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Vehicle routing problems with split deliveries

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Abstract

This paper is a survey on the vehicle routing problems with split deliveries, a class of routing problems where each customer may be served by more than one vehicle. Starting from the most classical routing problems, we introduce the split delivery vehicle routing problem (SDVRP). We review a formulation, the main properties and exact and heuristic solution approaches for the SDVRP. Then, we present a general overview of several variants of the SDVRP and of the literature available.

Keywords: traveling salesman problem; vehicle routing problem; split deliveries; survey

1. Introduction

Consider a set of customers that need to be served by means of an unlimited fleet of identical capacitated vehicles that start and end their routes at a depot. Each customer may be visited by several vehicles if beneficial, that is, split deliveries are allowed. The split delivery vehicle routing problem (SDVRP) is the problem of finding a set of routes that minimizes the total traveling cost. In the SDVRP, the traditional assumption made in the vehicle routing problem (VRP) that each customer is visited only once is relaxed.

The SDVRP was formally introduced by Dror and Trudeau (1989, 1990) and has received increasing attention over time. Some properties of the problem and complexity results have been proved. Moreover, the maximum saving that can be achieved by split deliveries has been studied. Heuristic and exact solution algorithms have been proposed. The SDVRP is a very challenging problem that at present can be solved to optimality in a systematic way only on instances with less than 30 customers.

Split deliveries are also allowed in other problems. While some of these problems are extensions of the SDVRP, others are more complex problems often motivated by applications. In Archetti and Speranza (2008), a survey on the SDVRP can be found, whereas in Golden et al. (2008), a more general overview of the literature where split deliveries are considered is presented.

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This paper is a survey on the SDVRP that also overviews its variants and in general all routing problems that consider split deliveries. In Section 2, we introduce the SDVRP, starting from the classical traveling salesman problem (TSP) and the VRP. In Section 3, the SDVRP is formally described and its mathematical programming formulation is presented. The main properties are presented in Section 4. Sections 5 and 6 review the heuristics and the exact approaches proposed for the SDVRP. Finally, Section 7 provides an overview of all the studied variants of the SDVRP and in general of the papers where problems that involve split deliveries are considered.

2. Introduction to the SDVRPs

Let us start this introductory section with the most classical routing problem, the TSP. In the TSP, the salesman has to visit all the customers, starting and ending the tour at the same point, at minimum cost. In the TSP, there is a single vehicle with unlimited capacity that has to visit all the customers, starting and ending the tour at the depot, at minimum cost. In Fig. 1, an example of TSP instance and solution are shown. The costs of traversing the edges are shown while no demand is specified for the customers, as a single vehicle can serve all the customers.

When the customers have a specified demand, a single capacitated vehicle may not be sufficient to serve all the customers. Then, in general, a fleet of capacitated vehicles must be available.

Figure 2 shows an example of an instance of the VRP, where a fleet of vehicles each of capacity $Q = 4$ is available. The number associated with a vertex represents the demand of the customer. In the VRP, it is assumed that each customer must be served only by one vehicle.

In Fig. 3a, a solution of the instance given in Fig. 2 is shown that makes use of four vehicles. Note that, although the total demand of the customers is 12 and the capacity of each vehicle is four, no solution exists with three vehicles only. As we need to visit each customer with one vehicle

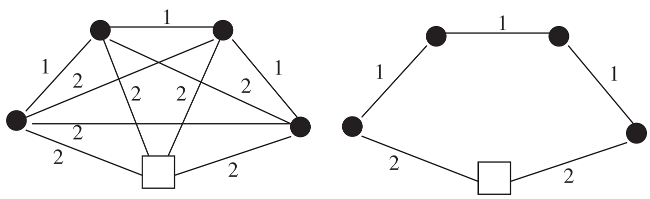

Fig. 1. Example with one vehicle.

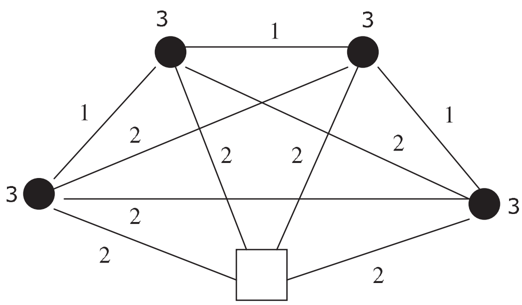

Fig. 2. Example: vehicles with $Q = 4$.

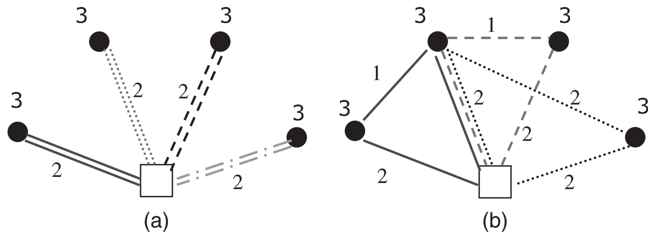

Fig. 3. Example: (a) vehicle routing problem and (b) split delivery vehicle routing problem solutions, cost = 16.

only, we need four vehicles to visit all the customers. Moreover, each vehicle serves one customer only because no pair of customers can be served together by the same vehicle. In the VRP, we aim at finding the set of routes, starting and ending at the depot, of total minimum cost. Determining the minimum number of vehicles necessary to serve all the customers is equivalent to solving a bin packing problem and thus is itself an NP-hard problem. Besides, the solution of the minimum total traveling cost may use a number of vehicles greater than the minimum possible number. Therefore, if the traveling cost is considered to be the most relevant performance measure, then an unlimited fleet of vehicles is assumed. The optimal solution will provide the optimum set of routes and, as a consequence, the optimum number of vehicles. Sometimes, a maximum number of vehicles available is given and the optimal solution, which minimizes the total cost of the routes, may make use of all the vehicles or of a smaller number. In this case, it may happen that no feasible solution exists as in the example of Fig. 2 if the maximum number of vehicles available would be fixed to three.

In the SDVRP, the assumption that a customer must be visited by one vehicle only is relaxed. The goal is still to find routes, starting and ending at the depot, of total minimum cost. In Fig. 3b, a solution of the SDVRP is shown, where one of the customers is visited three times. Note that this solution has the same cost, 16, of the VRP solution of Fig. 3a. A major difference between the two solutions is that the SDVRP solution makes use of only three vehicles. It is now interesting to see whether the optimal solution of the SDVRP may have a cost smaller than the cost of the optimal solution of the VRP. Figure 4a shows an instance where an unlimited fleet of vehicles is available, each with capacity five. Figures 4b and c show the optimal solution of the VRP and the SDVRP, respectively. The solution of the VRP has cost 24, whereas the solution of the SDVRP has cost 18. Moreover, the SDVRP solution uses two vehicles only, whereas two vehicles cannot serve all the customers in the VRP case, where the solution of minimum cost uses three vehicles.

In general, in the SDVRP, a solution always exists that makes use of the minimum possible number of vehicles. This number can be easily computed as it is the ratio between the overall demand and the vehicle capacity, rounded to the minimum integer greater than the ratio. A better solution, in terms of the minimum total cost of the routes, may, however, exist that makes use of a larger number of vehicles. Take the example shown in Fig. 5. The capacity of the vehicles is Q. The minimum number of vehicles is two and if we impose that two vehicles only must be used in the SDVRP, then one of the customers must be visited by two vehicles and the optimal solution has cost equal to eight. If we do not constrain the number of vehicles to be used, then the optimal solution uses three vehicles, one per customer, with a total cost equal to six. Note that this is the optimal solution of the SDVRP with unlimited fleet but also the optimal solution of the VRP with

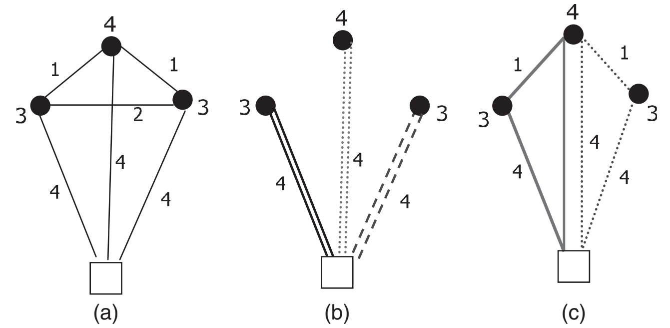

Fig. 4. Example: (a) problem instance, (b) vehicle routing problem and (c) split delivery routing problem solutions of different costs.

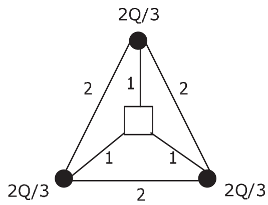

Fig. 5. Example: split delivery vehicle routing problem with limited and unlimited fleet.

unlimited fleet as each customer is served by one vehicle only. This example also shows that the optimal solution of the VRP with unlimited fleet may be better than the optimal solution of the SDVRP with minimum number of vehicles.

3. Description and formulation of the SDVRP

The SDVRP is defined over an undirected graph $G=(V,E)$ with vertex set $V=\{0,1,\ldots,n\}$, where 0 represents the depot and the other vertices represent the customers, and E is the set of edges. The cost (also called length) c_{ij} of an edge $(i,j)\in E$ is non-negative and satisfies the triangle inequality. A demand d_i is associated with each customer $i\in V-\{0\}$. An unlimited fleet of vehicles is available, each with capacity $Q>0$. A lower bound on the number of vehicles needed to serve the customers is given by $\left\lceil \sum_{i=1}^{n} \frac{d_i}{Q} \right\rceil$. In Archetti et al. (2009), it is shown that there always exists an optimal solution that uses a number of vehicles not greater than $2\left\lceil \sum_{i=1}^{n} \frac{d_i}{Q} \right\rceil$. Thus, in the following formulation, we fix $m=2\left\lceil \sum_{i=1}^{n} \frac{d_i}{Q} \right\rceil$. Each vehicle starts and ends its route at the depot. The demands of the customers must be fully satisfied, and the quantity delivered in each tour cannot exceed Q. The objective is to minimize the total distance traveled by the vehicles. A mixed integer linear programming (MILP) formulation for the SDVRP has been presented by Archetti et al. (2006a). We use the following notation:

x_{ij}^{v} is a binary variable that takes value 1 if vehicle v travels directly from i to j, and 0 otherwise,
y_{iv} is the quantity of the demand of i delivered by the vehicle v.

The SDVRP can be formulated as follows:

$$\min \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{v=1}^{m} c_{ij} x_{ij}^{v}, \tag{1}$$

subject to

$$\sum_{i=0}^{n} \sum_{v=1}^{m} x_{ij}^{v} \geqslant 1, \quad j = 0, \ldots, n, \tag{2}$$

$$\sum_{i=0}^{n} x_{ip}^{v} - \sum_{j=0}^{n} x_{pj}^{v} = 0, \quad p = 0, \ldots, n;\ v = 1, \ldots, m, \tag{3}$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij}^{v} \leqslant |S| - 1, \quad v = 1, \ldots, m;\ S \subseteq V - \{0\}, \tag{4}$$

$$y_{iv} \leqslant d_i \sum_{j=0}^{n} x_{ij}^{v}, \quad i = 1, \ldots, n;\ v = 1, \ldots, m, \tag{5}$$

$$\sum_{v=1}^{m} y_{iv} = d_i, \quad i = 1, \ldots, n, \tag{6}$$

$$\sum_{i=1}^{n} y_{iv} \leqslant Q, \quad v = 1, \ldots, m, \tag{7}$$

$$x_{ij}^{v} \in \{0, 1\}, \quad i = 0, \ldots, n;\ j = 0, \ldots, n;\ v = 1, \ldots, m, \tag{8}$$

$$y_{iv} \geqslant 0, \quad i = 1, \ldots, n;\ v = 1, \ldots, m. \tag{9}$$

Constraints (2)–(4) are the classical routing constraints. Constraints (2) impose that each vertex is visited at least once, while (3) are the flow conservation constraints and (4) are the subtour elimination constraints. Constraints (5) impose that customer i is served by vehicle v only if v visits i, constraints (6) ensure that the entire demand of each customer is satisfied, while constraints (7) guarantee that the quantity delivered by each vehicle does not exceed the vehicle capacity. It is worth noting that constraints (2) are not necessary to correctly model the problem. They are used to strengthen the formulation and improve the quality of the solution of the linear relaxation.

It has been shown by Archetti et al. (2006a) that there always exists an optimal solution to (1)–(9) where the y variables have an integer value when d_i and Q have integer values.

4. Properties

In this section, we summarize the known properties of the SDVRP.

4.1. Computational complexity

In Archetti et al. (2005), it was shown that the SDVRP with integer demands can be solved in polynomial time for $Q = 2$ while it is NP-hard for $Q > 2$. In Archetti et al. (2010), the computational complexity of both the VRP and the SDVRP on special graphs is studied, namely on a line, a star, a tree and a circle. It turned out that, on these special graphs, the SDVRP is never harder than the VRP.

4.2. Reducibility

In the SDVRP, the customer demands may be greater than the vehicle capacity. There exists a class of instances where we know how to optimally serve the part of demand exceeding the vehicle capacity.

Definition 1. An SDVRP instance is reducible if an optimal solution exists such that each vertex is served by as many direct trips as possible from the depot to the vertex, with a full load in each trip, until the demand of each vertex is lower than the vehicle capacity Q.

Archetti et al. (2005) have shown that the SDVRP with integer demands is reducible when $Q = 2$ and that this result does not extend to the case with $Q > 2$. Even in the case of Euclidean distances, the SDVRP is not reducible for $Q \geqslant 3$.

4.3. The k-split cycles

Dror and Trudeau (1989) have shown an interesting property of optimal solutions. To understand their result, we first need the following definition.

Definition 2. Consider a set $C = \{i_1, i_2, \ldots, i_k\}$ of customers and suppose that there exist k routes $r_1, \ldots, r_k$, $k \geqslant 2$, such that r_w contains customers i_w and i_{w+1}, $w = 1, \ldots, k-1$, and r_k contains customers i_1 and i_k. Such a configuration is called a k-split cycle.

Dror and Trudeau have shown that, if the distances satisfy the triangle inequality, then there always exists an optimal solution to the SDVRP that does not contain k-split cycles, $k \geqslant 2$.

4.4. Number of splits

On the basis of the k-split cycle property, Archetti et al. (2006b) have shown that there always exists an optimal solution to the SDVRP where the number of splits is less than the number of routes, where the number of splits is defined as the sum, over all customers, of the number of visits minus one. Moreover, no pair of routes has more than one customer in common.

4.5. Savings

In Archetti et al. (2006b), it is shown that the value of the optimal solution of the VRP can be as large as twice the value of the optimal solution of the SDVRP when the fleet of vehicles is unlimited. The same result holds if we consider the number of vehicles used, that is, the optimal solution of the SDVRP may use half the number of vehicles used in the VRP optimal solution. Gueguen (1999) showed instead that, when the fleet of vehicles is limited, the ratio between the optimal solution of the VRP and the optimal solution of the SDVRP may go to infinity. Archetti et al. (2008a) carried out an empirical study to see which are the factors that primarily influence the savings gained by allowing split deliveries. They show that the major factor is the value of customer demands with respect to the vehicle capacity, while customer locations do not seem to have a huge influence. The largest savings are obtained in the case where the average customer demand is just above half of the vehicle capacity and the variance of the customer demands is low.

5. Heuristic algorithms

A major research effort has been spent in designing effective heuristic solution approaches.

The first heuristic algorithm for the SDVRP has been proposed by Dror and Trudeau (1989, 1990). It is a local search algorithm where two kinds of moves are introduced. The first move, *k-split interchange*, splits the visit to a customer into different routes having enough residual capacity to serve the demand of the split customer. The second move, *route addition*, considers a split customer, removes it from all the routes where it is visited and creates a new route visiting that customer only. Note that Dror and Trudeau considered only the case with $d_i \leqslant Q$ for each i. Many of the subsequent approaches made the same assumption or were tested on instances satisfying it. In order to evaluate the savings achieved by split deliveries, Dror and Trudeau proposed new instances created on the basis of benchmark instances for the VRP and varying the customer demands. In particular, they generate the demands in the following interval:

$$d_i = \lfloor \alpha Q + \delta(\gamma - \alpha)Q \rfloor,$$

where δ is a random number in $(0, 1)$ while α and γ are parameters defining lower and upper bounds on customer demands. Apart from the original VRP instances, six new classes of instances have been generated with the following values of α and γ: $(0.01, 0.1)$, $(0.1, 0.3)$, $(0.1, 0.5)$, $(0.1, 0.9)$, $(0.3, 0.7)$ and $(0.7, 0.9)$. Dror and Trudeau (1989, 1990) showed that the main savings are achieved for large values of customer demands. The same instances were then used in successive works.

Sixteen years passed before a new heuristic for the SDVRP was proposed and all subsequent approaches were based on metaheuristics or hybrid schemes.

The first heuristic proposed after the local search of Dror and Trudeau is a tabu search algorithm presented in 2006 by Archetti et al. (2006a). It is based on a simple scheme where the neighborhood is constructed in the following way. Choose a customer i and a route r not visiting i. Then, insert i in r delivering a quantity $q = \min\{d_i, q_r\}$, where q_r is the residual capacity of route r. Then, remove a quantity q from the route/routes visiting i: if customer i is split, then order the routes visiting i on the basis of the savings gained when removing i and apply the removal on the basis of the ordered list. This first tabu search algorithm was proved to be much more effective than Dror and Trudeau's algorithm. It is also obviously much slower but even the fast version, run for 1 min only, proved to beat the previous local search algorithm.

A memetic algorithm with population management was proposed in 2007 by Boudia et al. (2007) consisting in a genetic algorithm, combined with a local search procedure for intensification, and using a distance measure to control population diversity. In the genetic algorithm, a standard crossover operator is implemented while the local search procedure tries to improve solutions by means of three moves that allow split deliveries. The first move is derived from the *k-split interchange* proposed by Dror and Trudeau, where a new procedure is inserted in order to find the best splitting of customer i. The two remaining moves swap pairs or triplets of customers where split deliveries are allowed. The algorithm was tested on the same set of instances used in Archetti et al. (2006a) (which is the set of instances proposed by Dror and Trudeau) and proved to beat the tabu search in many cases.

In the last few years, the use of hybrid heuristic algorithms to solve NP-hard problems has registered a remarkable increase. A hybrid heuristic algorithm is a solution procedure that combines different algorithmic ideas. In particular, several heuristics have been proposed that solve to optimality MILP models in the framework of a heuristic scheme. The reasons for the success of these approaches can be summarized in two major points:

- while, on the one hand, many NP-hard problems are too complex to be solved to optimality, on the other, solving to optimality subproblems, aggregated problems or other simpler optimization problems may be a reasonable task. The exact solution of simpler optimization problems can intensify the search on small portions of the solution space, where it is more likely that good solutions exist;
- commercial softwares for the solution of MILP problems are available and are extremely powerful nowadays.

Three hybrid algorithms have been proposed for the solution of the SDVRP. The first one was proposed in 2007 by Chen et al. (2007). An initial solution is built through the standard Clarke and Wright saving algorithm for the VRP. Then, a MILP model, called endpoint mixed integer program (EMIP), is applied on the current solution to optimally reallocate the endpoints of each route, i.e., the first and the last vertex visited by each route (excluding the depot). The reallocation of these vertices consists in moving them to other routes or splitting their demand into different routes (they will always be inserted into the routes as endpoints). The reason for considering only the endpoints of each route is that splitting the demand of vertices that are far from the depot would probably cause a relevant increase in the total distance traveled, while a split delivery is more beneficial for vertices that are close to the depot. The solution obtained with EMIP is then improved through a variable length record-to-record travel algorithm (VRTR). The algorithm was compared with the one proposed by Archetti et al. (2006a) on the same set of instances and was proved to be able to improve the previous results. Moreover, a new class of 21 instances was introduced with a size ranging from eight customers to 288 customers. Vehicle capacity is 100 units while customer demand is either 60 units or 90 units. Each instance has a geometric symmetry (star shape) with customers located in concentric circles around the depot. This allowed the authors to visually estimate a near-optimal solution to compare the solution of the EMIP+VRTR approach with. The gap between the visually estimated solution and the

EMIP+VRTR solution is always below 3% even if the CPU time tends to become quite high for large instances.

A second hybrid algorithm was proposed in 2008 by Archetti et al. (2008b). Here, the approach is completely different from the one presented in Chen et al. (2007). In particular, the SDVRP is first solved by applying the tabu search algorithm of Archetti et al. (2006a). The idea is that useful information can be collected by analyzing the solutions visited by the tabu search. If an edge is often traversed in these solutions, then it is reasonable to conclude that there is a high probability that this edge is included in high-quality solutions, and vice versa. Another information collected concerns customer splits: if a customer is never or rarely split in tabu search solutions, then it will probably not be split in high-quality solutions. This information is collected and then, once the tabu search ends, it is used to reduce the original graph by discarding those edges that are never or rarely traversed by tabu search solutions. Then, on the basis of this reduced graph, all possible routes are constructed for which the total demand of the customers visited is not greater than $Q+\delta/2$, where δ is the average customer demand. Note that this set of routes is much smaller than the set of all feasible routes for the original problem for two reasons:

- they are constructed on a reduced graph;
- only routes for which the sum of the demands of the customers visited is not greater than $Q+\delta/2$ are considered.

Obviously, the size of the set of generated routes depends on the number of discarded edges. Preliminary tests were performed in order to tune the parameters in such a way that a set of routes of reasonable size was generated (large enough to contain good solutions and not too large to be handled by an optimization model). A route-based formulation was presented that aims at identifying the best routes among the set of generated routes. The formulation makes use of the information collected by the tabu search about customer splits. In particular, when a customer is never or rarely split in the best solutions found by the tabu search, then a constraint imposing a single visit to this customer is introduced in the route-based formulation. However, because the set of generated routes is too large to be handled by an MILP model, first, the linear relaxation is solved. Then the routes are sorted on the basis of different "desirability" criteria based on the reduced costs. Finally, the MILP model is repeatedly solved on small subsets of routes, starting from the more "desirable" ones. The approach is tested on the same set of instances presented in Archetti et al. (2006a) and compared with their tabu search algorithm. The results improve the ones produced by the tabu search in many cases.

Finally, a third hybrid algorithm was proposed by Jin et al. (2008) and is based on a column generation approach similar to the one proposed by Sierksma and Tijssen (1998). In particular, the master problem consists in a set covering route-based formulation. The pricing problem is similar to a TSP with profits and is solved through a limited-search-with-bounds algorithm, where each route is constructed on the basis of the observation that, in an optimal solution of the pricing problem, at most one customer will be split. The pricing problem is iteratively solved until no negative reduced cost columns are found, thus obtaining a lower bound for the SDVRP. Afterwards, the set of columns created is used to obtain a feasible solution for the SDVRP. First, the column (route) with the highest value in the linear relaxation of the master problem and visiting the highest number of customers is chosen and the corresponding variable is fixed to one. At this point, a reduced problem is obtained: the customers who are completely served by the fixed column can be removed while, if there is a split customer, the corresponding demand is reduced by the amount delivered. Then, the procedure is iterated and the column generation algorithm is applied on the reduced problem. The approach is tested on the set of 12 instances proposed by Belenguer et al. (2000) and compared with the results obtained by the cutting plane algorithm presented in this paper. They were able to improve the lower bound on 10 instances and the upper bound on six instances.

We observed in the introductory part of this survey that, contrary to the VRP, feasible solutions to the SDVRP always exist when the number of vehicles available is limited to the minimum possible number $\left\lceil \frac{\sum_i d_i}{Q} \right\rceil$. A few papers have studied this particular case of the SDVRP. This case indirectly considers the cost of the vehicles and assumes that the cost of a vehicle dominates the routing costs. Thus, the size of the fleet is fixed to the minimum possible value.

Mota et al. (2007) proposed for the SDVRP with minimum possible number of vehicles a scatter search heuristic where classical procedures for VRP and TSP are adapted to allow split deliveries.

More recently, Aleman (2009), Aleman et al. (2010) and Aleman and Hill (2010) proposed new metaheuristic algorithms. In Aleman et al.(2010), an adaptive memory algorithm is proposed, where again, the use of the minimum number of vehicles is imposed. An initial solution is generated through a constructive approach and then a variable neighborhood descent (VND) scheme is applied to improve the initial solution. The approach was tested on the set of instances proposed by Archetti et al. (2006a), Belenguer et al. (2000), Chen et al. (2007) and Jin et al. (2007). The results show that this approach is comparable to the ones proposed by Mota et al. (2007) and Boudia et al. (2007). Afterwards, Aleman and Hill (2010) proposed a tabu search with vocabulary building approach, which is a population-based approach that constructs an initial set of solutions and then uses this set to find attractive solution attributes to construct new solutions. As the search progresses, the solution set evolves. Good solutions are added to the set while bad solutions are removed. To construct the initial set of solutions, they used a variant of the construction approach proposed in Aleman et al. (2010), while the search is guided through a variant of the VND algorithm presented in Aleman et al. (2010). This approach was able to remarkably improve the results of the previous adaptive memory algorithm. Aleman (2009) also proposed a ring-based diversification scheme that was beaten by the approach proposed in Aleman and Hill (2010).

Finally, Derigs et al. (2009) proposed a local search-based metaheuristic, where they adapted four moves for the standard VRP to the case with split deliveries. Different metaheuristic schemes, embedding the four previous moves, have been implemented: simulated annealing, threshold accepting, record-to-record travel, attribute-based hill climber (ABHC) and attribute-based local beam search. The best scheme proved to be the ABHC. This scheme proved to compare favorably with the approaches proposed in Archetti et al. (2008b) and in Chen et al. (2007).

6. Exact algorithms

The first exact approach was proposed in 1994 by Dror et al. (1994). In this paper, an arc flow formulation was proposed together with new classes of valid inequalities. Results were provided in order to show the strength of the inequalities proposed on instances with up to 20 customers. The inequalities were able to remarkably improve the lower bound, reducing the gap with respect to the upper bound of more than 30% in most cases. The upper bound was calculated using the Dror and Trudeau heuristic.

A cutting plane approach was then proposed in 2000 by Belenguer et al. (2000), where a polyhedral study of the problem is presented. Some facet-inducing inequalities and other valid inequalities are introduced and embedded in a cutting plane algorithm. The tests were conducted on 11 instances from the TSPLIB with a number of vertices ranging from 22 to 101, and 14 randomly generated instances with a size ranging from 51 to 101. They were able to solve five instances of the TSPLIB, the largest one having 51 nodes.

The interest in exact algorithms for the SDVRP has increased in the last few years. In 2006, Lee et al. (2006) studied the case where the minimum number of vehicles is used and proposed a shortest path approach. The SDVRP is formulated as a dynamic programming model where the routes are constructed sequentially through a labeling algorithm. The solution space and the corresponding states are reduced thanks to the k-split cycle property. Nevertheless, their number remains huge. They tested their approach on instances with up to seven customers. In 2007, Jin et al. (2007) proposed an exact approach for the case with minimum number of vehicles. Their approach is a two-stage algorithm, where the first stage corresponds to a clustering problem where an MILP model is solved to partition customers into clusters. The objective function aims at minimizing the clustering cost, where the cost of each cluster is initially set to zero and then updated through the values obtained in the second stage. In the second stage, a TSP is solved on each cluster in order to find the optimal route. The cost of each cluster obtained in the second stage is used to update the objective function in the first stage and the procedure is iterated. Valid inequalities are added to strengthen the MILP model. The approach was able to solve instances with up to 22 vertices but with a great computational effort. The largest instance solved is eil22, a VRP benchmark instance where demands have very large values.

Recently, the efforts for solving SDVRP to optimality have converged to the use of branch-and-price methods, the leading methodology for solving VRPs to optimality. Archetti et al. (2009) implemented a branch-and-price-and-cut algorithm, where each column generated by the subproblem represents a route together with the quantities delivered to each customer visited by the route. The master problem obtained finds the optimal combination of the routes identified by the subproblem. The subproblem is solved through a labeling algorithm. A set of valid inequalities is introduced to improve the quality of the lower bound. Moreover, a heuristic solution is obtained by solving an MILP model on the set of columns generated and choosing the best ones. Tests are performed on the instances proposed by Archetti et al. (2006a), Belenguer et al. (2000) and Chen et al. (2007). They considered both cases with an unlimited number of vehicles and with a number of vehicles equal to the minimum needed. In both cases, they were able to solve to optimality seven instances, one of which with 144 customers. They improved in most cases previous lower bounds and they also found some new best feasible solutions. A similar formulation is proposed in Baldacci et al. (2009) with a different bounding procedure and a slightly different approach for the pricing. Moreno et al. (2010) proposed a lower bounding procedure based on column and cut generation. The formulation of the master problem is different from the one presented in Archetti et al. (2009) and takes into account the unit of flows traversing each edge and is strengthened with the introduction of different valid inequalities. The pricing problem is solved with a dynamic programming algorithm and two heuristics are used in order to speed up its solution: the scaling and the sparsification heuristic. Specific valid inequalities for the SDVRP are introduced as well as general fractional cuts. The tests were conducted on the instances proposed by Belenguer et al. (2000), Chen et al. (2007) and on TSPLIB instances. The results show that in most cases, they were able to improve the previous lower bounds presented in Belenguer et al. (2000) and in Jin et al. (2008). In Moreno (2008), the lower bounding procedure proposed in Moreno et al. (2010) is extended to a branch-and-price algorithm that was able to solve three instances with 50 customers and one instance with 75 customers.

The benchmark instances proposed in Archetti et al. (2006a), Belenguer et al. (2000) and Chen et al. (2007) can be found at http://www-c.eco.unibs.it/ ~ archetti/sdvrp.zip.

7. Variants and applications

In this section, we review the several variants of the SDVRP that appeared in the literature. We organize this section structuring the contributions according to their major characteristic. Some of these characteristics, such as the presence of time windows and of pick-up and delivery operations, are well studied in the traditional VRPs. However, few contributions are available for the case when split deliveries are allowed. Whereas several of these variants are inspired by practical problems, some are strongly related to applications and present a number of specific features. To these variants, we dedicate a specific part of this section.

7.1. Time windows

One of the most interesting variants of the SDVRP is the case with time windows. This means that each customer can only be visited within a specified time interval. The presence of time windows associated with the customers is a traditional characteristic of the classical VRP and is clearly of interest also when split deliveries are allowed.

A first exact approach to solve the SDVRP with time windows is due to Gendreau et al. (2006). In this paper, a Dantzig–Wolfe decomposition of the problem is proposed and the problem is solved with a branch-and-price algorithm, where the master problem is solved by means of column generation. The construction of the routes is carried out in the subproblem while the choice of the routes and the associated quantities to be delivered to the customers is a task of the master problem. Instances with up to 50 customers are solved to optimality.

A different Dantzig–Wolfe decomposition is presented by Desaulniers (2010), who introduced the concept of extreme delivery patterns, which correspond to routes where all customers receive either no delivery or complete delivery, with the only exception of one customer at most. Extreme delivery patterns are obtained by means of the solution of the subproblem, while a complete solution that identifies and combines extreme delivery patterns is found in the master problem. The results are compared with the ones of Gendreau et al. (2006) showing that the new approach is much more efficient. A large number of instances with 50 customers is solved and one instance with 100 customers within 1 h of CPU time. An enhanced branch-and-price-and-cut method for the SDVRPTW is presented in Archetti et al. (2011), where enhancement procedures are applied to the algorithm proposed by Desaulniers (2010). In particular, a tabu search algorithm is implemented to solve the subproblem and new valid inequalities are applied, together with a new heuristic algorithm to find capacity cuts. The results show the effectiveness of these new procedures, as, within 1 h, they were able to solve to optimality 86 additional instances, including seven instances with 100 customers.

The only heuristic approach for the solution of the SDVRP with time windows can be found in Ho and Haugland (2004), where a solution method based on tabu search is proposed. Experimental results are shown on problem instances with up to 100 customers.

7.2. Pick-up and delivery

In routing problems, often pick-up and delivery operations are combined in the same route. While there is a large amount of literature on the classical pick-up and delivery VRP, the first paper on the VRP with split deliveries and pick-ups appeared in Mitra (2005). A homogeneous fleet of vehicles has to serve the delivery and pick-up operations of a set of customers. The problem is a single-commodity many-to-many pick-up and delivery problem, that is, items collected from a customer with a pick-up demand can be delivered to any customer with a delivery demand. Each vehicle leaves and returns to the depot only once. Each customer may have both delivery and pick-up demands. Any demand may exceed the vehicle capacity. Split deliveries and pick-ups are allowed, which implies that each customer may be visited by more than one vehicle and more than once by the same vehicle. A customer may prefer a single visit. There are no time windows and no restrictions on the maximum route length. The objective of the problem is to determine the minimum number of vehicles required to satisfy the delivery and pick-up demands and identify the routes so as to minimize the total route cost. In this paper, an MILP formulation is presented and a heuristic is proposed. The heuristic first determines the minimum required number of vehicles and then builds the routes on the basis of the cheapest insertion criterion. The same problem is also studied in Mitra (2008), where an alternative formulation is proposed together with a parallel clustering technique. The performance of the method is tested on the same instances tested in Mitra (2005).

The benefit of using split loads in the pick-up and delivery problem is quantified in Nowak et al. (2008). The problem analyzed is a single-commodity one-to-one pick-up and delivery problem, that is, any pick-up location is paired with a delivery location where the load has to be transported. A heuristic to solve the pick-up and delivery problem with split loads is described and a set of random large-scale problem instances are solved. The paper shows that, for a given set of origins and destinations, the largest benefit occurs when the size of the load is just above one-half of the vehicle capacity. In Nowak et al. (2009), an empirical analysis is developed that makes use of the previous heuristic. The goal is to evaluate how the benefit deriving from split loads is influenced by the mean load size and variance, by the number of origins and by the geographical distribution of origins and destinations.

A slight variant of the SDVRP with pick-up and delivery is analyzed in Thangiah et al. (2007), where also time windows are considered. As in Nowak et al. (2008), the problem is a single-commodity one-to-one pick-up and delivery problem. A heuristic algorithm is proposed and applied on both static and real-time data sets.

7.3. Profit maximization

The most common objective function in routing problems is the minimization of the length or the cost of the routes. Sometimes, the objective function includes the cost of the vehicles. The maximization of profit as objective appears in the class of routing problems with profits (see Feillet et al., 2005), where customers have to be selected among a set of potential customers on the basis of their profit. When split deliveries are allowed, the profit becomes an interesting measure of quality because splitting demand usually increases the overall profit.

In Korsvik et al. (2010), a ship routing and scheduling problem has to be solved on a daily basis. The mandatory contract cargoes are specified in long-term agreements between the shipping company and the cargo owners. Moreover, spot cargoes can be selected. There are pre-specified time windows during which the loading of the cargoes must start, and there may also exist time windows for unloading. The problem is similar to the pick-up and delivery problem with time windows. A major difference is that here the option of splitting loads is introduced. This means that each cargo can be transported by several ships. However, the total quantity transported must still be equal to a specified value. Moreover, the time required to load or unload a cargo is dependent on the quantity. The problem consists in selecting spot cargoes and determining routes and schedules in such a way that the profit is maximized. A large neighborhood search heuristic is proposed.

A similar problem that arises in the tramp shipping industry is described in Brønmo et al. (2007). This is a short-term scheduling problem that concerns pick-ups and deliveries of bulk cargoes. A cargo is either derived from a long-term contract or acquired in the spot market. Normally, there is a mix of mandatory contract cargoes and optional spot cargoes. Time windows for loading and unloading the cargoes are identified. A tramp shipping company aims at maximizing the profit from optional cargoes. For long-term contracts, the sizes of the cargoes are flexible. This means in particular that the cargo quantities may be given in an interval. The paper introduces flexible cargoes also in the short-term scheduling. A mathematical programming model is presented together with a set partitioning approach to solve it. It is shown that the increased flexibility increases profit.

7.4. Inventory and production

Inventory routing problems have received considerable attention in the last decades (see Bertazzi et al., 2008 for a recent survey). In these problems, time is an important component because inventory is built upon time and constraints on the inventory capacity are to be satisfied and/or the inventory cost has to be accounted for in the objective function. In these problems, the quantities to deliver to customers have to be decided. A customer is usually visited several times on the time horizon but only once in the same day. In some cases, inventory routing models also incorporate decisions concerning production.

In Chandra and Fisher (1994), an integrated multi-period multi-product production and distribution system model based on a multi-stop routing problem formulation with additional setup constraints and split delivery relaxation is presented. A computational study is presented to show the value of coordinating production and distribution by comparing a two-phase solution procedure, where the multi-item lot-sizing subproblem is first solved, and the distribution plan is then heuristically found, with an integrated solution approach that modifies the production plan on the basis of the solution of the distribution plan. The reduction in the total operating cost achieved with coordination ranges from 3% to 20%. Fumero and Vercellis (1999) presented a multi-period integrated model for a single plant logistical system, in which multiple items are manufactured and delivered to customers. Production, inventory and routing decisions are simultaneously considered and split deliveries are allowed. The model is solved by Lagrangian relaxation. A comparison with a decoupled solution approach, which is an approach where production and distribution phases are solved independently, is presented.

In Yu et al. (2006, 2008), an inventory routing problem with split delivery and a constraint on the fleet size is studied. Because of the complexity of the problem, different heuristics are proposed in the two papers, based on an approximate model of the problem and Lagrangian relaxation.

A different problem is considered in Bolduc et al. (2010), the so-called SDVRP with production and demand calendars. This problem consists in determining which customers to serve by a common carrier and the delivery routes for the customers served by a private fleet, in order to minimize the overall transportation and inventory costs. Split deliveries are allowed. A tabu search heuristic is proposed.

7.5. *Minimum fraction served*

Split deliveries may be complicated to handle by a company and companies may in some cases accept it only if the quantity involved in the delivery is large enough. This is the motivation of the model studied in Golden et al. (2010), where split deliveries are allowed only if a minimum fraction of a customer demand is served by a vehicle. A heuristic method is proposed.

7.6. *Heterogeneous fleet*

The VRP with split deliveries and a heterogenous fleet is considered in Tavakkoli-Moghaddam et al. (2007). The objective is to minimize the fleet cost and the total distance traveled. The fleet cost is dependent on the number of used vehicles and the unused capacity. An MILP model is proposed and solved by means of a simulated annealing method.

7.7. *Stochastic*

In Bouzaiene-Ayari et al. (1993), the SDVRP with stochastic demands is considered. The authors propose a heuristic algorithm that is an adaptation of a previous algorithm proposed by Dror and Trudeau (1986) for the stochastic VRP. The algorithm was tested on instances with up to 130 customers.

7.8. Discrete demands

In Nagao and Nagamochi (2007), the SDVRP with discrete demands is studied. In this problem, each customer requires the supply of different items. The demand of each item can be greater than one and, while a customer can be visited more than once, each item has to be supplied by exactly one vehicle. A dynamic programming algorithm is proposed and tested on instances derived from real data.

7.9. Arc routing

Only a few papers appeared in the literature that consider arc routing problem with split deliveries.

In Mullaseril et al. (1997), a feed distribution problems is studied. The problem can be seen as a collection of split delivery capacitated rural postman problem with time windows on arcs. Several heuristics are presented and compared with the working practices on a cattle ranch.

The capacitated arc routing problem (CARP) has various applications like mail delivery, waste collection and street maintenance. The problem is usually defined on an undirected graph. Some edges with known demands must be served by vehicles. The goal is to determine a set of routes that visit all the required edges, where each edge must be served by one vehicle only. This latter constraint is relaxed in the split delivery CARP studied in Labadi et al. (2008), where each edge may be served by different vehicles. An insertion heuristic and a memetic algorithm are proposed. Lower and upper bounds for the split delivery CARP are proposed in Belenguer et al. (2010).

7.10. Applications

In Frizzell and Giffin (1992) and Frizzell and Giffin (1995), a vehicle routing and scheduling problem is studied with numerous features added in an attempt to model a real problem. These features include the following: split deliveries are allowed; time windows are associated with customers; customer and depot locations are given on a grid network; the time to make a delivery is dependent on delivery size; bounds on the admissibility of any split deliveries are defined. Different heuristics are proposed.

The objective of the problem studied in Song et al. (2002) is to find an optimal allocation of newspaper agents and optimal routes and schedules to minimize the delivery cost while keeping the total delay time of delivery as small as possible. A practical approach is used to solve the problem that uses heuristic algorithms and a digital map. A split delivery scheme is also selected. The quantity of demanded newspapers is divided in smaller quantities and delivered at several times.

In Ambrosino and Sciomachen (2007), a food distribution problem is analyzed that can be seen as a generalization of the asymmetric capacitated VRP with split deliveries. A real application related to an Italian company that holds food markets along the national highway network is considered. The goal is to minimize the total cost, which includes fixed costs of the vehicles and the traveling costs, while satisfying operative and customer requirement constraints. The case of two products is studied. An MILP model and a two-step heuristic procedure are presented.

A real-life heterogeneous fleet VRP with time windows and split deliveries that occurs in a major Brazilian retail group is presented in Belfiore and Yoshizaki (2009). Heuristics are proposed to build initial solutions and then a scatter search approach is presented. The obtained solutions are compared with the solution actually implemented by the company.

In Sherali et al. (1999), models and algorithms for routing and scheduling ships in a maritime transportation system are presented. The paper is focused on a problem of the Kuwait Petroleum Corporation. The total demand for a product for each customer is partitioned into smaller quantities, and feasible delivery time windows are specified for each such quantity. The partitions are specified by the minimum and maximum quantities of the product that can be shipped to a destination within a continuous time interval. An MILP model is presented. An alternate aggregate model that retains the main features of the problem is formulated. A specialized rolling horizon heuristic is developed to solve it that improves the results obtained using the current scheduling practice.

In Sierksma and Tijssen (1998), a problem of determining a flight schedule for helicopters to off-shore platform locations for exchanging crew is considered. The demand of each off-shore location can be split among different helicopters. The problem is solved by means of a column generation-based heuristic. Computational results on real data are shown. A similar problem is considered in Moreno (2008) and solved again through a column generation-based heuristic.

McKay and Hartley (1974) considered the problem of scheduling seagoing tankers where different products have to be transported from their origin port to the destination port. Split deliveries are allowed. A mathematical formulation is proposed based on simplifying assumptions that make the problem manageable.

In Yi and Ozdamar (2007), the problem of coordinating disaster response activities is studied. In particular, they considered the transportation of commodities from major supply centers to distribution centers in affected areas and the transport of wounded people from affected areas to temporary and permanent emergency units. They proposed a flow formulation where wounded people and vehicles are treated as integer value commodities. Split deliveries for the transportation of goods to distribution centers are allowed. They proposed a two-stage algorithm and tested it on an earthquake scenario as well as on larger size hypothetical disaster scenarios. A similar problem is studied in Shen et al. (2009), where the problem of routing vehicles to service a large-scale bio-terrorism emergency is considered. The problem is decomposed into two stages. The first stage is a planning stage, where the routes are generated in advance with respect to the emergency. The second stage is an operational stage, where the planned routes and the information revealed at the time of the emergency are taken into account to decide the delivery quantity and any adjustment to the routes. The objective is to minimize the unmet demand of medication or antidotes. Split deliveries are allowed. Mathematical formulations and solution approaches are proposed for both stages.

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