Lab №7 "Controllability and Observability"

Report

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Содержание

1 Introduction

1.1 Objective

This work aims to study the controllability and observability of systems.

1.1.1 Software Implementation

The source code can be found on GitHub repository.

2 Main Part

2.1 Assignment 1

Consider the system:

$$\dot{x} = Ax + Bu$$
$$x(t_1) = \int_0^{t_1} Bu(t)dt$$

2.1.1 Controllability through Controllability Matrix

$U = [B|AB|\dots|A^{n-1}B]$ for $A \in R^{n\times n}$ and $B \in R^{n\times m}$ – the controllability matrix of the system. If its rank is equal to n, the system is controllable.

2.1.2 Controllability and Controllability Gramian

$$P(t_1) = \int_0^{t_1} e^{At}BB^Te^{A^Tt}dt$$

For a controllable system, the controllability Gramian is positive definite at any time t.

2.1.3 Controllability through Eigenvalues of the System Matrix

$$\forall\lambda \in spec(A) : rank(A - \lambda I|B) = n \iff \text{The system is controllable}$$

2.1.4 Controllability through Jordan Form

The Jordan form of matrix $A = PJP^{-1}$:

$$\dot{x} = PJP^{-1}x + Bu$$

Let $\hat{x} = P^{-1}x$, then the system's Jordan form becomes:

$$\dot{\hat{x}} = J\hat{x} + P^{-1}Bu = J\hat{x} + \hat{B}u$$

The system in Jordan form is fully controllable if:

- each eigenvalue corresponds to only one Jordan block.
- the elements of the input matrix corresponding to the last rows of the blocks are nonzero.

2.1.5 Software Control of the System

To compute the control necessary to reach the state x_1 at time t_1, it suffices to calculate:

$$u(t) = B^Te^{A^T(t_1-t)}(P(t_1))^{-1}x_1$$

2.1.6 Results

Assignment variant:

$$A = \begin{bmatrix} 3 & 4 & -1 \\ -10 & -11 & -4 \\ 10 & 10 & 3 \end{bmatrix}; B = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}; x_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}; t_1 = 3$$

The controllability matrix U:

$$U = \begin{bmatrix} -2 & 17 & -62 \\ 5 & -23 & -1 \\ -3 & 21 & 3 \end{bmatrix};$$

$$rankU = 3 = n \rightarrow \text{the system is controllable.}$$

Eigenvalues obtained $spec(A) = [-2+5j, -2-5j, -1+0j]$. Each of them satisfies the expression $rank(A - \lambda I|B) = n$ – thus controllable.

Let's also consider controllability of eigenvalues through Jordan form:

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2-5j & 0 \\ 0 & 0 & -2+5j \end{bmatrix}; \hat{B} = \begin{bmatrix} 2 \\ -1.5+1.5j \\ -1.5-1.5j \end{bmatrix};$$

As can be seen, the conditions are met – another confirmation that the pair (A, B) is controllable.

Since the matrix is fully controllable, any point belongs to the controllable subspace of the system, including x_1.

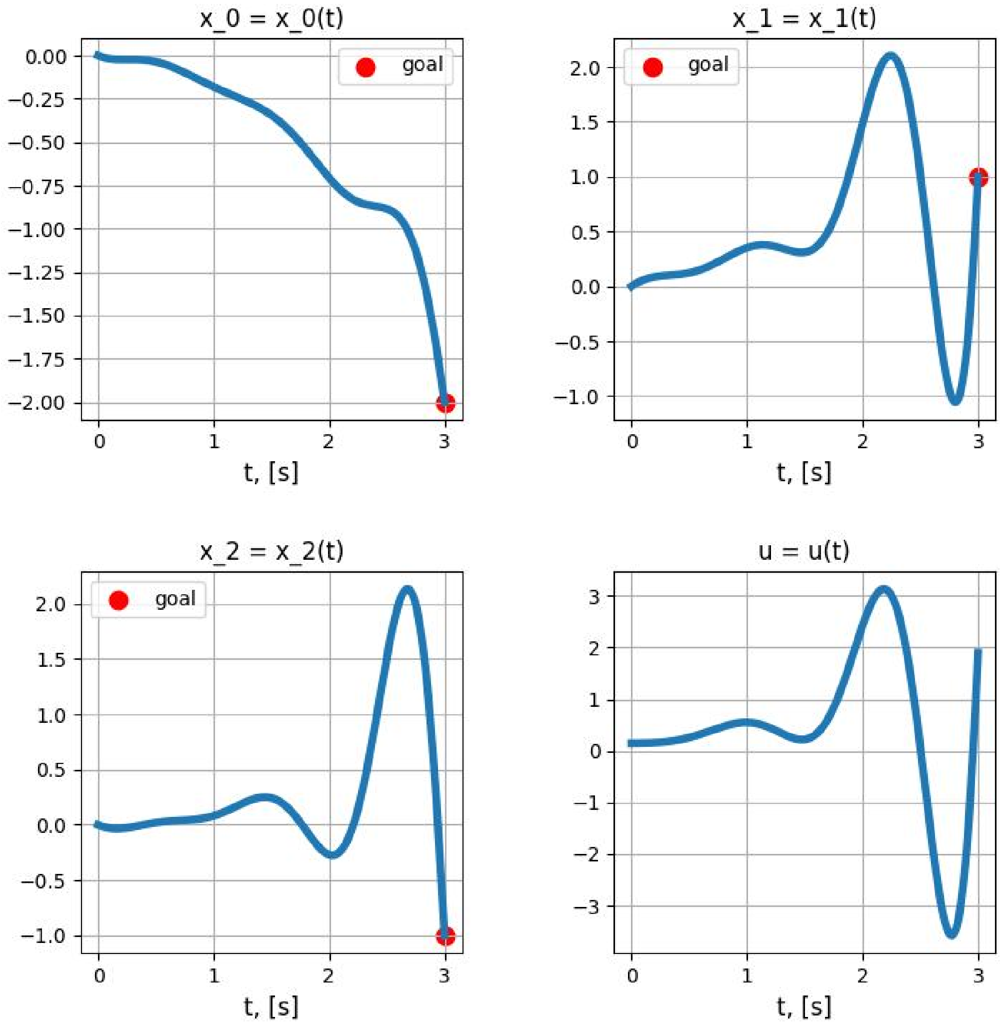
Controllability Gramian obtained:

$$P(t_1) = \begin{bmatrix} 1.20 & -1.34 & 0.23 \\ -1.34 & 2.76 & -1.12 \\ 0.23 & -1.12 & 1.47 \end{bmatrix};$$

$$spec(P(t_1)) = [4, 0.27, 1.15]$$

The Gramian is positive definite, indicating full controllability of the system.

Figure 2.1.6 illustrates the results of the system's simulation. As observed, it reaches the desired state within the specified time.



2.2 Assignment 2

2.2.1 Belonging of the State to the Controllable Subspace

In this assignment, it is additionally necessary to know how to check if the state belongs to the controllable subspace.

To do this, it is necessary to compare $rank(U)$ and $rank(U|x_1)$. If they match – the state belongs.

2.2.2 Results

Assignment variant:

$$A = \begin{bmatrix} 3 & 4 & -1 \\ -10 & -11 & -4 \\ 10 & 10 & 3 \end{bmatrix}; B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}; x_1' = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}; x_1'' = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}; t_1 = 3$$

The controllability matrix U:

$$U = \begin{bmatrix} 2 & 11 & -102 \\ 1 & -27 & 79 \\ -1 & 27 & -79 \end{bmatrix};$$

$$rankU = 2 = n \rightarrow \text{the system is incompletely controllable.}$$

Eigenvalues obtained $spec(A) = [-2+5j, -2-5j, -1+0j]$. The third one is uncontrollable.

Let's also consider controllability of eigenvalues through Jordan form:

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2-5j & 0 \\ 0 & 0 & -2+5j \end{bmatrix}; \hat{B} = \begin{bmatrix} 0 \\ -1.5+1.5j \\ -1.5-1.5j \end{bmatrix};$$

As can be seen, for the first eigenvalue, the corresponding element in the control vector is 0. This further confirms its uncontrollability.

Out of x_1' and x_1'', only the first state belongs to the controllable subspace.

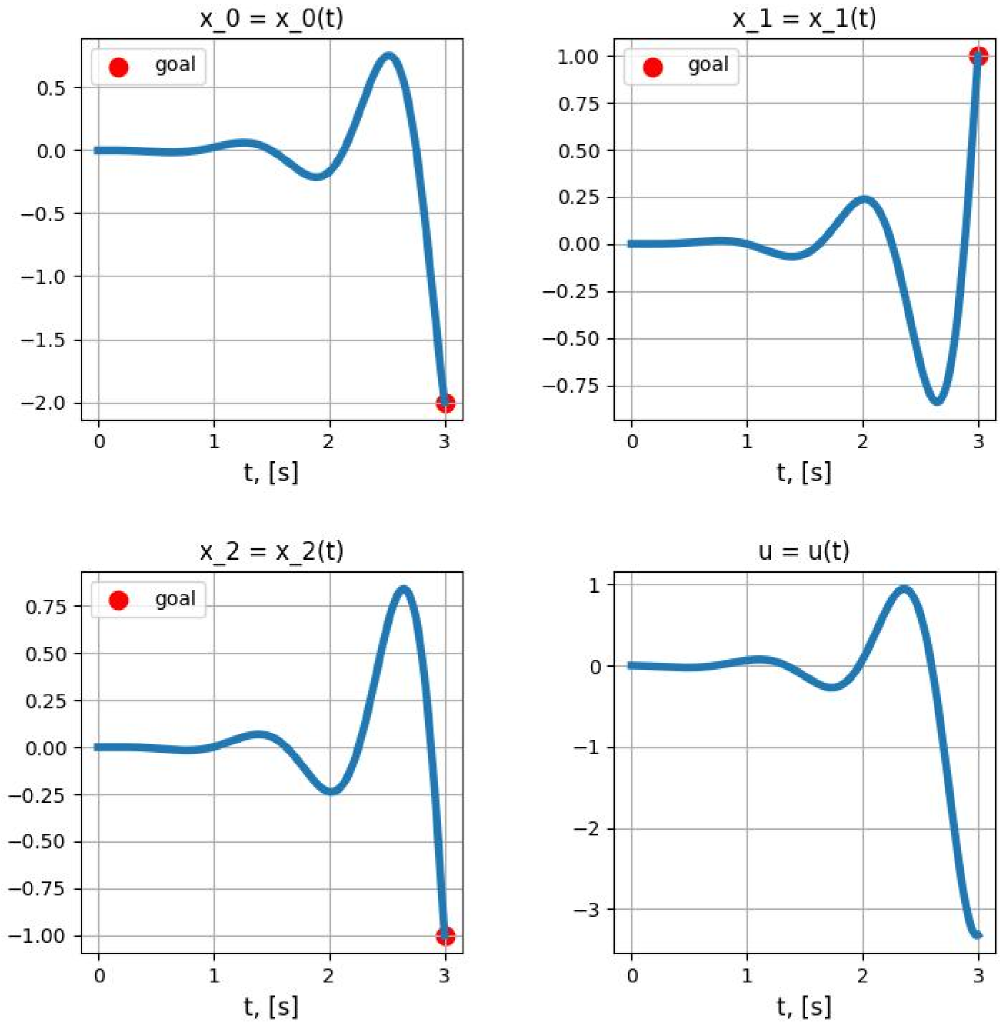
Controllability Gramian obtained:

$$P(t_1) = \begin{bmatrix} 2.05 & -1.63 & 1.63 \\ -1.63 & 2.40 & -2.40 \\ 1.63 & -2.40 & 2.40 \end{bmatrix};$$

$$spec(P(t_1)) = [0.74, 6.12, 0]$$

One of the eigenvalues of the Gramian is 0. To find the software control, it is necessary to use the pseudo-inverse matrix.

Figure 2.2.2 illustrates the results of the system's simulation. As observed, it reaches the desired state within the specified time.



2.3 Assignment 3

We have the system:

$$\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$$

$$y(t) = Ce^{At}x(0)$$

2.3.1 Observability through Observability Matrix

$V = [C|CA|\dots|CA^{n-1}]^T$ for $A \in R^{n\times n}$ and $C \in R^{k\times n}$ – the observability matrix of the system. If its rank is equal to n, the system is observable.

2.3.2 Observability and Observability Gramian

$$Q(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$$

For an observable system, the observability Gramian is positive definite at any time t.

2.3.3 Observability through Eigenvalues of the System Matrix

$$\forall \lambda \in spec(A) : rank(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}) = n \Longleftrightarrow \text{The system is observable}$$

2.3.4 Observability through Jordan Form

The Jordan form of matrix $A = PJP^{-1}$:

$$\begin{cases} \dot{x} = PJP^{-1}x \\ y = Cx \end{cases}$$

Let $\hat{x} = P^{-1}x$, then the system's Jordan form becomes:

$$\begin{cases} \dot{\hat{x}} = J\hat{x} \\ y = CP\hat{x} = \hat{C}\hat{x} \end{cases}$$

The system in Jordan form is fully observable if:

- each eigenvalue corresponds to only one Jordan block.
- the elements of the output matrix corresponding to the first columns of the blocks are nonzero.

2.3.5 Initial Conditions of the System

To compute the initial conditions of the system, it suffices to calculate:

$$x(0) = (P(t_1))^{-1} \int_0^{t_1} e^{A^T t} C^T y(t) dt$$

2.3.6 Results

Assignment variant:

$$A = \begin{bmatrix} -21 & -38 & 6 \\ 8 & 13 & -4 \\ -6 & -14 & -1 \end{bmatrix}; C = \begin{bmatrix} 9 & 18 & -2 \end{bmatrix}; y = 3e^{-5x} \cos 2x - e^{-5x} \sin 2x; t_1 = 3$$

The observability matrix V:

$$V = \begin{bmatrix} 9 & 18 & -2 \\ -33 & -80 & -16 \\ 149 & 438 & 138 \end{bmatrix};$$

$$rankU = V = n \rightarrow \text{the system is observable.}$$

Eigenvalues obtained $spec(A) = [-5+2j, -5-2j, 1+0j]$. Each of them is observable, which further demonstrates the observability of the system.

Let's also consider observability of eigenvalues through Jordan form:

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5-2j & 0 \\ 0 & 0 & -5+2j \end{bmatrix}; \hat{C} = \begin{bmatrix} -2 & 7 & 7 \end{bmatrix};$$

As can be seen, the conditions are met – another confirmation that the pair (A, C) is observable.

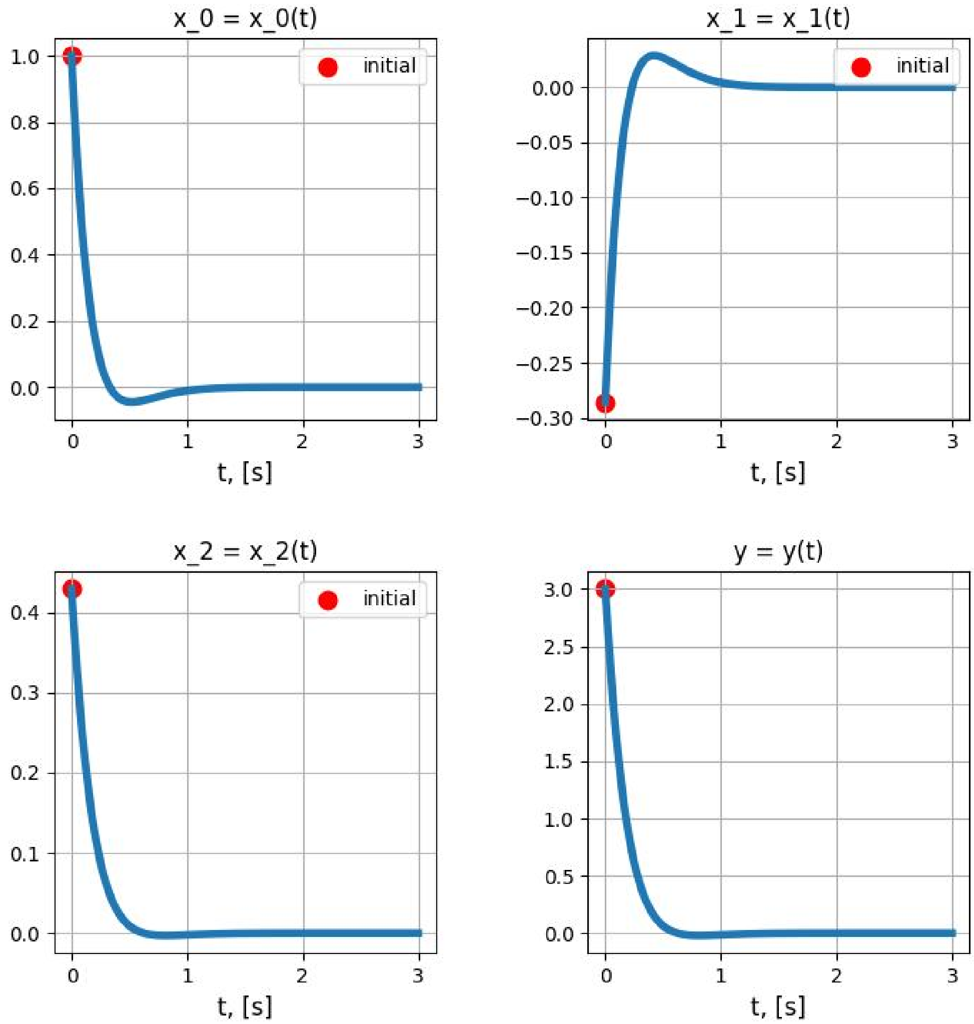
The observability Gramian obtained:

$$Q(t_1) = \begin{bmatrix} 815 & 1627 & -809 \\ 1627 & 3251 & -1618 \\ -809 & -1618 & 807 \end{bmatrix};$$

$$spec(P(t_1)) = [4872, 0.057, 2.37]$$

The Gramian is positive definite, indicating full observability of the system.

Figure 2.3.6 illustrates the results of the system's simulation. As observed, during the simulation, it indeed arrives at the desired observation vector within the specified time. Since the system is fully observable, there can't be any other initial states. The dimension of $Nullspace(V) = dimV - rankV = 0$.



2.4 Assignment 4

2.4.1 Belonging to the Unobservable Subspace

In this assignment, it's additionally necessary to know how to check the state's belonging to the unobservable subspace.

To do this, it's required to verify the equality $Vx = 0$. If this holds true, the state is unobservable.

Accordingly, the basis of the unobservable subspace is equal to $Nullspace(V)$.

2.4.2 Results

Assignment variant:

$$A = \begin{bmatrix} -21 & -38 & 6 \\ 8 & 13 & -4 \\ -6 & -14 & -1 \end{bmatrix}; C = \begin{bmatrix} 7 & 14 & 0 \end{bmatrix}; y = 3e^{-5x}\cos 2x - e^{-5x}\sin 2x; t_1 = 3$$

The observability matrix V:

$$V = \begin{bmatrix} 7 & 14 & 0 \\ -35 & -84 & -14 \\ 147 & 434 & 140 \end{bmatrix};$$

$$rank V = 2 = n \rightarrow \text{the system is partially observable.}$$

Eigenvalues obtained $spec(A) = [-5+2j, -5-2j, 1+0j]$. The third one is unobservable, which once again emphasizes the unobservability of the system.

Let's also consider the observability of eigenvalues through the Jordan form:

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5-2j & 0 \\ 0 & 0 & -5+2j \end{bmatrix}; \hat{C} = \begin{bmatrix} 0 & -3.5+3.5j & -3.5-3.5j \end{bmatrix};$$

As can be seen, the conditions are met – another confirmation that the pair (A, C) is unobservable.

The observability Gramian obtained:

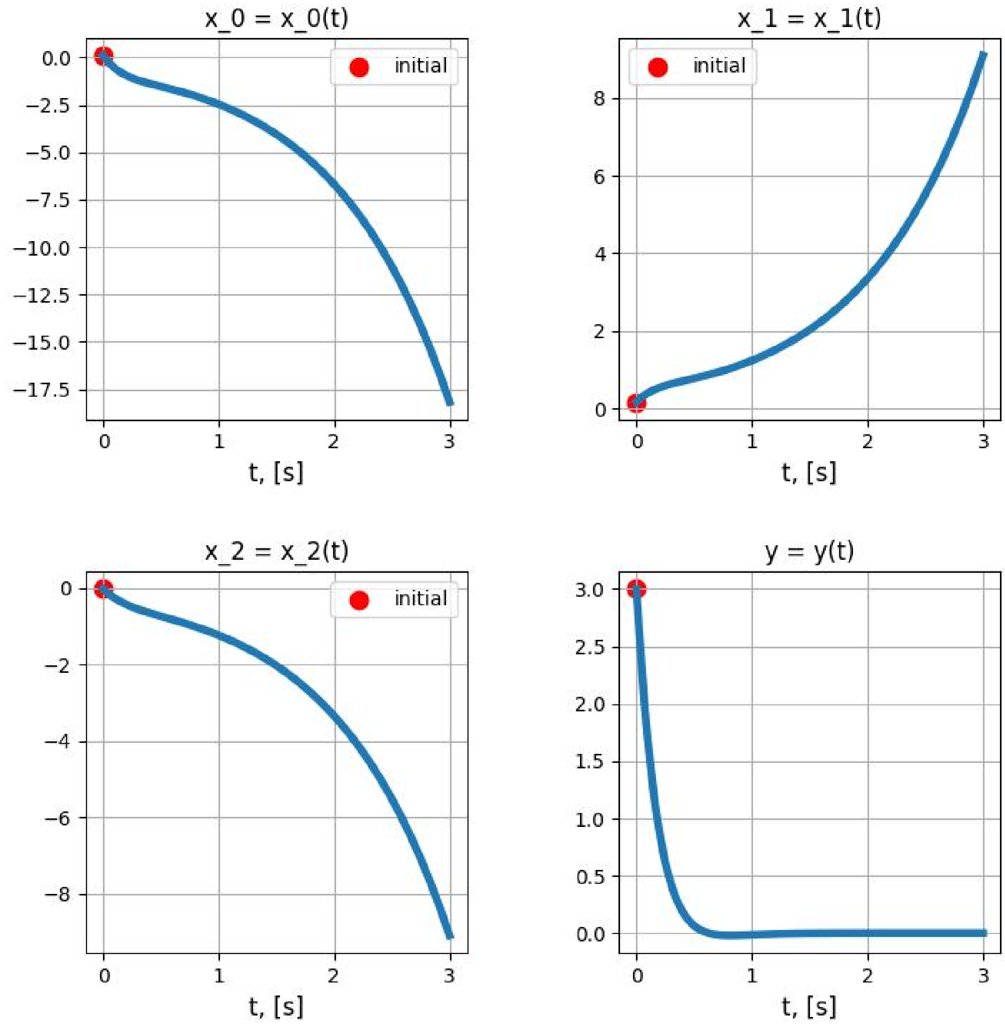
$$P(t_1) = \begin{bmatrix} 4.56 & 8.27 & -0.84 \\ 8.27 & 15.2 & -1.35 \\ -0.84 & -1.35 & 0.33 \end{bmatrix};$$

$$spec(P(t_1)) = [19.8, 0.25, 0]$$

One of the eigenvalues of the Gramian is 0, indicating that it's not positive definite, hence the pair is unobservable.

Figure 2.4.2 illustrates the results of the system's simulation. As observed, during the simulation, it indeed arrives at the desired observation vector within the specified time.

Since the system is only partially observable, there might be other initial states leading to the same observation. To find them, let's find the basis of $Nullspace(V)$. Since the rank of the observability matrix is 2, and the dimension of the system matrix is 3, the basis of the null space will represent a line, defined by one vector $[-0.81, 0.41, -0.41]$. As shown in Figure 2.4.2, despite different initial conditions and state behaviors, the observations completely coincide.

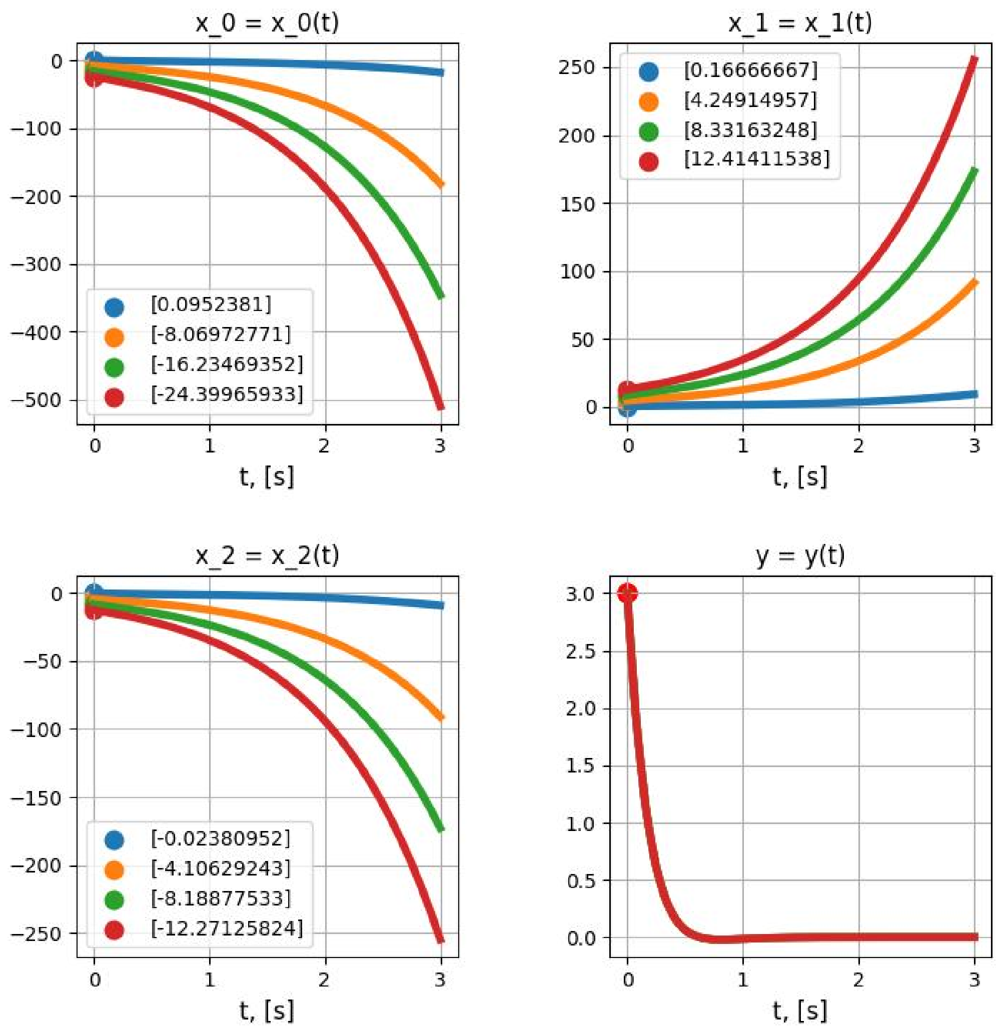


3 Conclusion

In this work, the controllability and observability of systems have been studied.

3.1 Summary

1. Fully controllable systems have been investigated.
2. Various controllability criteria have been tested.
3. Partially controllable systems have been examined, and methods for checking state membership in the controllable subspace have been verified.
4. Fully observable systems have been explored.



5. Various observability criteria have been checked.
6. Partially observable systems have been studied, and methods for verifying the uniqueness of obtained initial conditions have been confirmed.