Mathematical Foundations for Computer Applications

Unit 3 : Mathematical Logic

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Mathematical Foundations for Computer Applications

Fundamentals of Logic and Proofs

- Propositional Logic,
- Propositional Equivalences,
- Predicates and Quantifiers,
- Rules of Inference,
- Introduction to Proofs,
- Normal Forms

Mathematical Foundations for Computer Applications

Propositional Logic

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Logic

- **Logic** is the basis of all mathematical reasoning and it provides rules and methods to check whether a given argument is valid or not.
- **Logical reasoning** is used in developing algorithms needed for computer programs.
- **Logic** is applied to decide whether one statement follows from, or is a logical consequence of one or more statements.

Logic

- It has two parts – **propositional calculus** which deals with analysis of propositions and **predicate calculus** which deals with the analysis of predicates which are the propositions involving variables.

Propositional Logic

- A **proposition** is a declarative sentence (a sentence that declares a fact) that is either **true or false**, but not both.
- These values - true or false, are called truth values.

(i) True is denoted by T or 1.

(ii) False is denoted by F or 0.

- The propositions are denoted using **alphabets.(capitals or small)**
- **Examples:**

1. p: Chennai is the capital of Tamil Nadu (True)
2. q: 3 + 2 = 0 (False)

Propositional Logic

- The following are not propositions.

1. Oh! How beautiful it is! (**Exclamatory sentences** are not propositions.)
2. Ring the bell. (**Commands** are not propositions.)
3. Where are you going? (**Interrogative sentences** are not propositions.)
4. This statement is false. (**Self contradictory** sentences are not propositions.)

Propositional Logic

- Are the following sentences propositions?
    1. **Delhi is the capital of India.**
    2. **Read this carefully.**
    3. **1+2=3**
    4. **x+1=2**
    5. **What time is it?**

Propositional Logic

- Are the following sentences propositions?
  1. Delhi is the capital of India. (Yes)
  2. Read this carefully. (Not)
  3. 1+2=3 (Yes)
  4. x+1=2 (not) It is because unless we give a specific value of x, we cannot say whether the statement is true or false.
  5. What time is it? (Not)

Propositional Logic

- Are the following propositions or not:

(i) Shyam is rich.

(ii) May God bless you.

(iii) Oh! What a wonderful party.

(iv) I am a liar.

(v) Close the door.

Propositional Logic

- Are the following propositions or not:

(i) Shyam is rich. Ans: Yes

(ii) May God bless you. Ans: No

(iii) Oh! What a wonderful party. Ans: No

(iv) I am a liar. Ans: No

(v) Close the door. Ans: No

Propositional Logic

Propositions can either be

1. Primitive propositions or
2. Compound propositions.

PRIMITIVE PROPOSITIONS:

- A proposition is said to be primitive if it cannot be broken down into simpler propositions. They are also called as atomic propositions.

1. Sugar is sweet.
2. $6 + 4 = 10$.

Propositional Logic

COMPOUND PROPOSITIONS:

- A proposition which is composed of atomic propositions connected by ‘and’, ‘or’, etc., are called compound propositions. They are composite.

Eg--Sam studies well and also plays keyboard.

Here ‘Sam studies well’ and ‘Sam plays keyboard’ are atomic statements connected by the connective ‘and’.

Propositional Logic

- **Propositional Logic** – The area of logic that deals with propositions
- **Propositional Variables** – variables that represent propositions: **p, q, r, s**

  E.g. Proposition p – "Today is Friday."
- **Truth values** – **T, F**
- **Logical operators** are used to form new propositions from two or more existing propositions. The logical operators are also called **connectives**.

Propositional Logic

LOGICAL OPERATORS:

- The **three basic logical operators** are conjunction, disjunction and negation and they correspond to the English words ‘and’, ‘or’ and ‘not’.
- Conditional and biconditional operators are also logical operators. There are some more connectives NAND , NOR and exclusive-or.

Propositional Logic

- If p and q are propositions, new compound propositions can be formed by using connectives

Most common connectives:

- Conjunction AND. Symbol $\wedge$
- Disjunction OR Symbol $\vee$
- Negation Symbol $\sim$
- Implication Symbol $\rightarrow$
- Double implication Symbol $\leftrightarrow$

Propositional Logic

Definition 1 Let p be a proposition. The **negation** of p, denoted by ¬p, is the statement "It is not the case that p." The proposition ¬p is read "not p." The truth value of the negation of p, ¬p is the opposite of the truth value of p

p	¬p
T	F
F	T

Examples - Find the negation of the proposition "Today is Friday."

"It is not the case that today is Friday."

Or "Today is not Friday." or "It is not Friday today

Propositional Logic

Definition 2 Let p and q be propositions. The **conjunction** of p and q, denoted by **$p \land q$**, is the proposition "**p and q**". The conjunction $p \land q$ is true when both p and q are true and is false otherwise.

- The Truth Table for the Conjunction of Two Propositions

p	q	$p \land q$
F	F	F
F	T	F
T	F	F
T	T	T

Propositional Logic

Example--Find the conjunction of the propositions p and q where p is the proposition "**Today is Friday**." and q is the proposition "**It is raining today**.", and the truth value of the conjunction.

The **conjunction** is the proposition

"Today is Friday and it is raining today."

The proposition is true on rainy Fridays.

Propositional Logic

Definition 3 Let p and q be propositions. The **disjunction** of p and q, denoted by $\boldsymbol{p \vee q}$, is the proposition "***p* or *q*.**" The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

- The Truth Table for the disjunction of Two Propositions

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Propositional Logic

Note: **inclusive or** : The disjunction is true when at least one of the two propositions is true.

E.g. “Students who have taken calculus or computing essentials can take this class.” – those who take one or both classes.

Propositional Logic

Definition 4 Let p and q be propositions. The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition that is true when **exactly one of p and q is true** and is false otherwise.

- The truth table for the exclusive or of two propositions is

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Propositional Logic

- **Exclusive or** : The disjunction is true only when one of the proposition is true.

E.g. “Students who have taken calculus or competing essentials, but not both, can take this class.” – only those who take one of them.

NOTE– Odd number of ‘T’ then result is T, otherwise F.

Propositional Logic

Inputs			outputs
W	X	Y	Q = A⊕B⊕C
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Conditional Statements

Definition 5 Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition **"if p, then q."** The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Conditional Statements

- In the conditional statement $p \rightarrow q$, ***p* is called the *hypothesis*** (or *antecedent* or *premise*) and ***q* is called the *conclusion*** (or *consequence*).
- A conditional statement is also called an **implication**

Eg-- "If I am elected, then I will lower taxes."

Conditional Statements

The following ways to express this conditional statement:

- "if p, then q"
- "if p, q"
- "p is sufficient for q"
- "q if p"
- "q when p"
- "a necessary condition for p is q"
- "q unless $\neg p$"
- "p implies q"
- "p only if q"
- "a sufficient condition for q is p"
- "q whenever p"
- "q is necessary for p"
- "q follows from p"

Conditional Statements

Definition 6 Let p and q be propositions. The ***biconditional*** *statement* $p \leftrightarrow q$ is the proposition **"*p* if and only if *q*."** The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Conditional Statements

Common ways to express $p \leftrightarrow q$:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q." {can be expressed by "iff" }

- Biconditional statements are also called ***bi-implications***.
- $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$.

Conditional Statements

Eg- Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket."

- Then $p \leftrightarrow q$ is the statement
- "You can take the flight if and only if you buy a ticket."

Boolean Operations Summary:

Summarizing all the connectives (NAND, NOR, Ex-OR))

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$	$p \uparrow q$	$p \downarrow q$	$p \oplus q$
T	T	F	T	T	T	T	F	F	F
T	F	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	T	F	T
F	F	T	F	F	T	T	T	T	F

Precedence of Logical Operators

- **Precedence of Logical Operators.**

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Truth Tables of Compound proposition

1. Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Truth Tables of Compound proposition

2. Write truth table of the following proposition:

a) $\neg(p \rightarrow \neg q)$

b) $(p \wedge q) \rightarrow (p \vee q)$

c) $\neg (p \wedge q) \vee \neg (q \leftrightarrow p)$

d) $(p \rightarrow q) \vee (p \leftrightarrow \neg q)$

Truth Tables of Compound proposition

2. If p is true and *q* is false then find the truth value of the following:

a) $\neg(p \rightarrow \neg q)$

b) $(p \wedge q) \rightarrow (p \vee q)$

c) $\neg (p \wedge q) \vee \neg (q \leftrightarrow p)$

d) $(p \rightarrow q) \vee (p \leftrightarrow \neg q)$

THANK YOU

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