Tangent Planes to Surfaces

Let F be a differentiable function of three variables x, y, and z. For a constant k, the equation $F(x, y, z) = k$ represents a surface S in space. For example, the equation $x^2 + y^2 + z^2 = 9$ represents the sphere with radius 3 and center at the origin.

Let $P(x_0, y_0, z_0)$ be a point on S. We wish to find the tangent plane to the surface S at P. Let C be a smooth curve that lies on the surface S and passes through the point P. The curve C is described by a differentiable vector function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a < t < b.$$

There exists a parameter $t_0 \in (a, b)$ such that

$$\mathbf{r}(t_0) = (x_0, y_0, z_0).$$

Since the curve C lies on the surface S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F\big(x(t), y(t), z(t)\big) = k, \quad a < t < b.$$

We use the chain rule to differentiate both sides of the above equation as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

Recall that $\nabla F = \frac{\partial F}{\partial x}\,\mathbf{i} + \frac{\partial F}{\partial y}\,\mathbf{j} + \frac{\partial F}{\partial z}\,\mathbf{k}$. Moreover, $\mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$. Thus we obtain

$$\nabla F \cdot \mathbf{r}'(t) = 0, \quad a < t < b.$$

In particular, for $t = t_0$ we have

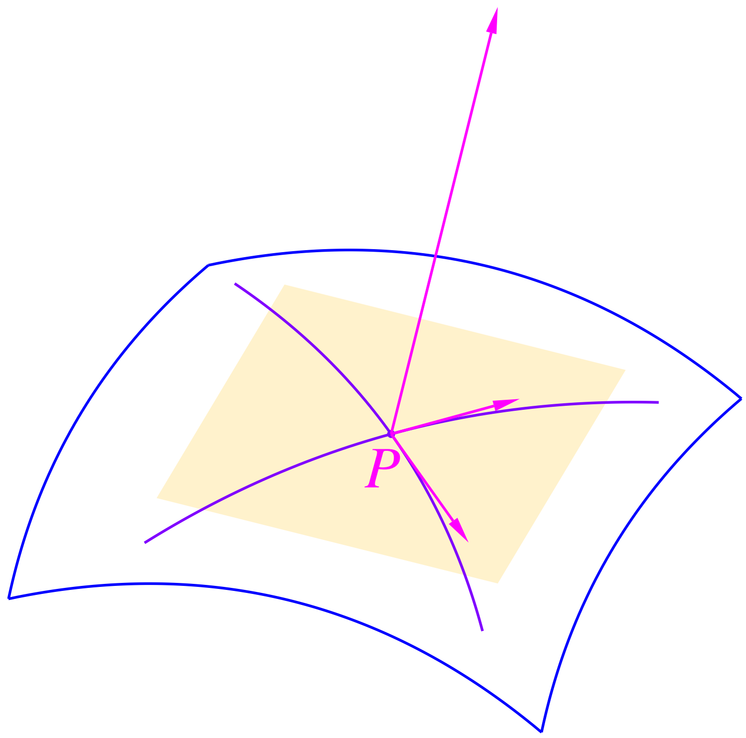
$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0.$$

Note that $\mathbf{r}'(t_0)$ is a tangent vector to the curve C at P.

Normal Vectors

The above equation shows that the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.

Conclusion. If the gradient $\nabla F(x_0, y_0, z_0) \neq 0$, then $\nabla F(x_0, y_0, z_0)$ is a normal vector to the tangent plane to the surface $F(x, y, z) = k$ at (x_0, y_0, z_0).



Equations of Tangent Planes

Let S be the surface represented by the equation $F(x,y,z) = k$, where k is a constant and F is a differentiable function. For a point $P = (x_0, y_0, z_0)$ on S, let

$$a = \left.\frac{\partial F}{\partial x}\right|_P, \quad b = \left.\frac{\partial F}{\partial y}\right|_P, \quad c = \left.\frac{\partial F}{\partial z}\right|_P.$$

If $\nabla F(x_0, y_0, z_0) = (a, b, c) \neq 0$, then the tangent plane to the surface S at the point $P(x_0, y_0, z_0)$ has an equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

If a surface S is represented by the equation $z = f(x, y)$, then we may rewrite it as $f(x, y) - z = 0$.

Suppose f has continuous partial derivatives. The tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ has an equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example. Find the tangent plane to the paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution. Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x, \quad f_y(x, y) = 2y,$$

$$f_x(1, 1) = 4, \quad f_y(1, 1) = 2.$$

Hence, the tangent plane at $(1, 1, 3)$ has an equation

$$z - 3 = 4(x - 1) + 2(y - 1).$$

This is simplified to

$$z = 4x + 2y - 3.$$

Example. Find an equation of the tangent plane and symmetric equations of the normal line to the surface $4x^2 + 9y^2 - z^2 = 16$ at the point $(2, 1, 3)$.

Solution. Let $F(x, y, z) = 4x^2 + 9y^2 - z^2$. We have

$$\nabla F(x, y, z) = 8x\,\mathbf{i} + 18y\,\mathbf{j} - 2z\,\mathbf{k}.$$

It follows that

$$\nabla F(2, 1, 3) = 16\,\mathbf{i} + 18\,\mathbf{j} - 6\,\mathbf{k}.$$

Hence, the tangent plane at $(2, 1, 3)$ has an equation

$$16(x - 2) + 18(y - 1) - 6(z - 3) = 0,$$

which is simplified to $8x + 9y - 3z = 16$. Moreover, the normal line to the surface at $(2, 1, 3)$ has symmetric equations

$$\frac{x-2}{8} = \frac{y-1}{9} = \frac{z-3}{-3}.$$