A5

1. Given:

$$\left.\begin{array}{l} \pi : \mathbb{R}^3 - \{0\} \to S^2 \\ \pi(cP) = \dfrac{cP}{\|cP\|} \end{array}\right\} - 1$$

a) $cP \in \mathbb{R}^3$

$$d\pi : T_{cP}(\mathbb{R}^3 - \{0\}) \to T_{\pi(cP)}(S^2)$$

To find: $\dfrac{d\pi}{dcP}$

Let $cP = [x, y, z]$

$$\Rightarrow \pi(cP) = \frac{cP}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{d\pi}{\partial cP} = \frac{1}{\sqrt{x^2+y^2+z^2}} \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \partial x/\partial z \\ \frac{\partial y}{\partial y} & \partial y/\partial y & \partial y/\partial z \\ \partial z/\partial x & \partial z/\partial y & \partial z/\partial z \end{bmatrix}$$

$$\approx \begin{bmatrix} \frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial y}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial z}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right] \\ \frac{\partial}{\partial x}\left[\frac{y}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial y}\left[\frac{y}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial z}\left[\frac{y}{\sqrt{x^2+y^2+z^2}}\right] \\ \frac{\partial}{\partial x}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial y}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] & \frac{\partial}{\partial z}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] \end{bmatrix}$$

CamScanner

b) ${}_wP \in \mathbb{R}^3$, $X_{cw} = (t_{cw}, R_{cw}) \in SE(3)$

X_{cw} can be used as transformation matrix that transforms the ${}_wP$ to ${}_cP$ (ie) point P wrt Camera's body centric frame.

$$\Rightarrow \quad {}_cP = \begin{bmatrix} R_{cw} & t_{cw} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}_wP \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad {}_cP = R_{cw}\,{}_wP + t_{cw}$$

c) To find $\dfrac{\partial\, {}_cP}{\partial X_{cw}}$ & $X_{cw} = (t_{cw}, R_{cw}) \in SE(3)$

$$\left[\begin{array}{l} vec(AB) = (B^T \otimes I)\, vec(A) \\ r_{cw} = vec(R_{cw}) \end{array}\right.$$

$${}_cP = R_{cw}\,{}_wP + t_{cw}$$

$${}_cP = ({}_wP^T \otimes I)\, r_{cw} + t_{cw}$$

$$\frac{\partial\, {}_cP}{\partial X_{cw}} = \begin{bmatrix} \dfrac{\partial\, {}_cP}{\partial R_{cw}} & \dfrac{\partial\, {}_cP}{\partial t_{cw}} \end{bmatrix}$$

$$\frac{\partial\, {}_cP}{\partial X_{cw}} = \begin{bmatrix} ({}_wP^T \otimes I) & I \end{bmatrix}$$

d) $\psi : \mathbb{R}^3 \times SE(3) \to S^2$

${}_wP \in \mathbb{R}^3$, $X_{cw} \in SE(3)$

ψ returns image $u \in S^2$ of point P in the camera

$$\Longrightarrow \quad \cancel{\psi = \pi ({}_wP}^T \dot{x}$$

$$\Rightarrow \quad \boxed{\psi = \pi\left(R_{cw}\,{}_wP + t_{cw} \right)} \text{ (or) } \psi = \pi({}_cP)$$

where ${}_cP = R_{cw}\,{}_wP + t_{cw}$

$$(\cos(xy))^2$$

$$2\cos(xy)\,x - \sin(xy)\; x\; y$$

e) $L : SE(3) \to \mathbb{R}$

$$L(X_{cw}) = \sum_{k=1}^{m} d_{s^2}(\psi({}^{\omega}P_k ; X_{cw}), \tilde{u}_k)^2 \quad — ②$$

$$\left[u_k \triangleq \psi({}^{\omega}P_k, X_{cw})\right]$$

distance fxn: $d_{s^2} : S^2 \times S^2 \to \mathbb{R}$

$$d_{s^2}(x, y) = \arccos(x^T y) \quad — ③$$

$$l_k(u) \triangleq d_{s^2}(u, \tilde{u}_k)^2 \quad — ④$$

To find:

$$\frac{d\, l_k(u)}{du}$$

from ③, ④

$$l_k(u) = (\arccos(u^T \tilde{u}_k))^2 \quad — ⑤ \qquad [\text{let } (u^T \tilde{u}_k) = t]$$

$$\frac{d l_k(u)}{du} = \frac{-2\tilde{u}_k(\arccos(u^T \tilde{u}_k))}{\sqrt{1-(u^T\tilde{u}_k)^2}} \qquad \left[\frac{d(\arccos(x))}{dx} = \frac{-1}{\sqrt{1-x^2}}\right]$$

f). to find: dL/dx_{cw}

$$L(X_{cw}) = \sum_{k=1}^{m} l_k(u_k) \quad — ⑥$$

from ⑤, ⑥ $\qquad [u_k = \psi({}^{\omega}P_k, X_{cw})]$

$$L(X_{cw}) = \sum_{k=1}^{m} (\arccos(u_k^T \tilde{u}_k))^2$$

$$\frac{dL}{dX_{cw}} = \left[\frac{\partial L}{\partial R_{cw}} \quad \frac{\partial L}{\partial t_{cw}}\right]$$

$$L(X_{cw}) = \sum_{k=1}^{3} l_k((\pi(R_{cw}\,{}_wP + t_{cw}), \tilde{u}_k)$$

$$\boxed{u_k = \pi({}_cP)}$$

$$\frac{\partial L}{\partial R_{cw}} = \sum_{k=1}^{3} \frac{\partial}{\partial R_{cw}} l_k((\pi(R_{cw}\,{}_wP + t_{cw}), \tilde{u}_k)$$

$$= \sum_{k=1}^{3} \frac{d l_k}{d u_k} \cdot \frac{d\pi}{d_cP} \cdot \frac{d_cP}{d R_{cw}}$$

$$\frac{\partial L}{\partial R_{cw}} = \sum_{k=1}^{3} \frac{d l_k}{d u_k} \cdot \frac{d\pi}{d_cP} \cdot ({}_wP^T \otimes I)$$

$$\frac{\partial L}{\partial t_{cw}} = \sum_{k=1}^{3} \frac{d l_k}{d u_k} \cdot \frac{d\pi}{d_cP} \cdot \frac{d_cP}{d t_{cw}}$$

$$\frac{\partial L}{\partial t_{cw}} = \sum_{k=1}^{3} \frac{d l_k}{d u_k} \cdot \frac{d\pi}{d_cP} \cdot I$$

$$\frac{dL}{dX_{cw}} = \left[\sum_{k=1}^{n} \frac{d l_k}{d u_k} \cdot \frac{d\pi}{d_cP} \cdot ({}_wP^T \otimes I) \quad \sum_{k=1}^{n} \frac{d l_k}{d u_k} \frac{d\pi}{d_cP} \, I \right]$$