1. Unfolding + Proofing 6+6 points

a) Solve the following recurrence using **unfolding**:

$T(1) = 6$
$T(n) = T(n / 4) + 10 \quad$ for $n \geq 2$

b) Proof (using **guess and proofing**) that your result from 1.a) is correct.

a) Unfolding:

$$
\begin{aligned}
T(n) &= T(n/4) + 10 = \\
&= (T((n/4)/4) + 10) + 10 = \\
&= T(n/16) + 20 = \\
&= (T((n/16)/4) + 20) + 10 = \\
&= T(n/64) + 30 = \ldots \\
T(n) &= T(n/4^i) + 10i
\end{aligned}
$$

max. recuvision depth at $i = \log_4(n)$

$$
\begin{aligned}
T(n) &= T(n/4^i) + 10i \\
&= T(n/5^{\log_4(n)}) + 10 \cdot \log_4(n) \\
&= T(n/n) + 10 \cdot \log_4(n) \\
&= T(1) + 10 \cdot \log_4(n) \\
&= 6 + 10 \cdot \log_4(n)
\end{aligned}
$$

$T(n) = O(\log(n))$

b) Proof:

Induction base: $T(1) = 6 + 10 \cdot \log_4 1 = 6 + 10 \cdot 0$

Induction assumption: $T(n/4) = 6 + 10 \cdot \log_4(n/4)$

Induction step: $T(n) = T(n/4) + 10 = (6 + 10 \cdot \log_4(n/4)) + 10$

$$
\begin{aligned}
T(n) &= 6 + 10(\log_4(n) - \log_4(4)) + 10 \\
T(n) &= 6 + 10(\log_4(n) - 1) + 10 \\
T(n) &= 6 + 10 \cdot \log_4(n) - 10 + 10 \\
T(n) &= 6 + 10 \cdot \log_4(n) \quad \checkmark
\end{aligned}
$$

2. Master Theorem 4+4+4 points

Solve the following recurrence using the **Master Theorem**:

a) $T(n) = 16\ T(n/4) + n^{0.5}$
b) $T(n) = 32\ T(n/4) + n^4/2$
c) $T(n) = 27\ T(n/3) + n^3$

Always provide your approach (Lösungsweg) and verify the additional condition for the case 3, that $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon > 0$.

$T(N) = a\ T(N/b) + f(N)$

case 1 $f(N) = O(N^{(\log_b a)-\varepsilon}),\ \varepsilon > 0$
$T(N) = \Theta(N^{\log_b a})$

case 2 $\exists\ k \geq 0$: $f(N) = \Theta(N^{\log_b a} \log^k N)$
$T(N) = \Theta(N^{\log_b a} \log^{k+1} N)$

case 3 $\exists\ \varepsilon > 0$, $c<1$, $f(N) = \Omega(N^{(\log_b a)+\varepsilon})$ and $a\,f(N/b) \leq c\,f(N)$ for N sufficiently large, then
$T(N) = \Theta(f(N))$

$f = O(g)$	**f does not grow faster than g**
$f = \Omega(g)$	**f grows at least as fast as g**
$f = \Theta(g)$	**f and g are of the same order**

a) $T(n) = 16\,T(n/4) + n^{0.5}$

$a = 16$

$b = 4$

$f(n) = n^{0,5}$

$N^{\log_b a} = n^{\log_4 16} = n^2$

f(n) compared wit $N^{\log_b a}$ results in case 1

$f(n) = O(n^{(\log_a b) - \varepsilon}) = \Theta(n^{2-\varepsilon})$

$\rightarrow$ $n^{0,5}$ dose not grow faster than $n^{2-\varepsilon}$

$\rightarrow$ is true for the constant $\varepsilon > 0$ (e.g.: $\varepsilon = 1$)

$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

b) $T(n) = 32\,T(n/4) + n^4/2$

$a = 32$

$b = 4$

$f(n) = n^4/2 = \frac{n^4}{2}$

$n^{\log_b a} = n^{\log_4 32} = \frac{1}{2}\log_2(32) = \frac{1}{2}\log_2(2^5) = \frac{1}{2} \cdot 5 = n^{2,5}$

f(n) compared with $n^{\log_b a}$ results in case 3

$f(n)\ \Omega(n^{(\log_b a)+\varepsilon}) = \Omega(n^{4+\varepsilon})$

$\frac{1}{2}n^4$ growes at least as fast as $n^{2,5+\varepsilon}$ it is true for a constance $\varepsilon > 0$ (e.g. 0,5)

$a\,f(n/b) = 32 f(n/4) = 32\,(n/4)^4 \cdot \frac{1}{2} = 16\,(n^4/256) =$

$\frac{16 \cdot n^4}{256} = \frac{16}{128} \cdot \frac{n^4}{2} = \frac{1}{8} f(n) \rightarrow c = \frac{1}{8} < 1$

$T(n) = \Theta(f(n)) = \Theta(\frac{n^4}{2})$

c) $T(n) = 27\,T(n/3) + n^3$

$a = 27$

$b = 3$

$f(n) = n^3$

$n^{\log_b a} = n^{\log_3 27} = n^{\log_3(3^3)} = n^3$

f(n) compare with $n^{\log_b a}$ $\rightarrow$ case 2

$f(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$

$\rightarrow$ for $k = 0$ n^3 and n^3 are the same order

$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^3 \log n)$