LAB-1.

1. OBJECT POSE ESTIMATION!

Given:

$$OP_1 = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}, \quad OP_2 = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}, \quad OP_3 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, \quad OP_4 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$$SP_1 = \begin{pmatrix} -1.3840 \\ 4.5620 \\ -0.1280 \end{pmatrix}, \quad SP_2 = \begin{pmatrix} -0.9608 \\ 1.3110 \\ -1.6280 \end{pmatrix}, \quad SP_3 = \begin{pmatrix} 1.3250 \\ -2.3890 \\ 1.7020 \end{pmatrix}, \quad SP_4 = \begin{pmatrix} -1.3140 \\ 0.2501 \\ -0.7620 \end{pmatrix}$$

To find: Pose $T_{SO} \in SE(3)$ of O wrt S.

WKT

$$S_P = T_{SO}\, O_P$$

$$\Rightarrow T_{SO} = S_P\, O_P^{-1}$$

$$O_P = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 3 & 0 & -2 & 0 \\ -3 & -3 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_P = \begin{bmatrix} -1.384 & -0.9608 & 1.325 & -1.314 \\ 4.562 & 1.311 & -2.389 & 0.2501 \\ -0.1280 & -1.628 & 1.702 & -0.762 \end{bmatrix}$$

$$O_P^{-1} = \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -3 & -3 & 2 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & -6 & 1 & -7 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$3R_1 - 2R_2$

$3R_1 + 2R_2$

$$\sim \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -6 & 1 & -7 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & -4 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & -4 & 4 & -2 & 0 & 0 \\ 0 & -6 & 0 & -4 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2/3 & 0 & -1/3 & -1/3 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & \cancel{2/3} & 0 & -1/3 & -1/3 & 0 \\ 0 & 0 & 1 & -3 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & \cancel{-2/3}\ 0 & -1/3 & -1/3 & -2/3 \\ 0 & 0 & 1 & 0 & 3 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$OP^{-1} = \begin{bmatrix} 2 & -1 & 0 & 2 \\ \cancel{-2/3}\ 0 & -1/3 & -1/3 & -2/3 \\ 3 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{SO} = \begin{bmatrix} -1.384 & -0.9608 & 1.325 & -1.314 \\ 4.562 & 1.3110 & -2.389 & 0.2501 \\ -0.128 & -1.628 & 1.702 & -0.762 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 2 \\ 0 & -1/3 & -1/3 & -2/3 \\ 3 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.207 & -0.94574 & +0.32027 & 0.5335 \\ 1.957 & -0.221 & -0.437 & 1.3331 \\ 4.85 & -2.734 & 0.5427 & 5.1734 \\ & & & \end{bmatrix}$$

2. Given:

G is a lie group with group operation $*$.

$L_g : G \to G$ $\qquad \varphi : T_e(G) \to Lie(G)$.

$L_g(x) \triangleq g * x$ $\qquad \varphi(\omega) = V_\omega$

$V_\omega(x) \triangleq d(L_x)_e(\omega)$

$V \in \mathbb{R}^n$

a) The left translation $L_v : \mathbb{R}^n \to \mathbb{R}^n$.

$$L_v(x) = V * x = v + x \qquad x \in \mathbb{R}^n \qquad [* \text{- addition}]$$

if $v = (v_1, v_2 \ldots v_n)$ & $x = (x_1, x_2, \ldots x_n)$

$$L_v(x) = (v_1, v_2, \ldots v_n) + (x_1, x_2, \ldots x_n).$$

b) $dL_v(x) = \frac{\partial}{\partial x} L_v(x) \; \frac{d L_v(x)}{dX}$

$$= \left[\frac{\partial L_v(x)}{\partial x_1} \quad \frac{\partial L_v(x)}{\partial x_2} \quad \cdots\cdots \quad \frac{\partial L_v(x)}{\partial x_n} \right]$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & & & 1 \end{pmatrix}_{n \times n}$$

c) $V_\omega(x) = d(L_x)_e(\omega)$

$\Rightarrow V_\xi = dL_v(x)\,\xi$

2. d) $A \in GL(n)$

$L_A : GL(n) \to GL(n)$.

[* - matrix multiplication].

$L_A(x) = A * x$

$= A x$

e) $dL_A = \frac{dL_A}{dx} = \frac{d(Ax)}{dx}$

f) $V_\Omega = d(L_A)_I \, \Omega$

$= dA I \, \Omega \in \mathbb{R}^{n \times n}$

3. Given:

$$\exp: R^{n \times n} \to GL(n)$$

$$\exp(X) = \sum_{k=0}^{\infty} X^k / k!$$

$$\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

a) Find expression for k^{th} power of Ω^k.

$$\Omega^2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$$

$$\Omega^3 = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}$$

$$\Omega^4 = \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix}$$

$$\Omega^5 = \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^5 \\ -\omega^5 & 0 \end{pmatrix} \quad [\Rightarrow \text{Sign similar to } \omega^1]$$

$$\Omega^6 = \begin{pmatrix} 0 & \omega^5 \\ -\omega^5 & 0 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^6 & 0 \\ 0 & -\omega^6 \end{pmatrix} \quad [= \text{Similar to } \omega^2]$$

This implies.

$$\Omega^{2n} = \begin{pmatrix} (-1)^n \times \omega^{2n} & 0 \\ 0 & (-1)^n \times \omega^{2n} \end{pmatrix} \qquad \Omega^{4n+1} = \begin{pmatrix} 0 & \omega^{4n+1} \\ -\omega^{4n+1} & 0 \end{pmatrix}$$

$$\Omega^{4n+3} = \begin{pmatrix} 0 & -\omega^{4n+3} \\ \omega^{4n+3} & 0 \end{pmatrix} \qquad [n \in \mathbb{R}]$$

b) $\exp(\Omega) = \sum_{k=0}^{\infty} \Omega^k / k!$

$$= \sum_{\substack{k=2n \\ (\text{even})}}^{\infty} \frac{\Omega^k}{k!} + \sum_{k=4n+1}^{\infty} \frac{\Omega^k}{k!} + \sum_{k=4n+3}^{\infty} \frac{\Omega^k}{k!}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n}}{(2n)!} & 0 \\ 0 & \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n}}{(2n)!} \end{pmatrix} + \begin{pmatrix} 0 & \sum_{n=0}^{\infty} \frac{\omega^{4n+1}}{(4n+1)!} \\ -\sum \frac{\omega^{4n+1}}{(4n+1)!} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sum_{n=0}^{\infty} \frac{\omega^{4n+3}}{(4n+3)!} \\ +\sum_{n=0}^{\infty} \frac{\omega^{4n+3}}{(4n+3)!} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & 0 \\ 0 & \cos\omega \end{pmatrix} + \begin{pmatrix} 0 & \sum_{m=0}^{\infty} (-1)^m \frac{\omega^{2m+1}}{(2m+1)!} \\ -\sum_{m=0}^{\infty} (-1)^m \frac{\omega^{2m+1}}{(2m+1)!} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & 0 \\ 0 & \cos\omega \end{pmatrix} + \begin{pmatrix} 0 & \sin\omega \\ -\sin\omega & 0 \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$$

$\exp(\Omega)$ geometrically represents a rotation along an axis 'Z' by an angle ω, in $\mathbb{R}^2$.

4. Given:

$$\gamma(t) = x \star \exp(t\omega)$$

a) $x = \exp(\log(x))$

$\gamma(0) = x$ where $t=0$

$\gamma(1) = y$ where $t=1$

① $t=0$

$$\gamma(t) = x \star \exp(0)$$

$$x = x \star \exp(0)$$

② $t=1$

$$\gamma(1) = x \star \exp(\omega)$$

$$y = x \star \exp(\omega)$$

$$x^{-1}y = \exp(\omega)$$

$$\Rightarrow \quad \omega = \log(x^{-1}y)$$

$$\Rightarrow \quad \cancel{\gamma(t) = x \star \exp(\log} \quad \gamma(t) = x \star \exp(t \log(x^{-1}y)) \quad - \text{Ⓐ}$$

b) $\exp : \mathbb{R}^n \to \mathbb{R}^n$

$$\exp(\xi) = \xi.$$

A becomes

$$\gamma(t) = x \star (t \log(x^{-1}y)) \qquad [\star - \text{Addition for } \mathbb{R}^n]$$

$$\Rightarrow \quad \gamma(t) = x + t\log(x^{-1}y) \quad - \text{Ⓑ}$$

c) $(t_1, R_1) * (t_2, R_2) = (R_1 t_2 + t_1, R_1 R_2)$ — ©

$$\begin{array}{cc} t_1 & R_1 \end{array}$$

$$x_0 = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix} \right)$$

$$\begin{array}{cc} t_2 & R_2 \end{array}$$

$$x_1 = \left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.7500 & -0.4874 & 0.6597 \\ 0.4330 & 0.7891 & -0.4356 \\ -0.5000 & 0.6124 & 0.6124 \end{pmatrix} \right)$$

© can be written as

$$\begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t_2 & R_2 \\ 1 & 0 \end{bmatrix} = \cancel{(R_1 R_2, R_1 t_2 + t_1)} = (R_1 t_2 + t_1, R_1 R_2).$$

$$\begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix}$$

from Ⓐ

$$\gamma(t) = x * \exp(\tfrac{t}{2} \log x^{-1} y)$$

$$x = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \quad y = \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix}$$

$$\gamma(t) = x_0 * \exp(t \log x_0^{-1} x_1)$$

$t = 1/2$

$$\gamma(1/2) = x_0 * \exp(\tfrac{1}{2} \log x_0^{-1} x_1)$$

$$x_0^{-1} = \begin{bmatrix} 0.4331 & 0.2500 & -0.866 & -0.6831 \\ 0.1767 & 0.9185 & 0.3535 & -1.0953 \\ 0.8838 & -0.3062 & 0.3535 & -0.5776 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_0^{-1} x = \begin{bmatrix} 0.8661 & -0.3536 & -0.3535 & -1.4149 \\ 0.3535 & 0.9329 & -0.067 & 3.9926 \\ 0.3535 & -0.0670 & 0.9329 & 1.0257 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\exp(1/2\log(x_0^{-1}x_1)) = \begin{bmatrix} 0.9659 & -0.183 & -0.183 & -0.4738 \\ 0.1829 & 0.9829 & -0.017 & 2.0622 \\ 0.1829 & -0.0170 & 0.9829 & 0.5787 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Y$$

$$\gamma(1/2) = x_0 * Y$$

$$= \begin{bmatrix} 0.6122 & 0.0795 & 0.7865 & 1.6709 \\ 0.3535 & 0.8623 & -0.3623 & 2.5986 \\ -0.7071 & 0.5000 & 0.5000 & 1.3441 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \gamma_{SE(3)}$$

R_3 t_3

d) $(t_1, R_1) * (t_2, R_2) = (t_1 + t_2, R_1R_2)$

from Ⓐ

$$\gamma(t) = x * \exp(\tfrac{1}{2}\log x^{-1}y)$$

$x = x_0$; $Y = \exp(\frac{1}{2}\log(x_0^{-1}x))$ => from Ⓒ

$$\gamma(1/2) = \left[\underbrace{\begin{pmatrix} 0.6122 & 0.0795 & 0.7865 \\ 0.3535 & 0.8623 & -0.3623 \\ -0.707 & 0.5000 & 0.5000 \end{pmatrix}}_{R_1R_2}, \underbrace{\begin{pmatrix} 0.5262 \\ 3.0622 \\ 0.5787 \end{pmatrix}}_{t_1+t_2}\right] = \gamma_P$$

e) Plot translation components of $\gamma_{SE(3)}$ & γ_P over

i) $t \in [0, 1]$

ii) $t \in [0, 30]$

during interval ⓘ => plot shows that the robot is gradually in x, y, z axis

during interval ⓘⓘ => plot reveals that the trajectory (or) translation of the robot follows a screw (or) helix motion.

Both $\gamma_{SE(3)}$ & γ_P are similar plots.