

EECE 5550 Mobile Robotics Lab #2

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Problem 1: Differential drive kinematics on SE(2)

Consider a differential-drive robot with track width w and wheel radius r (Fig. 1).

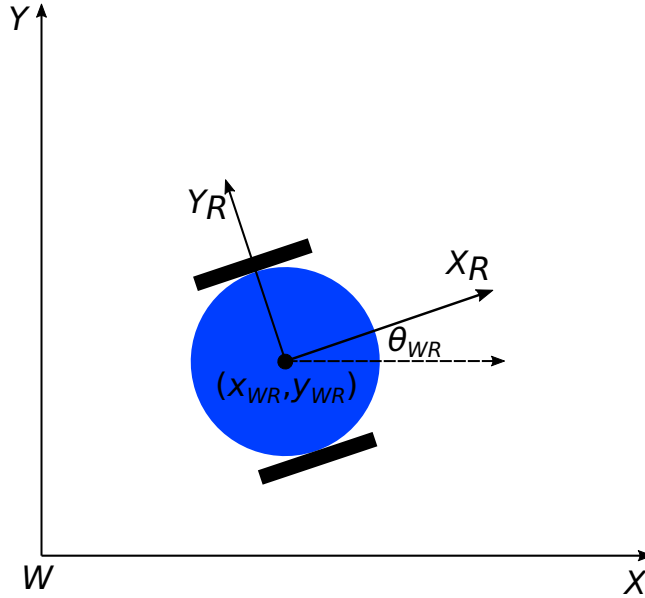


Figure 1: Schematic of a differential drive robot

Let $t_{WR} \triangleq (x_{WR}, y_{WR}) \in \mathbb{R}^2$ and $\theta_{WR} \in \mathbb{R}$ denote the position and orientation angle of the robot in the world coordinate frame W , as shown in Fig. 1. We showed in class that the equations of motion for this vehicle are:

$$\begin{pmatrix} \dot{x}_{WR} \\ \dot{y}_{WR} \\ \dot{\theta}_{WR} \end{pmatrix} = \begin{pmatrix} R(\theta_{WR}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{r}{2}(\dot{\varphi}_r + \dot{\varphi}_l) \\ 0 \\ \frac{r}{w}(\dot{\varphi}_r - \dot{\varphi}_l) \end{pmatrix}, \quad (1)$$

where $\dot{\varphi}_l$ and $\dot{\varphi}_r$ are the angular speeds of the left and right wheels, respectively, with positive values corresponding to forward motion (that is, motion along the robot's body-centric positive x -axis, $+x_R$), and

$$R: \mathbb{R} \rightarrow \text{SO}(2)$$

$$R(\theta) \triangleq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad (2)$$

is the mapping that assigns to each *angle* θ the corresponding *rotation matrix* $R(\theta)$.

In this exercise, we will reformulate the equations of motion (1) on the Lie group $\text{SE}(2)$:

$$\text{SE}(2) \cong \left\{ \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \mid R \in \text{SO}(2), t \in \mathbb{R}^2 \right\}, \quad (3)$$

and then apply Lie group theory to perform *forward simulation* of the vehicle's trajectory.

(a) Let's begin by defining the mapping:

$$\begin{aligned} \Psi: \mathbb{R}^2 \times \mathbb{R} &\rightarrow \text{SE}(2) \\ \Psi(t, \theta) &\triangleq \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (4)$$

that sends each pair (t, θ) to the element of $\text{SE}(2)$ obtained by replacing the *angle* θ by its corresponding *rotation matrix* $R(\theta)$.

Notice that

$$\Psi(0, 0) = \begin{pmatrix} R(0) & 0 \\ 0 & 1 \end{pmatrix} = I; \quad (5)$$

that is, the map Ψ sends the vector $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$ to the identity $I \in \text{SE}(2)$. This means that the *derivative map*:

$$d\Psi_{(0,0)}: T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}) \rightarrow \text{Lie}(\text{SE}(2)) \quad (6)$$

sends each tangent vector $(\dot{t}, \dot{\theta}) \in T_{(0,0)}(\mathbb{R}^2 \times \mathbb{R}) \cong \mathbb{R}^3$ to a tangent vector in $T_I(\text{SE}(2)) \cong \text{Lie}(\text{SE}(2))$, the *Lie algebra* of $\text{SE}(2)$. Derive a closed-form expression for the derivative map $d\Psi_{(0,0)}$ in (6).

(b) The differential drive kinematic equation (1) describes how the left and right wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ determine the robot's velocity $(\dot{t}, \dot{\theta}) \in T_{(t,\theta)}(\mathbb{R}^2 \times \mathbb{R})$. Suppose that the robot is at the origin $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$. Using the result of part (a), derive the corresponding mapping:

$$\dot{\Omega}: \mathbb{R}^2 \rightarrow \text{Lie}(\text{SE}(2)) \quad (7)$$

that sends the pair of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r)$ to the robot's velocity $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$ in the Lie group $\text{SE}(2)$ at I .

(c) The mapping $\dot{\Omega}$ from wheel speeds to velocities that you derived in part (b) was obtained under the assumption that the robot's pose in the world frame is $T_{WR} = I$.

Suppose now that the robot's pose in the world frame is $T_{WR} = X \in \text{SE}(2)$. Derive a closed-form expression for the map:

$$V: \text{SE}(2) \times \mathbb{R}^2 \rightarrow T(\text{SE}(2)) \quad (8)$$

that accepts as input the robot's pose $X \in \text{SE}(2)$ and wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ and returns its velocity $V(X, \dot{\varphi}_l, \dot{\varphi}_r) \in T_X(\text{SE}(2))$. You may write your answer in terms of the Lie algebra element $\dot{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$.

[Hint: you may find it convenient to introduce an auxiliary coordinate frame F that is *fixed* with respect to the world frame W , and is aligned with the robot's body-centric frame – i.e., frame F is *defined* so that the pose of the robot in frame F is $T_{FR} = I$. What is the velocity of the robot with respect to frame F ? How are the velocities of the robot in frames F and W related?]

- (d) Suppose that we *fix a specific choice* of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$. Given this choice, the map V that you derived in part (c) simplifies to a function $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X)$ that assigns to each pose $X \in \text{SE}(2)$ the velocity:

$$V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X) \triangleq V(X, \dot{\varphi}_l, \dot{\varphi}_r) \in T_X(\text{SE}(2)). \quad (9)$$

In other words: each choice of wheel speeds $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$ determines a *vector field* $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}$ on $\text{SE}(2)$ that accepts as input a robot pose $X \in \text{SE}(2)$, and returns the robot velocity $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}(X)$ determined by the selected wheel speeds at X .

Prove that $V_{(\dot{\varphi}_l, \dot{\varphi}_r)}$ is in fact a *left-invariant* vector field. [Hint: You may find Problem 2 from Lab #1 useful here.]

- (e) **Forward kinematics:** Suppose that the robot is at pose $X_0 \in \text{SE}(2)$ at time $t = 0$, and that we drive its wheels at constant velocity $(\dot{\varphi}_l, \dot{\varphi}_r) \in \mathbb{R}^2$. Given this initial data, write down a closed-form formula for the curve:

$$\gamma: \mathbb{R} \rightarrow \text{SE}(2) \quad (10)$$

that reports the robot's pose $\gamma(t)$ at time t . You may express your answer in terms of $\hat{\Omega}(\dot{\varphi}_l, \dot{\varphi}_r)$.

Question 2: PID altitude control

Consider the following second order dynamical system that models the altitude dynamics of a drone:

$$\ddot{h} = \frac{4k_T u}{m} - g, \quad IC: h(0) = 0, \dot{h} = 0, \quad (11)$$

where $m = 65g$ is the mass of the drone, $k_T = 5.276 \times 10^{-4}$ is the thrust coefficient, $g = 9.81 \text{ m/s}^2$ is the gravitational constant, $h \in \mathbb{R}$ is the altitude, and $u \in \mathbb{R}$ is the control input. Let u be designed as

$$u = PID + \frac{mg}{4k_T}, \quad (12)$$

where PID is the proportional-integral-derivative control architecture, and the last element is for the perfect gravity cancellation as we discussed in the class. The objective of the drone is to reach a reference altitude of $r = 1\text{m}$ and hover there.

- Design a P control using (2) for values of $K_p = 5, 15, 50$, plot h and \dot{h} , discuss the results.
- Design a PD control using (2) such that the closed-loop system is underdamped and the settling time is approximately 3 seconds, plot h and \dot{h} , discuss the results and justify why the system is underdamped.
- Design a PD control using (2) such that the closed-loop system is overdamped and the settling time is approximately 3 seconds, plot h and \dot{h} , discuss the results and justify why the system is overdamped.
- In this part, you will consider some uncertainty in the actuators so the actual control applied to the system will be $u' = 0.95u$. First, obtain the results of part (b) by considering u' . Then design a PID control using the same K_p, K_d gains as in (b) and some nonzero K_i gain (again considering u'). Plot h and \dot{h} , and discuss the differences between the responses obtained by the PD and PID controls.

Problem 3: Nonlinear feedback and stability analysis

In this problem you will devise a feedback controller to stabilize a damped driven pendulum in the upright position. Recall the equations of motion for a damped pendulum driven by an external torque τ :

$$ml^2\ddot{\theta} = -mgl \sin(\theta) - \mu\dot{\theta} + \tau, \quad (13)$$

where m is the mass of the bob, l is the length of the pendulum, μ is the damping parameter (due to friction), g is the gravitational acceleration, and τ is the external torque applied to the pendulum's pivot.

- (a) Defining $x = (\theta, \dot{\theta}) \in \mathbb{R}^2$, rewrite the second-order equations of motion (13) in the form of a *first-order* differential system for x :

$$\dot{x} = f(x, \tau). \quad (14)$$

Note that your function $f: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$ should express the derivative \dot{x} as a function of *both* the state x *and* the external torque $\tau \in \mathbb{R}$.

- (b) Let

$$\begin{aligned} g: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ g(x) &\triangleq f(x, 0) \end{aligned} \quad (15)$$

denote the equations of motion for the *free* system (i.e., the dynamics of the system when *no external torque* is applied). Show that the upright configuration $x^* = (\pi, 0)$ is an *unstable* stationary point for the free system.

Linear control synthesis: In the next part of this exercise, you will apply linearized stability analysis to devise a PD controller to stabilize the pendulum in the upright configuration.

- (c) We will assume a PD controller of the form:

$$\begin{aligned} \tau: \mathbb{R}^2 &\rightarrow \mathbb{R} \\ \tau(x) &\triangleq k_p \sin(x_1) + k_d x_2, \end{aligned} \quad (16)$$

where $k_p, k_d \in \mathbb{R}$ are controller gains (to be determined). Using the control law (16), derive an explicit expression for the *closed-loop* dynamics of the system in the form of an *autonomous* ODE:

$$\dot{x} = c(x) \triangleq f(x, \tau(x)). \quad (17)$$

- (d) Using your result in part (c), derive conditions on the controller gains k_p, k_d that are sufficient to guarantee that $x^* = (\pi, 0)$ is a (locally) asymptotically stable stationary point of the closed-loop system. [Hint: Show that x^* is a stationary point for the closed-loop system, and then apply linearized stability analysis to derive sufficient conditions on the controller gains.]

Controller design via linearized stability analysis [as you did in part (d)] is convenient in that it provides an easy method of constructing *locally* asymptotically stabilizing controllers using only a bit of linear algebra. However, because this approach is based upon (local) *linearization*, it doesn't directly provide any information about the *size* of the neighborhood around x^* over which the resulting controller works.

Nonlinear control synthesis: As an alternative approach, in the remainder of this question, you will apply the theory of Lyapunov functions to devise a *nonlinear* feedback controller whose *invariant subsets* we can explicitly characterize.

(e) Consider a candidate Lyapunov function of the form:

$$V: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$V(x) \triangleq -mgl(1 + \cos(x_1)) + \alpha mgl(1 - \cos^2(x_1)) + \frac{1}{2}ml^2x_2^2 \quad (18)$$

where $\alpha \in \mathbb{R}$ is a free parameter (to be determined). Calculate the gradient and Hessian of V , and derive a sufficient condition on α for the point $V(x^*) = 0$ to be an isolated local minimizer of V .

(f) Your results in part (e) show that there exists some neighborhood $U \in \mathbb{R}^2$ containing x^* such that $V(x) \geq 0$ for all $x \in U$, and x^* is the *unique* point in U satisfying $V(x^*) = 0$. In order to show that V is a valid Lyapunov function, we must identify a control law such that V is nonincreasing along the trajectories of the system. To that end, derive a closed-form expression for the time derivative \dot{V} of V :

$$\dot{V} = \frac{d}{dt}[V(x)] = \nabla V(x) \cdot f(x, \tau). \quad (19)$$

You may leave your answer in terms of the state x and the control τ (to be determined).

The result of part (f) suggests that we consider the following nonlinear feedback controller:

$$\tau: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\tau(x) \triangleq -2\alpha mgl \sin(x_1) \cos(x_1). \quad (20)$$

Our goal now will be to apply Lyapunov theory to show that this indeed stabilizes the upright position $x^* = (\pi, 0)$, and to characterize the *invariant subsets* around x^* .

(g) Using (20), derive an explicit expression for the closed-loop dynamics of the system:

$$\dot{x} = c(x) \triangleq f(x, \tau(x)) \quad (21)$$

under the control law (20).

(h) Using your result in part (g), find the set S of stationary points for the closed-loop system under the control law (20).

(i) Show that under the control law (20), the function V satisfies:

$$\dot{V} \leq 0. \quad (22)$$

The results of (e)–(i) show that the $x^* = (\pi, 0)$ is a stationary point of the closed-loop system under the nonlinear control law (20), and that V is a valid Lyapunov function for the closed-loop system in a neighborhood of x^* . We may therefore conclude that x^* is *stable in the sense of Lyapunov* for the closed-loop system. However, your result in (i) shows that \dot{V} is only negative *semidefinite* ($\dot{V} \leq 0$), but **not** negative *definite* ($\dot{V} \not\leq 0$). Therefore, we cannot conclude that x^* is *asymptotically* stable. To prove the (stronger) statement that x^* is *asymptotically* stable, we can apply the [LaSalle invariance principle](#).¹

¹**A word on the LaSalle invariance principle:** We saw in class that we can intuitively think of a Lyapunov function as measuring the “energy” of a system. If we can show that $\dot{V} \leq 0$ – that is, that the energy is *nonincreasing* along trajectories $x(t; x_0)$ of the system – then we know that the sublevel sets $L_c^-(V)$ of V must be *invariant* sets for our system. This is enough to show *stability in the sense of Lyapunov*, since any trajectory

- (j) Derive a closed-form expression for the set \mathcal{I} of points at which the time derivative of V is zero:

$$\mathcal{I} \triangleq \{x \in \mathbb{R}^2 \mid \dot{V}(x) = 0\}, \quad (23)$$

and argue that the only trajectories $x(t; x_0)$ contained *entirely* within this set are the stationary points S you found in part (h).

Altogether, your results show that there is a neighborhood U containing $x^* = (\pi, 0)$ on which V is a valid Lyapunov function for the closed-loop system under (20), and the only trajectory that is *entirely* contained in $\mathcal{I} \cap U$ is the stationary point $x^* = (\pi, 0)$. This proves that x^* is *asymptotically* stable.

As we saw in class, one of the primary advantages of the Lyapunov approach is that the sublevel sets:

$$L_c^-(V) \triangleq \{x \in \mathbb{R}^n \mid V(x) \leq c\} \quad (24)$$

of the Lyapunov function provide a great deal of information about the trajectories of the closed-loop system. In particular, we saw that any sublevel set $L_c^-(V)$ that is contained in a neighborhood U on which V is a valid Lyapunov function is an *invariant* set for the closed-loop system.

Let us consider the following choice for U :

$$U \triangleq \left\{x \in \mathbb{R}^2 \mid \cos(x_1) < -\frac{1}{2\alpha}, V(x) \geq 0, \dot{V}(x) \leq 0\right\}. \quad (25)$$

(note that here the restriction on the cosine of x_1 ensures that $x^* = (\pi, 0)$ is the *unique* stationary point contained in U).

- (k) Assuming the values $m = g = l = 1$, $\mu = .1$, and $\alpha = 2$, plot the following data over the portion of the phase plane satisfying $\frac{\pi}{2} \leq x_1 \leq \frac{3\pi}{2}$ and $-2 \leq x_2 \leq 2$:

- the Lyapunov function $V(x)$;
- the indicator function χ_U for the set U defined in (25);
- the phase portrait for the closed-loop dynamics under the control law (20);
- the level sets $V^{-1}(c)$ of the Lyapunov function $V(x)$ for values of c between .05 and 1.25 (inclusive), at increments of .1.

Submit these plots together with the code you used to produce them.

that *starts* in $L_c^-(V)$ will *stay* in $L_c^-(V)$.

However, we also saw that $\dot{V} \leq 0$ is not enough to prove *asymptotic stability*. The reason is that there may be a subset $\mathcal{I} \triangleq \{x \in U \mid \dot{V}(x) = 0\}$ of U in which the energy of the system is (locally) *constant* along trajectories. (For example, this is the case for the pendulum *without* friction – this system is conservative, and so in fact the energy is constant along *every* trajectory.) Any trajectory that is contained *entirely* in \mathcal{I} will have *constant* energy, and so need not “decay” over time to the stationary point x^* .

The *LaSalle invariance principle* is an *additional* criterion that we can use to prove that a fixed point x^* is locally *asymptotically* stable, even if our Lyapunov function only satisfies $\dot{V} \leq 0$. In brief, what we need to show is that *the only trajectory contained entirely within \mathcal{I} is just the constant trajectory $x(t) = x^*$ at the fixed point x^* itself*. Geometrically, what this means is that *any nonstationary trajectory that enters \mathcal{I} must eventually leave it*. In that case, even though the energy of the trajectory $x(t; x_0)$ does not decrease *while it is inside \mathcal{I}* , if it is nonstationary, then eventually it *will* leave \mathcal{I} , and then its energy *will* decrease again.

In summary: the LaSalle invariance principle states that if V is a Lyapunov function **and** the only trajectory $x(t; x_0)$ contained *entirely* in the subset $\mathcal{I} = \{x \in U \mid \dot{V}(x) = 0\}$ on which V is locally constant is the stationary trajectory $x^*(t) = x^*$, **then** x^* is in fact *asymptotically stable*.