Plane Polar Coordinats. (2 dimensional)

Plane Polar Coordinats that is (r, θ) Coordinate is useful when a particle is moving in a curved path & not in a straight line. Here we have two unit vectors $\hat{r}$ and $\hat{\theta}$ which are changing with positions & their time derivative is not zero. That is how they differ from Cartesian coord whose Unit vectors are $\hat{i}$ & $\hat{j}$ – they remain fixed & their time derivative is zero. In 3D we have Spherical Polar Coord & Cylindrical Polar Coord.

- Nots are for internal circulation only
- Two major book I consulted are Kleppner & Meriam. Some other materials are also consulted.
- If you find any mistake, or if you get Some good interesting example please let me know.

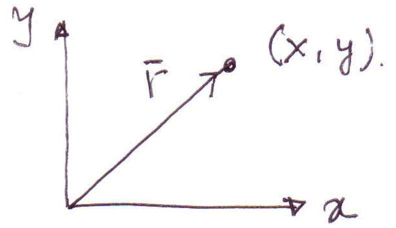
Amit.

(AMIT NEOGI)

14 Aug. 2016.

Motion in Cartesian Coordinate / Rectangular Coord.

Let us represent a vector in rectangular coord / Cartesian Coordinate.



$$\bar{r} = x\hat{i} + y\hat{j}.$$

Here a pt is represented as (x, y) and $\hat{i}$ and $\hat{j}$ are basis vectors / unit vectors along x and y axis.

Now if r is changing with time $r(t)$

Then Velocity

$$v = \frac{dr(t)}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt}$$

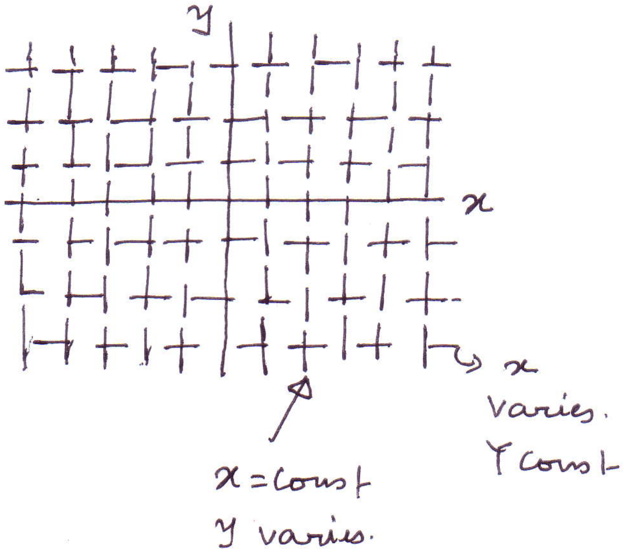
$$\frac{d\hat{i}}{dt} = 0 \qquad \frac{d\hat{j}}{dt} = 0.$$ Because they are fixed.

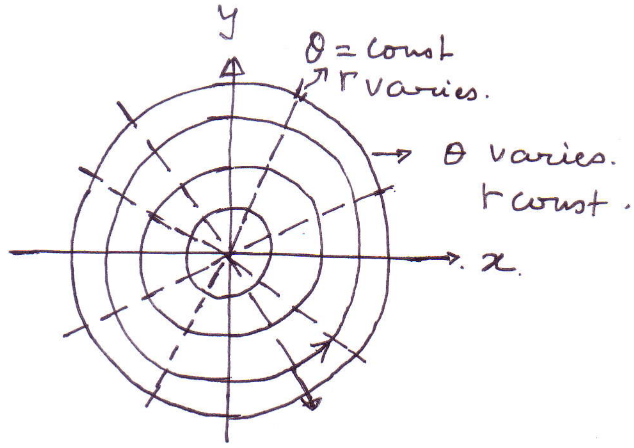
So $$\bar{v} = \dot{x}\hat{i} + \dot{y}\hat{j}.$$

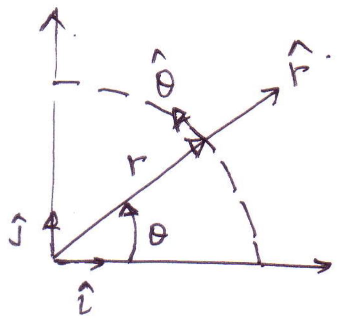
$$a = \frac{dv}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j}.$$

- Rectangular, or Cartesian Coordinates are well suited for describing motion in a straight line.
- If we orient the coordinate system so that one axis lies in the direction of motion, then only single coordinate changes as the point moves.
- Rectangular Coord are not suited for Circular motion

Plane Polar Coordinates.





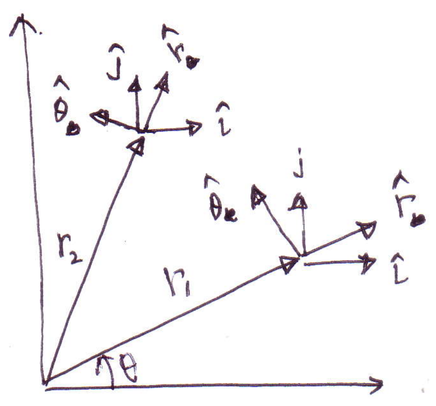


In the previous section vectors $\hat{i}$ and $\hat{j}$ are base vectors which point in the direction of increasing x and increasing y respectively.

Similarly two new unit vectors $\hat{r}$ and $\hat{\theta}$ which point in the direction of increasing r and increasing θ

~~and a line joining from origin~~

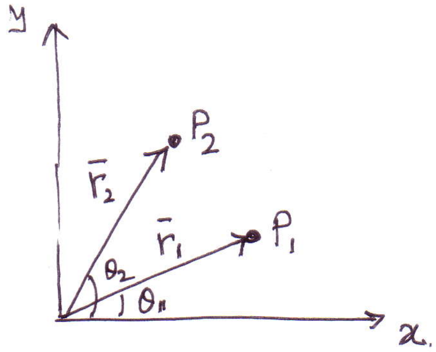
1. V. V. Imp. Diff. $\hat{i}$, $\hat{j}$ are fixed where as $\hat{r}$ and $\hat{\theta}$ are changing with position. For two positions :—



Although $\hat{r}$ and $\hat{\theta}$ vary with position, note that they depend on θ only, not on r.

One can think of $\hat{r}$ and $\hat{\theta}$ as being functionally dependent on θ.

2. Position Vectors. – if there are two of identical magnitude.



$$\bar{r}_1 = r\hat{r}$$

$$\bar{r}_2 = r\hat{r}$$

So two vectors – position vectors $\bar{r}_1$ and $\bar{r}_2$ are given the same expression.

Is it correct? How?

It is correct because $\hat{r}$ is changing with position. Actually here $\hat{r}$ is a function of θ and the right way will be $\hat{r}(\theta)$. Here θ is does not occur explicitly but it is implicit. So the position vector is actually not a function of r only but also of $\theta \Rightarrow$ Two Variables.

So

Position Vector in Cartesian Coordinate is

$$\bar{r} = x\hat{i} + y\hat{j}$$

In polar coordinate

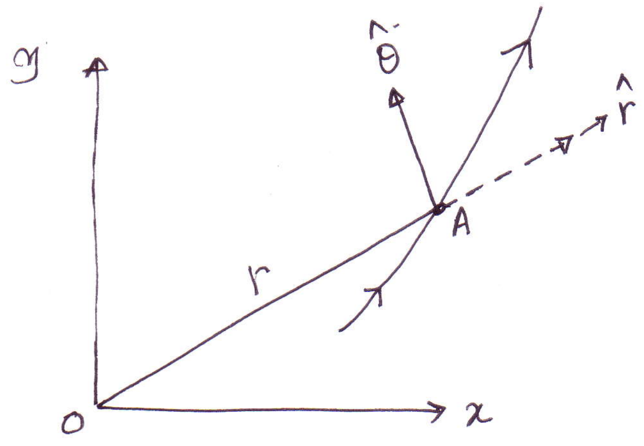
$$\bar{r} = r\hat{r}$$

How they are Related.

$$x\hat{i} + y\hat{j} = r(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

Equating. $x = r\cos\theta$ $\quad$ $y = r\sin\theta$.

Explaining polar coordinates in terms of a trajectory.

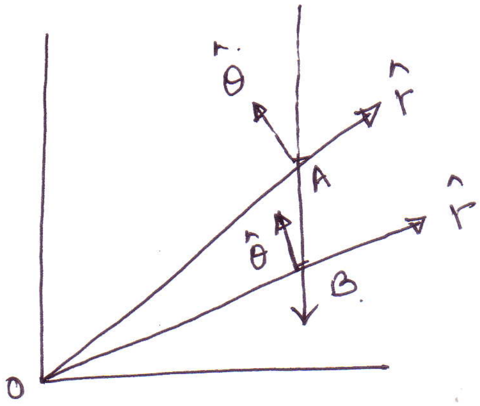


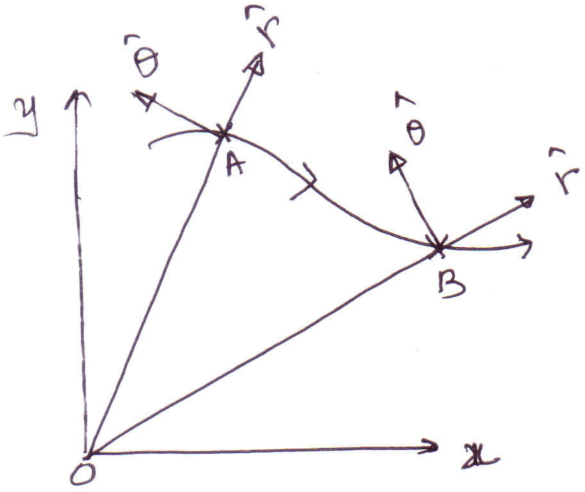
Suppose a particle is travelling through a path and pt A is somewhere in the path.

The position vector of pt A ⇒ Join pt A from the pole origin - Along the direction of r, it is $\hat{r}$ and perpendicular to it is $\hat{\theta}$.

⇒ Suppose I want to calculate now the velocity at A. Again resolve the velocity of pt A into 2 components One along $\hat{r}$ and other along $\hat{\theta}$.

⇒ acc - Again join from the origin & find two components along $\hat{r}$ & $\hat{\theta}$.





Whatever is the point, connect that point from origin. — along the line $\hat{r}$, ⊥ to line $\hat{\theta}$.

Velocity

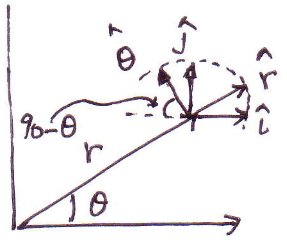
In Cartesian Coord. it is very easy because $\hat{i}, \hat{j}$ are const and their derivative is 0.

$$\bar{r} = r\,\hat{r}$$

$$\frac{d\bar{r}}{dt} = r \underline{\underline{\frac{d\hat{r}}{dt}}} + \hat{r}\frac{dr}{dt}$$

Finding $\frac{d\hat{r}}{dt}$ and $\frac{d\hat{\theta}}{dt}$

Expressing them in terms of Cartesian Coord.



$$\hat{r} = \hat{i}\cos\theta + \hat{j}\,\cancel{\cos\theta}\,\sin\theta$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\frac{d\hat{r}}{dt} = \hat{i}(-\sin\theta)\frac{d\theta}{dt} + \hat{j}\cos\theta\frac{d\theta}{dt}$$

$$= \left(-i\sin\theta + j\cos\theta\right)\frac{d\theta}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\hat{i}\cos\theta\frac{d\theta}{dt} + \hat{j}(-\sin\theta)\frac{d\theta}{dt}$$

$$= -\left[i\cos\theta + j\sin\theta\right] d\theta/dt = -\dot{\theta}\,\hat{r}$$

$$\boxed{\begin{aligned}\frac{d\hat{r}}{dt} &= \dot{\theta}\,\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\dot{\theta}\,\hat{r}\end{aligned}}$$

$$\frac{d\bar{r}}{dt} = r\dot{\theta}\hat{\theta} + \hat{r}\frac{dr}{dt}$$

$$\boxed{\bar{v} = \frac{d\bar{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\Rightarrow \frac{d\hat{r}}{d\theta} = ? = \frac{d\hat{r}}{dt}\cdot\frac{dt}{d\theta} = \cancel{\dot{\theta}}\,\hat{\theta}\cdot\frac{1}{\cancel{\dot{\theta}}} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = \frac{d\hat{\theta}}{dt}\cdot\frac{dt}{d\theta} = -\cancel{\dot{\theta}}\,\hat{r}\cdot\frac{1}{\cancel{\dot{\theta}}} = -\hat{r}.$$

Acceleration.

$$a = \frac{d\bar{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right).$$

$$= \ddot{r}\,\hat{r} + \frac{d\hat{r}}{dt}\dot{r} + \frac{d}{dt}(r\dot{\theta})\,\hat{\theta} + (r\dot{\theta})\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\,\hat{r} + \dot{\theta}\,\hat{\theta}\,\dot{r} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + (r\dot{\theta})(-\dot{\theta}\hat{r}).$$

$$\boxed{a = \hat{r}\left(\ddot{r} - r\dot{\theta}^2\right) + \hat{\theta}\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right).}$$

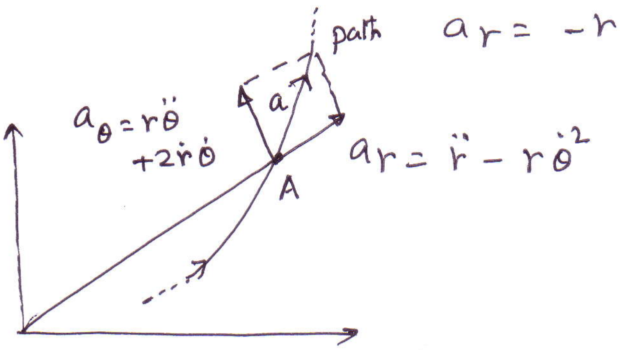
The terms.

$\hat{r}$
- $\ddot{r} \rightarrow$ radially outward.
- $-r\dot{\theta}^2 \rightarrow$ radially inward - Centripetal acceleration

$\hat{\theta}$
- $r\ddot{\theta} \rightarrow$ Tangential acceleration $r\alpha$.
- $2\dot{r}\dot{\theta} \rightarrow$ Consist of 2 terms. The first effect Coms Coms from that portion of Change in magnitude $d(r\dot{\theta})$ of V_θ due to change in r and the Second effect Comes from the change in direction of V_r.

for Circular path $\quad V_r = 0 \quad V_\theta = r\dot{\theta}$

$a_r = -r\dot{\theta}^2 \quad a_\theta = r\ddot{\theta}$



Calculate the two Comp Then square, add & find the sqrt

$$a = \sqrt{a_\theta^2 + a_r^2}$$

Centrifugal and Coriolis terms

See also: Mechanics of planar particle motion and Centrifugal force (rotating reference frame)

The term $r\dot{\theta}^2$ is sometimes referred to as the *centrifugal term*, and the term $2\dot{r}\dot{\theta}$ as the *Coriolis term*. For example, see Shankar.[20] Although these equations bear some resemblance in form to the centrifugal and Coriolis effects found in rotating reference frames, nonetheless there is not a necessary physical connection.[21] For example, the physical centrifugal and Coriolis forces appear only in non-inertial frames of reference. In contrast, these terms that appear when acceleration is expressed in polar coordinates are a mathematical consequence of differentiation; these terms appear wherever polar coordinates are used. In particular, these terms appear even when polar coordinates are used in inertial frames of reference, where the physical centrifugal and Coriolis forces never appear.

Co-rotating frame

For a particle in planar motion, one approach to attaching physical significance to these terms is based on the concept of an instantaneous *co-rotating frame of reference*.[22] To define a co-rotating frame, first an origin is selected from which the distance $r(t)$ to the particle is defined. An axis of rotation is set up that is perpendicular to the plane of motion of the particle, and passing through this origin. Then, at the selected moment t, the rate of rotation of the co-rotating frame Ω is made to match the rate of rotation of the particle about this axis, $d\theta/dt$. Next, the terms in the acceleration in the inertial frame are related to those in the co-rotating frame. Let the location of the particle in the inertial frame be $(r(t), \theta(t))$, and in the co-rotating frame be $(r(t), \theta'(t))$. *Because the co-rotating frame rotates at the same rate as the particle,* $d\theta'/dt = 0$. *The fictitious centrifugal force in the co-rotating frame is* $mr\Omega^2$, radially outward. The velocity of the particle in the co-rotating frame also is radially outward, because $d\theta'/dt = 0$, and has a value $-2m(dr/dt)\Omega$, pointed in the direction of θ. Thus, using these forces in Newton's second law we find:

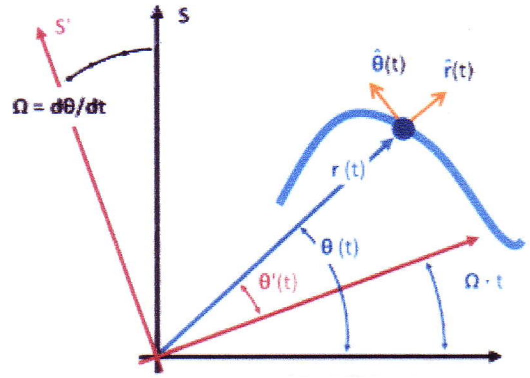
$$\boldsymbol{F} + \boldsymbol{F_{cf}} + \boldsymbol{F_{Cor}} = m\ddot{\boldsymbol{r}} \ ,$$

where over dots represent time differentiations, and **F** is the net real force (as opposed to the fictitious forces). In terms of components, this vector equation becomes:

$$F_r + mr\Omega^2 = m\ddot{r}$$
$$F_\theta - 2m\dot{r}\Omega = mr\ddot{\theta} \ .$$

which can be compared to the equations for the inertial frame:

$$F_r = m\ddot{r} - mr\dot{\theta}^2$$
$$F_\theta = mr\ddot{\theta} + 2m\dot{r}\dot{\theta} \ .$$



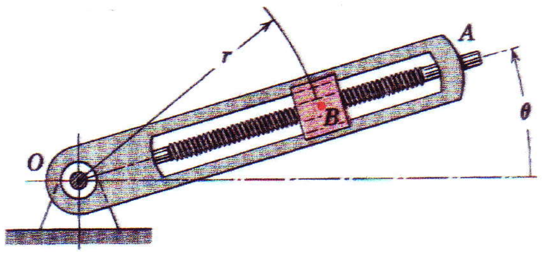
Inertial frame of reference S and instantaneous non-inertial co-rotating frame of reference S'. The co-rotating frame rotates at angular rate Ω equal to the rate of rotation of the particle about the origin of S' at the particular moment t. Particle is located at vector position $\mathbf{r}(t)$ and unit vectors are shown in the radial direction to the particle from the origin, and also in the direction of increasing angle θ normal to the radial direction. These unit vectors need not be related to the tangent and normal to the path. Also, the radial distance r need not be related to the radius of curvature of the path.

This comparison, plus the recognition that by the definition of the co-rotating frame at time t it has a rate of rotation $\Omega = d\theta/dt$, shows that we can interpret the terms in the acceleration (multiplied by the mass of the particle) as found in the inertial frame as the negative of the centrifugal and Coriolis forces that would be seen in the instantaneous, non-inertial co-rotating frame.

For general motion of a particle (as opposed to simple circular motion), the centrifugal and Coriolis forces in a particle's

Sample Problem 2/9

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$ where θ is in radians and t is in seconds. Simultaneously the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$ where r is in meters and t is in seconds. Calculate the velocity and acceleration of the slider for the instant when $t = 3$ s.



Solution. The coordinates and their time derivatives which appear ① in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for $t = 3$ s.

$$r = 0.2 + 0.04t^2 \qquad r_3 = 0.2 + 0.04(3^2) = 0.56 \text{ m}$$

$$\dot{r} = 0.08t \qquad \dot{r}_3 = 0.08(3) = 0.24 \text{ m/s}$$

$$\ddot{r} = 0.08 \qquad \ddot{r}_3 = 0.08 \text{ m/s}^2$$

$$\theta = 0.2t + 0.02t^3 \qquad \theta_3 = 0.2(3) + 0.02(3^3) = 1.14 \text{ rad}$$

$$\text{or } \theta = 1.14(180/\pi) = 65.3°$$

$$\dot{\theta} = 0.2 + 0.06t^2 \qquad \dot{\theta}_3 = 0.2 + 0.06(3^2) = 0.74 \text{ rad/s}$$

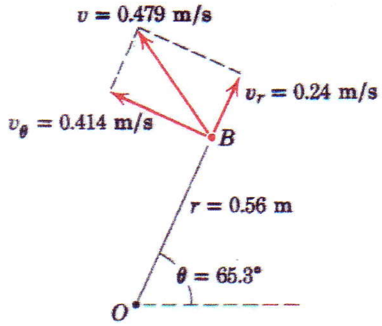
$$\ddot{\theta} = 0.12t \qquad \ddot{\theta}_3 = 0.12(3) = 0.36 \text{ rad/s}^2$$

The velocity components are obtained from Eq. 2/13 and for $t = 3$ s are

$[v_r = \dot{r}]$ $\qquad v_r = 0.24$ m/s

$[v_\theta = r\dot{\theta}]$ $\qquad v_\theta = 0.56(0.74) = 0.414$ m/s

$[v = \sqrt{v_r^2 + v_\theta^2}]$ $\qquad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479$ m/s $\qquad$ *Ans.*



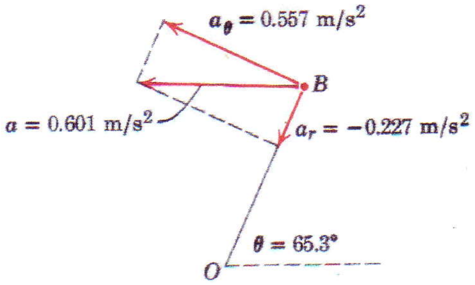
The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for $t = 3$ s are

$[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad a_r = 0.08 - 0.56(0.74)^2 = -0.227$ m/s^2

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557$ m/s^2

$[a = \sqrt{a_r^2 + a_\theta^2}]$ $\qquad a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601$ m/s^2 $\qquad$ *Ans.*

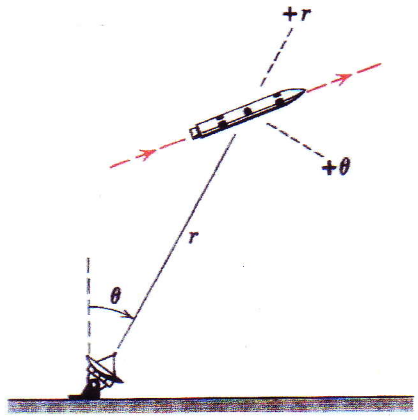


The acceleration and its components are also shown for the 65.3° position of the arm.

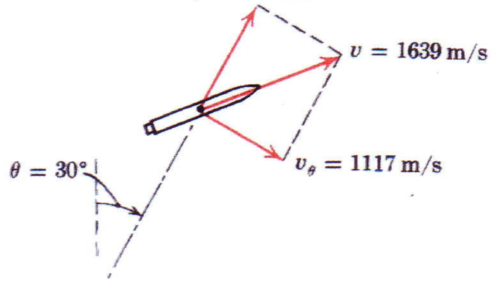
① We see that this problem is an example of constrained motion where the center B of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.

Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$ the tracking data give $r = 8(10^4)$ m, $\dot{r} = 1200$ m/s, and $\dot{\theta} = 0.80°$/s. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 9.20 m/s² vertically down. For these conditions determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



Solution. The components of velocity from Eq. 2/13 are

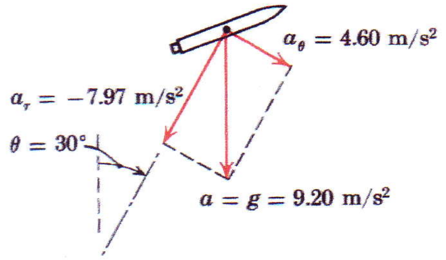


$[v_r = \dot{r}]$ $\qquad v_r = 1200$ m/s

① $[v_\theta = r\dot{\theta}]$ $\qquad v_\theta = 8(10^4)(0.80)\left(\frac{\pi}{180}\right) = 1117$ m/s

$[v = \sqrt{v_r^2 + v_\theta^2}]$ $\qquad v = \sqrt{(1200)^2 + (1117)^2} = 1639$ m/s $\qquad$ *Ans.*

Since the total acceleration of the rocket is $g = 9.20$ m/s² down, we can easily find its r- and θ-components for the given position. As shown in the figure they are



② $$a_r = -9.20 \cos 30° = -7.97 \text{ m/s}^2$$

$$a_\theta = 9.20 \sin 30° = 4.60 \text{ m/s}^2$$

We now equate these values to the polar-coordinate expressions for a_r and a_θ which contain the unknowns $\ddot{r}$ and $\ddot{\theta}$. Thus from Eq. 2/14

③ $[a_r = \ddot{r} - r\dot{\theta}^2]$ $\qquad -7.97 = \ddot{r} - 8(10^4)\left(0.80\frac{\pi}{180}\right)^2$

$$\ddot{r} = 7.63 \text{ m/s}^2$$

$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\qquad 4.60 = 8(10^4)\ddot{\theta} + 2(1200)\left(0.80\frac{\pi}{180}\right)$

$$\ddot{\theta} = -3.61(10^{-4}) \text{ rad/s}^2$$ *Ans.*

① We observe that the angle θ in polar coordinates need not always be taken positive in a counterclockwise sense.

② Note that the r-component of acceleration is in the negative r-direction, so it carries a minus sign.

③ We must be careful to convert $\dot{\theta}$ from °/s to rad/sec.

These two problems are taken from the book Dynamics by Meriam.