Q1. EXTENDED KALMAN FILTER.

Given
$$x_t = [P_x, V_x, P_y, V_y]$$

[P_x, V_x - Position + Velocity in x-axis
P_y, V_y - Position + Velocity in y-axis]

Landmarks:
$$l_i = [l_x^i, l_y^i]$$

a) Dynamical Model.

Robot travels with constant velocity

Process model: $x_{t+1} = g_t(x_t, u_t) + \epsilon_t$ $\quad \epsilon_t \sim N(0, R_t)$

for kalman filter.

$$x_{t+1} = A_t x_t + B_t u_t + \epsilon_t \Rightarrow A x_t + \epsilon_t$$

Since the robot travels with constant velocity, we can say that the next state $x^i_{t+1} = x^i_t + v^i_t \Delta t$

$$\therefore x_{t+1} = \begin{bmatrix} P_{t+1}[x] \\ V_{t+1}[x] \\ P_{t+1}[y] \\ V_{t+1}[y] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_t[x] \\ V_t[x] \\ P_t[y] \\ V_t[y] \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} 0 \\ \epsilon_x \\ 0 \\ \epsilon_y \end{bmatrix}}_{\epsilon_t} \quad [A \in R^{4\times 4}]$$

$$V^i_{t+1} = V^i_t + \epsilon^i_t \qquad \epsilon^i_t = \epsilon_i \ \forall t$$

$$\Rightarrow P^i_{t+1} = P^i_t + V^i_t + \epsilon^i_t$$

We can assume the the error ϵ belongs to a gaussian distribution with mean 0 + covariance R_t (ie) $\epsilon_t \sim N(0, R_t)$

b) Observation model.

$$Z_t = h_t(x_t) + \delta_t \qquad \delta_t \sim N(0, Q_t)$$

$$\Rightarrow Z_t = C_t x_t + \delta_t$$

We know that the observation made are the ~~eqbe~~ euclidean distance between the robot + land marks

Euclidean distance to land marks $z_l = \sqrt{[(l_x^i - P_x)^2 + (l_y^i - P_y)^2]}$

$$\Rightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sqrt{(l_1[x]-P[x])^2 + [l_1[y]-P[y]]^2} \\ \sqrt{(l_2[x]-P[x])^2 + [l_2[y]-P[y]]^2} \end{bmatrix} = h(x)$$

Let δ_t be some sensor noise.

$$= Z_t = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \delta_t \qquad \delta_t \sim N(0, Q_t)$$

c) Linearisation

$$H_t^1 = \frac{\partial h_t}{dx}(x) = \begin{bmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial y} & \frac{\partial z_1}{\partial v_x} & \frac{\partial z_1}{\partial v_y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial y} & \frac{\partial z_2}{\partial v_x} & \frac{\partial z_2}{\partial v_y} \end{bmatrix}$$

$$\frac{\partial z_1}{\partial x} = \frac{1}{2}\left[(l_1[x]-P[x])^2 + (l_1[y]-P[y])^2\right]^{-1/2} \cdot 2[l_1[x] - P[x]]$$

$$\frac{\partial z_1}{\partial y} = \frac{1}{2}\left[(l_1[x]-P[x])^2 + [l_1[y]-P[y]]^2\right]^{-1/2} \cdot 2[l_1[y]-P[y]]$$

$$\frac{\partial z_2}{\partial x} = \frac{1}{2}\left[(l_2[x]-P[x])^2 + (l_2[y]-P[y])^2\right]^{-1/2} \cdot 2[l_2[x]-P[x]]$$

$$\frac{\partial z_2}{\partial y} = \frac{1}{2}\left[(l_2[x]-P[x])^2 + (l_2[y]-P[y])^2\right]^{-1/2} \cdot 2[l_2[y]-P[y]]$$

$$H_t^1 = \begin{bmatrix} l_1[x]-P[x] & l_1[y]-P[y] \\ l_2[x]-P[x] & l_2[y]-P[y] \end{bmatrix} \frac{1}{\sqrt{(l_x^i - P_x)^2 + (l_y^i - P_y)^2}}$$

$$\left[\text{Let } \Delta = \sqrt{(l_x^i - P_x)^2 + (l_y^i - P_y)^2}\right]$$

$$\left[\text{Jacobian} = \begin{bmatrix} \frac{\partial z_1}{\partial x} & \frac{\partial z_1}{\partial v_x} & \frac{\partial z_1}{\partial y} & \frac{\partial z_1}{\partial v_y} \\ \frac{\partial z_2}{\partial x} & \frac{\partial z_2}{\partial v_x} & \frac{\partial z_2}{\partial y} & \frac{\partial z_2}{\partial v_y} \end{bmatrix} \right]$$

$\frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0$

$$\text{Jacobian} = \begin{bmatrix} l_1(x) - P_x & l_1(y) - P(y) & 0 & 0 \\ l_2(x) - P_x & l_2(y) - P(y) & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{(l_x^i - P_x)^2 + (l_y^i - P_y)^2}}$$

$\frac{1}{\Delta} \Leftarrow$

$$\text{Jacobian} \approx \frac{1}{\Delta} \begin{bmatrix} l_1(x) - P_x & 0 & l_1(y) - P(y) & 0 \\ l_2(x) - P_x & 0 & l_2(y) - P(y) & 0 \end{bmatrix}$$

d) Propogation model.

$x_t \sim N(\mu_t, \Sigma_t)$

$x_{t+1} \sim N(\mu_{t+1}, \Sigma_{t+1})$

$x_{t+1} = A_t \cdot x_t + \epsilon_t \qquad \epsilon_t \sim N(0, R_t)$

for Kalman Filter

$\mu_{t+1} = A\mu_t$

$\Sigma_{t+1} = A\Sigma_t A^{-1} + R_t \quad \Rightarrow x_{t+1} \sim N(A\mu_t, A\Sigma_t A^{-1} + R_t)$

for Extended Kalman Filter.

$\mu_{t+1} = A\mu_t$

$$\Sigma_t = \begin{bmatrix} \sigma_{Px}^2 & 0 & 0 & 0 \\ 0 & \sigma_{Vx}^2 & 0 & 0 \\ 0 & 0 & \sigma_{Py}^2 & 0 \\ 0 & 0 & 0 & \sigma_{Vy}^2 \end{bmatrix}$$

$\Sigma_{t+1} = G_t \Sigma_t G_t^T + R_t$

$$G_t = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} = A$$

$G_t = \frac{\partial g_t}{\partial x}(x, u)$

$$\Rightarrow x_{t+1} \sim N(A\mu_t, G_t \Sigma_t G_t^T + R_t)$$

$$\begin{array}{cc} \downarrow & \downarrow \\ \hat{\mu}_t & \hat{\Sigma}_t \\ \downarrow & \downarrow \\ \mu_{t+1} & \Sigma_{t+1} \end{array}$$

Update step.

Prior belief $x_t \sim N(\hat{\mu}_t, \hat{\Sigma}_t)$

$$Q_t = \begin{bmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_p^2 \end{bmatrix}$$

$z_t = C_t x_t + \delta_t$, $\delta_t \sim N(0, Q_t)$

$x_t | z_t \sim N(\mu_t, \Sigma_t)$

kalman gain.

$$K_t = \hat{\Sigma}_t H_t^T (H_t \hat{\Sigma}_t H_t^T + Q_t)^{-1}$$

$$\mu_t = \hat{\mu}_t + K_t (z_t - h_t(\hat{\mu}_t))$$

$$\Sigma_t = (I - K_t H_t) \hat{\Sigma}_t$$

$$x_t | z_t \sim N(\hat{\mu}_t + K_t(z_t - h_t(\hat{\mu}_t)), (I - K_t H_t)\hat{\Sigma}_t)$$

$$P(x_t | z_t) \propto P(\delta_t) P(x_t)$$

$$\propto \exp\left(\frac{-1}{2}(z_t - C_t x_t)^T Q_t^{-1} (z_t - C_t x_t)\right) \exp\left(\frac{-1}{2}(x_t - \hat{\mu}_t)^T \hat{\Sigma}_t^{-1} (x_t - \hat{\mu}_t)\right)$$

HW-4

Q.3. a) Given.

$\hat{\varphi}_l = \dot{\varphi}_l + \epsilon_l$, $\epsilon_l \sim \mathcal{N}(0, \sigma_l^2)$

$\hat{\varphi}_r = \dot{\varphi}_r + \epsilon_r$, $\epsilon_r \sim \mathcal{N}(0, \sigma_r^2)$

$x_{t_2} \in SE(2)$, σ_l, σ_r - Varience.

$x_{t_1} \in SE(2)$,

To find: Motion Model $p(x_{t_2} | x_{t_1}, \dot{\varphi}_l, \dot{\varphi}_r, r, \omega, \sigma_l, \sigma_r)$

We know that:

$$\gamma(t_2) = x(t_2) = x_{t_1} \exp(t_2 \, \dot{\Omega}(\hat{\varphi}_l, \hat{\varphi}_r)) \quad -①$$

$$x_{t_2} = x_{t_1} \exp\left[t_2 \cdot \begin{bmatrix} 0 & \frac{-r}{\omega}(\hat{\varphi}_r - \hat{\varphi}_l) & \frac{r}{2}(\hat{\varphi}_r + \hat{\varphi}_l) \\ \frac{r}{\omega}(\hat{\varphi}_r - \hat{\varphi}_l) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

(ie) $$\dot{\Omega}(\hat{\varphi}_l, \hat{\varphi}_r) = \begin{bmatrix} 0 & \frac{r}{\omega}(\hat{\varphi}_r - \hat{\varphi}_l) & \frac{r}{2}(\hat{\varphi}_r + \hat{\varphi}_l) \\ \frac{r}{\omega}(\hat{\varphi}_r - \hat{\varphi}_l) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

① is the motion model. ① $\sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$

b) Given.

$z_t = l_t + \epsilon_p$, $\epsilon_p \sim \mathcal{N}(0, \sigma_p^2 I_2)$ [let $\Sigma = \sigma_p^2 I_2$]

$\Rightarrow \epsilon_p = z_t - l_t$.

To find: The liklihood function $p(z_t | x_t)$

$$P(z_t | x_t) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(\frac{-1}{2} (z_t - l_t)^T \Sigma^{-1} (z_t - l_t)\right) \quad -②$$