Sequence and Series of function

Pointwise Convergence and Uniform Convergence

Pointwise Convergence

Definition of pointwise convergence:

A sequence of functions $f_1, f_2, \dots, f_n, \dots : E \to \mathbb{R}$ (where E is a subset of $\mathbb{R}$ is said to be converges pointwise on E to function $f: E \to \mathbb{R}$ if and only if

$$\lim_{n\to\infty} f_n(x) = f(x) \ \ for\ all\ \ x \in E$$

Similarly a series of function $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise to $S(x)$ on E if and only if

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} f_k(x)\right) = S(x) \ \ for\ all\ \ x \in E$$

Example 1 (Pointwise Convergence of sequence of functions)

Let two sequences of functions $f_n, g_n : [0,1] \to \mathbb{R}$ be defined as

$$f_n(x) = e^{\frac{x}{n}} \quad \text{and} \quad g_n(x) = x^n$$

Show that f, g both converge pointwise on $[0,1]$

Solution:

For $f(x)$, note that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{\frac{x}{n}} = e^0 = 1 \quad for\ x \in [0,1]$$

Thus $f_n(x)$ converges pointwise to $f(x) = 1$

For $g(x)$

By taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} x^n = \begin{cases} \lim_{n\to\infty} x^n = 0 & if\ x \in [0,1) \\ \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1 & if\ x = 1 \end{cases}$$

Thus $g_n(x)$ converges pointwise to $g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$

Example 2 (Pointwise Convergence of series of functions)

Discuss the pointwise convergence of series of functions

$$\sum_{n=1}^{\infty}\frac{n^2}{x^n}$$

on $(0,\infty)$

Solution:

By applying root test (or ratio test if you wish), we have

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{x^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{x}=\frac{1}{x}\quad \text{for } x\in(0,\infty)$$

We see the series converges (absolutely) if $\frac{1}{x}<1\Leftrightarrow x>1$ and diverges if $\frac{1}{x}>1\Leftrightarrow x<1$. Now it remains to check the case $\frac{1}{x}=1\Leftrightarrow x=1$ (which root test does not give any conclusion). For $x=1$, the series become

$$\sum_{n=1}^{\infty}n^2$$

Which clearly diverges by term test (as $\lim_{n\to\infty}n^2=+\infty$, thus when $x>1$ (or $x\in(1,\infty)$), the series converges pointwise.

(*Note: In the above example $(1,\infty)$ is also called <u>domain of convergence</u>)

Example 3

Discuss the pointwise convergence of series of functions

$$\sum_{n=1}^{\infty}\frac{e^{2n}}{(1+|x-1|)^n}$$

Solution:

By applying root test (or ratio test) again, we have

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\frac{e^{2n}}{(1+|x-1|)^n}}=\lim_{n\to\infty}\frac{e^2}{1+|x-1|}=\frac{e^2}{1+|x-1|}\quad \text{for } x\in(0,\infty)$$

Hence the series converges if $\frac{e^2}{1+|x-1|}<1\Leftrightarrow|x-1|>e^2-1\Leftrightarrow x>e^2$ or $x<2-e^2$

and diverges if $\frac{e^2}{1+|x-1|}>1\Leftrightarrow|x-1|<e^2-1\Leftrightarrow 2-e^2<x<e^2$

It remains to check the case $\frac{e^2}{1+|x-1|}=1\Leftrightarrow x=2-e^2$ and $x=e^2$

For $x=2-e^2$, both series become $\sum_{n=1}^{\infty}1$ which clearly diverges.

For $x=e^2$, both series become $\sum_{n=1}^{\infty}1$ which clearly diverges.

Hence the domain of convergence of this series is

$x<2-e^2$ and $x>e^2$ (or $x\in(-\infty,2-e^2)\cup(e^2,\infty)$)

In case when the series is a power series (i.e. $\sum_{n=1}^{\infty} a_n(x-c)^n$ for some constant $a_n \in \mathbb{R}$. Then one may have the following fact:

Given a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, the domain of convergence of the series is an <u>non-empty interval</u> (E) which $E \subseteq [c-R+c+R]$ where R is so called **<u>radius of convergence</u>** of the series

Example 4

Find the domain of convergence and radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x-1)^n$$

Solution:

We can apply root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{n^3}}{3}|x-1| = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^3}{3}|x-1| = \frac{|x-1|}{3}$$

The series converges when $\frac{|x-1|}{3} < 1 \rightarrow |x-1| < 3 \rightarrow -2 < x < 4$

The series diverges when $\frac{|x-1|}{3} > 1 \rightarrow |x-1| > 3 \rightarrow x < -2 \text{ and } x > 4$

At $x = -2$, the series become $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n n^3$ which diverges

At $x = 4$, the series become $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(3)^n = \sum_{n=1}^{\infty} n^3$ which diverges

Hence the domain of convergence is $(-2,4)$ which $c = 1$, the radius of convergence $R = 3$.

Example 5

Find the domain of convergence and radius of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$$

Solution:

Since the terms involves factorial, instead of using root test, it may better for us to use ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}x^{n+1}}{(n+1)!}}{\frac{2^n x^n}{n!}}\right| = \lim_{n\to\infty} \frac{2x}{n+1} = 0 < 1$$

So the series converges for all $x \in \mathbf{R}$, the domain of convergence is $(-\infty, \infty)$ and $R = \infty$

Uniform Convergence

Besides pointwise convergence, next we would like to introduce another type of convergence which is called uniform convergence. It allows us to do some operations like

$$\lim_{x\to a}\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty}\lim_{x\to a} f_n(x)$$

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx=\sum_{n=1}^{\infty}\int_a^b f_n(x)dx$$

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x)=\sum_{n=1}^{\infty}\frac{d}{dx}f_n(x)$$

Definition: (Uniform Convergence of Function)

Given a sequence of function $f_n: E\to\mathbb{R}$, we say f_n **converges uniformly** to f iff

$$\lim_{n\to\infty}\left(\sup_{x\in E}|f_n(x)-f(x)|\right)=0$$

In other word, $\sup_{x\in E}|f_n(x)-f(x)|\to 0$ as $n\to\infty$

(Note: $\sup_{x\in E}|g(x)|$ is called **sup-norm** of $g(x)$ on E)

Definition: (Uniform Convergence of Series of Function)

Let $g_n: E\to\mathbb{R}$ be a sequence of functions, we say the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to function $S(x)$ on E iff the partial sum $S_n(x)=\sum_{k=1}^{n} g_k(x)$ converges uniformly to $S(x)$ on E

Example 6 (Example 1 revisited)

Discuss the uniform convergence of

$$f_n(x)=e^{\frac{x}{n}} \quad\text{and}\quad g_n(x)=x^n$$

on $[0,1]$

Solution:

For $f_n(x)$

(Step 1: Find the Limit)

From example 1, we see the limit is $f(x)=\lim_{n\to\infty} f_n(x)=1$

(Step 2: Compute sup-norm)

$$\sup_{x\in[0,1]}|f_n(x)-f(x)|=\sup_{x\in[0,1]}\left|e^{\left(\frac{x}{n}\right)}-1\right|=\sup_{x\in[0,1]}(e^{\frac{x}{n}}-1)=e^{\frac{1}{n}}-1$$

Then

$$\lim_{n\to\infty}\left(\sup_{x\in E}|f_n(x)-f(x)|\right)=\lim_{n\to\infty}(e^{\frac{1}{n}}-1)=0$$

Thus the function is uniformly convergent on $[0,1]$

For $\mathbf{g_n(x)}$

(Step 1: Find the limit)

From example 1, we see that the limit is $g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$

(Step 2: Compute sup-norm)

Note that $|g_n(x) - g(x)| = \begin{cases} |x^n - 1| = |1^n - 1| = 0 & \text{for } x = 1 \\ |x^n - 0| = |x^n| & \text{for } x \in [0,1) \end{cases}$

Then

$$\sup_{x\in[0,1]} |g_n(x) - g(x)| = \sup\left\{\sup_{x\in[0,1)} x^n, 0\right\} = \sup\{1,0\} = 1$$

$$\Rightarrow \lim_{n\to\infty}\left(\sup_{x\in E}|f_n(x) - f(x)|\right) = \lim_{n\to\infty} 1 = 1 \neq 0$$

The function is not uniformly convergent on $[0,1]$

Remark: (Pointwise Convergence v.s. Uniformly Convergence)
These two convergences looks similar but there is a big fundamental difference:
Pointwise convergence only require for every $x \in E$, $f_n(x)$ converges to $f(x)$, but the "speed" of convergence can be varied among different points. Some points may converge faster and some points may converge slower.
But uniform convergence also requires the "speed" of convergence is "similar" among all points besides pointwise convergent so that the property of functions can be preserved when taking limit. (For example: limit of continuous function is continuous, limit of integrable function is integrable)

Let consider the two functions in Example 6, by plotting their graphs out, we can see

1. $f_n(x) = e^{\frac{x}{n}}$ $\qquad\qquad$ $f(x) = \lim_{n\to\infty} f_n(x) = 1$

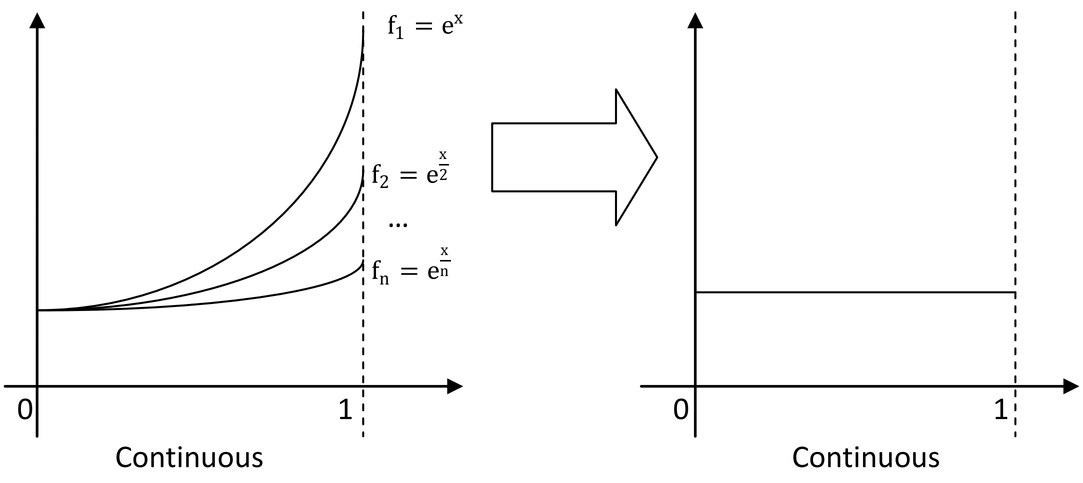

Fig 1: Graph of $f_n(x)$ (converges uniformly on $[0,1]$

2. $g_n(x) = x^n$ $\qquad$ $g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$

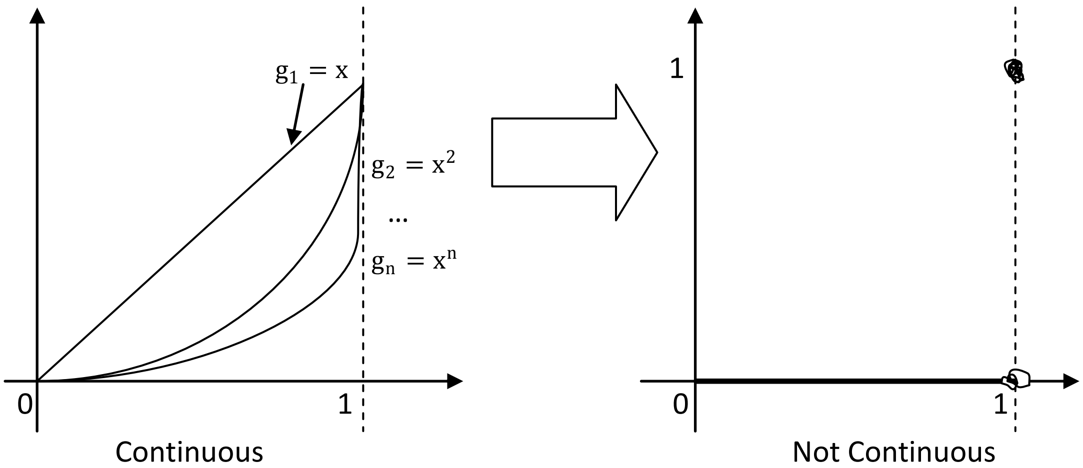

Fig 2: Graph of $g_n(x)$ (does not converges uniformly on $[0,1]$

From the above two examples, we see if $f_n(x)$ converges in similar speed, then the continuity of f_n can be preserved (in (1)). However if $g_n(x)$ converges very fast at some points and converges very slow in some points (say (2)). Then the property of $g_n(x)$ may be "destroyed" at $n \to \infty$.

<u>L-test (for sequence of functions)</u>

Let $f_n: E \to R$ be sequences of functions on set E, suppose

1) $\lim_{n\to\infty} f_n(x) = f(x)$ (Pointwise Limit)
2) For each $n = 1,2,3, \dots$ there is constant L_n such that
$$|f_n(x) - f(x)| \le L_n \text{ for all } x \in E$$
3) $\lim_{n\to\infty} L_n = 0$

Then $f_n(x)$ converges uniformly to $f(x)$

Example 7

Show that the following sequence of functions
$$f_n(x) = \frac{\sin nx}{1 + nx}$$
converges uniformly on $[c, \infty)$ where c is a positive number.

(Step 1: Find the limit first)

For any $x \in [c, \infty)$, we have
$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{1 + nx} = 0 \quad (\text{Since } 1 + nx \to \infty)$$

(Step 2: Compute the sup-norm)
$$0 \le |f_n(x) - f(x)| = \frac{|sinnx|}{|1 + nx|} \le \frac{1}{|1 + nx|} \le \frac{1}{1 + nc} = L_n \dots\dots for\ x \in [c, \infty)$$

Note that $\lim_{n\to\infty} \frac{1}{1+nc} = 0$,

By L-test, $f_n(x)$ converges uniformly on $[c, \infty)$.

In the following, there are some suggested exercises, you should try to do them in order to understand the material. If you have any questions about them, you are welcome to find me during office hours. You are also welcome to submit your work (complete or incomplete) to me and I can give some comments to your work.

☺Exercise 0

Determine whether the following statements are true or not. Give a brief explanation. (e.g. proof, provide counter-examples etc.)

*In the following problems, let $f_n(x)$ be a sequence of functions on E

a) If $f_n(x)$ *converges uniformly* to $f(x)$ on E, then $f_n(x)$ *converges pointwise* to $f(x)$ on E

b) If $f_n(x)$ *converges pointwise* to $f(x)$ on E, then $f_n(x)$ *converges uniformly* on E.

c) Given a series of functions $\sum_{n=1}^{\infty} f_n(x)$, then its domain of convergence MUST be an interval.

d) The domain of convergence of power series is always an interval WITH BOTH ENDPOINTS.

e) The domain of convergence of power series is always an interval WITHOUT BOTH ENDPOINTS.

f) Given a sequence of continuous functions $f_n(x)$ on E, then $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous on E

g) Given a sequence of integrable functions $g_n(x)$ on E, then $g(x) = \lim_{n\to\infty} g_n(x)$ is also integrable on E.

Answer:
a) is true and b), c), d), e), f), g) are all false.

(Hint: To disprove g), consider the following counter-example:

$\mathbb{Q}$ (set of rational numbers) is countable, let $\mathbb{Q} = \{r_1, r_2, r_3, \dots.\}$, construct $g_n(x)$ as

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \dots r_n \\ 0 & \text{otherwise} \end{cases}$$

$$\text{then} \quad g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

One can show g_n is Riemann integrable and g(x) is not integrable.

☺Exercise 1

Find the domain of convergence of following series of functions

a) $\sum_{n=1}^{\infty} (-1)^n x e^{-nx}$

b) $\sum_{n=1}^{\infty} \frac{1}{(1+|x-1|+|x-2|)^n}$

☺Exercise 2

Find the domain of convergence and radius of convergence for the following power series

a) $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{(n+1)(n+2)}$

b) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n x^n$

c) $\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}(x+1)^n$

☺Exercise 3

a) Show that the sequence of functions $f_n(x) = \frac{1}{nx+1}$ converges pointwise but not converge uniformly on $(0,1)$.

b) Show that the sequence of functions $g_n(x) = \frac{x}{nx+1}$ converges uniformly on $(0,1)$

☺Exercise 4

Show that if f_n and g_n converges uniformly to f, g respectively on a set E, then $f_n + g_n$ converges uniformly on E.

☺Exercise 5

a) Show that if f_n and g_n are <u>bounded</u> and converges uniformly to f, g respectively on a set E, then $f_n g_n$ converges uniformly on E.
(Hint: Note that $f_n g_n - fg = f_n g_n - f_n g + f_n g - fg$)

b) Is the statement still true if the condition "f_n and g_n are bounded" is omitted.

☺Exercise 6

Show that the following sequences of functions on indicated intervals.

a) $f_n(x) = \frac{x}{1+(nx^3)^{\frac{1}{3}}+(nx^3)^{\frac{2}{3}}}$ on $x \in [1, \infty)$

b) $f_n(x) = \frac{n^2x}{1+n^3x^2}$ on $x \in \mathbb{R}$

(Hint: Use Calculus!)

WEIERSTRASS M-TEST

Our goal is to prove the following result:

Weierstrass M-Test. Let $\sum_{n=1}^{\infty} f_n$ be a series of real-valued functions on a subset A of $\mathbb{R}$. Suppose that there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of nonnegative real numbers such that for all $n \in \mathbb{N}$ and $x \in A$ we have $|f_n(x)| \leq M_n$. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof of Weierstrass M-Test. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, is is Cauchy, so we can choose $k \in \mathbb{N}$ such that for all $l \geq j \geq k$ we have $\sum_{n=j}^{l} M_n < \epsilon$. Then for all $l \geq j \geq k$ and $x \in A$ we have

$$\left| \sum_{n=j}^{l} f_n(x) \right| \leq \sum_{n=j}^{l} |f_n(x)| < \epsilon.$$

Thus $\sum_{n=1}^{\infty} f_n$ satisfies the Uniform Cauchy Criterion, hence converges uniformly. **QED**