

Remark Of course the requirement 'countably separated' is essential here; for other measure spaces we can have ϕ and π actually equal without $g(x)$ and $f(x)$ being related for any particular x (see 343I and 343J).

387D Corollary Under the hypotheses of 387C, π and ϕ generate the same full subgroup of $\operatorname{Aut}\mathfrak{A}$ iff $\{f^n(x) : n \in \mathbb{Z}\} = \{g^n(x) : n \in \mathbb{Z}\}$ for almost every $x \in X$.

387E Induced automorphisms of principal ideals: Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. For $b \in \mathfrak{A}$ write $\mathfrak{A}_b$ for the principal ideal of $\mathfrak{A}$ generated by b.

(a) For any $b \in \mathfrak{A}$ we have a measure-preserving automorphism $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$ defined by saying that $\pi_b d = \pi^n d$ whenever $n \ge 1$ and $d \subseteq b \cap \pi^{-n} b \setminus \sup_{1\le i<n} \pi^{-i} b$.

(b) If $c \in \mathfrak{A}$ and $\pi c = c$ then $\pi_b(b \cap c) = b \cap c$ for every $b \in \mathfrak{A}$.

(c) If $c \subseteq b \in \mathfrak{A}$, then $\pi_c = (\pi_b)_c$, where $(\pi_b)_c$ is the automorphism of $\mathfrak{A}_c$ induced by $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$.

(d) If $0 \ne a \subseteq b \in \mathfrak{A}$ and $n \ge 1$ there are a non-zero $a' \subseteq a$ and an $m \ge 1$ such that $\pi_b^n d = \pi^m d$ for every $d \subseteq a'$.

(e) If π is aperiodic, so is π_b for every $b \in \mathfrak{A}$.

proof (a) For $n \ge 1$ set

$$d_n = b \cap \pi^{-n} b \setminus \sup_{1\le i<n} \pi^{-i} b.$$

If $1 \le m < n$ then

$$d_n \subseteq \pi^{-n} b \setminus \pi^{-m} b, \quad d_m \subseteq \pi^{-m} b$$

so $d_m \cap d_n = 0$. Also

$$d_m \subseteq b, \quad \pi^{n-m} d_n \cap b = \pi^{n-m}(d_n \cap \pi^{-(n-m)} b) = 0$$

so

$$\pi^n d_n \cap \pi^m d_m = \pi^m(\pi^{n-m} d_n \cap d_m) = 0.$$

Thirdly,

$$\sup_{n\ge 1} d_n = \sup_{n\ge 1} b\cap\pi^{-n}b = b$$

by 385D, applied to π^{-1}.

It follows that $\langle d_n\rangle_{n\ge 1}$ is a partition of unity in $\mathfrak{A}_b$. Since $\langle\pi^n d_n\rangle_{n\ge 1}$ is also a disjoint family in $\mathfrak{A}_b$, and

$$\sum_{n=1}^\infty \bar\mu\pi^n d_n = \sum_{n=1}^\infty \bar\mu d_n = \bar\mu b,$$

it is another partition of unity. So we have an automorphism $\pi_b:\mathfrak{A}_b\to\mathfrak{A}_b$ defined by setting $\pi_b d = \pi^n d$ if $d\subseteq d_n$ (381B). Of course π_b is measure-preserving, by 382C.

(b) If $\pi c = c$, then $\pi^n(d_n\cap c)\subseteq c$ for every n, so $\pi_b(b\cap c)\subseteq c$; because $\bar\mu\pi_b(b\cap c) = \bar\mu(b\cap c)$, $\pi_b(b\cap c) = b\cap c$.

(c) Set $D=\{d: d\in\mathfrak{A}_c,\ \pi_c d = (\pi_b)_c d\}$. Then D is order-dense in $\mathfrak{A}_c$. **P** Take any non-zero $a\in\mathfrak{A}_c$. Since $c\subseteq\sup_{n\ge 1}\pi^{-n}c$, there is an $n\in\mathbb{N}$ such that $a'=a\cap\pi^{-n}c\setminus\sup_{1\le i<n}\pi^{-i}c$ is non-zero. Next, there is a non-zero $d\subseteq a'$ such that for every $m\le n$ either $d\subseteq\pi^{-m}b$ or $d\cap\pi^{-m}b=0$. Enumerate $\{m: m\le n,\ d\subseteq\pi^{-m}b\}$ in ascending order as $(m_0,\dots,m_k)$ (note that as $a'\subseteq b\cap\pi^{-n}b$, we must have $m_0=0$ and $m_k=n$). Set $d_i=\pi^{m_i}d$ for $i\le k$, so that

$$d_0=d,\qquad \pi^{m_{i+1}-m_i}d_i = d_{i+1}\subseteq b,$$

while

$$\pi^j d_i = \pi^{m_i+j}d\subseteq 1\setminus b$$

for $1\le j<m_{i+1}-m_i$; that is, $d_{i+1}=\pi_b d_i$ for $i<k$. Thus

$$\pi_b^k d = \pi^{m_k}d = \pi^n d\subseteq c,$$

while

$$\pi_b^i d = d_i = \pi^{m_i}d\subseteq\pi^{m_i}a'\subseteq 1\setminus c$$

for every $i<k$, and

$$(\pi_b)_c d = \pi^n d = \pi_c d,$$

so that $d\in D$. As a is arbitrary, D is order-dense. **Q**

Because π_c and $(\pi_b)_c$ are both order-continuous Boolean homomorphisms on $\mathfrak{A}_c$, and every member of $\mathfrak{A}_c$ is a supremum of some subfamily of D (313K), they must be equal.

(d) Induce on n. We know that there are a non-zero $c\subseteq a$ and an $i\ge 1$ such that $\pi_b d=\pi^i d$ for every $d\subseteq c$. So if $n=1$ we can just take $a'=c$ and $m=i$. For the inductive step to $n+1$, take a' and $m\ge 1$ such that $0\ne a'\subseteq\pi_b c$ and $\pi_b^n d=\pi^m d$ for every $d\subseteq a'$; then

$$0\ne\pi_b^{-1}a'\subseteq c\subseteq a$$

and $\pi_b^{n+1}d=\pi^{m+i}d$ for every $d\subseteq\pi_b^{-1}a'$.

(e) If $0\ne a\subseteq b$ and $n\ge 1$, take a non-zero $a'\subseteq a$ and $m\ge 1$ such that $\pi^m d=\pi_b^n d$ for every $d\subseteq a'$ ((d) above). Now (because π is aperiodic) there is a $d\subseteq a'$ such that $\pi^m d\ne d$, so $\pi_b^n d\ne d$. As n and a are arbitrary, π_b is aperiodic.

387F Similar ideas lead us to the following fact.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism; let $\mathfrak{C}$ be the closed subalgebra $\{c:\pi c=c\}$. Let $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ be two disjoint families in $\mathfrak{A}$ such that $\bar\mu(c\cap d_i)=\bar\mu(c\cap e_i)$ for every $i\in I$ and $c\in\mathfrak{C}$. Then there is a $\phi\in G_\pi$, the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π, such that $\phi d_i=e_i$ for every $i\in I$.

proof Adding $d^*=1\setminus\sup_{i\in I}d_i$, $e^*=1\setminus\sup_{i\in I}e_i$ to the respective families, we may suppose that $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ are partitions of unity. Define $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively by the formula

$$a_n=\sup_{i\in I}(d_i\setminus\sup_{m<n}a_m)\cap\pi^{-n}(e_i\setminus\sup_{m<n}\pi^m a_m).$$

Then $a_n \cap d_i \cap a_m = 0$ whenever $m < n$ and $i \in I$, so $\langle a_n \rangle_{n\in\mathbb{N}}$ is disjoint. Also

$$\pi^n a_n \subseteq \sup_{i\in I} e_i \setminus \sup_{m<n} \pi^m a_m$$

for each n, so $\langle \pi^n a_n \rangle_{n\in\mathbb{N}}$ is disjoint. Note that as $\pi^n(a_n \cap d_j) \subseteq e_j$ for each j,

$$\begin{aligned}\pi^n a_n \cap e_i &= \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_i = \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_j \cap e_i \\ &= \pi^n(a_n \cap d_i) \cap e_i = \pi^n(a_n \cap d_i)\end{aligned}$$

for every $i \in I$, $n \in \mathbb{N}$.

? Suppose, if possible, that $a = 1 \setminus \sup_{n\in\mathbb{N}} a_n$ is non-zero. Then there is an $i \in I$ such that $a \cap d_i \neq 0$. Set $c = \sup_{n\in\mathbb{N}} \pi^n(a \cap d_i)$; then $\pi c \subseteq c$ so $c \in \mathfrak{C}$. Now

$$\begin{aligned}\sum_{n=0}^{\infty} \bar\mu(c \cap e_i \cap \pi^n a_n) &= \sum_{n=0}^{\infty} \bar\mu(c \cap \pi^n(a_n \cap d_i)) = \sum_{n=0}^{\infty} \bar\mu(\pi^n(c \cap a_n \cap d_i)) \\ &= \sum_{n=0}^{\infty} \bar\mu(c \cap a_n \cap d_i) = \bar\mu(c \cap d_i \setminus a) < \bar\mu(c \cap d_i) = \bar\mu(c \cap e_i).\end{aligned}$$

So $b = c \cap e_i \setminus \sup_{n\in\mathbb{N}} \pi^n a_n$ is non-zero, and there is an $n \in \mathbb{N}$ such that $b \cap \pi^n(a \cap d_i)$ is non-zero. But look at $a' = \pi^{-n}(b \cap \pi^n(a \cap d_i))$. We have $0 \neq a' \subseteq a \cap d_i$, so $a' \subseteq d_i \setminus \sup_{m<n} a_m$; while

$$\pi^n a' \subseteq b \subseteq e_i \setminus \sup_{m<n} \pi^m a_m.$$

But this means that $a' \subseteq a_n$, which is absurd. **X**

This shows that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Since

$$\textstyle\sum_{n=0}^{\infty} \bar\mu(\pi^n a_n) = \sum_{n=0}^{\infty} \bar\mu a_n = \bar\mu 1,$$

$\langle \pi^n a_n \rangle_{n\in\mathbb{N}}$ is also a partition of unity. We can therefore define $\phi \in G_\pi$ by setting $\phi d = \pi^n d$ whenever $n \in \mathbb{N}$ and $d \subseteq a_n$. Now, for any $i \in I$,

$$\phi d_i = \sup_{n\in\mathbb{N}} \phi(d_i \cap a_n) = \sup_{n\in\mathbb{N}} \pi^n(d_i \cap a_n) = \sup_{n\in\mathbb{N}} e_i \cap a_n = e_i.$$

So we have found a suitable ϕ.

387G von Neumann transformations: Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$ an automorphism. π is **weakly von Neumann** if there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $a_0 = 1$ and, for every n, $a_{n+1} \cap \pi^{2^n} a_{n+1} = 0$, $a_{n+1} \cup \pi^{2^n} a_{n+1} = a_n$. In this case, π is **von Neumann** if $\langle a_n \rangle_{n\in\mathbb{N}}$ can be chosen in such a way that $\{\pi^m a_n : m, n \in \mathbb{N}\}$ τ-generates $\mathfrak{A}$, and **relatively von Neumann** if $\langle a_n \rangle_{n\in\mathbb{N}}$ can be chosen so that $\{\pi^m a_n : m, n \in \mathbb{N}\} \cup \{c : \pi c = c\}$ τ-generates $\mathfrak{A}$.

(b) There is another way of looking at automorphisms of this type which will be useful. If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism, then a **dyadic cycle system** for π is a finite or infinite family $\langle d_{mi} \rangle_{m\le n, i<2^m}$ or $\langle d_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$ such that (α) for each m, $\langle d_{mi} \rangle_{i<2^m}$ is a partition of unity such that $\pi d_{mi} = d_{m,i+1}$ whenever $i < 2^m - 1$ (so that $\pi d_{m,2^m-1}$ must be d_{m0}) (β) $d_{m0} = d_{m+1,0} \cup d_{m+1,2^m}$ for every $m < n$ (in the finite case) or for every $m \in \mathbb{N}$ (in the infinite case). An easy induction on m shows that if $k \le m$ then

$$d_{ki} = \sup\{d_{mj} : j < 2^m,\ j \equiv i \bmod 2^k\}$$

for every $i < 2^k$.

Conversely, if d is such that $\langle \pi^j d \rangle_{j<2^n}$ is a partition of unity in $\mathfrak{A}$, then we can form a finite dyadic cycle system $\langle d_{mi} \rangle_{m\le n, i<2^m}$ by setting $d_{mi} = \sup\{\pi^j d : j < 2^n,\ j \equiv i \bmod 2^m\}$ whenever $m \le n$ and $j < 2^m$.

(c) Now an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is weakly von Neumann iff it has an infinite dyadic cycle system $\langle d_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$. (The a_m of (a) correspond to the d_{m0} of (b); starting from the definition in (a), you must check first, by induction on m, that $\langle \pi^i a_m \rangle_{i<2^m}$ is a partition of unity in $\mathfrak{A}$.) π is von Neumann iff it has a dyadic cycle system $\langle d_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$ which τ-generates $\mathfrak{A}$.

387H Example The following is the basic example of a von Neumann transformation – in a sense, the only example of a measure-preserving von Neumann transformation. Let μ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, Σ its domain, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Define $f : X \to X$ by setting

$$\begin{aligned} f(x)(n) &= 1 - x(n) \text{ if } x(i) = 0 \text{ for every } i < n, \\ &= x(n) \text{ otherwise.} \end{aligned}$$

Then f is a homeomorphism and a measure space automorphism. **P** (i) To see that f is a homeomorphism, perhaps the easiest way is to look at g, where

$$\begin{aligned} g(x)(n) &= 1 - x(n) \text{ if } x(i) = 1 \text{ for every } i < n, \\ &= x(n) \text{ otherwise,} \end{aligned}$$

and check that f and g are both continuous and that fg, gf are both the identity function. (ii) To see that f is inverse-measure-preserving, it is enough to check that $\mu\{x : f(x)(i) = z(i) \text{ for every } i \le n\} = 2^{-n-1}$ for every $n \in \mathbb{N}$, $z \in X$ (254G). But

$$\{x : f(x)(i) = z(i) \text{ for every } i \le n\} = \{x : x(i) = g(z)(i) \text{ for every } i \le n\}.$$

(iii) Similarly, g is inverse-measure-preserving, so f is a measure space automorphism. **Q**

If $n \in \mathbb{N}$, $x \in X$ then

$$\begin{aligned} f^{2^k}(x)(n) &= 1 - x(n) \text{ if } n \ge k \text{ and } x(i) = 0 \text{ whenever } k \le i < n, \\ &= x(n) \text{ otherwise.} \end{aligned}$$

(Induce on k. For the inductive step, observe that if we identify X with $\{0,1\} \times X$ then $f^2(\epsilon, y) = (\epsilon, f(y))$ for every $\epsilon \in \{0,1\}$ and $y \in X$.)

Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the corresponding automorphism, setting $\pi E^\bullet = f^{-1}[E]^\bullet$ for $E \in \Sigma$. Then π is a von Neumann transformation. **P** Set $E_n = \{x : x \in X, x(i) = 1 \text{ for every } i < n\}$, $a_n = E_n^\bullet$. Then $f^{-2^n}[E_{n+1}] = \{x : x(i) = 1 \text{ for } i < n, x(n) = 0\}$, so a_{n+1}, $\pi^{2^n} a_{n+1}$ split a_n for each n, and $\langle a_n \rangle_{n \in \mathbb{N}}$ witnesses that π is weakly von Neumann. Next, inducing on n, we find that $\{f^{-i}[E_n] : i < 2^n\}$ runs over the basic cylinder sets of the form $\{x : x(i) = z(i) \text{ for every } i < n\}$ determined by coordinates less than n. Since the equivalence classes of such sets τ-generate $\mathfrak{A}$ (see part (a) of the proof of 331K), π is a von Neumann transformation. **Q**

For another way of looking at the functions f and g, see 445Xq in Volume 4.

387I We are now ready to approach the main results of this section.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Let $\mathfrak{C}$ be the closed subalgebra $\{c : \pi c = c\}$. Then for any $a \in \mathfrak{A}$ there is a $b \subseteq a$ such that $\bar{\mu}(b \cap c) = \frac{1}{2}\bar{\mu}(a \cap c)$ for every $c \in \mathfrak{C}$ and π_b is a weakly von Neumann transformation, writing π_b for the induced automorphism of the principal ideal $\mathfrak{A}_b$, as in 387E.

Remark On first reading, there is something to be said for supposing here that π is ergodic, that is, that $\mathfrak{C} = \{0, 1\}$.

proof Set $\epsilon_n = \frac{1}{2}(1 + 2^{-n})$ for each $n \in \mathbb{N}$, so that $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ is strictly decreasing, with $\epsilon_0 = 1$ and $\lim_{n \to \infty} \epsilon_n = \frac{1}{2}$. Now there are $\langle b_n \rangle_{n \in \mathbb{N}}$, $\langle d_{ni} \rangle_{n \in \mathbb{N}, i < 2^n}$ such that, for each $n \in \mathbb{N}$,

$$b_{n+1} \subseteq b_n \subseteq a, \quad \bar{\mu}(b_n \cap c) = \epsilon_n \bar{\mu}(a \cap c) \text{ for every } c \in \mathfrak{C},$$

$$\langle d_{ni} \rangle_{i < 2^n} \text{ is disjoint}, \quad \sup_{i < 2^n} d_{ni} = b_n,$$

$$\pi_{b_n} d_{ni} = d_{n,i+1} \text{ for every } i < 2^n - 1,$$

$$b_{n+1} \cap d_{ni} = d_{n+1,i} \cup d_{n+1,i+2^n} \text{ for every } i < 2^n.$$

P Start with $b_0 = d_{00} = a$. To construct b_{n+1} and $\langle d_{n+1,i} \rangle_{i < 2^{n+1}}$, given $\langle d_{ni} \rangle_{i < 2^n}$, note first that (because π_{b_n} is measure-preserving and $\pi_{b_n}(b_n \cap c) = b_n \cap c$) $\bar{\mu}(d_{n0} \cap c) = \bar{\mu}(d_{ni} \cap c)$ whenever $c \in \mathfrak{C}$, $i < 2^n$, so

$$\bar{\mu}(d_{n0} \cap c) = 2^{-n}\bar{\mu}(b_n \cap c) = 2^{-n}\epsilon_n\bar{\mu}(a \cap c)$$

for every $c \in \mathfrak{C}$, and

$$d_{n0} = b_n \setminus \sup_{i<2^n-1} \pi_{b_n} d_{ni} = \pi_{b_n} d_{n,2^n-1} = \pi_{b_n}^{2^n} d_{n0}.$$

Now π_{b_n} is aperiodic (387Ee) so $\pi_{b_n}^{2^n}$ also is (385A), and there is a $d_{n+1,0} \subseteq d_{n0}$ such that

$$\pi_{b_n}^{2^n} d_{n+1,0} \cap d_{n+1,0} = 0, \quad \bar{\mu}(d_{n+1,0} \cap c) = 2^{-n-1}\epsilon_{n+1}\bar{\mu}(a \cap c) \text{ for every } c \in \mathfrak{C}$$

(applying 385E(iii) to $\pi_{b_n}^{2^n} \restriction \mathfrak{A}_{d_{n0}}$, with $\gamma = \epsilon_{n+1}/2\epsilon_n$). Set $d_{n+1,j} = \pi_{b_n}^j d_{n+1,0}$ for each $j < 2^{n+1}$. Because $\pi_{b_n}^{2^n} d_{n+1,0} \subseteq d_{n0} \setminus d_{n+1,0}$, while $\langle \pi_{b_n}^j d_{n0} \rangle_{j<2^n}$ is disjoint, $\langle \pi_{b_n}^j d_{n+1,0} \rangle_{j<2^{n+1}}$ is disjoint. Set $b_{n+1} = \sup_{i<2^{n+1}} \pi_{b_n}^i d_{n+1,0}$; then $b_{n+1} \subseteq b_n$ and $\bar{\mu}(b_{n+1} \cap c) = \epsilon_{n+1}\bar{\mu}(a \cap c)$ for every $c \in \mathfrak{C}$. For $j < 2^{n+1}$, $d_{n+1,j} \subseteq d_{ni}$ where i is either j or $j - 2^n$, so $b_{n+1} \cap d_{ni} = d_{n+1,i} \cup d_{n+1,i+2^n}$.

For $i < 2^{n+1} - 1$,

$$\pi_{b_n} d_{n+1,i} = d_{n+1,i+1} \subseteq b_{n+1},$$

so we must also have

$$\pi_{b_{n+1}} d_{n+1,i} = (\pi_{b_n})_{b_{n+1}} d_{n+1,i} = d_{n+1,i+1}$$

(using 387Ec). Thus the induction continues. **Q**

Set

$$b = \inf_{n\in\mathbb{N}} b_n, \quad e_{ni} = b \cap d_{ni} \text{ for } n \in \mathbb{N},\ i < 2^n.$$

Because $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-increasing,

$$\bar{\mu}(b \cap c) = \lim_{n\to\infty} \bar{\mu}(b_n \cap c) = \tfrac{1}{2}\bar{\mu}(a \cap c)$$

for every $c \in \mathfrak{C}$. Next,

$$e_{ni} = b \cap b_{n+1} \cap d_{ni} = b \cap (d_{n+1,i} \cup d_{n+1,i+2^n}) = e_{n+1,i} \cup e_{n+1,i+2^n}$$

whenever $i < 2^n$.

If $m \le n$, $j < 2^m$ then

$$b_n \cap d_{mj} = \sup\{d_{ni} : i < 2^n,\ i \equiv j \bmod 2^m\}$$

(induce on n). So

$$\bar{\mu}(b_n \cap d_{mj}) = 2^{n-m}\bar{\mu}d_{n0} = 2^{-m}\epsilon_n;$$

taking the limit as $n \to \infty$, $\bar{\mu}e_{mj} = 2^{-m}\bar{\mu}b$. Next,

$$\begin{aligned}\pi_{b_n}(b_n \cap d_{mj}) &= \sup\{d_{n,i+1} : i < 2^n,\ i \equiv j \bmod 2^m\} \\ &= \sup\{d_{ni} : i < 2^n,\ i \equiv j+1 \bmod 2^m\} = b_n \cap d_{m,j+1},\end{aligned}$$

here interpreting $d_{n,2^n}$ as d_{n0}, $d_{m,2^m}$ as d_{m0}. Consequently $\pi_b e_{mj} \subseteq e_{m,j+1}$. **P?** Otherwise, there are a non-zero $e \subseteq d_{mj} \cap b$ and $k \ge 1$ such that $\pi^i e \cap b = 0$ for $1 \le i < k$ and $\pi^k e \subseteq b \setminus d_{m,j+1}$. Take $n \ge m$ so large that $\bar{\mu}e > k\bar{\mu}(b_n \setminus b)$, so that

$$e' = e \setminus \sup_{1\le i<k} \pi^{-i}(b_n \setminus b) \ne 0;$$

now $\pi^i e' \cap b_n = 0$ for $1 \le i < k$, while $\pi^k e' \subseteq b_n$, and

$$\pi_{b_n} e' = \pi^k e' \subseteq 1 \setminus d_{m,j+1}.$$

But this means that $\pi_{b_n}(b_n \cap d_{mj}) \not\subseteq d_{m,j+1}$, which is impossible. **XQ**

Since $\bar{\mu}\pi_b e_{mj} = \bar{\mu}e_{m,j+1}$, we must have $\pi_b e_{mj} = e_{m,j+1}$. And this is true whenever $m \in \mathbb{N}$ and $j < 2^m$, if we identify $e_{m,2^m}$ with e_{m0}. Thus $\langle e_{mi} \rangle_{m\in\mathbb{N},i<2^m}$ is a dyadic cycle system for π_b and π_b is a weakly von Neumann transformation.

387J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and π, ψ two measure-preserving automorphisms of $\mathfrak{A}$. Suppose that ψ belongs to the full subgroup G_π generated by π and that there is a $b \in \mathfrak{A}$ such that $\sup_{n\in\mathbb{Z}} \psi^n b = 1$ and the induced automorphisms ψ_b, π_b on $\mathfrak{A}_b$ are equal. Then $G_\psi = G_\pi$.

proof (a) The first fact to note is that if $0 \neq b' \subseteq b$, $n \in \mathbb{Z}$ and $\pi^n b' \subseteq b$, then there are $m \in \mathbb{Z}$, $b'' \subseteq b'$ such that $b'' \neq 0$ and $\pi^n d = \psi^m d$ for every $d \subseteq b''$. **P** (α) If $n = 0$ take $b'' = b'$, $m = 0$. (β) Next, suppose that $n > 0$. Take a non-zero $b_1 \subseteq b'$ such that $\pi^i b_1$ is either included in b or disjoint from b for every $i \leq n$ (e.g., an atom of the finite subalgebra generated by $\{b'\} \cup \{\pi^{-i} b : 0 \leq i \leq n\}$). Enumerate $\{i : i \leq n, \pi^i b_1 \subseteq b\}$ in ascending order as $(l_0, \ldots, l_k)$. Then $\pi^{l_j} d = \pi_b^j d$ whenever $d \subseteq b_1$ and $j \leq k$ (compare part (c) of the proof of 387E); in particular, $\pi^n d = \pi_b^k d = \psi_b^k d$ for every $d \subseteq b_1$. But now 387Ed tells us that there must be a non-zero $b'' \subseteq b_1$ and an $m \in \mathbb{N}$ such that

$$\psi^m d = \psi_b^k d = \pi^n d$$

for every $d \subseteq b''$. (γ) If $n < 0$, then set $e' = \pi^n b'$, so that e', $\pi^{-n} e' \subseteq b$. By (β), there are a non-zero $e'' \subseteq e'$ and an $s \in \mathbb{N}$ such that $\pi^{-n} d = \psi^s d$ for every $d \subseteq e''$. Setting $b'' = \pi^{-n} e'' = \psi^s e''$, we have $0 \neq b'' \subseteq b'$ and $\pi^n d = \psi^{-s} d$ for every $d \subseteq b''$. **Q**

(b) Now take any non-zero $a \in \mathfrak{A}$. Then there are m, $n \in \mathbb{Z}$ such that $a_1 = a \cap \psi^m b \neq 0$, $a_2 = \pi a_1 \cap \psi^n b \neq 0$. Set $b_1 = \psi^{-m} \pi^{-1} a_2$. Because $\psi \in G_\pi$, there are a non-zero $b_2 \subseteq b_1$ and a $k \in \mathbb{Z}$ such that $\psi^{-n} \pi \psi^m d = \pi^k d$ for every $d \subseteq b_2$. Now

$$\pi^k b_2 = \psi^{-n} \pi \psi^m b_2 \subseteq \psi^{-n} \pi \psi^m b_1 = \psi^{-n} a_2 \subseteq b.$$

By (a), there are a non-zero $b_3 \subseteq b_2$ and an $r \in \mathbb{Z}$ such that $\pi^k d = \psi^r d$ for every $d \subseteq b_3$. Consider $a' = \psi^m b_3$. Then

$$0 \neq a' \subseteq \psi^m b_1 = \pi^{-1} a_2 \subseteq a_1 \subseteq a;$$

and, for $d \subseteq a'$, $\psi^{-m} d \subseteq b_3 \subseteq b_2$, so that

$$\pi d = \psi^n (\psi^{-n} \pi \psi^m) \psi^{-m} d = \psi^n \pi^k \psi^{-m} d = \psi^{n+r-m} d.$$

As a is arbitrary, this shows that $\pi \in G_\psi$, so that $G_\pi \subseteq G_\psi$ and the two are equal.

387K Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, and ϕ any member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π. Suppose that $\langle d_{mi} \rangle_{m \leq n, i < 2^m}$ is a finite dyadic cycle system for ϕ. Then there is a weakly von Neumann transformation ψ, with dyadic cycle system $\langle d'_{mi} \rangle_{m \in \mathbb{N}, i < 2^m}$, such that $G_\psi = G_\pi$, $\psi a = \phi a$ whenever $a \cap d_{n0} = 0$, and $d'_{mi} = d_{mi}$ whenever $m \leq n$ and $i < 2^m$.

proof Write $\mathfrak{C}$ for the closed subalgebra $\{c : \pi c = c\}$. By 387I there is a $b \subseteq d_{n0}$ such that $\bar{\mu}(b \cap c) = \frac{1}{2} \bar{\mu}(d_{n0} \cap c)$ for every $c \in \mathfrak{C}$ and $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$ is a weakly von Neumann transformation. Let $\langle e_{ki} \rangle_{k \in \mathbb{N}, i < 2^k}$ be a dyadic cycle system for π_b.

If we define $\psi_1 \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\psi_1 d = \pi_b d \text{ for } d \subseteq b, \quad \psi_1 d = \pi_{1 \setminus b} d \text{ for } d \subseteq 1 \setminus b,$$

then $\psi_1 \in G_\pi$. Next, for any $c \in \mathfrak{C}$,

$$\bar{\mu}(\phi^{-2^n+1} b \cap c) = \bar{\mu}\phi^{-2^n+1}(b \cap c) = \bar{\mu}(b \cap c) = \frac{1}{2}\bar{\mu}(d_{n0} \cap c) = \bar{\mu}((d_{n0} \setminus b) \cap c)$$

because $\phi^{-2^n+1} \in G_\pi$, so $\phi^{-2^n+1} c = c$. By 387F, there is a $\psi_2 \in G_\pi$ such that $\psi_2(d_{n0} \setminus b) = \phi^{-2^n+1} b$. Set $\psi_3 = \phi^{-2^n+1} \psi_2^{-1} \phi^{-2^n+1} \psi_1$, so that $\psi_3 \in G_\pi$ and

$$\psi_3 b = \phi^{-2^n+1} \psi_2^{-1} \phi^{-2^n+1} b = \phi^{-2^n+1}(d_{n0} \setminus b).$$

Thus $\psi_3 b$ and $\psi_2(d_{n0} \setminus b)$ are disjoint and have union $\phi^{-2^n+1} d_{n0} = d_{n1}$ (if $n = 0$, we must read d_{01} as $d_{00} = 1$). Accordingly we can define $\psi \in G_\pi$ by setting

$$\begin{aligned} \psi d &= \psi_3 d \text{ if } d \subseteq b, \\ &= \psi_2 d \text{ if } d \subseteq d_{n0} \setminus b, \\ &= \phi d \text{ if } d \cap d_{n0} = 0. \end{aligned}$$

Since $\psi d_{n0} = d_{n1}$, we have $\psi d_{ni} = \phi d_{ni}$ for every $i < 2^n$, and therefore $\psi^i d_{m0} = d_{mi}$ whenever $m \leq n$ and $i < 2^m$. Looking at ψ^{2^n}, we have

$$\psi^{2^n}d_{n0} = \phi^{2^n}d_{n0} = d_{n0}, \quad \psi^{2^n}b = \phi^{2^n-1}\psi_3 b = d_{n0}\setminus b,$$

so that $\psi^{2^n}(d_{n0}\setminus b) = b$ and $\psi^{2^{n+1}}b = b$. Accordingly

$$\psi^{2^{n+1}}d = \phi^{2^n-1}\psi_2\phi^{2^n-1}\psi_3 d = \psi_1 d = \pi_b d$$

for every $d \subseteq b$. Also $\sup_{i<2^{n+1}}\psi^i b = 1$, so 387J tells us that $G_\psi = G_\pi$.

Now define $\langle a_m\rangle_{m\in\mathbb{N}}$ as follows. For $m \le n$, $a_m = d_{m0}$; for $m > n$, $a_m = e_{m-n-1,0}$. Then for $m < n$ we have

$$\psi^{2^m}a_{m+1} = \psi^{2^m}d_{m+1,0} = \phi^{2^m}d_{m+1,0} = d_{m+1,2^m} = a_m \setminus a_{m+1},$$

for $m = n$ we have

$$\psi^{2^n}a_{n+1} = \psi^{2^n}e_{00} = \psi^{2^n}b = d_{n0}\setminus b = a_n\setminus a_{n+1},$$

and for $m > n$ we have

$$\begin{aligned}\psi^{2^m}a_{m+1} &= (\psi^{2^{n+1}})^{2^{m-n-1}}e_{m-n,0} = (\pi_b)^{2^{m-n-1}}e_{m-n,0}\\ &= e_{m-n,2^{m-n-1}} = e_{m-n-1,0}\setminus e_{m-n,0} = a_m\setminus a_{m+1}.\end{aligned}$$

Thus $\langle a_m\rangle_{m\in\mathbb{N}}$ witnesses that ψ is a weakly von Neumann transformation. If $d'_{mi} = \psi^i a_m$ for $m\in\mathbb{N}$, $i < 2^m$ then $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ will be a dyadic cycle system for ψ and $d'_{mi} = d_{mi}$ for $m \le n$, as required.

387L Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure space and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a\in\mathfrak{A}$ write $\mathfrak{C}_a = \{a\cap c : c\in\mathfrak{C}\}$.

(a) Suppose that $b\in\mathfrak{A}$, $w\in\mathfrak{C}$ and $\delta > 0$ are such that $\bar\mu(b\cap c)\ge\delta\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w$. Then there is an $e\in\mathfrak{A}$ such that $e\subseteq b\cap w$ and $\bar\mu(e\cap c) = \delta\bar\mu c$ whenever $c\in\mathfrak{C}_w$.

(b) Suppose that $k\ge 1$ and that $(b_0,\dots,b_r)$ is a finite partition of unity in $\mathfrak{A}$. Then there is a partition E of unity in $\mathfrak{A}$ such that

$$\bar\mu(e\cap c) = \frac{1}{k}\bar\mu c \text{ for every } e\in E,\ c\in\mathfrak{C},$$

$$\#(\{e : e\in E, \exists\, i\le r,\ b_i\cap e\notin\mathfrak{C}_e\}) \le r+1.$$

proof (a) Set $a = b\cap w$ and consider the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{A}$. We know that $(\mathfrak{A}_a,\bar\mu\restriction\mathfrak{A}_a)$ is a totally finite measure algebra (322H), and that $\mathfrak{C}_a$ is a closed subalgebra of $\mathfrak{A}_a$ (333Bc); and it is easy to see that $\mathfrak{A}_a$ is relatively atomless over $\mathfrak{C}_a$.

Let $\theta : \mathfrak{C}_w\to\mathfrak{C}_a$ be the Boolean homomorphism defined by setting $\theta c = c\cap b$ for $c\in\mathfrak{C}_w$. If $c\in\mathfrak{C}_w$ and $\theta c = 0$, then $c\in\mathfrak{C}$ and $\delta\bar\mu c\le\bar\mu(c\cap b) = 0$, so $c = 0$; thus θ is injective; since it is certainly surjective, it is a Boolean isomorphism. We can therefore define a functional $\nu = \bar\mu\theta^{-1} : \mathfrak{C}_a\to[0,\infty[$, and we shall have $\delta\nu d\le\bar\mu d$ for every $d\in\mathfrak{C}_a$. By 331B, there is an $e\in\mathfrak{A}_a$ such that $\delta\nu d = \bar\mu(d\cap e)$ for every $d\in\mathfrak{C}_a$, that is, $\delta\bar\mu c = \bar\mu(c\cap e)$ for every $c\in\mathfrak{C}_w$, as required.

(b)(i) Write D for the set of all those $e\in\mathfrak{A}$ such that $\bar\mu(c\cap e) = \frac{1}{k}\bar\mu c$ for every $c\in\mathfrak{C}$ and $b_i\cap e\in\mathfrak{C}_e$ for every $i\le r$. Then whenever $a\in\mathfrak{A}$ and $\gamma > \frac{r+1}{k}$ is such that $\mu(a\cap c) = \gamma\mu c$ for every $c\in\mathfrak{C}$, there is an $e\in D$ such that $e\subseteq a$. **P** For $d\in\mathfrak{A}$, $c\in\mathfrak{C}$ set $\nu_d(c) = \bar\mu(d\cap c)$, so that $\nu_d : \mathfrak{C}\to[0,\infty[$ is a completely additive functional. For $i\le r$ set $v_i = [\![\bar\mu\restriction\mathfrak{C} > k\nu_{a\cap b_i}]\!]$, in the notation of 326P; so that $v_i\in\mathfrak{C}$ and $\bar\mu c\ge k\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\subseteq v_i$, while $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\cap v_i = 0$. Setting $v = \inf_{i\le r}v_i$, we have

$$k\gamma\bar\mu v = k\bar\mu(a\cap v) = \textstyle\sum_{i=0}^r k\mu(a\cap b_i\cap v)\le(r+1)\bar\mu v.$$

Since $k\gamma > r+1$, $v = 0$. So if we now set $w_i = (\inf_{j<i}v_j)\setminus v_i$ for $i\le r$ (starting with $w_0 = 1\setminus v_0$), $(w_0,\dots,w_r)$ is a partition of unity in $\mathfrak{C}$, and $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\subseteq w_i$.

By (a), we can find for each $i\le r$ an $e_i\in\mathfrak{A}$ such that $e_i\subseteq a\cap b_i\cap w_i$ and $\bar\mu(c\cap e_i) = \frac{1}{k}\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w_i$. Set $e = \sup_{i\le r}e_i$, so that $e\subseteq a$,

$$e\cap b_i = e\cap w_i\cap b_i = e_i = e\cap w_i\in\mathfrak{C}_e$$

for each i, and

$$\begin{aligned}\bar\mu(c\cap e)&=\sum_{i=0}^r\bar\mu(c\cap e_i)=\sum_{i=0}^r\bar\mu(c\cap w_i\cap e_i)\\&=\sum_{i=0}^r\frac{1}{k}\bar\mu(c\cap w_i)=\frac{1}{k}\bar\mu c\end{aligned}$$

for every $c\in\mathfrak{C}$. So e has all the properties required. **Q**

(ii) Let $E_0\subseteq D$ be a maximal disjoint family, and set $m=\#(E_0)$, $a=1\setminus\sup E_0$. Then

$$\bar\mu(a\cap c)=\bar\mu c-\textstyle\sum_{e\in E_0}\bar\mu(c\cap e)=(1-\frac{m}{k})\bar\mu c$$

for every $c\in\mathfrak{C}$, while a does not include any member of D. By (i), $1-\frac{m}{k}\le\frac{r+1}{k}$, that is, $m\ge k-r-1$.

Applying (a) repeatedly, with $w=1$ and $\delta=\frac{1}{k}$, we can find disjoint $d_0,\dots,d_{k-m-1}\subseteq a$ such that $\bar\mu(c\cap d_i)=\frac{1}{k}\bar\mu c$ for every $c\in\mathfrak{C}$ and $i<k-m$. So if we set $E=E_0\cup\{d_i:i<k-m\}$ we shall have a partition of unity with the properties required.

387M Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an aperiodic measure-preserving automorphism. Write $\mathfrak{C}$ for the closed subalgebra $\{c:\pi c=c\}$. Suppose that ϕ is a member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π with a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n,i<2^m}$, and that $a\in\mathfrak{A}$ and $\epsilon>0$. Then there is a $\psi\in G_\pi$ such that

(i) ψ has a dyadic cycle system $\langle d'_{mi}\rangle_{m\le k,i<2^m}$, with $k\ge n$ and $d'_{mi}=d_{mi}$ for $m\le n$, $i<2^m$;

(ii) $\psi d=\phi d$ if $d\cap d_{n0}=0$;

(iii) there is an a' in the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{d'_{ki}:i<2^k\}$ such that $\bar\mu(a\,\triangle\,a')\le\epsilon$.

proof (a) Take $k\ge n$ so large that $2^k\epsilon\ge 2^n2^{2^n}\bar\mu1$. Let $\mathfrak{D}$ be the subalgebra of the principal ideal $\mathfrak{A}_{d_{n1}}$ generated by $\{d_{n1}\cap\phi^{-j}a:j<2^n\}$; then $\mathfrak{D}$ has atoms $b_0,\dots,b_r$ where $r<2^{2^n}$. (If $n=0$, take $d_{01}=d_{00}=1$.) Applying 387L to the closed subalgebra $\mathfrak{C}_{d_{n1}}$ of $\mathfrak{A}_{d_{n1}}$, we can find a partition of unity E of $\mathfrak{A}_{d_{n1}}$ such that

$$\bar\mu(e\cap c)=2^{-k}\bar\mu c\text{ for every }e\in E,\ c\in\mathfrak{C}$$

$$E_1=\{e:e\in E,\text{ there is some }i\le r\text{ such that }b_i\cap e\notin\mathfrak{C}_e\}$$

has cardinal at most $r+1\le2^{2^n}$. Of course $\bar\mu e=2^{-k}$ for every $e\in E$, so $\#(E)=2^{k-n}$ and $\bar\mu(\sup E_1)\le 2^{-k}2^{2^n}\le2^{-n}\epsilon$. Write e^* for $\sup E_1$.

(b) For $e\in E$ set $e'=\phi^{2^n-1}e$; then $\{e':e\in E\}$ is a disjoint family, of cardinal 2^{k-n}; enumerate it as $\langle v_i\rangle_{i<2^{k-n}}$. Note that

$$\sup\nolimits_{i<2^{k-n}}v_i=\phi^{2^n-1}(\sup E)=d_{n0},$$

$$\bar\mu(v_i\cap c)=\bar\mu(\phi^{-2^n+1}v_i\cap c)=2^{-k}\bar\mu c$$

for every $c\in\mathfrak{C}$, $i<2^{k-n}$. There is therefore a $\psi_1\in G_\pi$ such that

$$\psi_1v_i=\phi^{-2^n+1}v_{i+1}\text{ for }i<2^{k-n}-1,\quad\psi_1v_{2^{k-n}-1}=\phi^{-2^n+1}v_0$$

(387F). We have

$$\begin{aligned}\psi_1d_{n0}&=\psi_1(\sup_{i<2^{k-n}}v_i)=\sup_{i<2^{k-n}}\psi_1v_i=\sup_{i<2^{k-n}-1}\phi^{-2^n+1}v_{i+1}\cup\phi^{-2^n+1}v_0\\&=\sup_{i<2^{k-n}}\phi^{-2^n+1}v_i=\phi^{-2^n+1}d_{n0}=d_{n1}=\phi d_{n0}.\end{aligned}$$

So we may define $\psi\in G_\pi$ by setting

$$\begin{aligned}\psi d&=\psi_1d\text{ if }d\subseteq d_{n0},\\&=\phi d\text{ if }d\cap d_{n0}=0.\end{aligned}$$

(c) For each $i < 2^{k-n}$,

$$\psi^{2^n} v_i = \phi^{2^n-1}\psi_1 v_i = v_{i+1}$$

(identifying $v_{2^{k-n}}$ with v_0). Moreover, $\psi^j v_i \subseteq d_{nl}$ whenever $i < 2^{k-n}$ and $j \equiv l \bmod 2^n$. So $\langle \psi^j v_0\rangle_{j<2^k}$ is a partition of unity in $\mathfrak{A}$. What this means is that if we set

$$d'_{mj} = \sup\{\psi^i v_0 : i < 2^k,\ i \equiv j \bmod 2^m\}$$

for $m \le k$, then $\langle d'_{mj}\rangle_{m\le k, j<2^m}$ is a dyadic cycle system for ψ, with $d'_{mj} = d_{mj}$ if $m \le n$, $j < 2^m$.

(d) Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{kj} : j < 2^k\}$. Recall the definition of $\{v_i : i < 2^{k-n}\}$ as $\{\phi^{2^n-1}e : e \in E\}$; this implies that

$$\{\psi v_i : i < 2^{k-n}\} = \{\psi_1 v_i : i < 2^{k-n}\} = \{\phi^{-2^n+1} v_i : i < 2^{k-n}\} = E,$$

so that

$$\{\psi^{j+1} v_i : i < 2^{k-n}\} = \{\phi^j e : e \in E\}$$

for $j < 2^n$, and

$$\mathfrak{B} \supseteq \{d'_{kj} : j < 2^k\} = \{\psi^j v_i : i < 2^{k-n},\ j < 2^n\} = \{\phi^j e : e \in E,\ j < 2^n\}.$$

Set $E_0 = E \setminus E_1$. For $e \in E_0$ and $i \le r$ there is a $c_{ei} \in \mathfrak{C}$ such that $e \cap b_i = e \cap c_{ei}$. Set

$$K = \{(i,j) : 1 \le i \le r,\ j < 2^n,\ b_i \subseteq \phi^{-j}a\},$$

$$a' = \sup\{\phi^j e \cap c_{ei} : e \in E_0,\ (i,j) \in K\}.$$

Then a' is a supremum of (finitely many) members of $\mathfrak{B}$, so belongs to $\mathfrak{B}$. If $(i,j) \in K$ and $e \in E_0$, then

$$\phi^j e \cap c_{ei} = \phi^j(e \cap c_{ei}) = \phi^j(e \cap b_i) \subseteq a,$$

so $a' \subseteq a$. Next, $d_{n1} \cap \phi^{-j}(a \setminus a') \subseteq e^*$ for each $j < 2^n$. **P** Set

$$I = \{i : i \le r,\ (i,j) \in K\} = \{i : b_i \subseteq \phi^{-j}a\};$$

then $d_{n1} \cap \phi^{-j}a = \sup_{i\in I} b_i$. Now, for each $i \in I$,

$$b_i = \sup_{e\in E}(b_i \cap e) \subseteq \sup_{e\in E_0}(e \cap c_{ei}) \cup e^*,$$

so that

$$d_{n1} \cap \phi^{-j}a = \sup_{i\in I} b_i \subseteq \sup_{e\in E_0, i\in I}(e \cap c_{ei}) \cup e^* = (d_{n1} \cap \phi^{-j}a') \cup e^*. \text{ **Q**}$$

But this means that

$$\bar\mu(d_{n,j+1} \cap a \setminus a') = \bar\mu(d_{n1} \cap \phi^{-j}(a \setminus a')) \le \bar\mu e^* \le 2^{-n}\epsilon$$

for every $j < 2^n$ (interpreting $d_{n,2^n}$ as d_{n0}, as usual), and

$$\bar\mu(a \triangle a') = \sum_{j=1}^{2^n} \bar\mu(d_{nj} \cap a \setminus a') \le \epsilon,$$

so that the final condition of the lemma is satisfied.

387N Theorem Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, with Maharam type ω, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Then there is a relatively von Neumann transformation $\phi : \mathfrak{A} \to \mathfrak{A}$ such that ϕ and π generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

proof (a) The idea is to construct ϕ as the limit of a sequence $\langle\phi_n\rangle_{n\in\mathbb{N}}$ of weakly von Neumann transformations such that $G_{\phi_n} = G_\pi$. Each ϕ_n will have a dyadic cycle system $\langle d_{nmi}\rangle_{m\in\mathbb{N}, i<2^m}$; there will be a strictly increasing sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ such that

$$d_{n+1,m,i} = d_{n,m,i} \text{ whenever } m \le k_n,\ i < 2^m,$$

$$\phi_{n+1}a = \phi_n a \text{ whenever } a \cap d_{n,k_n,0} = 0.$$

Interpolated between the ϕ_n will be a second sequence $\langle\psi_n\rangle_{n\in\mathbb{N}}$ in G_π, with associated (finite) dyadic cycle systems $\langle d'_{nmi}\rangle_{m\le k'_n, i<2^m}$.

(b) Before starting on the inductive construction we must fix on a countable set $B \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$, and a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in B such that every member of B recurs cofinally often in the sequence. (For instance, take the sequence of first members of an enumeration of $B \times \mathbb{N}$.) As usual, I write $\mathfrak{C}$ for the closed subalgebra $\{c : \pi c = c\}$. The induction begins with $\psi_0 = \pi$, $k'_0 = 0$, $d'_{000} = 1$. Given $\psi_n \in G_\pi$ and its dyadic cycle system $\langle d'_{nmi} \rangle_{m\le k'_n, i<2^m}$, use 387K to find a weakly von Neumann transformation ϕ_n, with dyadic cycle system $\langle d_{nmi} \rangle_{m\in\mathbb{N}, i<2^m}$, such that $G_{\phi_n} = G_\pi$, $d_{nmi} = d'_{nmi}$ for $m \le k'_n$ and $i < 2^m$, and $\phi_n a = \psi_n a$ whenever $a \cap d'_{n,k'_n,0} = 0$.

(c) Given the weakly von Neumann transformation ϕ_n, with its dyadic cycle system $\langle d_{nmi} \rangle_{m\in\mathbb{N}, i<2^m}$, such that $G_{\phi_n} = G_\pi$, then we have a partition of unity $\langle e_{nj} \rangle_{j\in\mathbb{Z}}$ such that $\pi a = \phi_n^j a$ whenever $j \in \mathbb{Z}$ and $a \subseteq e_{nj}$ (387C(ii)). Take r_n such that $\bar\mu \tilde e_n \le 2^{-n}$, where $\tilde e_n = \sup_{|j|>r_n} e_{nj}$, and $k_n > k'_n$ such that $2^{-k_n}(2r_n + 1) \le 2^{-n}$. Set

$$e^*_n = \sup_{|j|\le r_n} \phi_n^{-j} d_{n,k_n,0},$$

so that $\bar\mu e^*_n \le 2^{-n+1}$.

Now use 387M to find a $\psi_{n+1} \in G_\pi$, with a dyadic cycle system $\langle d'_{n+1,m,i} \rangle_{m\le k'_{n+1}, i<2^m}$, such that $k'_{n+1} \ge k_n$, $d'_{n+1,m,i} = d_{nmi}$ if $m \le k_n$, $\psi_{n+1} a = \phi_n a$ if $a \cap d_{n,k_n,0} = 0$, and there is a b'_n in the algebra generated by $\mathfrak{C} \cup \{d'_{n+1,m,i} : m \le k'_{n+1},\ i < 2^m\}$ such that $\bar\mu(b_n \triangle b'_n) \le 2^{-n}$. Continue.

(d) The effect of this construction is to ensure that if $l < n$ in $\mathbb{N}$ then

$$d_{lmi} = d_{nmi} \text{ whenever } m \le k_l,\ i < 2^m,$$

$$\phi_n a = \phi_l a \text{ whenever } a \cap d_{l,k_l,0} = 0,$$

$$b'_l \text{ belongs to the subalgebra generated by } \mathfrak{C} \cup \{d_{nmi} : m \le k_n,\ i < 2^m\},$$

and, of course, $d_{n,k_n,0} \subseteq d_{l,k_l,0}$. Since $\langle k_n \rangle_{n\in\mathbb{N}}$ is strictly increasing, $\inf_{n\in\mathbb{N}} d_{n,k_n,0} = 0$. Now, for each $n \in \mathbb{N}$,

$$d_{n,k_n,1} = \phi_n d_{n,k_n,0} = \phi_{n+1} d_{n,k_n,0} \supseteq \phi_{n+1} d_{n+1,k_{n+1},0} = d_{n+1,k_{n+1},1},$$

so setting

$$a_0 = 1 \setminus d_{0,k_0,0}, \quad a_{n+1} = d_{n,k_n,0} \setminus d_{n+1,k_{n+1},0} \text{ for each } n,$$

we have

$$\phi_0 a_0 = 1 \setminus d_{0,k_0,1}, \quad \phi_{n+1} a_{n+1} = d_{n,k_n,1} \setminus d_{n+1,k_{n+1},1} \text{ for each } n,$$

and $\langle \phi_n a_n \rangle_{n\in\mathbb{N}}$ is a partition of unity. There is therefore a $\phi \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\phi a = \phi_n a$ if $a \subseteq a_n$; because G_π is full, $\phi \in G_\pi$.

(e) If $m \le n$, then $a_m \cap d_{n,k_n,0} = 0$, so $\phi_n a = \phi_m a = \phi a$ for every $a \subseteq a_m$. Thus $\phi_n a = \phi a$ for every $a \subseteq \sup_{m\le n} a_m = 1 \setminus d_{n,k_n,0}$. In particular, $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$, $1 \le i < 2^m$ (counting $d_{n,m,2^m}$ as d_{nm0}, as usual); so that in fact $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$, $i < 2^m$.

For each n, we have $d_{nmi} = d'_{nmi} = d_{n+1,m,i}$ whenever $m \le k_n$ and $i < 2^m$. We therefore have a family $\langle d^*_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$ defined by saying that $d^*_{mi} = d_{nmi}$ whenever $n \in \mathbb{N}$, $m \le k_n$ and $i < 2^m$. Now, for any $m \in \mathbb{N}$, there is a $k_n > m$, so that $\langle d^*_{mi} \rangle_{i<2^m} = \langle d_{nmi} \rangle_{i<2^m}$ is a partition of unity; and

$$d^*_{mi} = d_{nmi} = d_{n,m+1,i} \cup d_{n,m+1,i+2^m} = d^*_{m+1,i} \cup d^*_{m+1,i+2^m}$$

for each $i < 2^m$. Moreover,

$$\phi d^*_{m,i} = \phi_n d_{nmi} = d_{n,m,i+1} = d^*_{m,i+1}$$

at least for $1 \le i < 2^m$ (counting $d^*_{m,2^m}$ as $d^*_{m,0}$, as usual), so that in fact $\phi d^*_{mi} = d^*_{m,i+1}$ for every $i < 2^m$. Thus $\langle d^*_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$ is a dyadic cycle system for ϕ, and ϕ is a weakly von Neumann transformation.

Writing $\mathfrak{B}$ for the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d^*_{mi} : m \in \mathbb{N},\ i < 2^m\}$, then

$$\begin{aligned}\mathfrak{C} \cup \{d'_{nmi} : m \le k'_n,\ i < 2^m\} &= \mathfrak{C} \cup \{d_{n+1,m,i} : m \le k'_n,\ i < 2^m\} \\ &= \mathfrak{C} \cup \{d^*_{mi} : m \le k'_n,\ i < 2^m\} \subseteq \mathfrak{B}\end{aligned}$$

for any $n\in\mathbb{N}$. So $b'_n\in\mathfrak{B}$ for every n. If $b\in B$ and $\epsilon>0$, there is an $n\in\mathbb{N}$ such that $2^{-n}\le\epsilon$ and $b_n=b$, so that $\bar\mu(b\,\triangle\,b'_n)\le\epsilon$; as every b'_n belongs to $\mathfrak{B}$, and $\mathfrak{B}$ is closed, $b\in\mathfrak{B}$; as b is arbitrary, and B τ-generates $\mathfrak{A}$, $\mathfrak{B}=\mathfrak{A}$. Thus ϕ is a von Neumann transformation.

(f) If $n\in\mathbb{N}$ and $d\cap e^*_n=0$, then $\phi^jd=\phi^j_nd$ and $\phi^{-j}d=\phi_n^{-j}d$ whenever $0\le j\le r_n$. **P** Induce on j. For $j=0$ the result is trivial. For the inductive step to $j+1\le r_n$, note that if $d'\cap d_{n,k_n,1}=0$ then $\phi_n^{-1}d'\cap d_{n,k_n,0}=0$, so

$$\phi^{-1}d'=\phi^{-1}\phi_n(\phi_n^{-1}d')=\phi^{-1}\phi(\phi_n^{-1}d')=\phi_n^{-1}d'.$$

Now we have

$$\phi^{j+1}d=\phi(\phi^j_nd)=\phi_n(\phi^j_nd)=\phi_n^{j+1}d$$

because

$$\phi^j_nd\cap d_{n,k_n,0}=\phi^j_n(d\cap\phi_n^{-j}d_{n,k_n,0})=0,$$

while

$$\phi^{-j-1}d=\phi^{-1}(\phi_n^{-j}d)=\phi_n^{-1}(\phi_n^{-j}d)=\phi_n^{-j-1}d$$

because

$$\phi_n^{-j}d\cap d_{n,k_n,1}=\phi_n^{-j}(d\cap\phi_n^{j+1}d_{n,k_n,0})=0.\ \mathbf{Q}$$

Thus $\phi^jd=\phi^j_nd$ whenever $|j|\le r_n$.

(g) Finally, $G_\phi=G_\pi$. **P** I remarked in (d) that $\phi\in G_\pi$, so that $G_\phi\subseteq G_\pi$. To see that $\pi\in G_\phi$, take any non-zero $a\in\mathfrak{A}$. Because $\bar\mu(e^*_n\cup\tilde e_n)\le 2^{-n+1}$ for each n, there is an n such that $a'=a\setminus(e^*_n\cup\tilde e_n)\ne0$. Now there is some $j\in\mathbb{Z}$ such that $a''=a'\cap e_{nj}\ne0$; since $a'\cap\tilde e_n=0$, $|j|\le r_n$. If $d\subseteq a''$, then $\pi d=\phi^j_nd$, by the definition of e_{nj}. But also $\phi^j_nd=\phi^jd$, by (f), because $d\cap e^*_n=0$. So $\pi d=\phi^jd$ for every $d\subseteq a''$. As a is arbitrary, $\pi\in G_\phi$ and $G_\pi\subseteq G_\phi$. **Q**

This completes the proof.

387O Theorem Let $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$ be totally finite measure algebras of countable Maharam type, and $\pi_1:\mathfrak{A}_1\to\mathfrak{A}_1$, $\pi_2:\mathfrak{A}_2\to\mathfrak{A}_2$ measure-preserving automorphisms. For each i, let $\mathfrak{C}_i$ be the closed subalgebra $\{c:c\in\mathfrak{A}_i,\ \pi_ic=c\}$ and G_{π_i} the full subgroup of $\operatorname{Aut}\mathfrak{A}_i$ generated by π_i. If $(\mathfrak{A}_1,\bar\mu_1,\mathfrak{C}_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\mathfrak{C}_2)$ are isomorphic, so are $(\mathfrak{A}_1,\bar\mu_1,G_{\pi_1})$ and $(\mathfrak{A}_2,\bar\mu_2,G_{\pi_2})$.

proof (a) It is enough to consider the case in which $(\mathfrak{A}_1,\bar\mu_1,\mathfrak{C}_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\mathfrak{C}_2)$ are actually equal; I therefore delete the subscripts and speak of a structure $(\mathfrak{A},\bar\mu,\mathfrak{C})$, with two automorphisms π_1, π_2 of $\mathfrak{A}$ both with fixed-point subalgebra $\mathfrak{C}$.

(b) Suppose first that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$, that is, that both the π_i are aperiodic (385E). In this case, 387N tells us that there are von Neumann transformations ϕ_1 and ϕ_2 of $\mathfrak{A}$ such that $G_{\pi_1}=G_{\phi_1}$ and $G_{\pi_2}=G_{\phi_2}$. But $(\mathfrak{A},\bar\mu,\phi_1)$ and $(\mathfrak{A},\bar\mu,\phi_2)$ are isomorphic. **P** Let $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ and $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ be dyadic cycle systems for ϕ_1, ϕ_2 respectively such that $\mathfrak{C}\cup\{d_{mi}:m\in\mathbb{N},\ i<2^m\}$ and $\mathfrak{C}\cup\{d'_{mi}:m\in\mathbb{N},\ i<2^m\}$ both τ-generate $\mathfrak{A}$.

Writing $\mathfrak{B}_1$, $\mathfrak{B}_2$ for the subalgebras of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{d_{mi}:m\in\mathbb{N},\ i<2^m\}$ and $\mathfrak{C}\cup\{d'_{mi}:m\in\mathbb{N},\ i<2^m\}$ respectively, it is easy to see that these algebras are isomorphic: we just set $\theta_0c=c$ for $c\in\mathfrak{C}$, $\theta_0d_{mi}=d'_{mi}$ for $i<2^m$ to obtain a measure-preserving isomorphism $\theta_0:\mathfrak{B}_1\to\mathfrak{B}_2$. Because these are topologically dense subalgebras of $\mathfrak{A}$, there is a unique extension of θ_0 to a measure-preserving automorphism $\theta:\mathfrak{A}\to\mathfrak{A}$ (324O). Next, we see that

$$\theta\phi_1\theta^{-1}c=c=\phi_2c\text{ for every }c\in\mathfrak{C},$$

$$\theta\phi_1\theta^{-1}d'_{mi}=\theta\phi_1d_{mi}=\theta d_{m,i+1}=d'_{m,i+1}=\phi_2d'_{m,i+1}$$

for $m\in\mathbb{N}$, $i<2^m$ (as usual, taking $d_{m,2^m}$ to be d_{m0} and $d'_{m,2^m}$ to be d'_{m0}). But this means that $\theta\phi_1\theta^{-1}b=\phi_2b$ for every $b\in\mathfrak{B}_2$, so (again because $\mathfrak{B}_2$ is dense in $\mathfrak{A}$) $\theta\phi_1\theta^{-1}=\phi_2$. Thus θ is an isomorphism between $(\mathfrak{A},\bar\mu,\phi_1)$ and $(\mathfrak{A},\bar\mu,\phi_2)$. **Q**

Of course θ is now also an isomorphism between $(\mathfrak{A},\bar\mu,G_{\phi_1})=(\mathfrak{A},\bar\mu,G_{\pi_1})$ and $(\mathfrak{A},\bar\mu,G_{\phi_2})=(\mathfrak{A},\bar\mu,G_{\pi_2})$.

(c) Next, consider the case in which π_1 is periodic, with period n, for some $n \geq 1$. In this case $\pi_2 \in G_{\pi_1}$. **P** Let $(d_0, \dots, d_{n-1})$ be a partition of unity in $\mathfrak{A}$ such that $\pi_1 d_i = d_{i+1}$ for $i < n-1$ and $\pi_1 d_{n-1} = d_0$ (385B). If $d \subseteq d_j$, then $c = \sup_{i<n} \pi_1^i d \in \mathfrak{C}$ and $d = d_j \cap c$; so any member of $\mathfrak{A}$ is of the form $\sup_{j<n} d_j \cap c_j$ for some family $c_0, \dots, c_{n-1}$ in $\mathfrak{C}$.

If $a \in \mathfrak{A} \setminus \{0\}$, take $i, j < n$ such that $a' = a \cap d_i \cap \pi_2^{-1} d_j \neq 0$. Then any $d \subseteq a'$ is of the form

$$d_i \cap c_1 = \pi_2^{-1}(d_j \cap c_2) = c_2 \cap \pi_2^{-1} d_j$$

for some $c_1, c_2 \in \mathfrak{C}$; setting $c = c_1 \cap c_2$, we have

$$d = d_i \cap c, \quad \pi_2 d = d_j \cap c = \pi_1^{j-i} d.$$

As a is arbitrary, this shows that $\pi_2 \in G_{\pi_1}$. **Q**

Now $\sup_{n\in\mathbb{Z}} \pi_2^n d_0$ belongs to $\mathfrak{C}$ and includes d_0, so must be 1. Finally, if we look at the two induced automorphisms $(\pi_1)_{d_0}$, $(\pi_2)_{d_0}$ on $\mathfrak{A}_{d_0}$, they must both be the identity, by 387Eb, because every element of $\mathfrak{A}_{d_0}$ is of the form $d_0 \cap c$ for some $c \in \mathfrak{C}$. So 387J tells us that $G_{\pi_1} = G_{\pi_2}$.

(d) For the general case, we see from 385C that there is a partition of unity $\langle c_i \rangle_{1\leq i\leq\omega}$ in $\mathfrak{C}$ such that $\pi_1 \restriction \mathfrak{A}_{c_\omega}$ is aperiodic and if i is finite and $c_i \neq 0$ then $\pi_1 \restriction \mathfrak{A}_{c_i}$ is periodic with period i. For each i, let H_i be $\{\phi \restriction \mathfrak{A}_{c_i} : \phi \in G_{\pi_1}\}$; then H_i is a full subgroup of $\operatorname{Aut} \mathfrak{A}_{c_i}$, and

$$G_{\pi_1} = \{\phi : \phi \in \operatorname{Aut} \mathfrak{A}, \ \phi \restriction \mathfrak{A}_{c_i} \in H_i \text{ whenever } 1 \leq i \leq \omega\}.$$

Similarly, writing $H'_i = \{\phi \restriction \mathfrak{A}_{c_i} : \phi \in G_{\pi_2}\}$,

$$G_{\pi_2} = \{\phi : \phi \in \operatorname{Aut} \mathfrak{A}, \ \phi \restriction \mathfrak{A}_{c_i} \in H'_i \text{ whenever } 1 \leq i \leq \omega\}.$$

Note also that H_i, H'_i are the full subgroups of $\operatorname{Aut} \mathfrak{A}_{c_i}$ generated by $\pi_1 \restriction \mathfrak{A}_{c_i}$, $\pi_2 \restriction \mathfrak{A}_{c_i}$ respectively. By (b) and (c), $H_i = H'_i$ for finite i, while there is a measure-preserving automorphism $\theta : \mathfrak{A}_{c_\omega} \to \mathfrak{A}_{c_\omega}$ such that $\theta H_\omega \theta^{-1} = H'_\omega$. Now we can define a measure-preserving automorphism $\theta_1 : \mathfrak{A} \to \mathfrak{A}$ by setting $\theta_1 a = \theta a$ if $a \subseteq c_\omega$, $\theta_1 a = a$ if $a \cap c_\omega = 0$, and we shall have $\theta_1 G_{\pi_1} \theta_1^{-1} = G_{\pi_2}$. Thus $(\mathfrak{A}, \bar{\mu}, G_{\pi_1})$ and $(\mathfrak{A}, \bar{\mu}, G_{\pi_2})$ are isomorphic, as claimed.

387X Basic exercises >(a) Let (X, Σ, μ) be a localizable measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that π and ϕ are automorphisms of $\mathfrak{A}$, and that π is represented (in the sense of 343A) by a measure space automorphism $f : X \to X$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut} \mathfrak{A}$ generated by π; (ii) there is a function $g : X \to X$, representing ϕ, such that $g(x) \in \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$; (iii) there is a measure space automorphism $g : X \to X$, representing ϕ, such that $\{g^n(x) : n \in \mathbb{Z}\} \subseteq \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$.

(b) Let (X, Σ, μ) be a totally finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$; let $f : X \to X$ be a measure space isomorphism, and $\pi : \mathfrak{A} \to \mathfrak{A}$ the corresponding measure-preserving automorphism. Take $E \in \Sigma$ and $b = E^\bullet \in \mathfrak{A}$; let $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$ be the corresponding induced automorphism, as in 387E. (i) Set $E_1 = \{x : x \in E, \exists\, n \geq 1, \ f^n(x) \in E\}$; show that $\mu(E \setminus E_1) = 0$, so that $E_1^\bullet = b$. (ii) For $x \in E_1$ set $n_x = \min\{n : n \geq 1, \ f^{n_x}(x) \in E\}$, $g_E(x) = f^{n_x}(x)$. Show that $g_E : E_1 \to E$ induces $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$. (iii) Show that there is a measurable set $G \subseteq E_1$ such that $E_1 \setminus G$ is negligible and $g_E \restriction G = g_G$ is an automorphism for the induced measure on G. (*Hint*: try $G = E \cap \bigcap_{n\in\mathbb{N}} \bigcup_{m,r\geq n} f^m[E] \cap f^{-r}[E]$.) (iv) Show that, in (iii), $\{g_G^n(x) : n \in \mathbb{Z}\} = G \cap \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in G$. (v) Find an expression of 387Ec in this language.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. Let us say that a **pseudo-cycle** for π is a partition of unity $\langle a_i \rangle_{i<n}$, where $n \geq 1$, such that $\pi a_i = a_{i+1}$ for $i < n-1$ (so that $\pi a_{n-1} = a_0$). (i) Show that if we have pseudo-cycles $\langle a_i \rangle_{i<n}$ and $\langle b_j \rangle_{j<m}$, where m is a multiple of n, then we have a pseudo-cycle $\langle c_j \rangle_{j<m}$ with $c_0 \subseteq a_0$, so that $a_i = \sup\{c_j : j < m, \ j \equiv i \bmod n\}$ for every $i < n$. (ii) Show that π is weakly von Neumann iff it has a pseudo-cycle of length 2^n for any $n \in \mathbb{N}$.

(d) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be probability algebras, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving von Neumann automorphisms. Show that there is a measure-preserving Boolean isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\pi_2 = \theta \pi_2 \theta^{-1}$.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type, and $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ the group of measure-preserving Boolean automorphisms of $\mathfrak{A}$. Let $\pi \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ be a von Neumann transformation. (i) Show that for any ultrafilter $\mathcal{F}$ on $\mathbb{N}$ there is a $\phi_{\mathcal{F}} \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ defined by the formula $\phi_{\mathcal{F}}(a) = \lim_{n \to \mathcal{F}} \pi^n a$ for every $a \in \mathfrak{A}$, the limit being taken in the measure-algebra topology. (ii) Show that $\{\phi_{\mathcal{F}} : \mathcal{F}$ is an ultrafilter on $\mathbb{N}\}$ is a subgroup of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ homeomorphic to $\mathbb{Z}_2^{\mathbb{N}}$. (*Hint*: 387H.)

(f) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a weakly von Neumann automorphism. Show that π^n is a weakly von Neumann automorphism for every $n \in \mathbb{Z} \setminus \{0\}$. (*Hint*: consider $n = 2$, $n = -1$, odd $n \geq 3$ separately. The formula of 387H may be useful.)

(g) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a von Neumann automorphism. (i) Show that π is ergodic (in the sense that its fixed-point algebra is trivial) but π^2 is not ergodic. (ii) Show that π^2 is relatively von Neumann. (iii) Show that π^n is von Neumann for every odd $n \in \mathbb{Z} \setminus \{0\}$.

387Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Let us say that G is **full** if whenever $\phi : \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, and there is a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that for every $i \in I$ there is a $\pi_i \in G$ such that $\phi a = \pi_i a$ for every $a \subseteq a_i$, then $\phi \in G$. Show that if ϕ and π are order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself, then the following are equiveridical: (i) ϕ belongs to the full semigroup generated by ϕ; (ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{N}$ such that $\phi d = \pi^n d$ for every $d \subseteq b$; (iii) there is a partition of unity $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$ such that $\phi a = \pi^n a$ for every $n \in \mathbb{N}$, $a \subseteq a_n$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and Z its Stone space. Let π, $\phi : \mathfrak{A} \to \mathfrak{A}$ be Boolean automorphisms, and f, $g : Z \to Z$ the corresponding homeomorphisms. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut} \mathfrak{A}$ generated by π; (ii) $\{z : g(z) \notin \{f^n(z) : n \in \mathbb{Z}\}\}$ is nowhere dense in Z; (iii) $\{z : \{g^n(z) : n \in \mathbb{Z}\} \not\subseteq \{f^n(z) : n \in \mathbb{Z}\}\}$ is nowhere dense in Z.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ such that the quotient algebra $\mathfrak{A} = \Sigma/\mathcal{I}$ is Dedekind complete and there is a countable subset of Σ separating the points of X. Suppose that f and g are automorphisms of the structure $(X, \Sigma, \mathcal{I})$ inducing π, $\phi \in \operatorname{Aut} \mathfrak{A}$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut} \mathfrak{A}$ generated by π; (ii) $\{x : x \in X,\ f(x) \notin \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$; (iii) $\{x : x \in X,\ \{f^n(x) : n \in \mathbb{Z}\} \not\subseteq \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a relatively von Neumann transformation. Show that π is aperiodic and has zero entropy.

(e) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be atomless probability algebras, and $(\mathfrak{A}, \bar{\mu})$ their probability algebra free product. Let $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$ be a measure-preserving von Neumann automorphism and $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ a mixing measure-preserving automorphism. Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the measure-preserving automorphism such that $\pi(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$. Show that π is an ergodic weakly von Neumann automorphism which is not a relatively von Neumann automorphism.

(f) Let μ be Lebesgue measure on $[0,1]^2$, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra; let $\mathfrak{C}$ be the closed subalgebra of elements expressible as $(E \times [0,1])^\bullet$, where $E \subseteq [0,1]$ is measurable. Suppose that $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism such that $\mathfrak{C} = \{c : \pi c = c\}$. Show that there is a family $\langle f_x \rangle_{x \in [0,1]}$ of ergodic measure space automorphisms of $[0,1]$ such that $(x, y) \mapsto (x, f_x(y))$ is a measure space automorphism of $[0,1]^2$ representing π.

387 Notes and comments Dye's theorem (Dye 59) is actually Theorem 387O in the case in which π_1, π_2 are ergodic, that is, in which $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both trivial. I take the trouble to give the generalized form here (a simplified version of that in Krieger 76) because it seems a natural target, once we have a classification of the relevant structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ (333R). The essential mathematical ideas are the same in both cases. You can find the special case worked out in Hajian Ito & Kakutani 75, from which I have taken the argument used here; and you may find it useful to go through the version above, to check what kind of simplifications arise if each $\mathfrak{C}$ is taken to be $\{0,1\}$. Essentially the difference will be that every 'aperiodic' turns into 'ergodic' (with an occasional 'atomless' thrown in) and '331B' turns into '331C'. As far as I know, there is no simplification available in the structure of the argument; of course the details become a bit easier, but with the possible exception of 387L-387M I think there is little difference.

Of course modifying a general argument to give a simpler proof of a special case is a standard exercise in this kind of mathematics. What is much more interesting is the reverse process. What kinds of theorem about ergodic automorphisms will in fact be true of all automorphisms? A variety of very powerful approaches to such questions have been developed in the last half-century, and I hope to describe some of the ideas in Volumes 4 and 5. The methods used in this section are relatively straightforward and do not require any deep theoretical underpinning beyond Maharam's lemma 331B. But an alternative approach can be found using 387Yf: in effect (at least for the Lebesgue measure algebra) any measure-preserving automorphism can be disintegrated into ergodic measure space automorphisms (the fibre maps f_x of 387Yf). It is sometimes possible to guess which theorems about ergodic transformations are 'uniformisable' in the sense that they can be applied to such a family $\langle f_x\rangle_{x\in[0,1]}$, in a systematic way, to provide a structure which can be interpreted on the product measure. The details tend to be complex, which is one of the reasons why I do not attempt to work through them here; but such disintegrations can be a most valuable aid to intuition.

Dye's theorem itself is less significant than many results which I have omitted; I include it partly because it is relatively easy and partly because some of the ideas on the way are very important indeed. In particular, the concept of 'recurrence' is vital. I have already offered a couple of relevant exercises (382Yc-382Yd). The fundamental lemmas introduced in this section are 387E and 387F.

In this section I use von Neumann transformations as an auxiliary tool: the point is, first, that two von Neumann transformations are isomorphic – that is, the von Neumann transformations on a given totally finite measure algebra $(\mathfrak{A},\bar\mu)$ (necessarily isomorphic to the Lebesgue measure algebra, since we must have $\mathfrak{A}$ atomless and $\tau(\mathfrak{A})=\omega$) form a conjugacy class in the group $\mathrm{Aut}_{\bar\mu}\,\mathfrak{A}$ of measure-preserving automorphisms; and next, that for any ergodic measure-preserving automorphism π (on an atomless totally finite algebra of countable Maharam type) there is a von Neumann transformation ϕ such that $G_\pi=G_\phi$ (387N). But I think they are remarkable in themselves. A (weakly) von Neumann transformation has a 'pseudo-cycle' (387Xc) for every power of 2. For some purposes, existence is all we need to know; but in the arguments of 387K-387N we need to keep track of named pseudo-cycles in what I call 'dyadic cycle systems' (387G).

In this volume I have systematically preferred arguments which deal directly with measure algebras, rather than with measure spaces. I believe that such arguments can have a simplicity and clarity which repays the extra effort of dealing with more abstract structures. But undoubtedly it is necessary, if you are to have any hope of going farther in the subject, to develop methods of transferring intuitions and theorems between the two contexts. I offer 387Xb as an example. The description there of 'induced automorphism' requires a certain amount of manoeuvering around negligible sets, but in 387Xb(iv) gives a valuably graphic description. In the same way, 387C, 387Xa and 387Yb provide alternative ways of looking at full subgroups.

There are contexts in which it is useful to know whether an element of the full subgroup generated by π actually belongs to the full semigroup generated by π (387Ya); for instance, this happens in 387F.

Chapter 39

Measurable algebras

In the final chapter of this volume, I present results connected with the following question: which algebras can appear as the underlying Boolean algebras of measure algebras? Put in this form, there is a trivial answer (391A). The proper question is rather: which algebras can appear as the underlying Boolean algebras of semi-finite measure algebras? This is easily reducible to the question: which algebras can appear as the underlying Boolean algebras of probability algebras? Now in one sense Maharam's theorem (§332) gives us the answer exactly: they are the countable simple products of the measure algebras of $\{0,1\}^\kappa$ for cardinals κ. But if we approach from another direction, things are more interesting. Probability algebras share a very large number of very special properties. Can we find a selection of these properties which will be sufficient to force an abstract Boolean algebra to be a probability algebra when endowed with a suitable functional?

No fully satisfying answer to this question is known. But in exploring the possibilities we encounter some interesting and important ideas. In §391 I discuss algebras which have strictly positive additive real-valued functionals; for such algebras, weak (σ,∞)-distributivity is necessary and sufficient for the existence of a measure; so we are led to look for conditions sufficient to ensure that there is a strictly positive additive functional. A slightly different approach lies through the concept of 'submeasure'. Submeasures arise naturally in the theories of topological Boolean algebras, topological Riesz spaces and vector measures (see the second half of §393), and on any given algebra there is a strictly positive 'uniformly exhaustive' submeasure iff there is a strictly positive additive functional; this is the Kalton-Roberts theorem (392F). It is unknown whether the word 'uniformly' can be dropped; this is one of the forms of the Control Measure Problem, which I investigate at length in §393. In §394, I look at a characterization in terms of the special properties which the automorphism group of a measure algebra must have (Kawada's theorem, 394Q). §395 complements the previous section by looking briefly at the subgroups of an automorphism group $\operatorname{Aut}\mathfrak{A}$ which can appear as groups of measure-preserving automorphisms.

391 Kelley's theorem

In this section I introduce the notion of 'measurable algebra' (391B), which will be the subject of the whole chapter once the trivial construction of 391A has been dealt with. I show that for weakly (σ,∞)-distributive algebras countable additivity can be left to look after itself, and all we need to find is a strictly positive finitely additive functional (391D). I give Kelley's criterion for the existence of such a functional (391H-391J), and a version of Gaifman's example of a ccc Boolean algebra without such a functional (391N).

391A Proposition Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra. Then there is a function $\bar\mu:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar\mu)$ is a measure algebra.

proof Set $\bar\mu 0=0$, $\bar\mu a=\infty$ for $a\in\mathfrak{A}\setminus\{0\}$.

391B Definition I will call a Boolean algebra $\mathfrak{A}$ **measurable** if there is a functional $\bar\mu:\mathfrak{A}\to[0,\infty[$ such that $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra.

In this case, if $\bar\mu\ne 0$, then it has a scalar multiple with total mass 1. So a Boolean algebra $\mathfrak{A}$ is measurable iff either it is $\{0\}$ or there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

391C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) The following are equiveridical: (i) there is a functional $\bar\mu:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra; (ii) $\mathfrak{A}$ is Dedekind σ-complete and $\{a:a\in\mathfrak{A},\ \mathfrak{A}_a\text{ is measurable}\}$ is order-dense in $\mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal generated by a.

(b) The following are equiveridical: (i) there is a functional $\bar\mu:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra; (ii) $\mathfrak{A}$ is Dedekind complete and $\{a:a\in\mathfrak{A},\ \mathfrak{A}_a\text{ is measurable}\}$ is order-dense in $\mathfrak{A}$.

proof (a) (i)$\Rightarrow$(ii): if $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, then $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$ is order-dense in $\mathfrak{A}$ and $\mathfrak{A}_a$ is measurable for every $a\in\mathfrak{A}^f$.

(ii)⇒(i): setting $D = \{a : a \in \mathfrak{A},\ \mathfrak{A}_a$ is measurable$\}$, D is order-dense, so there is a partition of unity $C \subseteq D$ (313K). For each $c \in C$, choose $\bar\mu_c$ such that $(\mathfrak{A}_c, \bar\mu_c)$ is a totally finite measure algebra. Set $\bar\mu a = \sum_{c\in C} \bar\mu_c(a \cap c)$ for every $a \in \mathfrak{A}$; then it is easy to check that $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra.

(b) Follows immediately.

391D Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:
(i) $\mathfrak{A}$ is measurable;
(ii) $\mathfrak{A}$ is Dedekind σ-complete and weakly (σ,∞)-distributive, and there is a strictly positive finitely additive functional $\nu : \mathfrak{A} \to [0,\infty[$.

Remark An additive functional ν on a Boolean algebra $\mathfrak{A}$ is **strictly positive** if $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$.

proof (i)⇒(ii) Put the definition together with 322C(b)-(c) (for Dedekind completeness) and 322F (for weak (σ,∞)-distributivity).

(ii)⇒(i) Given that (ii) is satisfied, let $\bar\mu$ be the completely additive part of ν, as defined in 326Yq: that is,

$$\bar\mu a = \inf\{\sup_{d\in D} \nu d : D \text{ is a non-empty upwards-directed set with supremum } a\}$$

for every $a \in \mathfrak{A}$. Then $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra.

P$(\boldsymbol{\alpha})$ Of course $0 \le \bar\mu a \le \nu 1$ for every $a \in \mathfrak{A}$, so $\bar\mu$ is a function from $\mathfrak{A}$ to $[0,\infty[$.

$(\boldsymbol{\beta})$ Suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a. Let $\epsilon > 0$. For each n, choose an upwards-directed set D_n, with supremum a_n, such that $\sup_{d\in D_n} \nu d \le \bar\mu a_n + 2^{-n}\epsilon$; choose an upwards-directed set D, with supremum a, such that $\sup_{d\in D} \nu d \le \bar\mu a + \epsilon$.
Set

$$D^* = \{\sup_{i\le n} d_i : n \in \mathbb{N},\ d_i \in D_i \text{ for every } i \le n\}.$$

Then D^* is upwards-directed, $d \subseteq a$ for every $d \in D^*$, and any upper bound for D^* must be an upper bound for D_n for every n, so $\sup D^* = a$. This means that

$$\bar\mu a \le \sup_{d\in D^*} \nu d = \sup\{\nu(\sup_{i\le n} d_i) : d_i \in D_i \text{ for } i \le n\}$$
$$= \sup\{\sum_{i=0}^{n} \nu d_i : d_i \in D_i \text{ for } i \le n\}$$

(because $d_i \cap d_j \subseteq a_i \cap a_j = 0$ if $i \ne j$, $d_i \in D_i$, $d_j \in D_j$)

$$= \sum_{i=0}^{\infty} \sup_{d\in D_i} \nu d \le \sum_{i=0}^{\infty} \bar\mu a_i + 2^{-i}\epsilon = 2\epsilon + \sum_{i=0}^{\infty} \bar\mu a_i.$$

Next, set $D'_n = \{a_n \cap d : d \in D\}$ for each n. Then D'_n is upwards-directed and has supremum $a_n \cap a = a_n$ (313Ba), so

$$\sum_{i=0}^{\infty} \bar\mu a_i \le \sum_{i=0}^{\infty} \sup_{d\in D'_i} \nu d = \sup\{\sum_{i=0}^{n} \nu(d_i \cap a_i) : n \in \mathbb{N},\ d_i \in D \text{ for } i \le n\}.$$

But, given $n \in \mathbb{N}$ and $d_i \in D$ for $i \le n$, there is a $d \in D$ such that $\sup_{i\le n} d_i \subseteq d$ (because D is upwards-directed), so that (because $a_0, \dots, a_n$ are disjoint)

$$\sum_{i=0}^{n} \nu(d_i \cap a_i) \le \nu d \le \bar\mu a + \epsilon.$$

Accordingly $\sum_{i=0}^{\infty} \bar\mu a_i \le \bar\mu a + \epsilon$.
As ϵ is arbitrary, $\sum_{i=0}^{\infty} \bar\mu a_i = \bar\mu a$; as $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\bar\mu$ is countably additive.

$(\boldsymbol{\gamma})$ Now suppose that $a \in \mathfrak{A}$ and $\bar\mu a = 0$. Then for every $n \in \mathbb{N}$ there is an upwards-directed set D_n, with supremum a, such that $\nu d \le 2^{-n}$ for every $d \in D_n$. Set $B_n = \{a \setminus d : d \in D_n\}$, so that each B_n is downwards-directed and has infimum 0 (313A). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there is a set $B \subseteq \mathfrak{A}$, also with infimum 0, such that for every $b \in B$, $n \in \mathbb{N}$ there is a $b' \in B_n$ with $b' \subseteq b$. If $B = \emptyset$ then $\mathfrak{A} = \{0\}$ and surely $a = 0$. Otherwise, set $D = \{a \setminus b : b \in B\}$. Then D is upwards-directed and has supremum a. But if $d \in D$, then for every $n \in \mathbb{N}$ there is a $b' \in B_n$ such that $b' \subseteq a \setminus d$, that is, $d \subseteq a \setminus b' \in D_n$, and $\nu d \le 2^{-n}$. Accordingly $\nu d = 0$; but we are supposing that ν is strictly positive, so this means that $D = \{0\}$ and a must be 0.

This shows that $\bar{\mu}$ is strictly positive.

(δ) Thus $\bar{\mu}$ is a strictly positive countably additive functional; as we are also assuming that $\mathfrak{A}$ is Dedekind σ-complete, $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra. **Q**

Accordingly $\mathfrak{A}$ is measurable.

391E Thus we are led naturally to the question: which Boolean algebras carry strictly positive *finitely* additive functionals? The Hahn-Banach theorem, suitably applied, gives some sort of answer to this question. For the sake of applications later on, I give two general results on the existence of additive functionals related to given functionals.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\phi : \mathfrak{A} \to [0,1]$ a functional. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \le \phi a$ for every $a \in \mathfrak{A}$;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a finite indexed family in $\mathfrak{A}$, and $\sum_{i \in I} \chi a_i \ge m\chi 1$ in $S = S(\mathfrak{A})$ (definition: 361A), where $m \in \mathbb{N}$, then $\sum_{i \in I} \phi a_i \ge m$.

proof (a)(i)$\Rightarrow$(ii) If $\nu : \mathfrak{A} \to [0,1]$ is a finitely additive functional such that $\nu 1 = 1$ and $\nu a \le \phi a$ for every $a \in \mathfrak{A}$, let $h : S \to \mathbb{R}$ be the positive linear functional corresponding to ν (361G). Now if $\langle a_i \rangle_{i \in I}$ is a finite family in $\mathfrak{A}$ and $\sum_{i \in I} \chi a_i \ge m\chi 1$, then

$$\begin{aligned}\sum_{i \in I} \phi a_i \le \sum_{i \in I} \nu a_i &= \sum_{i \in I} h(\chi a_i) \\ &= h(\sum_{i \in I} \chi a_i) \ge h(m\chi 1) = m.\end{aligned}$$

As $\langle a_i \rangle_{i \in I}$ is arbitrary, (ii) is true.

(b)(ii)$\Rightarrow$(i) Now suppose that ϕ satisfies (ii). For $u \in S$, set

$$p(u) = \inf\{\sum_{i=0}^n \alpha_i \phi a_i : a_0, \dots, a_n \in \mathfrak{A},\ \alpha_0, \dots, \alpha_n \ge 0,\ \sum_{i=0}^n \alpha_i \chi a_i \ge u\}.$$

Then it is easy to check that $p(u+v) \le p(u) + p(v)$ for all u, $v \in S$, and that $p(\alpha u) = \alpha p(u)$ for all $u \in S$, $\alpha \ge 0$. Also $p(\chi 1) \ge 1$. **P?** If not, there are $a_0, \dots, a_n \in \mathfrak{A}$ and $\alpha_0, \dots, \alpha_n \ge 0$ such that $\chi 1 \le \sum_{i=0}^n \alpha_i \chi a_i$ but $\sum_{i=0}^n \alpha_i \phi a_i < 1$. Increasing each α_i slightly if necessary, we may suppose that every α_i is rational; let $m \ge 1$ and $k_0, \dots, k_n \in \mathbb{N}$ be such that $\alpha_i = k_i/m$ for each $i \le n$.

Set $K = \{(i,j) : 0 \le i \le n,\ 1 \le j \le k_i\}$, and for $(i,j) \in K$ set $a_{ij} = a_i$. Then

$$\sum_{(i,j) \in K} \chi a_{ij} = \sum_{i=0}^n k_i \chi a_i = m \sum_{i=0}^n \alpha_i \chi a_i \ge m\chi 1,$$

but

$$\sum_{(i,j) \in K} \phi a_{ij} = \sum_{i=0}^n k_i \phi a_i = m \sum_{i=0}^n \alpha_i \phi a_i < m,$$

which is supposed to be impossible. **XQ**

By the Hahn-Banach theorem, in the form 3A5Aa, there is a linear functional $h : S \to \mathbb{R}$ such that $h(\chi 1) = p(\chi 1) \ge 1$ and $h(u) \le p(u)$ for every $u \in S$. In particular, $h(\chi a) \le \phi b$ whenever $a \subseteq b \in \mathfrak{A}$. Set $\nu a = h(\chi a)$ for $a \in \mathfrak{A}$; then $\nu : \mathfrak{A} \to [0, \infty[$ is an additive functional, $\nu 1 \ge 1$ and $\nu a \le \phi b$ whenever $a \subseteq b$ in $\mathfrak{A}$. We do not know whether ν is positive, but if we define ν^+ as in 362Ab, we shall have a non-negative additive functional such that

$$\nu^+ a = \sup_{b \subseteq a} \nu b \le \phi a$$

for every $a \in \mathfrak{A}$, and

$$1 \le \nu 1 \le \nu^+ 1 \le \phi 1 = 1,$$

so ν^+ witnesses that (i) is true.

391F Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\psi : A \to [0,1]$ a functional, where $A \subseteq \mathfrak{A}$. Then the following are equiveridical:

(i) there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in \mathfrak{A}$;

(ii) whenever $\langle a_i \rangle_{i\in I}$ is a finite indexed family in $\mathfrak{A}$, there is a set $J \subseteq I$ such that $\#(J) \ge \sum_{i\in I} \psi a_i$ and $\inf_{i\in J} a_i \ne 0$.

Remark In (ii) here, we may have to interpret the infimum of the empty set in $\mathfrak{A}$ as 1.

proof We apply 391E to the functional ϕ, where $\phi a = 1 - \psi(1 \setminus a)$ for $a \in \mathfrak{A}$.

(a) If $\nu : \mathfrak{A} \to [0,1]$ is a non-negative finitely additive functional such that $\nu 1 = 1$, then

$$\nu a \ge \psi a \iff 1 - \nu a \le 1 - \psi a \iff \nu(1 \setminus a) \le \phi(1 \setminus a).$$

So (i) here is true of ψ iff 391E(i) is true of ϕ.

(b) Suppose that (ii) here is true of ψ, and that $\langle a_i \rangle_{i\in I}$ is a finite family in $\mathfrak{A}$ such that $\sum_{i\in I} \chi a_i \ge m\chi 1$, while $\sum_{i\in I} \phi a_i = \beta$. Then

$$\textstyle\sum_{i\in I} \psi(1 \setminus a_i) = \sum_{i\in I}(1 - \phi a_i) = \#(I) - \beta,$$

so there is a set $J \subseteq I$ such that $\#(J) \ge \#(I) - \beta$ and $\inf_{i\in J}(1 \setminus a_i) = c \ne 0$. Now $c \cap a_i = 0$ for $i \in J$, so

$$m\chi c \le \textstyle\sum_{i\in I} \chi(a_i \cap c) = \sum_{i\in I\setminus J} \chi(a_i \cap c) \le \#(I \setminus J)\chi c$$

and $m \le \#(I) - \#(J) \le \beta$. As $\langle a_i \rangle_{i\in I}$ is arbitrary, 391E(ii) is true of ϕ.

(c) Suppose that 391E(ii) is true of ϕ, and that $\langle a_i \rangle_{i\in I}$ is a finite family in $\mathfrak{A}$. Set

$$\beta = \textstyle\sum_{i\in I} \phi(1 \setminus a_i) = \#(I) - \sum_{i\in I} \psi a_i$$

and let k be the least integer greater than β. Since $\sum_{i\in I} \phi(1 \setminus a_i) < k$, $\sum_{i\in I} \chi(1 \setminus a_i) \not\ge k\chi 1$, that is, $\sum_{i\in I} \chi a_i \not\le (\#(I) - k)\chi 1$. But this means that there must be some $J \subseteq I$ such that $\#(J) > \#(I) - k$ and $\inf_{i\in J} a_i \ne 0$. Now

$$\textstyle\sum_{i\in I} \psi a_i = \#(I) - \beta \le \#(I) - (k-1) \le \#(J).$$

As $\langle a_i \rangle_{i\in I}$ is arbitrary, (ii) here is true of ψ.

(d) Since we know that 391E(i) $\iff$ 391E(ii), we can conclude that (i) and (ii) here are equivalent.

391G Corollary Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu_0 : \mathfrak{B} \to \mathbb{R}$ a non-negative finitely additive functional. Then there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending ν_0.

proof (a) Suppose first that $\nu_0 1 = 1$. Set $\psi b = \nu_0 b$ for every $b \in \mathfrak{B}$. Then ψ must satisfy the condition (ii) of 391F when regarded as a functional defined on a subset of $\mathfrak{B}$; but this means that it satisfies the same condition when regarded as a functional defined on a subset of $\mathfrak{A}$. So there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu 1 = 1$ and $\nu b \ge \nu_0 b$ for every $b \in \mathfrak{B}$. In this case

$$\nu b = 1 - \nu(1 \setminus b) \le 1 - \nu_0(1 \setminus b) = \nu_0 b \le \nu b$$

for every $b \in \mathfrak{B}$, so ν extends ν_0.

(b) For the general case, if $\nu_0 1 = 0$ then ν_0 must be the zero functional on $\mathfrak{B}$, so we can take ν to be the zero functional on $\mathfrak{A}$; and if $\nu_0 1 = \gamma > 0$, we apply (a) to $\gamma^{-1}\nu_0$.

391H Definition Let $\mathfrak{A}$ be a Boolean algebra, and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. The **intersection number** of A is the largest $\delta \ge 0$ such that whenever $\langle a_i \rangle_{i\in I}$ is a finite family in A, with $I \ne \emptyset$, there is a $J \subseteq I$ such that $\#(J) \ge \delta\#(I)$ and $\inf_{i\in J} a_i \ne 0$.

Remarks (a) It is essential to note that in the definition above the $\langle a_i \rangle_{i\in I}$ are indexed families, with repetitions allowed; see 391Xh.

(b) I spoke perhaps rather glibly of 'the largest δ such that . . . '; you may prefer to write

$$\delta = \inf\{\sup_{\emptyset \ne J \subseteq \{0,\dots,n\}, \inf_{j\in J} a_j \ne 0} \frac{\#(J)}{n+1} : a_0, \dots, a_n \in A\}.$$

391I Proposition Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. Write C for the set of non-negative finitely additive functionals $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$. Then the intersection number of A is precisely $\max_{\nu \in C} \inf_{a\in A} \nu a$.

proof Write δ for the intersection number of A, and δ' for $\sup_{\nu\in C} \inf_{a\in A} \nu a$.

(a) For any $\gamma < \delta'$, we can find a $\nu \in C$ such that $\nu a \ge \gamma$ for every $a \in A$. So if we set $\psi a = \gamma$ for every $a \in A$, ψ satisfies condition (i) of 391F. But this means that if $\langle a_i \rangle_{i\in I}$ is any finite family in A, there must be a $J \subseteq I$ such that $\inf_{i\in J} a_i \ne 0$ and $\#(J) \ge \gamma\#(I)$. Accordingly $\gamma \le \delta$; as γ is arbitrary, $\delta' \le \delta$.

(b) Define $\psi : A \to [0,1]$ by setting $\psi a = \delta$ for every $a \in A$. If $\langle a_i \rangle_{i\in I}$ is a finite indexed family in A, there is a $J \subseteq I$ such that $\#(J) \ge \delta\#(I)$ and $\inf_{i\in J} a_i \ne 0$; but $\delta\#(I) = \sum_{i\in I} \psi a_i$, so this means that condition (ii) of 391F is satisfied. So there is a $\nu \in C$ such that $\nu a \ge \delta$ for every $a \in A$; and ν witnesses not only that $\delta' \ge \delta$, but that the supremum is a maximum.

391J Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:
(i) there is a strictly positive finitely additive functional on $\mathfrak{A}$;
(ii) either $\mathfrak{A} = \{0\}$ or $\mathfrak{A}\setminus\{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof (i)⇒(ii) If there is a strictly positive finitely additive functional ν on $\mathfrak{A}$, and $\mathfrak{A} \ne \{0\}$, set $A_n = \{a : \nu a \ge 2^{-n}\nu 1\}$ for every $n \in \mathbb{N}$; then (applying 391I to the functional $\frac{1}{\nu 1}\nu$) we see that every A_n has intersection number at least 2^{-n}, while $\mathfrak{A} \setminus \{0\} = \bigcup_{n\in\mathbb{N}} A_n$ because ν is strictly positive, so (ii) is satisfied.

(ii)⇒(i) If $\mathfrak{A} \setminus \{0\}$ is expressible as $\bigcup_{n\in\mathbb{N}} A_n$, where each A_n has intersection number $\delta_n > 0$, then for each n choose a finitely additive functional ν_n on $\mathfrak{A}$ such that $\nu_n 1 = 1$, $\nu_n a \ge \delta_n$ for every $a \in A_n$. Setting $\nu a = \sum_{n=0}^{\infty} 2^{-n}\nu_n a$ for every $a \in \mathfrak{A}$, ν is a strictly positive additive functional on $\mathfrak{A}$, and (i) is true.

391K Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is measurable iff it is Dedekind σ-complete and weakly (σ,∞)-distributive and either $\mathfrak{A} = \{0\}$ or $\mathfrak{A} \setminus \{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof Put 391D and 391J together.

391L 391J-391K are due to Kelley 59; condition (ii) of 391J is called **Kelley's criterion**. It provides some sort of answer to the question 'which Boolean algebras carry strictly positive finitely additive functionals?', but leaves quite open the possibility that there is some more abstract criterion which is also necessary and sufficient. It is indeed a non-trivial exercise to find any ccc Boolean algebra which does not carry a strictly positive finitely additive functional. The first example published seems to have been that of Gaifman 64; I present this in a modified form, following Comfort & Negrepontis 82. First, a definition:

Definition Let $\mathfrak{A}$ be a Boolean algebra. A set $A \subseteq \mathfrak{A}$ is **linked** if $a \cap b \ne 0$ for all a, $b \in A$. $\mathfrak{A}$ is **σ-linked** if $\mathfrak{A} \setminus \{0\}$ is a countable union of linked sets.

Remark 'σ-linkedness' is one of a very long list of 'chain conditions' which have been studied; there is an extensive investigation in Comfort & Negrepontis 82. I hope to look at some of them in Volume 5. For the moment I will introduce them one by one as the occasion arises.

391M Lemma A σ-linked Boolean algebra is ccc.

proof If $\mathfrak{A}$ is σ-linked, take a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of linked sets with union $\mathfrak{A} \setminus \{0\}$. Then any disjoint subset of $\mathfrak{A}$ can meet each A_n in at most one element, so must be countable.

391N Example There is a σ-linked Boolean algebra $\mathfrak{A}$ such that there is no strictly positive finitely additive real-valued functional defined on $\mathfrak{A}$.

proof (a) Write $\mathcal{I}$ for the set of half-open intervals of the form $[q,q'[$ where q, $q'\in\mathbb{Q}$ and $q<q'$. Then $\mathcal{I}$ is countable; enumerate it as $\langle I_n\rangle_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$ let $\mathcal{J}_n\subseteq\mathcal{I}$ be a disjoint family of subintervals of I_n such that $\#(\mathcal{J}_n)=(n+1)^2$. Give $\{0,1\}^{\mathbb{R}}$ its usual product topology, so that it is a compact Hausdorff space (3A3K); for $x\in\{0,1\}^{\mathbb{R}}$ set $Q_x=\{t:t\in\mathbb{R},\,x(t)=1\}$. Now set

$$X=\bigcap_{n\in\mathbb{N}}\{x:x\in\{0,1\}^{\mathbb{R}},\,\#(\{J:J\in\mathcal{J}_n,\,Q_x\cap J\ne\emptyset\})\le n+1\}.$$

Take $\mathfrak{A}$ to be the algebra of open-and-closed subsets of X.

(b) I show first that there is no strictly positive finitely additive functional on $\mathfrak{A}$. **P?** If there were, there would be a sequence $\langle A_k\rangle_{k\in\mathbb{N}}$ with union $\mathfrak{A}\setminus\{\emptyset\}$ such that every A_k had strictly positive intersection number (391J). For each $t\in\mathbb{R}$, write $e_t=\{x:x\in X,\,x(t)=1\}$, so that $e_t\in\mathfrak{A}$. Note that if we set $x_t(t)=1$, $x_t(s)=0$ for $s\ne t$, then $Q_x=\{t\}$ so $x_t\in X$, $x_t\in e_t$ and $e_t\ne\emptyset$; so there must be some k such that $e_t\in A_k$. Set $T_k=\{t:e_t\in A_k\}$ for each k; then $\bigcup_{k\in\mathbb{N}}T_k=\mathbb{R}$. By Baire's theorem (3A3G) there is a $k\in\mathbb{N}$ such that $G=\operatorname{int}\overline{T}_k\ne\emptyset$.

Let $\delta>0$ be the intersection number of A_k. There must be some $n\ge 1/\delta$ such that $I_n\subseteq G$. Since $I\subseteq\overline{T}_k$, there is for every $J\in\mathcal{J}_n$ a point $t_J\in J\cap T_k$, so that $e_{t_J}\in A_k$.

Consider the family $\langle e_{t_J}\rangle_{J\in\mathcal{J}_n}$. Because the intersection number of A_k is δ, there must be a set $\mathcal{K}\subseteq\mathcal{J}_n$, of cardinal at least $\delta\#(\mathcal{J}_n)=\delta(n+1)^2>n+1$, such that $\inf_{J\in\mathcal{K}}e_{t_J}\ne\emptyset$, that is, $X\cap\bigcap_{J\in\mathcal{K}}e_{t_J}\ne\emptyset$. But if $x\in\bigcap_{J\in\mathcal{K}}e_{t_J}$ then Q_x contains t_J for every $J\in\mathcal{K}$, so

$$\#(\{J:J\in\mathcal{J}_n,\,Q_x\cap J\ne\emptyset\})\ \ge\#(\mathcal{K})>n+1$$

and $x\notin X$. **XQ**

Thus $\mathfrak{A}$ does not satisfy Kelley's criterion and there is no strictly positive finitely additive functional defined on $\mathfrak{A}$.

(c)(i) Write F for the set of functions f such that $\operatorname{dom}f$ is a finite subset of $\mathbb{R}$ and $f(t)\in\{0,1\}$ for every $t\in\operatorname{dom}f$; for $f\in F$ write

$$Q_f=\{t:t\in\operatorname{dom}f,\,f(t)=1\},$$

$$H_f=\{x:x\in\{0,1\}^{\mathbb{R}},\,x(t)=f(t)\text{ for every }t\in\operatorname{dom}f\},$$

$$c_f=X\cap H_f\in\mathfrak{A}.$$

(ii) For $k\in\mathbb{N}$ write $\mathfrak{K}_k$ for the set of all disjoint finite families $\mathcal{K}$ of non-empty half-open intervals with rational endpoints such that $\#(\mathcal{K})=k$ and whenever $n\le 2k$ and $J\in\mathcal{J}_n$ then every member of $\mathcal{K}$ is either included in J or disjoint from J. (This allows $\emptyset$ to belong to $\mathfrak{K}_0$.) For $\mathcal{K}\in\mathfrak{K}_k$ let $F_{\mathcal{K}}$ be the set of those $f\in F$ such that $Q_f=\bigcup\mathcal{K}\cap\operatorname{dom}f$ and $Q_f\cap K$ has just one member for every $K\in\mathcal{K}$. Set

$$A_{\mathcal{K}}=\{a:a\in\mathfrak{A},\text{ there is some }f\in F_{\mathcal{K}}\text{ such that }\emptyset\ne c_f\subseteq a\}.$$

I claim that $a\cap b\ne\emptyset$ for any a, $b\in A_{\mathcal{K}}$. **P** There are f, $g\in F_{\mathcal{K}}$ such that $a\supseteq c_f\ne\emptyset$, $b\supseteq c_g\ne\emptyset$. Now if $t\in\operatorname{dom}f\cap\operatorname{dom}g$,

$$f(t)=1\iff t\in\bigcup\mathcal{K}\iff g(t)=1;$$

that is, f and g agree on $\operatorname{dom}f\cap\operatorname{dom}g$, and therefore there is a function $h=f\cup g$, with domain $\operatorname{dom}f\cup\operatorname{dom}g$, extending both. Of course $h\in F$. Define $z\in\{0,1\}^{\mathbb{R}}$ by setting $z(t)=h(t)$ if $t\in\operatorname{dom}h$, 0 otherwise, so that $Q_z=Q_h=Q_f\cup Q_g$ and z extends both f and g. Note that $Q_z\subseteq\bigcup\mathcal{K}$, and $\#(Q_z)\le\#(Q_f)+\#(Q_g)=2k$.

Now we are supposing that c_f is non-empty; say $x\in c_f$. Then $Q_x\supseteq Q_f$, so Q_x meets every member of $\mathcal{K}$. If $n\le 2k$, then

$$\{J:J\in\mathcal{J}_n,\,Q_x\cap J\ne\emptyset\}\supseteq\{J:J\in\mathcal{J}_n,\,J\cap\bigcup\mathcal{K}\ne\emptyset\}$$

because for $K\in\mathcal{K}$, $J\in\mathcal{J}_n$ either $K\subseteq J$ or $K\cap J=\emptyset$. But this means that

$$\begin{aligned}\#(\{J:J\in\mathcal{J}_n,\,Q_z\cap J\ne\emptyset\})&=\#(\{J:J\in\mathcal{J}_n,\,J\cap\bigcup\mathcal{K}\ne\emptyset\})\\&\le\#(\{J:J\in\mathcal{J}_n,\,Q_x\cap J\ne\emptyset\})\le n+1\end{aligned}$$

for every $n\le 2k$; while also

$$\#(\{J : J \in \mathcal{J}_n,\ Q_z \cap J \neq \emptyset\}) \leq \#(Q_z) \leq 2k \leq n+1$$

for every $n \geq 2k$. So $z \in X$. As z extends both f and g, $z \in c_f \cap c_g \subseteq a \cap b$ and $a \cap b \neq \emptyset$. **Q**

Thus every $A_{\mathcal{K}}$ is linked.

(iii) For every non-zero $a \in \mathfrak{A}$ there is some $k \in \mathbb{N}$, $\mathcal{K} \in \mathfrak{K}_k$ such that $a \in A_{\mathcal{K}}$. **P** Take any $x \in a$. Because a is a relatively open subset of X, there is an open $G \subseteq \{0,1\}^{\mathbb{R}}$ such that $a = G \cap X$; now, by the definition of the topology of $\{0,1\}^{\mathbb{R}}$, there is an $f \in F$ such that $x \in H_f \subseteq G$, so that $\emptyset \neq c_f \subseteq a$.

Set $k = \#(Q_f)$. Then we can choose a disjoint family $\langle K_t \rangle_{t \in Q_f}$ in $\mathcal{I}$ such that, for each t, $K_t \cap \operatorname{dom} f = \{t\}$ and, for every $J \in \bigcup_{n \leq 2k} \mathcal{J}_n$, either $K_t \subseteq J$ or $K_t \cap J = \emptyset$. Set $\mathcal{K} = \{K_t : t \in Q_f\}$; then $\mathcal{K} \in \mathfrak{K}_k$, $f \in F_{\mathcal{K}}$ and $a \in A_{\mathcal{K}}$. **Q**

But $\bigcup_{k \in \mathbb{N}} \mathfrak{K}_k$ is countable, being a subset of the family of finite sets in the countable set $\mathcal{I}$, so the $A_{\mathcal{K}}$ form a countable family of linked sets covering $\mathfrak{A} \setminus \{0\}$, and $\mathfrak{A}$ is σ-linked.

391X Basic exercises For this series of exercises, I will call a Boolean algebra $\mathfrak{A}$ **chargeable** if there is a strictly positive finitely additive functional $\nu : \mathfrak{A} \to [0, \infty[$.

(a) Show that a chargeable Boolean algebra is ccc, so is Dedekind complete iff it is Dedekind σ-complete.

(b) Show (i) that any subalgebra of a chargeable Boolean algebra is chargeable (ii) that a countable simple product of chargeable Boolean algebras is chargeable (iii) that any free product of chargeable Boolean algebras is chargeable.

(c) (i) Let $\mathfrak{A}$ be a Boolean algebra with a chargeable order-dense subalgebra. Show that $\mathfrak{A}$ is chargeable. (ii) Show that the Dedekind completion of a chargeable Boolean algebra is chargeable.

(d) (i) Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is chargeable for any set I. (ii) Show that the regular open algebra of $\mathbb{R}$ is chargeable.

(e) (i) Show that any principal ideal of a chargeable Boolean algebra is chargeable. (ii) Let $\mathfrak{A}$ be a chargeable Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is chargeable.

>(f) Show that a Boolean algebra is chargeable iff it is isomorphic to a subalgebra of a measurable algebra. (*Hint*: if $\mathfrak{A}$ is a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ is a strictly positive additive functional, set $\rho(a, b) = \nu(a \triangle b)$ for a, $b \in \mathfrak{A}$. Show that ρ is a metric on $\mathfrak{A}$ and that the metric completion of $\mathfrak{A}$ under ρ is a measurable algebra under the natural operations (cf. 324O).)

(g) Explain how to use the Hahn-Banach theorem to prove 391G directly, without passing through 391F. (*Hint*: $S(\mathfrak{B})$ can be regarded as a subspace of $S(\mathfrak{A})$.)

>(h) Take $X = \{0, 1, 2, 3\}$, $\mathfrak{A} = \mathcal{P}X$, $A = \{\{0,1\}, \{0,2\}, \{0,3\}, \{1,2,3\}\}$. Show that the intersection number of A is $\frac{3}{5}$. (*Hint*: use 391I.) Show that if $a_0, \dots, a_n$ are *distinct* members of A then there is a set $J \subseteq \{0, \dots, n\}$, with $\#(J) \geq \frac{2}{3}(n+1)$, such that $\inf_{j \in J} a_j \neq 0$.

(i) Let $\mathfrak{A}$ be a Boolean algebra. For non-empty $A \subseteq \mathfrak{A} \setminus \{0\}$ write $\delta(A)$ for the intersection number of A. Show that for any non-empty $A \subseteq \mathfrak{A} \setminus \{0\}$, $\delta(A) = \sup\{\delta(I) : I \text{ is a non-empty finite subset of } A\}$.

(j) (i) Show that any subalgebra of a σ-linked Boolean algebra is σ-linked. (ii) Show that if $\mathfrak{A}$ is a Boolean algebra with a σ-linked order-dense subalgebra, then $\mathfrak{A}$ is σ-linked. (iii) Show that the simple product of a countable family of σ-linked Boolean algebras is σ-linked. (iv) Show that a principal ideal of a σ-linked Boolean algebra is σ-linked.

391Y Further exercises (a) Show that in 391D and 391K we can replace 'weakly (σ, ∞)-distributive' by 'weakly σ-distributive'.

(b) Show that $\mathcal{P}\mathbb{N}$ is σ-linked and chargeable but that the quotient algebra $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is not ccc, therefore neither σ-linked nor chargeable.

(c) (i) Show that if X is a separable topological space, then its regular open algebra is chargeable. (ii) Let $\langle X_i \rangle_{i \in I}$ be any family of topological spaces with chargeable regular open algebras. Show that their product has a chargeable regular open algebra.

(d) Show that any σ-linked Boolean algebra can have cardinal at most $\mathfrak{c}$. (*Hint*: take a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of linked sets covering $\mathfrak{A} \setminus \{0\}$. Set $A_n^* = \{a : \exists\, b \in A_n,\, b \subseteq a\}$. Show that if $a \not\subseteq b$ there is an n such that $a \in A_n^*$ and $b \notin A_n^*$.)

(e) Show that the free product of $\mathfrak{c}$ or fewer σ-linked Boolean algebras is σ-linked. (*Hint*: let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of σ-linked Boolean algebras, where $I \subseteq \mathbb{R}$. For each $i \in I$, let $\langle A_{in} \rangle_{n \in \mathbb{N}}$ be a sequence of linked sets with union $\mathfrak{A}_i \setminus \{0\}$. For $q_0 < q_1 < \ldots < q_r \in \mathbb{Q}$ and $n_1, \ldots, n_r \in \mathbb{N}$ let $B_{\mathbf{q},\mathbf{n}}$ be the set of elements expressible as $\inf_{i \in J} \varepsilon_i(a_i)$ where $J \subseteq I \cap [q_0, q_r[$ is a set meeting each interval $[q_{k-1}, q_k[$ in at most one point and $a_i \in A_{i,n_k}$ if $i \in J \cap [q_{k-1}, q_k[$; show that $B_{\mathbf{q},\mathbf{n}}$ is linked.)

(f) Let X be the space of 391N. (i) Show that X is closed in $\{0,1\}^{\mathbb{R}}$, therefore compact. (ii) Show that the regular open algebra of X is σ-linked but not chargeable.

(g) Let $\mathfrak{A}$ be the algebra of 391N. Show that $\mathfrak{A}$ is not weakly (σ, ∞)-distributive. (*Hint*: show that for any $t \in \mathbb{R}$ there is a strictly decreasing sequence $\langle t_i \rangle_{i \in \mathbb{N}}$, converging to t, such that $x \mapsto \langle x(t_i) \rangle_{i \in \mathbb{N}}$ is a surjection from X onto $\{0,1\}^{\mathbb{N}}$, and hence that the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ can be regularly embedded into $\mathfrak{A}$; now use 316Xq and 316Xm.)

(h) If $\mathfrak{A}$ is a Boolean algebra, a set $A \subseteq \mathfrak{A}$ is m**-linked**, where $m \geq 2$, if $a_1 \cap a_2 \cap \ldots \cap a_m \neq 0$ for all $a_1, \ldots, a_m \in A$; and $\mathfrak{A}$ is σ**-**m**-linked** if $\mathfrak{A} \setminus \{0\}$ can be expressed as the union of a sequence of m-linked sets. (Thus 'linked' is '2-linked' and 'σ-linked' is 'σ-2-linked'.) Show that the algebra of 391N is σ-m-linked for every $m \geq 2$. (*Hint*: in part (c) of the proof, replace each '$2k$' by 'mk'.)

391Z Problem Must a Dedekind complete σ-linked weakly (σ, ∞)-distributive Boolean algebra be measurable?

391 Notes and comments By the standards of this volume, this is an easy section; I note that I have hardly called on anything after Chapter 32, except for a reference to the construction $S(\mathfrak{A})$ in §361. I do ask for a bit of functional analysis (the Hahn-Banach theorem) in 391E. Kelley's criterion (391J) is a little unsatisfying. It is undoubtedly useful (see part (b) of the proof of 391N, or 392F below), but at the same time the structure of the criterion – a special sequence of subsets of $\mathfrak{A}$ – is rather close to the structure of the conclusion; after all, one is, or can be represented by, a function from $\mathfrak{A} \setminus \{0\}$ to $\mathbb{N}$, while the other is a function from $\mathfrak{A}$ to $\mathbb{R}$. Also the actual intersection number of a family $A \subseteq \mathfrak{A} \setminus \{0\}$ can be hard to calculate; as often as not, the best method is to look at the additive functionals on $\mathfrak{A}$ (see 391Xh).

I take the trouble to show that Gaifman's example (391N) is σ-linked in order to show that even conditions very much stronger than 'ccc' are not sufficient to guarantee the existence of suitable functionals. (See also 391Yh.) For other examples see Comfort & Negrepontis 82. But I note that none of the standard examples is weakly (σ, ∞)-distributive (see 391Yg), and I believe it is open, in the formal sense, whether any weakly (σ, ∞)-distributive σ-linked Boolean algebra must carry a strictly positive finitely additive functional (391Z). But I conjecture that the answer is 'no'.

Since every σ-linked algebra has cardinal at most $\mathfrak{c}$ (391Yd), not every measurable algebra is σ-linked. In fact it is known that a measurable algebra of cardinal $\mathfrak{c}$ or less is σ-m-linked (391Yh) for every $m \geq 2$ (Dow & Steprāns 93).

392 Submeasures

In §391 I looked at what we can deduce if a Boolean algebra carries a strictly positive finitely additive functional. There are important contexts in which we find ourselves with a subadditive, rather than additive, functional, and this is what I wish to investigate here. It turns out that, once we have found the right hypotheses, such functionals can also provide a criterion for measurability of an algebra (392J). The argument runs through a new idea, using a result in finite combinatorics (392D).

392A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **submeasure** on $\mathfrak{A}$ is a functional $\nu : \mathfrak{A} \to [0, \infty[$ such that

$\nu(a \cup b) \le \nu a + \nu b$ for all a, $b \in \mathfrak{A}$;

$\nu a \le \nu b$ whenever $a \subseteq b$;

$\nu 0 = 0$.

(In this context I do not allow ∞ as a value of a submeasure.) Any positive finitely additive functional is a submeasure (326Ba, 326Bf).

392B Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ a submeasure. Then

(a) ν is **strictly positive** if $\nu a > 0$ for every $a \ne 0$;

(b) ν is **exhaustive** if $\lim_{n\to\infty} \nu a_n = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$;

(c) ν is **uniformly exhaustive** if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that there is no disjoint family $a_0, \dots , a_n$ with $\nu a_i \ge \epsilon$ for every $i \le n$.

392C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If there is an exhaustive strictly positive submeasure on $\mathfrak{A}$, then $\mathfrak{A}$ is ccc.

(b) A uniformly exhaustive submeasure on $\mathfrak{A}$ is exhaustive.

(c) Any positive linear functional on $\mathfrak{A}$ is a uniformly exhaustive submeasure.

proof These are all elementary. If $\nu : \mathfrak{A} \to [0, \infty[$ is an exhaustive strictly positive submeasure, and $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A} \setminus \{0\}$, then $\{i : \nu a_i \ge 2^{-n}\}$ must be finite for each n, so I is countable. (Cf. 322G.) If $\nu : \mathfrak{A} \to [0, \infty[$ is a uniformly exhaustive submeasure and $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint in $\mathfrak{A}$, then $\{i : \nu a_i \ge 2^{-n}\}$ is finite for each n, so $\lim_{i\to\infty} \nu a_i = 0$. If $\nu : \mathfrak{A} \to [0, \infty[$ is a positive linear functional, and $\epsilon > 0$, then take $n \ge \frac{1}{\epsilon}\nu 1$; if $a_0, \dots , a_n$ are disjoint, then $\sum_{i=0}^{n} \nu a_i \le \nu 1$, so $\min_{i\le n} \nu a_i < \epsilon$.

392D Lemma Suppose that k, l, $m \in \mathbb{N}$ are such that $3 \le k \le l \le m$ and $18mk \le l^2$. Let L, M be sets of sizes l, m respectively. Then there is a set $R \subseteq M \times L$ such that (i) each vertical section of R has just three members (ii) $\#(R[E]) \ge \#(E)$ whenever $E \in [M]^{\le k}$; so that for every $E \in [M]^{\le k}$ there is an injective function $f : E \to L$ such that $(x, f(x)) \in R$ for every $x \in E$.

recall that $[M]^{\le k} = \{I : I \subseteq M, \#(I) \le k\}$ (3A1J).

proof (a) We need to know that $n! \ge 3^{-n}n^n$ for every $n \in \mathbb{N}$; this is immediate from the inequality

$$\sum_{i=2}^{n} \ln i \ge \int_1^n \ln x\,dx = n \ln n - n + 1 \text{ for every } n \ge 2.$$

(b) Let Ω be the set of those $R \subseteq M \times L$ such that each vertical section of R has just three members, so that

$$\#(\Omega) = \#([L]^3)^m = \Bigl(\frac{l!}{3!(l-3)!}\Bigr)^m.$$

(I write $[X]^j$ for the set of subsets of X with j members.) Let us regard Ω as a probability space with the uniform probability.

If $F \in [L]^n$, where $3 \le n \le k$, and $x \in M$, then

$$\Pr(R[\{x\}] \subseteq F) = \frac{\#([F]^3)}{\#([L]^3)}$$

(because $R[\{x\}]$ is a random member of $[L]^3$)

$$= \frac{n(n-1)(n-2)}{l(l-1)(l-2)} \le \frac{n^3}{l^3}.$$

So if $E \in [M]^n$ and $F \in [L]^n$, then

$$\Pr(R[E] \subseteq F) = \prod_{x\in E} \Pr(R[\{x\}] \subseteq F)$$

(because the sets $R[\{x\}]$ are chosen independently)

$$\le \frac{n^{3n}}{l^{3n}}.$$

Accordingly

$$\begin{aligned}&\Pr(\text{there is an } E \subseteq M \text{ such that } \#(R[E]) < \#(E) \le k)\\ &\quad\le \Pr(\text{there is an } E \subseteq M \text{ such that } \#(R[E]) \le \#(E) \le k)\\ &\quad= \Pr(\text{there is an } E \subseteq M \text{ such that } 3 \le \#(R[E]) \le \#(E) \le k)\end{aligned}$$

(because if $E \ne \emptyset$ then $\#(R[E]) \ge 3$)

$$\le \sum_{n=3}^{k} \sum_{E\in[M]^n} \sum_{F\in[L]^n} \Pr(R[E] \subseteq F) \le \sum_{n=3}^{k} \#([M]^n)\#([L]^n)\frac{n^{3n}}{l^{3n}}$$

$$= \sum_{n=3}^{k} \frac{m!}{n!(m-n)!}\frac{l!}{n!(l-n)!}\frac{n^{3n}}{l^{3n}} \le \sum_{n=3}^{k} \frac{m^n l^n n^{3n}}{n!n!l^{3n}} \le \sum_{n=3}^{k} \frac{m^n n^n 3^{2n}}{l^{2n}}$$

(using (a))

$$= \sum_{n=3}^{k} \Bigl(\frac{9mn}{l^2}\Bigr)^n \le \sum_{n=3}^{k} \frac{1}{2^n} < 1.$$

There must therefore be some $R \in \Omega$ such that $\#(R[E]) \ge \#(E)$ whenever $E \subseteq M$ and $\#(E) \le k$.

(c) If now $E \in [M]^{\le k}$, the restriction $R_E = R \cap (E \times L)$ has the property that $\#(R_E[I]) \ge \#(I)$ for every $I \subseteq E$. By Hall's Marriage Lemma (3A1K) there is an injective function $f : E \to L$ such that $(x, f(x)) \in R_E \subseteq R$ for every $x \in E$.

Remark Of course this argument can be widely generalized; see references in Kalton & Roberts 83.

392E Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a uniformly exhaustive submeasure. Then for any $\epsilon \in]0, \nu 1]$ the set $A = \{a : \nu a \ge \epsilon\}$ has intersection number greater than 0.

proof (a) If $\nu 1 = 0$ this is trivial, so we may assume that $\nu 1 > 0$; since neither the hypothesis nor the conclusion is affected if we multiply ν by a positive scalar, we may suppose that $\nu 1 = 1$. Because ν is uniformly exhaustive, there is an $r \ge 1$ such that whenever $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ then $\#(\{i : \nu c_i > \frac{1}{5}\epsilon\}) \le r$, so that $\sum_{i\in I} \nu c_i \le r + \frac{1}{5}\epsilon\#(I)$. Set $\delta = \epsilon/5r$, $\eta = \frac{1}{74}\delta^2$, so that

$$\delta - \eta \ge \tfrac{1}{18}(\delta - \eta)^2 \ge \tfrac{1}{18}(\delta^2 - 2\eta) = 4\eta.$$

(b) Let $\langle a_i\rangle_{i\in I}$ be a non-empty finite family in A. Let m be any multiple of $\#(I)$ greater than or equal to $1/\eta$. Then there are integers k, l such that

$$3\eta \le \frac{k}{m} \le 4\eta \le \frac{1}{18}(\delta - \eta)^2, \quad \delta - \eta \le \frac{l}{m} \le \delta,$$

in which case

$$3 \le k \le l \le m, \quad 18mk \le m^2(\delta - \eta)^2 \le l^2.$$

(c) Take a set M of the form $I \times S$ where $\#(S) = m/\#(I)$, so that $\#(M) = m$. For $x = (i, s) \in M$ set $d_x = a_i$. Let L be a set with l members. By 392D, there is a set $R \subseteq M \times L$ such that every vertical section of R has just three members and whenever $E \in [M]^{\leq k}$ there is an injective function $f_E : E \to L$ such that $(x, f_E(x)) \in R$ for every $x \in E$.

For $E \subseteq M$ set

$$b_E = \inf_{x \in E} d_x \setminus \sup_{x \in M \setminus E} d_x,$$

so that $\langle b_E \rangle_{E \subseteq M}$ is a partition of unity in $\mathfrak{A}$. For $x \in M$, $j \in L$ set

$$c_{xj} = \sup\{b_E : x \in E \in [M]^{\leq k},\ f_E(x) = j\}.$$

If x, y are distinct members of M and $j \in L$ then

$$c_{xj} \cap c_{yj} = \sup\{b_E : x,\ y \in E \in [M]^{\leq k},\ f_E(x) = f_E(y) = j\} = 0,$$

because every f_E is injective. Set

$$m_j = \#(\{x : x \in M,\ c_{xj} \neq 0\})$$

for each $j \in L$. Note that $c_{xj} = 0$ if $(x, j) \notin R$, so $\sum_{j \in L} m_j \leq \#(R) = 3m$.

We have

$$\textstyle\sum_{x \in M} \nu c_{xj} \leq r + \frac{1}{5}\epsilon m_j$$

for each j, by the choice of r; so

$$\begin{aligned}\sum_{x \in M, j \in L} \nu c_{xj} &\leq rl + \frac{1}{5}\epsilon \sum_{j \in L} m_j \leq rl + \frac{3}{5}m\epsilon \\ &\leq (r\delta + \frac{3}{5}\epsilon)m = \frac{4}{5}\epsilon m < \epsilon m\end{aligned}$$

by the choice of l and δ. There must therefore be some $x \in M$ such that

$$\nu(\textstyle\sup_{j \in L} c_{xj}) \leq \sum_{j \in L} \nu c_{xj} < \epsilon \leq \nu d_x,$$

and d_x cannot be included in

$$\textstyle\sup_{j \in L} c_{xj} = \sup\{b_E : x \in E \in [M]^{\leq k}\}.$$

But as $\sup\{b_E : x \in E \subseteq M\}$ is just d_x, there must be an $E \subseteq M$, of cardinal greater than k, such that $b_E \neq 0$.

Recall now that $M = I \times S$, and that

$$k \geq 3\eta m = 3\eta\#(I)\#(S).$$

The set $J = \{i : \exists\ s,\ (i, s) \in E\}$ must therefore have more than $3\eta\#(I)$ members, since $E \subseteq J \times S$. But also $d_{(i,s)} = a_i$ for each $(i, s) \in E$, so that $\inf_{i \in J} a_i \supseteq b_E \neq 0$.

(d) As $\langle a_i \rangle_{i \in I}$ is arbitrary, the intersection number of A is at least $3\eta > 0$.

392F Theorem Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive uniformly exhaustive submeasure. Then $\mathfrak{A}$ has a strictly positive finitely additive functional.

proof If $\mathfrak{A} = \{0\}$ this is trivial. Otherwise, let $\nu : \mathfrak{A} \to [0, \infty[$ be a strictly positive uniformly exhaustive submeasure. For each n, $A_n = \{a : \nu a \geq 2^{-n}\nu 1\}$ has intersection number greater than 0, and $\bigcup_{n \in \mathbb{N}} A_n = \mathfrak{A} \setminus \{0\}$ because ν is strictly positive; so $\mathfrak{A}$ has a strictly positive finitely additive functional, by Kelley's theorem (391J).

392G Since positive additive functionals are uniformly exhaustive submeasures, the condition of this theorem is necessary as well as sufficient. Thus we have a description, in terms of submeasures, of a condition equivalent to one part of the criterion for measurability of an algebra in 391D. The language of submeasures also provides a formulation of another part of this criterion, as follows.

Definition Let $\mathfrak{A}$ be a Boolean algebra. A **Maharam submeasure** or **continuous outer measure** on $\mathfrak{A}$ is a submeasure $\nu : \mathfrak{A} \to [0, \infty[$ such that $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

392H Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν a Maharam submeasure on $\mathfrak{A}$.

(a) ν is sequentially order-continuous.

(b) ν is 'countably subadditive', that is, whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a \in \mathfrak{A}$ is such that $a = \sup_{n\in\mathbb{N}} a \cap a_n$, then $\nu a \le \sum_{n=0}^{\infty} \nu a_n$.

(c) If $\mathfrak{A}$ is Dedekind σ-complete, then ν is exhaustive.

proof (a) (Of course ν is an order-preserving function, by the definition of 'submeasure'; so we can apply the ordinary definition of 'sequentially order-continuous' in 313Hb.) (i) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a, then $\langle a_n \setminus a \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty} \nu(a_n \setminus a) = 0$; but as

$$\nu a_n \le \nu a \le \nu a_n + \nu(a \setminus a_n)$$

for every n, it follows that $\nu a = \lim_{n\to\infty} \nu a_n$. (ii) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum a, then

$$\nu a \le \nu a_n \le \nu a + \nu(a_n \setminus a) \to \nu a$$

as $n \to \infty$.

(b) Set $b_n = \sup_{i\le n} a \cap a_i$; then $\nu b_n \le \sum_{i=0}^{n} \nu a_i$ for each n (inducing on n), so that

$$\nu a = \lim_{n\to\infty} \nu b_n \le \sum_{i=0}^{\infty} \nu a_i.$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, set $b_n = \sup_{i\ge n} a_i$ for each n; then $\inf_{n\in\mathbb{N}} b_n = 0$, so

$$\limsup_{n\to\infty} \nu a_n \le \lim_{n\to\infty} \nu b_n = 0.$$

392I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Then $\mathfrak{A}$ is ccc, Dedekind complete and weakly (σ, ∞)-distributive.

proof By 392Hc, ν is exhaustive; by 392Ca, $\mathfrak{A}$ is ccc; by 316Fa, $\mathfrak{A}$ is Dedekind complete.

Now suppose that we have a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Let A be the set

$$\{a : a \in \mathfrak{A}, \forall\ n \in \mathbb{N}\ \exists\ a' \in A_n \text{ such that } a' \le a\}.$$

By 316Fc and 392Ha, ν is order-continuous, so $\inf_{a\in A_n} \nu a = 0$ for each n. Given $\epsilon > 0$, we can choose $\langle a_n \rangle_{n\in\mathbb{N}}$ such that $a_n \in A_n$ and $\nu a_n \le 2^{-n}\epsilon$ for each n; now $a = \sup_{n\in\mathbb{N}} a_n \in A$ and $\nu a \le \sum_{n=0}^{\infty} \nu a_n \le 2\epsilon$. Thus $\inf_{a\in A} \nu a = 0$. Since ν is strictly positive, $\inf A = 0$. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ, ∞)-distributive.

392J Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is Dedekind σ-complete and carries a uniformly exhaustive strictly positive Maharam submeasure.

proof If $\mathfrak{A}$ is measurable, it surely satisfies the conditions, since any totally finite measure on $\mathfrak{A}$ is also a uniformly exhaustive strictly positive Maharam submeasure. If $\mathfrak{A}$ satisfies the conditions, then it is weakly (σ, ∞)-distributive, by 392I, and carries a strictly positive finitely additive functional, by 392F; so is measurable, by 391D.

392X Basic exercises (a) Show that the first two clauses of the definition 392A can be replaced by '$\nu a \le \nu(a \cup b) \le \nu a + \nu b$ whenever $a \cap b = 0$'.

(b) Let $\mathfrak{A}$ be any Boolean algebra and ν a submeasure on $\mathfrak{A}$. (i) Show that the following are equiveridical: (α) ν is order-continuous; (β) whenever $A \subseteq \mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a\in A} \nu a = 0$. (ii) Show that in this case ν is exhaustive. (*Hint*: if $\langle a_n \rangle_{n\in\mathbb{N}}$ is disjoint, then $\bigcup_{n\in\mathbb{N}}\{b : b \supseteq a_i$ for every $i \ge n\}$ has infimum 0.)

>(c) Let $\mathfrak{A}$ be the finite-cofinite algebra on an uncountable set (316Yk). (i) Set $\nu_1 0 = 0$, $\nu_1 a = 1$ for $a \in \mathfrak{A} \setminus \{0\}$. Show that ν_1 is a strictly positive Maharam submeasure but is not exhaustive. (ii) Set $\nu_2 a = 0$ for finite a, 1 for cofinite a. Show that ν_2 is a uniformly exhaustive Maharam submeasure but is not order-continuous.

>(d) Let $\mathfrak{A}$ be a Boolean algebra and ν a submeasure on $\mathfrak{A}$. Set $I = \{a : \nu a = 0\}$. Show that (i) I is an ideal of $\mathfrak{A}$ (ii) there is a submeasure $\bar{\nu}$ on $\mathfrak{A}/I$ defined by setting $\bar{\nu} a^\bullet = \nu a$ for every $a \in \mathfrak{A}$ (iii) if ν is exhaustive, so is $\bar{\nu}$ (iv) if ν is uniformly exhaustive, so is $\bar{\nu}$ (v) if ν is a Maharam submeasure, I is a σ-ideal (vi) if ν is a Maharam submeasure and $\mathfrak{A}$ is Dedekind σ-complete, $\bar{\nu}$ is a Maharam submeasure.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and ν an order-continuous submeasure on $\mathfrak{A}$. Show that ν has a unique support $a \in \mathfrak{A}$ such that $\nu \restriction \mathfrak{A}_a$ is strictly positive and $\nu \restriction \mathfrak{A}_{1 \setminus a}$ is identically zero.

(f) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a uniformly exhaustive Maharam submeasure on $\mathfrak{A}$. Show that there is a non-negative countably additive functional μ on $\mathfrak{A}$ such that $\{a : \mu a = 0\} = \{a : \nu a = 0\}$. (*Hint*: 392Xd(vi).)

392 Notes and comments Much of this section is a matter of generalizing earlier arguments. Thus 392C and 392H ought by now to be very easy, while 392I uses the methods of 322F-322G, and 392Xb recalls the elementary theory of τ-additive functionals.

The new ideas are in the combinatorics of 392D-392E. I have cast 392D in the form of an argument in probability theory. Of course there is nothing here but simple counting, since the probability measure simply puts the same mass on each point of Ω, and every statement of the form 'Pr$(R \dots) \le \dots$' is just a matter of counting the elements R of Ω with the given property. But I think many of us find that the probabilistic language makes the calculations more natural; in particular, we can use intuitions associated with the notion of independence of events. Indeed I strongly recommend the method. It has been used to very great effect in the last fifty years in a wide variety of combinatorial problems.

392F/392J constitute the **Kalton-Roberts theorem** (Kalton & Roberts 83).

It is not known whether every exhaustive submeasure is uniformly exhaustive; this is the Control Measure Problem, which I will treat in the next section.

393 The Control Measure Problem

I come now to a discussion of a classic problem of measure theory. Its importance derives to a great extent from the variety of forms in which it appears, and this section is devoted primarily to a description of some of these forms, with the arguments to show that they are indeed all the same problem, in that a solution to one of the questions will provide solutions to all the others. The syntax of the exposition seems to be simplest if I present each formulation as a statement 'CM_n'; the corresponding question being 'is CM_n true?', and the proof that all the questions are really the same question becomes a proof that $\mathrm{CM}_m \iff \mathrm{CM}_n$ for all m, n.

The propositions CM_* are listed in 393A, 393H, 393J, 393L and 393P, while the arguments that they are equiveridical form the rest of the material down to and including 393R. CM_1-$\mathrm{CM}_{3\mathrm{B}}$ all involve submeasures in one way or another, and much of the section amounts to a theory of submeasures. On the way I mention a description of the open-and-closed algebra of $\{0,1\}^{\mathbb{N}}$ (393F). The propositions CM_4, CM_5 and CM_6 concern topologies on Boolean algebras and L^0 spaces, and vector measures. They can be expressed more or less adequately with very little of the surrounding theories, but for completeness I include a basic theorem on vector measures (393S). At the end of the section I present two of the many examples of submeasures which have been described.

393A The problem The language I introduced in the last section is already sufficient for more than one formulation of the problem. Consider the statements

(CM_1) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, then $\mathfrak{A}$ is measurable.

(CM_1') Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a Maharam submeasure. Then there is a non-negative countably additive functional $\mu : \mathfrak{A} \to [0,\infty[$ such that, for $a \in \mathfrak{A}$, $\mu a = 0 \iff \nu a = 0$.

(CM_1'') Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0,\infty[$ a Maharam submeasure. Then there is a totally finite measure μ, with domain Σ, such that, for $E \in \Sigma$, $\nu E = 0 \iff \mu E = 0$.

(CM_2) Every exhaustive submeasure is uniformly exhaustive.

(CM_2') Every exhaustive submeasure on the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ is uniformly exhaustive.

The following lemmas assemble the new ideas we need in order to prove that these statements are equiveridical.

393B Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν a strictly positive submeasure on $\mathfrak{A}$.

(a) We have a metric ρ on $\mathfrak{A}$ defined by the formula

$$\rho(a,b) = \nu(a \triangle b)$$

for all a, $b \in \mathfrak{A}$.

(b) The Boolean operations $\cup$, $\cap$, $\triangle$, $\setminus$, and the function $\nu : \mathfrak{A} \to \mathbb{R}$, are all uniformly continuous for ρ.

(c) The metric space completion $(\widehat{\mathfrak{A}}, \hat\rho)$ of $(\mathfrak{A}, \rho)$ is a Boolean algebra under the natural continuous extensions of the Boolean operations, and ν has a unique continuous extension $\hat\nu$ to $\widehat{\mathfrak{A}}$ which is again a strictly positive submeasure.

(d) If ν is exhaustive, then $\widehat{\mathfrak{A}}$ is Dedekind complete and ccc and $\hat\nu$ is a Maharam submeasure.

(e) If ν is additive, then $(\widehat{\mathfrak{A}}, \hat\nu)$ is a totally finite measure algebra.

proof (a)-(b) This is just a generalization of 323A-323B; essentially the same formulae can be used. For the triangle inequality for ρ, we have $a \triangle c \subseteq (a \triangle b) \cup (b \triangle c)$, so

$$\rho(a,c) = \nu(a \triangle c) \le \nu(a \triangle b) + \nu(b \triangle c) = \rho(a,b) + \rho(b,c).$$

For the uniform continuity of the Boolean operations, we have

$$(b * c) \triangle (b' * c') \subseteq (b \triangle b') * (c \triangle c')$$

so that

$$\rho(b * c, b' * c') \le \rho(b,b') + \rho(c,c')$$

for each of the operations $* = \cup, \cap, \setminus$ and $\triangle$. For the uniform continuity of the function ν itself, we have

$$\nu b \le \nu c + \nu(b \setminus c) \le \nu c + \rho(b,c),$$

so that $|\nu b - \nu c| \le \rho(b,c)$.

(c) $\mathfrak{A} \times \mathfrak{A}$ is a dense subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, so the Boolean operations on $\mathfrak{A}$, regarded as uniformly continuous functions from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A} \subseteq \widehat{\mathfrak{A}}$, have unique extensions to continuous binary operations on $\widehat{\mathfrak{A}}$ (3A4G). If we look at

$$A = \{(a,b,c) : a \triangle (b \triangle c) = (a \triangle b) \triangle c\},$$

this is a closed subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, because the maps $(a,b,c) \mapsto a \triangle (b \triangle c)$, $(a,b,c) \mapsto (a \triangle b) \triangle c$ are continuous and the topology of $\widehat{\mathfrak{A}}$ is Hausdorff; since A includes the dense set $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$, it is the whole of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, that is, $a \triangle (b \triangle c) = (a \triangle b) \triangle c$ for all a, b, $c \in \widehat{\mathfrak{A}}$. All the other identities we need to show that $\widehat{\mathfrak{A}}$ is a Boolean algebra can be confirmed by the same method. Of course $\mathfrak{A}$ is now a subalgebra of $\widehat{\mathfrak{A}}$.

Because $\nu : \mathfrak{A} \to [0, \infty[$ is uniformly continuous, it has a unique continuous extension $\hat{\nu} : \widehat{\mathfrak{A}} \to [0, \infty[$. We have

$$\hat{\nu}0 = 0, \quad \hat{\nu}a \le \hat{\nu}(a \cup b) \le \hat{\nu}a + \hat{\nu}b, \quad \hat{\nu}a = \hat{\rho}(a, 0)$$

for every a, $b \in \mathfrak{A}$ and therefore for every a, $b \in \widehat{\mathfrak{A}}$, so $\hat{\nu}$ is a submeasure on $\widehat{\mathfrak{A}}$, and

$$\hat{\nu}a = 0 \Longrightarrow \hat{\rho}(a, 0) = 0 \Longrightarrow a = 0,$$

so $\hat{\nu}$ is strictly positive.

(d) Now suppose that ν is exhaustive.

(i) The point is that any non-increasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\widehat{\mathfrak{A}}$ is a Cauchy sequence for the metric $\hat{\rho}$. **P** Let $\epsilon > 0$. For each $n \in \mathbb{N}$, choose $b_n \in \mathfrak{A}$ such that $\hat{\rho}(a_n, b_n) \le 2^{-n}\epsilon$, and set $c_n = \inf_{i \le n} b_i$. Then

$$\hat{\rho}(a_n, c_n) = \hat{\rho}(\inf_{i \le n} a_i, \inf_{i \le n} b_i) \le \sum_{i=0}^{n} \hat{\rho}(a_i, b_i) \le 2\epsilon$$

for every n. Choose $\langle n(k) \rangle_{k \in \mathbb{N}}$ inductively so that, for each $k \in \mathbb{N}$, $n(k+1) \ge n(k)$ and

$$\nu(c_{n(k)} \setminus c_{n(k+1)}) \ge \sup_{i \ge n(k)} \nu(c_{n(k)} \setminus c_i) - \epsilon.$$

Then $\langle c_{n(k)} \setminus c_{n(k+1)} \rangle_{k \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, so

$$\begin{aligned}
\limsup_{k \to \infty} \sup_{i \ge n(k)} \hat{\rho}(a_{n(k)}, a_i) &\le 4\epsilon + \limsup_{k \to \infty} \sup_{i \ge n(k)} \hat{\rho}(c_{n(k)}, c_i) \\
&= 4\epsilon + \limsup_{k \to \infty} \sup_{i \ge n(k)} \nu(c_{n(k)} \setminus c_i) \\
&\le 4\epsilon + \limsup_{k \to \infty} \nu(c_{n(k)} \setminus c_{n(k+1)}) + \epsilon = 5\epsilon.
\end{aligned}$$

As ϵ is arbitrary, $\langle a_n \rangle_{n \in \mathbb{N}}$ is Cauchy. **Q**

(ii) It follows that $\widehat{\mathfrak{A}}$ is Dedekind σ-complete. **P** If $\langle a_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\widehat{\mathfrak{A}}$, $\langle b_n \rangle_{n \in \mathbb{N}} = \langle \inf_{i \le n} a_i \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence with a limit $b \in \widehat{\mathfrak{A}}$. For any $k \in \mathbb{N}$,

$$\hat{\nu}(b \setminus a_k) = \lim_{n \to \infty} \hat{\nu}(b_n \setminus a_k) = 0,$$

so $b \subseteq a_k$, because $\hat{\nu}$ is strictly positive. While if $c \in \widehat{\mathfrak{A}}$ is a lower bound for $\{a_n : n \in \mathbb{N}\}$, we have $c \subseteq b_n$ for every n, so

$$\hat{\nu}(c \setminus b) = \lim_{n \to \infty} \hat{\nu}(c \setminus b_n) = 0$$

and $c \subseteq b$. Thus $b = \inf_{n \in \mathbb{N}} a_n$; as $\langle a_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\widehat{\mathfrak{A}}$ is Dedekind σ-complete (314Bc). **Q**

(iii) We also find that $\hat{\nu}$ is a Maharam submeasure, because if $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\widehat{\mathfrak{A}}$ with infimum 0, it must have a limit a which (as in (ii) just above) must be its infimum, that is, $a = 0$; consequently

$$\lim_{n \to \infty} \hat{\nu}a_n = \hat{\nu}a = 0.$$

(iv) It follows at once that $\hat{\nu}$ is exhaustive (392Hc), so that $\widehat{\mathfrak{A}}$ is ccc (392Ca) and Dedekind complete (316Fa).

(e) If ν is additive, then (being finite-valued) it must be exhaustive, so that $\widehat{\mathfrak{A}}$ is Dedekind complete and $\hat{\nu}$ is a Maharam submeasure. But now observe that the function

$$(a, b) \mapsto \hat{\nu}(a \cup b) + \hat{\nu}(a \cap b) - \hat{\nu}a - \hat{\nu}b : \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \to \mathbb{R}$$

is continuous (by (b)) and zero on the dense set $\mathfrak{A} \times \mathfrak{A}$, so is zero everywhere on $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$. In particular, $\hat{\nu}(a \cup b) = \hat{\nu}a + \hat{\nu}b$ whenever $a \cap b = 0$ in $\widehat{\mathfrak{A}}$, and $\hat{\nu}$ is additive. Since it is also countably subadditive (392Hb) it is countably additive, and $(\widehat{\mathfrak{A}}, \hat{\nu})$ is a totally finite measure algebra.

393C Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. Then there are a Dedekind complete Boolean algebra $\mathfrak{B}$, a strictly positive Maharam submeasure λ on $\mathfrak{B}$, and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\nu = \lambda\pi$. If ν is additive, then $(\mathfrak{B}, \lambda)$ is a totally finite measure algebra.

proof (a) Set $I = \{a : \nu a = 0\}$. Then $I \lhd \mathfrak{A}$; let $\mathfrak{B}_0$ be the Boolean quotient algebra $\mathfrak{A}/I$ (312K). If a_1, $a_2 \in \mathfrak{A}$ and $a_1^\bullet \subseteq a_2^\bullet$ in $\mathfrak{B}$, then $a_1 \setminus a_2 \in I$, so

$$\nu a_1 \le \nu a_2 + \nu(a_1 \setminus a_2) = \nu a_2.$$

So if $a_1^\bullet = a_2^\bullet$ we must have $\nu a_1 = \nu a_2$; there is therefore a functional $\lambda_0 : \mathfrak{B}_0 \to [0, \infty[$ such that $\lambda_0 a^\bullet = \nu a$ for every $a \in \mathfrak{A}$. We have just seen that $\lambda_0 a_1^\bullet \le \lambda_0 a_2^\bullet$ if $a_1^\bullet \subseteq a_2^\bullet$; now of course $\lambda 0^\bullet = \nu 0 = 0$,

$$\lambda_0(a_1^\bullet \cup a_2^\bullet) = \lambda_0(a_1 \cup a_2)^\bullet = \nu(a_1 \cup a_2) \le \nu a_1 + \nu a_2 = \lambda_0 a_1^\bullet + \lambda_0 a_2^\bullet$$

for all a_1, $a_2 \in \mathfrak{A}$, so λ_0 is a submeasure. If $b \in \mathfrak{B} \setminus \{0\}$, then $b = a^\bullet$ where $a \notin I$, so $\lambda_0 b = \nu a > 0$; thus λ_0 is strictly positive.

If $\langle b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}_0$, express it as $\langle a_n^\bullet \rangle_{n\in\mathbb{N}}$ and set $\tilde{a}_n = a_n \setminus \sup_{i<n} a_i$ for each n, so that $\langle \tilde{a}_n \rangle_{n\in\mathbb{N}}$ is disjoint and

$$\lim_{n\to\infty} \lambda_0 b_n = \lim_{n\to\infty} \nu \tilde{a}_n = 0$$

because ν is exhaustive. So λ_0 is exhaustive.

(b) Let $\mathfrak{B}$ be the metric completion of $\mathfrak{B}_0$ under the metric associated with λ_0 (393Bc) and λ the associated strictly positive submeasure; because λ_0 is exhaustive, $\mathfrak{B}$ is Dedekind complete and λ is a Maharam submeasure (393Bd).

Writing $\pi a = a^\bullet$, interpreted as a member of $\mathfrak{B}$, for each $a \in \mathfrak{A}$, π is a Boolean homomorphism and

$$\lambda \pi a = \lambda_0 a^\bullet = \nu a$$

for every $a \in \mathfrak{A}$.

(c) If ν is additive, then $\nu(a_1 \cup a_2) + \nu(a_1 \cap a_2) = \nu a_1 + \nu a_2$ for all a_1, $a_2 \in \mathfrak{A}$, so $\lambda_0(b_1 \cup b_2) + \lambda_0(b_1 \cap b_2) = \lambda_0 b_1 + \lambda_0 b_2$ for all b_1, $b_2 \in \mathfrak{B}_0$, and λ_0 is additive; by 393Bd, $(\mathfrak{B}, \lambda)$ is a totally finite measure algebra. This completes the proof.

393D Definition Let $\mathfrak{A}$ be a Boolean algebra and ν, ν' two submeasures on $\mathfrak{A}$. Then ν is **absolutely continuous** with respect to ν' if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\nu a \le \epsilon$ whenever $\nu' a \le \delta$.

393E Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν, ν' two Maharam submeasures on $\mathfrak{A}$ such that $\nu a = 0$ whenever $\nu' a = 0$. Then ν is absolutely continuous with respect to ν'.

proof ? Otherwise, we can find a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu' a_n \le 2^{-n}$ for every n, but $\epsilon = \inf_{n\in\mathbb{N}} \nu a_n > 0$. Set $b_n = \sup_{i\ge n} a_i$ for each n, $b = \inf_{n\in\mathbb{N}} b_n$. Then $\nu' b_n \le \sum_{i=n}^\infty 2^{-i} \le 2^{-n+1}$ for each n (392Hb), so $\nu' b = 0$; but $\nu b_n \ge \epsilon$ for each n, so $\nu b \ge \epsilon$ (392Ha), contrary to the hypothesis. **X**

393F Lemma Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then a Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ iff it is atomless, countable and not $\{0\}$.

proof (a) I must check that $\mathfrak{B}$ has the declared properties. The point is that it is the subalgebra $\mathfrak{B}'$ of $\mathcal{P}X$ generated by $\{b_i : i \in \mathbb{N}\}$, where I write $X = \{0,1\}^{\mathbb{N}}$, $b_i = \{x : x \in X, x(i) = 1\}$. **P** Of course b_i and its complement $\{x : x(i) = 0\}$ are open, so $b_i \in \mathfrak{B}$ for each i, and $\mathfrak{B}' \subseteq \mathfrak{B}$. In the other direction, the open cylinder sets of X are all of the form $c_z = \{x : x(i) = z(i) \text{ for every } i \in J\}$, where $J \subseteq I$ and $z \in \{0,1\}^J$; now

$$c_z = X \cap \bigcap_{z(i)=1} b_i \setminus \bigcup_{z(i)=0} b_i \in \mathfrak{B}'.$$

If $b \in \mathfrak{B}$ then b is expressible as a union of such cylinder sets, because it is open; but also it is compact, so is the union of finitely many of them, and must belong to $\mathfrak{B}'$. Thus $\mathfrak{B} = \mathfrak{B}'$, as claimed. **Q**

Because $\mathfrak{B} = \mathfrak{B}'$ is generated by a countable set, it is countable (331Gc). Next, if $b \in \mathfrak{B}$ is non-empty, there are a finite $J \subseteq I$ and a $z \in \{0,1\}^J$ such that $c_z \subseteq b$; now if we take any $i \in \mathbb{N} \setminus J$, and look at the two extensions z_0, z_1 of z to $J \cup \{i\}$, then c_{z_0}, c_{z_1} are disjoint non-empty members of $\mathfrak{B}$ included in b. So $\mathfrak{B}$ is atomless.

(b) Now suppose that $\mathfrak{A}$ is another algebra with the same properties. Enumerate $\mathfrak{A}$ as $\langle a_n \rangle_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{b_i : i < n\}$ (so that $\mathfrak{B}_0 = \{0,1\}$). Then $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$

is an increasing sequence of subalgebras of $\mathfrak{B}$ with union $\mathfrak{B}$; also $b \cap b_n$, $b \setminus b_n$ are non-zero for every $n \in \mathbb{N}$, $b \in \mathfrak{B}_n$.

Choose finite subalgebras $\mathfrak{A}_n \subseteq \mathfrak{A}$ and isomorphisms $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ as follows. $\mathfrak{A}_0 = \{0, 1\}$, $\pi_0 0 = 0$, $\pi_0 1 = 1$. Given $\mathfrak{A}_n$ and π_n, let A_n be the set of atoms of $\mathfrak{A}_n$. For $a \in A_n$, choose $a' \in \mathfrak{A}$ such that

if $a_n \cap a$, $a_n \setminus a$ are both non-zero, then $a' = a_n \cap a$;

otherwise, $a' \subseteq a$ is any element such that a', $a \setminus a'$ are both non-zero.

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Set $a'_n = \sup_{a \in A_n} a'$. Then we see that $a \cap a'_n$, $a \setminus a'_n$ are non-zero for every $a \in A_n$ and therefore for every non-zero $a \in \mathfrak{A}_n$, that is, that

$$\sup\{a : a \in \mathfrak{A}_n,\ a \subseteq a'_n\} = 0, \quad \inf\{a : a \in \mathfrak{A}_n,\ a \supseteq a'_n\} = 1.$$

Let $\mathfrak{A}_{n+1}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n \cup \{a'_n\}$. Since we have

$$\sup\{b : b \in \mathfrak{B}_n,\ b \subseteq b_n\} = 0, \quad \inf\{b : b \in \mathfrak{B}_n,\ b \supseteq b_n\} = 1,$$

there is a (unique) extension of $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ to a homomorphism $\pi_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{B}_{n+1}$ such that $\pi_{n+1} a'_n = b_n$ (312N). Since we similarly have an extension ϕ of π_n^{-1} to a homomorphism from $\mathfrak{B}_{n+1}$ to $\mathfrak{A}_{n+1}$ with $\phi b_n = a'_n$, and since $\phi\pi_{n+1}$, $\pi_{n+1}\phi$ must be the respective identity homomorphisms, π_{n+1} is an isomorphism, and the induction continues.

Since π_{n+1} extends π_n for each n, these isomorphisms join together to give us an isomorphism

$$\pi : \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n \to \bigcup_{n \in \mathbb{N}} \mathfrak{B}_n = \mathfrak{B}.$$

Observe next that the construction ensures that $a_n \in \mathfrak{A}_{n+1}$ for each n, since $a_n \cap a$ is either 0 or a or $a'_n \cap a$ for every $a \in A_n$, and in all cases belongs to $\mathfrak{A}_{n+1}$. So $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ contains every a_n and (by the choice of $\langle a_n \rangle_{n \in \mathbb{N}}$) must be the whole of $\mathfrak{A}$. Thus $\pi : \mathfrak{A} \to \mathfrak{B}$ witnesses that $\mathfrak{A} \cong \mathfrak{B}$.

393G Theorem If one of CM_1, CM'_1, CM''_1, CM_2, CM'_2 is true, so are the others.

proof $\mathbf{CM_1 \Rightarrow CM'_1}$ Assume that CM_1 is true, and that $\mathfrak{A}$ and ν are as in the statement of CM'_1. Set $I = \{a : \nu a = 0\}$; because ν is countably subadditive, I is a σ-ideal, $\mathfrak{B} = \mathfrak{A}/I$ is Dedekind σ-complete and the quotient map $a \mapsto a^\bullet$ is sequentially order-continuous (313Qb, 314C). Now we have a functional $\lambda : \mathfrak{B} \to [0, \infty[$ defined by saying that $\lambda a^\bullet = \nu a$ for every $a \in \mathfrak{A}$, and λ is a strictly positive submeasure (as in part (a) of the proof of 393C). In fact λ is a Maharam submeasure. **P** Suppose that $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{B}$ with infimum 0. Choose $a_n \in \mathfrak{A}$ such that $a_n^\bullet = b_n$ for each n, and set

$$c_n = \inf_{i \le n} a_i \setminus \inf_{i \in \mathbb{N}} a_i;$$

then $\langle c_n \rangle_{n \in \mathbb{N}}$ is non-increasing and has infimum 0, while $c_n^\bullet = b_n$ for each n, so

$$\lim_{n \to \infty} \lambda b_n = \lim_{n \to \infty} \nu c_n = 0.$$

As $\langle b_n \rangle_{n \in \mathbb{N}}$ is arbitrary, λ is a Maharam submeasure. **Q**

Now CM_1 tells us that $\mathfrak{B}$ must be a measurable algebra, that is, carries a totally finite measure $\bar\mu$. Setting $\mu a = \bar\mu a^\bullet$ for $a \in \mathfrak{A}$, we see that μ is a non-negative countably additive functional, and that

$$\mu a = 0 \iff a^\bullet = 0 \iff \lambda a = 0,$$

as required.

$\mathbf{CM'_1 \Rightarrow CM''_1}$ is trivial; allowing for the change in notation, CM''_1 is just a special case of CM'_1.

$\mathbf{CM''_1 \Rightarrow CM_1}$ Assume CM''_1, and let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure ν. Then we can express $\mathfrak{A}$ as $\Sigma/\mathcal{I}$ for some σ-algebra Σ of subsets of a set X and a σ-ideal $\mathcal{I}$ of Σ (314M). Set $\lambda E = \nu E^\bullet$ for $E \in \Sigma$; then X, Σ, λ satisfy the conditions of CM''_1, so there is a totally finite measure μ on X, with domain Σ, such that

$$\mu E = 0 \iff \lambda E = 0 \iff \nu E^\bullet = 0 \iff E^\bullet = 0.$$

Consequently we can identify $\mathfrak{A}$ with the measure algebra of (X, Σ, μ), and $\mathfrak{A}$ is measurable.

$\mathbf{CM_1 \Rightarrow CM_2}$ Now suppose that CM_1 is true, and that ν is an exhaustive submeasure on a Boolean algebra $\mathfrak{A}$. By 393B, we can find a Dedekind complete Boolean algebra $\mathfrak{B}$ with a strictly positive Maharam submeasure λ and a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $\nu a=\lambda\pi a$ for every $a\in\mathfrak{A}$. CM_1 assures us that $\mathfrak{B}$ is measurable; let $\bar\mu$ be a totally finite measure on $\mathfrak{B}$.

Let $\epsilon>0$. By 393E, λ is absolutely continuous with respect to $\bar\mu$, so there is a $\delta>0$ such that $\lambda b\le\epsilon$ whenever $\bar\mu a\le\delta$. Take $n\ge\frac{1}{\delta}\bar\mu 1$. If $a_0,\dots,a_n$ are disjoint in $\mathfrak{A}$, then $\pi a_0,\dots,\pi a_n$ are disjoint in $\mathfrak{B}$, so there is some i such that $\bar\mu\pi a_i\le\delta$, in which case $\nu a_i=\lambda\pi a_i\le\epsilon$. As ϵ is arbitrary, ν is uniformly exhaustive; as $\mathfrak{A}$ and ν are arbitrary, CM_2 is true.

$\mathbf{CM_2\Rightarrow CM_2'}$ is trivial.

$\mathbf{CM_2'\Rightarrow CM_1}$ Suppose that CM_2' is true, that is, every exhaustive submeasure on the algebra $\mathfrak{B}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ is uniformly exhaustive, and that ν is a strictly positive Maharam submeasure on a Dedekind complete Boolean algebra $\mathfrak{A}$. Let E be the set of atoms of $\mathfrak{A}$, and set $a=1\setminus\sup E$, so that the principal ideal $\mathfrak{A}_a$ is atomless, and $\nu\restriction\mathfrak{A}_a$ is still a strictly positive Maharam submeasure.

? Suppose, if possible, that $\nu\restriction\mathfrak{A}_a$ is not uniformly exhaustive. Then there is a family $\langle a_{ni}\rangle_{i\le n\in\mathbb{N}}$ in $\mathfrak{A}_a$ such that $\langle a_{ni}\rangle_{i\le n}$ is disjoint for each n but $\inf_{i\le n\in\mathbb{N}}\nu a_{ni}>0$. There is a countable atomless subalgebra $\mathfrak{D}$ of $\mathfrak{A}_a$ containing every a_{ni}. **P** For each $d\in\mathfrak{A}_a$ fix on a $d'\subseteq d$ such that d' and $d\setminus d'$ are both non-zero. Define $\langle D_n\rangle_{n\in\mathbb{N}}$ by setting

$$D_0=\{0,a\}\cup\{a_{ni}:i\le n\in\mathbb{N}\},$$

$$D_{n+1}=D_n\cup\{d_1\cap d_2:d_1,\,d_2\in D_n\}\cup\{a\setminus d:d\in D_n\}\cup\{d':d\in D_n\}.$$

Then every D_n is countable, so $\mathfrak{D}=\bigcup_{n\in\mathbb{N}}D_n$ is countable. Because $\mathfrak{D}$ includes D_0, it contains all the a_{ni}; also $d_1\cap d_2$, $1\setminus d_1$ and d_1' belong to $\mathfrak{D}$ for every d_1, $d_2\in\mathfrak{D}$, so $\mathfrak{D}$ is an atomless subalgebra. **Q**

Because ν is an exhaustive submeasure on $\mathfrak{A}$, $\nu\restriction\mathfrak{D}$ is an exhaustive submeasure on $\mathfrak{D}$. Because every a_{ni} belongs to $\mathfrak{D}$ and $\inf_{i\le n\in\mathbb{N}}\nu a_{ni}>0$, $\nu\restriction\mathfrak{D}$ is not uniformly exhaustive.

Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. By 393F, there is a Boolean isomorphism $\pi:\mathfrak{B}\to\mathfrak{D}$. Set $\lambda b=\nu\pi b$ for every $b\in\mathfrak{B}$; then $(\mathfrak{B},\lambda)$ is isomorphic to $(\mathfrak{D},\nu\restriction\mathfrak{D})$, and, in particular, λ is an exhaustive submeasure on $\mathfrak{B}$ which is not uniformly exhaustive, which we are supposing to be impossible. **X**

Thus $\mathfrak{A}_a$ has a uniformly exhaustive Maharam submeasure; since it is surely Dedekind complete, it is measurable (392J). Let $\bar\mu_1$ be a totally finite measure on $\mathfrak{A}_a$. Next, because $\lim_{n\to\infty}\nu e_n=0$ for any sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ of distinct elements of E, and $\nu e>0$ for every $e\in E$, E must be countable, and there is a summable family $\langle\alpha_e\rangle_{e\in E}$ of strictly positive real numbers. Setting $\bar\mu c=\bar\mu_1(c\cap a)+\sum_{e\in E,e\subseteq c}\alpha_e$, we get a totally finite measure $\bar\mu$ on $\mathfrak{A}$, and $\mathfrak{A}$ is measurable.

393H Variations on $\mathbf{CM_2}$ The following modifications of CM_2 are interesting because they display some general properties of exhaustive submeasures. Consider the statements

($\mathrm{CM}_{3\mathrm{A}}$) If $\mathfrak{A}$ is a Boolean algebra and ν is a non-zero exhaustive submeasure on $\mathfrak{A}$, then there is a non-zero finitely additive functional μ on $\mathfrak{A}$ such that $0\le\mu a\le\nu a$ for every $a\in\mathfrak{A}$.

($\mathrm{CM}_{3\mathrm{A}}'$) If $\mathfrak{A}$ is a Boolean algebra and ν is a non-zero exhaustive submeasure on $\mathfrak{A}$, then there is a non-zero non-negative finitely additive functional μ on $\mathfrak{A}$ such that μ is absolutely continuous with respect to ν.

($\mathrm{CM}_{3\mathrm{B}}$) If $\mathfrak{A}$ is a Boolean algebra and ν is an exhaustive submeasure on $\mathfrak{A}$, then there is a non-negative finitely additive functional μ on $\mathfrak{A}$ such that ν is absolutely continuous with respect to μ.

393I Proposition If one of $\mathrm{CM}_1,\dots,\mathrm{CM}_2'$, $\mathrm{CM}_{3\mathrm{A}}$, $\mathrm{CM}_{3\mathrm{A}}'$, $\mathrm{CM}_{3\mathrm{B}}$ is true, so are the others.

proof $\mathbf{CM_1\Rightarrow CM_{3A}}$ & $\mathbf{CM_{3B}}$ Assume CM_1, and let ν be a non-zero exhaustive submeasure on a Boolean algebra $\mathfrak{A}$. By 393C, there are a Dedekind complete Boolean algebra $\mathfrak{B}$, a strictly positive Maharam submeasure λ on $\mathfrak{B}$, and a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $\nu=\lambda\pi$. Since $\lambda 1=\nu 1\ne 0$, $\mathfrak{B}\ne\{0\}$. CM_1 tells us that $\mathfrak{B}$ is measurable; let $\bar\mu$ be a strictly positive countably additive functional on $\mathfrak{B}$, and set $\mu_1=\bar\mu\pi$, so that μ_1 is a finitely additive functional on $\mathfrak{A}$. By 393E, $\bar\mu$ is absolutely continuous with respect to λ and λ is absolutely continuous with respect to $\bar\mu$; it follows from the latter that ν is absolutely continuous with respect to μ_1.

Set $\delta = \frac{1}{2}\lambda 1/\bar{\mu}1$, and consider $D = \{b : b \in \mathfrak{B}, \lambda b \le \delta\bar{\mu}d\}$. **?** If D is order-dense in $\mathfrak{B}$, there is a disjoint $C \subseteq D$ such that $\sup C = 1$. Because $\mathfrak{B}$ is measurable, it is ccc, and C is countable. If C is infinite, enumerate it as $\langle c_n\rangle_{n\in\mathbb{N}}$; if it is finite, enumerate it as $\langle c_i\rangle_{i\le n}$ and set $c_i = 0$ for $i > n$. In either case, $\langle c_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in C with supremum 1, so that

$$\lambda 1 \le \sum_{n=0}^{\infty} \lambda c_n \le \delta \sum_{n=0}^{\infty} \bar{\mu} c_n = \delta\bar{\mu}1 < \lambda 1$$

by the choice of δ, which is absurd. **X**

There is therefore a non-zero $b \in \mathfrak{B}$ such that $\lambda c > \delta\bar{\mu}c$ for every non-zero $c \subseteq b$. Set

$$\mu_2 a = \delta\bar{\mu}(b \cap \pi a) \le \lambda(b \cap \pi a) \le \lambda\pi a = \nu a$$

for $a \in \mathfrak{A}$; then μ_2 is a non-negative finitely additive functional on $\mathfrak{A}$, $\mu_2 \le \nu$, and $\mu_2 1 = \delta\bar{\mu}b \ne 0$, so μ_2 is non-zero.

As $\mathfrak{A}$ and ν are arbitrary, $\mathrm{CM}_{3\mathrm{A}}$ and $\mathrm{CM}_{3\mathrm{B}}$ are both true.

(The statement of $\mathrm{CM}_{3\mathrm{B}}$ does not assume ν to be non-zero, but of course the case $\nu = 0$ is trivial.)

$\mathbf{CM_{3A}} \Rightarrow \mathbf{CM'_{3A}}$ is trivial.

$\mathbf{CM'_{3A}} \Rightarrow \mathbf{CM_1}$ Assume $\mathrm{CM}'_{3\mathrm{A}}$, and let ν be a strictly positive Maharam submeasure on a Dedekind complete Boolean algebra $\mathfrak{A}$. Note that $\mathfrak{A}$ is ccc (392Ca). Now $C = \{a : a \in \mathfrak{A}, \mathfrak{A}_a \text{ is measurable}\}$ is order-dense in $\mathfrak{A}$. **P** Take any $a \in \mathfrak{A} \setminus \{0\}$. Set $\nu_a b = \nu(a \cap b)$ for $b \in \mathfrak{A}$. It is easy to check that ν_a is a non-zero Maharam submeasure on $\mathfrak{A}$; in particular, it is exhaustive. So there is a non-zero non-negative finitely additive functional μ which is absolutely continuous with respect to ν_a. If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty} \nu_a b_n = 0$, so $\lim_{n\to\infty} \mu b_n = 0$; accordingly μ is countably additive (326Ga), therefore completely additive, because $\mathfrak{A}$ is ccc (326L). Set $I = \{b : \mu b = 0\}$; then I is an order-closed ideal, so contains its supremum b_0 say. Since $\mu b = 0$ whenever $\nu_a b = 0$, $b_0 \supseteq 1 \setminus a$, and $c \subseteq a$, where $c = 1 \setminus b_0$. But $\mathfrak{A}_c \cap I = \{0\}$, so $\mu{\restriction}\mathfrak{A}_c$ is strictly positive, and witnesses that $\mathfrak{A}_c$ is measurable. Also $\mu c = \mu 1 \ne 0$, so c is a non-zero member of C included in a. As a is arbitrary, C is order-dense. **Q**

By 391Cb, there is a function $\bar{\mu}_1$ such that $(\mathfrak{A}, \bar{\mu}_1)$ is a localizable measure algebra. Because $\mathfrak{A}$ is ccc, $\mathfrak{A}$ is measurable (322G).

$\mathbf{CM_{3B}} \Rightarrow \mathbf{CM_1}$ Assume $\mathrm{CM}_{3\mathrm{B}}$, and let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. By $\mathrm{CM}_{3\mathrm{B}}$, there is a non-negative finitely additive functional μ on $\mathfrak{A}$ such that ν is absolutely continuous with respect to μ. But this means that

$$\mu a = 0 \Longrightarrow \nu a = 0 \Longrightarrow a = 0,$$

that is, μ is strictly positive. Also $\mathfrak{A}$ is Dedekind complete and weakly (σ,∞)-distributive, by 392I. So $\mathfrak{A}$ is measurable, by 391D. As $\mathfrak{A}$ and ν are arbitrary, CM_1 is true.

393J The first published version of the Control Measure Problem (Maharam 47), in the form 'is CM_1 true?', was a re-formulation of a question about topologies on Boolean algebras, which I now describe. Consider the statement

(CM_4) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra with a Hausdorff topology $\mathfrak{T}$ such that (i) the Boolean operation $\cup : \mathfrak{A}\times\mathfrak{A} \to \mathfrak{A}$ is continuous at $(0,0)$ (ii) if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n\rangle_{n\in\mathbb{N}}$ converges to 0 for $\mathfrak{T}$. Then $\mathfrak{A}$ is measurable.

393K Proposition CM_4 is true iff $\mathrm{CM}_1, \dots, \mathrm{CM}_{3\mathrm{B}}$ are true.

proof $\mathbf{CM_1} \Rightarrow \mathbf{CM_4}$ Assume CM_1, and take $\mathfrak{A}$, $\mathfrak{T}$ as in the statement of CM_4.

(i) For any $e \in \mathfrak{A} \setminus \{0\}$, there is a Maharam submeasure ν on $\mathfrak{A}$ such that $\nu e > 0$.

$\mathbf{P}(\boldsymbol{\alpha})$ Choose a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0, as follows. Because $\mathfrak{T}$ is Hausdorff, there is a neighbourhood G_0 of 0 not containing e. Given G_n, choose a neighbourhood G_{n+1} of 0 such that $G_{n+1} \subseteq G_n$ and $a \cup b \cup c \in G_n$ whenever a, b, $c \in G_{n+1}$. (Take neighbourhoods H, H' of 0 such that $a \cup b \in G_n$ for a, $b \in H$, $b \cup c \in H$ for b, $c \in H'$ and set $G_{n+1} = H \cap H' \cap G_n$.) Define $\nu_0 : \mathfrak{A} \to [0,1]$ by setting

$$\begin{aligned}\nu_0 a &= 1 \text{ if } a \notin G_0,\\ &= 2^{-n} \text{ if } a \in G_n \setminus G_{n+1},\\ &= 0 \text{ if } a \in \bigcap_{n\in\mathbb{N}} G_n.\end{aligned}$$

Then whenever $a_0, \dots , a_r \in \mathfrak{A}$, $n \in \mathbb{N}$ and $\sum_{i=0}^{r} \nu_0 a_i < 2^{-n}$, $\sup_{i\le r} a_i \in G_n$. To see this, induce on r. If $r = 0$ then we have $\nu_0 a_0 < 2^{-n}$ so $a_0 \in G_{n+1} \subseteq G_n$. For the inductive step to $r \ge 1$, there must be a $k \le r$ such that $\sum_{i<k} \nu_0 a_i < 2^{-n-1}$, $\sum_{k<i\le n} \nu_0 a_i < 2^{-n-1}$ (allowing $k = 0$ or $k = n$, in which case one of the sums will be zero). (If $\sum_{i=0}^{r} \nu_0 a_i < 2^{-n-1}$, take $k = n$; otherwise, take k to be the least number such that $\sum_{i=0}^{k} \nu_0 a_i \ge 2^{-n-1}$.) By the inductive hypothesis, and because 0 certainly belongs to G_{n+1}, $b = \sup_{i<k} a_i$ and $c = \sup_{k<i\le r} a_i$ both belong to G_{n+1}; but also $\nu_0 a_k < 2^{-n}$ so $a_k \in G_{n+1}$. Accordingly, by the choice of G_{n+1},

$$\sup_{i\le r} a_i = b \cup a_k \cup c$$

belongs to G_n, and the induction continues.

(β) Set

$$\nu_1 a = \inf\{\textstyle\sum_{i=0}^{r} \nu_0 a_i : a_0, \dots , a_r \in \mathfrak{A},\ a = \sup_{i\le r} a_i\}$$

for every $a \in \mathfrak{A}$. It is easy to see that $\nu_1(a \cup b) \le \nu_1 a + \nu_1 b$ for all $a, b \in \mathfrak{A}$; also $a \in G_n$ whenever $\nu_1 a < 2^{-n}$, so, in particular, $\nu_1 e \ge 1$, because $e \notin G_0$.

Set

$$\nu a = \inf\{\nu_1 b : a \cap e \subseteq b \subseteq e\}$$

for every $a \in \mathfrak{A}$. Then of course $0 \le \nu a \le \nu b$ whenever $a \subseteq b$, and

$$\nu 0 \le \nu_1 0 \le \nu_0 0 = 0,$$

so $\nu 0 = 0$. If $a, b \in \mathfrak{A}$ and $\epsilon > 0$, there are a', b' such that $a \cap e \subseteq a' \subseteq e$, $b \cap e \subseteq b' \subseteq e$, $\nu_1 a' \le \nu a + \epsilon$ and $\nu_1 b' \le \nu b + \epsilon$; so that $(a \cup b) \cap e \subseteq a' \cup b' \subseteq e$ and

$$\nu(a \cup b) \le \nu_1(a' \cup b') \le \nu_1 a' + \nu_1 b' \le \nu a + \nu b + 2\epsilon.$$

As ϵ, a and b are arbitrary, ν is a submeasure. Next, if $\langle a_i\rangle_{i\in\mathbb{N}}$ is any non-increasing sequence in $\mathfrak{A}$ with infimum 0, $\langle a_i \cap e\rangle_{i\in\mathbb{N}}$ is another, so converges to 0 for $\mathfrak{T}$. If $n \in \mathbb{N}$ there is an m such that $a_i \cap e \in G_n$ for every $i \ge m$, so that

$$\nu a_i \le \nu_1(a_i \cap e) \le \nu_0(a_i \cap e) \le 2^{-n}$$

for every $i \ge m$. As n is arbitrary, $\lim_{i\to\infty} \nu a_i = 0$; as $\langle a_i\rangle_{i\in\mathbb{N}}$ is arbitrary, ν is a Maharam submeasure. Finally,

$$\nu e = \nu_1 e \ge 1,$$

so $\nu e \ne 0$. **Q**

(ii) Write C for the set of those $c \in \mathfrak{A}$ such that $\mathfrak{A}_c$ is a measurable algebra. Then C is order-dense in $\mathfrak{A}$. **P** Take any $e \in \mathfrak{A} \setminus \{0\}$. By (i), there is a Maharam submeasure ν such that $\nu e > 0$. Set $I = \{a : \nu a = 0\}$, $a^* = \sup I$. Because $\mathfrak{A}$ is ccc, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in I such that $a^* = \sup_{n\in\mathbb{N}} a_n$; because ν is countably subadditive, $\nu a^* = 0$. Consider $c = e \setminus a^*$. Then $\nu\restriction\mathfrak{A}_c$ is a strictly positive Maharam submeasure, so CM_1 assures us that $\mathfrak{A}_c$ is measurable; also $c \ne 0$ because $\nu c = \nu e \ne 0$, so $0 \ne c \subseteq e$ and $c \in C$. As e is arbitrary, C is order-dense. **Q**

Now $\mathfrak{A}$ is a ccc, Dedekind complete Boolean algebra in which the measurable principal ideals are order-dense, so $\mathfrak{A}$ is measurable, just as in the proof of $\mathrm{CM}'_{3\mathrm{A}} \Rightarrow \mathrm{CM}_1$ (393I)

$\mathbf{CM_4 \Rightarrow CM_1}$ Now assume that CM_4 is true, and that $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure ν. Then the associated metric ρ on $\mathfrak{A}$ (393B) defines a topology $\mathfrak{T}$ satisfying the conditions of CM_4. **P** The continuity of $\cup$ is mentioned in 393B. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, then $\rho(a_n, 0) = \nu a_n \to 0$ so $a_n \to 0$ for $\mathfrak{T}$. **Q** By CM_4, $\mathfrak{A}$ is measurable.

393L My own first encounter with the Control Measure Problem was in the course of investigating topological Riesz spaces. For various questions depending on this problem, see FREMLIN 75. I give the following as a sample.

(CM_5) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra, and suppose that there is a Hausdorff linear space topology $\mathfrak{T}$ on $L^0(\mathfrak{A})$ such that for every neighbourhood G of 0 there is a neighbourhood H of 0 such that $u \in G$ whenever $v \in H$ and $|u| \le |v|$. Then $\mathfrak{A}$ is measurable.

393M Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ a strictly positive Maharam submeasure. For $u \in L^0 = L^0(\mathfrak{A})$ set

$$\tau(u) = \inf\{\alpha : \alpha \ge 0,\, \nu[\![|u| > \alpha]\!] \le \alpha\}.$$

Then τ defines a metrizable linear space topology $\mathfrak{T}$ on L^0 such that for every neighbourhood G of 0 there is a neighbourhood H of 0 such that $u \in G$ whenever $v \in H$ and $|u| \le |v|$.

proof (a) The point is that

$$\tau(u+v) \le \tau(u) + \tau(v), \quad \tau(\alpha u) \le \tau(u) \text{ if } |\alpha| \le 1, \quad \lim_{\alpha\to 0} \tau(\alpha u) = 0$$

for every u, $v \in L^0$. **P** (i) It will save a moment if we observe that whenever $\beta > \tau(u)$ there is an $\alpha \le \beta$ such that $\nu[\![|u| > \alpha]\!] \le \alpha$, so that

$$\nu[\![|u| > \beta]\!] \le \nu[\![|u| > \alpha]\!] \le \alpha \le \beta.$$

Also, because ν is sequentially order-continuous,

$$\nu[\![|u| > \tau(u)]\!] = \lim_{n\to\infty} \nu[\![|u| > \tau(u) + 2^{-n}]\!] \le \lim_{n\to\infty} \tau(u) + 2^{-n} = \tau(u).$$

(ii) So

$$\nu[\![|u+v| > \tau(u) + \tau(v)]\!] \le \nu[\![|u| + |v| > \tau(u) + \tau(v)]\!]) \le \nu([\![|u| > \tau(u)]\!] \cup [\![|v| > \tau(v)]\!])$$

(364Fa)

$$\le \nu[\![|u| > \tau(u)]\!] + \nu[\![|v| > \tau(v)]\!] \le \tau(u) + \tau(v),$$

and $\tau(u+v) \le \tau(u) + \tau(v)$. (iii) If $|\alpha| \le 1$ then

$$\nu[\![|\alpha u| > \tau(u)]\!] \le \nu[\![|u| > \tau(u)]\!] \le \tau(u),$$

and $\tau(\alpha u) \le \tau(u)$. (iv) $\lim_{n\to\infty} \nu[\![|u| > n]\!] = 0$ because $\langle [\![|u| > n]\!] \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0. So if $\epsilon > 0$, there is an $n \ge 1$ such that $\nu[\![|u| > n\epsilon]\!] \le \epsilon$, in which case $\nu[\![|\alpha u| > \epsilon]\!] \le \epsilon$ whenever $|\alpha| \le \frac{1}{n}$, so that $\tau(\alpha u) \le \epsilon$ whenever $|\alpha| \le \frac{1}{n}$. As ϵ is arbitrary, $\lim_{\alpha\to 0} \tau(\alpha u) = 0$. **Q**

(b) Accordingly we have a metric $(u, v) \mapsto \tau(u-v)$ which defines a linear space topology $\mathfrak{T}$ on L^0 (2A5B). Now let G be an open set containing 0. Then there is an $\epsilon > 0$ such that $H = \{u : \tau(u) < \epsilon\}$ is included in G. If $v \in H$ and $|u| \le |v|$, then

$$\nu[\![|u| > \tau(v)]\!] \le \nu[\![|v| > \tau(v)]\!] \le \tau(v),$$

so $\tau(u) \le \tau(v)$ and $u \in H \subseteq G$. So $\mathfrak{T}$ satisfies all the conditions.

393N Proposition CM_5 is true iff $CM_1, \dots, CM_4$ are true.

proof $CM_4 \Rightarrow CM_5$ Assume CM_4. Let $\mathfrak{A}$, $\mathfrak{T}$ be as in the statement of CM_5. Let $\mathfrak{S}$ be the topology on $\mathfrak{A}$ induced by $\mathfrak{T}$ and the function $\chi : \mathfrak{A} \to L^0$; that is, $\mathfrak{S} = \{\chi^{-1}[G] : G \in \mathfrak{T}\}$. Then $\mathfrak{S}$ satisfies the conditions of CM_4. **P** (i) Because $\mathfrak{T}$ is Hausdorff and χ is injective, $\mathfrak{S}$ is Hausdorff. (ii) If $0 \in G \in \mathfrak{S}$, there is an $H \in \mathfrak{T}$ such that $G = \chi^{-1}[H]$. Now 0 (the zero of L^0) belongs to H, so there is an open set H_1 containing 0 such that $u \in H$ whenever $v \in H_1$ and $|u| \le |v|$. Next, addition on L^0 is continuous for $\mathfrak{T}$, so there is an open set H_2 containing 0 such that $u + v \in H_1$ whenever u, $v \in H_2$. Consider $G' = \chi^{-1}[H_2]$. This is an open set in $\mathfrak{A}$ containing 0, and if a, $b \in G'$ then

$$|\chi(a \cup b)| \le \chi a + \chi b \in H_2 + H_2 \subseteq H_1,$$

so $\chi(a\cup b)\in H$ and $a\cup b\in G$. As G is arbitrary, $\cup$ is continuous at $(0,0)$. (iii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, $u_0=\sup_{n\in\mathbb{N}} n\chi a_n$ is defined in L^0 (use the criterion of 364Ma:

$$\inf_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}}[\![n\chi a_n>m]\!]=\inf_{m\in\mathbb{N}}a_{m+1}=0.)$$

If $0\in G\in\mathfrak{S}$, take $H\in\mathfrak{T}$ such that $G=\chi^{-1}[H]$, and $H_1\in\mathfrak{T}$ such that $0\in H_1$ and $u\in H$ whenever $v\in H_1$, $|u|\le|v|$. Because scalar multiplication is continuous for $\mathfrak{T}$, there is a $k\ge 1$ such that $\frac{1}{k}u_0\in H_1$. For any $n\ge k$, $\chi a_n\le\frac{1}{k}u_0$ so $\chi a_n\in H$ and $a_n\in G$. As G is arbitrary, $\langle a_n\rangle_{n\in\mathbb{N}}\to 0$ for $\mathfrak{S}$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, condition (ii) in the statement of CM_4 is satisfied. **Q**

Since CM_4 is true, $\mathfrak{A}$ must be a measurable algebra.

$\mathbf{CM_5\Rightarrow CM_1}$ Assume that CM_5 is true, and let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure ν. By 393M, $L^0=L^0(\mathfrak{A})$ has a topology satisfying the conditions of CM_5, so $\mathfrak{A}$ is measurable.

393O The phrase ‘control measure’ derives, in fact, from none of the formulations above; it belongs to the theory of vector measures, as follows.

Definitions (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Hausdorff linear topological space. (The idea is intended to apply, in particular, when $\mathfrak{A}$ is a σ-algebra of subsets of a set.) A function $\theta:\mathfrak{A}\to U$ is a **vector measure** if $\sum_{n=0}^{\infty}\theta a_n=\lim_{n\to\infty}\sum_{i=0}^{n}\theta a_i$ is defined in U and equal to $\theta(\sup_{n\in\mathbb{N}}a_n)$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

(b) In this case, a non-negative countably additive functional $\mu:\mathfrak{A}\to[0,\infty[$ is a **control measure** for θ if $\theta a=0$ whenever $\mu a=0$.

393P Now I can formulate the last of this string of statements equiveridical to CM_1. Consider the statement

(CM_6) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, U is a metrizable linear topological space, and $\theta:\mathfrak{A}\to U$ is a vector measure, then θ has a control measure.

393Q Lemma If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, U a Hausdorff linear topological space, and $\theta:\mathfrak{A}\to U$ a vector measure, then $\lim_{n\to\infty}\theta a_n=0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

proof $\theta a_n=\sum_{i=n}^{\infty}\theta(a_i\setminus a_{i+1})$ for every n.

393R Proposition CM_6 is true iff $\mathrm{CM}_1,\dots,\mathrm{CM}_5$ are true.

proof $\mathbf{CM_1'\Rightarrow CM_6}$ **(i)** Assume CM_1', and let $\mathfrak{A}$, U and θ be as in the statement of CM_6. Then there is a functional $\tau:U\to[0,\infty[$ such that, for u, $v\in U$, $\tau(u+v)\le\tau(u)+\tau(v)$, $\tau(\alpha u)\le\tau(u)$ whenever $|\alpha|\le 1$, $\lim_{\alpha\to 0}\tau(\alpha u)=0$, $\tau(u)=0$ iff $u=0$, and the topology of U is defined by the metric $(u,v)\mapsto\tau(u-v)$ (2A5Cb). For $a\in\mathfrak{A}$ set $\nu a=\sup_{b\subseteq a}\min(1,\tau(\theta b))$.

(ii) Consider the functional $\nu:\mathfrak{A}\to[0,\infty[$. (α)

$$\nu 0=\tau(\theta 0)=\tau(0)=0.$$

(To see that $\theta 0=0$ set $a_n=0$ for every n in the definition 393Oa.) (β) If $a\subseteq b$ then of course $\nu a\le\nu b$. (γ) If a, $b\in\mathfrak{A}$ and $c\subseteq a\cup b$, then

$$\begin{aligned}\min(1,\tau(\theta c))&=\min(1,\tau(\theta(c\cap a)+\theta(c\setminus a)))\\&\le\min(1,\tau(\theta(c\cap a))+\tau(\theta(c\setminus a)))\le\nu a+\nu b.\end{aligned}$$

As c is arbitrary, $\nu(a\cup b)\le\nu a+\nu b$. Thus ν is a submeasure. (δ) **?** Suppose, if possible, that there is a non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, with infimum 0, such that $\langle\nu(a_n)\rangle_{n\in\mathbb{N}}$ does not converge to 0. Because $\langle\nu(a_n)\rangle_{n\in\mathbb{N}}$ is non-increasing, $\epsilon=\frac{1}{3}\inf_{n\in\mathbb{N}}\nu a_n$ is greater than 0. Now for each $n\in\mathbb{N}$ there are $m>n$, b such that $b\subseteq a_n\setminus a_m$ and $\tau(\theta b)\ge\epsilon$. **P** As $\nu a_n\ge 3\epsilon$ there is a $c\subseteq a_n$ such that $\tau(\theta c)\ge 2\epsilon$. Now $\langle\theta(a_m\cap c\rangle_{m\in\mathbb{N}}$ converges to 0 in U, by 393Q, so there is an $m>n$ such that

$$\tau(\theta(c \setminus a_m)) = \tau(\theta c - \theta(a_m \cap c)) \geq \tau(\theta c) - \tau(\theta(a_m \cap c)) \geq \epsilon,$$

and we can take $b = c \setminus a_m$. **Q**

We may therefore choose $\langle b_k \rangle_{k\in\mathbb{N}}$, $\langle n(k) \rangle_{k\in\mathbb{N}}$ such that $n(k+1) > n(k)$, $b_k \subseteq a_{n(k)} \setminus a_{n(k+1)}$ and $\tau(\theta b_k) \geq \epsilon$ for every k. But $\langle b_k \rangle_{k\in\mathbb{N}}$ is disjoint, so we ought to be able to form $\sum_{k=0}^{\infty} \theta b_k = \theta(\sup_{k\in\mathbb{N}} b_k)$ in U, and $\lim_{k\to\infty} \theta b_k = 0$ in U, that is, $\lim_{k\to\infty} \tau(\theta b_k) = 0$. **X**

Thus $\lim_{n\to\infty} \nu a_n = 0$ for every non-increasing sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ with infimum 0, and ν is a Maharam submeasure.

(iii) By CM_1', there is a non-negative countably additive functional $\mu : \mathfrak{A} \to [0, \infty[$ such that $\nu a = 0$ whenever $\mu a = 0$. In particular, if $\mu a = 0$, then $\tau(\theta a) = 0$ and $\theta a = 0$. So μ is a control measure for θ. As $\mathfrak{A}$, U and θ are arbitrary, CM_6 is true.

$\mathbf{CM_6} \Rightarrow \mathbf{CM_1}$ Assume CM_6, and let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure ν. Give $L^0 = L^0(\mathfrak{A})$ the topology of 393M. Then $\chi : \mathfrak{A} \to L^0$ is a vector measure in the sense of 393O. **P** If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n = \sup_{i\leq n} a_i$, so that $\chi b_n = \sum_{i=0}^{n} \chi a_i$ for each n. We have $\nu(a \setminus b_n) \to 0$, so that

$$\tau(\chi a - \chi b_n)) = \min(1, \nu(a \setminus b_n)) \to 0,$$

where τ is the functional of 393M, and $\chi a = \sum_{i=0}^{\infty} \chi a_i$ in L^0. **Q**

CM_6 now assures us that there is a non-negative countably additive functional μ on $\mathfrak{A}$ such that

$$\mu a = 0 \Longrightarrow \chi a = 0 \Longrightarrow a = 0,$$

so that μ is strictly positive and $\mathfrak{A}$ is measurable. As $\mathfrak{A}$ and ν are arbitrary, CM_1 is true.

***393S** I must not go any farther without remarking that the generality of the phrase 'metrizable linear topological space' in CM_6 is essential. If we look only at normed spaces we do not need to know anything about the Control Measure Problem, as the following theorem shows.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, U a normed space and $\theta : \mathfrak{A} \to U$ a vector measure. Then θ has a control measure.

proof (a) Since U can certainly be embedded in a Banach space $\hat{U}$ (3A5Ib), and as θ will still be a vector measure when regarded as a map from $\mathfrak{A}$ to $\hat{U}$, we may assume from the beginning that U itself is complete.

(b) θ is bounded (that is, $\sup_{a\in\mathfrak{A}} \|\theta a\|$ is finite). **P?** Suppose, if possible, otherwise. Choose $\langle a_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 1$. Given that $\sup_{a\subseteq a_n} \|\theta a\| = \infty$, choose $b \subseteq a_n$ such that $\|\theta b\| \geq \|\theta a_n\| + 1$. Then $\|\theta(a_n \setminus b)\| \geq 1$. Also

$$\sup_{a\subseteq a_n} \|\theta a\| \leq \sup_{a\subseteq a_n} \|\theta(a \cap b)\| + \|\theta(a \setminus b)\|,$$

so at least one of $\sup_{a\subseteq b} \|\theta a\|$, $\sup_{a\subseteq a_n\setminus b} \|\theta a\|$ must be infinite. We may therefore take a_{n+1} to be either b or $a_n \setminus b$ and such that $\sup_{a\subseteq a_{n+1}} \|\theta a\| = \infty$. Observe that in either case we shall have $\|\theta(a_n \setminus a_{n+1})\| \geq 1$. Continue.

At the end of the induction we shall have a disjoint sequence $\langle a_n \setminus a_{n+1} \rangle_{n\in\mathbb{N}}$ such that $\|\theta(a_n \setminus a_{n+1})\| \geq 1$ for every n, so that $\sum_{n=0}^{\infty} \theta(a_n \setminus a_{n+1})$ cannot be defined in U; which is impossible. **XQ**

(c) Accordingly we have a bounded linear operator $T : L^\infty \to U$, where $L^\infty = L^\infty(\mathfrak{A})$, such that $T\chi = \theta$ (363Ea).

Now the key to the proof is the following fact: if $\langle u_n \rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $(L^\infty)^+$, $\langle Tu_n \rangle_{n\in\mathbb{N}} \to 0$ in U. **P** Let γ be such that $u_n \leq \gamma\chi 1$ for every n. Let $\epsilon > 0$, and let k be the integer part of γ/ϵ. For $n \in \mathbb{N}$, $i \leq k$ set $a_{ni} = [\![u_n > \epsilon(i+1)]\!]$; then $\langle a_{ni} \rangle_{n\in\mathbb{N}}$ is disjoint for each i, and if we set $v_n = \epsilon \sum_{i=0}^{k} \chi a_{ni}$, we get $v_n \leq u_n \leq v_n + \epsilon\chi 1$, so $\|u_n - v_n\|_\infty \leq \epsilon$.

Because $\langle a_{ni} \rangle_{n\in\mathbb{N}}$ is disjoint, $\sum_{n=0}^{\infty} \theta a_{ni}$ is defined in U, and $\langle \theta a_{ni} \rangle_{n\in\mathbb{N}} \to 0$, for each $i \leq k$. Consequently

$$Tv_n = \epsilon \sum_{i=0}^{k} \theta a_{ni} \to 0$$

as $n \to \infty$. But

$$\|Tu_n - Tv_n\| \leq \|T\| \|u_n - v_n\|_\infty \leq \epsilon \|T\|$$

for each n, so $\limsup_{n\to\infty}\|Tu_n\|\le\epsilon\|T\|$. As ϵ is arbitrary, $\lim_{n\to\infty}\|Tu_n\|=0$. **Q**

(d) Consider the adjoint operator $T':U^*\to(L^\infty)^*$. Recall that L^∞ is an M-space (363B) so that its dual is an L-space (356N). Write

$$A=\{T'g:g\in U^*,\,\|g\|\le 1\}\subseteq(L^\infty)^*=(L^\infty)^\sim.$$

If $u\in L^\infty$, then

$$\sup_{f\in A}|f(u)|=\sup_{\|g\|\le 1}|(T^*g)(u)|=\sup_{\|g\|\le 1}|g(Tu)|=\|Tu\|.$$

Now A is uniformly integrable. **P** I use the criterion of 356O. Of course $\|f\|\le\|T'\|$ for every $f\in A$, so A is norm-bounded. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded disjoint sequence in $(L^\infty)^+$, then

$$\sup_{f\in A}|f(u_n)|=\|Tu_n\|\to 0$$

as $n\to\infty$. So A is uniformly integrable. **Q**

(e) Next, $A\subseteq(L^\infty)^\sim_c$. **P** If $f\in A$, it is of the form $T'g$ for some $g\in U^*$, that is,

$$f(\chi a)=(T'g)(\chi a)=gT(\chi a)=g(\theta a)$$

for every $a\in\mathfrak{A}$. If now $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a,

$$f(\chi a)=g(\theta(\sup_{n\in\mathbb{N}}a_n))=g(\sum_{n=0}^\infty\theta a_n)=\sum_{n=0}^\infty g(\theta a_n)=\sum_{n=0}^\infty f(\chi a_n).$$

So $f\chi$ is countably additive. By 363K, $f\in(L^\infty)^\sim_c$. **Q**

(f) Because A is uniformly integrable, there is for each $m\in\mathbb{N}$ an $f_m\ge 0$ in $(L^\infty)^*$ such that $\|(|f|-f_m)^+\|\le 2^{-m}$ for every $f\in A$; moreover, we can suppose that f_m is of the form $\sup_{i\le k_m}|f_{mi}|$ where every f_{mi} belongs to A (354R(b-iii)), so that $f_m\in(L^\infty)^\sim_c$ and $\mu_m=f_m\chi$ is countably additive. Set

$$\gamma_m=1+\mu_m 1\text{ for each }m,\quad \mu=\sum_{m=0}^\infty\frac{1}{2^m\gamma_m}\mu_m;$$

then $\mu:\mathfrak{A}\to[0,\infty[$ is a non-negative countably additive functional.

Now μ is a control measure for θ. **P** If $\mu a=0$, then $\mu_m a=0$, that is, $f_m(\chi a)=0$, for every $m\in\mathbb{N}$. But this means that if $g\in U^*$ and $\|g\|\le 1$,

$$|g(\theta a)|=|(T'g)(\chi a)|\le f_m(\chi a)+\|(|T'g|-f_m)^+\|\le 2^{-m}$$

for every m, by the choice of f_m; so that $g(\theta a)=0$. As g is arbitrary, $\theta a=0$; as a is arbitrary, μ is a control measure for θ. **Q**

393T This concludes the list of 'positive' results I wish to present in this section. I now devote a few pages to significant examples of submeasures. The first example is a classic formulation (taken from Talagrand 80) which shows in clear relief some of the fundamental ways in which submeasures differ from measures.

Examples (a) Fix $n\ge 1$, and let I be the set $\{0,1,\dots,2n-1\}$, $X=[I]^n$, so that X is a finite set. For each $i\in I$ set $A_i=\{a:i\in a\in X\}$. For $E\subseteq X$ set

$$\nu E=\frac{1}{n+1}\inf\{\#(J):J\subseteq I,\,E\subseteq\bigcup_{i\in J}A_i\}$$

$$=\frac{1}{n+1}\inf\{\#(J):a\cap J\ne\emptyset\text{ for every }a\in E\}.$$

It is elementary to check that $\nu:\mathcal{P}X\to[0,\infty[$ is a strictly positive submeasure, therefore (because $\mathcal{P}X$ is finite) a Maharam submeasure.

The essential properties of ν are twofold: (i) $\nu X=1$; (ii) for any non-negative additive functional μ such that $\mu E\le\nu E$ for every $E\subseteq X$, $\mu X\le\frac{2}{n+1}$. **P** (i)(α) If $J\subseteq I$ and $\#(J)\le n$, there is an $a\in[I\setminus J]^n$, so that $a\in X\setminus\bigcup_{i\in J}A_i$ and $X\not\subseteq\bigcup_{i\in J}A_i$. This means that X cannot be covered by fewer than $n+1$ of the sets A_i, so that νX must be at least 1. (β) On the other hand, if $J\subseteq I$ is any set of cardinal $n+1$, $a\cap J\ne\emptyset$ for every $a\in X$, so that $X=\bigcup_{i\in J}A_i$ and $\nu X\le 1$. (ii) Every member of X belongs to just n of the sets A_i, so

$$n\mu X = \sum_{i\in I}\mu A_i \le \frac{\#(I)}{n+1} = \frac{2n}{n+1},$$

and $\mu X \le \frac{2}{n+1}$. **Q**

(b) This example shows at least that any proof of $\mathrm{CM}_{3\mathrm{A}}$ cannot work through any generally valid inequality of the form 'if ν is a Maharam submeasure there is an additive functional μ with $\delta\nu \le \mu \le \nu$'. If we take a sequence of these spaces we can form a result which in one direction is stronger, as follows.

For each $n \ge 1$ set $I_n = \{0,\dots,2n-1\}$ and define $X_n = [I_n]^n$, $A_{ni} = \{a : i \in a \in X_n\}$, $\nu_n : \mathcal{P}X_n \to [0,1]$ as in (a) above. Set $Z = \prod_{n=1}^{\infty} X_n$, $K = \{(n,i) : n \ge 1,\ i \in I_n\}$ and

$$C_{ni} = \{x : x \in Z,\ x(n) \in A_{ni}\}$$

for $(n,i) \in K$. Now define $\theta : \mathcal{P}Z \to [0,1]$ by setting

$$\theta W = \inf\{\sum_{(n,i)\in J}\frac{1}{n+1} : J \subseteq K,\ W \subseteq \bigcup_{(n,i)\in J} C_{ni}\}$$

for every $W \subseteq Z$. Then θ is an outer measure on Z, by arguments we have been familiar with since 114D. Also $\theta Z = 1$. **P** (i) Because (for instance) X_1 is covered by the two sets A_{10} and A_{11}, Z is covered by C_{10} and C_{11}, so that $\theta Z \le 1$. (ii) If $J \subseteq K$ and $\sum_{(n,i)\in J}\frac{1}{n+1} < 1$, set $J_n = \{i : (n,i) \in J\}$ for each n; then $\#(J_n) < n+1$, so we can choose an $x(n) \in X_n \setminus \bigcup_{i\in J_n} A_{ni}$. This defines a sequence $x \in Z$ such that $x \notin C_{ni}$ for every $(n,i) \in J$, and $Z \ne \bigcup_{(n,i)\in J} C_{ni}$. As J is arbitrary, $\theta Z \ge 1$. **Q**

Finally, if Σ is any subalgebra of $\mathcal{P}Z$ containing every C_{ni}, and $\mu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\mu E \le \theta E$ for every $E \in \Sigma$, then $\mu = 0$. **P** For each n, every point of x belongs to n different C_{ni}, just as in (a) above; so that

$$n\mu Z \le \sum_{i\in I_n}\mu C_{ni} \le \frac{2n}{n+1}.$$

As this is true for every $n \ge 1$, $\mu Z = 0$. **Q**

(c) A non-zero submeasure ν on a Boolean algebra $\mathfrak{A}$ is called **pathological** if the only additive functional μ such that $0 \le \mu a \le \nu a$ for every $a \in \mathfrak{A}$ is the zero functional. Thus the submeasure θ of (b) above is pathological, and $\mathrm{CM}_{3\mathrm{A}}$ can be read 'an exhaustive submeasure cannot be pathological'.

(d) It is important to note that Fubini's theorem fails catastrophically for submeasures. As a simple example, consider the space (X,ν) of (a) above, for some fixed $n \ge 1$. If we define $\theta : \mathcal{P}(X\times X) \to [0,1]$ by setting

$$\theta W = \inf\{\sum_{i=0}^{\infty}\nu E_i\cdot\nu F_i : W \subseteq \bigcup_{i\in\mathbb{N}} E_i\times F_i\}$$

for each $W \subseteq X\times X$, then we obtain an outer measure, just as if ν itself were a measure; but $\theta(X\times X) \le 4n/(n+1)^2$ is small compared with $(\nu X)^2$. **P** Give $\{0,\dots,2n-1\} = \mathbb{Z}_{2n}$ its usual group operation $+_{2n}$ of addition mod $2n$. Then

$$X\times X \subseteq \bigcup_{i<2n}(A_i\times A_i) \cup \bigcup_{i<2n}(A_i\times A_{i+_{2n}1}),$$

because if a, $b \in X$ either $a\cap b \ne \emptyset$ and there is some i such that a, b both belong to A_i, or $b = \{0,\dots,2n-1\}\setminus a$ and there is some i such that $i \in a$, $i +_{2n} 1 \in b$. So

$$\theta(X\times X) \le \sum_{i<2n}\nu A_i(\nu A_i + \nu A_{i+_{2n}1}) = \frac{4n}{(n+1)^2}. \ \mathbf{Q}$$

393U For the next example, I present a much deeper idea from Roberts 93.

Example Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then for any $\epsilon > 0$ we can find a submeasure $\nu : \mathfrak{B} \to [0,1]$ such that

(i) for every $n \in \mathbb{N}$ there is a disjoint sequence $S_{n0},\dots,S_{nn}$ in $\mathfrak{B}$ such that $\nu S_{ni} = 1$ for every $i \le n$;

(ii) if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathfrak{B}$ then $\limsup_{n\to\infty}\nu E_n \le \epsilon$.

proof (a) For each $n \in \mathbb{N}$ let I_n be the finite set $\{0,\dots,n\}$, given its discrete topology; set $X = \prod_{n\in\mathbb{N}} I_n$, with the product topology; let $\mathfrak{C}$ be the algebra of subsets of X generated by sets of the form $S_{ij} = \{x :$

$x(i) = j\}$, where $i \in \mathbb{N}$ and $j \le i$. Note that X is compact and Hausdorff and that every member of $\mathfrak{C}$ is open-and-closed (because all the S_{ij} are). Also $\mathfrak{C}$ is atomless, countable and non-zero, so is isomorphic to $\mathfrak{B}$, by 393F. It will therefore be enough if I can describe a submeasure $\nu : \mathfrak{C} \to [0,1]$ with the properties (i) and (ii) above, and this is what I will do.

(b) For each $n \in \mathbb{N}$ let $\mathcal{A}_n$ be the set of non-empty members of $\mathfrak{C}$ determined by coordinates in $\{0,\dots,n\}$; note that $\mathcal{A}_n$ is finite. For $k \le l \in \mathbb{N}$, say that $E \in \mathfrak{C}$ is (k,l)**-thin** if for every $A \in \mathcal{A}_k$ there is an $A' \in \mathcal{A}_l$ such that $A' \subseteq A \setminus E$. Note that if $k' \le k \le l \le l'$ then $\mathcal{A}_{k'} \subseteq \mathcal{A}_k$ and $\mathcal{A}_l \subseteq \mathcal{A}_{l'}$, so any (k,l)-thin set is also (k',l')-thin.

Say that every $E \in \mathfrak{C}$ is $(k,0)$**-small** for every $k \in \mathbb{N}$, and that for k, $r \in \mathbb{N}$ a set $E \in \mathfrak{C}$ is $(k,r+1)$**-small** if there is some $l \ge k$ such that E is (k,l)-thin and (l,r)-small. Observe that E is $(k,1)$-small iff it is (k,l)-thin for some $l \ge k$, that is, there is no member of $\mathcal{A}_k$ included in E. Observe also that if E is (k,r)-small then it is (k',r)-small for every $k' \le k$.

Write $\mathcal{S} = \{S_{ij} : j \le i \in \mathbb{N}\}$.

(c) Suppose that $E \in \mathfrak{C}$ and $k \le l \le m$ are such that E is both (k,l)-thin and (l,m)-thin. Then whenever $A \in \mathcal{A}_k$, $S \in \mathcal{S}$ and $A \cap S \ne \emptyset$, there is an $A' \in \mathcal{A}_m$ such that $A' \subseteq A \setminus E$ and $A' \cap S \ne \emptyset$. **P** Take $S = S_{ni}$ where $i \le n$. (i) If $n \le l$, then $A \cap S \in \mathcal{A}_l$; because E is (l,m)-thin, there is an $A' \in \mathcal{A}_m$ such that $A' \subseteq (A \cap S) \setminus E$. (ii) If $n > l$, there is an $A' \in \mathcal{A}_l$ such that $A' \subseteq A \setminus E$, because E is (k,l)-thin; now $A' \in \mathcal{A}_m$, and $A' \cap S$ is non-empty because A' is determined by coordinates less than n. **Q**

(d) It follows that if $S \in \mathcal{S}$, $k \in \mathbb{N}$, $A \in \mathcal{A}_k$, $A \cap S \ne \emptyset$, $r \in \mathbb{N}$ and $E_0,\dots,E_{r-1}$ are $(k,2r)$-small, then $A \cap S$ is not covered by $E_0,\dots,E_{r-1}$. **P** Induce on r. The case $r = 0$ demands only that $A \cap S$ should not be covered by the empty sequence, that is, $A \cap S \ne \emptyset$, which is one of the hypotheses. For the inductive step to $r+1$, we know that for each $j \le r$ there are l_j, m_j such that $k \le l_j \le m_j$ and E_j is (k,l_j)-thin and (l_j,m_j)-thin and $(m_j,2r)$-small. Rearranging $E_0,\dots,E_r$ if necessary we may suppose that $m_r \le m_j$ for every $j \le r$; set $m = m_r$. By (c), there is an $A' \in \mathcal{A}_m$ such that $A' \cap S \ne \emptyset$ and $A' \subseteq A \setminus E_r$. Now every E_j, for $j < r$, is $(m_j,2r)$-small, therefore $(m,2r)$-small, so by the inductive hypothesis $A' \cap S \not\subseteq \bigcup_{j<r} E_j$. Accordingly $A \cap S \not\subseteq \bigcup_{j \le r} E_j$ and the induction continues. **Q**

(e) Now suppose that $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{C}$. Then for any $k \in \mathbb{N}$ there are l, $n^* \in \mathbb{N}$ such that E_n is (k,l)-thin for every $n \ge n^*$. **P** Consider $G_n = \bigcup_{j\ge n} E_j$ for each $n \in \mathbb{N}$. Then every G_n is open and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$. If $A \in \mathcal{A}_k$, then A, with its subspace topology, is compact, so Baire's theorem (3A3G) tells us that there is an n_A such that $G_{n_A} \cap A$ is not dense in A; let l_A be such that $A \setminus G_{n_A}$ includes a member of $\mathcal{A}_{l_A}$. Set $n^* = \max\{n_A : A \in \mathcal{A}_k\}$, $l = \max\{l_A : A \in \mathcal{A}_k\}$. If $n \ge n^*$, $A \in \mathcal{A}_k$ there is an $A' \in \mathcal{A}_{l_A} \subseteq \mathcal{A}_l$ such that

$$A' \subseteq A \setminus G_{n_A} \subseteq A \setminus E_n.$$

As A is arbitrary, E_n is (k,l)-thin. **Q**

It follows at once that for any $r \in \mathbb{N}$ we can find n_r^*, $k_0 < k_1 < \ldots < k_r \in \mathbb{N}$ such that E_n is (k_j,k_{j+1})-thin for every $j < r$ and $n \ge n_r^*$; so that E_n is $(0,r)$-small for every $n \ge n_r^*$.

(f) Take an integer $r \ge 1/\epsilon$. Let $\mathcal{U}$ be the set of $(0,2r)$-small members of $\mathfrak{C}$. Set

$$\nu E = \frac{1}{r}\min\{m : E \subseteq E_1 \cup \ldots \cup E_m \text{ for some } E_1,\dots,E_m \in \mathcal{U}\}$$

if E can be covered by r or fewer members of $\mathcal{U}$, 1 otherwise. It is easy to check that $\nu : \mathfrak{C} \to [0,1]$ is a submeasure. By (d), no member of $\mathcal{S}$ can be covered by r or fewer members of $\mathcal{U}$, so $\nu S_{ni} = 1$ whenever $i \le n \in \mathbb{N}$. By (e), if $\langle E_n \rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathfrak{C}$, E_n belongs to $\mathcal{U}$ for all but finitely many n, so that $\nu E_n \le \frac{1}{r} \le \epsilon$ for all but finitely many n. Thus ν has the required properties.

393X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Show that a subset F of $\mathfrak{A}$ is closed for the topology defined by ν iff the limit of any order*-convergent sequence in F belongs to F.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with a strictly positive Maharam submeasure ν. Show that $\mathfrak{A}$ is complete under the metric defined from ν (393B).

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and ν, ν' two strictly positive Maharam submeasures on $\mathfrak{A}$. Show that the corresponding metrics on $\mathfrak{A}$ (393B) are uniformly equivalent.

(d) Let $\mathfrak{A}$ be a countable Boolean algebra, not $\{0\}$. Show that $\mathfrak{A}$ is isomorphic to an order-closed subalgebra of the algebra $\mathfrak{B}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. (*Hint*: show that $\mathfrak{A} \otimes \mathfrak{B} \cong \mathfrak{B}$.)

(e) Suppose that CM_1-CM_6 are false. Show that there is a Dedekind complete Boolean algebra $\mathfrak{A}$, with a strictly positive Maharam submeasure, such that (i) the only countably additive real-valued functional on $\mathfrak{A}$ is the zero functional (ii) $\tau(\mathfrak{A}) = \omega$.

(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure. Show that if $\tau(\mathfrak{A}) \le \omega$ then the associated topology on $\mathfrak{A}$ (393B) is separable and $\mathfrak{A}$ is σ-linked.

(g) Consider the statement

(CM_6') If X is a set, Σ a σ-algebra of subsets of X, U a metrizable linear topological space, and $\theta : \Sigma \to U$ a vector measure, then θ has a control measure.

Show that $\mathrm{CM}_6' \iff \mathrm{CM}_6$.

(h) Consider the outer measures ν_n, θ of 393Ta-b. Give every X_n its discrete topology and Z the corresponding product topology. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z. (i) Show that $\mathcal{E}$ is the subalgebra of $\mathcal{P}Z$ generated by sets of the form $\{x : x \in Z,\, x(n) = a\}$ for $n \ge 1$, $a \in X_n$. (ii) Take $n \ge 1$ and set $A = \{a : a \in X_n,\, \sum_{i \in a} i \text{ is even}\}$. Show that $\nu_n A = \nu_n(X \setminus A) = \frac{n}{n+1}$. (iii) Show that for any $\alpha < 1$ there is a disjoint sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that $\theta E_n \ge \alpha$ for every n.

393Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure. Show that if $\tau(\mathfrak{A}) \le \omega$ then $\mathfrak{A}$ is σ-m-linked (391Yh) for every m.

(b) Let $\mathfrak{A}$ be a Boolean algebra with just four elements, and let N^+ be the set of submeasures on $\mathfrak{A}$, $N = N^+ - N^-$ the set of functionals from $\mathfrak{A}$ to $\mathbb{R}$ expressible as the difference of two submeasures. (i) Show that N is just the three-dimensional space of functionals $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu 0 = 0$. (ii) Show that there is a partial order on N under which N is a partially ordered linear space with positive cone N^+, but that N is now not a lattice.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that there is at most one Hausdorff topology $\mathfrak{T}$ on $\mathfrak{A}$ such that (i) the Boolean operations $\cap$, $\triangle$ are continuous (ii) for every open set G containing 0 there is an open set H containing 0 such that $a \in G$ whenever $a \subseteq b \in H$ (iii) $0 \in \overline{A}$ whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and has infimum 0. (*Hint*: Given such a topology $\mathfrak{T}$, show that the construction in 393K produces enough order-continuous submeasures on $\mathfrak{A}$ to define $\mathfrak{T}$. Now if ν is any order-continuous submeasure, let a be its support; because $\mathfrak{A}_a$ is ccc, there is a $\mathfrak{T}$-continuous submeasure with support a; use 393E to see that ν is continuous.)

(d) Let $\mathfrak{A}$ be a Boolean algebra. Let $\mathfrak{T}$ be the topology on $\mathfrak{A}$ such that the closed sets for $\mathfrak{T}$ are just those sets $F \subseteq \mathfrak{A}$ such that $a \in F$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a sequence in F which order*-converges to a (367Yc). Show that $a \mapsto a \cup c$, $a \mapsto a \cap c$, $a \mapsto a \triangle c$, $a \mapsto a \setminus c$, $a \mapsto c \setminus a$ are all $\mathfrak{T}$-continuous, for any $c \in \mathfrak{A}$. Write $\mathcal{U}$ for the family of open sets G containing 0 such that $[0, a] \subseteq G$ for every $a \in G$; for $A \subseteq \mathfrak{A}$ set $A' = \{a \cup b : a,\, b \in A\}$. Show that $\overline{G} \subseteq G' \in \mathcal{U}$ for every $G \in \mathcal{U}$.

Suppose now that $\mathfrak{A}$ is ccc and Dedekind complete and that $\mathfrak{T}$ is Hausdorff. Show that (i) $\mathfrak{A}$ is weakly (σ, ∞)-distributive; (ii) whenever $A \subseteq \mathfrak{A}$ and $a \in \overline{A}$ there is a sequence in A which order*-converges to a (see 367Yl); (iii) whenever $0 \in G \in \mathfrak{T}$ there is an $H \in \mathcal{U}$ such that $H \subseteq G$ (*hint*: $H = \operatorname{int}\{a : [0, a] \subseteq G\} \in \mathcal{U}$); (iv) for any $a \in \mathfrak{A} \setminus \{0\}$ there is a $G \in \mathcal{U}$ such that $a \notin G'$; (v) for every $G \in \mathcal{U}$ there is an $H \in \mathcal{U}$ such that $H'' \subseteq G'$; (vi) for any $a \in \mathfrak{A} \setminus \{0\}$ there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{U}$ such that $a \not\subseteq \sup(\bigcap_{n \in \mathbb{N}} \overline{G_n})$; (vii) there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{U}$ such that $\bigcap_{n \in \mathbb{N}} \overline{G_n} = \{0\}$; (viii) for every $G \in \mathcal{U}$ there is an $H \in \mathcal{U}$ such that $H' \subseteq G$; (ix) there is a strictly positive Maharam submeasure on $\mathfrak{A}$ (*hint*: 393K).

(See Balcar Główczyński & Jech 98.)

(e) Consider the statement

(CM_4') Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra with a Hausdorff topology $\mathfrak{T}$ such that if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ which order*-converges to $a \in \mathfrak{A}$, then $\langle a_n\rangle_{n\in\mathbb{N}}$ converges to 0 for the topology $\mathfrak{T}$. Then $\mathfrak{A}$ is measurable.

Show that CM_4' is true iff CM_1 is true.

(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with a strictly positive Maharam submeasure, and let $\mathfrak{S}$, $\mathfrak{T}$ be the associated metrizable topologies on $\mathfrak{A}$, $L^0(\mathfrak{A})$ (393B, 393N). Write $d(\mathfrak{A})$, $d(L^0)$ for the topological densities of these spaces (331Yf). Show that

$$\max(\tau(\mathfrak{A}),\omega) = \max(d(\mathfrak{A}),\omega) = \max(d(L^0),\omega).$$

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\lambda : \mathfrak{A} \to [0,\infty[$ a non-negative finitely additive functional, and ν a Maharam submeasure on $\mathfrak{A}$ such that $\nu a = 0$ whenever $\lambda a = 0$. Show that ν is absolutely continuous with respect to λ and therefore also with respect to the countably additive part of λ as defined in 362Bc.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, U a Hausdorff linear topological space and $\theta : \mathfrak{A} \to U$ a vector measure. Suppose that there is a bounded finitely additive functional $\lambda : \mathfrak{A} \to \mathbb{R}$ such that $\theta a = 0$ whenever $\lambda a = 0$. Show that θ has a control measure.

(i) A linear topological space U is **locally convex** if the topology has a base consisting of convex sets. Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, U is a metrizable locally convex linear topological space, and $\theta : \mathfrak{A} \to U$ is a vector measure, then θ has a control measure.

(h) Consider the outer measure θ of 393Tb. (i) Show that there is a non-decreasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ of subsets of Z such that $\theta(\bigcup_{n\in\mathbb{N}} W_n) > \sup_{n\in\mathbb{N}} \theta W_n$. (ii) Show that if $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of *open* subsets of Z then $\theta(\bigcup_{n\in\mathbb{N}} W_n) = \sup_{n\in\mathbb{N}} \theta W_n$.

393 Notes and comments This section is long in pages, but I hope has been reasonably easy reading, because so many of the arguments amount to re-written versions of ideas already presented. Thus 393Ba-b can be thought of as an elementary extension of 323A-323B, and the completion procedure in 393Bc uses some of the same ideas as the proof of 325C. The notion of 'absolute continuity' for submeasures is taken directly from that for measures, and 393E is a simple translation of 232Ba. Other fragments of the argument for the Radon-Nikodým theorem appear in the proof that CM_1 implies $\mathrm{CM}_{3\mathrm{A}}$ (393I) and in 393S. The argument of 393K takes a fair bit of space, but is mostly a recapitulation of the proof that a uniformity can be defined from pseudometrics (Engelking 89, 8.1.10; Bourbaki 66, IX.1.4). If there is a new idea there, it is in the formula

$$\nu a = \inf\{\nu_1 b : a \cap e \subseteq b \subseteq e\}$$

in the middle of the proof. (Contrast this with the formula

$$\nu a = \sup_{b\subseteq a} \min(1, \tau(\theta b))$$

in the proof of 393R.) 393M is just a matter of picking the right description of the topology of convergence in measure to generalize.

I do think it is a little surprising that $\mathrm{CM}_{3\mathrm{A}}$ and $\mathrm{CM}_{3\mathrm{B}}$ should be equiveridical; one is asking for an additive functional dominated by ν, and the other for an additive functional weakly dominating ν. Of course it is not exactly the same ν in both cases. It is tempting to look for a theory of spaces of submeasures corresponding to the theory of spaces of additive functionals (§326), but it seems that, at the very least, there are new obstacles (393Yb).

The arguments of this section tend to be based on hypotheses of the form 'CM_n is true'; I hope that this will not lead to any presumption that the Control Measure Problem has a positive answer. I am myself inclined to believe that all the statements CM_n are false, and that one day all this material will have to be re-written with the arguments in reverse, proving $\neg\mathrm{CM}_n \Rightarrow \neg\mathrm{CM}_m$ instead of $\mathrm{CM}_m \Rightarrow \mathrm{CM}_n$, or using reductio ad absurdum at every point where it is not used above.

I include CM_2' because it seems to make the problem more accessible, and indeed it is significant for theoretical reasons. In this form it is possible to show that the Control Measure Problem cannot depend on any of the special axioms that have been introduced in the last thirty-five years; that is, if there is a solution using such an axiom as the continuum hypothesis, or Martin's axiom, or the axiom of constructibility, then there must be a solution using only the ordinary axioms of Zermelo-Fraenkel set theory (including the axiom of choice). For elucidation of this remark, which I fear may be somewhat cryptic, look for 'Shoenfield's Absoluteness Theorem' in a book on mathematical logic and forcing (e.g., KUNEN 80). An incidental benefit from including CM_2' is that it has forced me, at last, to spell out the characterization of the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (393F).

I mention CM_1'' and CM_6' (393Xg) as well as CM_1' and CM_6 because some natural versions of the Control Measure Problem arise in the context of σ-algebras of sets, rather than Dedekind σ-complete Boolean algebras, and a reminder that this makes no difference to the problem may be helpful.

This is not a book about vector measures, but having introduced vector measures in CM_6 it would be disgraceful to leave you unaware that there is no control measure problem for measures taking values in normed spaces, and since the proof is no more than an assembly of ideas already covered in their scattered locations, it seems right to set it out (393S).

I have said already that my own prejudice is in favour of believing that all the statements CM_* are false; that is, that there is a Dedekind complete Boolean algebra $\mathfrak{A}$, with a strictly positive Maharam submeasure, such that $\mathfrak{A}$ is not measurable. Ordinarily such a presumption makes a problem seem less interesting; only rarely is there as much honour and profit in constructing a recondite counter-example as there is in proving a substantial theorem. But in this case I believe that it could be the starting point of a new theory of 'submeasurable algebras', being Dedekind complete Boolean algebras carrying strictly positive Maharam submeasures. These would be ccc weakly (σ,∞)-distributive algebras (392I), some of them σ-m-linked for every m (393Ya), carrying metrics for which the Boolean operations were uniformly continuous and order-closed sets were closed (393B); there would be associated theories of topologies of 'convergence in submeasure' on algebras and L^0 spaces (393B, 393M). We already have a remarkable characterization of such algebras in terms of order*-convergence (393Yd). But the real prize would be a new algebra for use in forcing, conceivably leading to new models of set theory.

The examples in 393T and 393U are there for different purposes. In 393Ta we have submeasures on finite algebras, which are therefore necessarily uniformly exhaustive, in which dominated measures are cruelly dominated; the submeasures are 'almost pathological', and can readily be assembled into a submeasure which is pathological in the strict sense (393Tb), but is now very far from being exhaustive (393Xh). Of course the method of 393Tb is extraordinarily crude, but as far as I know nothing else works either (see 393Td).

I called 393T a 'classic' construction; I am sure that whatever resolution is at last found for the Control Measure Problem, 393Ta at least will always be of interest. It is less clear that 393U will endure in the same way, but for the moment it is the best example known of a submeasure which is almost exhaustive while being far from uniformly exhaustive.

394 Kawada's theorem

I now describe a completely different characterization of (homogeneous) measurable algebras, based on the special nature of their automorphism groups. The argument depends on the notion of 'non-paradoxical' group of automorphisms; this is an idea of great importance in other contexts, and I therefore aim at a fairly thorough development, with proofs which are adaptable to other circumstances.

394A Definitions Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **full local semigroup generated by** G if there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i\rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-τ-equidecomposable**.

I will write $a \preccurlyeq^\tau_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-τ-equidecomposable.

For any function f with domain $\mathfrak{A}$, I will say that f is **G-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

394B The notion of 'full local semigroup' is of course an extension of the idea of 'full subgroup' (381M, 387A). The word 'semigroup' is justified by (c) of the following lemma, and the word 'full' by (e).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) Suppose that a, $b \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism. Then the following are equiveridical:

(i) $\phi \in G^*_\tau$;

(ii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\pi \in G$ such that $\phi c = \pi c$ for every $c \subseteq c_1$;

(iii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\psi \in G^*_\tau$ such that $\phi c = \psi c$ for every $c \subseteq c_1$.

(b) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_τ.

(c) Suppose that a, b, a', $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_τ. Then $\psi\phi \in G^*_\tau$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(d) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi{\restriction}\mathfrak{A}_c \in G^*_\tau$ for any $c \subseteq a$.

(e) Suppose that a, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i \rangle_{i \in I}$ in G^*_τ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\tau$.

proof (a) (Compare 387B.)

(i)$\Rightarrow$(iii) is trivial, since of course $G \subseteq G^*_\tau$.

(iii)$\Rightarrow$(ii) Suppose that ϕ satisfies (iii), and that $0 \ne c_0 \subseteq a$. Then we can find a $\psi \in G^*_\tau$ and a non-zero $c_1 \subseteq c_0$ such that ϕ agrees with ψ on $\mathfrak{A}_{c_1}$. Suppose that $\operatorname{dom}\psi = \mathfrak{A}_d$, where necessarily $d \supseteq c_1$. Then there are a partition of unity $\langle d_i \rangle_{i \in I}$ in $\mathfrak{A}_d$ and a family $\langle \pi_i \rangle_{i \in I}$ such that $\psi c = \pi_i c$ whenever $c \subseteq d_i$. There is some $i \in I$ such that $c_2 = c_1 \cap d_i \ne 0$, and we see that $\phi c = \psi c = \pi_i c$ for every $c \subseteq c_2$. As c_0 is arbitrary, ϕ satisfies (ii).

(ii)$\Rightarrow$(i) If ϕ satisfies (ii), set

$$D = \{d : d \subseteq a, \text{ there is some } \pi \in G \text{ such that } \pi c = \phi c \text{ for every } c \subseteq d\}.$$

The hypothesis is that D is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle a_i \rangle_{i \in I}$ of $\mathfrak{A}_a$ lying within D (313K); for each $i \in I$ take $\pi_i \in G$ such that $\phi c = \pi_i c$ for $c \subseteq a_i$; then $\langle a_i \rangle_{i \in I}$ and $\langle \pi_i \rangle_{i \in I}$ witness that $\phi \in G^*_\tau$.

(b) This is elementary; if $\langle a_i \rangle_{i \in I}$, $\langle \pi_i \rangle_{i \in I}$ witness that $\phi \in G^*_\tau$, then $\langle \phi a_i \rangle_{i \in I} = \langle \pi_i a_i \rangle_{i \in I}$, $\langle \pi_i^{-1} \rangle_{i \in I}$ witness that $\phi^{-1} \in G^*_\tau$.

(c) I ought to start by computing the domain of $\psi\phi$:

$$\begin{aligned} d \in \operatorname{dom}(\psi\phi) &\iff d \in \operatorname{dom}\phi,\ \phi d \in \operatorname{dom}\psi \\ &\iff d \subseteq a,\ \phi d \subseteq b \iff d \subseteq \phi^{-1}(a' \cap b) = c. \end{aligned}$$

So the domain of $\psi\phi$ is indeed $\mathfrak{A}_c$; now $\phi{\restriction}\mathfrak{A}_c$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\phi c}$, where $\phi c = a' \cap b \in \mathfrak{A}_b$, so $\psi\phi$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\psi\phi c} = \mathfrak{A}_{c'}$. Let $\langle a_i \rangle_{i \in I}$, $\langle b_j \rangle_{j \in J}$ be partitions of unity in $\mathfrak{A}_a$, $\mathfrak{A}_b$ respectively, and $\langle \pi_i \rangle_{i \in I}$, $\langle \theta_j \rangle_{j \in J}$ families in G such that $\phi d = \pi_i d$ for $d \subseteq a_i$, $\psi e = \theta_j e$ for $e \subseteq b_j$. Set $c_{ij} = a_i \cap \pi_i^{-1} b_j$; then $\langle c_{ij} \rangle_{i \in I, j \in J}$ is a partition of unity in $\mathfrak{A}_c$ and $\psi\phi d = \theta_j \pi_i d$ for $d \subseteq c_{ij}$, so $\psi\phi \in G^*_\tau$ (because all the $\theta_j \pi_i$ belong to G).

(d) This is nearly trivial; use the definition of G^*_τ or the criteria of (a), or apply (c) with the identity map on $\mathfrak{A}_c$ as one of the factors.

(e) This follows at once from the criterion (a-iii) above, or otherwise.

394C Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) For a, $b\in\mathfrak{A}$, $a\preccurlyeq^\tau_G b$ iff there is a $\phi\in G^*_\tau$ such that $a\in\operatorname{dom}\phi$ and $\phi a\subseteq b$.

(b)(i) $\preccurlyeq^\tau_G$ is transitive and reflexive;

(ii) if $a\preccurlyeq^\tau_G b$ and $b\preccurlyeq^\tau_G a$ then a and b are G-τ-equidecomposable.

(c) G-τ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i\preccurlyeq^\tau_G b_i$ for every $i\in I$, then $\sup_{i\in I}a_i\preccurlyeq^\tau_G\sup_{i\in I}b_i$.

proof (a) This is immediate from the definition of 'G-τ-equidecomposable' and 394Bd.

(b)(i) $a\preccurlyeq^\tau_G a$ because the identity homomorphism belongs to G^*_τ. If $a\preccurlyeq^\tau_G b\preccurlyeq^\tau_G c$ there are ϕ, $\psi\in G^*_\tau$ such that $\phi a\subseteq b$, $\psi b\subseteq c$ so that $\psi\phi a\subseteq c$; as $\psi\phi\in G^*_\tau$ (394Bc), $a\preccurlyeq^\tau_G c$.

(ii) (This is of course a Schröder-Bernstein theorem, and the proof is the usual one.) Take ϕ, $\psi\in G^*_\tau$ such that $\phi a\subseteq b$, $\psi b\subseteq a$. Set $a_0=a$, $b_0=b$, $a_{n+1}=\psi b_n$ and $b_{n+1}=\phi a_n$ for each n. Then $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences; set $a_\infty=\inf_{n\in\mathbb{N}}a_n$, $b_\infty=\inf_{n\in\mathbb{N}}b_n$. For each n,

$$\phi\restriction\mathfrak{A}_{a_{2n}\setminus a_{2n+1}}:\mathfrak{A}_{a_{2n}\setminus a_{2n+1}}\to\mathfrak{A}_{b_{2n+1}\setminus b_{2n+2}},$$

$$\psi\restriction\mathfrak{A}_{b_{2n}\setminus b_{2n+1}}:\mathfrak{A}_{b_{2n}\setminus b_{2n+1}}\to\mathfrak{A}_{a_{2n+1}\setminus a_{2n+2}}$$

are isomorphisms, while

$$\phi\restriction\mathfrak{A}_{a_\infty}:\mathfrak{A}_{a_\infty}\to\mathfrak{A}_{b_\infty}$$

is another. So we can define an isomorphism $\theta:\mathfrak{A}_a\to\mathfrak{A}_b$ by setting

$$\begin{aligned}\theta c&=\phi c\text{ if }c\subseteq a_\infty\cup\sup_{n\in\mathbb{N}}a_{2n}\setminus a_{2n+1},\\&=\psi^{-1}c\text{ if }c\subseteq\sup_{n\in\mathbb{N}}a_{2n+1}\setminus a_{2n+2}.\end{aligned}$$

By 394Be, $\theta\in G^*_\tau$, so a and b are G-τ-equidecomposable.

(c) This is easy to prove directly from the results in 394B, but also follows at once from (b-i); any transitive reflexive relation gives rise to an equivalence relation.

(d) We may suppose that I is well-ordered by a relation $\le$. For $i\in I$, set $a'_i=a_i\setminus\sup_{j<i}a_j$. Set $a=\sup_{i\in I}a_i=\sup_{i\in I}a'_i$, $b=\sup_{i\in I}b_i$. For each $i\in I$, we have a $b'_i\subseteq b_i$ and a $\phi_i\in G^*_\tau$ such that $\phi_i a'_i=b'_i$. Set $b'=\sup_{i\in I}b'_i\subseteq b$; then we have an isomorphism $\psi:\mathfrak{A}_a\to\mathfrak{A}_{b'}$ defined by setting $\psi d=\phi_i d$ if $d\subseteq a'_i$, and $\psi\in G^*_\tau$, so a and b' are G-τ-equidecomposable and $a\preccurlyeq^\tau_G b$.

394D Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a\ne 1$ such that a is G-τ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are non-zero G-τ-equidecomposable a, b, $c\in\mathfrak{A}$ such that $a\cap b=0$ and $a\cup b\subseteq c$;

(iv) there are G-τ-equidecomposable a, $b\in\mathfrak{A}$ such that $a\subset b$.

proof Write G^*_τ for the full local semigroup generated by G.

(i)$\Rightarrow$(ii) Assume (i). There is a $\phi\in G^*_\tau$ such that $\phi 1=a$. Set $a_n=\phi^n(1\setminus a)$ for each $n\in\mathbb{N}$; because every ϕ^n belongs to G^*_τ (counting ϕ^0 as the identity operator on $\mathfrak{A}$, and using 394Bc), with $\operatorname{dom}\phi^n=\mathfrak{A}$, a_n is G-τ-equidecomposable with $a_0=1\setminus a$ for every n. Also $a_n=\phi^n 1\setminus\phi^{n+1}1$ for each n, while $\langle\phi^n 1\rangle_{n\in\mathbb{N}}$ is non-increasing, so $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Thus (ii) is true.

(ii)$\Rightarrow$(iii) Assume (ii). Set $a=\sup_{n\in\mathbb{N}}a_{2n}$, $b=\sup_{n\in\mathbb{N}}a_{2n+1}$, $c=\sup_{n\in\mathbb{N}}a_n$, so that $a\cap b=0$ and $a\cup b=c$. For each n we have a $\phi_n\in G^*_\tau$ such that $\phi_n a_0=a_n$. So if we set

$$\psi d=\sup_{n\in\mathbb{N}}\phi_n\phi_{2n}^{-1}(d\cap a_{2n})\text{ for }d\subseteq a,$$

ψ belongs to G^*_τ (using 394B) and witnesses that a and c are G-τ-equidecomposable. Similarly, b and c are G-τ-equidecomposable, so (iii) is true.

(iii)$\Rightarrow$(iv) is trivial.

(iv)⇒(i) Take $\phi \in G^*_\tau$ such that $\phi b = a$. Set

$$\psi d = \phi(d \cap b) \cup (d \setminus b)$$

for $d \in \mathfrak{A}$; then $\psi \in G^*_\tau$ witnesses that 1 is G-τ-equidecomposable with $a \cup (1 \setminus b) \neq 1$.

394E Definition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **fully non-paradoxical** if the statements of 394D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-τ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are no non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if $a \subseteq b \in \mathfrak{A}$ and a, b are G-τ-equidecomposable then $a = b$.

Note that if G is fully non-paradoxical, and H is a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that $H \supseteq G$, then H is also fully non-paradoxical, because if $a \preccurlyeq^\tau_G b$ then $a \preccurlyeq^\tau_H b$, so that a and b are H-τ-equidecomposable whenever they are G-τ-equidecomposable.

394F Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms of $\mathfrak{A}$. Then G is fully non-paradoxical.

proof If $\phi : \mathfrak{A} \to \mathfrak{A}_a$ belongs to the full local subgroup generated by G, then we have a partition of unity $\langle a_i \rangle_{i\in I}$ and a family $\langle \pi_i \rangle_{i\in I}$ in G such that $\phi a_i = \pi_i a_i$ for every i; but this means that

$$\bar\mu a = \sum_{i\in I} \bar\mu\phi_i a_i = \sum_{i\in I} \bar\mu\pi_i a_i = \sum_{i\in I} \bar\mu a_i = \bar\mu 1.$$

As $\bar\mu 1 < \infty$, we can conclude that $a = 1$, so that G satisfies the condition (i) of 394E.

394G The fixed-point subalgebra of a group Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(a) By the **fixed-point subalgebra** of G I mean

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\}.$$

(I looked briefly at this construction in 333R, and again in §387 in the special case of a group generated by a single element.) This is a subalgebra of $\mathfrak{A}$, and is order-closed, because every $\pi \in G$ is order-continuous.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. In this case $\mathfrak{C}$ is Dedekind complete (314Ea), and we have, for any $a \in \mathfrak{A}$, an element $\operatorname{upr}(a, \mathfrak{C})$ of $\mathfrak{C}$, defined by setting

$$\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\}$$

(314V). Now $\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}$. **P** Set $c_1 = \operatorname{upr}(a, \mathfrak{C})$, $c_2 = \sup\{\pi a : \pi \in G\}$. (i) Because $a \subseteq c_1 \in \mathfrak{C}$, $\pi a \subseteq \pi c_1 = c_1$ for every $\pi \in G$, and $c_2 \subseteq c_1$. (ii) For any $\phi \in G$,

$$\phi c_2 = \sup_{\pi\in G} \phi\pi a = \sup_{\pi\in G} \pi a = c_2$$

because $G = \{\phi\pi : \pi \in G\}$. So $c_2 \in \mathfrak{C}$; since also $a \subseteq c_2$, $c_1 \subseteq c_2$, and $c_1 = c_2$, as claimed. **Q**

(c) Again supposing that $\mathfrak{A}$ is Dedekind complete, write G^*_τ for the full local semigroup generated by G. Then $\phi(a \cap c) = \phi a \cap c$ whenever $\phi \in G^*_\tau$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$. **P** We have $\phi a = \sup_{i\in I} \pi_i a_i$, where $a = \sup_{i\in I} a_i$ and $\pi_i \in G$ for every i. Now

$$\phi(a \cap c) = \sup_{i\in I} \pi_i(a_i \cap c) = \sup_{i\in I} \pi_i a_i \cap c = \phi a \cap c. \text{ **Q**}$$

Consequently $\operatorname{upr}(\phi a, \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C})$ whenever $\phi \in G^*_\tau$ and $a \in \operatorname{dom}\phi$. **P** For $c \in \mathfrak{C}$,

$$a \subseteq c \iff a \cap c = a \iff \phi(a \cap c) = \phi a \iff \phi a \cap c = \phi a \iff \phi a \subseteq c. \text{ **Q**}$$

It follows that $\operatorname{upr}(a, \mathfrak{C}) \subseteq \operatorname{upr}(b, \mathfrak{C})$ whenever $a \preccurlyeq^\tau_G b$.

(d) Still supposing that $\mathfrak{A}$ is Dedekind complete, we also find that if $a \preccurlyeq^\tau_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\tau_G b \cap c$. **P** There is a $\phi \in G^*_\tau$ such that $\phi a \subseteq b$; now $\phi(a \cap c) = \phi a \cap c \subseteq b \cap c$. **Q** Hence, or otherwise, $a \cap c$ and $b \cap c$ are G-τ-equidecomposable whenever a and b are G-τ-equidecomposable and $c \in \mathfrak{C}$.

(e) Of course the case $\mathfrak{C} = \{0, 1\}$ is particularly significant; when this happens I will call G **ergodic**. Thus an automorphism π is 'ergodic' in the sense of 372P iff the group $\{\pi^n : n \in \mathbb{Z}\}$ it generates is ergodic.

394H I now embark on a series of lemmas leading to the main theorem (394N).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b \in \mathfrak{A}$. Set $c_0 = \sup\{c : c \in \mathfrak{C},\, a \cap c \preccurlyeq^\tau_G b\}$; then $a \cap c_0 \preccurlyeq^\tau_G b$ and $b \setminus c_0 \preccurlyeq^\tau_G a$.

proof Enumerate G as $\langle \pi_\xi \rangle_{\xi<\kappa}$, where $\kappa = \#(G)$. Define $\langle a_\xi \rangle_{\xi<\kappa}$, $\langle b_\xi \rangle_{\xi<\kappa}$ inductively, setting

$$a_\xi = (a \setminus \sup_{\eta<\xi} a_\eta) \cap \pi_\xi^{-1}(b \setminus \sup_{\eta<\xi} b_\eta), \quad b_\xi = \pi_\xi a_\xi.$$

Then $\langle a_\xi \rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_a$ and $\langle b_\xi \rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_b$, and $\sup_{\xi<\kappa} a_\xi$ is G-τ-equidecomposable with $\sup_{\xi<\kappa} b_\xi$. Set $a' = a \setminus \sup_{\xi<\kappa} a_\xi$, $b' = b \setminus \sup_{\xi<\kappa} b_\xi$,

$$\tilde{c}_0 = 1 \setminus \operatorname{upr}(a', \mathfrak{C}) = \sup\{c : c \in \mathfrak{C},\, c \cap a' = 0\}.$$

Then

$$a \cap \tilde{c}_0 \subseteq \sup_{\xi<\kappa} a_\xi \preccurlyeq^\tau_G b,$$

so $\tilde{c}_0 \subseteq c_0$.

Now $b' \subseteq \tilde{c}_0$. **P?** Otherwise, because $\tilde{c}_0 = 1 \setminus \sup_{\xi<\kappa} \pi_\xi a'$ (394Gb), there must be a $\xi < \kappa$ such that $\pi_\xi a' \cap b' \neq 0$. But in this case $d = a' \cap \pi_\xi^{-1} b' \neq 0$, and we have

$$d \subseteq (a \setminus \sup_{\eta<\xi} a_\eta) \cap \pi_\xi^{-1}(b \setminus \sup_{\eta<\xi} b_\eta),$$

so that $d \subseteq a_\xi$, which is absurd. **XQ** Consequently

$$b \setminus \tilde{c}_0 \subseteq \sup_{\xi<\kappa} b_\xi \preccurlyeq^\tau_G a.$$

Now take any $c \in \mathfrak{C}$ such that $a \cap c \preccurlyeq^\tau_G b$, and consider $c' = c \setminus \tilde{c}_0$. Then $b' \cap c' = 0$, that is, $b \cap c' = \sup_{\xi<\kappa} b_\xi \cap c'$, which is G-τ-equidecomposable with $\sup_{\xi<\kappa} a_\xi \cap c' = (a \setminus a') \cap c'$ (394Gd). But now

$$a \cap c' = a \cap c \cap c' \preccurlyeq^\tau_G b \cap c' \preccurlyeq^\tau_G (a \cap c') \setminus (a' \cap c');$$

because G is fully non-paradoxical, $a' \cap c'$ must be 0, that is, $c' \subseteq \tilde{c}_0$ and $c' = 0$. As c' is arbitrary, $c_0 \subseteq \tilde{c}_0$ and $c_0 = \tilde{c}_0$. So c_0 has the required properties.

Remark By analogy with the notation I used in discussing the Hahn decomposition of countably additive functionals (326O), we might denote c_0 as '$[\![a \preccurlyeq^\tau_G b]\!]$', or perhaps '$[\![a \preccurlyeq^\tau_G b]\!]_{\mathfrak{C}}$', 'the region (in $\mathfrak{C}$) where $a \preccurlyeq^\tau_G b$'. The same notation would write $\operatorname{upr}(a, \mathfrak{C})$ as '$[\![a \neq 0]\!]_{\mathfrak{C}}$'.

394I The construction I wish to use depends essentially on L^0 spaces as described in §364. The next step is the following.

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$. Then there are non-negative u, $v \in L^0 = L^0(\mathfrak{C})$ such that

$$\begin{aligned}
[\![u \ge n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a disjoint family } \langle d_i \rangle_{i<n} \\
&\text{such that } c \cap a \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i < n\}, \\
[\![v \le n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a family } \langle d_i \rangle_{i<n} \\
&\text{such that } d_i \preccurlyeq^\tau_G a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}
\end{aligned}$$

for every $n \in \mathbb{N}$. Moreover, we can arrange that

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v>0]\!]=\operatorname{upr}(b,\mathfrak{C})$,
(iii) $u\le v\le u+\chi 1$.

Remark By writing 'max' in the formulae above, I mean to imply that the elements $[\![u\ge n]\!]$, $[\![v\le n]\!]$ belong to the sets described.

proof (a) Choose $\langle c_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ as follows. Given $\langle b_i\rangle_{i<n}$, set $b'_n=b\setminus\sup_{i<n}b_i$,

$$c_n=\sup\{c:a\cap c\preccurlyeq^\tau_G b'_n\},$$

so that $a\cap c_n\preccurlyeq^\tau_G b'_n$ (394H); choose $b_n\subseteq b'_n$ such that $a\cap c_n$ is G-τ-equidecomposable with b_n, and continue. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$.

For each n, we have $b'_n\setminus c_n\preccurlyeq^\tau_G a$, by 394H; while $a\cap c\not\preccurlyeq^\tau_G b'_n$ whenever $c\in\mathfrak{C}$ and $c\not\subseteq c_n$. Note also that, because $\operatorname{upr}(a,\mathfrak{C})=1$,

$$c_n=\operatorname{upr}(a\cap c_n,\mathfrak{C})=\operatorname{upr}(b_n,\mathfrak{C})\subseteq\operatorname{upr}(b'_n,\mathfrak{C}).$$

(b) Now $c_\infty=\inf_{n\in\mathbb{N}}c_n=0$. **P** $\langle b_n\cap c_\infty\rangle_{n\in\mathbb{N}}$ is a disjoint sequence, all G-τ-equidecomposable with $a\cap c_\infty$, so $a\cap c_\infty=0$; because $\operatorname{upr}(a,\mathfrak{C})=1$, it follows that $c_\infty=0$. **Q** Accordingly, if we set $u=\sup_{n\in\mathbb{N}}(n+1)\chi c_n$, $u\in L^0$ and $[\![u\ge n]\!]=c_{n-1}$ for $n\ge1$. The construction ensures that $[\![u\in\mathbb{N}]\!]$, as defined in 364H, is equal to 1.

(c) Consider next $c'_0=\operatorname{upr}(b,\mathfrak{C})$, $c'_n=c_{n-1}\cap\operatorname{upr}(b'_n,\mathfrak{C})$ for $n\ge1$. Then $\langle c'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with zero infimum, so again we can define $v\in L^0$ by setting $v=\sup_{n\in\mathbb{N}}(n+1)\chi c'_n$. Once again, $[\![v\in\mathbb{N}]\!]=1$, and $[\![v\le n]\!]=1\setminus c'_n$ for each n.

Of course $[\![v>0]\!]=c'_0=\operatorname{upr}(b,\mathfrak{C})$. Because $c_n\subseteq c'_n\subseteq c_{n-1}$,

$$(n+1)\chi c_n\le(n+1)\chi c'_n\le n\chi c_{n-1}+\chi1$$

for each $n\ge1$, and $u\le v\le u+\chi1$.

(d) Now set

$$C_n=\{c:c\in\mathfrak{C},\text{ there is a disjoint family }\langle d_i\rangle_{i<n}\text{ such that }c\cap a\preccurlyeq^\tau_G d_i\subseteq b\text{ for every }i<n\}.$$

Then $c_n=\max C_{n+1}$.

P(α) Because $c_n\subseteq c_{n-1}\subseteq\ldots\subseteq c_0$, $a\cap c_n\preccurlyeq^\tau_G b_i$ for every $i\le n$, so that $\langle b_i\rangle_{i\le n}$ witnesses that $c_n\in C_{n+1}$.

(β) Suppose that $c\in C_{n+1}$; let $\langle d_i\rangle_{i\le n}$ be a disjoint family such that $c\cap a\preccurlyeq^\tau_G d_i\subseteq b$ for every i. Set $c'=c\setminus c_n$. For each $i<n$, $b_i\preccurlyeq^\tau_G a$, so

$$b_i\cap c'\preccurlyeq^\tau_G a\cap c'\preccurlyeq^\tau_G d_i\cap c',$$

while also

$$b'_n\cap c'\preccurlyeq^\tau_G a\cap c'\preccurlyeq^\tau_G d_n\cap c'.$$

Take $d\subseteq d_n\cap c'$ such that $b'_n\cap c'$ is G-τ-equidecomposable with d. Then

$$b\cap c'=(b'_n\cap c')\cup\sup_{i<n}(b_i\cap c')\preccurlyeq^\tau_G d\cup\sup_{i<n}(d_i\cap c')\subseteq b\cap c'.$$

Because G is fully non-paradoxical, $d\cup\sup_{i<n}(d_i\cap c')$ must be exactly $b\cap c'$, so d must be the whole of $d_n\cap c'$, and

$$a\cap c'\preccurlyeq^\tau_G d_n\cap c'=d\preccurlyeq^\tau_G b'_n.$$

But this means that $c'\subseteq c_n$. Thus $c'=0$ and $c\subseteq c_n$. So $c_n=\sup C_{n+1}=\max C_{n+1}$. **Q**

Accordingly

$$[\![u\ge n]\!]=c_{n-1}=\max C_n$$

for $n\ge1$. For $n=0$ we have $[\![u\ge0]\!]=1=\max C_0$. So $[\![u\ge n]\!]=\max C_n$ for every n, as required.

(e) Similarly, if we set

$$C'_n = \{c : c \in \mathfrak{C},\text{ there is a family } \langle d_i\rangle_{i<n}$$
$$\text{such that } d_i \preccurlyeq^\tau_G a \text{ for every } i<n \text{ and } b\cap c \subseteq \sup_{i<n} d_i\}$$

then $1\setminus c'_n = \max C'_n$ for every n.

P$(\boldsymbol{\alpha})$ If $n=0$, then of course (interpreting $\sup\emptyset$ as 0) $1\setminus c'_0 \in C'_0$ because $b\subseteq c'_0$. For each $n\in\mathbb{N}$, set

$$\tilde b_n = b_n \cup (b'_n\setminus c_n) = (b_n\cap c_n)\cup(b'_n\setminus c_n).$$

Because $b_n \preccurlyeq^\tau_G a$ and $b'_n\setminus c_n \preccurlyeq^\tau_G a$, we have $b_n\cap c_n \preccurlyeq^\tau_G a\cap c_n$ and $b'_n\setminus c_n \preccurlyeq^\tau_G a\setminus c_n$, so $\tilde b_n \preccurlyeq^\tau_G a$ (394Cd). If we look at

$$\sup_{i<n}\tilde b_i \supseteq \sup_{i<n} b_i \cup (b'_{n-1}\setminus c_{n-1}),$$

we see that, for $n\ge 1$,

$$b\setminus \sup_{i<n}\tilde b_i \subseteq b'_n\cap c_{n-1}\subseteq c'_n,$$

so that $b\setminus c'_n \subseteq \sup_{i<n}\tilde b_i$ and $\{\tilde b_i : i<n\}$ witnesses that $1\setminus c'_n\in C'_n$.

$(\boldsymbol{\beta})$ Now take any $c\in C'_n$ and a corresponding family $\langle d_i\rangle_{i<n}$ such that $d_i \preccurlyeq^\tau_G a$ for every $i<n$ and $b\cap c\subseteq \sup_{i<n} d_i$.

Set $c' = c\cap c'_n$. For each $i<n$,

$$c'\cap d_i \preccurlyeq^\tau_G c'\cap a \preccurlyeq^\tau_G b_i$$

because $c'\subseteq c_i$. So (by 394Cd, as usual)

$$c'\cap b \preccurlyeq^\tau_G c'\cap \sup_{i<n} b_i \subseteq c'\cap b,$$

and (again because G is fully non-paradoxical) $c'\cap b = c'\cap\sup_{i<n} b_i$, that is, $c'\cap b'_n = 0$. But $c'\subseteq c'_n\subseteq \operatorname{upr}(b'_n,\mathfrak{C})$, so c' must be 0, which means that $c\subseteq 1\setminus c'_n$. As c is arbitrary, $1\setminus c'_n = \sup C'_n = \max C'_n$. **Q**

Thus $[\![v\le n]\!] = \sup C'_n$, as declared.

394J Notation In the context of 394I, I will write $\lfloor b:a\rfloor$ for u, $\lceil b:a\rceil$ for v.

394K Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, b, b_1, $b_2\in\mathfrak{A}$ and that $\operatorname{upr}(a,\mathfrak{C})=1$.

(a) $\lfloor 0:a\rfloor = \lceil 0:a\rceil = 0$, $\lfloor 1:a\rfloor \ge \chi 1$ and $\lfloor 1:1\rfloor = \chi 1$.

(b) If $b_1 \preccurlyeq^\tau_G b_2$ then $\lfloor b_1:a\rfloor \le \lfloor b_2:a\rfloor$ and $\lceil b_1:a\rceil\le\lceil b_2:a\rceil$.

(c) $\lceil b_1\cup b_2 : a\rceil \le \lceil b_1:a\rceil + \lceil b_2:a\rceil$.

(d) If $b_1\cap b_2=0$, $\lfloor b_1:a\rfloor + \lfloor b_2:a\rfloor \le \lfloor b_1\cup b_2:a\rfloor$.

(e) If $c\in\mathfrak{C}$ is such that $a\cap c$ is a relative atom over $\mathfrak{C}$ (definition: 331A), then $c\subseteq [\![\lceil b:a\rceil - \lfloor b:a\rfloor = 0]\!]$.

proof (a)-(b) are immediate from the definitions and the basic properties of $\preccurlyeq^\tau_G$, $\lceil\ldots\rceil$ and $\lfloor\ldots\rfloor$, as listed in 394C, 394E and 394I.

(c) For j, $k\in\mathbb{N}$, set $c_{jk} = [\![\lceil b_1:a\rceil = j]\!]\cap[\![\lceil b_2:a\rceil = k]\!]$. Then

$$c_{jk}\subseteq [\![\lceil b_1\cup b_2:a\rceil \le j+k]\!]\cap[\![\lceil b_1:a\rceil + \lceil b_2:a\rceil = j+k]\!].$$

P We may suppose that $c_{jk}\ne 0$. Of course

$$c_{jk}\subseteq[\![\lceil b_1:a\rceil+\lceil b_2:a\rceil = j+k]\!].$$

Next, there are sets J, $J'\subseteq\mathfrak{A}$ such that $d\preccurlyeq^\tau_G a$ for every $d\in J\cup J'$, $\#(J)\le j$, $\#(J')\le k$, $\sup J\supseteq b_1\cap c_{jk}$ and $\sup J'\supseteq b_2\cap c_{jk}$. So $\sup(J\cup J')\supseteq(b_1\cup b_2)\cap c_{jk}$ and $J\cup J'$ witnesses that $c_{jk}\subseteq[\![\lceil b_1\cup b_2:a\rceil\le j+k]\!]$. **Q**

Accordingly

$$c_{jk}\subseteq[\![\lceil b_1:a\rceil + \lceil b_2:a\rceil - \lceil b_1\cup b_2:a\rceil \ge 0]\!].$$

Now as $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, we must have $\lceil b_1\cup b_2:a\rceil \le \lceil b_1:a\rceil + \lceil b_2:a\rceil$.

(d) This time, set $c_{jk} = [\![\lfloor b_1 : a\rfloor = j]\!] \cap [\![\lfloor b_2 : a\rfloor = k]\!]$ for j, $k \in \mathbb{N}$. Then

$$c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor \ge j+k]\!] \cap [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!]$$

for every j, $k \in \mathbb{N}$. **P** Once again, we surely have

$$c_{jk} \subseteq [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!].$$

Next, we can find a family $\langle d_i\rangle_{i<j+k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a\cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_1 \text{ for every } i<k,$$

$$\langle d_i\rangle_{j\le i<j+k} \text{ is disjoint, } a\cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_2 \text{ for } j\le i<j+k.$$

As $b_1\cap b_2 = 0$, the whole family $\langle d_i\rangle_{i<j+k}$ is disjoint and witnesses that $c_{jk} \subseteq [\![\lfloor b_1\cup b_2 : a\rfloor \ge j+k]\!]$. **Q**

So

$$c_{jk} \subseteq [\![\lfloor b_1\cup b_2 : a\rfloor - \lfloor b_1 : a\rfloor - \lfloor b_2 : a\rfloor \ge 0]\!]$$

Since $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, as before, we must have $\lfloor b_1\cup b_2 : a\rfloor \ge \lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor$.

(e) **?** Otherwise, there must be some $k\in\mathbb{N}$ such that

$$c_0 = c\cap[\![\lfloor b:a\rfloor = k]\!] \cap [\![\lceil b:a\rceil > k]\!] \ne 0.$$

Let $\langle d_i\rangle_{i<k}$ be a disjoint family in $\mathfrak{A}_b$ such that $a\cap c_0 \preccurlyeq^\tau_G d_i$ for each i; cutting the d_i down if necessary, we may suppose that $a\cap c_0$ is G-τ-equidecomposable with d_i for each i. As $c_0 \not\subseteq [\![\lceil b:a\rceil \le k]\!]$, $b\cap c_0 \not\subseteq \sup_{i<k} d_i$; set $d = b\cap c_0\setminus \sup_{i<k} d_i \ne 0$. If $c'\in\mathfrak{C}$ is non-zero and $c'\subseteq c_0$, then $a\cap c' \preccurlyeq^\tau_G d_i$ for every $i<k$, while $c' \not\subseteq [\![\lfloor b:a\rfloor \ge k+1]\!]$, so $a\cap c' \not\preccurlyeq^\tau_G d$; by 394H, $d\cap c_0 \preccurlyeq^\tau_G a$ and $d = d\cap c_0 \preccurlyeq^\tau_G a\cap c_0$. There is therefore a non-zero $\tilde a \subseteq a\cap c_0$ such that $\tilde a \preccurlyeq^\tau_G d$. But now remember that $a\cap c$ is supposed to be a relative atom over $\mathfrak{C}$, so $\tilde a = a\cap\tilde c$ for some $\tilde c\in\mathfrak{C}$ such that $\tilde c\subseteq c_0$. In this case, $a\cap\tilde c \preccurlyeq^\tau_G d_i$ for every $i<k$ and also $a\cap\tilde c \preccurlyeq^\tau_G d$, so $0\ne\tilde c\subseteq [\![\lfloor b:a\rfloor \ge k+1]\!]$, which is absurd. **X**

394L Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a_1, a_2, $b\in\mathfrak{A}$ and that $\operatorname{upr}(a_1,\mathfrak{C}) = \operatorname{upr}(a_2,\mathfrak{C}) = 1$. Then

$$\lfloor b:a_2\rfloor \ge \lfloor b:a_1\rfloor \times \lfloor a_1:a_2\rfloor, \quad \lceil b:a_2\rceil \le \lceil b:a_1\rceil \times \lceil a_1:a_2\rceil.$$

proof I use the same method as in 394K. As usual, write G^*_τ for the full local semigroup generated by G.

(a) For j, $k\in\mathbb{N}$ set

$$c_{j,k} = [\![\lfloor b:a_1\rfloor = j]\!] \cap [\![\lfloor a_1:a_2\rfloor = k]\!].$$

Then

$$c_{j,k} \subseteq [\![\lfloor b:a_1\rfloor \times \lfloor a_1:a_2\rfloor = jk]\!] \cap [\![\lfloor b:a_2\rfloor \ge jk]\!].$$

P Write c for $c_{j,k}$. As in parts (c) and (d) of the proof of 394K, it is elementary that c is included in $[\![\lfloor b:a_1\rfloor \times \lfloor a_1:a_2\rfloor = jk]\!]$; what we need to check is that $c\subseteq [\![\lfloor b:a_2\rfloor \ge jk]\!]$. Again, we may suppose that $c\ne 0$. There are families $\langle d_i\rangle_{i<j}$, $\langle d^*_l\rangle_{l<k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a_1\cap c \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i<j,$$

$$\langle d^*_l\rangle_{l<k} \text{ is disjoint, } a_2\cap c \preccurlyeq^\tau_G d^*_l \subseteq a_1 \text{ for every } l<k.$$

For each $i<j$, let $\phi_i\in G^*_\tau$ be such that $\phi_i(a_1\cap c)\subseteq d_i$. If $i<j$ and $l<k$, then

$$a_2\cap c \preccurlyeq^\tau_G d^*_l\cap c \preccurlyeq^\tau_G \phi_i(d^*_l\cap c) \subseteq \phi_i(a_1\cap c) \subseteq d_i \subseteq b.$$

Also $\langle\phi_i(d^*_l\cap c)\rangle_{i<j,l<k}$ is disjoint because $\langle\phi_i(a_1\cap c)\rangle_{i<j}$ and $\langle d^*_l\rangle_{l<k}$ are, so witnesses that c is included in $[\![\lfloor b:a_2\rfloor \ge jk]\!]$. **Q**

Now, just as in 394K, it follows from the fact that $\sup_{j,k\in\mathbb{N}} c_{j,k} = 1$ that $\lfloor b:a_1\rfloor \times \lfloor a_1:a_2\rfloor \le \lfloor b:a_2\rfloor$.

(b) For j, $k\in\mathbb{N}$ set

$$c_{j,k} = [\![\lceil b:a_1\rceil = j]\!] \cap [\![\lceil a_1:a_2\rceil = k]\!].$$

Then

$$c_{j,k} \subseteq [\![\lceil b : a_1 \rceil \times \lceil a_1 : a_2 \rceil = jk]\!] \cap [\![\lceil b : a_2 \rceil \leq jk]\!].$$

P Write c for $c_{j,k}$. Then $c \subseteq [\![\lceil b : a_1 \rceil \times \lceil a_1 : a_2 \rceil = jk]\!]$. There are families $\langle d_i \rangle_{i<j}$, $\langle d_l^* \rangle_{l<k}$ such that $d_i \preccurlyeq_G^\tau a_1$ for every $i < j$, $d_l^* \preccurlyeq_G^\tau a_2$ for every $l < k$, $b \cap c \subseteq \sup_{i<j} d_i$ and $a_1 \cap c \subseteq \sup_{l<k} d_l^*$. For each $i < j$, let $d_i' \subseteq a_1$ be G-τ-equidecomposable with d_i, and take $\phi_i \in G_\tau^*$ such that $\phi_i d_i' = d_i$. Then

$$\phi_i(d_i' \cap d_l^*) \preccurlyeq_G^\tau d_l^* \preccurlyeq_G^\tau a_2 \text{ for every } i < j,\, l < k,$$

$$\begin{aligned}\sup_{i<j,l<k} \phi_i(d_i' \cap d_l^*) &= \sup_{i<j} \phi_i(d_i' \cap \sup_{l<k} d_l^*) \supseteq \sup_{i<j} \phi_i(d_i' \cap c) \\ &= \sup_{i<j} d_i \cap c \supseteq b \cap c.\end{aligned}$$

So $\langle \phi_i(d_i' \cap d_l^*) \rangle_{i<j,l<k}$ witnesses that $c \subseteq [\![\lceil b : a_2 \rceil \leq jk]\!]$. **Q**

Once again, it follows easily that $\lceil b : a_1 \rceil \times \lceil a_1 : a_2 \rceil \geq \lceil b : a_2 \rceil$.

394M Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$.

(a) For any $a \in \mathfrak{A}$, there is a $b \subseteq a$ such that $b \preccurlyeq_G^\tau a \setminus b$ and $a' = a \setminus \operatorname{upr}(b, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$.

(b) Now suppose that G is fully non-paradoxical. Then for any $\epsilon > 0$ there is an $a \in \mathfrak{A}$ such that $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $\lceil b : a \rceil \leq \lfloor b : a \rfloor + \epsilon \lfloor 1 : a \rfloor$ for every $b \in \mathfrak{A}$.

proof (a) Set $B = \{d : d \subseteq a,\ d \preccurlyeq_G^\tau a \setminus d\}$ and let $D \subseteq B$ be a maximal subset such that $\operatorname{upr}(d, \mathfrak{C}) \cap \operatorname{upr}(d', \mathfrak{C}) = 0$ for all distinct d, $d' \in \mathfrak{D}$. Set $b = \sup D$. For any $d \in D$, $d \preccurlyeq_G^\tau a \setminus d$, so

$$b \cap \operatorname{upr}(d, \mathfrak{C}) = d \cap \operatorname{upr}(d, \mathfrak{C}) \preccurlyeq_G^\tau (a \setminus d) \cap \operatorname{upr}(d, \mathfrak{C}) = (a \setminus b) \cap \operatorname{upr}(d, \mathfrak{C}) \subseteq a \setminus b$$

by 394Gc. By 394H,

$$b = b \cap \sup\nolimits_{d \in D} \operatorname{upr}(d, \mathfrak{C}) \preccurlyeq_G^\tau a \setminus b.$$

? Suppose, if possible, that $a' = a \setminus \operatorname{upr}(b, \mathfrak{C})$ is not a relative atom over $\mathfrak{C}$. Let $d_0 \subseteq a'$ be an element not expressible as $a' \cap c$ for any $c \in \mathfrak{C}$; then $d_0 \neq a \cap \operatorname{upr}(d_0, \mathfrak{C})$ and there must be a $\pi \in G$ such that $d_1 = \pi d_0 \cap a \setminus d_0$ is non-zero. In this case

$$d_1 \preccurlyeq_G^\tau \pi^{-1} d_1 \subseteq d_0 \subseteq a \setminus d_1,$$

so $d_1 \in B$; but also

$$d_1 \cap \operatorname{upr}(d, \mathfrak{C}) \subseteq d_1 \cap \operatorname{upr}(b, \mathfrak{C}) = 0,$$

so $\operatorname{upr}(d_1, \mathfrak{C}) \cap \operatorname{upr}(d, \mathfrak{C}) = 0$, for every $d \in D$, and we ought to have put d_1 into D. **X**

Thus b has the required properties.

(b)(i) For every $n \in \mathbb{N}$ we can find $a_n \in \mathfrak{A}$ and $c_n \in \mathfrak{C}$ such that $\operatorname{upr}(a_n, \mathfrak{C}) = 1$, $a_n \setminus c_n$ is a relative atom over $\mathfrak{C}$, and $\lfloor 1 : a_n \rfloor \geq 2^n \chi c_n$. **P** Induce on n. The induction starts with $a_0 = c_0 = 1$, because $\lfloor 1 : 1 \rfloor = \chi 1$. For the inductive step, having found a_n and c_n, let $d \subseteq a_n \cap c_n$ be such that $d \preccurlyeq_G^\tau a_n \cap c_n \setminus d$ and $a_n \cap c_n \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$, as in (a). Set $c_{n+1} = \operatorname{upr}(d, \mathfrak{C})$, $a_{n+1} = (a_n \setminus c_{n+1}) \cup d$; then

$$\begin{aligned}\operatorname{upr}(a_{n+1}, \mathfrak{C}) &= \operatorname{upr}(a_n \setminus c_{n+1}, \mathfrak{C}) \cup \operatorname{upr}(d, \mathfrak{C}) \\ &= (\operatorname{upr}(a_n, \mathfrak{C}) \setminus c_{n+1}) \cup c_{n+1} = (1 \setminus c_{n+1}) \cup c_{n+1} = 1\end{aligned}$$

by 314Vb-314Vc and the inductive hypothesis.

We have $c_{n+1} \cap d \preccurlyeq_G^\tau c_{n+1} \cap a_n \setminus d$, so

$$c_{n+1} \cap a_{n+1} = d \subseteq a_n, \quad c_{n+1} \cap a_{n+1} \preccurlyeq_G^\tau a_n \setminus d,$$

and $\lfloor a_n : a_{n+1} \rfloor \geq 2\chi c_{n+1}$; by 394L,

$$\lfloor 1 : a_{n+1} \rfloor \geq \lfloor 1 : a_n \rfloor \times \lfloor a_n : a_{n+1} \rfloor \geq 2^n \chi c_n \times 2\chi c_{n+1} = 2^{n+1} \chi c_{n+1}.$$

If

$$b \subseteq a_{n+1} \setminus c_{n+1} = (a_n \setminus c_n) \cup (a_n \cap c_n \setminus c_{n+1}),$$

then, because both terms on the right are relative atoms over $\mathfrak{C}$, there are c', $c'' \in \mathfrak{C}$ such that

$$\begin{aligned} b &= (b \cap a_n \setminus c_n) \cup (b \cap a_n \cap c_n \setminus c_{n+1}) \\ &= (c' \cap a_n \setminus c_n) \cup (c'' \cap a_n \cap c_n \setminus c_{n+1}) = c \cap a_{n+1} \setminus c_{n+1} \end{aligned}$$

where $c = (c' \setminus c_n) \cup (c'' \cap c_n)$ belongs to $\mathfrak{C}$. So $a_{n+1} \setminus c_{n+1}$ is a relative atom over $\mathfrak{C}$.

Thus the induction continues. **Q**

(ii) Now suppose that $\epsilon > 0$. Take n such that $2^{-n} \le \epsilon$, and consider a_n, c_n taken from (i) above. Let $b \in \mathfrak{A}$. Set

$$c = [\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor - \epsilon \lfloor 1 : a_n \rfloor > 0]\!] \in \mathfrak{C}.$$

Since we know that

$$\epsilon \lfloor 1 : a_n \rfloor \ge 2^{-n} 2^n \chi c_n = \chi c_n, \quad \lceil b : a_n \rceil \le \lfloor b : a_n \rfloor + \chi 1,$$

we must have $c \cap c_n = 0$. But this means that $a_n \cap c$ is a relative atom over $\mathfrak{C}$. By 394Ke, c is included in $[\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor = 0]\!]$; as also $\lfloor 1 : a_n \rfloor \ge \chi 1$ (394Ka), c must be zero, that is, $\lceil b : a_n \rceil \le \lfloor b : a_n \rfloor + \epsilon \lfloor 1 : a_n \rfloor$.

394N We are at last ready for the theorem.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then there is a unique function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and order-continuous;

(ii) $[\![\theta a > 0]\!] = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$; in particular, $\theta a = 0$ iff $a = 0$;

(iii) $\theta 1 = \chi 1$;

(iv) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(v) If a, $b \in \mathfrak{A}$ are G-τ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so I suppose henceforth that $\mathfrak{A} \ne \{0\}$.

(a) Set $A^* = \{a : a \in \mathfrak{A}, \operatorname{upr}(a, \mathfrak{C}) = 1\}$ and for $a \in A^*$, $b \in \mathfrak{A}$ set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0 = L^0(\mathfrak{C});$$

the first thing to note is that because $\lfloor 1 : a \rfloor \ge \chi 1$, we can always do the divisions to obtain elements $\theta_a(b)$ of $L^0(\mathfrak{A})$ (364P). Set

$$\theta b = \inf_{a \in A^*} \theta_a b$$

for $b \in \mathfrak{A}$. (Note that $L^0(\mathfrak{C})$ is Dedekind complete (364O), so the infimum is defined.)

(b) The formulae of 394K tell us that, for $a \in A^*$ and b_1, $b_2 \in \mathfrak{A}$,

$$\theta_a 0 = 0, \quad \theta_a b_1 \le \theta_a b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta_a(b_1 \cup b_2) \le \theta_a b_1 + \theta_a b_2,$$

$$\theta_a 1 \ge \chi 1.$$

It follows at once that

$$\theta 0 = 0, \quad \theta b_1 \le \theta b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta 1 \ge \chi 1.$$

(c) For each $n \in \mathbb{N}$ there is an $e_n \in A^*$ such that $\lceil b : e_n \rceil \le \lfloor b : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$ for every $b \in \mathfrak{A}$ (394M). Now $\theta_{e_n} b \le \theta_a b + 2^{-n} \lceil b : a \rceil$ for every $a \in A^*$, $b \in \mathfrak{A}$. **P** $\lceil a : e_n \rceil \le \lfloor a : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$, so

$$\begin{aligned}\lceil a:e_n\rceil\times\lfloor 1:a\rfloor&\le\lfloor a:e_n\rfloor\times\lfloor 1:a\rfloor+2^{-n}\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor\\&\le\lfloor 1:e_n\rfloor+2^{-n}\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor\end{aligned}$$

(by 394L); accordingly

$$\lceil b:e_n\rceil\times\lfloor 1:a\rfloor\le\lceil b:a\rceil\times\lceil a:e_n\rceil\times\lfloor 1:a\rfloor$$

(by the other half of 394L)

$$\le\lceil b:a\rceil\times\lfloor 1:e_n\rfloor+2^{-n}\lceil b:a\rceil\times\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor$$

and, dividing by $\lfloor 1:a\rfloor\times\lfloor 1:e_n\rfloor$, we get $\theta_{e_n}b\le\theta_a b+2^{-n}\lceil b:a\rceil$. **Q**

(d) Now θ is additive. **P** Taking $\langle e_n\rangle_{n\in\mathbb{N}}$ from (c), observe first that

$$\inf_{n\in\mathbb{N}}\theta_{e_n}b\le\theta_a b+\inf_{n\in\mathbb{N}}2^{-n}\lceil b:a\rceil=\theta_a b$$

for every $a\in A^*$, $b\in\mathfrak{A}$, so that $\theta b=\inf_{n\in\mathbb{N}}\theta_{e_n}b$ for every b. Now suppose that b_1, $b_2\in\mathfrak{A}$ and $b_1\cap b_2=0$. Then, for any $n\in\mathbb{N}$,

$$\begin{aligned}\lceil b_1:e_n\rceil+\lceil b_2:e_n\rceil&\le\lfloor b_1:e_n\rfloor+\lfloor b_2:e_n\rfloor+2^{-n+1}\lfloor 1:e_n\rfloor\\&\le\lfloor b_1\cup b_2:e_n\rfloor+2^{-n+1}\lfloor 1:e_n\rfloor\end{aligned}$$

(by 394Kd)

$$\le\lceil b_1\cup b_2:e_n\rceil+2^{-n+1}\lfloor 1:e_n\rfloor.$$

Dividing by $\lfloor 1:e_n\rfloor$, we have

$$\theta b_1+\theta b_2\le\theta_{e_n}b_1+\theta_{e_n}b_2\le\theta_{e_n}(b_1\cup b_2)+2^{-n+1}\chi 1.$$

Taking the infimum over n, we get

$$\theta b_1+\theta b_2\le\theta(b_1\cup b_2).$$

In the other direction, if a, $a'\in A^*$ and $n\in\mathbb{N}$,

$$\begin{aligned}\theta(b_1\cup b_2)&\le\theta_{e_n}(b_1\cup b_2)\le\theta_{e_n}(b_1)+\theta_{e_n}(b_2)\\&\le\theta_a(b_1)+2^{-n}\lceil b_1:a\rceil+\theta_{a'}(b_2)+2^{-n}\lceil b_2:a'\rceil.\end{aligned}$$

As n is arbitrary, $\theta(b_1\cup b_2)\le\theta_a(b_1)+\theta_{a'}(b_2)$; as a and a' are arbitrary, $\theta(b_1\cup b_2)\le\theta b_1+\theta b_2$ (using 351Dc). As b_1 and b_2 are arbitrary, θ is additive. **Q**

We see also that $\lceil 1:e_n\rceil\le(1+2^{-n})\lfloor 1:e_n\rfloor$, so that $\theta_{e_n}1\le(1+2^{-n})\chi 1$ for each n; since we already know that $\theta 1\ge\chi 1$, we have $\theta 1=\chi 1$ exactly.

(e) If $c\in\mathfrak{C}$ then

$$[\![\theta c>0]\!]\subseteq[\![\theta_1 c>0]\!]\subseteq[\![\lceil c:1\rceil>0]\!]=\operatorname{upr}(c,\mathfrak{C})=c$$

(394I(ii)). It follows that

$$\theta(b\cap c)\le\theta b\wedge\theta c\le\theta b\times\chi c$$

for any $b\in\mathfrak{A}$, $c\in\mathfrak{C}$. Similarly, $\theta(b\setminus c)\le\theta b\times\chi(1\setminus c)$; adding, we must have equality in both, and $\theta(b\cap c)=\theta b\times\chi c$.

Rather late, I point out that

$$0\le\theta a\le\theta 1=\chi 1\in L^\infty=L^\infty(\mathfrak{C})$$

for every $a\in\mathfrak{A}$, so that $\theta a\in L^\infty$ for every a.

(f) If $b\in\mathfrak{A}\setminus\{0\}$, then

$$[\![\theta b>0]\!]\subseteq[\![\theta_1 b>0]\!]\subseteq[\![\lceil b:1\rceil>0]\!]=\operatorname{upr}(b,\mathfrak{C})$$

by 394I(ii) again. **?** Suppose, if possible, that $[\![\theta b>0]\!]\ne\operatorname{upr}(b,\mathfrak{C})$. Set $c_0=\operatorname{upr}(b,\mathfrak{C})\setminus[\![\theta b>0]\!]$, $a_0=b\cup(1\setminus\operatorname{upr}(b,\mathfrak{C}))\in A^*$. Let $k\ge 1$ be such that $c_1=c_0\cap[\![\lceil 1:a_0\rceil\le k]\!]\ne 0$. Then $a_0\cap c_1=b\cap c_1$, so

$$\theta a_0\times\chi c_1=\theta(a_0\cap c_1)=\theta(b\cap c_1)=\theta b\times\chi c_1=0.$$

By 364Nb, there is an $a\in A^*$ such that $c_1\not\subseteq[\![\theta_a a_0\times\chi c_1\ge\frac{1}{k}]\!]$, that is, $c_2=c_1\cap[\![\theta_a a_0<\frac{1}{k}]\!]\ne 0$. Now

$$c_2\subseteq[\![\lfloor 1:a\rfloor-k\lceil a_0:a\rceil>0]\!]\subseteq[\![\lceil 1:a_0\rceil\times\lceil a_0:a\rceil-k\lceil a_0:a\rceil>0]\!]\subseteq[\![\lceil 1:a_0\rceil>k]\!],$$

which is impossible, as $c_2\subseteq c_1$. **X**

Thus $[\![\theta b>0]\!]=\operatorname{upr}(b,\mathfrak{C})$. In particular, $\theta b=0$ iff $b=0$.

(g) If b, $b'\in\mathfrak{A}$ and $b\preccurlyeq_G^\tau b'$, then $\theta b\le\theta b'$. **P** For every $a\in A^*$, $\lceil b:a\rceil\le\lceil b':a\rceil$ (394Kb) so $\theta_a b\le\theta_a b'$. **Q** So if b, $b'\in\mathfrak{A}$ and $c=[\![\theta b-\theta b'>0]\!]$, $b'\cap c\preccurlyeq_G^\tau b$. **P?** Otherwise, by 394H, there is a non-zero $c'\subseteq c$ such that $b\cap c'\preccurlyeq_G^\tau b'$. But in this case $\theta b\times\chi c'=\theta(b\cap c')\le\theta b'$ and $c'\subseteq[\![\theta b'-\theta b\ge 0]\!]$. **XQ**

(h) If $\langle a_i\rangle_{i\in I}$ is any disjoint family in $\mathfrak{A}$ with supremum a, $\theta a=\sum_{i\in I}\theta a_i$, where the sum is to be interpreted as $\sup_{J\subseteq I\text{ is finite}}\sum_{i\in J}\theta a_i$. **P** Induce on $\#(I)$. If $\#(I)$ is finite, this is just finite additivity ((d) above). For the inductive step to $\#(I)=\kappa\ge\omega$, we may suppose that I is actually equal to the cardinal κ. Of course

$$\theta a\ge\theta(\sup_{\xi\in J}a_\xi)=\sum_{\xi\in J}\theta a_\xi$$

for every finite $J\subseteq\kappa$, so (because $L^\infty(\mathfrak{C})$ is Dedekiond complete) $u=\sum_{\xi<\kappa}\theta a_\xi$ is defined, and $u\le\theta a$.

For $\zeta<\kappa$, set $b_\zeta=\sup_{\xi<\zeta}a_\xi$. By the inductive hypothesis,

$$\theta b_\zeta=\sum_{\xi<\zeta}\theta a_\xi=\sup_{J\subseteq\zeta\text{ is finite}}\sum_{\xi\in J}\theta a_\xi\le u.$$

At the same time, if $J\subseteq\kappa$ is finite, there is some $\zeta<\kappa$ such that $J\subseteq\zeta$, so that $\sum_{\xi\in J}\theta a_\xi\le\theta b_\zeta$; accordingly $\sup_{\zeta<\kappa}\theta b_\zeta=u$.

? Suppose, if possible, that $u<\theta a$; set $v=\theta a-u$. Take $\delta>0$ such that $c_0=[\![v>\delta]\!]\ne 0$. Let $\zeta<\kappa$ be such that $c_1=c_0\setminus[\![u-\theta b_\zeta>\delta]\!]$ is non-zero (cf. 364Nb). Now $u-\theta b_\zeta\le\theta(a\setminus b_\zeta)$, so

$$c_1\subseteq[\![\theta(a\setminus b_\zeta)>0]\!]=\operatorname{upr}(a\setminus b_\zeta,\mathfrak{C}),$$

and $c_1\cap(a\setminus b_\zeta)\ne 0$; there is therefore an $\eta'\ge\zeta$ such that $d=c_1\cap a_{\eta'}\ne 0$. Since $\theta d\le u-\theta b_\zeta$ and $c_1\subseteq[\![u-\theta b_\zeta\le\delta]\!]\cap[\![v>\delta]\!]$, $[\![v-\theta d>0]\!]\supseteq c_1$.

Choose $\langle d_\xi\rangle_{\xi<\kappa}$ inductively, as follows. Given that $\langle d_\eta\rangle_{\eta<\xi}$ is a disjoint family in $\mathfrak{A}_{a\setminus d}$ such that d_η is G-τ-equidecomposable with $a_\eta\cap c_1$ for every $\eta<\xi$, then $e_\xi=\sup_{\eta<\xi}d_\eta$ is G-τ-equidecomposable with $b_\xi\cap c_1$, so that $\theta e_\xi\le\theta b_\xi$, and

$$\begin{aligned}[\![\theta(a\setminus(d\cup e_\xi))-\theta a_\xi>0]\!]&=[\![\theta a-\theta d-\theta e_\xi-\theta a_\xi>0]\!]\supseteq[\![\theta a-\theta d-\theta b_\xi-\theta a_\xi>0]\!]\\&=[\![\theta a-\theta d-\theta b_{\xi+1}>0]\!]\supseteq[\![v-\theta d>0]\!]\supseteq c_1.\end{aligned}$$

By (g), $a_\xi\cap c_1\preccurlyeq_G^\tau a\setminus(d\cup e_\xi)$; take $d_\xi\subseteq a\setminus(d\cup e_\xi)$ G-τ-equidecomposable with $a_\xi\cap c_1$, and continue.

At the end of this induction, we have a disjoint family $\langle d_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}_{a\setminus d}$ such that d_ξ is G-τ-equidecomposable with $a_\xi\cap c_1$ for every ξ. But this means that $a'=\sup_{\xi<\kappa}d_\xi$ is G-τ-equidecomposable with $a\cap c_1$, while $a'\subseteq(a\setminus d)\cap c_1$; since $d\cap c_1\ne 0$, G cannot be fully non-paradoxical. **X**

Thus $\theta a=u=\sum_{\xi<\kappa}\theta a_\xi$ and the induction continues. **Q**

(i) It follows that θ is order-continuous. **P** (α) If $B\subseteq\mathfrak{A}$ is non-empty and upwards-directed and has supremum e, then $\bigcup_{b\in B}\mathfrak{A}_b$ is order-dense in $\mathfrak{A}_e$, so includes a partition of unity A of $\mathfrak{A}_e$; now (h) tells us that

$$\theta e=\sum_{a\in A}\theta a\le\sup_{b\in B}\theta b.$$

Since of course $\theta b\le\theta e$ for every $b\in B$, $\theta e=\sup_{b\in B}\theta b$. (β) If $B\subseteq\mathfrak{A}$ is non-empty and downwards-directed and has infimum e, then, using (α), we see that

$$\theta 1-\theta e=\theta(1\setminus e)=\sup_{b\in B}\theta(1\setminus b)=\sup_{b\in B}\theta 1-\theta b,$$

so that $\theta e=\inf_{b\in B}\theta b$. **Q**

(j) I still have to show that θ is unique. Let $\theta':\mathfrak{A}\to L^\infty$ be any non-negative order-continuous G-invariant additive function such that $\theta'c=\chi c$ for every $c\in\mathfrak{C}$.

(i) Just as in (e) of this proof, but more easily, we see that $\theta'(b\cap c)=\theta'b\times\chi c$ for every $b\in\mathfrak{A}$, $c\in\mathfrak{C}$.

(ii) If $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a, then $\langle\sup_{i\in J}a_i\rangle_{J\subseteq I\text{ is finite}}$ is an upwards-directed family with supremum a, so that

$$\theta'a=\sup_{J\subseteq I\text{ is finite}}\theta'(\sup_{i\in J}a_i)=\sup_{J\subseteq I\text{ is finite}}\sum_{i\in J}\theta'a_i=\sum_{i\in I}\theta'a_i.$$

(iii) $\theta'a=\theta'b$ whenever a and b are G-τ-equidecomposable. **P** Take a partition $\langle a_i\rangle_{i\in I}$ of a and a family $\langle\pi_i\rangle_{i\in I}$ in G such that $\langle\pi_i a_i\rangle_{i\in I}$ is a partition of b. Then

$$\theta'a=\sum_{i\in I}\theta'a_i=\sum_{i\in I}\theta'\pi_i a_i=\theta'b.\ \mathbf{Q}$$

Consequently $\theta'a\le\theta'b$ whenever $a\preccurlyeq_G^\tau b$.

(iv) Take $a\in A^*$, $b\in\mathfrak{A}$ and for j, $k\in\mathbb{N}$ set $c_{jk}=[\![\lfloor 1:a\rfloor=j]\!]\cap[\![\lceil b:a\rceil=k]\!]$. Then

$$\lceil b:a\rceil\times\chi c_{jk}\ge\theta'b\times\lfloor 1:a\rfloor\times\chi c_{jk}.$$

P If $c_{jk}=0$ this is trivial; suppose $c_{jk}\ne 0$. Now we have sets I, J such that $\#(I)=j$, $\#(J)\le k$, $a\cap c_{jk}\preccurlyeq_G^\tau d$ for every $d\in I$, $e\preccurlyeq_G^\tau a$ for every $e\in J$, I is disjoint, and $b\cap c_{jk}\subseteq\sup J$. So

$$\begin{aligned}\theta'b\times\lfloor 1:a\rfloor\times\chi c_{jk}&=j\theta'b\times\chi c_{jk}=j\theta'(b\cap c_{jk})\le j\sum_{e\in J}\theta'(e\cap c_{jk})\\&\le jk\theta'(a\cap c_{jk})\le k\sum_{d\in I}\theta'(d\cap c_{jk})\le k\theta'c_{jk}\\&=k\chi c_{jk}=\lceil b:a\rceil\times\chi c_{jk}.\ \mathbf{Q}\end{aligned}$$

Summing over j and k, $\lceil b:a\rceil\ge\theta'b\times\lfloor 1:a\rfloor$, that is, $\theta_a b\ge\theta'b$. Taking the infimum over a, $\theta b\ge\theta'b$. But also

$$\theta b=\chi 1-\theta(1\setminus b)\le\chi 1-\theta'(1\setminus b)=\theta'b,$$

so $\theta b=\theta'b$. As b is arbitrary, $\theta=\theta'$. This completes the proof.

394O We have reached the summit. The rest of the section is a list of easy corollaries.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Then there is a G-invariant additive functional $\nu:\mathfrak{A}\to[0,1]$ such that $\nu 1=1$.

proof Let $\mathfrak{C}$ be the fixed-point subalgebra of G, and $\theta:\mathfrak{A}\to L^\infty(\mathfrak{C})$ the function of 394N. By 311D, there is a ring homomorphism $\nu_0:\mathfrak{C}\to\{0,1\}$ such that $\nu_0 1=1$; now ν_0 can also be regarded as an additive functional from $\mathfrak{C}$ to $\mathbb{R}$. Let $f_0:L^\infty(\mathfrak{C})\to\mathbb{R}$ be the corresponding positive linear functional (363K). Set $\nu=f_0\theta$. Then ν is order-preserving and additive because f_0 and θ are, $\nu 1=f_0(\chi 1)=\nu_0 1>0$, and ν is G-invariant because θ is.

394P Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a measurable algebra;
(ii) $\mathfrak{C}$ is a measurable algebra;
(iii) there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

proof (iii)$\Rightarrow$(i)$\Rightarrow$(ii) are trivial. For (ii)$\Rightarrow$(iii), let $\theta:\mathfrak{A}\to L^\infty(\mathfrak{C})$ be the function of 394N, and $\bar\nu:\mathfrak{C}\to\mathbb{R}$ a strictly positive countably additive functional. Let $f:L^\infty(\mathfrak{C})\to\mathbb{R}$ be the corresponding linear operator; then f is sequentially order-continuous (363K again). Set $\bar\mu=f\theta$. Then $\bar\mu$ is additive and order-preserving and sequentially order-continuous because f and θ are. It is also strictly positive, because if $a\in\mathfrak{A}\setminus\{0\}$ then $\theta a>0$ (394N(ii)), that is, there is some $\delta>0$ such that $[\![\theta a>\delta]\!]\ne 0$, so that

$$\bar\mu a\ge\delta\bar\nu[\![\theta a>\delta]\!]>0.$$

Finally, $\bar\mu$ is G-invariant because θ is.

394Q Corollary: Kawada's Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ is ergodic and fully non-paradoxical. Then $\mathfrak{A}$ is measurable.

proof This is the case $\mathfrak{C} = \{0, 1\}$ of 394P.

394R Thus the existence of an ergodic fully non-paradoxical subgroup is a sufficient condition for a Dedekind complete Boolean algebra to be measurable. It is not quite necessary, because if a measure algebra $\mathfrak{A}$ is not homogeneous then its automorphism group is not ergodic. But for homogeneous algebras the condition is necessary as well as sufficient, by the following result.

Proposition If $(\mathfrak{A}, \bar{\mu})$ is a homogeneous totally finite measure algebra, the group $G = \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ of measure-preserving automorphisms of $\mathfrak{A}$ is ergodic.

proof If $\mathfrak{A} = \{0, 1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. If $a \in \mathfrak{A} \setminus \{0, 1\}$, set $\gamma = \min(\bar{\mu}a, \bar{\mu}(1 \setminus a)) > 0$; then there are $b \subseteq a$, $d \subseteq 1 \setminus a$ such that $\bar{\mu}b = \bar{\mu}d = \gamma$. By 382Fb, there is a $\pi \in G$ such that $\pi b = d$, so that $\pi a \neq a$. As a is arbitrary, the fixed-point subalgebra of G is $\{0, 1\}$.

394X Basic exercises (a) Re-write the section on the assumption that every group G is ergodic, that is, $\mathfrak{C}$ is always $\{0, 1\}$, so that $L^0(\mathfrak{C})$ may be identified with $\mathbb{R}$, the functions $\lceil \dots \rceil$ and $\lfloor \dots \rfloor$ become real-valued, the functionals θ_a (394N) become submeasures and θ becomes a measure.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\langle c_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{C}$ and that a, $b \in \mathfrak{A}$ are such that $a \cap c_i \preccurlyeq_G^\tau b$ for every $i \in I$. Show that $a \preccurlyeq_G^\tau b$.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ iff the full local semigroup generated by G has many involutions (definition: 381P).

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) there is a strictly positive real-valued additive functional on $\mathfrak{A}$; (ii) there is a strictly positive real-valued additive functional on $\mathfrak{C}$; (iii) there is a strictly positive G-invariant real-valued additive functional on $\mathfrak{A}$.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) there is a non-zero completely additive functional on $\mathfrak{A}$; (ii) there is a non-zero completely additive functional on $\mathfrak{C}$; (iii) there is a non-zero G-invariant completely additive functional on $\mathfrak{A}$.

(f) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Show that it is a measurable algebra iff there is a fully non-paradoxical subgroup G of $\operatorname{Aut}\mathfrak{A}$ such that the fixed-point subalgebra of G is purely atomic.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that the following are equiveridical: (i) $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic; (ii) $\mathfrak{A}$ is quasi-homogeneous in the sense of 374G; (iii) whenever a, $b \in \mathfrak{A}$ and $\mu a = \mu b$ then the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$ are isomorphic.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is fully non-paradoxical iff (i) for every infinite cardinal κ, the Maharam-type-κ component of $\mathfrak{A}$ (definition: 332G) has finite measure (ii) for every $\gamma \in \left]0, \infty\right[$ there are only finitely many atoms of measure γ.

394Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. For ϕ, $\psi \in G^*_\tau$, say that $\phi \leq \psi$ if ψ extends ϕ. (i) Show that every member of G^*_τ can be extended to a maximal member of G^*_τ. (ii) Show that G is fully non-paradoxical iff every maximal member of G^*_τ is actually a Boolean automorphism of $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that $\mathfrak{A}$ is ccc iff $\mathfrak{C}$ is ccc. (*Hint*: if $\mathfrak{C}$ is ccc, $L^\infty(\mathfrak{C})$ has the countable sup property (363Yb).)

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $\mathfrak{C}$ is.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G an ergodic subgroup of $\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. Suppose that there is a non-zero $a \in \mathfrak{A}$ for which there is no $\phi \in G^*_\tau$ such that $\phi a \subset a$. Show that there is a measure $\bar{\mu}$ such that $(\mathfrak{A},\bar{\mu})$ is a localizable measure algebra. (*Hint*: show that $\mathfrak{A}_a$ is a measurable algebra.)

394Z Problem Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that whenever $\langle a_i\rangle_{i\le n}$ is a finite partition of unity in $\mathfrak{A}$ and we are given π_i, $\pi'_i \in G$ for every $i \le n$, then the elements $\pi_0 a_0$, $\pi' a_0$, $\pi_1 a_1$, $\pi'_1 a_1, \dots , \pi'_n a_n$ are not all disjoint. Must there be a non-zero non-negative G-invariant finitely additive functional θ on $\mathfrak{A}$?

(See 'Tarski's theorem' in the notes below.)

394 Notes and comments Regarded as a sufficient condition for measurability, Kawada's theorem suffers from the obvious defect that it is going to be rather rarely that we can verify the existence of an ergodic fully non-paradoxical group of automorphisms without having some quite different reason for supposing that our algebra is measurable. If we think of it as a criterion for the existence of a G-invariant measure, rather than as a criterion for measurability in the abstract, it seems to make better sense. But if we know from the start that the algebra $\mathfrak{A}$ is measurable, the argument short-circuits, as we shall see in §395.

I take the trouble to include the 'τ' in every 'G-τ-equidecomposable', 'G^*_τ' and '$\preccurlyeq^\tau_G$' because there are important variations on the concept, in which the partitions $\langle a_i\rangle_{i\in I}$ of 394A are required to be finite or countable. Indeed **Tarski's theorem** relies on one of these. I spell it out because it is close to Kawada's in spirit, though there are significant differences in the ideas needed in the proof:

> Let X be a set and G a subgroup of $\operatorname{Aut}\mathcal{P}X$. Then the following are equiveridical: (i) there is a G-invariant additive functional $\theta : \mathcal{P}X \to [0,1]$ such that $\theta A = 1$; (ii) there are no $A_0, \dots , A_n$, $\pi_0, \dots , \pi_n$, $\pi'_0, \dots , \pi'_n$ such that $A_0, \dots , A_n$ are subsets of X covering X, $\pi_0, \dots , \pi'_n$ all belong to G, and $\pi_0 A_0$, $\pi'_0 A_0$, $\pi_1 A_1$, $\pi'_1 A_1, \dots , \pi'_n A_n$ are all disjoint.

For a proof, see 449I in Volume 4; for an illuminating discussion of this theorem, see Wagon 85, Chapter 9. But it seems to be unknown whether the natural translation of this result is valid in all Dedekind complete Boolean algebras (394Z). Note that we are looking for theorems which do not depend on any special properties of the group G or the Boolean algebra $\mathfrak{A}$. For abelian or 'amenable' groups, or weakly (σ,∞)-distributive algebras, for instance, much more can be done, as described in 395Ya and §449.

The methods of this section can, however, be used to prove similar results for *countable* groups of automorphisms on Dedekind σ-complete Boolean algebras; I will return to such questions in §448.

As noted, Kawada (Kawada 44) treated the case in which the group G of automorphisms is ergodic, that is, the fixed-point subalgebra $\mathfrak{C}$ is trivial. Under this hypothesis the proof is of course very much simpler. (You may find it useful to reconstruct the original version, as suggested in 394Xa.) I give the more general argument partly for the sake of 394O, partly to separate out the steps which really need ergodicity from those which depend only on non-paradoxicality, partly to prepare the ground for the countable version in the next volume, partly to show off the power of the construction in §364, and partly to get you used to 'Boolean-valued' arguments. A bolder use of language could indeed simplify some formulae slightly by writing (for instance) $[\![k\lceil a_0 : a\rceil < \lfloor 1 : a\rfloor]\!]$ in place of $[\![\lfloor 1 : a\rfloor - k\lceil a_0 : a\rceil > 0]\!]$ (see part (f) of the proof of 394N). As in §387, the differences involved in the extension to non-ergodic groups are, in a sense, just a matter of technique; but this time the technique is more obtrusive. The presentation here owes a good deal to Nadkarni 90 and something to Becker & Kechris 96.

395 The Hajian-Ito theorem

In the notes to the last section, I said that the argument there short-circuits if we are told that we are dealing with a measurable algebra. The point is that in this case there is a much simpler criterion for the existence of a G-invariant measure (395B(ii)), with a proof which is independent of §394 in all its non-trivial parts, which makes it easy to prove that non-paradoxicality is sufficient as well as necessary.

395A Lemma Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra.

(a) Let $\pi \in \operatorname{Aut}\mathfrak{A}$ be a Boolean automorphism (not necessarily measure-preserving). Let T_π be the corresponding Riesz homomorphism from $L^0 = L^0(\mathfrak{A})$ to itself (364R). Then there is a unique $w_\pi \in (L^0)^+$ such that $\int w_\pi \times v = \int T_\pi v$ for every $v \in (L^0)^+$.

(b) If ϕ, $\pi \in \operatorname{Aut}\mathfrak{A}$ then $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c) For each $\pi \in \operatorname{Aut}\mathfrak{A}$ we have a norm-preserving isomorphism U_π from $L^2 = L^2(\mathfrak{A}, \bar\mu)$ to itself defined by setting

$$U_\pi v = T_\pi v \times \sqrt{w_{\pi^{-1}}}$$

for every $v \in L^2$, and $U_{\pi\phi} = U_\pi U_\phi$ for all π, $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Applying 365T with $\bar\nu a = \bar\mu(\pi a)$, we see that there is a unique $w_\pi \in (L^0)^+$ such that $\int_a w_\pi = \bar\mu(\pi a)$ for every $a \in \mathfrak{A}$. If we look at

$$W = \{v : v \in (L^0)^+, \int v \times w_\pi = \int T_\pi v\},$$

we see that W contains χa for every $a \in \mathfrak{A}$, that $v + v' \in W$ and $\alpha v \in W$ whenever v, $v' \in W$ and $\alpha \ge 0$, and that $\sup_{n\in\mathbb{N}} v_n \in W$ whenever $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in W which is bounded above in L^0. By 364Kd, $W = (L^0)^+$, as required.

(b) For any $v \in (L^0)^+$,

$$\int w_{\pi\phi} \times v = \int T_{\pi\phi} v = \int T_\pi T_\phi v$$

(364Re)

$$= \int w_\pi \times T_\phi v = \int T_\phi (T_{\phi^{-1}} w_\pi \times v)$$

(recalling that T_ϕ is multiplicative)

$$= \int w_\phi \times T_{\phi^{-1}} w_\pi \times v.$$

As v is arbitrary (and $(\mathfrak{A}, \bar\mu)$ is semi-finite), $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c)(i) For any $v \in L^0$,

$$\int (T_\pi v \times \sqrt{w_{\pi^{-1}}})^2 = \int T_\pi v^2 \times w_{\pi^{-1}} = \int T_{\pi^{-1}} T_\pi v^2 = \int v^2.$$

So $U_\pi v \in L^2$ and $\|U_\pi v\|_2 = \|v\|_2$ whenever $v \in L^2$, and U_π is a norm-preserving operator on L^2.

(ii) Now consider $U_{\pi\phi}$. For any $v \in L^2$, we have

$$U_\pi U_\phi v = T_\pi (T_\phi v \times \sqrt{w_{\phi^{-1}}}) \times \sqrt{w_{\pi^{-1}}}$$
$$= T_\pi T_\phi v \times \sqrt{T_\pi w_{\phi^{-1}} \times w_{\pi^{-1}}}$$

(using 364Rd)

$$= T_{\pi\phi} v \times \sqrt{w_{\phi^{-1}\pi^{-1}}}$$

(by (b) above)

$$= U_{\pi\phi} v.$$

So $U_{\pi\phi} = U_\pi U_\phi$.

(iii) Writing ι for the identity operator on $\mathfrak{A}$, we see that T_ι is the identity operator on L^0, $w_\iota = \chi 1$ and U_ι is the identity operator on L^2. Since $U_{\pi^{-1}}U_\pi = U_\pi U_{\pi^{-1}} = U_\iota$, $U_\pi : L^2 \to L^2$ is an isomorphism, with inverse $U_{\pi^{-1}}$, for every $\pi \in \operatorname{Aut}\mathfrak{A}$.

395B Theorem (Hajian & Ito 69) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is a G-invariant functional $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a totally finite measure algebra;

(ii) whenever $a \in \mathfrak{A} \setminus \{0\}$ and $\langle \pi_n \rangle_{n\in\mathbb{N}}$ is a sequence in G, $\langle \pi_n a \rangle_{n\in\mathbb{N}}$ is not disjoint;

(iii) G is fully non-paradoxical (definition: 394E).

proof (a) (i)$\Rightarrow$(iii) by the argument of 394F, and (iii)$\Rightarrow$(ii) by the criterion (ii) of 394E. So for the rest of the proof I assume that (ii) is true and seek to prove (i).

(b) Let $\bar\mu$ be such that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra. If $a \in \mathfrak{A} \setminus \{0\}$, then $\inf_{\pi\in G} \bar\mu(\pi a) > 0$. **P?** Otherwise, let $\langle \pi_n \rangle_{n\in\mathbb{N}}$ be a sequence in G such that $\bar\mu\pi_n a \le 2^{-n}$ for each $n \in \mathbb{N}$. Set $b_n = \sup_{k\ge n} \pi_k a$ for each n; then $\inf_{n\in\mathbb{N}} b_n = 0$, so that

$$\inf_{n\in\mathbb{N}} \pi b_n = 0, \quad \lim_{n\to\infty} \bar\mu(\pi\pi_n a) = 0$$

for every $\pi \in \operatorname{Aut}\mathfrak{A}$. Choose $\langle n_i \rangle_{i\in\mathbb{N}}$ inductively so that

$$\bar\mu(\pi_{n_i}^{-1}\pi_{n_j} a) \le 2^{-j-2}\bar\mu a$$

whenever $i < j$. Set

$$c = a \setminus \sup_{i<j} \pi_{n_i}^{-1}\pi_{n_j} a.$$

Because

$$\sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \bar\mu(\pi_{n_i}^{-1}\pi_{n_j} a) < \bar\mu a,$$

$c \ne 0$, while $\pi_{n_i} c \cap \pi_{n_j} c = 0$ whenever $i < j$, contrary to the hypothesis (ii). **XQ**

(c) For each $\pi \in G$, define $w_\pi \in L^0 = L^0(\mathfrak{A})$ and $U_\pi : L^2 \to L^2$ as in 395A, where $L^2 = L^2(\mathfrak{A}, \bar\mu)$. If $a \in \mathfrak{A} \setminus \{0\}$, then $\inf_{\pi\in G} \int_a \sqrt{w_\pi} > 0$. **P?** Otherwise, there is a sequence $\langle \pi_n \rangle_{n\in\mathbb{N}}$ in G such that $\int_a v_n \le 4^{-n-2}\bar\mu a$ for every n, where $v_n = \sqrt{w_{\pi_n}}$. In this case, $\bar\mu(a \cap [\![v_n \ge 2^{-n}]\!]) \le 2^{-n-2}\bar\mu a$ for every n, so that $b = a \setminus \sup_{n\in\mathbb{N}} [\![v_n \ge 2^{-n}]\!]$ is non-zero. But now

$$\bar\mu(\pi_n b) = \int_b w_{\pi_n} = \int_b v_n^2 \le 4^{-n}\bar\mu b \to 0$$

as $n \to \infty$, contradicting (b) above. **XQ**

(d) Write $e = \chi 1$ for the standard weak order unit of L^0 or L^2. Let $C \subseteq L^2$ be the convex hull of $\{U_\pi e : \pi \in G\}$. Then C and its norm closure $\overline{C}$ are G-invariant in the sense that $U_\pi v \in C$, $U_\pi v' \in \overline{C}$ whenever $v \in C$, $v' \in \overline{C}$ and $\pi \in G$. By 3A5Ld, there is a unique $u_0 \in \overline{C}$ such that $\|u_0\|_2 \le \|u\|_2$ for every $u \in \overline{C}$. Now if $\pi \in G$, $U_\pi u_0 \in \overline{C}$, while $\|U_\pi u_0\|_2 = \|u_0\|_2$; so $U_\pi u_0 = u_0$. Also, if $a \in \mathfrak{A} \setminus \{0\}$,

$$\int_a u_0 \ge \inf_{u\in\overline{C}} \int_a u = \inf_{u\in C} \int_a u$$

(because $u \mapsto \int_a u$ is $\|\;\|_2$-continuous)

$$= \inf_{\pi\in G} \int_a U_\pi e = \inf_{\pi\in G} \int_a T_\pi e \times \sqrt{w_{\pi^{-1}}}$$

$$= \inf_{\pi\in G} \int_a \sqrt{w_{\pi^{-1}}} = \inf_{\pi\in G} \int_a \sqrt{w_\pi} > 0$$

by (c). So $[\![u_0 > 0]\!] = 1$.

(e) For $a \in \mathfrak{A}$, set $\bar\nu a = \int_a u_0^2$. Because $u_0 \in L^2$, $\bar\nu$ is a non-negative countably additive functional on $\mathfrak{A}$; because $[\![u_0^2 > 0]\!] = [\![u_0 > 0]\!] = 1$, $\bar\nu$ is strictly positive, and $(\mathfrak{A}, \bar\nu)$ is a totally finite measure algebra. Finally, $\bar\nu$ is G-invariant. **P** If $a \in \mathfrak{A}$ and $\pi \in G$, then

$$\begin{aligned}\bar\nu(\pi a) &= \int_{\pi a} u_0^2 = \int u_0^2 \times \chi(\pi a) = \int T_\pi(T_{\pi^{-1}}u_0^2 \times \chi a)\\ &= \int w_\pi \times T_{\pi^{-1}}u_0^2 \times \chi a = \int_a (T_{\pi^{-1}}u_0 \times \sqrt{w_\pi})^2\\ &= \int_a (U_{\pi^{-1}}u_0)^2 = \int_a u_0^2 = \bar\nu a. \ \mathbf{Q}\end{aligned}$$

So (i) is true.

395C Remark If $\mathfrak{A}$ is a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$, a non-zero element a of $\mathfrak{A}$ is called **weakly wandering** if there is a sequence $\langle\pi_n\rangle_{n\in\mathbb{N}}$ in G such that $\langle\pi_n a\rangle_{n\in\mathbb{N}}$ is disjoint. Thus condition (ii) of 395B may be read as 'there is no weakly wandering element of $\mathfrak{A}$'.

395X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ an order-continuous Boolean homomorphism. Let $T_\pi : L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ be the corresponding Riesz homomorphism. Show that there is a unique $w_\pi\in L^1(\mathfrak{A},\bar\mu)$ such that $\int T_\pi v = \int v\times w_\pi$ for every $v\in L^0(\mathfrak{A})^+$.

(b) In 395A, show that the map $\pi\mapsto U_\pi : \operatorname{Aut}\mathfrak{A}\to \mathsf{B}(L^2;L^2)$ is injective.

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ an ergodic measure-preserving automorphism, and $\phi\in\operatorname{Aut}\mathfrak{A}$ an automorphism. Suppose that $\phi\pi\phi^{-1}$ is measure-preserving. Show that ϕ is measure-preserving. (*Hint*: compare w_ϕ, $w_{\pi\phi}$ and $w_{\phi\pi}$ in 395A.)

(d) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Suppose that there is a strictly positive G-invariant finitely additive functional on $\mathfrak{A}$. Show that there is a G-invariant $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra.

395Y Further exercises (a) Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b\in\mathfrak{A}$, say that a and b are **G-equidecomposable** if there are *finite* partitions of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}_b$, and a family $\langle\pi_i\rangle_{i\in I}$ in G, such that $\pi_i a_i = b_i$ for every $i\in I$. Show that the following are equiveridical: (i) G is fully non-paradoxical in the sense of 394E; (ii) if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of mutually G-equidecomposable elements of $\mathfrak{A}$, they must all be 0.

395 Notes and comments I have separated these few pages from §394 partly because §394 was already up to full weight and partly in order that the ideas here should not be entirely overshadowed by those of the earlier section. It will be evident that the construction of the U_π in 395A, providing us with a faithful representation, acting on a Hilbert space, of the whole group $\operatorname{Aut}\mathfrak{A}$, is a basic tool for the study of that group.

Appendix to Volume 3

Useful Facts

This volume assumes a fairly wide-ranging competence in analysis, a solid understanding of elementary set theory and some straightforward Boolean algebra. As in previous volumes, I start with a few pages of revision in set theory, but the absolutely essential material is in §3A2, on commutative rings, which is the basis of the treatment of Boolean rings in §311. I then give three sections of results in analysis: topological spaces (§3A3), uniform spaces (§3A4) and normed spaces (§3A5). Finally, I add six sentences on group theory (§3A6).

3A1 Set Theory

3A1A The axioms of set theory This treatise is based on arguments within, or in principle reducible to, 'ZFC', meaning 'Zermelo-Fraenkel set theory, including the Axiom of Choice'. For discussions of this system, see, for instance, Krivine 71, Jech 78 or Kunen 80. As I remarked in §2A1, I believe that it is helpful, as a matter of general principle, to distinguish between results dependent on the axiom of choice and those which can be proved without it, or with some relatively weak axiom such as 'countable choice'. (See 134C.) In Volumes 1 and 2, such a distinction is useful in appreciating the special features of different ideas. In the present volume, however, most of the principal theorems require something close to the full axiom of choice, and there are few areas where it seems at present appropriate to work with anything weaker. Indeed, at many points we shall approach questions which are, or may be, undecidable in ZFC; but I postpone discussion of these to Volume 5. In particular, I specifically exclude, for the time being, results dependent on such axioms as the continuum hypothesis.

3A1B Definition Let X be a set. By an **enumeration** of X I mean a bijection $f : \kappa \to X$ where $\kappa = \#(X)$ (2A1Kb); more often than not I shall express such a function in the form $\langle x_\xi\rangle_{\xi<\kappa}$. In this case I say that the function f, or the family $\langle x_\xi\rangle_{\xi<\kappa}$, **enumerates** X. You will see that I am tacitly assuming that $\#(X)$ is always defined, that is, that the axiom of choice is true.

3A1C Calculation of cardinalities The following formulae are basic.

(a) For any sets X and Y, $\#(X \times Y) \le \max(\omega, \#(X), \#(Y))$. (Enderton 77, p. 64; Jech 78, p. 42; Krivine 71, p. 33; Kunen 80, 10.13.)

(b) For any $r \in \mathbb{N}$ and any family $\langle X_i\rangle_{i\le r}$ of sets, $\#(\prod_{i=0}^r X_i) \le \max(\omega, \max_{i\le r} \#(X_i))$. (Induce on r.)

(c) For any family $\langle X_i\rangle_{i\in I}$ of sets, $\#(\bigcup_{i\in I} X_i) \le \max(\omega, \#(I), \sup_{i\in I} \#(X_i))$. (Jech 78, p. 43; Krivine 71, p. 33; Kunen 80, 10.21.)

(d) For any set X, the set $[X]^{<\omega}$ of finite subsets of X has cardinal at most $\max(\omega, \#(X))$. (There is a surjection from $\bigcup_{r\in\mathbb{N}} X^r$ onto $[X]^{<\omega}$. For the notation $[X]^{<\omega}$ see 3A1J below.)

3A1D Cardinal exponentiation For a cardinal κ, I write 2^κ for $\#(\mathcal{P}\kappa)$. So $2^\omega = \mathfrak{c}$, and $\kappa^+ \le 2^\kappa$ for every κ. (Enderton 77, p. 132; Lipschutz 64, p. 139; Jech 78, p. 24; Krivine 71, p. 25; Halmos 60, p. 93.)

3A1E Definition The class of infinite initial ordinals, or cardinals, is a subclass of the class On of all ordinals, so is itself well-ordered; being unbounded, it is a proper class; consequently there is a unique increasing enumeration of it as $\langle\omega_\xi\rangle_{\xi\in\mathrm{On}}$. We have $\omega_0 = \omega$, $\omega_{\xi+1} = \omega_\xi^+$ for every ξ (compare 2A1Fc), $\omega_\xi = \bigcup_{\eta<\xi} \omega_\eta$ for non-zero limit ordinals ξ. (Enderton 77, pp. 213-214; Jech 78, p. 25; Krivine 71, p. 31.)

3A1F Cofinal sets (a) If P is any partially ordered set (definition: 2A1A), a subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \le q$.

(b) If P is any partially ordered set, the **cofinality** of P, $\mathrm{cf}(P)$, is the least cardinal of any cofinal subset of P. Note that $\mathrm{cf}(P) = 0$ iff $P = \emptyset$, and that $\mathrm{cf}(P) = 1$ iff P has a greatest element.

(c) Observe that if P is upwards-directed and $\mathrm{cf}(P)$ is finite, then $\mathrm{cf}(P)$ is either 0 or 1; for if Q is a finite, non-empty cofinal set then it has an upper bound, which must be the greatest element of P.

(d) If P is a totally ordered set of cofinality κ, then there is a strictly increasing family $\langle p_\xi\rangle_{\xi<\kappa}$ in P such that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **P** If $\kappa = 0$ then $P = \emptyset$ and this is trivial. Otherwise, let Q be a cofinal subset of P of cardinal κ, and $\{q_\xi : \xi < \kappa\}$ an enumeration of Q. Define $\langle p_\xi\rangle_{\xi<\kappa}$ inductively, as follows. Start with $p_0 = q_0$. Given $\langle p_\eta\rangle_{\eta<\xi}$, where $\xi < \kappa$, then if $p_\eta < q_\xi$ for every $\eta < \xi$, take $p_\xi = q_\xi$; otherwise, because $\#(\xi) \le \xi < \kappa$, $\{p_\eta : \eta < \xi\}$ cannot be cofinal with P, so there is a $p_\xi \in P$ such that $p_\xi \not\le p_\eta$ for every $\eta < \xi$, that is, $p_\eta < p_\xi$ for every $\eta < \xi$. Note that there is some $\eta < \xi$ such that $q_\xi \le p_\eta$, so that $q_\xi \le p_\xi$. Continue.

Now $\langle p_\xi\rangle_{\xi<\kappa}$ is a strictly increasing family in P such that $q_\xi \le p_\xi$ for every ξ; it follows at once that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **Q**

(e) In particular, for a totally ordered set P, $\mathrm{cf}(P) = \omega$ iff there is a cofinal strictly increasing sequence in P.

3A1G Zorn's Lemma In Volume 2 I used Zorn's Lemma only once or twice, giving the arguments in detail. In the present volume I feel that continuing in such a manner would often be tedious; but nevertheless the arguments are not always quite obvious, at least until you have gained a good deal of experience. I therefore take a paragraph to comment on some of the standard forms in which they appear.

The statement of Zorn's Lemma, as quoted in 2A1M, refers to arbitrary partially ordered sets P. A large proportion of the applications can in fact be represented more or less naturally by taking P to be a family $\mathfrak{P}$ of sets ordered by $\subseteq$; in such a case, it will be sufficient to check that (i) $\mathfrak{P}$ is not empty (ii) $\bigcup\mathfrak{Q} \in \mathfrak{P}$ for every non-empty totally ordered $\mathfrak{Q} \subseteq \mathfrak{P}$. More often than not, this will in fact be true for all non-empty upwards-directed sets $\mathfrak{Q} \subseteq \mathfrak{P}$, and the line of the argument is sometimes clearer if phrased in this form.

Within this class of partially ordered sets, we can distinguish a special subclass. If A is any set and $\perp$ any relation on A, we can consider the collection $\mathfrak{P}$ of sets $I \subseteq A$ such that $a \perp b$ for all distinct a, $b \in I$. In this case we need look no farther before declaring '$\mathfrak{P}$ has a maximal element'; for $\emptyset$ necessarily belongs to $\mathfrak{P}$, and if $\mathfrak{Q}$ is any upwards-directed subset of $\mathfrak{P}$, then $\bigcup\mathfrak{Q} \in \mathfrak{P}$. **P** If a, b are distinct elements of $\bigcup\mathfrak{Q}$, there are I_1, $I_2 \in \mathfrak{Q}$ such that $a \in I_1$, $b \in I_2$; because $\mathfrak{Q}$ is upwards-directed, there is an $I \in \mathfrak{Q}$ such that $I_1 \cup I_2 \subseteq I$, so that a, b are distinct members of $I \in \mathfrak{P}$, and $a \perp b$. **Q** So $\bigcup\mathfrak{Q}$ is an upper bound of $\mathfrak{Q}$ in $\mathfrak{P}$; as $\mathfrak{Q}$ is arbitrary, $\mathfrak{P}$ satisfies the conditions of Zorn's Lemma, and must have a maximal element.

Another important type of partially ordered set in this context is a family Φ of functions, ordered by saying that $f \le g$ if g is an extension of f. In this case, for any non-empty upwards-directed $\Psi \subseteq \Phi$, we shall have a function h defined by saying that

$$\operatorname{dom} h = \bigcup_{f\in\Psi} \operatorname{dom} f, \quad h(x) = f(x) \text{ whenever } f \in \Psi,\ x \in \operatorname{dom} f,$$

and the usual attack is to seek to prove that any such h belongs to Φ.

I find that at least once I wish to use Zorn's Lemma 'upside down': that is, I have a non-empty partially ordered set P in which every non-empty totally ordered subset has a *lower* bound. In this case, of course, P has a *minimal* element. The point is that the definition of 'partial order' is symmetric, so that $(P, \ge)$ is a partially ordered set whenever $(P, \le)$ is; and we can seek to apply Zorn's Lemma to either.

3A1H Natural numbers and finite ordinals I remarked in 2A1De that the first few ordinals

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \quad \ldots$$

may be identified with the natural numbers $0, 1, 2, 3, \ldots$; the idea being that $n = \{0, 1, \ldots, n-1\}$ is a set with n elements. If we do this, then the set $\mathbb{N}$ of natural numbers becomes identified with the first infinite ordinal ω. This convention makes it possible to present a number of arguments in a particularly elegant form. A typical example is in 344H. There I wish to describe an inductive construction for a family $\langle K_z\rangle_{z\in S}$ where $S=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$. If we think of n as the set of its predecessors, then $z\in\{0,1\}^n$ becomes a function from n to $\{0,1\}$; since the set n has just n members, this corresponds well to the idea of z as the list of its n coordinates, except that it would now be natural to index them as $z(0),\dots,z(n-1)$ rather than as $\zeta_1,\dots,\zeta_n$, which was the language I favoured in Volume 2. An extension of z to a member of $\{0,1\}^{n+1}$ is of the form $v=z^\frown i$ where $v(k)=z(k)$ for $k<n$ and $v(n)=i$. If $w\in\{0,1\}^{\mathbb{N}}$, then we can identify the initial segment $(w(0),w(1),\dots,w(n-1))$ of its first n coordinates with the restriction $w\restriction n$ of w to the set $n=\{0,\dots,n-1\}$.

3A1I Definitions (a) If P and Q are lattices (2A1Ad), a **lattice homomorphism** from P to Q is a function $f:P\to Q$ such that $f(p\wedge p')=f(p)\wedge f(p')$ and $f(p\vee p')=f(p)\vee f(p')$ for all p, $p'\in P$. Such a homomorphism is surely order-preserving (313H), for if $p\le p'$ in P then $f(p')=f(p\vee p')=f(p)\vee f(p')$ and $f(p)\le f(p')$.

(b) If P is a lattice, a **sublattice** of P is a set $Q\subseteq P$ such that $p\vee q$ and $p\wedge q$ belong to Q for all p, $q\in Q$.

(c) A lattice P is **distributive** if

$$(p\wedge q)\vee r=(p\vee r)\wedge(q\vee r),\quad (p\vee q)\wedge r=(p\wedge r)\vee(q\wedge r)$$

for all p, q, $r\in P$.

3A1J Subsets of given size The following concepts are used often enough for a special notation to be helpful. If X is a set and κ is a cardinal, write

$$[X]^\kappa=\{A:A\subseteq X,\ \#(A)=\kappa\},$$

$$[X]^{\le\kappa}=\{A:A\subseteq X,\ \#(A)\le\kappa\},$$

$$[X]^{<\kappa}=\{A:A\subseteq X,\ \#(A)<\kappa\}.$$

Thus

$$[X]^0=[X]^{\le 0}=[X]^{<1}=\{\emptyset\},$$

$[X]^2$ is the set of doubleton subsets of X, $[X]^{<\omega}$ is the set of finite subsets of X, $[X]^{\le\omega}$ is the set of countable subsets of X, and so on.

3A1K The next result is one of the fundamental theorems of combinatorics. In this volume it is used in the proofs of Ornstein's theorem (§386) and the Kalton-Roberts theorem (§392).

Hall's Marriage Lemma Suppose that X and Y are finite sets and $R\subseteq X\times Y$ is a relation such that $\#(R[I])\ge\#(I)$ for every $I\subseteq X$. Then there is an injective function $f:X\to Y$ such that $(x,f(x))\in R$ for every $x\in X$.

Remark Recall that $R[I]$ is the set $\{y:\exists\, x\in I,\ (x,y)\in R\}$ (1A1Bc).

proof Bollobás 79, p. 54, Theorem 7; Anderson 87, 2.2.1; Bose & Manvel 84, §10.2.

3A2 Rings

I give a very brief outline of the indispensable parts of the elementary theory of (commutative) rings. I assume that you have seen at least a little group theory.

3A2A Definition A **ring** is a triple $(R,+,.)$ such that

$(R,+)$ is a commutative group; its identity will always be denoted 0 or 0_R;

$(R,.)$ is a semigroup, that is, $ab\in R$ for all a, $b\in R$ and $a(bc)=(ab)c$ for all a, b, $c\in R$;

$a(b+c)=ab+ac$, $(a+b)c=ac+bc$ for all a, b, $c\in R$.

A **commutative ring** is one in which multiplication is commutative, that is, $ab=ba$ for all a, $b\in R$.

3A2B Elementary facts Let R be a ring.

(a) $a0 = 0a = 0$ for every $a \in R$. **P**

$$a0 = a(0+0) = a0 + a0, \quad 0a = (0+0)a = 0a + 0a;$$

because $(R,+)$ is a group, we may subtract $a0$ or $0a$ from each side of the appropriate equation to see that $0 = a0$, $0 = 0a$. **Q**

(b) $(-a)b = a(-b) = -(ab)$ for all a, $b \in R$. **P**

$$ab + ((-a)b) = (a + (-a))b = 0b = 0 = a0 = a(b + (-b)) = ab + a(-b);$$

subtracting ab from each term, we get $(-a)b = -(ab) = a(-b)$. **Q**

3A2C Subrings If R is a ring, a **subring** of R is a set $S \subseteq R$ such that $0 \in S$ and $a+b$, ab, $-a$ belong to S for all a, $b \in S$. In this case S, together with the addition and multiplication induced by those of R, is a ring in its own right.

3A2D Homomorphisms (a) Let R, S be two rings. A function $\phi : R \to S$ is a **ring homomorphism** if $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in R$. The **kernel** of ϕ is $\{a : a \in R,\ \phi(a) = 0_S\}$.

(b) Note that if $\phi : R \to S$ is a ring homomorphism, then it is also a group homomorphism from $(R,+)$ to $(S,+)$, so that $\phi(0_R) = 0_S$ and $\phi(-a) = -\phi(a)$ for every $a \in R$; moreover, $\phi[R]$ is a subring of S, and ϕ is injective iff its kernel is $\{0_R\}$.

(c) If R, S and T are rings, and $\phi : R \to S$, $\psi : S \to T$ are ring homomorphisms, then $\psi\phi : R \to T$ is a ring homomorphism, because

$$(\psi\phi)(a * b) = \psi(\phi(a * b)) = \psi(\phi(a) * \phi(b)) = \psi(\phi(a)) * \psi(\phi(b))$$

for all a, $b \in R$, taking $*$ to be either addition or multiplication. If ϕ is bijective, then $\phi^{-1} : S \to R$ is a ring homomorphism, because

$$\phi^{-1}(c * d) = \phi^{-1}(\phi(\phi^{-1}(c)) * \phi(\phi^{-1}(d))) = \phi^{-1}\phi(\phi^{-1}(c) * \phi^{-1}(d)) = \phi^{-1}(c) * \phi^{-1}(d)$$

for all c, $d \in S$, again taking $*$ to be either addition or multiplication.

3A2E Ideals (a) Let R be a ring. An **ideal** of R is a subring I of R such that $ab \in I$ and $ba \in I$ whenever $a \in I$ and $b \in R$. this case we write $I \triangleleft R$.

Note that R and $\{0\}$ are always ideals of R.

(b) If R and S are rings and $\phi : R \to S$ is a ring homomorphism, then the kernel I of ϕ is an ideal of R. **P** (i) Because ϕ is a group homomorphism, I is a subgroup of $(R,+)$. (ii) If $a \in I$, $b \in R$ then

$$\phi(ab) = \phi(a)\phi(b) = 0_S\phi(b) = 0_S, \quad \phi(ba) = \phi(b)\phi(a) = \phi(b)0_S = 0_S$$

so ab, $ba \in I$. **Q**

3A2F Quotient rings (a) Let R be a ring and I an ideal of R. A **coset** of I is a set of the form $a + I = \{a + x : x \in I\}$ where $a \in R$. (Because $+$ is commutative, we do not need to distinguish between 'left cosets' $a + I$ and 'right cosets' $I + a$.) Let R/I be the set of cosets of I in R.

(b) For A, $B \in R/I$, set

$$A + B = \{x + y : x \in A,\ y \in B\}, \quad A \cdot B = \{xy + z : x \in A,\ y \in B,\ z \in I\}.$$

Then $A + B$, $A \cdot B$ both belong to R/I; moreover, if $A = a + I$ and $B = b + I$, then $A + B = (a + b) + I$ and $A \cdot B = ab + I$. **P** (i)

$$\begin{aligned}A+B &= (a+I)+(b+I)\\ &= \{(a+x)+(b+y) : x,\, y \in I\}\\ &= \{(a+b)+(x+y) : x,\, y \in I\}\end{aligned}$$

(because addition is associative and commutative)

$$\subseteq \{(a+b)+z : z \in I\} = (a+b)+I$$

(because $I+I \subseteq I$)

$$\begin{aligned}&= \{(a+b)+(z+0) : z \in I\}\\ &\subseteq (a+I)+(b+I) = A+B\end{aligned}$$

because $0 \in I$. (ii)

$$\begin{aligned}A\cdot B &= \{(a+x)(b+y)+z : x,\, y,\, z \in I\}\\ &= \{ab+(ay+xb+z) : x,\, y,\, z \in I\}\\ &\subseteq \{ab+w : w \in I\} = ab+I\end{aligned}$$

(because ay, $xb \in I$ for all x, $y \in I$, and I is closed under addition)

$$\begin{aligned}&= \{(a+0)(b+0)+w : w \in I\}\\ &\subseteq A\cdot B.\ \mathbf{Q}\end{aligned}$$

(c) It is now an elementary exercise to check that $(R/I,+,\cdot)$ is a ring, with zero $0+I = I$ and additive inverses $-(a+I) = (-a)+I$.

(d) Moreover, the map $a \mapsto a+I : R \to R/I$ is a ring homomorphism.

(e) Note that for a, $b \in R$, the following are equiveridical: (i) $a \in b+I$; (ii) $b \in a+I$; (iii) $(a+I)\cap(b+I) \neq \emptyset$; (iv) $a+I = b+I$; (v) $a-b \in I$. Thus the cosets of I are just the equivalence classes in R under the equivalence relation $a \sim b \iff a+I = b+I$; accordingly I shall generally write $a^\bullet$ for $a+I$, if there seems no room for confusion. In particular, the kernel of the canonical map from R to R/I is just $\{a : a+I = I\} = I = 0^\bullet$.

(f) If R is commutative so is R/I, since

$$a^\bullet b^\bullet = (ab)^\bullet = (ba)^\bullet = b^\bullet a^\bullet$$

for all a, $b \in R$.

3A2G Factoring homomorphisms through quotient rings: Proposition Let R and S be rings, I an ideal of R, and $\phi : R \to S$ a homomorphism such that I is included in the kernel of ϕ. Then we have a ring homomorphism $\pi : R/I \to S$ such that $\pi(a^\bullet) = \phi(a)$ for every $a \in R$. π is injective iff I is precisely the kernel of ϕ.

proof If a, $b \in R$ and $a^\bullet = b^\bullet$ in R/I, then $a-b \in I$ (3A2Fe), so $\phi(a)-\phi(b) = \phi(a-b) = 0$, and $\phi(a) = \phi(b)$. This means that the formula offered does indeed define a function π from R/I to S. Now if a, $b \in R$ and $*$ is either multiplication or addition,

$$\pi(a^\bullet * b^\bullet) = \pi((a*b)^\bullet) = \phi(a*b) = \phi(a)*\phi(b) = \pi(a^\bullet)*\pi(b^\bullet).$$

So π is a ring homomorphism.

The kernel of π is $\{a^\bullet : \phi(a) = 0\}$, which is $\{0\}$ iff $\phi(a) = 0 \iff a^\bullet = 0 \iff a \in I$.

3A2H Product rings (a) Let $\langle R_i\rangle_{i\in I}$ be any family of rings. Set $R=\prod_{i\in I}R_i$ and for a, $b\in R$ define $a+b$, $ab\in R$ by setting

$$(a+b)(i)=a(i)+b(i),\quad (ab)(i)=a(i)b(i)$$

for every $i\in I$. It is easy to check from the definition in 3A2A that R is a ring; its zero is given by the formula

$$0_R(i)=0_{R_i}\text{ for every }i\in I,$$

and its additive inverses by the formula

$$(-a)(i)=-a(i)\text{ for every }i\in I.$$

(b) Now let S be any other ring. Then it is easy to see that a function $\phi:S\to R$ is a ring homomorphism iff $s\mapsto\phi(s)(i):S\to R_i$ is a ring homomorphism for every $i\in I$.

(c) Note that R is commutative iff R_i is commutative for every i.

3A3 General topology

In §2A3, I looked at a selection of topics in general topology in some detail, giving proofs; the point was that an ordinary elementary course in the subject would surely go far beyond what we needed there, and at the same time might omit some of the results I wished to quote. It seemed therefore worth taking a bit of space to cover the requisite material, giving readers the option of delaying a proper study of the subject until a convenient opportunity arose. In the context of the present volume, this approach is probably no longer appropriate, since we need a much greater proportion of the fundamental ideas, and by the time you have reached familiarity with the topics here you will be well able to find your way about one of the many excellent textbooks on the subject. This time round, therefore, I give most of the results without proofs (as in §§2A1 and 3A1), hoping that some of the references I offer will be accessible in all senses. I do, however, give a full set of definitions, partly to avoid ambiguity (since even in this relatively mature subject, there are some awkward divergences remaining in the usage of different authors), and partly because many of the proofs are easy enough for even a novice to fill in with a bit of thought, once the meaning of the words is clear. In fact this happens so often that I will mark with a $*$ those points where a proof needs an idea not implicit in the preceding work.

3A3A Taxonomy of topological spaces I begin with the handful of definitions we need in order to classify the different types of topological space used in this volume. A couple have already been introduced in Volume 2, but I repeat them because the list would look so odd without them.

Definitions Let $(X,\mathfrak{T})$ be a topological space.

(a) X is **Hausdorff** or $\mathbf{T_2}$ if for any distinct points x, $y\in X$ there are disjoint open sets G, $H\subseteq X$ such that $x\in G$ and $y\in H$.

(b) X is **regular** if whenever $F\subseteq X$ is closed and $x\in X\setminus F$ there are disjoint open sets G, $H\subseteq X$ such that $x\in G$ and $F\subseteq H$. (Note that in this definition I do not require X to be Hausdorff, following James 87 but not Engelking 89, Bourbaki 66, Dugundji 66, Schubert 68 or Gaal 64.)

(c) X is **completely regular** if whenever $F\subseteq X$ is closed and $x\in X\setminus F$ there is a continuous function $f:X\to[0,1]$ such that $f(x)=1$ and $f(y)=0$ for every $y\in F$. (Note that many authors restrict the phrase 'completely regular' to Hausdorff spaces. The terms **Tychonoff space** and $\mathbf{T}_{3\frac{1}{2}}$ **space** are also used for Hausdorff completely regular spaces.)

(d) X is **zero-dimensional** if whenever $G\subseteq X$ is an open set and $x\in G$ then there is an open-and-closed set H such that $x\in H\subseteq G$.

(e) X is **extremally disconnected** if the closure of every open set in X is open.

(f) X is **compact** if every open cover of X has a finite subcover.

(g) X is **locally compact** if for every $x \in X$ there is a set $K \subseteq X$ such that $x \in \operatorname{int} K$ and K is compact (in its subspace topology, as defined in 2A3C).

(h) If every subset of X is open, we call $\mathfrak{T}$ the **discrete topology** on X.

3A3B Elementary relationships (a) A completely regular space is regular. (Engelking 89, p. 39; Dugundji 66, p. 154; Schubert 68, p. 104.)

(b) A locally compact Hausdorff space is completely regular, therefore regular. $*$ (Engelking 89, 3.3.1; Dugundji 66, p. 238; Gaal 64, p. 149.)

(c) A compact Hausdorff space is locally compact, therefore completely regular and regular.

(d) A regular extremally disconnected space is zero-dimensional. (Engelking 89, 6.2.25.)

(e) Any topology defined by pseudometrics (2A3F), in particular the weak topology of a normed space (2A5I), is completely regular, therefore regular. (Bourbaki 66, IX.1.5; Dugundji 66, p. 200.)

(f) If X is a completely regular Hausdorff space (in particular, if X is (locally) compact and Hausdorff), and x, y are distinct points in X, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$. (Apply 3A3Ac with $F = \{y\}$, which is closed because X is Hausdorff.)

(g) An open set in a locally compact Hausdorff space is locally compact in its subspace topology. (Engelking 89, 3.3.8; Bourbaki 66, I.9.7.)

3A3C Continuous functions Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces.

(a) If $f : X \to Y$ is a function and $x \in X$, we say that f is **continuous at** x if $x \in \operatorname{int} f^{-1}[H]$ whenever $H \subseteq Y$ is an open set containing $f(x)$.

(b) Now a function $f : X \to Y$ is continuous iff it is continuous at every point of X. (Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 183; James 87, p. 26.)

(c) If $f : X \to Y$ is continuous at $x \in X$, and $A \subseteq X$ is such that $x \in \overline{A}$, then $f(x) \in \overline{f[A]}$. (Bourbaki 66, I.2.1; Schubert 68, p. 23.)

(d) If $f : X \to Y$ is continuous, then $f[\overline{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$. (Engelking 89, 1.4.1; Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 184; James 87, p. 27.)

(e) A function $f : X \to Y$ is a **homeomorphism** if it is a continuous bijection and its inverse is also continuous; that is, if $\mathfrak{S} = \{f[G] : G \in \mathfrak{T}\}$ and $\mathfrak{T} = \{f^{-1}[H] : H \in \mathfrak{S}\}$.

(f) A function $f : X \to [-\infty, \infty]$ is **lower semi-continuous** if $\{x : x \in X,\ f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. (Cf. 225H.)

3A3D Compact spaces Any extended series of applications of general topology is likely to involve some new features of compactness. I start with the easy bits, continuing from 2A3Nb.

(a) The first is just a definition of compactness in terms of closed sets instead of open sets. A family $\mathcal{F}$ of sets has the **finite intersection property** if $\bigcap \mathcal{F}_0$ is non-empty for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$. Now a topological space X is compact iff $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property. (Engelking 89, 3.1.1; Bourbaki 66, I.9.1; Dugundji 66, p., 223; Schubert 68, p. 68; Gaal 64, p. 127.)

(b) A marginal generalization of this is the following. Let X be a topological space and $\mathcal{F}$ a family of closed subsets of X with the finite intersection property. If $\mathcal{F}$ contains a compact set then $\bigcap\mathcal{F}\ne\emptyset$. (Apply (a) to $\{K\cap F:F\in\mathcal{F}\}$ where $K\in\mathcal{F}$ is compact.)

(c) In a Hausdorff space, compact subsets are closed. (Engelking 89, 3.1.8; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 70; Gaal 64, p. 138; James 87, p. 77.)

(d) If X is compact, Y is Hausdorff and $\phi:X\to Y$ is continuous and injective, then ϕ is a homeomorphism between X and $\phi[X]$ (where $\phi[X]$ is given the subspace topology). (Engelking 89, 3.1.12; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 71; Gaal 64, p. 207.)

(e) Let X be a regular topological space and A a subset of X. Then the following are equiveridical: (i) A is relatively compact in X (that is, A is included in some compact subset of X, as in 2A3Na); (ii) $\overline{A}$ is compact; (iii) every ultrafilter on X which contains A has a limit in X. **P**(ii)⇒(i) is trivial, and (i)⇒(iii) is a consequence of 2A3R; neither of these requires X to be regular. Now assume (iii) and let $\mathcal{F}$ be an ultrafilter on X containing $\overline{A}$. Set

$$\mathcal{H}=\{B:B\subseteq X,\text{ there is an open set }G\in\mathcal{F}\text{ such that }A\cap G\subseteq B\}.$$

Then $\mathcal{H}$ does not contain $\emptyset$ and $B_1\cap B_2\in\mathcal{H}$ whenever B_1, $B_2\in\mathcal{H}$, so $\mathcal{H}$ is a filter on X, and it contains A. Let $\mathcal{H}^*\supseteq\mathcal{H}$ be an ultrafilter (2A1O). By hypothesis, $\mathcal{H}^*$ has a limit x say. Because $A\in\mathcal{H}^*$, $X\setminus\overline{A}$ is an open set not belonging to $\mathcal{H}^*$, and cannot be a neighbourhood of x; thus x must belong to $\overline{A}$. Let G be an open set containing x. Then there is an open set H such that $x\in H\subseteq\overline{H}\subseteq G$ (this is where I use the hypothesis that X is regular). Because $\mathcal{H}^*\to x$, $H\in\mathcal{H}^*$ so $X\setminus\overline{H}$ does not belong to $\mathcal{H}^*$ and therefore does not belong to $\mathcal{H}$. But $X\setminus\overline{H}$ is open, so by the definition of $\mathcal{H}$ it cannot belong to $\mathcal{F}$. As $\mathcal{F}$ is an ultrafilter, $\overline{H}\in\mathcal{F}$ and $G\in\mathcal{F}$. As G is arbitrary, $\mathcal{F}\to x$. As $\mathcal{F}$ is arbitrary, $\overline{A}$ is compact (2A3R). Thus (iii)⇒(ii). **Q**

3A3E Dense sets Recall that a set D in a topological space X is **dense** if $\overline{D}=X$, and that X is **separable** if it has a countable dense subset (2A3Ud).

(a) If X is a topological space, $D\subseteq X$ is dense and $G\subseteq X$ is dense and open, then $G\cap D$ is dense. (Engelking 89, 1.3.6.) Consequently the intersection of finitely many dense open sets is always dense.

(b) If X and Y are topological spaces, $D\subseteq A\subseteq X$, D is dense in A and $f:X\to Y$ is a continuous function, then $f[D]$ is dense in $f[A]$. (Use 3A3Cd.)

3A3F Meager sets Let X be a topological space.

(a) A set $A\subseteq X$ is **nowhere dense** or **rare** if $\operatorname{int}\overline{A}=\emptyset$, that is, $\operatorname{int}(X\setminus A)=X\setminus\overline{A}$ is dense, that is, for every non-empty open set G there is a non-empty open set $H\subseteq G\setminus A$.

(b) A set $M\subseteq X$ is **meager**, or **of first category**, if it is expressible as the union of a sequence of nowhere dense sets.

(c) Any subset of a nowhere dense set is nowhere dense; the union of finitely many nowhere dense sets is nowhere dense. (3A3Ea.)

(d) Any subset of a meager set is meager; the union of countably many meager sets is meager. (314L.)

3A3G Baire's theorem for locally compact Hausdorff spaces Let X be a locally compact Hausdorff space and $\langle G_n\rangle_{n\in\mathbb{N}}$ a sequence of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}}G_n$ is dense. $*$ (Engelking 89, 3.9.4; Bourbaki 66, IX.5.3; Dugundji 66, p. 249; Schubert 68, p. 148.)

3A3H Corollary (a) Let X be a compact Hausdorff space. Then a non-empty open subset of X cannot be meager. (Dugundji 66, p. 250; Schubert 68, p. 147.)

(b) Let X be a non-empty locally compact Hausdorff space. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets covering X, then there is some $n\in\mathbb{N}$ such that $\operatorname{int}\overline{A}_n$ is non-empty. (Dugundji 66, p. 250.)

3A3I Product spaces (a) Definition Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and $X=\prod_{i\in I}X_i$ their Cartesian product. We say that a set $G\subseteq X$ is open for the **product topology** if for every $x\in G$ there are a finite $J\subseteq I$ and a family $\langle G_j\rangle_{j\in J}$ such that every G_j is an open set in the corresponding X_j and

$$\{y:y\in X,\,y(j)\in G_j\text{ for every }j\in J\}$$

contains x and is included in G.

(Of course we must check that this does indeed define a topology; see Engelking 89, 2.3.1; Bourbaki 66, I.4.1; Schubert 68, p. 38; Gaal 64, p. 144.)

(b) If $\langle X_i\rangle_{i\in I}$ is a family of topological spaces, with product X, and Y another topological space, a function $\phi:Y\to X$ is continuous iff $\pi_i\phi$ is continuous for every $i\in I$, where $\pi_i(x)=x(i)$ for $x\in X$, $i\in I$. (Engelking 89, 2.3.6; Bourbaki 66, I.4.1; Dugundji 66, p. 101; Schubert 68, p. 62; James 87, p. 31.)

(c) Let $\langle X_i\rangle_{i\in I}$ be any family of non-empty topological spaces, with product X. If $\mathcal{F}$ is a filter on X and $x\in X$, then $\mathcal{F}\to x$ iff $\pi_i[[\mathcal{F}]]\to x(i)$ for every i, where $\pi_i(y)=y(i)$ for $y\in X$, and $\pi_i[[\mathcal{F}]]$ is the image filter on X_i (2A1Ib). (Bourbaki 66, I.7.6; Schubert 68, p. 61; James 87, p. 32.

(d) The product of any family of Hausdorff spaces is Hausdorff. (Engelking 89, 2.3.11; Bourbaki 66, I.8.2; Schubert 68, p. 62; James 87, p. 87.)

(e) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If D_i is a dense subset of X_i for each i, then $\prod_{i\in I}D_i$ is dense in $\prod_{i\in I}X_i$. (Engelking 89, 2.3.5.).

(f) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If F_i is a closed subset of X_i for each i, then $\prod_{i\in I}F_i$ is closed in $\prod_{i\in I}X_i$. (Engelking 89, 2.3.4; Bourbaki 66I.4.3.)

(g) Let $\langle(X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces with product $(X,\mathfrak{T})$. Suppose that each $\mathfrak{T}_i$ is defined by a family P_i of pseudometrics on X_i (2A3F). Then $\mathfrak{T}$ is defined by the family $\mathrm{P}=\{\tilde\rho_i:i\in I,\rho\in\mathrm{P}_i\}$ of pseudometrics on X, where I write $\tilde\rho_i(x,y)=\rho(\pi_i(x),\pi_i(y))$ whenever $i\in I$, $\rho\in\mathrm{P}_i$ and x, $y\in X$, taking π_i to be the coordinate map from X to X_i, as in (b)-(c). **P** (Compare 2A3Tb). (i) It is easy to check that every $\tilde\rho_i$ is a pseudometric on X. Write $\mathfrak{T}_{\mathrm{P}}$ for the topology generated by P. (ii) If $x\in G\in\mathfrak{T}_{\mathrm{P}}$, let $\mathrm{P}'\subseteq\mathrm{P}$ and $\delta>0$ be such that P' is finite and $\{y:\tau(y,x)\le\delta\text{ for every }\tau\in\mathrm{P}'\}$ is included in G. Express P' as $\{\tilde\rho_j:j\in J,\,\rho\in\mathrm{P}'_j\}$ where $J\subseteq I$ is finite and $\mathrm{P}'_j\subseteq\mathrm{P}_j$ is finite for each $j\in J$. Set

$$G_j=\{t:t\in X_j,\,\rho(t,\pi_j(x))<\delta\text{ for every }\rho\in\mathrm{P}'_j\}$$

for every $j\in J$. Then $G'=\{y:\pi_j(y)\in G_j\text{ for every }j\in J\}$ contains x, belongs to $\mathfrak{T}$ and is included in G. As x is arbitrary, $G\in\mathfrak{T}$; as G is arbitrary, $\mathfrak{T}_{\mathrm{P}}\subseteq\mathfrak{T}$. (iii) Now every π_i is $(\mathfrak{T}_{\mathrm{P}},\mathfrak{T}_i)$-continuous, by 2A3H; by (b) above, the identity map from X to itself is $(\mathfrak{T}_{\mathrm{P}},\mathfrak{T})$-continuous, that is, $\mathfrak{T}_{\mathrm{P}}\subseteq\mathfrak{T}$ and $\mathfrak{T}_{\mathrm{P}}=\mathfrak{T}$, as claimed. **Q**

3A3J Tychonoff's theorem The product of any family of compact topological spaces is compact.

proof Engelking 89, 3.2.4; Bourbaki 66, I.9.5; Dugundji 66, p. 224; Schubert 68, p. 72; Gaal 64, p. 146 and p. 272; James 87, p. 67.

3A3K The spaces $\{0,1\}^I$, $\mathbb{R}^I$ For any set I, we can think of $\{0,1\}^I$ as the product $\prod_{i\in I}X_i$ where $X_i=\{0,1\}$ for each i. If we endow each X_i with its discrete topology, the product topology is the **usual topology** on $\{0,1\}^I$. Being a product of Hausdorff spaces, it is Hausdorff; by Tychonoff's theorem, it is compact. A subset G of $\{0,1\}^I$ is open iff for every $x\in G$ there is a finite $J\subseteq I$ such that $\{y:y\in\{0,1\}^I,\,y{\restriction}J=x{\restriction}J\}\subseteq G$.

Similarly, the 'usual topology' of $\mathbb{R}^I$ is the product topology when each factor is given its Euclidean topology (cf. 2A3Tc).

3A3L Cluster points of filters (a) Let X be a topological space and $\mathcal{F}$ a filter on X. A point x of X is a **cluster point** (or **adherence point** or **accumulation point**) of $\mathcal{F}$ if $x\in\overline{A}$ for every $A\in\mathcal{F}$.

(b) For any topological space X, filter $\mathcal{F}$ on X and $x \in X$, x is a cluster point of $\mathcal{F}$ iff there is a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$. (Engelking 89, 1.6.8; Bourbaki 66, I.7.2; Gaal 64, p. 260; James 87, p. 22.)

(c) If $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}$, $\alpha \in \mathbb{R}$ and $\lim_{n \to \mathcal{H}} \alpha_n = \alpha$ for every non-principal ultrafilter $\mathcal{H}$ on $\mathbb{N}$ (definition: 2A3Sb), then $\lim_{n \to \infty} \alpha_n = \alpha$. **P?** If $\langle \alpha_n \rangle_{n \in \mathbb{N}} \not\to \alpha$, there is some $\epsilon > 0$ such that $I = \{n : |\alpha_n - \alpha| \ge \epsilon\}$ is infinite. Now $\mathcal{F}_0 = \{F : F \subseteq \mathbb{N}, I \setminus F \text{ is finite}\}$ is a filter on $\mathbb{N}$, so there is an ultrafilter $\mathcal{F} \supseteq \mathcal{F}_0$. But now α cannot be $\lim_{n \to \mathcal{F}} \alpha_n$. **XQ**

3A3M Topology bases (a) If X is a set and $\mathcal{A}$ is any family of subsets of X, the **topology generated by** $\mathcal{A}$ is the smallest topology on X including $\mathcal{A}$.

(b) If X is a set and $\mathfrak{T}$ is a topology on X, a **base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that whenever $x \in G \in \mathfrak{T}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$; that is, such that $\mathfrak{T} = \{\bigcup \mathcal{G} : \mathcal{G} \subseteq \mathcal{U}\}$. In this case, of course, $\mathcal{U}$ generates $\mathfrak{T}$.

(c) If X is a set and $\mathcal{E}$ is a family of subsets of X, then $\mathcal{E}$ is a base for a topology on X iff (i) whenever $E_1, E_2 \in \mathcal{E}$ and $x \in E_1 \cap E_2$ then there is an $E \in \mathcal{E}$ such that $x \in E \subseteq E_1 \cap E_2$ (ii) $\bigcup \mathcal{E} = X$. (Engelking 89, p. 12.)

3A3N Uniform convergence (a) Let X be a set, (Y, ρ) a metric space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of functions from X to Y. We say that $\langle f_n \rangle_{n \in \mathbb{N}}$ converges **uniformly** to a function $f : X \to Y$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) \le \epsilon$ whenever $n \ge n_0$ and $x \in X$.

(b) Let X be a topological space and (Y, ρ) a metric space. Suppose that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence of continuous functions from X to Y converging uniformly to $f : X \to Y$. Then f is continuous. $*$ (Engelking 89, 1.4.7/4.2.19; Gaal 64, p. 202.)

3A3O One-point compactifications Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. Take any object x_∞ not belonging to X and set $X^* = X \cup \{x_\infty\}$. Let $\mathfrak{T}^*$ be the family of those sets $H \subseteq X^*$ such that $H \cap X \in \mathfrak{T}$ and either $x_\infty \notin H$ or $X \setminus H$ is compact (for $\mathfrak{T}$). Then $\mathfrak{T}^*$ is the unique compact Hausdorff topology on X^* inducing $\mathfrak{T}$ as the subspace topology on X; $(X^*, \mathfrak{T}^*)$ is the **one-point compactification** or **Alexandroff compactification** of $(X, \mathfrak{T})$. (Engelking 89, 3.5.11; Bourbaki 66, I.9.8; Dugundji 66, p. 246.)

3A3P Miscellaneous definitions Let X be a topological space.

(a) A subset F of X is a **zero set** or **functionally closed** if it is of the form $f^{-1}[\{0\}]$ for some continuous function $f : X \to \mathbb{R}$. A subset G of X is a **cozero set** or **functionally open** if its complement is a zero set.

(b) An **isolated point** of X is a point $x \in X$ such that the singleton set $\{x\}$ is open.

3A4 Uniformities

I continue the work of §3A3 with some notes on uniformities, so as to be able to discuss completeness and the extension of uniformly continuous functions in non-metrizable contexts (3A4F-3A4H). As in §3A3, most of the individual steps are elementary; I mark exceptions with a $*$.

3A4A Uniformities (a) Let X be any set. A **uniformity** on X is a filter $\mathcal{W}$ on $X \times X$ such that

(i) $(x, x) \in W$ for every $x \in X$, $W \in \mathcal{W}$;
(ii) for every $W \in \mathcal{W}$, $W^{-1} = \{(y, x) : (x, y) \in W\} \in \mathcal{W}$;
(iii) for every $W \in \mathcal{W}$, there is a $V \in \mathcal{W}$ such that

$$V \circ V = \{(x, z) : \exists\, y,\ (x, y) \in V \,\&\, (y, z) \in V\} \subseteq W.$$

It is convenient to allow the special case $X = \emptyset$, $\mathcal{W} = \{\emptyset\}$, even though this is not properly speaking a filter.

The pair $(X, \mathcal{W})$ is now a **uniform space**.

(b) If $\mathcal{W}$ is a uniformity on a set X, the associated topology $\mathfrak{T}$ is the set of sets $G \subseteq X$ such that

$$\text{for every } x \in G \text{ there is a } W \in \mathcal{W} \text{ such that } W[\{x\}] = \{y : (x,y) \in W\} \subseteq G.$$

(Engelking 89, 8.1.1; Bourbaki 66, II.1.2; Gaal 64, p. 48; Schubert 68, p. 115; James 87, p. 101.)

(c) We say that a uniformity is **Hausdorff** if the associated topology is Hausdorff.

(d) If U is a linear topological space, then it has an associated uniformity

$$\mathcal{W} = \{W : W \subseteq U \times U, \text{ there is an open set } G \text{ containing } 0 \text{ such that } (u,v) \in W \text{ whenever } u - v \in G\}.$$

(Schaefer 66, I.1.4.)

3A4B Uniformities and pseudometrics (a) If P is a family of pseudometrics on a set X, then the associated uniformity is the smallest uniformity on X containing all the sets $W(\rho;\epsilon) = \{(x,y) : \rho(x,y) < \epsilon\}$ as ρ runs through P, ϵ through $]0,\infty[$. (Engelking 89, 8.1.18; Bourbaki 66, IX.1.2.)

(b) If $\mathcal{W}$ is the uniformity defined by a family P of pseudometrics, then the topology associated with $\mathcal{W}$ is the topology defined from P (2A3F). (Dugundji 66, p. 203.)

(c) A uniformity $\mathcal{W}$ is **metrizable** if it can be defined by a single metric.

(d) If U is a linear space with a topology defined from a family of functionals $\tau : U \to [0,\infty[$ such that $\tau(u+v) \le \tau(u) + \tau(v)$, $\tau(\alpha u) \le \tau(u)$ when $|\alpha| \le 1$, and $\lim_{\alpha\to 0} \tau(\alpha u) = 0$ (2A5B), the uniformity defined from the topology (3A4Ad) coincides with the uniformity defined from the pseudometrics $\rho_\tau(u,v) = \tau(u-v)$. (Immediate from the definitions.)

3A4C Uniform continuity (a) If $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces, a function $\phi : X \to Y$ is **uniformly continuous** if $\{(x,y) : (\phi(x),\phi(y)) \in V\} \in \mathcal{W}$ for every $V \in \mathcal{V}$.

(b) The composition of uniformly continuous functions is uniformly continuous. (Bourbaki 66, II.2.1; Schubert 68, p. 118.)

(c) If uniformities $\mathcal{W}$, $\mathcal{V}$ on sets X, Y are defined by non-empty families P, Θ of pseudometrics, then a function $\phi : X \to Y$ is uniformly continuous iff for every $\theta \in \Theta$, $\epsilon > 0$ there are $\rho_0, \dots, \rho_n \in$ P and $\delta > 0$ such that $\theta(\phi(x),\phi(x')) \le \epsilon$ whenever x, $x' \in X$ and $\max_{i\le n} \rho_i(x,x') \le \delta$. (Elementary verification.)

(d) A uniformly continuous function is continuous for the associated topologies. (Bourbaki 66, II.2.1; Schubert 68, p. 118; James 87, p. 102.)

3A4D Subspaces (a) If $(X,\mathcal{W})$ is a uniform space and Y is any subset of X, then $\mathcal{W}_Y = \{W \cap (Y\times Y) : W \in \mathcal{W}\}$ is a uniformity on Y; it is the **subspace uniformity**. (Bourbaki 66, II.2.4; Schubert 68, p. 122.)

(b) If $\mathcal{W}$ defines a topology $\mathfrak{T}$ on X, then the topology defined by $\mathcal{W}_Y$ is the subspace topology on Y, as defined in 2A3C. (Schubert 68, p. 122; James 87, p. 103.)

(c) If $\mathcal{W}$ is defined by a family P of pseudometrics on X, then $\mathcal{W}_Y$ is defined by $\{\rho\restriction Y \times Y : \rho \in \text{P}\}$. (Elementary verification.)

3A4E Product uniformities (a) If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces, the **product uniformity** is the smallest uniformity $\mathcal{W}$ on $X \times Y$ containing all sets of the form

$$\{((x,y),(x',y')) : (x,x') \in U,\ (y,y') \in V\}$$

as U runs through $\mathcal{U}$ and V through $\mathcal{V}$. (Engelking 89, §8.2; Bourbaki 66, II.2.6; Schubert 68, p. 124; James 87, p. 93.)

(b) If $\mathcal{U}$, $\mathcal{V}$ are defined from families P, Θ of pseudometrics, then $\mathcal{W}$ will be defined by the family $\{\tilde{\rho} : \rho \in \mathrm{P}\} \cup \{\bar{\theta} : \theta \in \Theta\}$, writing

$$\tilde{\rho}((x,y),(x',y')) = \rho(x,x'), \quad \bar{\theta}((x,y),(x',y')) = \theta(y,y')$$

as in 2A3Tb. (Elementary verification.)

(c) If $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, a map $\phi : Z \to X \times Y$ is uniformly continuous iff the coordinate maps $\phi_1 : Z \to X$ and $\phi_2 : Z \to Y$ are uniformly continuous. (Engelking 89, 8.2.1; Bourbaki 66, II.2.6; Schubert 68, p. 125; James 87, p. 93.)

3A4F Completeness (a) If $\mathcal{W}$ is a uniformity on a set X, a filter $\mathcal{F}$ on X is **Cauchy** if for every $W \in \mathcal{W}$ there is an $F \in \mathcal{F}$ such that $F \times F \subseteq W$.

Any convergent filter in a uniform space is Cauchy. (Bourbaki 66, II.3.1; Gaal 64, p. 276; Schubert 68, p. 134; James 87, p. 109.)

(b) A uniform space is **complete** if every Cauchy filter is convergent.

(c) If $\mathcal{W}$ is defined from a family P of pseudometrics, then a filter $\mathcal{F}$ on X is Cauchy iff for every $\rho \in \mathrm{P}$ and $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that $\rho(x,y) \le \epsilon$ for all x, $y \in F$; equivalently, for every $\rho \in \mathrm{P}$, $\epsilon > 0$ there is an $x \in X$ such that $U(x;\rho;\epsilon) \in \mathcal{F}$. (Elementary verification.)

(d) A complete subspace of a Hausdorff uniform space is closed. (Engelking 89, 8.3.6; Bourbaki 66, II.3.4; Schubert 68, p. 135; James 87, p. 148.)

(e) A metric space is complete iff every Cauchy sequence converges (cf. 2A4Db). (Schubert 68, p. 141; Gaal 64, p. 276; James 87, p. 150.)

(f) If (X,ρ) is a complete metric space, $D \subseteq X$ a dense subset, (Y,σ) a metric space and $f : X \to Y$ is an **isometry** (that is, $\sigma(f(x),f(x')) = \rho(x,x')$ for all x, $x' \in X$), then $f[X]$ is precisely the closure of $f[D]$ in Y. (For it must be complete, and and we use (d).)

(g) If U is a linear space with a linear space topology and the associated uniformity (3A4Ad), then a filter $\mathcal{F}$ on U is Cauchy iff for every open set G containing 0 there is an $F \in \mathcal{F}$ such that $F - F \subseteq G$ (cf. 2A5F). (Immediate from the definitions.)

3A4G Extension of uniformly continuous functions: Theorem If $(X,\mathcal{W})$ is a uniform space, $(Y,\mathcal{V})$ is a complete uniform space, $D \subseteq X$ is a dense subset of X, and $\phi : D \to Y$ is uniformly continuous (for the subspace uniformity of D), then there is a uniformly continuous $\hat{\phi} : X \to Y$ extending ϕ. If Y is Hausdorff, the extension is unique. * (Engelking 89, 8.3.10; Bourbaki 66, II.3.6; Gaal 64, p. 300; Schubert 68, p. 137; James 87, p. 152.)

In particular, if (X,ρ) is a metric space, (Y,σ) is a complete metric space, $D \subseteq X$ is a dense subset, and $\phi : D \to Y$ is an isometry, then there is a unique isometry $\hat{\phi} : X \to Y$ extending ϕ.

3A4H Completions (a) Theorem If $(X,\mathcal{W})$ is any Hausdorff uniform space, then we can find a complete Hausdorff uniform space $(\hat{X},\hat{\mathcal{W}})$ in which X is embedded as a dense subspace; moreover, any two such spaces are essentially unique. * (Engelking 89, 8.3.12; Bourbaki 66, II.3.7; Gaal 64, p. 297 & p. 300; Schubert 68, p. 139; James 87, p. 156.)

(b) Such a space $(\hat{X},\hat{\mathcal{W}})$ is called a **completion** of $(X,\mathcal{W})$. Because it is unique up to isomorphism as a uniform space, we may call it 'the' completion.

(c) If $\mathcal{W}$ is the uniformity defined by a metric ρ on a set X, then there is a unique extension of ρ to a metric $\hat{\rho}$ on $\hat{X}$ defining the uniformity $\hat{\mathcal{W}}$. (Bourbaki 66, IX.1.3.)

3A4I A note on metric spaces I mention some elementary facts which are used in §386. Let (X,ρ) be a metric space. If $x \in X$ and $A \subseteq X$ is non-empty, set

$$\rho(x,A)=\inf_{y\in A}\rho(x,y).$$

Then $\rho(x,A)=0$ iff $x\in\overline{A}$ (2A3Kb). If $B\subseteq X$ is another non-empty set, then

$$\rho(x,B)\le\rho(x,A)+\sup_{y\in A}\rho(y,B).$$

In particular, $\rho(x,\overline{A})=\rho(x,A)$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-empty sets with union A, then

$$\rho(x,A)=\lim_{n\to\infty}\rho(x,A_n).$$

3A5 Normed spaces

I run as quickly as possible over the results, nearly all of them standard elements of any introductory course in functional analysis, which I find myself calling on in this volume. As in the corresponding section of Volume 2 (§2A4), a large proportion of these are valid for both real and complex normed spaces, but as the present volume is almost exclusively concerned with real linear spaces I leave this unsaid, except in 3A5L, and if in doubt you should suppose that scalars belong to the field $\mathbb{R}$.

3A5A The Hahn-Banach theorem: analytic forms This is one of the central ideas of functional analysis, both finite- and infinite-dimensional, and appears in a remarkable variety of forms. I list those formulations which I wish to quote, starting with those which are more or less 'analytic', according to the classification of Bourbaki 87. Recall that if U is a normed space I write U^* for the Banach space of bounded linear functionals on U.

(a) Let U be a linear space and $p:U\to[0,\infty[$ a functional such that $p(u+v)\le p(u)+p(v)$ and $p(\alpha u)=\alpha p(u)$ whenever u, $v\in U$ and $\alpha\ge0$. Then for any $u_0\in U$ there is a linear functional $f:U\to\mathbb{R}$ such that $f(u_0)=p(u_0)$ and $f(u)\le p(u)$ for every $u\in U$. Rudin 91, 3.2; Dunford & Schwartz 57, II.3.10.

(b) Let U be a normed space and V a linear subspace of U. Then for any $f\in V^*$ there is a $g\in U^*$, extending f, with $\|g\|=\|f\|$. (363R; Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Lang 93, p. 69; Wilansky 64, p. 66; Taylor 64, 3.7-B & 4.3-A.)

(c) If U is a normed space and $u\in U$ there is an $f\in U^*$ such that $\|f\|\le1$ and $f(u)=\|u\|$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.14; Wilansky 64, p. 67; Taylor 64, 3.7-C & 4.3-B.)

(d) If U is a normed space and $V\subseteq U$ is a linear subspace which is not dense, then there is a non-zero $f\in U^*$ such that $f(v)=0$ for every $v\in V$. (Rudin 91, 3.5; Dunford & Schwartz 57, II.3.12; Taylor 64, 4.3-D.)

(e) If U is a normed space, U^* separates the points of U. (Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, II.3.14.)

3A5B Cones (a) Let U be a linear space. A **convex cone** (with apex 0) is a set $C\subseteq U$ such that $\alpha u+\beta v\in C$ whenever u, $v\in C$ and α, $\beta\ge0$. The intersection of any family of convex cones is a convex cone, so for every subset A of U there is a smallest convex cone including A.

(b) Let U be a normed space. Then the closure of a convex cone is a convex cone, and the closure of a linear subspace is a linear subspace. (Bourbaki 87, II.2.6; Dunford & Schwartz 57, V.2.1.)

3A5C Hahn-Banach theorem: geometric forms (a) Let U be a normed space and $C\subseteq U$ a convex set such that $\|u\|\ge1$ for every $u\in C$. Then there is an $f\in U^*$ such that $\|f\|\le1$ and $f(u)\ge1$ for every $u\in C$. (Dunford & Schwartz 57, V.1.12.)

(b) Let U be a normed space and $B \subseteq U$ a non-empty convex set such that $0 \notin \overline{B}$. Then there is an $f \in U^*$ such that $\inf_{u \in B} f(u) > 0$. (Bourbaki 87, II.4.1; Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, V.2.12.)

(c) Let U be a normed space, B a closed convex subset of U containing 0, and u a point of $U \setminus B$. Then there is an $f \in U^*$ such that $f(u) > 1$ and $f(v) \le 1$ for every $v \in B$. (Apply (b) to $B - u$ to find a $g \in U^*$ such that $g(u) < \inf_{v \in B} g(v)$ and now set $f = -\frac{1}{\alpha} g$ where $g(u) < \alpha < \inf_{v \in B} g(v)$).

3A5D Separation from finitely-generated cones Let U be a linear space over $\mathbb{R}$ and $u, v_0, \dots, v_n$ points of U such that u does not belong to the convex cone generated by $\{v_0, \dots, v_n\}$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(v_i) \ge 0$ for every i and $f(u) < 0$.

proof (a) If U is finite-dimensional this is covered by Gale 60, p. 56.

(b) For the general case, let V be the linear subspace of U generated by $u, v_0, \dots, v_n$. Then there is a linear functional $f_0 : V \to \mathbb{R}$ such that $f_0(u) < 0 \le f_0(v_i)$ for every i. By Zorn's Lemma, there is a maximal linear subspace $W \subseteq U$ such that $W \cap V = \{0\}$. Now $W + V = U$ (for if $u \notin W + V$, the linear subspace W' generated by $W \cup \{u\}$ still has trivial intersection with V), so we have an extension of f_0 to a linear functional $f : U \to \mathbb{R}$ defined by setting $f(v + w) = f_0(v)$ whenever $v \in V$ and $w \in W$. Now $f(u) < 0 \le \min_{i \le n} f(v_i)$, as required.

3A5E Weak topologies (a) Let U be any linear space over $\mathbb{R}$ and W a linear subspace of the space U' of all linear functionals from U to $\mathbb{R}$. Then I write $\mathfrak{T}_s(U, W)$ for the linear space topology defined by the method of 2A5B from the functionals $u \mapsto |f(u)|$ as f runs through W. (Bourbaki 87, II.6.2; Rudin 91, 3.10; Dunford & Schwartz 57, V.3.2; Taylor 64, 3.81.)

(b) I note that the weak topology of a normed space U (2A5Ia) is $\mathfrak{T}_s(U, U^*)$, while the weak* topology of U^* (2A5Ig) is $\mathfrak{T}_s(U^*, W)$ where W is the canonical image of U in U^{**}. (Rudin 91, 3.14.)

(c) Let U and V be linear spaces over $\mathbb{R}$ and $T : U \to V$ a linear operator. If $W \subseteq U'$ and $Z \subseteq V'$ are such that $gT \in W$ for every $g \in Z$, then T is continuous for $\mathfrak{T}_s(U, W)$ and $\mathfrak{T}_s(V, Z)$. (Bourbaki 87, II.6.4.)

(d) If U and V are normed spaces and $T : U \to V$ is a bounded linear operator then we have an **adjoint** (or **conjugate**, or **dual**) operator $T' : V^* \to U^*$ defined by saying that $T'g = gT$ for every $g \in V^*$. T' is linear and is continuous for the weak* topologies of U^* and V^*. (Bourbaki 87, II.6.4; Dunford & Schwartz 57, §VI.2; Taylor 64, 4.5.)

(e) If U is a normed space and $A \subseteq U$ is convex, then the closure of A for the norm topology is the same as the closure of A for the weak topology of U. In particular, norm-closed convex subsets (for instance, norm-closed linear subspaces) of U are closed for the weak topology. (Rudin 91, 3.12; Lang 93, p. 88; Dunford & Schwartz 57, V.3.13.)

3A5F Weak* topologies: Theorem If U is a normed space, the unit ball of U^* is compact and Hausdorff for the weak* topology. (Rudin 91, 3.15; Lang 93, p. 71; Dunford & Schwartz 57, V.4.2; Taylor 64, 4.61-A.)

3A5G Reflexive spaces (a) A normed space U is **reflexive** if every member of U^{**} is of the form $f \mapsto f(u)$ for some $u \in U$.

(b) A normed space is reflexive iff bounded sets are relatively weakly compact. (Dunford & Schwartz 57, V.4.8; Taylor 64, 4.61-C.)

(c) If U is a reflexive space, $\langle u_n \rangle_{n \in \mathbb{N}}$ is a bounded sequence in U and $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $\lim_{n \to \mathcal{F}} u_n$ is defined in U for the weak topology. (Use (b) and the argument of 2A3Se.)

3A5H (a) Uniform Boundedness Theorem Let U be a Banach space, V a normed space, and $A \subseteq \mathsf{B}(U;V)$ a set such that $\{Tu : T \in A\}$ is bounded in V for every $u \in U$. Then A is bounded in $\mathsf{B}(U;V)$. (Rudin 91, 2.6; Dunford & Schwartz 57, II.3.21; Taylor 64, 4.4-E.)

(b) Corollary If U is a normed space and $A \subseteq U$ is such that $f[A]$ is bounded for every $f \in U^*$, then A is bounded. (Wilansky 64, p. 117; Taylor 64, 4.4-AS.) Consequently any relatively weakly compact set in U is bounded. (Rudin 91, 3.18.)

3A5I Completions Let U be a normed space.

(a) Recall that U has a metric ρ associated with the norm (2A4Bb), and that the topology defined by ρ is a linear space topology (2A5D, 2A5B). This topology defines a uniformity $\mathcal{W}$ (3A4Ad) which is also the uniformity defined by ρ (3A4Bd). The norm itself is a uniformly continuous function from U to $\mathbb{R}$ (because $|\|u\| - \|v\|| \le \|u - v\|$ for all u, $v \in U$).

(b) Let $(\hat{U}, \hat{\mathcal{W}})$ be the uniform space completion of $(U, \mathcal{W})$ (3A4H). Then addition and scalar multiplication and the norm extend uniquely to make $\hat{U}$ a Banach space. (Schaefer 66, I.1.5; Lang 93, p. 78.)

(c) If U and V are Banach spaces with dense linear subspaces U_0 and V_0, then any norm-preserving isomorphism between U_0 and V_0 extends uniquely to a norm-preserving isomorphism between U and V (use 3A4G).

3A5J Normed algebras If U is a normed algebra (2A4J), its multiplication, regarded as a function from $U \times U$ to U, is continuous. (Wilansky 64, p. 259.)

3A5K Compact operators Let U and V be Banach spaces.

(a) A linear operator $T : U \to V$ is **compact** or **completely continuous** if $\{Tu : \|u\| \le 1\}$ is relatively compact in V for the topology defined by the norm of V.

(b) A linear operator $T : U \to V$ is **weakly compact** if $\{Tu : \|u\| \le 1\}$ is relatively weakly compact in V.

3A5L Hilbert spaces I mentioned the phrases 'inner product space', 'Hilbert space' briefly in 244N-244O, without explanation, as I did not there rely on any of the abstract theory of these spaces. For the main result of §395 we need one of their fundamental properties, so I now skim over the definitions.

(a) An **inner product space** is a linear space U over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ together with an operator $(\ |\ ) : U \times U \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ such that

$$(u_1 + u_2|v) = (u_1|v) + (u_2|v), \quad (\alpha u|v) = \alpha(u|v), \quad (u|v) = \overline{(v|u)}$$

(the complex conjugate of $(u|v)$),

$$(u|u) \ge 0, \quad u = 0 \text{ whenever } (u|u) = 0$$

for all u, u_1, u_2, $v \in U$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(b) If U is any inner product space, we have a norm on U defined by setting $\|u\| = \sqrt{(u|u)}$ for every $u \in U$. (Taylor 64, 3.2-B.)

(c) A **Hilbert space** is an inner product space which is a Banach space under the norm of (b) above, that is, is complete in the metric defined from its norm.

(d) If U is a Hilbert space, $C \subseteq U$ is a non-empty closed convex set, and $u \in U$, then there is a unique $v \in C$ such that $\|u - v\| = \inf_{w \in C} \|u - w\|$. (Taylor 64, 4.81-A.)

3A6 Group Theory

For Chapter 38 we need four definitions and two results from elementary abstract group theory.

3A6A Definition If G is a group, I will say that an element g of G is an **involution** if its order is 2, that is, $g^2 = e$, the identity of G, but $g \neq e$.

3A6B Definition If G is a group, the set $\operatorname{Aut} G$ of **automorphisms** of G (that is, bijective homomorphisms from G to itself) is a group. For $g \in G$ define $\hat{g} : G \to G$ by writing $\hat{g}(h) = ghg^{-1}$ for every $h \in G$; then $\hat{g} \in \operatorname{Aut} G$, and the map $g \mapsto \hat{g}$ is a homomorphism from G onto a normal subgroup J of $\operatorname{Aut} G$ (Rotman 84, p. 130). We call J the group of **inner automorphisms** of G. Members of $(\operatorname{Aut} G) \setminus J$ are called **outer automorphisms**.

3A6C Normal subgroups For any group G, the family of normal subgroups of G, ordered by $\subseteq$, is a Dedekind complete lattice, with $H \vee K = HK$ and $H \wedge K = H \cap K$. (Davey & Priestley 90, 2.8 & 2.19.)

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Index to volumes 1, 2 and 3

Principal topics and results

The general index below is intended to be comprehensive. Inevitably the entries are voluminous to the point that they are often unhelpful. I am therefore preparing a shorter, better-annotated, index which will, I hope, help readers to focus on particular areas. It does not mention definitions, as the bold-type entries in the main index are supposed to lead efficiently to these; and if you draw blank here you should always, of course, try again in the main index. Entries in the form of mathematical assertions frequently omit essential hypotheses and should be checked against the formal statements in the body of the work.

General index

This index covers the 'full' version of the work. References in **bold** type refer to definitions; references in *italics* are passing references. Definitions marked with **>** are those in which my usage is dangerously at variance with that of some other author.