Node Ranking Dynamics: Non Preferential Patterns

During the last decades, significant research efforts were directed towards the understanding of complex systems and the dynamics they undergo. To date, most studies concentrated on analyzing the degree-based *absolute popularity* of nodes, demonstrating evident preferential attachment [1] and detachment [2] patterns. However, numerous domains assign greater importance to node *relative popularity* exhibited by their ranked degrees. Indeed, from science [3, 4] to pop music, through economy [5, 6], industry [7, 8] and online search engines [9], ranking items is in the heart of all social activities. While node ranking attracted some attention from the scientific community (e.g. [10–13]), a modeling of the entire system's ranking evolution is still lacking. In particular, the degree-based preferential hypotheses raise some fundamental questions that have yet to be addressed. First, do preferential attachment dynamics apply also to node ranking? Namely, is the rate at which nodes gain or lose their rank a monotonic function of their prior ranks? Second, if preferential patterns do not apply to ranking evolution, what is its functional form and which forces govern its demeanor?

In our study we attend to the above stated questions by performing both theoretical and empirical analyses. We show for the first time that preferential principles do not apply to relative popularity evolution and that ranking dynamics follows a non-monotonous, inverse U-shaped, curve. In order to better comprehend these dynamical patterns, we propose an agents-based model which depends on the number of nodes (agents, N) and edges (agents' competitions, H) within the system. Employing this model, we derive the average change in rank $R_\updownarrow$ as a function of the current rank r:

$$\begin{aligned} R_\updownarrow(r) = & \sum_{j=r+1}^{N}\sum_{l=1}^{H}\sum_{i=0}^{\frac{H-l}{2}} \binom{H}{i} p_r^i \times (1-p_r)^{H-i} \times \binom{H-i}{l+i} \times p_j^{l+i} \times (1-p_j)^{H-l-2i} \\ & - \sum_{j=1}^{r-1}\sum_{l=1}^{H}\sum_{i=0}^{\frac{H-l}{2}} \binom{H}{i} \times p_j^i \times (1-p_j)^{H-i} \times \binom{H-i}{l+i} p_r^{l+i} \times (1-p_r)^{H-l-2i} \end{aligned} \quad (1)$$

where p_r is the probability of a node ranked r to attract a new edge. Panel A in Fig. 1 demonstrates the comparison between the dynamics established in the empirical data and our agents-based model. The encouragingly strong fit substantiates our hypothesis regarding the non-monotonic dynamics of relative node popularity. Panel B in Fig. 1 presents the evaluation of $R_\updownarrow$ examining various parameter settings. While the exact numerical characteristics of each plot vary with the chosen parameters, they all present an inverse-U shaped curve.

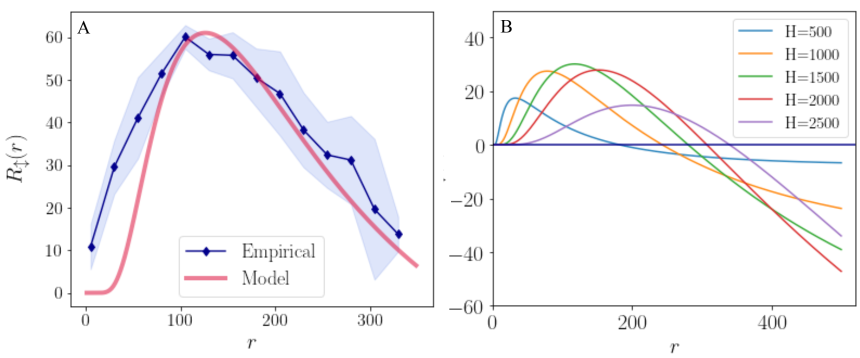

Fig. 1: Expected rank change in the next time-stamp $R_\updownarrow$ as a function of the current rank r. Panel A presents a comparison between the average rank change $R_\updownarrow$ established for the eToro's empirical data (blue curve) and the theoretical model (red curve), evaluated using the extracted empirical parameters: $H = 3100, N = 1000, \gamma = 2.1$. Panel B presents $R_\updownarrow$ evaluations given fixed $\gamma = 2.5$ and $N = 500$ with varying H values. All constellations manifest evident inverse-U patterns, each with different curve properties.

Apart from revealing an inherent difference between the evolution of the absolute and relative popularity measures, the exhibited non-monotonic evolution of node ranking may shed a new light on observed yet hitherto unexplained phenomena. Specifically, they might elucidate the enabling conditions for the formation of dynamic or stationary systems, manifested by the extent of ranking fortification by the top-ranked nodes and the extent of ranking volatility in the slightly lower-ranked nodes. These results not only deepen our comprehension of complex networks' popularity dynamics, but might also provide policy-makers and regulators with intervention means for tuning systems' properties in order to control the extent of their internal mobility.

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