Pricing Perpetual Barrier Options on Moderna, Pfizer, and Johnson & Johnson Stocks Using Microsoft Excel

by

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Abstract

The COVID-19 pandemic has resulted in millions of casualties worldwide, global shutdowns, and the advent of a "virtual reality". To hinder the continued viral spread and eliminate casualties, many pharmaceutical companies have developed and are in the process of distributing vaccines against this virus worldwide, returning the globe to a "normal" way of life. Most of these companies are publicly traded, leading to a rare opportunity for investors who are looking to expand and diversify their portfolios. This project looks at the process and equations underlying investment decisions, as they relate to well-known American biotechnology companies Pfizer Inc., Moderna Inc., and Johnson and Johnson Inc., who are leaders in vaccine development. The theory and application of the Black – Scholes Formula, multiperiod binomial models, the pricing of exotic options, calls, and barrier options will be explored in relation to the companies aforementioned and as it relates to making an investment into those vaccine stocks.

Dedication

I dedicate this thesis to my family and many friends. I want to begin by giving a special thanks to my family first. A special feeling of gratitude to my loving parents, whose words of encouragement and push for tenacity ring in my ears. Also, for always loving me unconditionally and whose good examples have taught me to work hard for the things that I aspire to achieve. I want to give special thanks to my mother who has been a constant source of support and encouragement during the challenges of my undergraduate school and life. My brother and sister who have never left my side and provide continual support. Lastly, my dad who challenges me to do better but also reminds to consider my mental health as well.

I also dedicate this thesis to many of my friends who have supported me throughout the process and journey. I will always appreciate all they have done, especially Mammie Barry for being an amazing friend and sister throughout my time at Bryn Mawr. As well as, always providing support and love. Also, Shaylin Chaney-Williams and Marye Cruz for their support in the math field. Additionally, to my fellow partners in the 2021 summer research program on math finance at Carnegie Mellon University. I enjoyed working with them to create this original project so much that I decided to expand upon the work and make it the focus of my thesis. Overall, I truly appreciate all my family and friends who have helped to support me through this journey, and I am thankful for having them in my life.

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Contents

Chapter 1

Introduction

Currently in the world, we are in a COVID-19 pandemic. Coronavirus disease 2019 (COVID-19), also known as COVID or the coronavirus , is a contagious disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). The first case was identified in Wuhan, China in December 2019.

Covid-19 transmits when people breath in air contaminated by droplets and small airborne particles containing the virus. Several testing methods have been developed to diagnose whether someone is infected with the disease. Also, vaccines have been approved and distributed to many countries. There are three major companies that were approved to give vaccinations to the population: Moderna, Pfizer, and Johnson & Johnson.

Moderna is a biotechnology company founded in 2010 that is based in Cambridge, Massachusetts. At Moderna they are pioneering a class of medicines based on messenger RNA (mRNA). They are currently developing mRNA-based vaccines and medicines for a variety of conditions such as COVID-19.

Pfizer is an American pharmaceutical and biotechnology corporation where the headquarters is at 42nd Street in Manhattan, New York City. Pfizer was established in 1849 in NY by two German immigrants Charles Pfizer and his cousin Charles F. Erhart. Pfizer develops and produces medicines and vaccines for immunology, oncology, etc. Also in response to the coronavirus pandemic, Pfizer partnered with BioNTech in April 2020 to begin developing a vaccine for COVID. On August 23, 2021, the Food and Drug Administration (FDA) approved the Pfizer COVID-19 vaccine for individuals sixteen years of age and older.

Lastly, Johnson & Johnson is an American multinational corporation founded in 1866 that develops medical devices, pharmaceuticals and consumer packaged goods. The Johnson & Johnson headquarter is in New Brunswick, New Jersey. In April 2020, Johnson & Johnson partnered with Catalent to produce a COVID-19 vaccine. On February 27, 2021, it was announced that the U.S. Food and Drug Administration issued for Emergency Use Authorization (EUA) for a single dose of COVID-19 vaccine in individuals eighteen years and older.

In this paper, I will begin by providing background knowledge on pricing options such as the Vanilla (American) options, European options and exotic options like Asian, Lookback and barrier option. For this thesis I will mainly be focusing on a barrier perpetual option. Then, we will begin by looking at the historical volatility definition and general equation. Next, I will define the historical volatility for Moderna, Pfizer, and Johnson & Johnson stocks and describe the comparisons. After I will discuss about the black scholes model and apply it to barrier perpetual options. I will examine the barrier option on all three companies using an price engine created in Excel.

Chapter 2

Options

In the present world of finance there are many types of financial instruments which go by the name of options. At its simplest, an **option** is the right to buy or sell an asset at a specified price (known as the strike price) on a predetermined date in the future. Options come with a time limit by which they must be exercised or else they expire and become worthless. The deadline is known as **exercise time, strike time, or expiration of the option**. The agreed upon price for buying or selling the security is known as the **strike price**. An option that gives the holder the right to buy an asset is called a **call option**. A call option goes up in price when the price of the underlying asset rises. We can think of a call option as a down payment on a future purchase. An option that gives the holder the right to sell an asset is known as a **put option**. A put option goes up in price when the price of the underlying asset goes down. For both put and call, one doesn't have to own the stock in order to profit from the price rise of the stock.

2.1 Vanilla Option

The types of options that can also be distinguished by their handling of expiry date. The **European** option are options that can only be exercised on the expiration date of the option. While an **American option** can be exercised at or before the strike time. Of the types of vanilla options, the European Option is simpler to treat mathematically, and will be the focus of the paper. However, in practice the American options are commonly traded. There is a mathematical price to be paid in terms of complexity for the added flexibility of the American-style option.

2.2 Exotic Option

Exotic options are the classes of option contracts with structures and features that are different from plain-vanilla options such as the American or European options. Exotic options are different from regular options in their expiration dates, exercise prices, payoffs, and underlying assets. All the features make the valuation of exotic options more sophisticated relative to the valuation of plain-vanilla options. The most common types of exotic options include the following: Asian option, Lookback option, Barrier option, etc.

The Asian option is one of the most commonly encountered types of exotic options. An **Asian** option is the payoff based on the average price rather than final price. They are option contracts whose payoffs are determined by the average price of the underlying security over several predetermined periods of time. These options allow the buyer to purchase (or sell) the underlying asset at the average price instead of the spot price.

A **Lookback** option is pays the holder of the option the maximum value of a stock price over time at a specified expiration date. A lookback option initially does not have a specified exercise price. However, on the maturity date, the holder of lookback options has the right to select the most favorable stock price among the prices that have occurred during the lifetime of the option. A lookback option allows the holder the advantage of knowing history when determining when to exercise their option. This type of option reduces uncertainties associated with the timing of market entry and reduces the chances the option will expire worthless. Lookback options are expensive to execute, so these advantages come at a cost.

A **barrier option** depends on whether or not the underlying asset has reached or exceeded a predetermined price called the **barrier**. Different types of barrier options include **knock-out options**, which are options that become worthless if the stock price goes below the barrier. Another type of barrier option is a **knock-in option**, which is the type of contract that is not an option until the stock price reaches the barrier. The last type of barrier option is a **perpetual option**, and this is an option with no expiration date. A barrier option can all be puts and calls. For this paper, we will focus on barrier options, specifically a perpetual barrier option.

Chapter 3

Historical Volatility

In this chapter we shall explain what we mean by historical volatility and then take a look at historical volatility for Moderna, Pfizer, and Johnson & Johnson stocks.

3.1 Definition of Historical Volatility

Historical volatility (HV) is a statistical measure of the dispersion of returns for a given security or market index over a given period of time. A **market index** is an index that measures the stock market. The calculation of the index value comes from the prices of the underlying holdings. Generally, this measure is calculated by determining the average deviation from the average price of a financial instrument in the given time period. Using standard deviation is the most common, but not the only way to calculate historical volatility. The higher the historical volatility value, the riskier the security. However, that is not necessarily a bad result as risk works both ways – bullish and bearish. Bullish is when an investor believes a stock or the overall market will go higher. Bearish is the opposite when an investor believes the stock will go down or underperform.

Historical volatility does not specifically measure the likelihood of loss, although it can be used to do so. What it does measure is how far a security's price moves away from its mean value.

For trending markets, historical volatility measures how far traded prices move away from a central average, or moving average, price. This is how a strongly trending but smooth market can have low volatility, even though prices change dramatically over time. Its value does not fluctuate dramatically from day to day but changes in value at a steady pace over time.

This measure is frequently compared with implied volatility to determine if options prices are over or undervalued. Implied volatility represents the expected volatility of a stock over the life of the option. The historical volatility is also used in all types of risk valuations. Stocks with a high historical volatility usually require a higher risk tolerance. And high volatility markets also require wider stop-loss levels and possibly higher margin requirements.

Aside from options pricing, HV is often used as an input in other technical studies, such as Bollinger Bands. These bands narrow and expand around a central average in response to changes in volatility, as measured by standard deviations.

3.1.1 Using Historical Volatility

Volatility has a bad connotation, but many traders and investors can make higher profits when volatility is higher. After all, if a stock or other security does not move, it has low volatility, but it also has a low potential to make capital gains. And on the other side of that argument, a stock or other security with a very high volatility level can have tremendous profit potential but at a huge cost. Its loss potential would also be tremendous. Timing of any trades must be perfect, and even a correct market call could end up losing money if the security's wide price swings trigger a stop-loss or margin call.

3.1.2 General Equations Involving Historical Volatility

Let S_n represent the stock price on day n where $n \in \mathbb{N}$ (we shall let S_0 be the stock price today). For $n \in \mathbb{N}$, let

$$R_n = \frac{S_n - S_{n-1}}{S_{n-1}}.$$

Also, let

$$X_n = ln(1 + R_n).$$

We notice that

$$1 - R_n = \frac{S_n}{S_n - 1}.$$

Then

$$\ln(1 + R_n) = \ln\left(\frac{S_n}{S_n - 1}\right).$$

The sample mean equation is

$$\mu_d = \frac{1}{N}\sum_{n=1}^{N}[X_n].$$

The sample variance equation is

$$\sigma^2 = \frac{1}{N-1}\sum_{n=1}^{N}[X_n - \mu_d].$$

The Historical Volatility is $\sigma = \sqrt{252}\ \sigma_d$. The 252 represents that there is 252 trading days in a year.

3.2 Apply Historical Volatility to Stocks

I started by taking the data of Moderna and Pfizer stock prices on Yahoo Finance. Where I was able to model the Sample Mean, Sample Variance, and Historical

Volatility during the past 2 months, 4 months, 8 months, and 1 year on Excel. This is by using the adjusted close data on Yahoo finance and then finding the daily returns, logarithmic returns and sample variance work. Now, this process was done for Johnson and Johnson as well.

In order to find the historical volatility, I began by creating a Google Spreadsheet. This is by first using Yahoo Finance to find the stock price of Moderna, Pfizer, and Johnson and Johnson. This is by specifically looking at the adjusted close. The adjusted close data has the stock price for one year from May 1, 2021 - May 1 2022. Then upload the adjusted close numbers in the second column. The first column has the dates for 2, 4, 8 month or 1 year of the stock prices. Using the adjusted close, I found the daily returns. I found the daily returns by subtracting the current adjusted close by the previous adjusted close, then divided by the previous adjusted close. After, I found the Logarithmic Returns using the daily returns. The Logarithmic Returns is the log of daily return plus one. The last column was Sample Variance Work, but I first had had to determine the sample mean. The sample mean is the sum of the Logarithmic Returns over the number of dates for each month. Thus, the Sample Variance Work is Logarithmic Returns minus the sample mean squared. Now, using the Sample Variance Work, we can find the Sample Variance. the Sample Variance is the sum of the Sample Variance Work over the total dates. Lastly, in order to find the historical volatility, it is the Sample Variance to the power of 0.5 multiplied by 252 (represents the number of trading days in a year) to the power 0.5. Ultimately, this is how I discovered the historical volatility. So, I found the historical volatility for Moderna, Pfizer and Johnson and Johnson.

3.2.1 Moderna

For the Moderna data set of 2 months, the sample mean was -0.0025 and the sample variance was 0.0023. The historical volatility is 0.7654. Next, for the 4 months the sample mean is -0.0076 and the sample variance is 0.0025. The historical volatility is 0.7988. For 8 months the sample mean is 0.0060, the sample variance work is 0.0028 and historical volatility is 0.8435. Lastly, for the 1 year of Moderna, the sample mean is -0.0012 and sample variance is 0.0027. The historical volatility is 0.8216.

3.2.2 Pfizer

For the Pfizer data set of 2 months the sample mean was -0.0017 and the sample variance was 0.0004. The historical volatility is 0.3001. Next, for the 4 months the sample mean is -0.0020 and the sample variance is 0.0003. The historical volatility is 0.2952. For 8 months the sample mean is 0.0003, the sample variance work is 0.0004 and historical volatility is 0.3238. Lastly, for the 1 year of Pfizer, the sample mean is 0.0001 and sample variance is 0.0003. The historical volatility is 0.2951.

3.2.3 Johnson and Johnson

For the Johnson and Johnson data set of 2 months is that the sample mean was 0.0023 and the sample variance was 0.0001. The historical volatility is 0.1917. Next, for the 4 months the sample mean is 0.0006 and the sample variance is 0.0001. The Historical Volatility is 0.1916. For 8 months the sample mean is 0.0003, the sample variance work is 0.0001 and historical volatility is 0.1703. Lastly, for the 1 year of Johnson and Johnson, the sample mean is 0.0005 and sample variance is 0.0001. The historical volatility is 0.1544.

3.2.4 Volatility Comparison

During these calculations, a surprising discovery was made. Initially, I assumed that the volatility for Moderna and Pfizer would be similar or that Pfizer would have greater volatility than Moderna. However, this was not the case. The historical volatility that I collected in the summer we looked at the adjusted closed data from April to June. The historical volatility data we discovered for those 2 months for Pfizer was 0.17 and for Moderna it was 0.6. The Moderna volatility was almost 3.5 times bigger than that of Pfizer. Now, looking at the adjusted data from March to April, the historical volatility for Pfizer is 0.30 and for Moderna is 0.77. The Moderna volatility is almost 2.5 times bigger than Pfizer's volatility. This is probably do to the due changes in stock price of Moderna and Pfizer.

Additionally, for Johnson and Johnson, I assumed the historical volatility would be smaller than both Moderna and Pfizer. My assumption was correct. The current historical volatility for Johnson and Johnson from March to April is 0.19. Which is the smallest historical volatility out of all three pharmaceutical companies.

We also observed that after Pfizer and Moderna became a part of the S&P 500, with Moderna entering the S&P 500 on July 21st, the stock prices of both showed an increase that will definitely increase the historical volatility of both stocks. There was an increase in the historical volatility in Moderna and Pfizer stocks from previously.

Chapter 4

Black-Scholes Model

4.1 History

The Black-Scholes model, also known as the Black-Scholes-Merton model developed for calculating the premium of an option was introduced in an article published in the Journal of Political Economy in 1973 in a paper entitled, "The Pricing of Options and Corporate Liabilities". The authors of this financial theory were, namely, Fischer S. Black, Myron S. Scholes and Robert C. Merton. Later that year, he published his own article titled "Theory of Rational Option Pricing", in The Bell Journal of Economics and Management Science, expanding the mathematical understanding and applications of the model, and coining the term Black–Scholes theory of options pricing. In 1997, Scholes and Merton were awarded the Nobel Memorial Prize in Economic Sciences for their work in finding "a new method to determine the value of derivatives". Black had passed away two years earlier, and so could not be a recipient, as Nobel Prizes are not given posthumously; however, the Nobel committee acknowledged his role in the Black-Scholes model.

4.2 Definition and Example

Black-Scholes is a pricing model used to determine the fair price or theoretical value for a call or a put option based on six variables such as volatility (σ), type of option (call or put) , underlying stock price (S_0), time (t), strike price (X), and risk-free rate (r). Then the value, V, of the call is given by the Black-Scholes Formula:

$$V = S_t N(d_1) - Xe^{-rt} N(d_2).$$

In this formula, $N(x)$ denotes the standard normal distribution function random variables d_1 and d_2. That is, $N(x) = P[Z \leq x]$. This distributed function is evaluated at the two points $d_1 = \frac{ln\frac{S_0}{X} + (\frac{r+\sigma^2}{2})t}{\sqrt{t\sigma}}$ and $d_2 = d_1 - \sigma\sqrt{t}$.

Example: We take Intel on May 22, 1998, when

$S_0 = \$74.625$

$X = \$100$

$t = 1.646$ years (expiration is January 2000)

$r = 0.05$

$\sigma = 0.375$

Then $d_1 = \frac{ln(74.625/100) + (0.05 + 0.375^2/2)1.646}{\sqrt{1.646 * 0.375}} = -0.207$

and $d_2 = -0.207 - 0.375\sqrt{1.646} = -0.688$.

So, $N(d_1) = 0.4164$ and $N(d_2) = 0.2451$.

Finally, $V = 74.625 * (0.4164) - 100e^{-0.05(1.646)}0.2451 = \$8.37$.

In other words, the fair price of this European call option on Intel stock with the given parameters is $8.37 in the Black-Scholes Model.

4.3 Black-Scholes Derivation

In this section, we present a derivation of the Black-Scholes formula. The reader will be surprised to discover how remarkably simple this argument turns out to be. Most of the argument is identifying expressions. That depends on a few definitions, some algebra, and a change of variables. Now we will do the derivation of the Black-Scholes formula in four steps:

Step 1: Find a stock model that allows us to compute the initial portfolio value without depending on the number of shares of stock bought.

We shall begin with the following investment portfolio.

Investment Portfolio:

Buy a shares of stock at S_d dollars per share and invest b dollars in a money market. Let π_t = value of portfolio at time t. Then initially our portfolio value is $\boxed{\pi_t = aS_0 + b}$ $\Diamond$.

At time $t = 0$, $\pi_t = aS_t + be^{rt}$. The present value of π_t is

$$\begin{aligned} e^{-rt}\pi_t &= e^{-rt}[aS_t + be^{rt}] \\ &= ae^{-rt}S_t + be^{-rt}e^{rt} \\ &= ae^{-rt}S_t + b \\ &= ae^{-rt}S_t + (\pi_0 - aS_0) \;\; \text{solving for for b in } \Diamond \\ &= a(e^{-rt}S_t - S_0) + \pi_0 \\ &= \pi_0. \end{aligned}$$

Then $\pi_0 = e^{-rt}$.

Thus taking expected value of both sides of the above equation, we obtain:

$$\begin{aligned} E[\pi_0] &= E[e^{-rt}\pi_t] \\ &= e^{-rt}E[\pi_t] \;\; \text{using } E[cx] = cE[x] \text{ for any constant c} \\ &\qquad\qquad \text{and any random variable } X \end{aligned}$$

Hence, $\pi_0 = e^{-rt}E[\pi_t]$, since E[constant] = constant (so $E[\pi_0] = \pi_0$).

But we need $E[e^{-rt}S_t - S_0] = 0$.

i.e. need $E[e^{-rt}S_t] - E[S_0] = 0$.

i.e. need $e^{-rt}E[S_t] - S_0 = 0$.

i.e. need $\boxed{\mathrm{S}_0 = e^{-rt}E[S_t]}$ $\triangle$.

We need to adjust Geometric Brownian Motion (GBM) Stock Model so that this additional condition holds.

Variance will remain the same but the expected value of $\ln\left(\dfrac{s(t)}{s(0)}\right)$ will be re-calibrated so that $\triangle$ is satisfied.

We will determine m in $\boxed{S_t = S_0 e^{\sigma B_t + mt}}$ $\heartsuit$ so that $\triangle$ holds true.

Substituting $\heartsuit$ into $\triangle$, we get

$$S_0 = e^{-rt}E[S_0 e^{\sigma B_t + mt}]$$
$$= S_0 e^{-rt}E[e^{\sigma B_t}e^{mt}].$$

i.e. $\mathrm{S}_0 = S_0 e^{-rt}E[e^{\sigma B_t}e^{mt}]$

Dividing by $S_0 \neq 0$, we have $1 = e^{-rt}E[e^{\sigma B_t}e^{mt}]$.

Then $1 = e^{-rt}e^{mt}E[e^{\sigma B_t}]$ using $E[cx] = cE[x]$ for any constant c and any random variable X.

i.e. $\boxed{1 = \mathrm{e}^{-rt}e^{mt}E[e^{\sigma B_t}]}$ ★ where B_t is a random normal variable with $E[B_t] = 0$ and $Var(B_t) = t$.

Let

$$Z = \frac{B_t - E[B_t]}{\sqrt{Var(B_t)}}$$
$$= \frac{B_t - 0}{\sqrt{t}}$$
$$= \frac{B_t}{\sqrt{t}}.$$

Then $Z = \dfrac{B_t}{\sqrt{t}}$ is a standard normal random variable and $B_t = \sqrt{t}Z$.

Substituting this into ★, we get

$1 = e^{(m-r)t}E[e^{\sigma\sqrt{t}z}]$

$= e^{(m-r)t}e^{\frac{1}{2}[\sigma\sqrt{t}]^2}$ using $E[e^{cz}] = e^{\frac{c^2}{2}}$ for any constant c and any s.n.r.v Z

$= e^{(m-r)t}e^{\frac{1}{2}\sigma^2 t}$ where s.n.r.v. is an abbreviation for a standard normal random variable.

Hence $1 = e^{(m-r)t+\frac{1}{2}\sigma^2 t}$. Then $(m-r)t + \frac{1}{2}\sigma^2 t = 0$. Dividing by $t > 0$, we have $(m-r) + \frac{1}{2}\sigma^2 = 0$. Then $m = r - \frac{1}{2}\sigma^2$. Substituting this into ♡, the updated stock model is $S_t = S_0 e^{\sigma B_t + (r - \frac{1}{2}\sigma^2)t}$ with $B_t = \sqrt{t}Z$.

This allows us to compute the initial portfolio value without depending on the number of shares of stock being bought.

Step 2: Find a formula for V_0.

Recall that one would exercise a European call option if $S_T > X$ (the stock price at expiration is greater than the strike price). The payoff of this option is

$$max\{S_T - X, 0\}.$$

From Step 1, $\pi_0 = e^{-rt}E[\pi_t]$. Taking $t = T > 0$ (expiration of option),

$$\begin{aligned} \pi_0 &= e^{-rt}E[\pi_t] \\ &= e^{-rt}E[max\{S_t - X, 0\}]. \end{aligned}$$

i.e. $\pi_0 = e^{-rt}E[max\{S_T - X, 0\}]$. Let $Y = E[max\{S_T - X, 0\}]$.

To calculate expected value in continuous model, we use

$$E[Y] = \int_{-\infty}^{\infty} Y f(x)\, dx$$

where $f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ (probability density function for s.n.r.v z).

Hence,

$$\begin{aligned} V_0 &= \pi_0 \\ &= e^{-rt}E[max\{S_t - X, 0\}] \\ &= e^{-rt}\int_{-\infty}^{\infty} max\{S_T - X, 0\} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx \end{aligned}$$

i.e. $V_0 = e^{-rt}\int_{-\infty}^{\infty} max\{S_T - X, 0\}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\,dx$

Step 3: Simplify formula for V_0.

Our goal is to obtain

$$V_0 = S_0\Phi(d_1) - Xe^{-rt}\Phi(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_T}{X}\right) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Note that $max\{S_T - X, 0\} = S_T - X$ if and only if $S_T - X > 0$.

This is true if and only if $S_0 e^{\sigma\sqrt{T}X + (r - \frac{1}{2}\sigma^2)T} - X > 0$ using results from Step 1 with $t = T > 0$ and replacing Z with the variable of integration x.

The above inequality holds if and only if $S_0 e^{\sigma\sqrt{T}x + (r-\frac{1}{2}\sigma^2)T} > X$, which is true if and only if $e^{\sigma\sqrt{T}x + (r-\frac{1}{2}\sigma^2)T} > \frac{x}{S_0}$ dividing by $S_0 > 0$.

Taking the natural logarithm, we have $\ln(e^{\sigma\sqrt{T}x + (r-\frac{1}{2}\sigma^2)T}) > \ln(\frac{x}{S_0})$, which is true

if and only if $\sigma\sqrt{T}x + (r - \frac{1}{2}\sigma^2)T > \ln(\frac{x}{S_0})$

if and only if $\sigma\sqrt{T}x > \ln(\frac{x}{S_0}) > (r - \frac{1}{2}\sigma^2)T$

if and only if $x > \boxed{\frac{\ln(\frac{x}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \bowtie$.

Hence $S_T - X > 0$ if and only if $x > \bowtie$ where $\bowtie = \frac{\ln(\frac{x}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

Thus

$$
\begin{aligned}
V_0 &= e^{-rt} \int_{-\infty}^{\infty} max\{S_T - X, 0\} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= e^{-rt} \int_{\infty}^{\infty} (S_T - X) \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= e^{-rt} \int_{\bowtie}^{\infty} \left[S_T \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} - x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \right] dx \\
&= e^{-rt} \Big[\int_{\bowtie}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx - \int_{\bowtie}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \Big] \\
&= e^{-rt} \int_{\bowtie}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx + e^{-rt} \int_{\bowtie}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= I + II,
\end{aligned}
$$

Hence $V_0 = I + II$ where $I = e^{-rt} \int_{\bowtie}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$ and $II = e^{-rt} \int_{\bowtie}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$.

Step 4: Simplify I and II.

We have

$$
\begin{aligned}
II &= -e^{-rt} X \int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= -X e^{-rt} [1 - \Phi(\bowtie)]
\end{aligned}
$$

i.e. $II = -Xe^{-rt}[1 - \Phi(\bowtie)]$.

Moreover, we have

$$
\begin{aligned}
I &= e^{-rt}\int_{\bowtie}^{\infty} S_T \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= e^{-rt}\int_{\bowtie}^{\infty} S_0 e^{\sigma\sqrt{T}x+(r-\frac{1}{2}\sigma^2)T} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \ \text{ using results from Step 1} \\
&= S_0 e^{-rt}\int_{\bowtie}^{\infty} e^{\sigma\sqrt{T}x} e^{(r-\frac{1}{2}\sigma^2)T} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \\
&= S_0 e^{-rt} e^{rT-\frac{1}{2}\sigma^2 T}\int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}+\sigma\sqrt{T}x} dx \\
&= S_0 e^{-\frac{1}{2}\sigma^2 T}\int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2\sigma\sqrt{T}x)} dx.
\end{aligned}
$$

i.e. $I = S_0 e^{-\frac{1}{2}\sigma^2 T}\int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-2\sigma\sqrt{T}x)} dx.$

Now let's complete the square for $x^2 + (-2\sigma\sqrt{T}x)$:

$$
\begin{aligned}
&= x^2 + (-2\sigma\sqrt{T}x) + (-\sigma\sqrt{T})^2 - (-\sigma\sqrt{T})^2 \\
&= (x^2 - 2\sigma\sqrt{T}x + \sigma^2 T) - \sigma^2 T \\
&= (x-\sigma\sqrt{T})(x-\sigma\sqrt{T}) - \sigma^2 T
\end{aligned}
$$

i.e. $\boxed{x^2 + (-2\sigma\sqrt{T}x) = (x-\sigma\sqrt{T})^2 - \sigma^2 T}$ $\triangledown$

$$
\begin{aligned}
&= S_0 e^{-\frac{1}{2}\sigma^2 T} \int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} e^{\frac{1}{2}\sigma^2 T} dx \\
&= S_0 e^{-\frac{1}{2}\sigma^2 T} e^{\frac{1}{2}\sigma^2 T} \int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} dx \\
&= S_0 \int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} dx \ \text{ Let } \ y = x - \sigma\sqrt{T} \text{and then } \ dy = dx \\
&= S_0 \int_{\bowtie}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= S_0 [1 - \Phi(\bowtie - \sigma\sqrt{T})] \\
&= S_0 \Phi(-(\bowtie - \sigma\sqrt{T})) \ \text{ using } \ \Phi(-x) = 1 - \Phi(x) \ \text{ for } \ x \in \mathbb{R}.
\end{aligned}
$$

Thus I=$S_0\Phi(-(\bowtie - \sigma\sqrt{T}))$.

Let us look at $-(\bowtie - \sigma\sqrt{T})$

$$
\begin{aligned}
&= - \bowtie + \sigma\sqrt{T} \\
&= -\left(\frac{\ln(\frac{X}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T} \\
&= -\frac{\ln(\frac{X}{S_0}) + (r - \frac{1}{2}\sigma^2)T + \sigma^2 T}{\sigma\sqrt{T}} \\
&= \frac{\ln((\frac{X}{S_0})^{-1}) + rT - \frac{1}{2}\sigma^2 T + \sigma^2 T}{\sigma\sqrt{T}} \ \text{ using} \ln(x^c) = c\ln(x) \ \text{ for all } \ c \in \mathbb{R} \ \text{ and } \ x > 0 \\
&= \frac{\ln(\frac{S_0}{x}) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\
&= d_1.
\end{aligned}
$$

Hence $\boxed{\text{-}(\bowtie - \sigma\sqrt{T}) = d_1}$ ◯

Thus I= $S_0\Phi(d_1)$. Hence

$$
\begin{aligned}
V_0 &= I + II \\
&= S_0\Phi(d_1) + -Xe^{-rT}[1 - \Phi(\bowtie)] \\
&= S_0\Phi(d_1) - Xe^{-rT}\Phi(-\bowtie) \;\; \text{using} \;\; \Phi(-x) = 1 - \Phi(x) \;\; \text{for all} \;\; x \in \mathbb{R}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T} \\
&= -(\bowtie - \sigma\sqrt{T}) - \sigma\sqrt{T} \;\; \text{using} \;\; \bigcirc \\
&= -\bowtie + \sigma\sqrt{T} - \sigma\sqrt{T} \\
&= -\bowtie .
\end{aligned}
$$

i.e. $d_2 = -\bowtie$

Thus

$$V_0 = S_0\Phi(d_1) - Xe^{-rT}\Phi(d_2)$$

where $d_1 = \dfrac{\ln\left(\frac{S_0}{x}\right) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

This completes the derivation for the Black-Scholes Formula for pricing a European call option. We can use the Put-Call Parity to find a formula for pricing a European put option. The Put-Call Parity is introduced in the next section.

4.4 Put - Call Parity

Put-Call Parity refers to a principle that defines the relationship between the price of European put and call options of the same class. This concept highlights the consistencies of these same classes. The European put and call options must have the same strike price and expiration date in order to be in the same class. This is because if the price at the expiration date is above the strike price, the call will be exercised. While if it is below, the put will be exercised. Thus in either case one unit of the asset will be purchased for the strike price. The put-call parity does not apply to American options because you can exercise the American option before the expiration date. The equation that expresses put-call parity is:

$$C + PV(x) = P + S \ \text{ or } \ C + Ke^{-rt} = S + P$$

where the C represents the price of the European call option, $PV(x) = \dfrac{K}{(1+r)^t}$ represents the present value of the strike price (x) discounted from the value on the expiration date at the risk free rate, P represents the price of the European put option, and S is the spot price or the current market value of the underlying asset. The term $\dfrac{K}{(1+r)^t}$ (also can be written as Ke^{-rt}) where K represents the strike price, r is the risk-free interest rate, and t is the time until the option expires.
Now, if we rearrange the equation, the price of an European put option can be obtained using the formula:

$$P = C + \frac{K}{(1+r)^t} - S_t.$$

4.4.1 Put - Call Parity Example

Let us now consider a question involving the put-call parity. Suppose a European call option on a barrel of crude oil with a strike price of \$50 and a maturity of one-month, trade for \$5. What is the price of the put premium with identical strike price an d time until expiration, if the one-month risk free rate is 2% and the spot price of the underlying asset is \$52?

Here is how calculate the put premium. By the Put-Call Parity we have

$$C + PV(x) = P + S.$$

Solving for P, we get:

$$P = C + PV(x) - S.$$

Substituting $PV(x) = \dfrac{X}{(1+r)^T}$, we obtain

$$\begin{aligned} P &= C + \frac{X}{(1+r)^T} - S \\ &= 5 + \frac{50}{(1+0.02)^{\frac{1}{12}}} - 52 \\ &\approx \$2.92. \end{aligned}$$

Thus the put premium of the European call option on a barrel of crude oil is \$50.

4.5 Black-Scholes European Call and Put Option Example

In this section, we will demonstrate how to apply the Black Scholes model by looking at another example.

Example: Suppose we want to value a TSLA November 1, 2020 100 call, which means the strike price on a call option on Tesla stock that will expire on November 1st is \$100. Tesla closed at \$117.25 on August 1, which is 92 days before the expiration date. The risk free interest rate is 8.5%. To determine the volatility of returns, we need to take the logarithm of returns and determine their volatility. We can find the 90-day historical volatility online, which was 0.445 [18].

So we have the required data to calculate the call price. Let us recall the Black-Scholes formula:

$$C = N(d_1)S_t - N(d_2)Ke^{-rt}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{t}.$$

In this example we have $S_t = \$117.25$,

$K = \$100$

$t = 92$ days (time to maturity in years, $\dfrac{92}{365} = 0.2520$)

$r = 8.5\% = 0.085$

$\sigma = 0.8445$

$$d_1 = \frac{ln\frac{117.25}{100} + (0.085 + \frac{0.8445^2}{2})0.2520}{0.8445\sqrt{0.2520}} = 0.6369,$$

$$d_2 = 0.6369 - 0.8445\sqrt{0.2520} = 0.2109.$$

Now, we just need to find the normal distribution of d_1 and d_2. The results are:

$$N(d_1) = 0.728 \text{ and } N(d_2) = 0.584.$$

If we put everything together using the Black-Scholes formula, we have:

$$C = 0.728 \times 0.2520 - 100e^{-0.2520*0.05}0.584 = 29.40.$$

So the theoretical price of this Tesla call option is about \$29.40. Now if we want to calculate the price of a Tesla put option of the same class, we can use put-call parity formula:

$$P = 29.4 + \frac{100}{(1+0.085)^{0.2520}} - 117.25 = 10.03.$$

Thus, the theoretical price of this Tesla put option is about \$10.03.

Chapter 5

Perpetual Barrier Options

5.1 Perpetual Option

The binomial model uses an iterative procedure with discrete times steps to price options. At each time n, a coin toss is simulated. For each head, the stock price at the next time goes up by a factor of u at the next time and for each tail the stock price goes down by a factor of d. Prices from the binomial model converge to the Black-Scholes prices as the time-step tends to zero if the parameters u and d in the model are scaled suitably. One advantage of the binomial model is that complicated types of options can be priced without having to use a partial differential equation.

For a perpetual option (i.e. one that never expires), the price of the option satisfies the Black-Scholes differential equation

$$\frac{1}{2}\sigma^2 x^2 v''(x) + rxv'(x) - rv(x) = 0.$$

Here $v(x)$ represents the price of the option when the stock price is x.

5.1.1 Finding Alpha

We know that the price of the option satisfies the Black-Scholes differential equation is

$$\frac{1}{2}\sigma^2 x^2 v''(x) + rxv'(x) - rv(x) = 0 \ \circledast$$

where σ is the volatility, r is interest rate, and x is the initial stock price.
When $r \neq 0$, we should let

$$v(x) = x^\alpha$$

$$v'(x) = \alpha x^{\alpha-1}$$

$$v''(x) = \alpha(\alpha)x^{\alpha-2}.$$

Now, we need to solve for α by plugging $\circledast$ into the quadratic formula. Let us find the two roots α_1 and α_2. We have

$$\frac{1}{2}\sigma^2 x^2 v''(x) + rxv'(x) - rv(x) = 0, \text{which implies}$$
$$\frac{1}{2}\sigma^2 x^2 \alpha(\alpha)x^{\alpha-2} + rx\alpha x^{\alpha-1} - rx^\alpha = 0$$

if we substitute expressions for $v(x), v'(x)$, and $v''(x)$ found above. Then

$$x^\alpha(\frac{1}{2}\sigma^2(\alpha^2 - \alpha) + r\alpha - r = 0$$

$$2(\frac{1}{2}\sigma^2 + (r - \frac{1}{2}\sigma^2)\alpha = 0$$

$$\sigma^2\alpha^2 + (r - \frac{1}{2}\sigma^2)2\alpha - 2r = 0$$

$$\sigma^2\alpha^2 + 2\alpha r - \alpha\sigma^2 - 2r = 0$$

<u>ASIDE:</u> Quadratic Formula -

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

so $a = \sigma^2$, $x^2 = \alpha^2$, $b = 2r - \sigma^2$, $x = \alpha$ and $c = -2r$

Solving for α_1

$$\begin{aligned}
\alpha_1 &= \frac{-(2r - \sigma^2) + \sqrt{(2r - \sigma^2)^2 - 4(\sigma^2)(-2r)}}{2(\sigma^2)} \\
&= \frac{-(2r - \sigma^2) + \sigma^2 + 2r}{2\sigma^2} \\
&= \frac{-2r + \sigma^2 + \sigma^2) + 2r}{2\sigma^2} \\
&= \frac{\sigma^2 + \sigma^2}{2\sigma^2} \\
&= \frac{2\sigma^2}{2\sigma^2} \\
&= 1.
\end{aligned}$$

i.e. $\alpha_1 = 1$

Now, solving for α_2, we obtain

$$\begin{aligned}
\alpha_2 &= \frac{-(2r - \sigma^2) - \sqrt{(2r - \sigma^2)^2 - 4(\sigma^2)(-2r)}}{2(\sigma^2)} \\
&= \frac{-(2r - \sigma^2) - (\sigma^2 + 2r)}{2\sigma^2} \\
&= -\frac{2r + \sigma^2 - \sigma^2) + 2r}{2\sigma^2} \\
&= \frac{-4r}{2\sigma^2} \\
&= -\frac{2r}{\sigma^2}
\end{aligned}$$

i.e. $\alpha_2 = -\frac{2r}{\sigma^2}$

Hence, the two roots are $\alpha_1 = 1$ and $\alpha_2 = -\frac{2r}{\sigma^2}$.

5.1.2 Solve for C_1 and C_2

$$\frac{1}{2}\sigma^2 x^2 v''(x) + rxv'(x) - rv(x) = 0$$

In order to solve for C_1 and C_2, we know that

$$\frac{1}{2}\sigma^2 x^2 v''(x) = 0$$

$$v''(x) = 0 \longrightarrow v'(x) = C_1$$

$$v(x) = C_1 X + C_2$$

First we are solving for C_1 and C_2 of the pays at the upper barrier such that $v(U) = A$ and $v(L) = 0$ where $U =$ upper barrier and $L =$lower barrier when $r \neq 0$:

$$v(x) = C_1 X + C_2 X^{\frac{-2r}{\sigma^2}}$$

$$v(L) = C_1 U + C_2 L^{\frac{-2r}{\sigma^2}-1} = 0.$$

Subtracting $C_2 L^{\frac{-2r}{\sigma^2}-1}$ we get

$$C_1 = -C_2 L^{\frac{-2r}{\sigma^2}-1} = 0.$$

Then

$$v(U) = C_1U + C_2U^{\frac{-2r}{\sigma^2}-1} = A$$
$$= -C_2L^{\frac{-2r}{\sigma^2}-1}U + C_2L^{\frac{-2r}{\sigma^2}-1} = A$$
$$= C_2(U^{\frac{-2r}{\sigma^2}} - UL^{\frac{-2r}{\sigma^2}-1}) = A$$
$$C_2 = \frac{A}{U^{\frac{-2r}{\sigma^2}} - UL^{\frac{-2r}{\sigma^2}-1}}$$
$$C_2 = \frac{A}{U(U^{\frac{-2r}{\sigma^2}-1} - L^{\frac{-2r}{\sigma^2}-1})}.$$

Thus we plug C_2 back into the C_1 equation to get

$$C_1 = -\frac{AL^{\frac{-2r}{\sigma^2}-1}}{U(U^{\frac{-2r}{\sigma^2}-1} - L^{\frac{-2r}{\sigma^2}-1})}.$$

Hence,

$$C_1 = -\frac{AL^{\frac{-2r}{\sigma^2}-1}}{U(U^{\frac{-2r}{\sigma^2}-1} - L^{\frac{-2r}{\sigma^2}-1})}$$

and

$$C_2 = \frac{A}{U(U^{\frac{-2r}{\sigma^2}-1} - L^{\frac{-2r}{\sigma^2}-1})}$$

Now, we are solving for C_1 and C_2 of the pays at the lower barrier such that $v(U) = 0$ and $v(L) = A$ when $r \neq 0$;

$$v(x) = C_1X + C_2X^{\frac{-2r}{\sigma^2}}$$

$$v(U) = C_1U + C_2U^{\frac{-2r}{\sigma^2}-1} = 0$$

Subtracting $C_2U^{\frac{-2r}{\sigma^2}-1}$ we get

$$C_1 = -C_2U^{\frac{-2r}{\sigma^2}-1} = 0.$$

Then

$$v(L) = C_1 L + C_2 L^{\frac{-2r}{\sigma^2}-1} = A$$
$$= -C_2 U^{\frac{-2r}{\sigma^2}-1} L + C_2 L^{\frac{-2r}{\sigma^2}-1} = A$$
$$= C_2 (L^{\frac{-2r}{\sigma^2}} - L U^{\frac{-2r}{\sigma^2}-1}) = A.$$

Then

$$C_2 = \frac{A}{L^{\frac{-2r}{\sigma^2}} - L U^{\frac{-2r}{\sigma^2}-1}}$$

$$C_2 = \frac{A}{L(L^{\frac{-2r}{\sigma^2}-1} - U^{\frac{-2r}{\sigma^2}-1})}$$

Thus we plug C_2 back into the C_1 equation to get

$$C_1 = -\frac{A U^{\frac{-2r}{\sigma^2}-1}}{L(L^{\frac{-2r}{\sigma^2}-1} - U^{\frac{-2r}{\sigma^2}-1})}.$$

Hence,

$$C_1 = -\frac{A U^{\frac{-2r}{\sigma^2}-1}}{L(L^{\frac{-2r}{\sigma^2}-1} - U^{\frac{-2r}{\sigma^2}-1})}$$

and

$$C_2 = \frac{A}{L(L^{\frac{-2r}{\sigma^2}-1} - U^{\frac{-2r}{\sigma^2}-1})}.$$

5.1.3 Stocks Upper and Lower Barrier Price

Moderna

Let $\sigma = 0.3$, $r = 0.01$, $X = 134$, $U = 170$, $L = 100$, and $A = 100$

Pays at Upper Barrier

$$v(X) = C_1 X + C_2 X^{\frac{r}{\sigma^2}}$$

$$v(X) = C_1 X + C_2 X^{\frac{-2(0.01)}{(0.7654)^2}}$$

We are trying to prove $U(175) = 100$. We have $U(175)$

$$= C_1(175) + C_2(175)^{\frac{-2(0.01)}{(0.7654)^2}-1}$$

$$= -\frac{100(100)^{\frac{-2(0.01)}{0.7654^2}-1}}{175\left(175^{\frac{-2(0.01)}{0.7654^2}-1} - 100^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(175) + \frac{100}{175\left(175^{\frac{-2(0.01)}{0.7654^2}-1} - 100^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(175)^{\frac{-2(0.01)}{(0.7654)^2}}$$

$$= 1.3005(175) + (-152.19319)(175)^{\frac{-2(0.01)}{(0.7654)^2}} \text{where} \;\; C_1 = 1.3005 \;\; \text{and} \;\; C_2 = -152.19$$

$$= 227.6 + (-127.6)$$

$$= 100.$$

Thus, $U(175)$ does equal 100.

Now, we are trying to prove $L(100) = 0$

$$= C_1(100) + C_2(100)^{\frac{-2(0.01)}{(0.7654)^2}-1}$$

$$= -\frac{100(100)^{\frac{-2(0.01)}{0.7654^2}-1}}{175\left(175^{\frac{-2(0.01)}{0.7654^2}-1} - 100^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(100) + \frac{100}{175\left(175^{\frac{-2(0.01)}{0.7654^2}-1} - 100^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(100)^{\frac{-2(0.01)}{(0.7654)^2}}$$

$$= 1.3005(100) + (-152.19319)(100)^{\frac{-2(0.01)}{(0.7654)^2}}$$

$$= 130.05 + (-130.05)$$

$$= 0.$$

Thus, $L(100)$ does equal 0.

Hence, since we know C_1 and C_2, we can find the pays at the upper barrier of the Moderna stock price ($S_0 = 134$):

$$v(134) = 1.3005(134) + (-152.19319)(134)^{\frac{-2(0.01)}{(0.7654)^2}} = \$45.51$$

<u>Pays at Lower Barrier</u>

We are trying to prove $U(175) = 0$. We have $U(175)$

$= C_1(175) + C_2(175)^{\frac{-2(0.01)}{(0.7654)^2}-1}$

$= -\dfrac{100(175)^{\frac{-2(0.01)}{0.7654^2}-1}}{100\left(100^{\frac{-2(0.01)}{0.7654^2}-1} - 175^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(175) + \dfrac{100}{100\left(100^{\frac{-2(0.01)}{0.7654^2}-1} - 175^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(175)^{\frac{-2(0.01)}{(0.7654)^2}}$

$= -1.2759(175) + (266.3381)(175)^{\frac{-2(0.01)}{(0.7654)^2}}$ where $C_1 = -1.2759$ and $C_2 = 266.3381$

$= -223.28 + (-223.28)$

$= 0.$

Thus, $U(175)$ does equal 0.

Now, we are trying to prove $L(100) = 100$. We have $L(100)$

$= C_1(100) + C_2(100)^{\frac{-2(0.01)}{(0.7654)^2}-1}$

$= -\dfrac{100(175)^{\frac{-2(0.01)}{0.7654^2}-1}}{100\left(100^{\frac{-2(0.01)}{0.7654^2}-1} - 175^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(100) + \dfrac{100}{100\left(100^{\frac{-2(0.01)}{0.7654^2}-1} - 175^{\frac{-2(0.01)}{0.7654^2}-1}\right)}(100)^{\frac{-2(0.01)}{(0.7654)^2}}$

$= -1.2759(100) + (266.3381)(100)^{\frac{-2(0.01)}{(0.7654)^2}}$

$= 127.59 + (-27.59)$

$= 100.$

Thus, $L(100)$ does equal 100.

Hence, since we know C_1 and C_2, we can find the pays at the lower barrier of the Moderna stock price ($S_0 = 134$):

$$v(134) = -1.2759(134) + (266.3381)(134)^{\frac{-2(0.01)}{(0.7654)^2}} = \$54.36.$$

Pfizer

Let $\sigma = 0.3$, $r = 0.01$, $X = 49$, $U = 60$, $L = 30$, and $A = 30$.

Pays at Upper Barrier

We are trying to prove $U(60) = 30$. We have $U(60)$

$$= C_1(60) + C_2(60)^{\frac{-2(0.01)}{(0.3)^2}-1}$$

$$= -\frac{30(30)^{\frac{-2(0.01)}{0.3^2}-1}}{60\left(60^{\frac{-2(0.01)}{0.3^2}-1} - 30^{\frac{-2(0.01)}{0.3^2}-1}\right)}(60) + \frac{30}{60\left(60^{\frac{-2(0.01)}{0.3^2}-1} - 30^{\frac{-2(0.01)}{0.3^2}-1}\right)}(60)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= 0.8751(60) + (-55.9006)(60)^{\frac{-2(0.01)}{(0.3)^2}} \text{ where } C_1 = 0.8751 \text{ and } C_2 = -55.9006$$

$$= 52.5 + (-22.5)$$

$$= 30.$$

Thus, $U(60)$ does equal 30.

Now, we are trying to prove $L(30) = 0$. We have $L(30)$

$$= C_1(30) + C_2(30)^{\frac{-2(0.01)}{(0.7654)^2}-1}$$

$$= -\frac{30(30)^{\frac{-2(0.01)}{0.3^2}-1}}{60\left(60^{\frac{-2(0.01)}{0.3^2}-1} - 30^{\frac{-2(0.01)}{0.3^2}-1}\right)}(30) + \frac{30}{60\left(60^{\frac{-2(0.01)}{0.3^2}-1} - 30^{\frac{-2(0.01)}{0.3^2}-1}\right)}(30)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= 0.8751(30) + (-55.9006)(30)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= 26.253 + (-26.253)$$

$$= 0.$$

Thus, $L(30)$ does equal 0.

Hence, since we know C_1 and C_2, we can find the pays at the upper barrier of the Pfizer stock price ($S_0 = 49$):

$$v(49) = 0.8751(49) + (-55.9006)(49)^{\frac{-2(0.01)}{(0.3)^2}} = \$19.34.$$

<u>Pays at Lower Barrier</u>

We are trying to prove $U(60) = 0$. We have $U(60)$

$$= C_1(60) + C_2(60)^{\frac{-2(0.01)}{(0.3)^2}-1}$$

$$= -\frac{30(60)^{\frac{-2(0.01)}{0.3^2}-1}}{30\left(30^{\frac{-2(0.01)}{0.3^2}-1} - 60^{\frac{-2(0.01)}{0.3^2}-1}\right)}(60) + \frac{30}{30\left(30^{\frac{-2(0.01)}{0.3^2}-1} - 60^{\frac{-2(0.01)}{0.3^2}-1}\right)}(60)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= -0.7501(60) + (111.8013)(60)^{\frac{-2(0.01)}{(0.3)^2}} \text{ where } C_1 = -0.7501 \text{ and } C_2 = 111.8013$$

$$= -45.00 + (45.0)$$

$$= 0.$$

Thus, $U(60)$ does equal 0.

Now, we are trying to prove $U(30) = 30$. We have $U(30)$

$$= C_1(30) + C_2(30)^{\frac{-2(0.01)}{(0.3)^2}-1}$$

$$= -\frac{30(60)^{\frac{-2(0.01)}{0.3^2}-1}}{30\left(30^{\frac{-2(0.01)}{0.3^2}-1} - 0^{\frac{-2(0.01)}{0.3^2}-1}\right)}(30) + \frac{30}{30\left(30^{\frac{-2(0.01)}{0.3^2}-1} - 60^{\frac{-2(0.01)}{0.3^2}-1}\right)}(30)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= -0.7501(30) + (111.8013)(30)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= -22.50 + (-52.50)$$

$$= 30.$$

Thus, $L(30)$ does equal 30.

Hence, since we know C_1 and C_2, we can find the pays at the lower barrier of the Pfizer stock price ($S_0 = 49$):

$$v(49) = -0.7501(49) + (111.8013)(49)^{\frac{-2(0.01)}{(0.3)^2}} = \$10.33.$$

Johnson and Johnson

Let $\sigma = 0.19$, $r = 0.01$, $X = 180$, $U = 200$, $L = 160$, and $A = 160$.

Pays at Upper Barrier

We are trying to prove $U(200) = 160$. We have $U(200)$

$$= C_1(200) + C_2(200)^{\frac{-2(0.01)}{(0.19)^2}-1}$$

$$= -\frac{160(160)^{\frac{-2(0.01)}{0.19^2}-1}}{200\left(200^{\frac{-2(0.01)}{0.19^2}-1} - 160^{\frac{-2(0.01)}{0.19^2}-1}\right)}(200) + \frac{160}{200\left(200^{\frac{-2(0.01)}{0.19^2}-1} - 160^{\frac{-2(0.01)}{0.19^2}-1}\right)}(200)^{\frac{-2(0.01)}{(0.19)^2}}$$

$$= 2.7301(200) + (-7268.0041)(200)^{\frac{-2(0.01)}{(0.3)^2}} \text{ where } C_1 = 2.7300 \text{ and } C_2 = -7268.0041$$

$$= 546.0 + (-386.0)$$

$$= 160.$$

Thus, $U(200)$ does equal 160.

Now, we are trying to prove $L(160) = 0$. We have $L(160)$

$$= C_1(160) + C_2(160)^{\frac{-2(0.01)}{(0.19)^2}-1}$$

$$= -\frac{160(160)^{\frac{-2(0.01)}{0.19^2}-1}}{200\left(200^{\frac{-2(0.01)}{0.19^2}-1} - 160^{\frac{-2(0.01)}{0.19^2}-1}\right)}(160) + \frac{160}{200\left(200^{\frac{-2(0.01)}{0.19^2}-1} - 160^{\frac{-2(0.01)}{0.19^2}-1}\right)}(160)^{\frac{-2(0.01)}{(0.19)^2}}$$

$$= 2.7301(160) + (-7268.0041)(160)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= 436.8 + (-436.8)$$

$$= 0.$$

Thus, $L(160)$ does equal 0.

Hence, since we know C_1 and C_2, we can find the pays at the upper barrier of the Johnson and Johnson stock price ($S_0 = 180$):

$$v(180) = 2.7301(180) + (-7268.0041)(180)^{\frac{-2(0.01)}{(0.3)^2}} = \$82.19.$$

Pays at Lower Barrier

We are trying to prove $U(200) = 0$. We have $U(200)$

$$= C_1(200) + C_2(200)^{\frac{-2(0.01)}{(0.19)^2}-1}$$

$$= -\frac{160(200)^{\frac{-2(0.01)}{0.19^2}-1}}{160\left(160^{\frac{-2(0.01)}{0.19^2}-1} - 200^{\frac{-2(0.01)}{0.19^2}-1}\right)}(200) + \frac{160}{160\left(160^{\frac{-2(0.01)}{0.19^2}-1} - 200^{\frac{-2(0.01)}{0.19^2}-1}\right)}(200)^{\frac{-2(0.01)}{(0.19)^2}}$$

$$= -2.4126(200) + (9085.0051)(200)^{\frac{-2(0.01)}{(0.3)^2}} \text{ where } C_1 = -2.4126 \text{ and } C_2 = 9085.0051$$

$$= -45.00 + (45.0)$$

$$= 0.$$

Thus, $U(200)$ does equal 0.

Now, we are trying to prove $L(160) = 160$. We have $L(160)$

$$= C_1(160) + C_2(160)^{\frac{-2(0.01)}{(0.19)^2}-1}$$

$$= -\frac{160(200)^{\frac{-2(0.01)}{0.19^2}-1}}{160\left(160^{\frac{-2(0.01)}{0.19^2}-1} - 200^{\frac{-2(0.01)}{0.19^2}-1}\right)}(160) + \frac{160}{160\left(160^{\frac{-2(0.01)}{0.19^2}-1} - 200^{\frac{-2(0.01)}{0.19^2}-1}\right)}(160)^{\frac{-2(0.01)}{(0.19)^2}}$$

$$= -2.4126(160) + (9085.0051)(160)^{\frac{-2(0.01)}{(0.3)^2}}$$

$$= -386.016 + (546.016)$$

$$= 160.$$

Thus, v(160) does equal 160.

Hence, since we know C_1 and C_2, we can find the pays at the lower barrier of the Pfizer stock price ($S_0 = 160$):

$$v(180) = -0.7501(49) + (111.8013)(49)^{\frac{-2(0.01)}{(0.3)^2}} = \$77.26.$$

5.1.4 Price Engine - Excel

Using the information above from section 5.1.1 and 5.1.2 I created an Excel Price engine. Below are the Excel Price Engine links.

Excel Price Engine Links:

- Moderna - `https://docs.google.com/spreadsheets/d/11PsSMPqubBatn4OHTanVo8IVdBCXw` `edit?usp=sharing`
- Pfizer - `https://docs.google.com/spreadsheets/d/1ngHsdyCHxhLQoNuETx_` `CenhkedGqjQuCjMMgKTrl-Y8/edit?usp=sharing`
- Johnson and Johnnson - `https://docs.google.com/spreadsheets/d/1EVX7diPsg1wAq-QVJs` `edit?usp=sharing`

The goal for this part is to create a barrier option for Moderna, Pfizer, Johnson and Johnson. A barrier option is an option whose payoff depends on whether or not the underlying asset price has reached or surpassed the pre-established price. Using Excel, I created a barrier option looking at the pays at the upper and lower barriers for Moderna, Pfizer and Johnson. We assumed the interest rate is not zero. This code can allow us to see what the price would be if you use different rates, lower and upper barrier prices. For the perpetual options, there is a Black-Scholes ordinary differential equation that was used to compute the prices.

Chapter 6

Conclusion

6.1 Summary

Overall, all three US companies are major providers of COVID-19 vaccines. Due to this, similar historical data should be expected for the companies. However, upon analysis, it has been realized that Moderna has a higher volatility than Pfizer. Both Moderna and Pfizer has higher volatility than Johnson and Johnson. The Moderna historical volatility was previously 0.6 and it increased to 0.77. The Pfizer stock priced increased because the historical volatility went from 0.17 to 0.30. Overall, the Pfizer increased by double. This could be due to present potential catalysts that affect Moderna and not Pfizer. For example, Pfizer can be used in children twelve and over, while Moderna is not yet been approved for children. We would expect to see an increase in both stocks as patients who need a third shot have been identified (persons with extreme kidney damage); Pfizer also petitioned for everyone to have to get the third shot (booster shot). This is probably why Pfizer increased a lot more the Moderna.

Additionally, I created a price engine that looks at the barrier option for each pharmaceutical company. The price engine looks at the pay at both the upper and lower barrier for each company. As, well as what the stock price would be if a upper and lower barrier was applied. This is using the perpetual barrier option calculated through the Black-scholes framework.

6.2 Future Questions

Some future questions that I could examine include the following:

- What the barrier perpetual option would be for Moderna, Pfizer and Johnson and Johnson if r was equal to zero.
- The perpetual barrier options were created on the basis that the barrier is infinite. What happens when the expiration date of the option is finite, after which the option pays nothing?
- How could we employ models with non-constant volatility?

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