Computational Physics
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Wave Equation

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1 Introduction

For a wave travelling with wave speed c, the wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

For a string wave, the wave speed c is given by:

$$c = \sqrt{\frac{T}{\mu}}$$

where T is the tension in the string and μ is its mass density.

To find a solution of the equation (1), it is necessary to define the boundary conditions as well as the initial state of the string. To solve it via Leapfrog algorithm, we approximate both the temporal and spatial derivatives in equation (1) with centered finite differences. Assuming uniform time step t, and grid spacing h, the Leapfrog algorithm can be written as

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} + \frac{c^2}{c'^2}(u_{i+1}^j + u_{i-1}^j - 2u_i^j) \tag{2}$$

where spatial position is indicated by a subscript i and temporal position by superscript j and $c' = \frac{h}{dt}$ is the lattice velocity.

Equation (2) requires us to keep track of two arrays: u to store positions at current time u_i^j as well as $uold$ to store positions at previous time u_i^{j-1}. At time $t = 0$, $uold$ i.e. u_i^{-1} can be created using the initial velocity v_0 as:

$$v_0(x_i) = \frac{u_i^1 - u_i^{-1}}{2t} \tag{3}$$

From, equation (2),

$$u_i^{-1} = u_i^0 - v_0(x_i)t + \frac{c^2}{2c'^2}(u_{i+1}^0 + u_{i-1}^0 - 2u_i^0) \tag{4}$$

2 Setup and Simulation

The following initial parameters were used:

$$u(x,t) = \begin{cases} 0 & \text{if x = 0} \\ 0 & \text{if x = 1} \\ e^{-200(x-0.5)^2} & \text{if 0 < x < 1} \end{cases}$$

$$\frac{\partial(x_i, 0)}{\partial t} = v_0(x_i) = 0$$

With grid spacing $h = 0.01$ m, the following motion of the string was obtained [1]:

[1] Please find animation "wave.gif" in parent folder

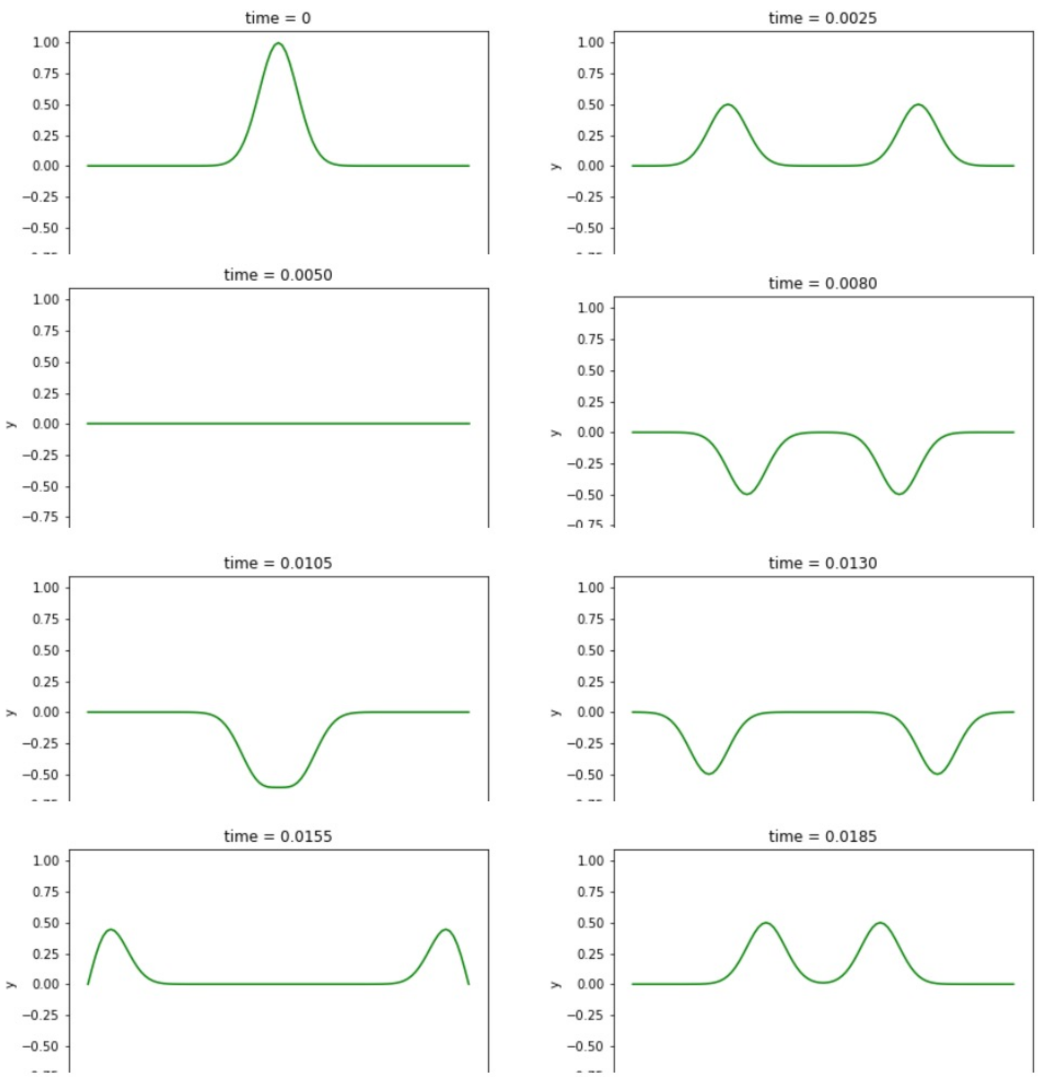

Figure 1: Time evolution of wave on a string

When the grid spacing was reduced to half, the algorithm blows up spectacularly as $t > \frac{h}{c}$. [2]

[2] Please find animation "half_grid.gif" in parent folder

2.1 Motion of a plucked string

The following initial conditions were implied to replicate the motion of a plucked string:

$$u(x,t) = \begin{cases} 0.08 & \text{if x = 0} \\ 0 & \text{if x = 1} \\ 0 & \text{if } 0 < \text{x} < 1 \end{cases}$$

The motion of the string can be described with the following plot [3]:

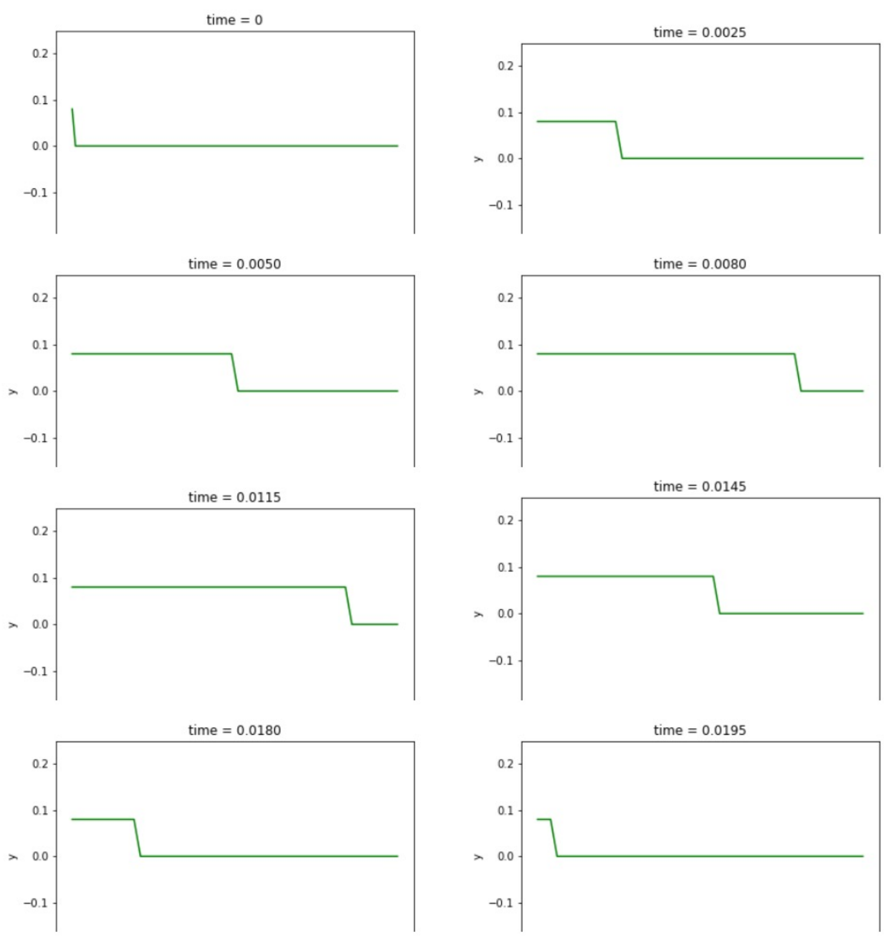

Figure 2: Time evolution of wave on a string when plucked at one end

[3]Please find animation "plucked.gif" in parent folder

3 Result

As seen from figure 1, the wave generated from initial pulse splits into two, gets reflected at each ends and again add up when they pass through each other. However, When a string is plucked near its end, a pulse reflects off the ends and bounces back and forth as seen in figure (2). It was also seen that upon reducing the grid spacing below $(\frac{h}{c})$, the algorithm blew up producing non-meaningful motion. Whereas, upon reducing the time step, we get more detailed motion between each time step.