EECE 5550 Mobile Robotics Lab #2

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Problem 1: Differential drive kinematics on SE(2)

Consider a differential-drive robot with track width w and wheel radius r (Fig. 1).

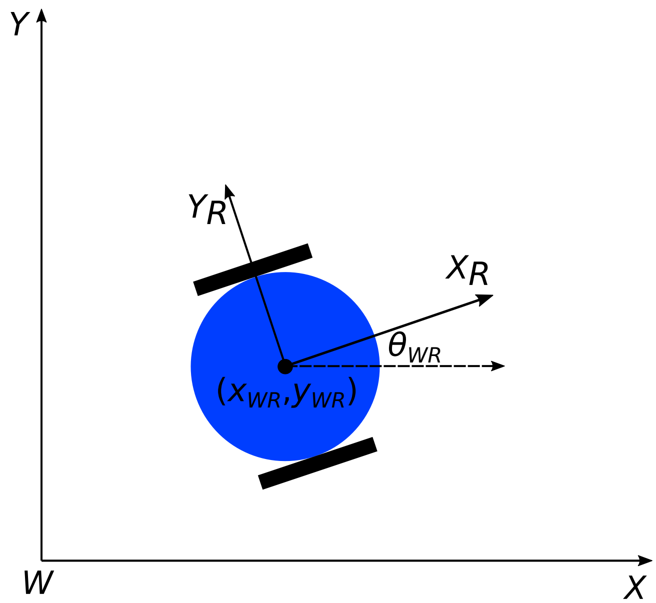

Figure 1: Schematic of a differential drive robot

Let $t_{WR} \triangleq (x_{WR}, y_{WR}) \in \mathbb{R}^2$ and $\theta_{WR} \in \mathbb{R}$ denote the position and orientation angle of the robot in the world coordinate frame W, as shown in Fig. 1. We showed in class that the equations of motion for this vehicle are:

$$\begin{pmatrix} \dot{x}_{WR} \\ \dot{y}_{WR} \\ \dot{\theta}_{WR} \end{pmatrix} = \begin{pmatrix} R(\theta_{WR}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{r}{2}(\dot{\varphi}_r + \dot{\varphi}_l) \\ 0 \\ \frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) \end{pmatrix}, \tag{1}$$

where $\dot{\varphi}_l$ and $\dot{\varphi}_r$ are the angular speeds of the left and right wheels, respectively, with positive values corresponding to forward motion (that is, motion along the robot's body-centric positive x-axis, $+x_R$), and

$$\begin{gathered} R\colon \mathbb{R} \to \mathrm{SO}(2) \\ R(\theta) \triangleq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \end{gathered} \tag{2}$$

is the mapping that assigns to each *angle* θ the corresponding *rotation matrix* $R(\theta)$.

In this exercise, we will reformulate the equations of motion (1) on the Lie group SE(2):

$$\mathrm{SE}(2) \cong \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \mid R \in \mathrm{SO}(2),\ t \in \mathbb{R}^2 \right\}, \tag{3}$$

and then apply Lie group theory to perform *forward simulation* of the vehicle's trajectory.

(a) Let's begin by defining the mapping:

$$\begin{aligned} \Psi &\colon \mathbb{R}^2 \times \mathbb{R} \to \mathrm{SE}(2) \\ \Psi(t,\theta) &\triangleq \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4}$$

that sends each pair (t,θ) to the element of SE(2) obtained by replacing the *angle* θ by its corresponding *rotation matrix* $R(\theta)$.

Notice that

$$\Psi(0,0) = \begin{pmatrix} R(0) & 0 \\ 0 & 1 \end{pmatrix} = I; \tag{5}$$

that is, the map Ψ sends the vector $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$ to the identity $I \in \mathrm{SE}(2)$. This means that the *derivative map*:

$$d\Psi_{(0,0)} \colon T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}) \to \mathrm{Lie}(\mathrm{SE}(2)) \tag{6}$$

sends each tangent vector $(\dot{t},\dot{\theta}) \in T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}) \cong \mathbb{R}^3$ to a tangent vector in $T_I(\mathrm{SE}(2)) \cong \mathrm{Lie}(\mathrm{SE}(2))$, the *Lie algebra* of SE(2). Derive a closed-form expression for the derivative map $d\Psi_{(0,0)}$ in (6).

(b) The differential drive kinematic equation (1) describes how the left and right wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ determine the robot's velocity $(\dot{t},\dot{\theta}) \in T_{(t,\theta)}(\mathbb{R}^2 \times \mathbb{R})$. Suppose that the robot is at the origin $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$. Using the result of part (a), derive the corresponding mapping:

$$\dot{\Omega} \colon \mathbb{R}^2 \to \mathrm{Lie}(\mathrm{SE}(2)) \tag{7}$$

that sends the pair of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r)$ to the robot's velocity $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$ in the Lie group SE(2) at I.

(c) The mapping $\dot{\Omega}$ from wheel speeds to velocities that you derived in part (b) was obtained under the assumption that the robot's pose in the world frame is $T_{WR} = I$.

Suppose now that the robot's pose in the world frame is $T_{WR} = X \in \mathrm{SE}(2)$. Derive a closed-form expression for the map:

$$V \colon \mathrm{SE}(2) \times \mathbb{R}^2 \to T(\mathrm{SE}(2)) \tag{8}$$

that accepts as input the robot's pose $X \in \mathrm{SE}(2)$ and wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ and returns its velocity $V(X, \dot{\varphi}_l, \dot{\varphi}_r) \in T_X(\mathrm{SE}(2))$. You may write your answer in terms of the Lie algebra element $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$.

[Hint: you may find it convenient to introduce an auxiliary coordinate frame F that is *fixed* with respect to the world frame W, and is aligned with the robot's body-centric frame – i.e., frame F is *defined* so that the pose of the robot in frame F is $T_{FR} = I$. What is the velocity of the robot with respect to frame F? How are the velocities of the robot in frames F and W related?]

(d) Suppose that we *fix a specific choice* of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$. Given this choice, the map V that you derived in part (c) simplifies to a function $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X)$ that assigns to each pose $X \in \mathrm{SE}(2)$ the velocity:

$$V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X) \triangleq V(X, \dot{\varphi}_l, \dot{\varphi}_r) \in T_X(\mathrm{SE}(2)). \tag{9}$$

In other words: each choice of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ determines a *vector field* $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}$ on $\mathrm{SE}(2)$ that accepts as input a robot pose $X \in \mathrm{SE}(2)$, and returns the robot velocity $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X)$ determined by the selected wheel speeds at X.

Prove that $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}$ is in fact a *left-invariant* vector field. [Hint: You may find Problem 2 from Lab #1 useful here.]

(e) **Forward kinematics:** Suppose that the robot is at pose $X_0 \in \mathrm{SE}(2)$ at time $t = 0$, and that we drive its wheels at constant velocity $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$. Given this initial data, write down a closed-form formula for the curve:

$$\gamma \colon \mathbb{R} \to \mathrm{SE}(2) \tag{10}$$

that reports the robot's pose $\gamma(t)$ at time t. You may express your answer in terms of $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$.

Question 2: PID altitude control

Consider the following second order dynamical system that models the altitude dynamics of a drone:

$$\ddot{h} = \frac{4k_T u}{m} - g, \qquad IC : h(0) = 0, \dot{h} = 0, \tag{11}$$

where $m = 65g$ is the mass of the drone, $k_T = 5.276 \times 10^{-4}$ is the thrust coefficient, $g = 9.81m/s^2$ is the gravitational constant, $h \in \mathbb{R}$ is the altitude, and $u \in \mathbb{R}$ is the control input. Let u be designed as

$$u = PID + \frac{mg}{4k_T}, \tag{12}$$

where PID is the proportional-integral-derivative control architecture, and the last element is for the perfect gravity cancellation as we discussed in the class. The objective of the drone is to reach a reference altitude of $r = 1m$ and hover there.

(a) Design a P control using (48) for values of $K_p = 5, 15, 50$, plot h and $\dot{h}$, discuss the results.

(b) Design a PD control using (48) such that the closed-loop system is underdamped and the settling time is approximately 3 seconds, plot h and $\dot{h}$, discuss the results and justify why the system is underdamped.

(c) Design a PD control using (48) such that the closed-loop system is overdamped and the settling time is approximately 3 seconds, plot h and $\dot{h}$, discuss the results and justify why the system is overdamped.

(d) In this part, you will consider some uncertainty in the actuators so the actual control applied to the system will be $u' = 0.95u$. First, obtain the results of part (b) by considering u'. Then design a PID control using the same K_p, K_d gains as in (b) and some nonzero K_i gain (again considering u'). Plot h and $\dot{h}$, and discuss the differences between the responses obtained by the PD and PID controls.

Problem 3: Nonlinear feedback and stability analysis

In this problem you will devise a feedback controller to stabilize a damped driven pendulum in the upright position. Recall the equations of motion for a damped pendulum driven by an external torque τ:

$$ml^2\ddot{\theta} = -mgl\sin(\theta) - \mu\dot{\theta} + \tau, \tag{13}$$

where m is the mass of the bob, l is the length of the pendulum, μ is the damping parameter (due to friction), g is the gravitational acceleration, and τ is the external torque applied to the pendulum's pivot.

(a) Defining $x = (\theta, \dot{\theta}) \in \mathbb{R}^2$, rewrite the second-order equations of motion (13) in the form of a *first*-order differential system for x:

$$\dot{x} = f(x, \tau). \tag{14}$$

Note that your function $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ should express the derivative $\dot{x}$ as a function of *both* the state x *and* the external torque $\tau \in \mathbb{R}$.

(b) Let

$$\begin{aligned} g&: \mathbb{R}^2 \to \mathbb{R}^2 \\ g(x) &\triangleq f(x, 0) \end{aligned} \tag{15}$$

denote the equations of motion for the *free* system (i.e., the dynamics of the system when *no external torque* is applied). Show that the upright configuration $x^* = (\pi, 0)$ is an *unstable* stationary point for the free system.

Linear control synthesis: In the next part of this exercise, you will apply linearized stability analysis to devise a PD controller to stabilize the pendulum in the upright configuration.

(c) We will assume a PD controller of the form:

$$\begin{aligned} \tau&: \mathbb{R}^2 \to \mathbb{R} \\ \tau(x) &\triangleq k_p \sin(x_1) + k_d x_2, \end{aligned} \tag{16}$$

where $k_p, k_d \in \mathbb{R}$ are controller gains (to be determined). Using the control law (16), derive an explicit expression for the *closed-loop* dynamics of the system in the form of an *autonomous* ODE:

$$\dot{x} = c(x) \triangleq f(x, \tau(x)). \tag{17}$$

(d) Using your result in part (c), derive conditions on the controller gains k_p, k_d that are sufficient to guarantee that $x^* = (\pi, 0)$ is a (locally) asymptotically stable stationary point of the closed-loop system. [Hint: Show that x^* is a stationary point for the closed-loop system, and then apply linearized stability analysis to derive sufficient conditions on the controller gains.]

Controller design via linearized stability analysis [as you did in part (d)] is convenient in that it provides an easy method of constructing *locally* asymptotically stabilizing controllers using only a bit of linear algebra. However, because this approach is based upon (local) *linearization*, it doesn't directly provide any information about the *size* of the neighborhood around x^* over which the resulting controller works.

Nonlinear control synthesis: As an alternative approach, in the remainder of this question, you will apply the theory of Lyapunov functions to devise a *nonlinear* feedback controller whose *invariant subsets* we can explicitly characterize.

(e) Consider a candidate Lyapunov function of the form:

$$\begin{gathered} V : \mathbb{R}^2 \to \mathbb{R} \\ V(x) \triangleq -mgl(1+\cos(x_1)) + \alpha mgl(1-\cos^2(x_1)) + \frac{1}{2}ml^2x_2^2 \end{gathered} \tag{18}$$

where $\alpha \in \mathbb{R}$ is a free parameter (to be determined). Calculate the gradient and Hessian of V, and derive a sufficient condition on α for the point $V(x^*) = 0$ to be an isolated local minimizer of V.

(f) Your results in part (e) show that there exists some neighborhood $U \in \mathbb{R}^2$ containing x^* such that $V(x) \geq 0$ for all $x \in U$, and x^* is the *unique* point in U satisfying $V(x^*) = 0$. In order to show that V is a valid Lyapunov function, we must identify a control law such that V is nonincreasing along the trajectories of the system. To that end, derive a closed-form expression for the time derivative $\dot{V}$ of V:

$$\dot{V} = \frac{d}{dt}\left[V(x)\right] = \nabla V(x) \cdot f(x,\tau). \tag{19}$$

You may leave your answer in terms of the state x and the control τ (to be determined).

The result of part (f) suggests that we consider the following nonlinear feedback controller:

$$\begin{gathered} \tau : \mathbb{R}^2 \to \mathbb{R} \\ \tau(x) \triangleq -2\alpha mgl\sin(x_1)\cos(x_1). \end{gathered} \tag{20}$$

Our goal now will be to apply Lyapunov theory to show that this indeed stabilizes the upright position $x^* = (\pi, 0)$, and to characterize the *invariant subsets* around x^*.

(g) Using (20), derive an explicit expression for the closed-loop dynamics of the system:

$$\dot{x} = c(x) \triangleq f(x, \tau(x)) \tag{21}$$

under the control law (20).

(h) Using your result in part (g), find the set S of stationary points for the closed-loop system under the control law (20).

(i) Show that under the control law (20), the function V satisfies:

$$\dot{V} \leq 0. \tag{22}$$

The results of (e)–(i) show that the $x^* = (\pi, 0)$ is a stationary point of the closed-loop system under the nonlinear control law (20), and that V is a valid Lyapunov function for the closed-loop system in a neighborhood of x^*. We may therefore conclude that x^* is *stable in the sense of Lyapunov* for the closed-loop system. However, your result in (i) shows that $\dot{V}$ is only negative *semidefinite* ($\dot{V} \leq 0$), but **not** negative *definite* ($\dot{V} \nless 0$). Therefore, we cannot conclude that x^* is *asymptotically* stable. To prove the (stronger) statement that x^* is *asymptotically* stable, we can apply the LaSalle invariance principle.[1]

[1] **A word on the LaSalle invariance principle:** We saw in class that we can intuitively think of a Lyapunov function as measuring the "energy" of a system. If we can show that $\dot{V} \leq 0$ – that is, that the energy is *nonincreasing* along trajectories $x(t; x_0)$ of the system – then we know that the sublevel sets $L_c^-(V)$ of V must be *invariant* sets for our system. This is enough to show *stability in the sense of Lyapunov*, since any trajectory

(j) Derive a closed-form expression for the set $\mathcal{I}$ of points at which the time derivative of V is zero:

$$\mathcal{I} \triangleq \{x \in \mathbb{R}^2 \mid \dot{V}(x) = 0\}, \tag{23}$$

and argue that the only trajectories $x(t; x_0)$ contained *entirely* within this set are the stationary points S you found in part (h).

Altogether, your results show that there is a neighborhood U containing $x^* = (\pi, 0)$ on which V is a valid Lyapunov function for the closed-loop system under (20), and the only trajectory that is *entirely* contained in $\mathcal{I} \cap U$ is the stationary point $x^* = (\pi, 0)$. This proves that x^* is *asymptotically* stable.

As we saw in class, one of the primary advantages of the Lyapunov approach is that the sublevel sets:

$$L_c^-(V) \triangleq \{x \in \mathbb{R}^n \mid V(x) \leq c\} \tag{24}$$

of the Lyapunov function provide a great deal of information about the trajectories of the closed-loop system. In particular, we saw that any sublevel set $L_c^-(V)$ that is contained in a neighborhood U on which V is a valid Lyapunov function is an *invariant* set for the closed-loop system.

Let us consider the following choice for U:

$$U \triangleq \left\{ x \in \mathbb{R}^2 \mid \cos(x_1) < -\frac{1}{2\alpha},\ V(x) \geq 0,\ \dot{V}(x) \leq 0 \right\}. \tag{25}$$

(note that here the restriction on the cosine of x_1 ensures that $x^* = (\pi, 0)$ is the *unique* stationary point contained in U).

(k) Assuming the values $m = g = l = 1$, $\mu = .1$, and $\alpha = 2$, plot the following data over the portion of the phase plane satisfying $\frac{\pi}{2} \leq x_1 \leq \frac{3\pi}{2}$ and $-2 \leq x_2 \leq 2$:

- the Lyapunov function $V(x)$;
- the indicator function χ_U for the set U defined in (25);
- the phase portrait for the closed-loop dynamics under the control law (20);
- the level sets $V^{-1}(c)$ of the Lyapunov function $V(x)$ for values of c between .05 and 1.25 (inclusive), at increments of .1.

Submit these plots togther with the code you used to produce them.

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that *starts* in $L_c^-(V)$ will *stay* in $L_c^-(V)$.

However, we also saw that $\dot{V} \leq 0$ is not enough to prove *asymptotic stability*. The reason is that there may be a subset $\mathcal{I} \triangleq \{x \in U \mid \dot{V}(x) = 0\}$ of U in which the energy of the system is (locally) *constant* along trajectories. (For example, this is the case for the pendulum *without* friction – this system is conservative, and so in fact the energy is constant along *every* trajectory.) Any trajectory that is contained *entirely* in $\mathcal{I}$ will have *constant* energy, and so need not "decay" over time to the stationary point x^*.

The *LaSalle invariance principle* is an *additional* criterion that we can use to prove that a fixed point x^* is locally *asymptotically* stable, even if our Lyapunov function only satisfies $\dot{V} \leq 0$. In brief, what we need to show is that *the only trajectory contained* ***entirely*** *within $\mathcal{I}$ is just the constant trajectory $x(t) = x^*$ at the fixed point x^* itself.* Geometrically, what this means that that *any nonstationary trajectory that enters $\mathcal{I}$ must eventually leave it.* In that case, even though the energy of the trajectory $x(t; x_0)$ does not decrease *while it is inside* $\mathcal{I}$, if it is nonstationary, then eventually it *will* leave $\mathcal{I}$, and then its energy *will* decrease again.

In summary: the LaSalle invariance principle states that if V is a Lyapunov function **and** the only trajectory $x(t; x_0)$ contained *entirely* in the subset $\mathcal{I} = \{x \in U \mid \dot{V}(x) = 0\}$ on which V is locally constant is the stationary trajectory $x^*(t) = x^*$, **then** x^* is in fact *asymptotically stable.*

Solutions

Problem 1

(a) To compute the derivative $d\Psi_{(0,0)}$, we simply differentiate the map (4) at $(0,0)$:

$$\begin{aligned} d\Psi_{(0,0)} &\colon \mathbb{R}^2 \times \mathbb{R} \to T_I(\mathrm{SE}(2)) \\ d\Psi_{(0,0)}[\dot{t}, \dot{\theta}] &= \begin{pmatrix} dR_0[\dot{\theta}] & \dot{t} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{26}$$

where $dR_\theta[\dot{\theta}]$ is the derivative of the rotation mapping $R(\theta)$ in (2):

$$\begin{aligned} dR_\theta &\colon \mathbb{R} \to \mathrm{SO}(2) \\ dR_\theta[\dot{\theta}] &= \dot{\theta} \begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}. \end{aligned} \tag{27}$$

It follows from (26) and (27) that the derivative of Ψ at $(0,0)$ is:

$$\begin{aligned} d\Psi_{(0,0)} &\colon T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}) \to T_I(\mathrm{SE}(2)) \\ d\Psi_{(0,0)}(\dot{t}, \dot{\theta}) &= \begin{pmatrix} \dot{\theta} S & \dot{t} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{28}$$

where:

$$S \triangleq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Skew}(2) = \mathrm{Lie}(\mathrm{SO}(2)). \tag{29}$$

(b) If the robot is at the origin [i.e., $(t, \theta) = (0,0) \in \mathbb{R}^2 \times \mathbb{R}$], then the kinematic equation (1) simplifies to:

$$\begin{pmatrix} \dot{x}_{WR} \\ \dot{y}_{WR} \\ \dot{\theta}_{WR} \end{pmatrix} = \begin{pmatrix} \frac{r}{2}(\dot{\varphi}_r + \dot{\varphi}_l) \\ 0 \\ \frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) \end{pmatrix} \in T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}). \tag{30}$$

Substituting (30) into the derivative mapping $d\Psi_{(0,0)}$ then gives an expression for the robot's velocity at $I \in \mathrm{SE}(2)$ as a function of the wheel speeds:

$$\begin{aligned} \dot{\Omega} &\colon \mathbb{R}^2 \to \mathrm{Lie}(\mathrm{SE}(2)) \\ \dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r) &= \begin{pmatrix} 0 & -\frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) & \frac{r}{2}(\dot{\varphi}_r + \dot{\varphi}_l) \\ \frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{31}$$

(c) To answer this question, let's introduce an auxiliary frame F that is *fixed* with respect to the world frame W, and is (initially) aligned with the robot's body-centric frame; i.e., we *define* F through the property that the robot's pose in frame F is $T_{FR} = I \in \mathrm{SE}(2)$.

We know that the robot's pose in frame F and the robot's pose in frame W are related by:

$$T_{WR} = T_{WF} T_{FR}, \tag{32}$$

where $T_{WF} \in \mathrm{SE}(2)$ is the (*constant*) transformation that sends frame F onto frame W. Note that we can equivalently write (32) as:

$$T_{WR} = L_{T_{WF}}(T_{FR}). \tag{33}$$

Differentiating (33), we find that the robot's velocities in frames F and W are related by:

$$\dot{T}_{WR} = dL_{T_{WF}}[\dot{T}_{FR}] = T_{WF}\dot{T}_{FR}, \tag{34}$$

where the last equality follows from Problem 2(e) on Lab #1. Furthermore, since we chose frame F so that $T_{FR} = I$, then we can apply the result of part (b) to determine the robot's velocity in frame F:

$$\dot{T}_{FR} = \dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r). \tag{35}$$

Thus, to determine the robot's velocity $\dot{T}_{WR}$ in the world frame, we need only identify the transformation T_{WF} that relates the frames W and F. Right-multiplying both sides of (32) by T_{FR}^{-1}, we find:

$$T_{WF} = T_{WR}T_{FR}^{-1}. \tag{36}$$

But $T_{WR} = X$, the pose of the robot in the world frame, and $T_{FR} = I$ by definition; therefore:

$$T_{WF} = X. \tag{37}$$

Substituting (35) and (37) into (34) thus gives an expression for the velocity of the robot in the world frame W as a function of both its current pose X and wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r)$:

$$V(X, \dot{\varphi}_l, \dot{\varphi}_r) = X\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r). \tag{38}$$

(d) If we think of the wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ as *fixed*, then (38) shows that the vector field $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X)$ simply assigns to each pose $X \in \mathrm{SE}(2)$ the *left-translation* of the (*constant*) Lie algebra element $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$ determined by $(\dot{\varphi}_l, \dot{\varphi}_r)$:

$$V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X) \triangleq X\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r) = d(L_X)_I\left[\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)\right]. \tag{39}$$

The right-hand side of (39) is precisely the left-invariant vector field on SE(2) determined by the Lie algebra element $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$ (cf. Problem 2(f) from Lab #1).

(e) We saw in part (d) that the velocity field $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}$ on SE(2) determined by the wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r)$ is the *left-invariant* vector field determined by the Lie group element $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$. Thus, the robot's trajectory $\gamma(t)$ will be the *integral curve* of this vector field that starts at the pose $X_0 \in \mathrm{SE}(2)$ at time $t = 0$. But we know that the exponential map provides a closed-form solution for integral curves of left-invariant vector fields on Lie groups; specifically:

$$\gamma(t) = X_0 \exp\left(t\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)\right). \tag{40}$$

Problem 2

(a) Define a P control using (48):

$$u = K_p(r - h) + \frac{mg}{4k_T}. \tag{41}$$

Substitute (41) into (11),

$$\ddot{h} = \frac{4k_T(K_p(r-h) + \frac{mg}{4k_T})}{m} - g, \qquad IC : h(0) = 0, \dot{h} = 0, \tag{42}$$

After reorganizing the terms, the second order differential equation of the closed-loop model becomes:

$$\ddot{h} + \frac{4K_T K_p}{m} h = \frac{4K_T K_p}{m} r. \tag{43}$$

Note that a second-order system cannot be stabilized with P control (to ensure a negative real part for the roots of the characteristic equation). So we expect to see a not stable behavior in the plots.

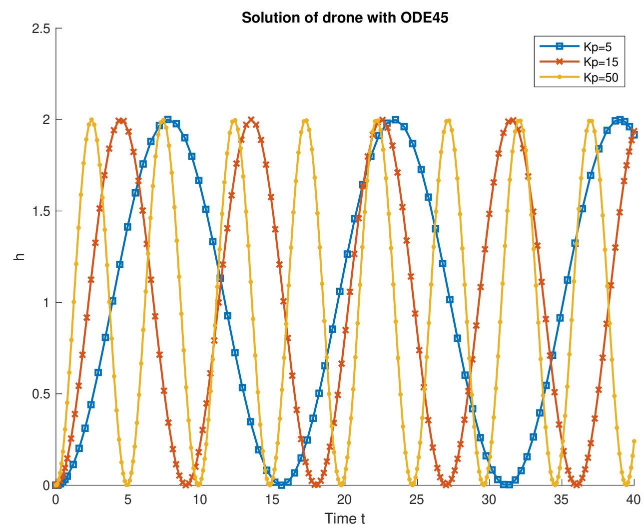


Figure 2: Response of the closed loop system (altitude) under varying K_p gains.

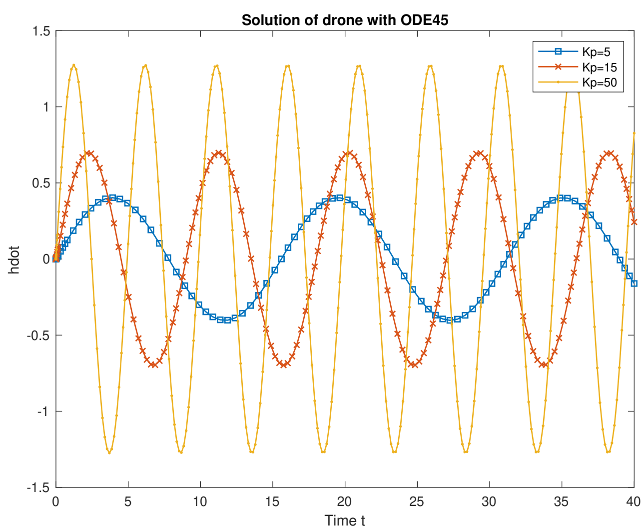


Figure 3: Response of the closed loop system (velocity) under varying K_p gains.

(b) Define a PD control using (48):

$$u = K_p(r-h) - K_d\dot{h} + \frac{mg}{4k_T}. \tag{44}$$

Substitute (44) into (11),

$$\ddot{h} = \frac{4k_T(K_p(r-h) - K_d\dot{h} + \frac{mg}{4k_T})}{m} - g, \qquad IC: h(0)=0, \dot{h}=0, \tag{45}$$

After reorganizing the terms, the second order differential equation of the closed-loop model becomes:

$$\ddot{h} + \frac{4K_TK_d}{m}\dot{h} + \frac{4K_TK_p}{m}h = \frac{4K_TK_p}{m}r. \tag{46}$$

Recall that ω_n and ζ appear in stable second order systems as follows:

$$\ddot{h} + 2\zeta\omega_n\dot{h} + \zeta^2 h = b_0 r, \tag{47}$$

and the settling time can be approximated by $\frac{3}{\zeta\omega_n}$. Since the question asks for an approximate settling time of 3 sec for an underdamped system, then we can choose ω and ζ such that $\zeta\omega_n = 1$ and $0 < \zeta < 1$. Let's choose $\zeta = 0.5$ and $\omega_n = 2$ and substitute them into (47), then we can find the corresponding K_p and K_d gains by equalizing (47) and (46).

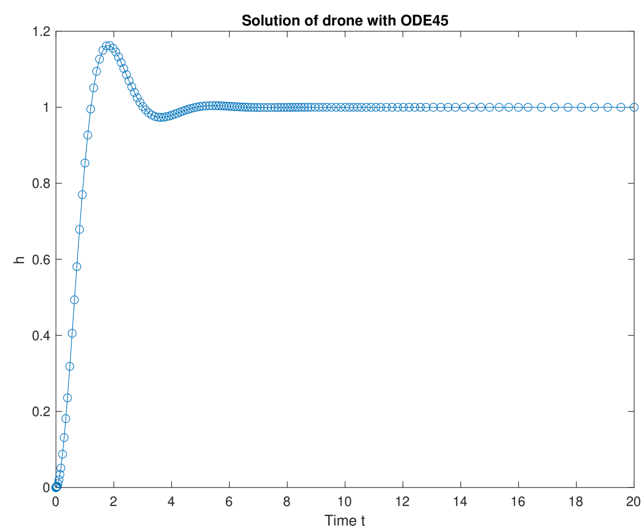


Figure 4: Response of an underdamped closed loop system (altitude) for $\zeta = 0.5$ and $\omega_n = 2$ with approximate settling time 3 sec.

(c) By using the results on part (b), an approximate settling time of 3 sec for an overdamped system can be achieved by choosing ω and ζ such that $\zeta > 1$. Let's choose $\zeta = 1.5$ and $\omega_n = 2.5$[2]. Then we can calculate the corresponding K_p and K_d gains as above.

[2]Note that $\frac{3}{\zeta\omega_n}$ is not a good settling time approximation for overdamped systems because the dominant root influences the settling time. In this case, one should tune the parameters to obtain a desired approximate settling time in the plots

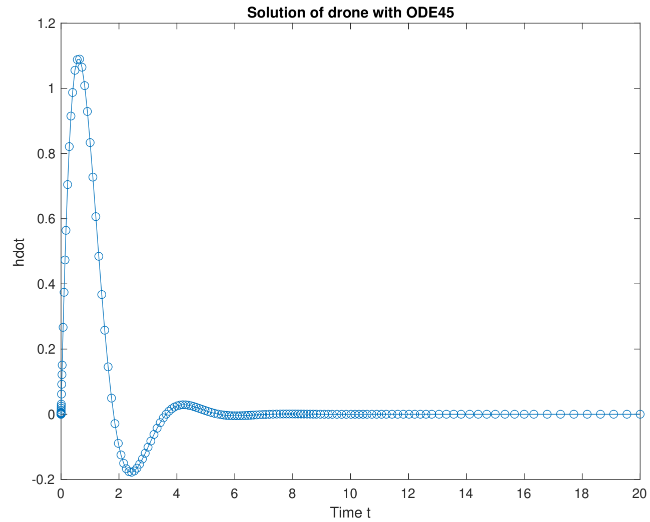


Figure 5: Response of an underdamped closed loop system (velocity) for $\zeta = 0.5$ and $\omega_n = 2$ with approximate settling time 3 sec.

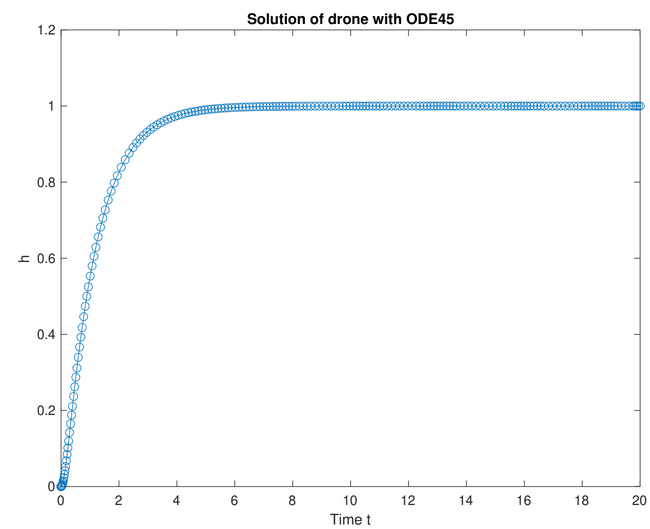


Figure 6: Response of an overdamped closed loop system (altitude) for $\zeta = 1.5$ and $\omega_n = 2.5$ with approximate settling time 3 sec.

(d) Due to the uncertainty in the actuators, this time, the response of the drone under a PD control will present a major steady-state error as shown in the following figures. Depending on the uncertainty, this steady-state error can be arbitrarily large. In order to diminish this error, we will need a PID control.

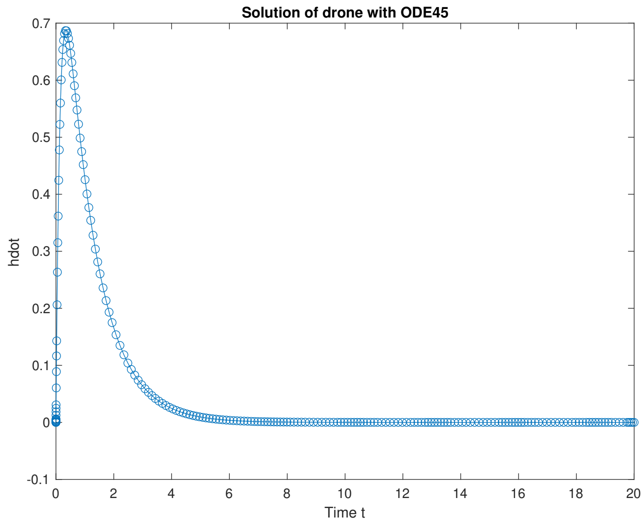


Figure 7: Response of an overdamped closed loop system (velocity) for $\zeta = 1.5$ and $\omega_n = 2.5$ with approximate settling time 3 sec.

Now, let's design the PID control using (48):

$$u = K_p(r-h) - K_d\dot{h} + K_i\int_0^t (r-h)dt + \frac{mg}{4k_T}. \tag{48}$$

Substitute (48) into (11),

$$\ddot{h} = \frac{4k_T(K_p(r-h) - K_d\dot{h} + K_i\int_0^t (r-h)dt + \frac{mg}{4k_T})}{m} - g, \qquad IC: h(0) = 0, \dot{h} = 0, \tag{49}$$

Let's define $y_1 = h$, $y_2 = \dot{h}$, $y_3 = \int_0^t (r - y_1)dt$. Then,

$$\begin{aligned} \dot{y}_1 &= y_2 && (50) \\ \dot{y}_2 &= \frac{4k_T}{m}(K_p(r-y_1) - K_d y_2 + K_i y_3) && (51) \\ \dot{y}_3 &= r - y_1 && (52) \\ & && (53) \end{aligned}$$

If we solve this system of equations, then the response under PID control is obtained. As seen from the figures, a non-zero K_i gain helps to clear out the steady state error.

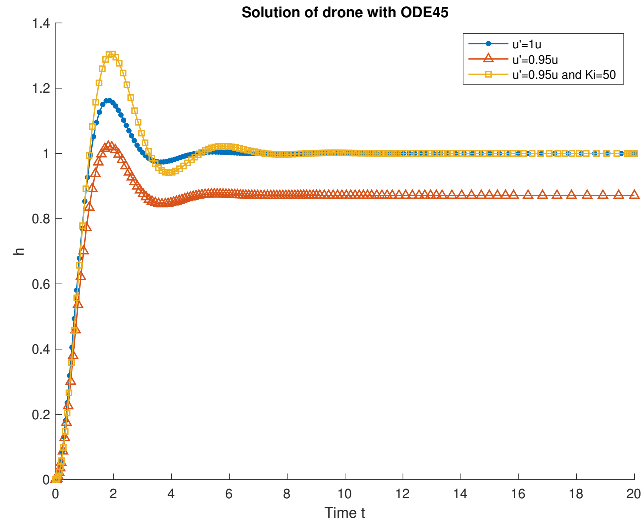


Figure 8: Response of an underdamped closed loop system (altitude) for $\zeta = 0.5$ and $\omega_n = 2$ with approximate settling time 3 sec.

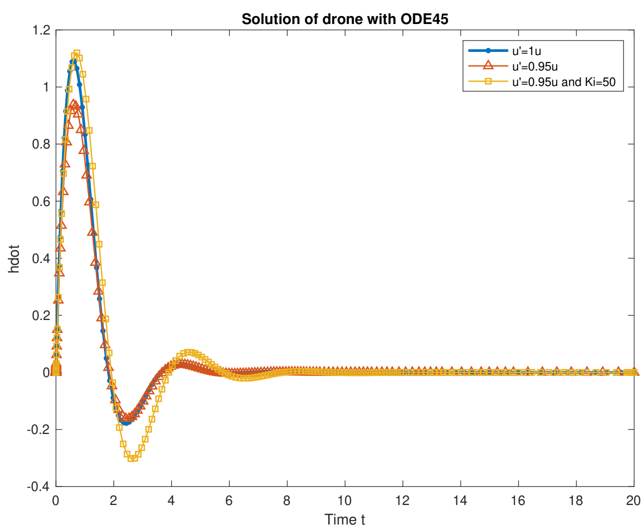


Figure 9: Response of an underdamped closed loop system (velocity) for $\zeta = 0.5$ and $\omega_n = 2$ with approximate settling time 3 sec.

Problem 3

(a) Solving (13) for $\ddot{\theta}$, we obtain:

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{\mu}{ml^2}\dot{\theta} + \frac{\tau}{ml^2}. \tag{54}$$

Letting $x \triangleq (\theta, \dot{\theta})$, it follows that:

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{\tau}{ml^2} \end{pmatrix}}_{\triangleq f(x,u)}. \tag{55}$$

(b) Substituting $x^* = (\pi, 0)$ into g, we obtain:

$$g(x^*) = f(x^*, 0) = \begin{pmatrix} 0 \\ -\frac{g}{l}\sin(\pi) - \frac{\mu}{ml^2} \cdot 0 + \frac{0}{ml^2}, \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{56}$$

which shows that x^* is a fixed point of the free system.

To prove that x^* is an *unstable* fixed point, we consider the linearization of the free system response $g(x)$ about x^*. The Jacobian of g at x^* is:

$$\frac{\partial g}{\partial x}(x) = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\mu}{ml^2} \end{pmatrix}. \tag{57}$$

Writing $A \triangleq \frac{\partial g}{\partial x}(x)$ for brevity, the characteristic polynomial of A is:

$$p_A(\lambda) = \det\big(\lambda I - A\big) = \lambda^2 + \frac{\mu}{ml^2}\lambda - \frac{g}{l}, \tag{58}$$

Applying the quadratic formula to (58), we find:

$$\lambda = \frac{-\frac{\mu}{ml^2} \pm \sqrt{\frac{\mu^2}{m^2l^4} + 4\frac{g}{l}}}{2}. \tag{59}$$

Observing that the quantity under the radical is always positive and strictly greater than $(\mu/ml^2)^2$, it follows from (59) that the eigenvalues of (59) are real, and

$$\lambda = \frac{-\frac{\mu}{ml^2} + \sqrt{\frac{\mu^2}{m^2l^4} + 4\frac{g}{l}}}{2} > 0 \tag{60}$$

is strictly positive. Therefore we conclude that $x^* = (\pi, 0)$ is an *unstable* fixed point.

(c) Substituting (16) into (55), we obtain:

$$\begin{aligned} c(x) = f(x, \tau(x)) &= \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{k_p\sin(x_1) + k_d x_2}{ml^2} \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ \left(\frac{k_p}{ml^2} - \frac{g}{l}\right)\sin(x_1) + \frac{k_d - \mu}{ml^2}x_2 \end{pmatrix}. \end{aligned} \tag{61}$$

(d) Substituting $x^* = (\pi, 0)$ into (61), we find:

$$c(\pi, 0) = \begin{pmatrix} 0 \\ \left(\frac{k_p}{ml^2} - \frac{g}{l}\right)\sin(\pi) + \frac{k_d - \mu}{ml^2} \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{62}$$

so that x^* is a stationary point of the closed-loop system. Therefore, we may study its local stability properties using linearized stability analysis. The Jacobian of c at x^* is:

$$\frac{\partial c}{\partial x}(x^*) = \begin{pmatrix} 0 & 1 \\ -(\frac{k_p}{ml^2} - \frac{g}{l}) & \frac{k_d - \mu}{ml^2} \end{pmatrix}. \tag{63}$$

Writing $A \triangleq \frac{\partial c}{\partial x}(x^*)$ for brevity, the characteristic polynomial of A is:

$$p_A(\lambda) = \lambda^2 - \left(\frac{k_d - \mu}{ml^2}\right)\lambda + \left(\frac{k_p}{ml^2} - \frac{g}{l}\right). \tag{64}$$

Applying the quadratic formula, the roots of p_A are thus:

$$\lambda = \frac{\left(\frac{k_d-\mu}{ml^2}\right) \pm \sqrt{\left(\frac{k_d-\mu}{ml^2}\right)^2 - 4\left(\frac{k_p}{ml^2} - \frac{g}{l}\right)}}{2}. \tag{65}$$

Now in order to guarantee that x^* is stable, *both* of the roots in (65) must have *strictly* negative real part. Observe that this requires:

$$k_d < \mu. \tag{66}$$

To see this, suppose (for contradiction) that (66) does *not* hold; i.e., that $k_d \geq \mu$. Then the numerator of the root:

$$\lambda_1 = \frac{\left(\frac{k_d-\mu}{ml^2}\right) + \sqrt{\left(\frac{k_d-\mu}{ml^2}\right)^2 - 4\left(\frac{k_p}{ml^2} - \frac{g}{l}\right)}}{2} \tag{67}$$

is the sum of a nonnegative real nonnegative number $(\frac{k_d-\mu}{ml^2})$ and a term that is either nonnegative real or purely imaginary (the radical); in either case, the sum of these terms will have a nonnegative real part, and therefore the linearization (63) will *not* be asymptotically stable.

On the other hand, if (66) holds, then the root:

$$\lambda_2 = \frac{\left(\frac{k_d-\mu}{ml^2}\right) - \sqrt{\left(\frac{k_d-\mu}{ml^2}\right)^2 - 4\left(\frac{k_p}{ml^2} - \frac{g}{l}\right)}}{2} \tag{68}$$

is guaranteed to have negative real part (by the same logic as in the previous paragraph). Therefore, it remains only to show that λ_1 also has strictly negative real part; i.e., that:

$$\left(\frac{k_d - \mu}{ml^2}\right) + \sqrt{\left(\frac{k_d - \mu}{ml^2}\right)^2 - 4\left(\frac{k_p}{ml^2} - \frac{g}{l}\right)} < 0. \tag{69}$$

Adding $-(\frac{k_d-\mu}{ml^2}) > 0$ to both sides of (69), squaring both sides, and simplifying, we obtain:

$$-4\left(\frac{k_p}{ml^2} - \frac{g}{l}\right) < 0, \tag{70}$$

or equivalently:

$$k_p > mgl. \tag{71}$$

In summary, in order for the linearized system (63) to be asymptotically stable, it is necessary and sufficient to have:

$$k_d < \mu, \quad k_p > mgl. \tag{72}$$

Finally, if (72) holds (so that the *linearized* system is asymptotically stable), the Poincare-Lyapunov Theorem then guarantees that the original (*nonlinear*) system is (locally) asymptotically stable about x^*.

(e) Direct calculation shows that:

$$\nabla V(x) = \begin{pmatrix} mgl\sin(x_1)\,(1+2\alpha\cos(x_1)) \\ ml^2 x_2 \end{pmatrix}, \tag{73a}$$

$$\nabla^2 V(x) = \begin{pmatrix} mgl\cos(x_1)\,(1+2\alpha\cos(x_1)) - 2\alpha\sin^2(\theta) & 0 \\ 0 & ml^2 \end{pmatrix}. \tag{73b}$$

Substituting $x^* = (\pi, 0)$ into (74), we find that:

$$\nabla V(x^*) = 0, \tag{74a}$$

$$\nabla^2 V(x^*) = \begin{pmatrix} mgl(2\alpha-1) & 0 \\ 0 & ml^2 \end{pmatrix}. \tag{74b}$$

Equation (74a) shows that x^* is a stationary point of V, and (74b) shows that the eigenvalues of the Hessian are:

$$\lambda_1 = ml^2, \quad \lambda_2 = mgl(2\alpha - 1). \tag{75}$$

Now recall from elementary calculus that $\nabla^2 V(x^*) \succ 0$ is sufficient to ensure that x^* is an isolated local minimizer. In light of (74b), this is equivalent to requiring that

$$\alpha > 1/2. \tag{76}$$

(f) Combining (55) with (73a), a direct computation shows that:

$$\begin{aligned} \frac{d}{dt}\left[V(x)\right] &= \nabla V(x)\cdot\dot{x} = \nabla V(x)\cdot f(x,u) \\ &= \begin{pmatrix} mgl\sin(x_1)(1+2\alpha\cos(x_1)) \\ ml^2 x_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{\tau}{ml^2} \end{pmatrix} \\ &= 2\alpha mgl\sin(x_1)\cos(x_1)x_2 - \mu x_2^2 + \tau x_2. \end{aligned} \tag{77}$$

(g) Substituting (20) into (55), we obtain:

$$\begin{aligned} c(x) \triangleq f(x,\tau(x)) &= \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{1}{ml^2}\left(-2\alpha mgl\sin(x_1)\cos(x_1)\right) \end{pmatrix} \\ &= \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1)(1+2\alpha\cos(x_1)) - \frac{\mu}{ml^2}x_2 \end{pmatrix}. \end{aligned} \tag{78}$$

(h) In light of (78), we must find all values of $x \in \mathbb{R}^2$ such that:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1)(1+2\alpha\cos(x_1)) - \frac{\mu}{ml^2}x_2 \end{pmatrix}. \tag{79}$$

It is immediate from the first row of (79) that we must have $x_2 = 0$, in which case the second row simplifies to:

$$-\frac{g}{l}\sin(x_1)(1+2\alpha\cos(x_1)) = 0. \tag{80}$$

This implies that either $\sin(x_1) = 0$ or $1 + 2\alpha\cos(x_1) = 0$. Note that $\sin(x_1) = 0$ if and only if $x_1 = k\pi$ for $k \in \mathbb{Z}$, which corresponds to the pendulum in the vertically upright or vertically hanging position, as expected. Alternatively, since $\alpha > 1/2$, then:

$$x_1 \in \cos^{-1}\left(-\frac{1}{2\alpha}\right) \tag{81}$$

also provides a pair of valid solutions on the unit circle, and we observe that these approach $\pi/2$ and $3\pi/2$ (i.e. corresponding to the pendulum in a horizontal position) as the control gain $\alpha \to \infty$.

In summary, the complete set of stationary points of the closed-loop system is given by:

$$S = \left\{ (x_1, 0) \in \mathbb{R}^2 \mid \sin(x_1) = 0 \text{ or } \cos(x_1) = -\frac{1}{2\alpha} \right\}. \tag{82}$$

(i) Substituting (20) into (77), we obtain:

$$\begin{aligned} \dot{V} &= 2\alpha mgl \sin(x_1)\cos(x_1)x_2 - \mu x_2^2 + \left[-2\alpha mgl \sin(x_1)\cos(x_1)\right] x_2 \\ &= -\mu x_x^2 \leq 0, \end{aligned} \tag{83}$$

so that V is negative semidefinite.

(j) It is immediate from (83) that:

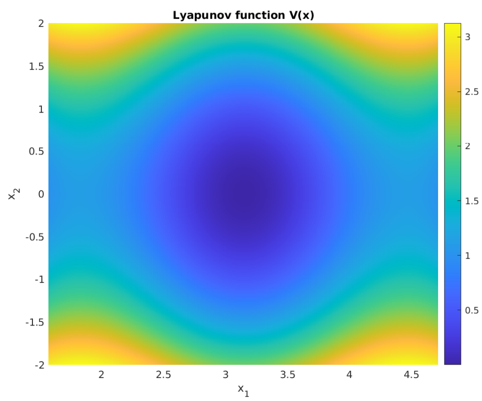
$$\mathcal{I} = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\}; \tag{84}$$

that is, $\mathcal{I}$ is simply the x_1-axis. Thus, in order for a trajectory $x(t; x_0)$ to be *entirely* contained in $\mathcal{I}$, it is necessary that $x_2(t; x_0) = 0$ and $\dot{x}_2(t; x_0) = 0$ for all t. But note that since $\dot{x}_1(t; x_0) = x_2(t; x_0)$ [by (55)], this also implies that $\dot{x}_1(t; x_0) = 0$ for all t. Together, these require:

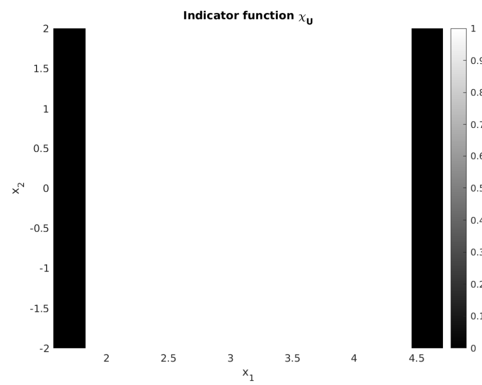
$$\dot{x}(t; x_0) = 0 \quad \forall t. \tag{85}$$

In other words, the only trajectories contained *entirely* within $\mathcal{I}$ are the stationary points S of the system.

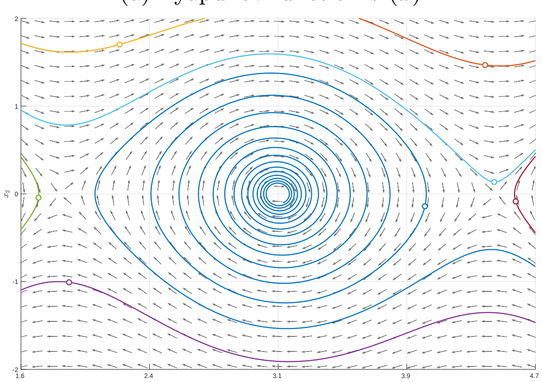
(k) See the below figures. Note that comparing Figs. 10c and 10d, we see that any trajectory $x(t; x_0)$ *starting* at a point x_0 whose contour is entirely contained in the region where V is a valid Lyapunov function *stays* inside that contour for all $t > 0$. That is, the sublevel sets bounded by the contours of the Lyapunov function shown in Fig. 10d are *invariant subsets* for the closed-loop nonlinear system.



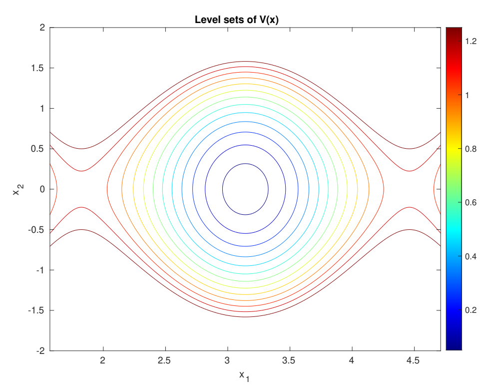
(a) Lyapunov function $V(x)$



(b) Indicator function $\chi_U(x)$.



(c) Phase portrait for the closed-loop system under the control law (20).



(d) Level sets of the Lyapunov function $V(x)$.