CHAPTER

2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. These are some of the topics we will study in later chapters.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways. Useful structures such as sequences and strings are special types of functions. In this chapter, we will introduce the notion of a sequence, which represents ordered lists of elements. Furthermore, we will introduce some important types of sequences and we will show how to define the terms of a sequence using earlier terms. We will also address the problem of identifying a sequence from its first few terms.

In our study of discrete mathematics, we will often add consecutive terms of a sequence of numbers. Because adding terms from a sequence, as well as other indexed sets of numbers, is such a common occurrence, a special notation has been developed for adding such terms. In this chapter, we will introduce the notation used to express summations. We will develop formulae for certain types of summations that appear throughout the study of discrete mathematics. For instance, we will encounter such summations in the analysis of the number of steps used by an algorithm to sort a list of numbers so that its terms are in increasing order.

The relative sizes of infinite sets can be studied by introducing the notion of the size, or cardinality, of a set. We say that a set is countable when it is finite or has the same size as the set of positive integers. In this chapter we will establish the surprising result that the set of rational numbers is countable, while the set of real numbers is not. We will also show how the concepts we discuss can be used to show that there are functions that cannot be computed using a computer program in any programming language.

Matrices are used in discrete mathematics to represent a variety of discrete structures. We will review the basic material about matrices and matrix arithmetic needed to represent relations and graphs. The matrix arithmetic we study will be used to solve a variety of problems involving these structures.

2.1 Sets

Introduction

In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such

collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

DEFINITION 1

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$. ◀

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$. ◀

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, {a, 2, Fred, New Jersey} is the set containing the four elements a, 2, Fred, and New Jersey. ◀

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (. . .) are used when the general pattern of the elements is obvious.

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$. ◀

Extra Examples

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbf{Q}^+$ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

Beware that mathematicians disagree whether 0 is a natural number. We consider it quite natural.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

(Note that some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Recall the notation for **intervals** of real numbers. When a and b are real numbers with $a < b$, we write

$$[a, b] = \{x \mid a \leq x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$(a, b) = \{x \mid a < x < b\}$$

Note that $[a, b]$ is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b.

Sets can have other sets as members, as Example 5 illustrates.

EXAMPLE 5 The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are $\mathbf{N}$, the set of natural numbers; $\mathbf{Z}$, the set of integers; $\mathbf{Q}$, the set of rational numbers; and $\mathbf{R}$, the set of real numbers. ◀

Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0, 1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

DEFINITION 2 Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

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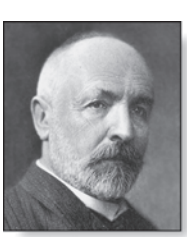

GEORG CANTOR (1845–1918) Georg Cantor was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker. He received his doctor's degree in 1867, after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death.

Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics.

Cantor married in 1874 and had five children. His melancholy temperament was balanced by his wife's happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor's views on set theory. Cantor suffered from mental illness throughout the later years of his life. He died in 1918 from a heart attack.

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$. The empty set can also be denoted by $\{\ \}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

{∅} has one more element than ∅.

A set with one element is called a **singleton set**. A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set. The single element of the set $\{\emptyset\}$ is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Links

NAIVE SET THEORY Note that the term *object* has been used in the definition of a set, Definition 1, without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated in 1895 by the German mathematician Georg Cantor. The theory that results from this intuitive definition of a set, and the use of the intuitive notion that for any property whatever, there is a set consisting of exactly the objects with this property, leads to **paradoxes**, or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902 (see Exercise 46 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory beginning with axioms. However, we will use Cantor's original version of set theory, known as **naive set theory**, in this book because all sets considered in this book can be treated consistently using Cantor's original theory. Students will find familiarity with naive set theory helpful if they go on to learn about axiomatic set theory. They will also find the development of axiomatic set theory much more abstract than the material in this text. We refer the interested reader to [Su72] to learn more about axiomatic set theory.

Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the **universal set** U, which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in Example 7.

Assessment

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure 1). ◀

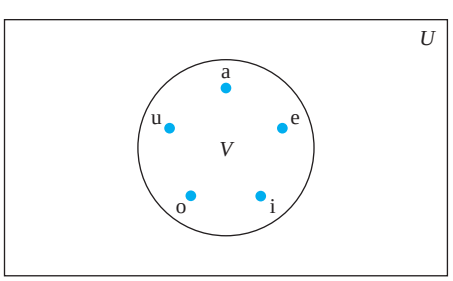

FIGURE 1 Venn Diagram for the Set of Vowels.

Subsets

It is common to encounter situations where the elements of one set are also the elements of a second set. We now introduce some terminology and notation to express such relationships between sets.

DEFINITION 3 The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

We have these useful rules for determining whether one set is a subset of another:

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

EXAMPLE 8 The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). Each of these facts follows immediately by noting that an element that belongs to the first set in each pair of sets also belongs to the second set in that pair. ◀

EXAMPLE 9 The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 100$], but not the later set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major. ◀

Links

BERTRAND RUSSELL (1872–1970) Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica*, written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for literature in 1950.

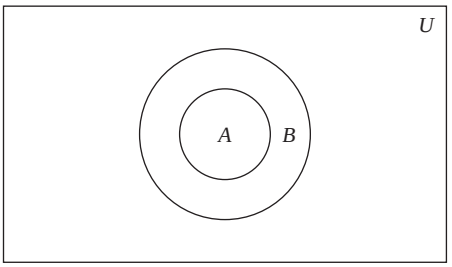

FIGURE 2 **Venn Diagram Showing that A Is a Subset of B.**

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1 For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Proof: We will prove (i) and leave the proof of (ii) as an exercise.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof. ◁

When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true. Venn diagrams can be used to illustrate that a set A is a subset of a set B. We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B. Because A is a subset of B, we draw the circle for A within the circle for B. This relationship is shown in Figure 2.

A useful way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. That is, $A = B$ if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$, which is what it means for the A and B to be equal. Because this method of showing two sets are equal is so useful, we highlight it here.

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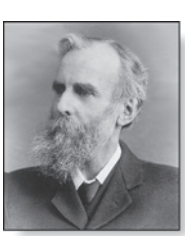

JOHN VENN (1834–1923) John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn's book *Symbolic Logic* clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as *Venn diagrams*. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}$.

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

The Size of a Set

Sets are used extensively in counting problems, and for such applications we need to discuss the sizes of sets.

DEFINITION 4 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Remark: The term *cardinality* comes from the common usage of the term *cardinal number* as the size of a finite set.

EXAMPLE 10 Let A be the set of odd positive integers less than 10. Then $|A| = 5$. ◀

EXAMPLE 11 Let S be the set of letters in the English alphabet. Then $|S| = 26$. ◀

EXAMPLE 12 Because the null set has no elements, it follows that $|\emptyset| = 0$. ◀

We will also be interested in sets that are not finite.

DEFINITION 5 A set is said to be *infinite* if it is not finite.

EXAMPLE 13 The set of positive integers is infinite. ◀

Extra Examples

We will extend the notion of cardinality to infinite sets in Section 2.5, a challenging topic full of surprising results.

Power Sets

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S, we build a new set that has as its members all the subsets of S.

DEFINITION 6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

EXAMPLE 14 What is the power set of the set $\{0, 1, 2\}$?

Extra Examples

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

EXAMPLE 15 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$ ◀

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

DEFINITION 7 The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, $\ldots$, and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$. In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

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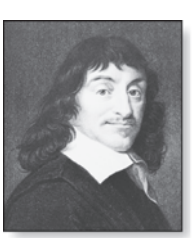


RENÉ DESCARTES (1596–1650) René Descartes was born into a noble family near Tours, France, about 200 miles southwest of Paris. He was the third child of his father's first wife; she died several days after his birth. Because of René's poor health, his father, a provincial judge, let his son's formal lessons slide until, at the age of 8, René entered the Jesuit college at La Flèche. The rector of the school took a liking to him and permitted him to stay in bed until late in the morning because of his frail health. From then on, Descartes spent his mornings in bed; he considered these times his most productive hours for thinking.

Descartes left school in 1612, moving to Paris, where he spent 2 years studying mathematics. He earned a law degree in 1616 from the University of Poitiers. At 18 Descartes became disgusted with studying and decided to see the world. He moved to Paris and became a successful gambler. However, he grew tired of bawdy living and moved to the suburb of Saint-Germain, where he devoted himself to mathematical study. When his gambling friends found him, he decided to leave France and undertake a military career. However, he never did any fighting. One day, while escaping the cold in an overheated room at a military encampment, he had several feverish dreams, which revealed his future career as a mathematician and philosopher.

After ending his military career, he traveled throughout Europe. He then spent several years in Paris, where he studied mathematics and philosophy and constructed optical instruments. Descartes decided to move to Holland, where he spent 20 years wandering around the country, accomplishing his most important work. During this time he wrote several books, including the *Discours*, which contains his contributions to analytic geometry, for which he is best known. He also made fundamental contributions to philosophy.

In 1649 Descartes was invited by Queen Christina to visit her court in Sweden to tutor her in philosophy. Although he was reluctant to live in what he called "the land of bears amongst rocks and ice," he finally accepted the invitation and moved to Sweden. Unfortunately, the winter of 1649–1650 was extremely bitter. Descartes caught pneumonia and died in mid-February.

Many of the discrete structures we will study in later chapters are based on the notion of the *Cartesian product* of sets (named after René Descartes). We first define the Cartesian product of two sets.

DEFINITION 8

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 16 Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Extra Examples

Solution: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university. ◀

EXAMPLE 17 What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$ ◀

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal, unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$ (see Exercises 31 and 38). This is illustrated in Example 18.

EXAMPLE 18 Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 17.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 17. ◀

The Cartesian product of more than two sets can also be defined.

DEFINITION 9

The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 19 What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2),$$
$$(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}.$$ ◀

Remark: Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$ (see Exercise 39).

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 20 Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. ◀

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. A relation from a set A to itself is called a relation on A.

EXAMPLE 21 What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, the ordered pairs in R are (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3). ◀

We will study relations and their properties at length in Chapter 9.

Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

EXAMPLE 22 What do the statements $\forall x \in \mathbf{R}\ (x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\ (x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

EXAMPLE 23 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbf{N}$, the set of nonnegative integers. ◀

Note that $\forall x\, P(x)$ is true over the domain U if and only if the truth set of P is the set U. Likewise, $\exists x\, P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Exercises

1. List the members of these sets.
- **a)** $\{x \mid x$ is a real number such that $x^2 = 1\}$
- **b)** $\{x \mid x$ is a positive integer less than 12$\}$
- **c)** $\{x \mid x$ is the square of an integer and $x < 100\}$
- **d)** $\{x \mid x$ is an integer such that $x^2 = 2\}$

2. Use set builder notation to give a description of each of these sets.
- **a)** $\{0, 3, 6, 9, 12\}$
- **b)** $\{-3, -2, -1, 0, 1, 2, 3\}$
- **c)** $\{m, n, o, p\}$

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
- **a)** the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
- **b)** the set of people who speak English, the set of people who speak Chinese
- **c)** the set of flying squirrels, the set of living creatures that can fly

4. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
- **a)** the set of people who speak English, the set of people who speak English with an Australian accent
- **b)** the set of fruits, the set of citrus fruits
- **c)** the set of students studying discrete mathematics, the set of students studying data structures

5. Determine whether each of these pairs of sets are equal.
- **a)** $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
- **b)** $\{\{1\}\}$, $\{1, \{1\}\}$
- **c)** $\emptyset$, $\{\emptyset\}$

6. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

7. For each of the following sets, determine whether 2 is an element of that set.
- **a)** $\{x \in \mathbf{R} \mid x$ is an integer greater than 1$\}$
- **b)** $\{x \in \mathbf{R} \mid x$ is the square of an integer$\}$
- **c)** $\{2, \{2\}\}$
- **d)** $\{\{2\}, \{\{2\}\}\}$
- **e)** $\{\{2\}, \{2, \{2\}\}\}$
- **f)** $\{\{\{2\}\}\}$

8. For each of the sets in Exercise 7, determine whether $\{2\}$ is an element of that set.

9. Determine whether each of these statements is true or false.
- **a)** $0 \in \emptyset$
- **b)** $\emptyset \in \{0\}$
- **c)** $\{0\} \subset \emptyset$
- **d)** $\emptyset \subset \{0\}$
- **e)** $\{0\} \in \{0\}$
- **f)** $\{0\} \subset \{0\}$
- **g)** $\{\emptyset\} \subseteq \{\emptyset\}$

10. Determine whether these statements are true or false.
- **a)** $\emptyset \in \{\emptyset\}$
- **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
- **c)** $\{\emptyset\} \in \{\emptyset\}$
- **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
- **e)** $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
- **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
- **g)** $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

11. Determine whether each of these statements is true or false.
- **a)** $x \in \{x\}$
- **b)** $\{x\} \subseteq \{x\}$
- **c)** $\{x\} \in \{x\}$
- **d)** $\{x\} \in \{\{x\}\}$
- **e)** $\emptyset \subseteq \{x\}$
- **f)** $\emptyset \in \{x\}$

12. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

13. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

14. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

15. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

17. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

18. Find two sets A and B such that $A \in B$ and $A \subseteq B$.

19. What is the cardinality of each of these sets?

a) $\{a\}$ **b)** $\{\{a\}\}$

c) $\{a, \{a\}\}$ **d)** $\{a, \{a\}, \{a, \{a\}\}\}$

20. What is the cardinality of each of these sets?

a) $\emptyset$ **b)** $\{\emptyset\}$

c) $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

21. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

22. Can you conclude that $A = B$ if A and B are two sets with the same power set?

23. How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

24. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$ **b)** $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ **d)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

25. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

26. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

27. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$. **b)** $B \times A$.

28. What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

29. What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

30. Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

31. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. **b)** $C \times B \times A$.

c) $C \times A \times B$. **d)** $B \times B \times B$.

33. Find A^2 if

a) $A = \{0, 1, 3\}$. **b)** $A = \{1, 2, a, b\}$.

34. Find A^3 if

a) $A = \{a\}$. **b)** $A = \{0, a\}$.

35. How many different elements does $A \times B$ have if A has m elements and B has n elements?

36. How many different elements does $A \times B \times C$ have if A has m elements, B has n elements, and C has p elements?

37. How many different elements does A^n have when A has m elements and n is a positive integer?

38. Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

40. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

41. Translate each of these quantifications into English and determine its truth value.

a) $\forall x \in \mathbf{R}\ (x^2 \neq -1)$ **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$

c) $\forall x \in \mathbf{Z}\ (x^2 > 0)$ **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$

42. Translate each of these quantifications into English and determine its truth value.

a) $\exists x \in \mathbf{R}\ (x^3 = -1)$ **b)** $\exists x \in \mathbf{Z}\ (x + 1 > x)$

c) $\forall x \in \mathbf{Z}\ (x - 1 \in \mathbf{Z})$ **d)** $\forall x \in \mathbf{Z}\ (x^2 \in \mathbf{Z})$

43. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^2 < 3$ **b)** $Q(x)$: $x^2 > x$

c) $R(x)$: $2x + 1 = 0$

44. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^3 \geq 1$ **b)** $Q(x)$: $x^2 = 2$

c) $R(x)$: $x < x^2$

***45.** The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [*Hint:* First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

***46.** This exercise presents **Russell's paradox**. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.

a) Show the assumption that S is a member of S leads to a contradiction.

b) Show the assumption that S is not a member of S leads to a contradiction.

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

***47.** Describe a procedure for listing all the subsets of a finite set.

2.2 Set Operations

Introduction

Two, or more, sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

Links

DEFINITION 1 Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B. The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B.

We will give some examples of the union of sets.

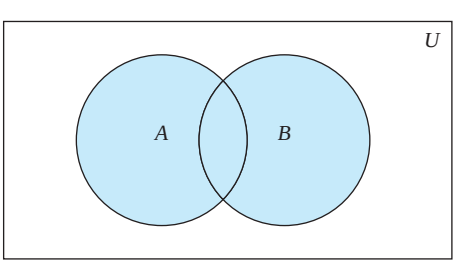
EXAMPLE 1 The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$. ◀

EXAMPLE 2 The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both). ◀

DEFINITION 2 Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

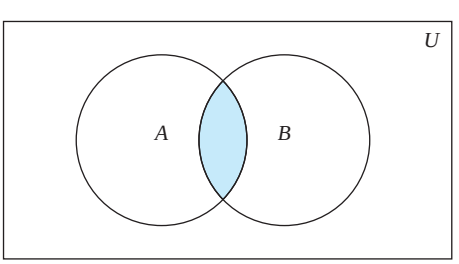
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram of the Union of A and B.



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram of the Intersection of A and B.

The Venn diagram shown in Figure 2 represents the intersection of two sets A and B. The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B.

We give some examples of the intersection of sets.

EXAMPLE 3 The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

EXAMPLE 4 The intersection of the set of all computer science majors at your school and the set of all mathematics majors is the set of all students who are joint majors in mathematics and computer science. ◀

DEFINITION 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

Be careful not to overcount!

We are often interested in finding the cardinality of a union of two finite sets A and B. Note that $|A| + |B|$ counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion–exclusion**. The principle of inclusion–exclusion is an important technique used in enumeration. We will discuss this principle and other counting techniques in detail in Chapters 6 and 8.

There are other important ways to combine sets.

DEFINITION 4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

Remark: The difference of sets A and B is sometimes denoted by $A \setminus B$.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

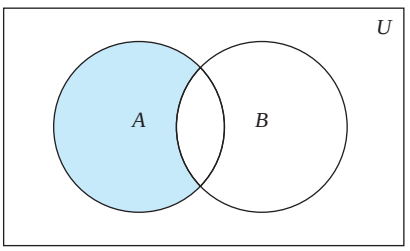
$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

The Venn diagram shown in Figure 3 represents the difference of the sets A and B. The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents $A - B$.

We give some examples of differences of sets.

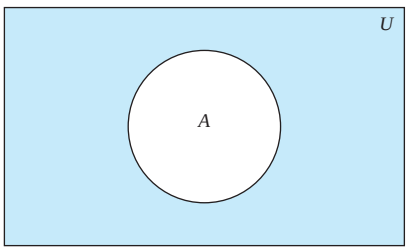
EXAMPLE 6 The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

EXAMPLE 7 The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not also mathematics majors. ◀



$A - B$ is shaded.

FIGURE 3 **Venn Diagram for the Difference of A and B.**



$\overline{A}$ is shaded.

FIGURE 4 **Venn Diagram for the Complement of the Set A.**

Once the universal set U has been specified, the **complement** of a set can be defined.

DEFINITION 5 Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

In Figure 4 the shaded area outside the circle representing A is the area representing $\overline{A}$.

We give some examples of the complement of a set.

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. ◀

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ◀

It is left to the reader (Exercise 19) to show that we can express the difference of A and B as the intersection of A and the complement of B. That is,

$$A - B = A \cap \overline{B}.$$

Set Identities

Table 1 lists the most important set identities. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of the remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.3. (Compare Table 6 of Section 1.6 and Table 1.) In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 12).

Set identities and propositional equivalences are just special cases of identities for Boolean algebra.

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this. We illustrate this type of proof by establishing the first of De Morgan's laws.

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

EXAMPLE 10 Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

This identity says that the complement of the intersection of two sets is the union of their complements.

Solution: We will prove that the two sets $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Extra Examples

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. We do this by showing that if x is in $\overline{A} \cup \overline{B}$, then it must also be in $\overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, we know that $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, the proposition $\neg(x \in A) \vee \neg(x \in B)$ is true.

By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved. ◀

We can more succinctly express the reasoning used in Example 10 using set builder notation, as Example 11 illustrates.

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement}\\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol}\\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection}\\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences}\\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol}\\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement}\\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union}\\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the second De Morgan law for logical equivalences. ◀

Proving a set identity involving more than two sets by showing each side of the identity is a subset of the other often requires that we keep track of different cases, as illustrated by the proof in Example 12 of one of the distributive laws for sets.

EXAMPLE 12 Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ◀

Set identities can also be proved using **membership tables**. We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used. (The reader should note the similarity between membership tables and truth tables.)

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid. ◀

Additional set identities can be established using those that we have already proved. Consider Example 14.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

EXAMPLE 14 Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

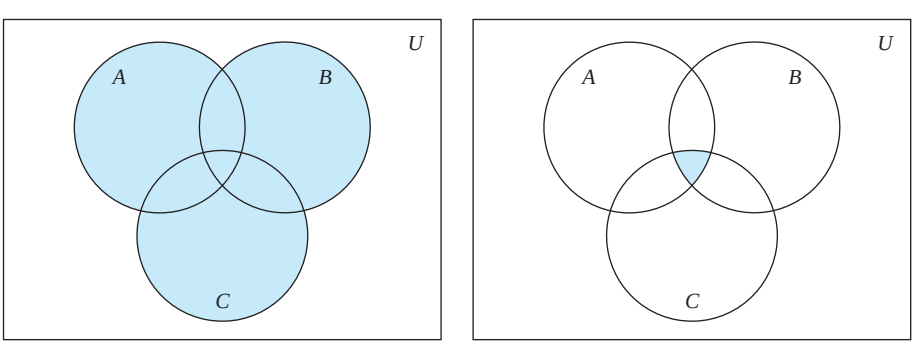
Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

◀

Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C. These combinations of the three sets, A, B, and C, are shown in Figure 5.



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

FIGURE 5 The Union and Intersection of A, B, and C.

EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,

$$A \cap B \cap C = \{0\}.$$ ◀

We can also consider unions and intersections of an arbitrary number of sets. We introduce these definitions.

DEFINITION 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

DEFINITION 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$. We illustrate generalized unions and intersections with Example 16.

EXAMPLE 16 For $i = 1, 2, \ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i + 1, i + 2, \ldots\} = \{n, n + 1, n + 2, \ldots\} = A_n.$$ ◀

We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n, \ldots$. Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\ (x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\ (x \in A_i)\}$.

EXAMPLE 17 Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer n is in at least one of the sets, because it belongs to $A_n = \{1, 2, \ldots, n\}$, and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set $\{1\}$, note that the only element that belongs to all the sets $A_1, A_2, \ldots$ is 1. To see this note that $A_1 = \{1\}$ and $1 \in A_i$ for $i = 1, 2, \ldots$. ◀

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A. Example 18 illustrates this technique.

EXAMPLE 18 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading.) Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000. ◀

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin \overline{A}$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

EXAMPLE 19 We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. ◀

To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the ith position of the bit string of the union is 1 if either of the bits in the ith position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise *OR* of the bit strings for the two sets. The bit in the ith position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are). Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

EXAMPLE 20 The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set $\{1, 3, 5\}$. ◀

Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.
a) $A \cap B$ **b)** $A \cup B$
c) $A - B$ **d)** $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.
a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that
a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that
a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that
a) $A \cup A = A$. **b)** $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that
a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

10. Show that
a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that
a) $A \cup B = B \cup A$.
b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$
a) by showing each side is a subset of the other side.
b) using a membership table.

16. Let A and B be sets. Show that
a) $(A \cap B) \subseteq A$. **b)** $A \subseteq (A \cup B)$.
c) $A - B \subseteq A$. **d)** $A \cap (B - A) = \emptyset$.
e) $A \cup (B - A) = A \cup B$.

17. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$
a) by showing each side is a subset of the other side.
b) using a membership table.

18. Let A, B, and C be sets. Show that
a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

19. Show that if A and B are sets, then
a) $A - B = A \cap \overline{B}$.
b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

20. Show that if A and B are sets with $A \subseteq B$, then
a) $A \cup B = B$.
b) $A \cap B = A$.

21. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

22. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

23. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.
c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

26. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.
a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

27. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.
a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

28. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.
a) $(A \cap B) \cup (C \cap D)$ **b)** $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$
c) $A - (B \cap C \cap D)$

29. What can you say about the sets A and B if we know that
a) $A \cup B = A$? **b)** $A \cap B = A$?
c) $A - B = A$? **d)** $A \cap B = B \cap A$?
e) $A - B = B - A$?

30. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?

c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

31. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

32. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

34. Draw a Venn diagram for the symmetric difference of the sets A and B.

35. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

36. Show that $A \oplus B = (A - B) \cup (B - A)$.

37. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.

c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

38. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$. **b)** $(A \oplus B) \oplus B = A$.

39. What can you say about the sets A and B if $A \oplus B = A$?

***40.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***41.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

42. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

43. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

44. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

45. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***46.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

47. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

48. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

49. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

50. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i + 1, i + 2, \ldots\}$.

b) $A_i = \{0, i\}$.

c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.

d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

51. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i + 1, \ldots, -1, 0, 1, \ldots, i - 1, i\}$.

b) $A_i = \{-i, i\}$.

c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \leq x \leq i$.

d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \geq i$.

52. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$

b) $\{1, 3, 6, 10\}$

c) $\{2, 3, 4, 7, 8, 9\}$

53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.

a) 11 1100 1111

b) 01 0111 1000

c) 10 0000 0001

54. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros

b) the string with all ones

55. What is the bit string corresponding to the difference of two sets?

56. What is the bit string corresponding to the symmetric difference of two sets?

57. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$

c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

58. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

59. Find the successors of the following sets.

a) $\{1, 2, 3\}$ **b)** $\emptyset$

c) $\{\emptyset\}$ **d)** $\{\emptyset, \{\emptyset\}\}$

60. How many elements does the successor of a set with n elements have?

Sometimes the number of times that an element occurs in an unordered collection matters. **Multisets** are unordered collections of elements where an element can occur as a member more than once. The notation $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers m_i, $i = 1, 2, \ldots, r$ are called the **multiplicities** of the elements a_i, $i = 1, 2, \ldots, r$.

Let P and Q be multisets. The **union** of the multisets P and Q is the multiset where the multiplicity of an element is the maximum of its multiplicities in P and Q. The **intersection** of P and Q is the multiset where the multiplicity of an element is the minimum of its multiplicities in P and Q. The **difference** of P and Q is the multiset where the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0. The **sum** of P and Q is the multiset where the multiplicity of an element is the sum of multiplicities in P and Q. The union, intersection, and difference of P and Q are denoted by $P \cup Q$, $P \cap Q$, and $P - Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of P and Q is denoted by $P + Q$.

61. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find

a) $A \cup B$. **b)** $A \cap B$. **c)** $A - B$.
d) $B - A$. **e)** $A + B$.

62. Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset $\{107 \cdot$ personal computers, $44 \cdot$ routers, $6 \cdot$ servers$\}$ and B could be the multiset $\{14 \cdot$ personal computers, $6 \cdot$ routers, $2 \cdot$ mainframes$\}$.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with $R =$ {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.

63. The **complement** of a fuzzy set S is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in S. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

64. The **union** of two fuzzy sets S and T is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cup R$ of rich or famous people.

65. The **intersection** of two fuzzy sets S and T is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cap R$ of rich and famous people.

2.3 Functions

Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure 1.

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions,

43. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ **b)** $\prod_{i=5}^{8} i$

c) $\prod_{i=1}^{100} (-1)^i$ **d)** $\prod_{i=1}^{10} 2$

Recall that the value of the factorial function at a positive integer n, denoted by $n!$, is the product of the positive integers from 1 to n, inclusive. Also, we specify that $0! = 1$.

44. Express $n!$ using product notation.

45. Find $\sum_{j=0}^{4} j!$.

46. Find $\prod_{j=0}^{4} j!$.

2.5 Cardinality of Sets

Introduction

In Definition 4 of Section 2.1 we defined the cardinality of a finite set as the number of elements in the set. We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other. In this section we extend this notion to infinite sets. That is, we will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.

We will be particularly interested in countably infinite sets, which are sets with the same cardinality as the set of positive integers. We will establish the surprising result that the set of rational numbers is countably infinite. We will also provide an example of an uncountable set when we show that the set of real numbers is not countable.

The concepts developed in this section have important applications to computer science. A function is called uncomputable if no computer program can be written to find all its values, even with unlimited time and memory. We will use the concepts in this section to explain why uncomputable functions exist.

We now define what it means for two sets to have the same size, or cardinality. In Section 2.1, we discussed the cardinality of finite sets and we defined the size, or cardinality, of such sets. In Exercise 79 of Section 2.3 we showed that there is a one-to-one correspondence between any two finite sets with the same number of elements. We use this observation to extend the concept of cardinality to all sets, both finite and infinite.

DEFINITION 1

The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

For infinite sets the definition of cardinality provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set. We can also define what it means for one set to have a smaller cardinality than another set.

DEFINITION 2

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Countable Sets

We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with a different cardinality.

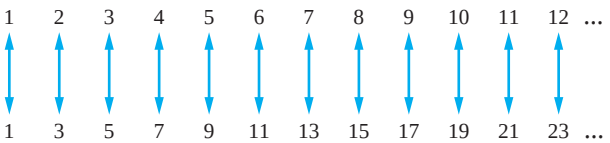


FIGURE 1 **A One-to-One Correspondence Between $\mathbf{Z}^+$ and the Set of Odd Positive Integers.**

DEFINITION 3

A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

We illustrate how to show a set is countable in the next example.

EXAMPLE 1 Show that the set of odd positive integers is a countable set.

Solution: To show that the set of odd positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers. Consider the function

$$f(n) = 2n - 1$$

from $\mathbf{Z}^+$ to the set of odd positive integers. We show that f is a one-to-one correspondence by showing that it is both one-to-one and onto. To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n - 1 = 2m - 1$, so $n = m$. To see that it is onto, suppose that t is an odd positive integer. Then t is 1 less than an even integer $2k$, where k is a natural number. Hence $t = 2k - 1 = f(k)$. We display this one-to-one correspondence in Figure 1. ◀

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers). The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$.

You can always get a room at Hilbert's Grand Hotel!

HILBERT'S GRAND HOTEL We now describe a paradox that shows that something impossible with finite sets may be possible with infinite sets. The famous mathematician David Hilbert invented the notion of the **Grand Hotel**, which has a countably infinite number of rooms, each occupied by a guest. When a new guest arrives at a hotel with a finite number of rooms, and all rooms are occupied, this guest cannot be accommodated without evicting a current guest. However, we can always accommodate a new guest at the Grand Hotel, even when all rooms are already occupied, as we show in Example 2. Exercises 5 and 8 ask you to show that we can accommodate a finite number of new guests and a countable number of new guests, respectively, at the fully occupied Grand Hotel.

DAVID HILBERT (1862–1943) Hilbert, born in Königsberg, the city famous in mathematics for its seven bridges, was the son of a judge. During his tenure at Göttingen University, from 1892 to 1930, he made many fundamental contributions to a wide range of mathematical subjects. He almost always worked on one area of mathematics at a time, making important contributions, then moving to a new mathematical subject. Some areas in which Hilbert worked are the calculus of variations, geometry, algebra, number theory, logic, and mathematical physics. Besides his many outstanding original contributions, Hilbert is remembered for his famous list of 23 difficult problems. He described these problems at the 1900 International Congress of Mathematicians, as a challenge to mathematicians at the birth of the twentieth century. Since that time, they have spurred a tremendous amount and variety of research. Although many of these problems have now been solved, several remain open, including the Riemann hypothesis, which is part of Problem 8 on Hilbert's list. Hilbert was also the author of several important textbooks in number theory and geometry.

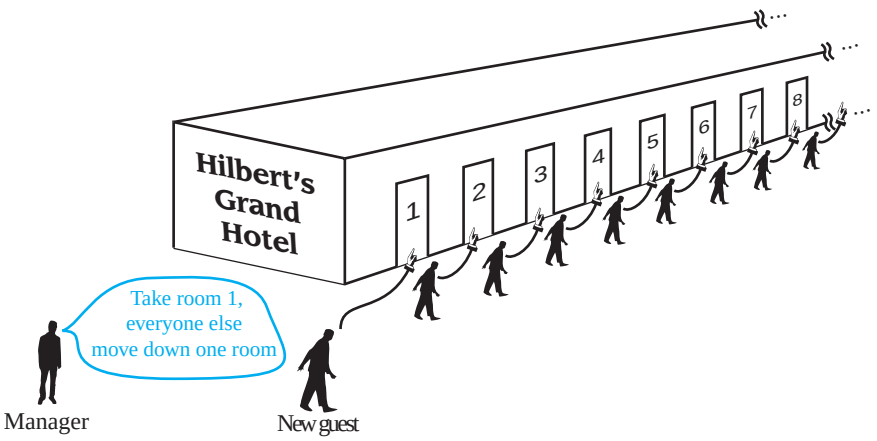


FIGURE 2 **A New Guest Arrives at Hilbert's Grand Hotel.**

EXAMPLE 2 How can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?

Solution: Because the rooms of the Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general, the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms. We illustrate this situation in Figure 2. ◀

When there are finitely many room in a hotel, the notion that all rooms are occupied is equivalent to the notion that no new guests can be accommodated. However, Hilbert's paradox of the Grand Hotel can be explained by noting that this equivalence no longer holds when there are infinitely many room.

EXAMPLES OF COUNTABLE AND UNCOUNTABLE SETS We will now show that certain sets of numbers are countable. We begin with the set of all integers. Note that we can show that the set of all integers is countable by listing its members.

EXAMPLE 3 Show that the set of all integers is countable.

Solution: We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: $0, 1, -1, 2, -2, \ldots$. Alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers. We leave it to the reader to show that the function $f(n) = n/2$ when n is even and $f(n) = -(n-1)/2$ when n is odd is such a function. Consequently, the set of all integers is countable. ◀

It is not surprising that the set of odd integers and the set of all integers are both countable sets (as shown in Examples 1 and 3). Many people are amazed to learn that the set of rational numbers is countable, as Example 4 demonstrates.

EXAMPLE 4 Show that the set of positive rational numbers is countable.

Solution: It may seem surprising that the set of positive rational numbers is countable, but we will show how we can list the positive rational numbers as a sequence $r_1, r_2, \ldots, r_n, \ldots$. First, note that every positive rational number is the quotient p/q of two positive integers. We can

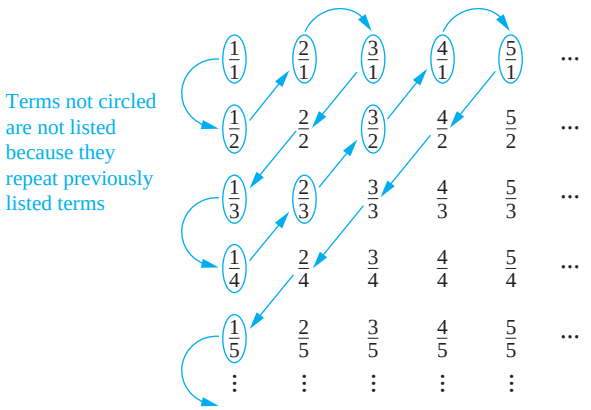


FIGURE 3 **The Positive Rational Numbers Are Countable.**

arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on, as displayed in Figure 3.

The key to listing the rational numbers in a sequence is to first list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on, following the path shown in Figure 3. Whenever we encounter a number p/q that is already listed, we do not list it again. For example, when we come to $2/2 = 1$ we do not list it because we have already listed $1/1 = 1$. The initial terms in the list of positive rational numbers we have constructed are $1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5$, and so on. These numbers are shown circled; the uncircled numbers in the list are those we leave out because they are already listed. Because all positive rational numbers are listed once, as the reader can verify, we have shown that the set of positive rational numbers is countable. ◀

An Uncountable Set

Not all infinite sets have the same size!

We have seen that the set of positive rational numbers is a countable set. Do we have a promising candidate for an uncountable set? The first place we might look is the set of real numbers. In Example 5 we use an important proof method, introduced in 1879 by Georg Cantor and known as the **Cantor diagonalization argument**, to prove that the set of real numbers is not countable. This proof method is used extensively in mathematical logic and in the theory of computation.

Links

EXAMPLE 5 Show that the set of real numbers is an uncountable set.

Extra Examples

Solution: To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable; see Exercise 16). Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r_1, r_2, r_3, \ldots$. Let the decimal representation of these real numbers be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}\ldots \\ r_4 &= 0.d_{41}d_{42}d_{43}d_{44}\ldots \\ &\vdots \end{aligned}$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (For example, if $r_1 = 0.23794102\ldots$, we have $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 7$, and so on.) Then, form a new real number with decimal expansion

$r = 0.d_1d_2d_3d_4\ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 \text{ if } d_{ii} \neq 4 \\ 5 \text{ if } d_{ii} = 4. \end{cases}$$

(As an example, suppose that $r_1 = 0.23794102\ldots$, $r_2 = 0.44590138\ldots$, $r_3 = 0.09118764\ldots$, $r_4 = 0.80553900\ldots$, and so on. Then we have $r = 0.d_1d_2d_3d_4\ldots = 0.4544\ldots$, where $d_1 = 4$ because $d_{11} \neq 4$, $d_2 = 5$ because $d_{22} = 4$, $d_3 = 4$ because $d_{33} \neq 4$, $d_4 = 4$ because $d_{44} \neq 4$, and so on.)

A number with a decimal expansion that terminates has a second decimal expansion ending with an infinite sequence of 9s because $1 = 0.999\ldots$.

Every real number has a unique decimal expansion (when the possibility that the expansion has a tail end that consists entirely of the digit 9 is excluded). Therefore, the real number r is not equal to any of $r_1, r_2, \ldots$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Because there is a real number r between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable (see Exercise 15). Hence, the set of real numbers is uncountable. ◀

RESULTS ABOUT CARDINALITY We will now discuss some results about the cardinality of sets. First, we will prove that the union of two countable sets is also countable.

THEOREM 1 If A and B are countable sets, then $A \cup B$ is also countable.

Proof: Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint. (If they are not, we can replace B by $B - A$, because $A \cap (B - A) = \emptyset$ and $A \cup (B - A) = A \cup B$.) Furthermore, without loss of generality, if one of the two sets is countably infinite and other finite, we can assume that B is the one that is finite.

This proof uses WLOG and cases.

There are three cases to consider: (i) A and B are both finite, (ii) A is infinite and B is finite, and (iii) A and B are both countably infinite.

Case (i): Note that when A and B are finite, $A \cup B$ is also finite, and therefore, countable.

Case (ii): Because A is countably infinite, its elements can be listed in an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ and because B is finite, its terms can be listed as $b_1, b_2, \ldots, b_m$ for some positive integer m. We can list the elements of $A \cup B$ as $b_1, b_2, \ldots, b_m, a_1, a_2, a_3, \ldots, a_n, \ldots$. This means that $A \cup B$ is countably infinite.

Case (iii): Because both A and B are countably infinite, we can list their elements as $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$, respectively. By alternating terms of these two sequences we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$. This means $A \cup B$ must be countably infinite.

We have completed the proof, as we have shown that $A \cup B$ is countable in all three cases. ◁

Because of its importance, we now state a key theorem in the study of cardinality.

THEOREM 2 **SCHRÖDER-BERNSTEIN THEOREM** If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Because Theorem 2 seems to be quite straightforward, we might expect that it has an easy proof. However, even though it can be proved without using advanced mathematics, no known proof is easy to explain. Consequently, we omit a proof here. We refer the interested reader to [AiZiHo09] and [Ve06] for a proof. This result is called the Schröder-Bernstein theorem after Ernst Schröder who published a flawed proof of it in 1898 and Felix Bernstein, a student of Georg Cantor, who presented a proof in 1897. However, a proof of this theorem was found in notes of Richard Dedekind dated 1887. Dedekind was a German mathematician who made important contributions to the foundations of mathematics, abstract algebra, and number theory.

We illustrate the use of Theorem 2 with an example.

EXAMPLE 6 Show that the $|(0, 1)| = |(0, 1]|$.

Solution: It is not at all obvious how to find a one-to-one correspondence between $(0, 1)$ and $(0, 1]$ to show that $|(0, 1)| = |(0, 1]|$. Fortunately, we can use the Schröder-Bernstein theorem instead. Finding a one-to-one function from $(0, 1)$ to $(0, 1]$ is simple. Because $(0, 1) \subset (0, 1]$, $f(x) = x$ is a one-to-one function from $(0, 1)$ to $(0, 1]$. Finding a one-to-one function from $(0, 1]$ to $(0, 1)$ is also not difficult. The function $g(x) = x/2$ is clearly one-to-one and maps $(0, 1]$ to $(0, 1/2] \subset (0, 1)$. As we have found one-to-one functions from $(0, 1)$ to $(0, 1]$ and from $(0, 1]$ to $(0, 1)$, the Schröder-Bernstein theorem tells us that $|(0, 1)| = |(0, 1]|$. ◀

UNCOMPUTABLE FUNCTIONS We will now describe an important application of the concepts of this section to computer science. In particular, we will show that there are functions whose values cannot be computed by any computer program.

DEFINITION 4 We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.

To show that there are uncomputable functions, we need to establish two results. First, we need to show that the set of all computer programs in any particular programming language is countable. This can be proved by noting that a computer programs in a particular language can be thought of as a string of characters from a finite alphabet (see Exercise 37). Next, we show that there are uncountably many different functions from a particular countably infinite set to itself. In particular, Exercise 38 shows that the set of functions from the set of positive integers to itself is uncountable. This is a consequence of the uncountability of the real numbers between 0 and 1 (see Example 5). Putting these two results together (Exercise 39) shows that there are uncomputable functions.

THE CONTINUUM HYPOTHESIS We conclude this section with a brief discussion of a famous open question about cardinality. It can be shown that the power set of $\mathbf{Z}^+$ and the set of real numbers $\mathbf{R}$ have the same cardinality (see Exercise 38). In other words, we know that $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}| = \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the set of real numbers.

An important theorem of Cantor (Exercise 40) states that the cardinality of a set is always less than the cardinality of its power set. Hence, $|\mathbf{Z}^+| < |\mathcal{P}(\mathbf{Z}^+)|$. We can rewrite this as $\aleph_0 < 2^{\aleph_0}$, using the notation $2^{|S|}$ to denote the cardinality of the power set of the set S. Also, note that the relationship $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$ can be expressed as $2^{\aleph_0} = \mathfrak{c}$.

This leads us to the famous **continuum hypothesis**, which asserts that there is no cardinal number X between $\aleph_0$ and $\mathfrak{c}$. In other words, the continuum hypothesis states that there is no set A such that $\aleph_0$, the cardinality of the set of positive integers, is less than $|A|$ and $|A|$ is less than $\mathfrak{c}$, the cardinality of the set of real numbers. It can be shown that the smallest infinite cardinal numbers form an infinite sequence $\aleph_0 < \aleph_1 < \aleph_2 < \cdots$. If we assume that the continuum hypothesis is true, it would follow that $\mathfrak{c} = \aleph_1$, so that $2^{\aleph_0} = \aleph_1$.

$\mathfrak{c}$ is the lowercase Fraktur c.

The continuum hypothesis was stated by Cantor in 1877. He labored unsuccessfully to prove it, becoming extremely dismayed that he could not. By 1900, settling the continuum hypothesis was considered to be among the most important unsolved problems in mathematics. It was the first problem posed by David Hilbert in his famous 1900 list of open problems in mathematics.

The continuum hypothesis is still an open question and remains an area for active research. However, it has been shown that it can be neither proved nor disproved under the standard set theory axioms in modern mathematics, the Zermelo-Fraenkel axioms. The Zermelo-Fraenkel axioms were formulated to avoid the paradoxes of naive set theory, such as Russell's paradox, but there is much controversy whether they should be replaced by some other set of axioms for set theory.

Exercises

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
- **a)** the negative integers
- **b)** the even integers
- **c)** the integers less than 100
- **d)** the real numbers between 0 and $\frac{1}{2}$
- **e)** the positive integers less than 1,000,000,000
- **f)** the integers that are multiples of 7

2. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
- **a)** the integers greater than 10
- **b)** the odd negative integers
- **c)** the integers with absolute value less than 1,000,000
- **d)** the real numbers between 0 and 2
- **e)** the set $A \times \mathbf{Z}^+$ where $A = \{2, 3\}$
- **f)** the integers that are multiples of 10

3. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
- **a)** all bit strings not containing the bit 0
- **b)** all positive rational numbers that cannot be written with denominators less than 4
- **c)** the real numbers not containing 0 in their decimal representation
- **d)** the real numbers containing only a finite number of 1s in their decimal representation

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
- **a)** integers not divisible by 3
- **b)** integers divisible by 5 but not by 7
- **c)** the real numbers with decimal representations consisting of all 1s
- **d)** the real numbers with decimal representations of all 1s or 9s

5. Show that a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

6. Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.

7. Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

8. Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

***9.** Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Show that all the arriving guests can be accommodated without evicting any current guest.

10. Give an example of two uncountable sets A and B such that $A - B$ is
- **a)** finite.
- **b)** countably infinite.
- **c)** uncountable.

11. Give an example of two uncountable sets A and B such that $A \cap B$ is
- **a)** finite.
- **b)** countably infinite.
- **c)** uncountable.

12. Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.

13. Explain why the set A is countable if and only if $|A| \leq |\mathbf{Z}^+|$.

14. Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$.

☞ **15.** Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

☞ **16.** Show that a subset of a countable set is also countable.

17. If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

18. Show that if A and B are sets $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

19. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

20. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

21. Show that if A, B, and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

22. Suppose that A is a countable set. Show that the set B is also countable if there is an onto function f from A to B.

23. Show that if A is an infinite set, then it contains a countably infinite subset.

24. Show that there is no infinite set A such that $|A| < |\mathbf{Z}^+| = \aleph_0$.

25. Prove that if it is possible to label each element of an infinite set S with a finite string of keyboard characters, from a finite list characters, where no two elements of S have the same label, then S is a countably infinite set.

26. Use Exercise 25 to provide a proof different from that in the text that the set of rational numbers is countable. [*Hint:* Show that you can express a rational number as a string of digits with a slash and possibly a minus sign.]

***27.** Show that the union of a countable number of countable sets is countable.

28. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

***29.** Show that the set of all finite bit strings is countable.

***30.** Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.

***31.** Show that $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

***32.** Show that when you substitute $(3n + 1)^2$ for each occurrence of n and $(3m + 1)^2$ for each occurrence of m in the right-hand side of the formula for the function $f(m, n)$ in Exercise 31, you obtain a one-to-one polynomial function $\mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$. It is an open question whether there is a one-to-one polynomial function $\mathbf{Q} \times \mathbf{Q} \rightarrow \mathbf{Q}$.

33. Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality

34. Show that $(0, 1)$ and $\mathbf{R}$ have the same cardinality. [*Hint:* Use the Schröder-Bernstein theorem.]

35. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [*Hint:* Assume that there is such a one-to-one correspondence. Represent a subset of the set of positive integers as an infinite bit string with ith bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with its ith bit equal to the complement of the ith bit of the ith string in the list. Show that this new bit string cannot appear in the list.]

***36.** Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. [*Hint:* Look at the first part of the hint for Exercise 35.]

***37.** Show that the set of all computer programs in a particular programming language is countable. [*Hint:* A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.]

***38.** Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2 \ldots d_n \ldots$ the function f with $f(n) = d_n$.]

***39.** We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 37 and 38 to show that there are functions that are not computable.

***40.** Show that if S is a set, then there does not exist an onto function f from S to $\mathcal{P}(S)$, the power set of S. Conclude that $|S| < |\mathcal{P}(S)|$. This result is known as **Cantor's theorem**. [*Hint:* Suppose such a function f existed. Let $T = \{s \in S \mid s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$.]

2.6 Matrices

Introduction

Matrices are used throughout discrete mathematics to express relationships between elements in sets. In subsequent chapters we will use matrices in a wide variety of models. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models. This section reviews matrix arithmetic that will be used in these algorithms.

DEFINITION 1

A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

EXAMPLE 1 The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix. ◀

We now introduce some terminology about matrices. Boldface uppercase letters will be used to represent matrices.

DEFINITION 2

Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The ith *row* of $\mathbf{A}$ is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The jth *column* of $\mathbf{A}$ is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}.$$

The (i, j)th *element* or *entry* of $\mathbf{A}$ is the element a_{ij}, that is, the number in the ith row and jth column of $\mathbf{A}$. A convenient shorthand notation for expressing the matrix $\mathbf{A}$ is to write $\mathbf{A} = [a_{ij}]$, which indicates that $\mathbf{A}$ is the matrix with its (i, j)th element equal to a_{ij}.

Matrix Arithmetic

The basic operations of matrix arithmetic will now be discussed, beginning with a definition of matrix addition.

DEFINITION 3

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The *sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because the sum of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

EXAMPLE 2 We have $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$. ◀

We now discuss matrix products. A product of two matrices is defined only when the number of columns in the first matrix equals the number of rows of the second matrix.

DEFINITION 4

Let $\mathbf{A}$ be an $m \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix. The *product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is the $m \times n$ matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. In other words, if $\mathbf{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

In Figure 1 the colored row of $\mathbf{A}$ and the colored column of $\mathbf{B}$ are used to compute the element c_{ij} of $\mathbf{AB}$. The product of two matrices is not defined when the number of columns in the first matrix and the number of rows in the second matrix are not the same.

We now give some examples of matrix products.

EXAMPLE 3 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find $\mathbf{AB}$ if it is defined.

Extra Examples

Solution: Because $\mathbf{A}$ is a 4×3 matrix and $\mathbf{B}$ is a 3×2 matrix, the product $\mathbf{AB}$ is defined and is a 4×2 matrix. To find the elements of $\mathbf{AB}$, the corresponding elements of the rows of $\mathbf{A}$ and the columns of $\mathbf{B}$ are first multiplied and then these products are added. For instance, the element in the (3, 1)th position of $\mathbf{AB}$ is the sum of the products of the corresponding elements of the third row of $\mathbf{A}$ and the first column of $\mathbf{B}$; namely, $3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3 = 7$. When all the elements of $\mathbf{AB}$ are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

◀

Matrix multiplication is *not* commutative. That is, if $\mathbf{A}$ and $\mathbf{B}$ are two matrices, it is not necessarily true that $\mathbf{AB}$ and $\mathbf{BA}$ are the same. In fact, it may be that only one of these two products is defined. For instance, if $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×4, then $\mathbf{AB}$ is defined and is 2×4; however, $\mathbf{BA}$ is not defined, because it is impossible to multiply a 3×4 matrix and a 2×3 matrix.

In general, suppose that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $r \times s$ matrix. Then $\mathbf{AB}$ is defined only when $n = r$ and $\mathbf{BA}$ is defined only when $s = m$. Moreover, even when $\mathbf{AB}$ and $\mathbf{BA}$ are

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

FIGURE 1 **The Product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$.**

both defined, they will not be the same size unless $m = n = r = s$. Hence, if both **AB** and **BA** are defined and are the same size, then both **A** and **B** must be square and of the same size. Furthermore, even with **A** and **B** both $n \times n$ matrices, **AB** and **BA** are not necessarily equal, as Example 4 demonstrates.

EXAMPLE 4 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$. ◀

Transposes and Powers of Matrices

We now introduce an important matrix with entries that are zeros and ones.

DEFINITION 5 The *identity matrix of order n* is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when **A** is an $m \times n$ matrix, we have

$$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

Powers of square matrices can be defined. When **A** is an $n \times n$ matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r \text{ times}}.$$

The operation of interchanging the rows and columns of a square matrix arises in many contexts.

DEFINITION 6

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. In other words, if $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

EXAMPLE 5 The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. ◀

Matrices that do not change when their rows and columns are interchanged are often important.

DEFINITION 7

A square matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

Note that a matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the ith row and ith column for some i). This symmetry is displayed in Figure 2.

EXAMPLE 6 The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric. ◀

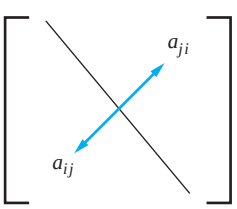


FIGURE 2 A Symmetric Matrix.

Zero–One Matrices

A matrix all of whose entries are either 0 or 1 is called a **zero–one matrix**. Zero–one matrices are often used to represent discrete structures, as we will see in Chapters 9 and 10. Algorithms using these structures are based on Boolean arithmetic with zero–one matrices. This arithmetic is based on the Boolean operations $\wedge$ and $\vee$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 8

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero–one matrices. Then the *join* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \vee \mathbf{B}$. The *meet* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \wedge \mathbf{B}$.

EXAMPLE 7 Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ ◀

We now define the **Boolean product** of two matrices.

DEFINITION 9

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Note that the Boolean product of $\mathbf{A}$ and $\mathbf{B}$ is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation $\vee$ and with multiplication replaced with the operation $\wedge$. We give an example of the Boolean products of matrices.

EXAMPLE 8 Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$ ◀

We can also define the Boolean powers of a square zero–one matrix. These powers will be used in our subsequent studies of paths in graphs, which are used to model such things as communications paths in computer networks.

DEFINITION 10

Let $\mathbf{A}$ be a square zero–one matrix and let r be a positive integer. The rth *Boolean power* of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. The rth Boolean product of $\mathbf{A}$ is denoted by $\mathbf{A}^{[r]}$. Hence

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

EXAMPLE 9 Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀

Exercises

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.

a) What size is $\mathbf{A}$?
b) What is the third column of $\mathbf{A}$?
c) What is the second row of $\mathbf{A}$?
d) What is the element of $\mathbf{A}$ in the (3, 2)th position?
e) What is $\mathbf{A}^t$?

2. Find $\mathbf{A} + \mathbf{B}$, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$.

3. Find $\mathbf{AB}$ if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$.

4. Find the product **AB**, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$.

5. Find a matrix **A** such that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

[*Hint:* Finding **A** requires that you solve systems of linear equations.]

6. Find a matrix **A** such that

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 0 & 3 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 7 & 1 & 3 \\ 1 & 0 & 3 \\ -1 & -3 & 7 \end{bmatrix}.$$

7. Let **A** be an $m \times n$ matrix and let **0** be the $m \times n$ matrix that has all entries equal to zero. Show that $\mathbf{A} = \mathbf{0} + \mathbf{A} = \mathbf{A} + \mathbf{0}$.

8. Show that matrix addition is commutative; that is, show that if **A** and **B** are both $m \times n$ matrices, then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

9. Show that matrix addition is associative; that is, show that if **A**, **B**, and **C** are all $m \times n$ matrices, then $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

10. Let **A** be a 3×4 matrix, **B** be a 4×5 matrix, and **C** be a 4×4 matrix. Determine which of the following products are defined and find the size of those that are defined.

a) **AB** **b)** **BA** **c)** **AC**

d) **CA** **e)** **BC** **f)** **CB**

11. What do we know about the sizes of the matrices **A** and **B** if both of the products **AB** and **BA** are defined?

12. In this exercise we show that matrix multiplication is distributive over matrix addition.

a) Suppose that **A** and **B** are $m \times k$ matrices and that **C** is a $k \times n$ matrix. Show that $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

b) Suppose that **C** is an $m \times k$ matrix and that **A** and **B** are $k \times n$ matrices. Show that $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$.

13. In this exercise we show that matrix multiplication is associative. Suppose that **A** is an $m \times p$ matrix, **B** is a $p \times k$ matrix, and **C** is a $k \times n$ matrix. Show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

14. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ when $i \neq j$. Show that the product of two $n \times n$ diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

15. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a formula for $\mathbf{A}^n$, whenever n is a positive integer.

16. Show that $(\mathbf{A}^t)^t = \mathbf{A}$.

17. Let **A** and **B** be two $n \times n$ matrices. Show that

a) $(\mathbf{A} + \mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.

b) $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$.

If **A** and **B** are $n \times n$ matrices with $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, then **B** is called the **inverse** of **A** (this terminology is appropriate because such a matrix **B** is unique) and **A** is said to be **invertible**. The notation $\mathbf{B} = \mathbf{A}^{-1}$ denotes that **B** is the inverse of **A**.

18. Show that

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}.$$

19. Let **A** be the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

20. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}.$$

a) Find $\mathbf{A}^{-1}$. [*Hint:* Use Exercise 19.]

b) Find $\mathbf{A}^3$.

c) Find $(\mathbf{A}^{-1})^3$.

d) Use your answers to (b) and (c) to show that $(\mathbf{A}^{-1})^3$ is the inverse of $\mathbf{A}^3$.

21. Let **A** be an invertible matrix. Show that $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ whenever n is a positive integer.

22. Let **A** be a matrix. Show that the matrix $\mathbf{AA}^t$ is symmetric. [*Hint:* Show that this matrix equals its transpose with the help of Exercise 17b.]

23. Suppose that **A** is an $n \times n$ matrix where n is a positive integer. Show that $\mathbf{A} + \mathbf{A}^t$ is symmetric.

Remark: Because $\mathbf{Z}_m$ with the operations of addition and multiplication modulo m satisfies the properties listed, $\mathbf{Z}_m$ with modular addition is said to be a **commutative group** and $\mathbf{Z}_m$ with both of these operations is said to be a **commutative ring**. Note that the set of integers with ordinary addition and multiplication also forms a commutative ring. Groups and rings are studied in courses that cover abstract algebra.

Remark: In Exercise 30, and in later sections, we will use the notations $+$ and $\cdot$ for $+_m$ and $\cdot_m$ without the subscript m on the symbol for the operator whenever we work with $\mathbf{Z}_m$.

Exercises

1. Does 17 divide each of these numbers?
a) 68 **b)** 84 **c)** 357 **d)** 1001

2. Prove that if a is an integer other than 0, then
a) 1 divides a. **b)** a divides 0.

3. Prove that part (*ii*) of Theorem 1 is true.

4. Prove that part (*iii*) of Theorem 1 is true.

5. Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

6. Show that if a, b, c, and d are integers, where $a \neq 0$, such that $a \mid c$ and $b \mid d$, then $ab \mid cd$.

7. Show that if a, b, and c are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$.

8. Prove or disprove that if $a \mid bc$, where a, b, and c are positive integers and $a \neq 0$, then $a \mid b$ or $a \mid c$.

9. What are the quotient and remainder when
a) 19 is divided by 7?
b) -111 is divided by 11?
c) 789 is divided by 23?
d) 1001 is divided by 13?
e) 0 is divided by 19?
f) 3 is divided by 5?
g) -1 is divided by 3?
h) 4 is divided by 1?

10. What are the quotient and remainder when
a) 44 is divided by 8?
b) 777 is divided by 21?
c) -123 is divided by 19?
d) -1 is divided by 23?
e) -2002 is divided by 87?
f) 0 is divided by 17?
g) 1,234,567 is divided by 1001?
h) -100 is divided by 101?

11. What time does a 12-hour clock read
a) 80 hours after it reads 11:00?
b) 40 hours before it reads 12:00?
c) 100 hours after it reads 6:00?

12. What time does a 24-hour clock read
a) 100 hours after it reads 2:00?
b) 45 hours before it reads 12:00?
c) 168 hours after it reads 19:00?

13. Suppose that a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \leq c \leq 12$ such that
a) $c \equiv 9a \pmod{13}$.
b) $c \equiv 11b \pmod{13}$.
c) $c \equiv a + b \pmod{13}$.
d) $c \equiv 2a + 3b \pmod{13}$.
e) $c \equiv a^2 + b^2 \pmod{13}$.
f) $c \equiv a^3 - b^3 \pmod{13}$.

14. Suppose that a and b are integers, $a \equiv 11 \pmod{19}$, and $b \equiv 3 \pmod{19}$. Find the integer c with $0 \leq c \leq 18$ such that
a) $c \equiv 13a \pmod{19}$.
b) $c \equiv 8b \pmod{19}$.
c) $c \equiv a - b \pmod{19}$.
d) $c \equiv 7a + 3b \pmod{19}$.
e) $c \equiv 2a^2 + 3b^2 \pmod{19}$.
f) $c \equiv a^3 + 4b^3 \pmod{19}$.

15. Let m be a positive integer. Show that $a \equiv b \pmod{m}$ if $a \bmod m = b \bmod m$.

16. Let m be a positive integer. Show that $a \bmod m = b \bmod m$ if $a \equiv b \pmod{m}$.

17. Show that if n and k are positive integers, then $\lceil n/k \rceil = \lfloor (n-1)/k \rfloor + 1$.

18. Show that if a is an integer and d is an integer greater than 1, then the quotient and remainder obtained when a is divided by d are $\lfloor a/d \rfloor$ and $a - d\lfloor a/d \rfloor$, respectively.

19. Find a formula for the integer with smallest absolute value that is congruent to an integer a modulo m, where m is a positive integer.

20. Evaluate these quantities.
a) $-17 \bmod 2$ **b)** $144 \bmod 7$
c) $-101 \bmod 13$ **d)** $199 \bmod 19$

21. Evaluate these quantities.
a) $13 \bmod 3$ **b)** $-97 \bmod 11$
c) $155 \bmod 19$ **d)** $-221 \bmod 23$

22. Find $a \operatorname{div} m$ and $a \bmod m$ when
a) $a = -111$, $m = 99$.
b) $a = -9999$, $m = 101$.
c) $a = 10299$, $m = 999$.
d) $a = 123456$, $m = 1001$.

23. Find a **div** m and a **mod** m when
a) $a = 228$, $m = 119$.
b) $a = 9009$, $m = 223$.
c) $a = -10101$, $m = 333$.
d) $a = -765432$, $m = 38271$.

24. Find the integer a such that
a) $a \equiv 43 \pmod{23}$ and $-22 \leq a \leq 0$.
b) $a \equiv 17 \pmod{29}$ and $-14 \leq a \leq 14$.
c) $a \equiv -11 \pmod{21}$ and $90 \leq a \leq 110$.

25. Find the integer a such that
a) $a \equiv -15 \pmod{27}$ and $-26 \leq a \leq 0$.
b) $a \equiv 24 \pmod{31}$ and $-15 \leq a \leq 15$.
c) $a \equiv 99 \pmod{41}$ and $100 \leq a \leq 140$.

26. List five integers that are congruent to 4 modulo 12.

27. List all integers between -100 and 100 that are congruent to -1 modulo 25.

28. Decide whether each of these integers is congruent to 3 modulo 7.
a) 37 **b)** 66
c) -17 **d)** -67

29. Decide whether each of these integers is congruent to 5 modulo 17.
a) 80 **b)** 103
c) -29 **d)** -122

30. Find each of these values.
a) $(177 \textbf{ mod } 31 + 270 \textbf{ mod } 31) \textbf{ mod } 31$
b) $(177 \textbf{ mod } 31 \cdot 270 \textbf{ mod } 31) \textbf{ mod } 31$

31. Find each of these values.
a) $(-133 \textbf{ mod } 23 + 261 \textbf{ mod } 23) \textbf{ mod } 23$
b) $(457 \textbf{ mod } 23 \cdot 182 \textbf{ mod } 23) \textbf{ mod } 23$

32. Find each of these values.
a) $(19^2 \textbf{ mod } 41) \textbf{ mod } 9$
b) $(32^3 \textbf{ mod } 13)^2 \textbf{ mod } 11$
c) $(7^3 \textbf{ mod } 23)^2 \textbf{ mod } 31$
d) $(21^2 \textbf{ mod } 15)^3 \textbf{ mod } 22$

33. Find each of these values.
a) $(99^2 \textbf{ mod } 32)^3 \textbf{ mod } 15$
b) $(3^4 \textbf{ mod } 17)^2 \textbf{ mod } 11$
c) $(19^3 \textbf{ mod } 23)^2 \textbf{ mod } 31$
d) $(89^3 \textbf{ mod } 79)^4 \textbf{ mod } 26$

34. Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a, b, c, d, and m are integers with $m \geq 2$, then $a - c \equiv b - d \pmod{m}$.

35. Show that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

☞ **36.** Show that if a, b, c, and m are integers such that $m \geq 2$, $c > 0$, and $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{mc}$.

37. Find counterexamples to each of these statements about congruences.
a) If $ac \equiv bc \pmod{m}$, where a, b, c, and m are integers with $m \geq 2$, then $a \equiv b \pmod{m}$.
b) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a, b, c, d, and m are integers with c and d positive and $m \geq 2$, then $a^c \equiv b^d \pmod{m}$.

38. Show that if n is an integer then $n^2 \equiv 0$ or $1 \pmod{4}$.

39. Use Exercise 38 to show that if m is a positive integer of the form $4k + 3$ for some nonnegative integer k, then m is not the sum of the squares of two integers.

40. Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod{8}$.

41. Show that if a, b, k, and m are integers such that $k \geq 1$, $m \geq 2$, and $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$.

42. Show that $\mathbf{Z}_m$ with addition modulo m, where $m \geq 2$ is an integer, satisfies the closure, associative, and commutative properties, 0 is an additive identity, and for every nonzero $a \in \mathbf{Z}_m$, $m - a$ is an inverse of a modulo m.

43. Show that $\mathbf{Z}_m$ with multiplication modulo m, where $m \geq 2$ is an integer, satisfies the closure, associative, and commutativity properties, and 1 is a multiplicative identity.

44. Show that the distributive property of multiplication over addition holds for $\mathbf{Z}_m$, where $m \geq 2$ is an integer.

45. Write out the addition and multiplication tables for $\mathbf{Z}_5$ (where by addition and multiplication we mean $+_5$ and $\cdot_5$).

46. Write out the addition and multiplication tables for $\mathbf{Z}_6$ (where by addition and multiplication we mean $+_6$ and $\cdot_6$).

47. Determine whether each of the functions $f(a) = a$ **div** d and $g(a) = a$ **mod** d, where d is a fixed positive integer, from the set of integers to the set of integers, is one-to-one, and determine whether each of these functions is onto.

4.2 Integer Representations and Algorithms

Introduction

Integers can be expressed using any integer greater than one as a base, as we will show in this section. Although we commonly use decimal (base 10), representations, binary (base 2), octal (base 8), and hexadecimal (base 16) representations are often used, especially in computer science. Given a base b and an integer n, we will show how to construct the base b representation of this integer. We will also explain how to quickly covert between binary and octal and between binary and hexadecimal notations.

As mentioned in Section 3.1, the term *algorithm* originally referred to procedures for performing arithmetic operations using the decimal representations of integers. These algorithms, adapted for use with binary representations, are the basis for computer arithmetic. They provide good illustrations of the concept of an algorithm and the complexity of algorithms. For these reasons, they will be discussed in this section.

We will also introduce an algorithm for finding a **div** d and a **mod** d where a and d are integers with $d > 1$. Finally, we will describe an efficient algorithm for modular exponentiation, which is a particularly important algorithm for cryptography, as we will see in Section 4.6.

Representations of Integers

In everyday life we use decimal notation to express integers. For example, 965 is used to denote $9 \cdot 10^2 + 6 \cdot 10 + 5$. However, it is often convenient to use bases other than 10. In particular, computers usually use binary notation (with 2 as the base) when carrying out arithmetic, and octal (base 8) or hexadecimal (base 16) notation when expressing characters, such as letters or digits. In fact, we can use any integer greater than 1 as the base when expressing integers. This is stated in Theorem 1.

THEOREM 1 Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0,$$

where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

A proof of this theorem can be constructed using mathematical induction, a proof method that is discussed in Section 5.1. It can also be found in [Ro10]. The representation of n given in Theorem 1 is called the **base *b* expansion of *n***. The base b expansion of n is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$. For instance, $(245)_8$ represents $2 \cdot 8^2 + 4 \cdot 8 + 5 = 165$. Typically, the subscript 10 is omitted for base 10 expansions of integers because base 10, or **decimal expansions**, are commonly used to represent integers.

BINARY EXPANSIONS Choosing 2 as the base gives **binary expansions** of integers. In binary notation each digit is either a 0 or a 1. In other words, the binary expansion of an integer is just a bit string. Binary expansions (and related expansions that are variants of binary expansions) are used by computers to represent and do arithmetic with integers.

EXAMPLE 1 What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Solution: We have

$$\begin{aligned}(1\ 0101\ 1111)_2 &= 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 \\ &\quad + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 351.\end{aligned}$$

◀

OCTAL AND HEXADECIMAL EXPANSIONS Among the most important bases in computer science are base 2, base 8, and base 16. Base 8 expansions are called **octal** expansions and base 16 expansions are **hexadecimal** expansions.

EXAMPLE 2 What is the decimal expansion of the number with octal expansion $(7016)_8$?

Solution: Using the definition of a base b expansion with $b = 8$ tells us that

$$(7016)_8 = 7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8 + 6 = 3598.$$ ◀

Sixteen different digits are required for hexadecimal expansions. Usually, the hexadecimal digits used are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F, where the letters A through F represent the digits corresponding to the numbers 10 through 15 (in decimal notation).

EXAMPLE 3 What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Solution: Using the definition of a base b expansion with $b = 16$ tells us that

$$(2AE0B)_{16} = 2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16 + 11 = 175627.$$ ◀

Each hexadecimal digit can be represented using four bits. For instance, we see that $(1110\ 0101)_2 = (E5)_{16}$ because $(1110)_2 = (E)_{16}$ and $(0101)_2 = (5)_{16}$. **Bytes**, which are bit strings of length eight, can be represented by two hexadecimal digits.

BASE CONVERSION We will now describe an algorithm for constructing the base b expansion of an integer n. First, divide n by b to obtain a quotient and remainder, that is,

$$n = bq_0 + a_0, \qquad 0 \leq a_0 < b.$$

The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b to obtain

$$q_0 = bq_1 + a_1, \qquad 0 \leq a_1 < b.$$

We see that a_1 is the second digit from the right in the base b expansion of n. Continue this process, successively dividing the quotients by b, obtaining additional base b digits as the remainders. This process terminates when we obtain a quotient equal to zero. It produces the base b digits of n from the right to the left.

EXAMPLE 4 Find the octal expansion of $(12345)_{10}$.

Extra Examples

Solution: First, divide 12345 by 8 to obtain

$$12345 = 8 \cdot 1543 + 1.$$

Successively dividing quotients by 8 gives

$$\begin{aligned} 1543 &= 8 \cdot 192 + 7, \\ 192 &= 8 \cdot 24 + 0, \\ 24 &= 8 \cdot 3 + 0, \\ 3 &= 8 \cdot 0 + 3. \end{aligned}$$

The successive remainders that we have found, 1, 7, 0, 0, and 3, are the digits from the right to the left of 12345 in base 8. Hence,

$$(12345)_{10} = (30071)_8.$$ ◀

EXAMPLE 5 Find the hexadecimal expansion of $(177130)_{10}$.

Solution: First divide 177130 by 16 to obtain

$$177130 = 16 \cdot 11070 + 10.$$

Successively dividing quotients by 16 gives

$$\begin{aligned} 11070 &= 16 \cdot 691 + 14, \\ 691 &= 16 \cdot 43 + 3, \\ 43 &= 16 \cdot 2 + 11, \\ 2 &= 16 \cdot 0 + 2. \end{aligned}$$

The successive remainders that we have found, 10, 14, 3, 11, 2, give us the digits from the right to the left of 177130 in the hexadecimal (base 16) expansion of $(177130)_{10}$. It follows that

$$(177130)_{10} = (2\text{B3EA})_{16}.$$

(Recall that the integers 10, 11, and 14 correspond to the hexadecimal digits A, B, and E, respectively.) ◀

EXAMPLE 6 Find the binary expansion of $(241)_{10}$.

Solution: First divide 241 by 2 to obtain

$$241 = 2 \cdot 120 + 1.$$

Successively dividing quotients by 2 gives

$$\begin{aligned} 120 &= 2 \cdot 60 + 0, \\ 60 &= 2 \cdot 30 + 0, \\ 30 &= 2 \cdot 15 + 0, \\ 15 &= 2 \cdot 7 + 1, \\ 7 &= 2 \cdot 3 + 1, \\ 3 &= 2 \cdot 1 + 1, \\ 1 &= 2 \cdot 0 + 1. \end{aligned}$$

The successive remainders that we have found, 1, 0, 0, 0, 1, 1, 1, 1, are the digits from the right to the left in the binary (base 2) expansion of $(241)_{10}$. Hence,

$$(241)_{10} = (1111\ 0001)_2.$$ ◀

The pseudocode given in Algorithm 1 finds the base b expansion $(a_{k-1} \ldots a_1 a_0)_b$ of the integer n.

TABLE 1 Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

ALGORITHM 1 Constructing Base *b* Expansions.

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while $q \neq 0$
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return $(a_{k-1}, \ldots, a_1, a_0)$ {$(a_{k-1} \ldots a_1 a_0)_b$ is the base b expansion of n}

In Algorithm 1, q represents the quotient obtained by successive divisions by b, starting with $q = n$. The digits in the base b expansion are the remainders of these divisions and are given by q **mod** b. The algorithm terminates when a quotient $q = 0$ is reached.

Remark: Note that Algorithm 1 can be thought of as a greedy algorithm, as the base b digits are taken as large as possible in each step.

CONVERSION BETWEEN BINARY, OCTAL, AND HEXADECIMAL EXPANSIONS

Conversion between binary and octal and between binary and hexadecimal expansions is extremely easy because each octal digit corresponds to a block of three binary digits and each hexadecimal digit corresponds to a block of four binary digits, with these correspondences shown in Table 1 without initial 0s shown. (We leave it as Exercises 13–16 to show that this is the case.) This conversion is illustrated in Example 7.

EXAMPLE 7 Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$ and the binary expansions of $(765)_8$ and $(A8D)_{16}$.

Solution: To convert $(11\ 1110\ 1011\ 1100)_2$ into octal notation we group the binary digits into blocks of three, adding initial zeros at the start of the leftmost block if necessary. These blocks, from left to right, are 011, 111, 010, 111, and 100, corresponding to 3, 7, 2, 7, and 4, respectively. Consequently, $(11\ 1110\ 1011\ 1100)_2 = (37274)_8$. To convert $(11\ 1110\ 1011\ 1100)_2$ into hexadecimal notation we group the binary digits into blocks of four, adding initial zeros at the start of the leftmost block if necessary. These blocks, from left to right, are 0011, 1110, 1011, and 1100, corresponding to the hexadecimal digits 3, E, B, and C, respectively. Consequently, $(11\ 1110\ 1011\ 1100)_2 = (3EBC)_{16}$.

To convert $(765)_8$ into binary notation, we replace each octal digit by a block of three binary digits. These blocks are 111, 110, and 101. Hence, $(765)_8 = (1\ 1111\ 0101)_2$. To convert $(A8D)_{16}$ into binary notation, we replace each hexadecimal digit by a block of four binary digits. These blocks are 1010, 1000, and 1101. Hence, $(A8D)_{16} = (1010\ 1000\ 1101)_2$. ◀

Algorithms for Integer Operations

The algorithms for performing operations with integers using their binary expansions are extremely important in computer arithmetic. We will describe algorithms for the addition and the multiplication of two integers expressed in binary notation. We will also analyze the computational complexity of these algorithms, in terms of the actual number of bit operations used. Throughout this discussion, suppose that the binary expansions of a and b are

$$a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2,\ b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2,$$

so that a and b each have n bits (putting bits equal to 0 at the beginning of one of these expansions if necessary).

We will measure the complexity of algorithms for integer arithmetic in terms of the number of bits in these numbers.

ADDITION ALGORITHM Consider the problem of adding two integers in binary notation. A procedure to perform addition can be based on the usual method for adding numbers with pencil and paper. This method proceeds by adding pairs of binary digits together with carries, when they occur, to compute the sum of two integers. This procedure will now be specified in detail.

To add a and b, first add their rightmost bits. This gives

$$a_0 + b_0 = c_0 \cdot 2 + s_0,$$

where s_0 is the rightmost bit in the binary expansion of $a + b$ and c_0 is the **carry**, which is either 0 or 1. Then add the next pair of bits and the carry,

$$a_1 + b_1 + c_0 = c_1 \cdot 2 + s_1,$$

where s_1 is the next bit (from the right) in the binary expansion of $a + b$, and c_1 is the carry. Continue this process, adding the corresponding bits in the two binary expansions and the carry, to determine the next bit from the right in the binary expansion of $a + b$. At the last stage, add a_{n-1}, b_{n-1}, and c_{n-2} to obtain $c_{n-1} \cdot 2 + s_{n-1}$. The leading bit of the sum is $s_n = c_{n-1}$. This procedure produces the binary expansion of the sum, namely, $a + b = (s_n s_{n-1} s_{n-2} \ldots s_1 s_0)_2$.

EXAMPLE 8 Add $a = (1110)_2$ and $b = (1011)_2$.

Solution: Following the procedure specified in the algorithm, first note that

$$a_0 + b_0 = 0 + 1 = 0 \cdot 2 + 1,$$

so that $c_0 = 0$ and $s_0 = 1$. Then, because

$$a_1 + b_1 + c_0 = 1 + 1 + 0 = 1 \cdot 2 + 0,$$

it follows that $c_1 = 1$ and $s_1 = 0$. Continuing,

$$a_2 + b_2 + c_1 = 1 + 0 + 1 = 1 \cdot 2 + 0,$$

so that $c_2 = 1$ and $s_2 = 0$. Finally, because

$$a_3 + b_3 + c_2 = 1 + 1 + 1 = 1 \cdot 2 + 1,$$

follows that $c_3 = 1$ and $s_3 = 1$. This means that $s_4 = c_3 = 1$. Therefore, $s = a + b = (1\,1001)_2$. This addition is displayed in Figure 1, where carries are shown in blue. ◀

```
 1 1 1
   1 1 1 0
 + 1 0 1 1
 ---------
 1 1 0 0 1
```

FIGURE 1 **Adding $(1110)_2$ and $(1011)_2$.**

The algorithm for addition can be described using pseudocode as follows.

ALGORITHM 2 Addition of Integers.

procedure *add*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$
and $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$, respectively}
$c := 0$
for $j := 0$ **to** $n - 1$
    $d := \lfloor (a_j + b_j + c)/2 \rfloor$
    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$
return $(s_0, s_1, \ldots, s_n)$ {the binary expansion of the sum is $(s_n s_{n-1}\ldots s_0)_2$}

Next, the number of additions of bits used by Algorithm 2 will be analyzed.

EXAMPLE 9 How many additions of bits are required to use Algorithm 2 to add two integers with n bits (or less) in their binary representations?

Solution: Two integers are added by successively adding pairs of bits and, when it occurs, a carry. Adding each pair of bits and the carry requires two additions of bits. Thus, the total number of additions of bits used is less than twice the number of bits in the expansion. Hence, the number of additions of bits used by Algorithm 2 to add two n-bit integers is $O(n)$. ◀

MULTIPLICATION ALGORITHM Next, consider the multiplication of two n-bit integers a and b. The conventional algorithm (used when multiplying with pencil and paper) works as follows. Using the distributive law, we see that

$$\begin{aligned} ab &= a(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1} 2^{n-1}) \\ &= a(b_0 2^0) + a(b_1 2^1) + \cdots + a(b_{n-1} 2^{n-1}). \end{aligned}$$

We can compute ab using this equation. We first note that $ab_j = a$ if $b_j = 1$ and $ab_j = 0$ if $b_j = 0$. Each time we multiply a term by 2, we shift its binary expansion one place to the left and add a zero at the tail end of the expansion. Consequently, we can obtain $(ab_j)2^j$ by **shifting** the binary expansion of ab_j j places to the left, adding j zero bits at the tail end of this binary expansion. Finally, we obtain ab by adding the n integers $ab_j 2^j$, $j = 0, 1, 2, \ldots, n - 1$.

Algorithm 3 displays this procedure for multiplication.

ALGORITHM 3 Multiplication of Integers.

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}a_{n-2}\ldots a_1a_0)_2$ and $(b_{n-1}b_{n-2}\ldots b_1b_0)_2$, respectively}
for $j := 0$ **to** $n-1$
    if $b_j = 1$ **then** $c_j := a$ shifted j places
    else $c_j := 0$
{$c_0, c_1, \ldots, c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ **to** $n-1$
    $p := p + c_j$
return p {p is the value of ab}

Example 10 illustrates the use of this algorithm.

EXAMPLE 10 Find the product of $a = (110)_2$ and $b = (101)_2$.

Solution: First note that

$$ab_0 \cdot 2^0 = (110)_2 \cdot 1 \cdot 2^0 = (110)_2,$$
$$ab_1 \cdot 2^1 = (110)_2 \cdot 0 \cdot 2^1 = (0000)_2,$$

and

$$ab_2 \cdot 2^2 = (110)_2 \cdot 1 \cdot 2^2 = (11000)_2.$$

To find the product, add $(110)_2$, $(0000)_2$, and $(11000)_2$. Carrying out these additions (using Algorithm 2, including initial zero bits when necessary) shows that $ab = (1\ 1110)_2$. This multiplication is displayed in Figure 2. ◀

```
   1 1 0
 × 1 0 1
 -------
   1 1 0
 0 0 0
1 1 0
-------
1 1 1 1 0
```

FIGURE 2 Multiplying $(110)_2$ and $(101)_2$.

Next, we determine the number of additions of bits and shifts of bits used by Algorithm 3 to multiply two integers.

EXAMPLE 11 How many additions of bits and shifts of bits are used to multiply a and b using Algorithm 3?

Solution: Algorithm 3 computes the products of a and b by adding the partial products $c_0, c_1, c_2, \ldots$, and c_{n-1}. When $b_j = 1$, we compute the partial product c_j by shifting the binary expansion of a by j bits. When $b_j = 0$, no shifts are required because $c_j = 0$. Hence, to find all n of the integers ab_j2^j, $j = 0, 1, \ldots, n-1$, requires at most

$$0 + 1 + 2 + \cdots + n - 1$$

shifts. Hence, by Example 5 in Section 3.2 the number of shifts required is $O(n^2)$.

To add the integers ab_j from $j = 0$ to $j = n-1$ requires the addition of an n-bit integer, an $(n+1)$-bit integer, ..., and a $(2n)$-bit integer. We know from Example 9 that each of these additions requires $O(n)$ additions of bits. Consequently, a total of $O(n^2)$ additions of bits are required for all n additions. ◀

Surprisingly, there are more efficient algorithms than the conventional algorithm for multiplying integers. One such algorithm, which uses $O(n^{1.585})$ bit operations to multiply n-bit numbers, will be described in Section 8.3.

ALGORITHM FOR div AND mod Given integers a and d, $d > 0$, we can find $q = a$ **div** d and $r = a$ **mod** d using Algorithm 4. In this brute-force algorithm, when a is positive we subtract d from a as many times as necessary until what is left is less than d. The number of times we perform this subtraction is the quotient and what is left over after all these subtractions is the remainder. Algorithm 4 also covers the case where a is negative. This algorithm finds the quotient q and remainder r when $|a|$ is divided by d. Then, when $a < 0$ and $r > 0$, it uses these to find the quotient $-(q + 1)$ and remainder $d - r$ when a is divided by d. We leave it to the reader (Exercise 59) to show that, assuming that $a > d$, this algorithm uses $O(q \log a)$ bit operations.

ALGORITHM 4 Computing div and mod.

procedure *division algorithm*(a: integer, d: positive integer)
$q := 0$
$r := |a|$
while $r \geq d$
    $r := r - d$
    $q := q + 1$
if $a < 0$ and $r > 0$ **then**
    $r := d - r$
    $q := -(q + 1)$
return (q, r) {$q = a$ **div** d is the quotient, $r = a$ **mod** d is the remainder}

There are more efficient algorithms than Algorithm 4 for determining the quotient $q = a$ **div** d and the remainder $r = a$ **mod** d when a positive integer a is divided by a positive integer d (see [Kn98] for details). These algorithms require $O(\log a \cdot \log d)$ bit operations. If both of the binary expansions of a and d contain n or fewer bits, then we can replace $\log a \cdot \log d$ by n^2. This means that we need $O(n^2)$ bit operations to find the quotient and remainder when a is divided by d.

Modular Exponentiation

In cryptography it is important to be able to find b^n **mod** m efficiently, where b, n, and m are large integers. It is impractical to first compute b^n and then find its remainder when divided by m because b^n will be a huge number. Instead, we can use an algorithm that employs the binary expansion of the exponent n.

Before we present this algorithm, we illustrate its basic idea. We will explain how to use the binary expansion of n, say $n = (a_{k-1} \ldots a_1 a_0)_2$, to compute b^n. First, note that

$$b^n = b^{a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \cdots b^{a_1 \cdot 2} \cdot b^{a_0}.$$

This shows that to compute b^n, we need only compute the values of b, b^2, $(b^2)^2 = b^4$, $(b^4)^2 = b^8, \ldots, b^{2^k}$. Once we have these values, we multiply the terms b^{2^j} in this list, where $a_j = 1$. (For efficiency, after multiplying by each term, we reduce the result modulo m.) This gives us b^n. For example, to compute 3^{11} we first note that $11 = (1011)_2$, so that $3^{11} = 3^8 3^2 3^1$. By successively squaring, we find that $3^2 = 9$, $3^4 = 9^2 = 81$, and $3^8 = (81)^2 = 6561$. Consequently, $3^{11} = 3^8 3^2 3^1 = 6561 \cdot 9 \cdot 3 = 177{,}147$.

The algorithm successively finds b **mod** m, b^2 **mod** m, b^4 **mod** $m, \ldots, b^{2^{k-1}}$ **mod** m and multiplies together those terms b^{2^j} **mod** m where $a_j = 1$, finding the remainder of the product when divided by m after each multiplication. Pseudocode for this algorithm is shown in Algorithm 5. Note that in Algorithm 5 we can use the most efficient algorithm available to compute values of the **mod** function, not necessarily Algorithm 4.

Be sure to reduce modulo m after each multiplication!

ALGORITHM 5 Modular Exponentiation.

procedure *modular exponentiation*(b: integer, $n = (a_{k-1}a_{k-2}\ldots a_1a_0)_2$,
m: positive integers)
$x := 1$
$power := b$ **mod** m
for $i := 0$ **to** $k - 1$
  if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m
  $power := (power \cdot power)$ **mod** m
return x {x equals b^n **mod** m}

We illustrate how Algorithm 5 works in Example 12.

EXAMPLE 12 Use Algorithm 5 to find 3^{644} **mod** 645.

Solution: Algorithm 5 initially sets $x = 1$ and $power = 3$ **mod** $645 = 3$. In the computation of 3^{644} **mod** 645, this algorithm determines 3^{2^j} **mod** 645 for $j = 1, 2, \ldots, 9$ by successively squaring and reducing modulo 645. If $a_j = 1$ (where a_j is the bit in the jth position in the binary expansion of 644, which is $(1010000100)_2$), it multiplies the current value of x by 3^{2^j} **mod** 645 and reduces the result modulo 645. Here are the steps used:

$i = 0$: Because $a_0 = 0$, we have $x = 1$ and $power = 3^2$ **mod** $645 = 9$ **mod** $645 = 9$;
$i = 1$: Because $a_1 = 0$, we have $x = 1$ and $power = 9^2$ **mod** $645 = 81$ **mod** $645 = 81$;
$i = 2$: Because $a_2 = 1$, we have $x = 1 \cdot 81$ **mod** $645 = 81$ and $power = 81^2$ **mod** $645 = 6561$ **mod** $645 = 111$;
$i = 3$: Because $a_3 = 0$, we have $x = 81$ and $power = 111^2$ **mod** $645 = 12{,}321$ **mod** $645 = 66$;
$i = 4$: Because $a_4 = 0$, we have $x = 81$ and $power = 66^2$ **mod** $645 = 4356$ **mod** $645 = 486$;
$i = 5$: Because $a_5 = 0$, we have $x = 81$ and $power = 486^2$ **mod** $645 = 236{,}196$ **mod** $645 = 126$;
$i = 6$: Because $a_6 = 0$, we have $x = 81$ and $power = 126^2$ **mod** $645 = 15{,}876$ **mod** $645 = 396$;
$i = 7$: Because $a_7 = 1$, we find that $x = (81 \cdot 396)$ **mod** $645 = 471$ and $power = 396^2$ **mod** $645 = 156{,}816$ **mod** $645 = 81$;
$i = 8$: Because $a_8 = 0$, we have $x = 471$ and $power = 81^2$ **mod** $645 = 6561$ **mod** $645 = 111$;
$i = 9$: Because $a_9 = 1$, we find that $x = (471 \cdot 111)$ **mod** $645 = 36$.

This shows that following the steps of Algorithm 5 produces the result 3^{644} **mod** $645 = 36$. ◀

Algorithm 5 is quite efficient; it uses $O((\log m)^2 \log n)$ bit operations to find b^n **mod** m (see Exercise 58).

Exercises

1. Convert the decimal expansion of each of these integers to a binary expansion.
a) 231 **b)** 4532 **c)** 97644

2. Convert the decimal expansion of each of these integers to a binary expansion.
a) 321 **b)** 1023 **c)** 100632

3. Convert the binary expansion of each of these integers to a decimal expansion.
a) $(1\ 1111)_2$ **b)** $(10\ 0000\ 0001)_2$
c) $(1\ 0101\ 0101)_2$ **d)** $(110\ 1001\ 0001\ 0000)_2$

4. Convert the binary expansion of each of these integers to a decimal expansion.
a) $(1\ 1011)_2$ **b)** $(10\ 1011\ 0101)_2$
c) $(11\ 1011\ 1110)_2$ **d)** $(111\ 1100\ 0001\ 1111)_2$

5. Convert the octal expansion of each of these integers to a binary expansion.
a) $(572)_8$ **b)** $(1604)_8$
c) $(423)_8$ **d)** $(2417)_8$

6. Convert the binary expansion of each of these integers to an octal expansion.
a) $(1111\ 0111)_2$
b) $(1010\ 1010\ 1010)_2$
c) $(111\ 0111\ 0111\ 0111)_2$
d) $(101\ 0101\ 0101\ 0101)_2$

7. Convert the hexadecimal expansion of each of these integers to a binary expansion.
a) $(80E)_{16}$ **b)** $(135AB)_{16}$
c) $(ABBA)_{16}$ **d)** $(DEFACED)_{16}$

8. Convert $(BADFACED)_{16}$ from its hexadecimal expansion to its binary expansion.

9. Convert $(ABCDEF)_{16}$ from its hexadecimal expansion to its binary expansion.

10. Convert each of the integers in Exercise 6 from a binary expansion to a hexadecimal expansion.

11. Convert $(1011\ 0111\ 1011)_2$ from its binary expansion to its hexadecimal expansion.

12. Convert $(1\ 1000\ 0110\ 0011)_2$ from its binary expansion to its hexadecimal expansion.

13. Show that the hexadecimal expansion of a positive integer can be obtained from its binary expansion by grouping together blocks of four binary digits, adding initial zeros if necessary, and translating each block of four binary digits into a single hexadecimal digit.

14. Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

15. Show that the octal expansion of a positive integer can be obtained from its binary expansion by grouping together blocks of three binary digits, adding initial zeros if necessary, and translating each block of three binary digits into a single octal digit.

16. Show that the binary expansion of a positive integer can be obtained from its octal expansion by translating each octal digit into a block of three binary digits.

17. Convert $(7345321)_8$ to its binary expansion and $(10\ 1011\ 1011)_2$ to its octal expansion.

18. Give a procedure for converting from the hexadecimal expansion of an integer to its octal expansion using binary notation as an intermediate step.

19. Give a procedure for converting from the octal expansion of an integer to its hexadecimal expansion using binary notation as an intermediate step.

20. Explain how to convert from binary to base 64 expansions and from base 64 expansions to binary expansions and from octal to base 64 expansions and from base 64 expansions to octal expansions.

21. Find the sum and the product of each of these pairs of numbers. Express your answers as a binary expansion.
a) $(100\ 0111)_2$, $(111\ 0111)_2$
b) $(1110\ 1111)_2$, $(1011\ 1101)_2$
c) $(10\ 1010\ 1010)_2$, $(1\ 1111\ 0000)_2$
d) $(10\ 0000\ 0001)_2$, $(11\ 1111\ 1111)_2$

22. Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansion.
a) $(112)_3$, $(210)_3$
b) $(2112)_3$, $(12021)_3$
c) $(20001)_3$, $(1111)_3$
d) $(120021)_3$, $(2002)_3$

23. Find the sum and product of each of these pairs of numbers. Express your answers as an octal expansion.
a) $(763)_8$, $(147)_8$
b) $(6001)_8$, $(272)_8$
c) $(1111)_8$, $(777)_8$
d) $(54321)_8$, $(3456)_8$

24. Find the sum and product of each of these pairs of numbers. Express your answers as a hexadecimal expansion.
a) $(1AE)_{16}$, $(BBC)_{16}$
b) $(20CBA)_{16}$, $(A01)_{16}$
c) $(ABCDE)_{16}$, $(1111)_{16}$
d) $(E0000E)_{16}$, $(BAAA)_{16}$

25. Use Algorithm 5 to find 7^{644} **mod** 645.

26. Use Algorithm 5 to find 11^{644} **mod** 645.

27. Use Algorithm 5 to find 3^{2003} **mod** 99.

28. Use Algorithm 5 to find 123^{1001} **mod** 101.

29. Show that every positive integer can be represented uniquely as the sum of distinct powers of 2. [*Hint:* Consider binary expansions of integers.]

30. It can be shown that every integer can be uniquely represented in the form

$$e_k 3^k + e_{k-1}3^{k-1} + \cdots + e_1 3 + e_0,$$

where $e_j = -1, 0,$ or 1 for $j = 0, 1, 2, \ldots, k$. Expansions of this type are called **balanced ternary expansions**. Find the balanced ternary expansions of

a) 5. **b)** 13. **c)** 37. **d)** 79.

31. Show that a positive integer is divisible by 3 if and only if the sum of its decimal digits is divisible by 3.

32. Show that a positive integer is divisible by 11 if and only if the difference of the sum of its decimal digits in even-numbered positions and the sum of its decimal digits in odd-numbered positions is divisible by 11.

33. Show that a positive integer is divisible by 3 if and only if the difference of the sum of its binary digits in even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3.

One's complement representations of integers are used to simplify computer arithmetic. To represent positive and negative integers with absolute value less than 2^{n-1}, a total of n bits is used. The leftmost bit is used to represent the sign. A 0 bit in this position is used for positive integers, and a 1 bit in this position is used for negative integers. For positive integers, the remaining bits are identical to the binary expansion of the integer. For negative integers, the remaining bits are obtained by first finding the binary expansion of the absolute value of the integer, and then taking the complement of each of these bits, where the complement of a 1 is a 0 and the complement of a 0 is a 1.

34. Find the one's complement representations, using bit strings of length six, of the following integers.

a) 22 **b)** 31 **c)** -7 **d)** -19

35. What integer does each of the following one's complement representations of length five represent?

a) 11001 **b)** 01101

c) 10001 **d)** 11111

36. If m is a positive integer less than 2^{n-1}, how is the one's complement representation of $-m$ obtained from the one's complement of m, when bit strings of length n are used?

37. How is the one's complement representation of the sum of two integers obtained from the one's complement representations of these integers?

38. How is the one's complement representation of the difference of two integers obtained from the one's complement representations of these integers?

39. Show that the integer m with one's complement representation $(a_{n-1}a_{n-2}\ldots a_1a_0)$ can be found using the equation $m = -a_{n-1}(2^{n-1} - 1) + a_{n-2}2^{n-2} + \cdots + a_1 \cdot 2 + a_0$.

Two's complement representations of integers are also used to simplify computer arithmetic and are used more commonly than one's complement representations. To represent an integer x with $-2^{n-1} \le x \le 2^{n-1} - 1$ for a specified positive integer n, a total of n bits is used. The leftmost bit is used to represent the sign. A 0 bit in this position is used for positive integers, and a 1 bit in this position is used for negative integers, just as in one's complement expansions. For a positive integer, the remaining bits are identical to the binary expansion of the integer. For a negative integer, the remaining bits are the bits of the binary expansion of $2^{n-1} - |x|$. Two's complement expansions of integers are often used by computers because addition and subtraction of integers can be performed easily using these expansions, where these integers can be either positive or negative.

40. Answer Exercise 34, but this time find the two's complement expansion using bit strings of length six.

41. Answer Exercise 35 if each expansion is a two's complement expansion of length five.

42. Answer Exercise 36 for two's complement expansions.

43. Answer Exercise 37 for two's complement expansions.

44. Answer Exercise 38 for two's complement expansions.

45. Show that the integer m with two's complement representation $(a_{n-1}a_{n-2}\ldots a_1a_0)$ can be found using the equation $m = -a_{n-1} \cdot 2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_1 \cdot 2 + a_0$.

46. Give a simple algorithm for forming the two's complement representation of an integer from its one's complement representation.

47. Sometimes integers are encoded by using four-digit binary expansions to represent each decimal digit. This produces the **binary coded decimal** form of the integer. For instance, 791 is encoded in this way by 011110010001. How many bits are required to represent a number with n decimal digits using this type of encoding?

A **Cantor expansion** is a sum of the form

$$a_n n! + a_{n-1}(n-1)! + \cdots + a_2 2! + a_1 1!,$$

where a_i is an integer with $0 \le a_i \le i$ for $i = 1, 2, \ldots, n$.

48. Find the Cantor expansions of

a) 2. **b)** 7.

c) 19. **d)** 87.

e) 1000. **f)** 1,000,000.

***49.** Describe an algorithm that finds the Cantor expansion of an integer.

***50.** Describe an algorithm to add two integers from their Cantor expansions.

51. Add $(10111)_2$ and $(11010)_2$ by working through each step of the algorithm for addition given in the text.

52. Multiply $(1110)_2$ and $(1010)_2$ by working through each step of the algorithm for multiplication given in the text.

53. Describe an algorithm for finding the difference of two binary expansions.

54. Estimate the number of bit operations used to subtract two binary expansions.

55. Devise an algorithm that, given the binary expansions of the integers a and b, determines whether $a > b$, $a = b$, or $a < b$.

56. How many bit operations does the comparison algorithm from Exercise 55 use when the larger of a and b has n bits in its binary expansion?

57. Estimate the complexity of Algorithm 1 for finding the base b expansion of an integer n in terms of the number of divisions used.

***58.** Show that Algorithm 5 uses $O((\log m)^2 \log n)$ bit operations to find b^n **mod** m.

59. Show that Algorithm 4 uses $O(q \log a)$ bit operations, assuming that $a > d$.

4.3 Primes and Greatest Common Divisors

Introduction

In Section 4.1 we studied the concept of divisibility of integers. One important concept based on divisibility is that of a prime number. A prime is an integer greater than 1 that is divisible by no positive integers other than 1 and itself. The study of prime numbers goes back to ancient times. Thousands of years ago it was known that there are infinitely many primes; the proof of this fact, found in the works of Euclid, is famous for its elegance and beauty.

We will discuss the distribution of primes among the integers. We will describe some of the results about primes found by mathematicians in the last 400 years. In particular, we will introduce an important theorem, the fundamental theorem of arithmetic. This theorem, which asserts that every positive integer can be written uniquely as the product of primes in nondecreasing order, has many interesting consequences. We will also discuss some of the many old conjectures about primes that remain unsettled today.

Primes have become essential in modern cryptographic systems, and we will develop some of their properties important in cryptography. For example, finding large primes is essential in modern cryptography. The length of time required to factor large integers into their prime factors is the basis for the strength of some important modern cryptographic systems.

In this section we will also study the greatest common divisor of two integers, as well as the least common multiple of two integers. We will develop an important algorithm for computing greatest common divisors, called the Euclidean algorithm.

Primes

Every integer greater than 1 is divisible by at least two integers, because a positive integer is divisible by 1 and by itself. Positive integers that have exactly two different positive integer factors are called **primes**.

DEFINITION 1 An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called *composite*.

Remark: The integer n is composite if and only if there exists an integer a such that $a \mid n$ and $1 < a < n$.

EXAMPLE 1 The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3. ◀

The primes are the building blocks of positive integers, as the fundamental theorem of arithmetic shows. The proof will be given in Section 5.2.

THEOREM 1 **THE FUNDAMENTAL THEOREM OF ARITHMETIC** Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Example 2 gives some prime factorizations of integers.

EXAMPLE 2 The prime factorizations of 100, 641, 999, and 1024 are given by

Extra Examples

$$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 5^2,$$
$$641 = 641,$$
$$999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37,$$
$$1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}.$$

◀

Trial Division

It is often important to show that a given integer is prime. For instance, in cryptology, large primes are used in some methods for making messages secret. One procedure for showing that an integer is prime is based on the following observation.

THEOREM 2 If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

Proof: If n is composite, by the definition of a composite integer, we know that it has a factor a with $1 < a < n$. Hence, by the definition of a factor of a positive integer, we have $n = ab$, where b is a positive integer greater than 1. We will show that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. If $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > \sqrt{n} \cdot \sqrt{n} = n$, which is a contradiction. Consequently, $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. Because both a and b are divisors of n, we see that n has a positive divisor not exceeding $\sqrt{n}$. This divisor is either prime or, by the fundamental theorem of arithmetic, has a prime divisor less than itself. In either case, n has a prime divisor less than or equal to $\sqrt{n}$. ◁

From Theorem 2, it follows that an integer is prime if it is not divisible by any prime less than or equal to its square root. This leads to the brute-force algorithm known as **trial division**. To use trial division we divide n by all primes not exceeding $\sqrt{n}$ and conclude that n is prime if it is not divisible by any of these primes. In Example 3 we use trial division to show that 101 is prime.

EXAMPLE 3 Show that 101 is prime.

Solution: The only primes not exceeding $\sqrt{101}$ are 2, 3, 5, and 7. Because 101 is not divisible by 2, 3, 5, or 7 (the quotient of 101 and each of these integers is not an integer), it follows that 101 is prime. ◀

Because every integer has a prime factorization, it would be useful to have a procedure for finding this prime factorization. Consider the problem of finding the prime factorization of n. Begin by dividing n by successive primes, starting with the smallest prime, 2. If n has a prime factor, then by Theorem 3 a prime factor p not exceeding $\sqrt{n}$ will be found. So, if no prime

factor not exceeding $\sqrt{n}$ is found, then n is prime. Otherwise, if a prime factor p is found, continue by factoring n/p. Note that n/p has no prime factors less than p. Again, if n/p has no prime factor greater than or equal to p and not exceeding its square root, then it is prime. Otherwise, if it has a prime factor q, continue by factoring $n/(pq)$. This procedure is continued until the factorization has been reduced to a prime. This procedure is illustrated in Example 4.

EXAMPLE 4 Find the prime factorization of 7007.

Solution: To find the prime factorization of 7007, first perform divisions of 7007 by successive primes, beginning with 2. None of the primes 2, 3, and 5 divides 7007. However, 7 divides 7007, with $7007/7 = 1001$. Next, divide 1001 by successive primes, beginning with 7. It is immediately seen that 7 also divides 1001, because $1001/7 = 143$. Continue by dividing 143 by successive primes, beginning with 7. Although 7 does not divide 143, 11 does divide 143, and $143/11 = 13$. Because 13 is prime, the procedure is completed. It follows that $7007 = 7 \cdot 1001 = 7 \cdot 7 \cdot 143 = 7 \cdot 7 \cdot 11 \cdot 13$. Consequently, the prime factorization of 7007 is $7 \cdot 7 \cdot 11 \cdot 13 = 7^2 \cdot 11 \cdot 13$. ◀

Links Prime numbers were studied in ancient times for philosophical reasons. Today, there are highly practical reasons for their study. In particular, large primes play a crucial role in cryptography, as we will see in Section 4.6.

The Sieve of Eratosthenes

Note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes less than 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7.

Links The **sieve of Eratosthenes** is used to find all primes not exceeding a specified positive integer. For instance, the following procedure is used to find the primes not exceeding 100. We begin with the list of all integers between 1 and 100. To begin the sieving process, the integers that are divisible by 2, other than 2, are deleted. Because 3 is the first integer greater than 2 that is left, all those integers divisible by 3, other than 3, are deleted. Because 5 is the next integer left after 3, those integers divisible by 5, other than 5, are deleted. The next integer left is 7, so those integers divisible by 7, other than 7, are deleted. Because all composite integers not exceeding 100 are divisible by 2, 3, 5, or 7, all remaining integers except 1 are prime. In Table 1, the panels display those integers deleted at each stage, where each integer divisible by 2, other than 2, is underlined in the first panel, each integer divisible by 3, other than 3, is underlined in the second panel, each integer divisible by 5, other than 5, is underlined in the third panel, and each integer divisible by 7, other than 7, is underlined in the fourth panel. The integers not underlined are the primes not exceeding 100. We conclude that the primes less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

Links THE INFINITUDE OF PRIMES It has long been known that there are infinitely many primes. This means that whenever $p_1, p_2, \ldots, p_n$ are the n smallest primes, we know there is a larger

ERATOSTHENES (276 B.C.E.–194 B.C.E.) It is known that Eratosthenes was born in Cyrene, a Greek colony west of Egypt, and spent time studying at Plato's Academy in Athens. We also know that King Ptolemy II invited Eratosthenes to Alexandria to tutor his son and that later Eratosthenes became chief librarian at the famous library at Alexandria, a central repository of ancient wisdom. Eratosthenes was an extremely versatile scholar, writing on mathematics, geography, astronomy, history, philosophy, and literary criticism. Besides his work in mathematics, he is most noted for his chronology of ancient history and for his famous measurement of the size of the earth.

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

prime not listed. We will prove this fact using a proof given by Euclid in his famous mathematics text, *The Elements*. This simple, yet elegant, proof is considered by many mathematicians to be among the most beautiful proofs in mathematics. It is the first proof presented in the book *Proofs from THE BOOK* [AiZi10], where THE BOOK refers to the imagined collection of perfect proofs that the famous mathematician Paul Erdős claimed is maintained by God. By the way, there are a vast number of different proofs than there are an infinitude of primes, and new ones are published surprisingly frequently.

THEOREM 3 There are infinitely many primes.

Proof: We will prove this theorem using a proof by contradiction. We assume that there are only finitely many primes, $p_1, p_2, \ldots, p_n$. Let

$$Q = p_1 p_2 \cdots p_n + 1.$$

By the fundamental theorem of arithmetic, Q is prime or else it can be written as the product of two or more primes. However, none of the primes p_j divides Q, for if $p_j \mid Q$, then p_j divides

$Q - p_1 p_2 \cdots p_n = 1$. Hence, there is a prime not in the list $p_1, p_2, \ldots, p_n$. This prime is either Q, if it is prime, or a prime factor of Q. This is a contradiction because we assumed that we have listed all the primes. Consequently, there are infinitely many primes. ◁

Remark: Note that in this proof we do *not* state that Q is prime! Furthermore, in this proof, we have given a nonconstructive existence proof that given any n primes, there is a prime not in this list. For this proof to be constructive, we would have had to explicitly give a prime not in our original list of n primes.

Because there are infinitely many primes, given any positive integer there are primes greater than this integer. There is an ongoing quest to discover larger and larger prime numbers; for almost all the last 300 years, the largest prime known has been an integer of the special form $2^p - 1$, where p is also prime. (Note that $2^n - 1$ cannot be prime when n is not prime; see Exercise 9.) Such primes are called **Mersenne primes**, after the French monk Marin Mersenne, who studied them in the seventeenth century. The reason that the largest known prime has usually been a Mersenne prime is that there is an extremely efficient test, known as the Lucas–Lehmer test, for determining whether $2^p - 1$ is prime. Furthermore, it is not currently possible to test numbers not of this or certain other special forms anywhere near as quickly to determine whether they are prime.

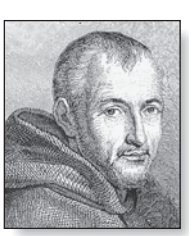
EXAMPLE 5 The numbers $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are Mersenne primes, while $2^{11} - 1 = 2047$ is not a Mersenne prime because $2047 = 23 \cdot 89$. ◀

Links

Progress in finding Mersenne primes has been steady since computers were invented. As of early 2011, 47 Mersenne primes were known, with 16 found since 1990. The largest Mersenne prime known (again as of early 2011) is $2^{43,112,609} - 1$, a number with nearly 13 million decimal digits, which was shown to be prime in 2008. A communal effort, the Great Internet Mersenne Prime Search (GIMPS), is devoted to the search for new Mersenne primes. You can join this search, and if you are lucky, find a new Mersenne prime and possibly even win a cash prize. By the way, even the search for Mersenne primes has practical implications. One quality control test for supercomputers has been to replicate the Lucas–Lehmer test that establishes the primality of a large Mersenne prime. (See [Ro10] for more information about the quest for finding Mersenne primes.)

THE DISTRIBUTION OF PRIMES Theorem 3 tells us that there are infinitely many primes. However, how many primes are less than a positive number x? This question interested mathematicians for many years; in the late eighteenth century, mathematicians produced large tables

Links



MARIN MERSENNE (1588–1648) Mersenne was born in Maine, France, into a family of laborers and attended the College of Mans and the Jesuit College at La Flèche. He continued his education at the Sorbonne, studying theology from 1609 to 1611. He joined the religious order of the Minims in 1611, a group whose name comes from the word *minimi* (the members of this group were extremely humble; they considered themselves the least of all religious orders). Besides prayer, the members of this group devoted their energy to scholarship and study. In 1612 he became a priest at the Place Royale in Paris; between 1614 and 1618 he taught philosophy at the Minim Convent at Nevers. He returned to Paris in 1619, where his cell in the Minims de l'Annociade became a place for meetings of French scientists, philosophers, and mathematicians, including Fermat and Pascal. Mersenne corresponded extensively with scholars throughout Europe, serving as a clearinghouse for mathematical and scientific knowledge, a function later served by mathematical journals (and today also by the Internet). Mersenne wrote books covering mechanics, mathematical physics, mathematics, music, and acoustics. He studied prime numbers and tried unsuccessfully to construct a formula representing all primes. In 1644 Mersenne claimed that $2^p - 1$ is prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$ but is composite for all other primes less than 257. It took over 300 years to determine that Mersenne's claim was wrong five times. Specifically, $2^p - 1$ is not prime for $p = 67$ and $p = 257$ but is prime for $p = 61$, $p = 87$, and $p = 107$. It is also noteworthy that Mersenne defended two of the most famous men of his time, Descartes and Galileo, from religious critics. He also helped expose alchemists and astrologers as frauds.

of prime numbers to gather evidence concerning the distribution of primes. Using this evidence, the great mathematicians of the day, including Gauss and Legendre, conjectured, but did not prove, Theorem 4.

THEOREM 4

THE PRIME NUMBER THEOREM The ratio of the number of primes not exceeding x and $x/\ln x$ approaches 1 as x grows without bound. (Here $\ln x$ is the natural logarithm of x.)

The prime number theorem was first proved in 1896 by the French mathematician Jacques Hadamard and the Belgian mathematician Charles-Jean-Gustave-Nicholas de la Vallée-Poussin using the theory of complex variables. Although proofs not using complex variables have been found, all known proofs of the prime number theorem are quite complicated.

We can use the prime number theorem to estimate the odds that a randomly chosen number is prime. The prime number theorem tells us that the number of primes not exceeding x can be approximated by $x/\ln x$. Consequently, the odds that a randomly selected positive integer less than n is prime are approximately $(n/\ln n)/n = 1/\ln n$. Sometimes we need to find a prime with a particular number of digits. We would like an estimate of how many integers with a particular number of digits we need to select before we encounter a prime. Using the prime number theorem and calculus, it can be shown that the probability that an integer n is prime is also approximately $1/\ln n$. For example, the odds that an integer near 10^{1000} is prime are approximately $1/\ln 10^{1000}$, which is approximately $1/2300$. (Of course, by choosing only odd numbers, we double our chances of finding a prime.)

Using trial division with Theorem 2 gives procedures for factoring and for primality testing. However, these procedures are not efficient algorithms; many much more practical and efficient algorithms for these tasks have been developed. Factoring and primality testing have become important in the applications of number theory to cryptography. This has led to a great interest in developing efficient algorithms for both tasks. Clever procedures have been devised in the last 30 years for efficiently generating large primes. Moreover, in 2002, an important theoretical discovery was made by Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. They showed there is a polynomial-time algorithm in the number of bits in the binary expansion of an integer for determining whether a positive integer is prime. Algorithms based on their work use $O((\log n)^6)$ bit operations to determine whether a positive integer n is prime.

However, even though powerful new factorization methods have been developed in the same time frame, factoring large numbers remains extraordinarily more time-consuming than primality testing. No polynomial-time algorithm for factoring integers is known. Nevertheless, the challenge of factoring large numbers interests many people. There is a communal effort on the Internet to factor large numbers, especially those of the special form $k^n \pm 1$, where k is a small positive integer and n is a large positive integer (such numbers are called *Cunningham numbers*). At any given time, there is a list of the "Ten Most Wanted" large numbers of this type awaiting factorization.

PRIMES AND ARITHMETIC PROGRESSIONS Every odd integer is in one of the two arithmetic progressions $4k + 1$ or $4k + 3, k = 1, 2, \ldots$. Because we know that there are infinitely many primes, we can ask whether there are infinitely many primes in both of these arithmetic progressions. The primes 5, 13, 17, 29, 37, 41, ... are in the arithmetic progression $4k + 1$; the primes 3, 7, 11, 19, 23, 31, 43, ... are in the arithmetic progression $4k + 3$. Looking at the evidence hints that there may be infinitely many primes in both progressions. What about other arithmetic progressions $ak + b$, $k = 1, 2, \ldots$, where no integer greater than one divides both a and b? Do they contain infinitely many primes? The answer was provided by the German mathematician G. Lejeune Dirichlet, who proved that every such arithmetic progression contains infinitely many primes. His proof, and all proofs found later, are beyond the scope of this book.

However, it is possible to prove special cases of Dirichlet's theorem using the ideas developed in this book. For example, Exercises 54 and 55 ask for proofs that there are infinitely many primes in the arithmetic progressions $3k + 2$ and $4k + 3$, where k is a positive integer. (The hint for each of these exercises supplies the basic idea needed for the proof.)

We have explained that every arithmetic progression $ak + b$, $k = 1, 2, \ldots$, where a and b have no common factor greater than one, contains infinitely many primes. But are there long arithmetic progressions made up of just primes? For example, some exploration shows that 5, 11, 17, 23, 29 is an arithmetic progression of five primes and 199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089 is an arithmetic progression of ten primes. In the 1930s, the famous mathematician Paul Erdős conjectured that for every positive integer n greater than two, there is an arithmetic progression of length n made up entirely of primes. In 2006, Ben Green and Terence Tao were able to prove this conjecture. Their proof, considered to be a mathematical tour de force, is a nonconstructive proof that combines powerful ideas from several advanced areas of mathematics.

Conjectures and Open Problems About Primes

Number theory is noted as a subject for which it is easy to formulate conjectures, some of which are difficult to prove and others that remained open problems for many years. We will describe some conjectures in number theory and discuss their status in Examples 6–9.

EXAMPLE 6 Extra Examples

It would be useful to have a function $f(n)$ such that $f(n)$ is prime for all positive integers n. If we had such a function, we could find large primes for use in cryptography and other applications. Looking for such a function, we might check out different polynomial functions, as some mathematicians did several hundred years ago. After a lot of computation we may encounter the polynomial $f(n) = n^2 - n + 41$. This polynomial has the interesting property that $f(n)$ is prime for all positive integers n not exceeding 40. [We have $f(1) = 41$, $f(2) = 43$, $f(3) = 47$, $f(4) = 53$, and so on.] This can lead us to the conjecture that $f(n)$ is prime for all positive integers n. Can we settle this conjecture?

Solution: Perhaps not surprisingly, this conjecture turns out to be false; we do not have to look far to find a positive integer n for which $f(n)$ is composite, because $f(41) = 41^2 - 41 + 41 = 41^2$. Because $f(n) = n^2 - n + 41$ is prime for all positive integers n with $1 \leq n \leq 40$, we might

TERENCE TAO (BORN 1975) Tao was born in Australia. His father is a pediatrician and his mother taught mathematics at a Hong Kong secondary school. Tao was a child prodigy, teaching himself arithmetic at the age of two. At 10, he became the youngest contestant at the International Mathematical Olympiad (IMO); he won an IMO gold medal at 13. Tao received his bachelors and masters degrees when he was 17, and began graduate studies at Princeton, receiving his Ph.D. in three years. In 1996 he became a faculty member at UCLA, where he continues to work.

Tao is extremely versatile; he enjoys working on problems in diverse areas, including harmonic analysis, partial differential equations, number theory, and combinatorics. You can follow his work by reading his blog where he discusses progress on various problems. His most famous result is the Green-Tao theorem, which says that there are arbitrarily long arithmetic progressions of primes. Tao has made important contributions to the applications of mathematics, such as developing a method for reconstructing digital images using the least possible amount of information.
Tao has an amazing reputation among mathematicians; he has become a Mr. Fix-It for researchers in mathematics. The well-known mathematician Charles Fefferman, himself a child prodigy, has said that "if you're stuck on a problem, then one way out is to interest Terence Tao." In 2006 Tao was awarded a Fields Medal, the most prestigious award for mathematicians under the age of 40. He was also awarded a MacArthur Fellowship in 2006, and in 2008, he received the Allan T. Waterman award, which came with a $500,000 cash prize to support research work of scientists early in their career. Tao's wife Laura is an engineer at the Jet Propulsion Laboratory.

be tempted to find a different polynomial with the property that $f(n)$ is prime for *all* positive integers n. However, there is no such polynomial. It can be shown that for every polynomial $f(n)$ with integer coefficients, there is a positive integer y such that $f(y)$ is composite. (See Exercise 23 in the Supplementary Exercises.) ◀

Many famous problems about primes still await ultimate resolution by clever people. We describe a few of the most accessible and better known of these open problems in Examples 7–9. Number theory is noted for its wealth of easy-to-understand conjectures that resist attack by all but the most sophisticated techniques, or simply resist all attacks. We present these conjectures to show that many questions that seem relatively simple remain unsettled even in the twenty-first century.

EXAMPLE 7 **Goldbach's Conjecture** In 1742, Christian Goldbach, in a letter to Leonhard Euler, conjectured that every odd integer $n, n > 5$, is the sum of three primes. Euler replied that this conjecture is equivalent to the conjecture that every even integer n, $n > 2$, is the sum of two primes (see Exercise 21 in the Supplementary Exercises). The conjecture that every even integer $n, n > 2$, is the sum of two primes is now called **Goldbach's conjecture**. We can check this conjecture for small even numbers. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $10 = 7 + 3$, $12 = 7 + 5$, and so on. Goldbach's conjecture was verified by hand calculations for numbers up to the millions prior to the advent of computers. With computers it can be checked for extremely large numbers. As of mid 2011, the conjecture has been checked for all positive even integers up to $1.6 \cdot 10^{18}$.

Links

Although no proof of Goldbach's conjecture has been found, most mathematicians believe it is true. Several theorems have been proved, using complicated methods from analytic number theory far beyond the scope of this book, establishing results weaker than Goldbach's conjecture. Among these are the result that every even integer greater than 2 is the sum of at most six primes (proved in 1995 by O. Ramaré) and that every sufficiently large positive integer is the sum of a prime and a number that is either prime or the product of two primes (proved in 1966 by J. R. Chen). Perhaps Goldbach's conjecture will be settled in the not too distant future. ◀

EXAMPLE 8 There are many conjectures asserting that there are infinitely many primes of certain special forms. A conjecture of this sort is the conjecture that there are infinitely many primes of the form $n^2 + 1$, where n is a positive integer. For example, $5 = 2^2 + 1$, $17 = 4^2 + 1$, $37 = 6^2 + 1$, and so on. The best result currently known is that there are infinitely many positive integers n such that $n^2 + 1$ is prime or the product of at most two primes (proved by Henryk Iwaniec in 1973 using advanced techniques from analytic number theory, far beyond the scope of this book). ◀

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EXAMPLE 9 **The Twin Prime Conjecture** **Twin primes** are pairs of primes that differ by 2, such as 3 and 5, 5 and 7, 11 and 13, 17 and 19, and 4967 and 4969. The twin prime conjecture asserts that there are infinitely many twin primes. The strongest result proved concerning twin primes is that there are infinitely many pairs p and $p + 2$, where p is prime and $p + 2$ is prime or the product of two primes (proved by J. R. Chen in 1966). The world's record for twin primes, as of mid 2011, consists of the numbers $65{,}516{,}468{,}355 \cdot 2^{333{,}333} \pm 1$, which have 100,355 decimal digits. ◀

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CHRISTIAN GOLDBACH (1690–1764) Christian Goldbach was born in Königsberg, Prussia, the city noted for its famous bridge problem (which will be studied in Section 10.5). He became professor of mathematics at the Academy in St. Petersburg in 1725. In 1728 Goldbach went to Moscow to tutor the son of the Tsar. He entered the world of politics when, in 1742, he became a staff member in the Russian Ministry of Foreign Affairs. Goldbach is best known for his correspondence with eminent mathematicians, including Euler and Bernoulli, for his famous conjectures in number theory, and for several contributions to analysis.

Greatest Common Divisors and Least Common Multiples

The largest integer that divides both of two integers is called the **greatest common divisor** of these integers.

DEFINITION 2 Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b. The greatest common divisor of a and b is denoted by $\gcd(a, b)$.

The greatest common divisor of two integers, not both zero, exists because the set of common divisors of these integers is nonempty and finite. One way to find the greatest common divisor of two integers is to find all the positive common divisors of both integers and then take the largest divisor. This is done in Examples 10 and 11. Later, a more efficient method of finding greatest common divisors will be given.

EXAMPLE 10 What is the greatest common divisor of 24 and 36?

Solution: The positive common divisors of 24 and 36 are 1, 2, 3, 4, 6, and 12. Hence, $\gcd(24, 36) = 12$. ◀

EXAMPLE 11 What is the greatest common divisor of 17 and 22?

Solution: The integers 17 and 22 have no positive common divisors other than 1, so that $\gcd(17, 22) = 1$. ◀

Because it is often important to specify that two integers have no common positive divisor other than 1, we have Definition 3.

DEFINITION 3 The integers a and b are *relatively prime* if their greatest common divisor is 1.

EXAMPLE 12 By Example 11 it follows that the integers 17 and 22 are relatively prime, because $\gcd(17, 22) = 1$. ◀

Because we often need to specify that no two integers in a set of integers have a common positive divisor greater than 1, we make Definition 4.

DEFINITION 4 The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

EXAMPLE 13 Determine whether the integers 10, 17, and 21 are pairwise relatively prime and whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because $\gcd(10, 17) = 1$, $\gcd(10, 21) = 1$, and $\gcd(17, 21) = 1$, we conclude that 10, 17, and 21 are pairwise relatively prime.

Because $\gcd(10, 24) = 2 > 1$, we see that 10, 19, and 24 are not pairwise relatively prime. ◀

Another way to find the greatest common divisor of two positive integers is to use the prime factorizations of these integers. Suppose that the prime factorizations of the positive integers a and b are

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, \; b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in the prime factorization of either a or b are included in both factorizations, with zero exponents if necessary. Then $\gcd(a, b)$ is given by

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_n, b_n)},$$

where $\min(x, y)$ represents the minimum of the two numbers x and y. To show that this formula for $\gcd(a, b)$ is valid, we must show that the integer on the right-hand side divides both a and b, and that no larger integer also does. This integer does divide both a and b, because the power of each prime in the factorization does not exceed the power of this prime in either the factorization of a or that of b. Further, no larger integer can divide both a and b, because the exponents of the primes in this factorization cannot be increased, and no other primes can be included.

EXAMPLE 14 Because the prime factorizations of 120 and 500 are $120 = 2^3 \cdot 3 \cdot 5$ and $500 = 2^2 \cdot 5^3$, the greatest common divisor is

$$\gcd(120, 500) = 2^{\min(3, 2)} 3^{\min(1, 0)} 5^{\min(1, 3)} = 2^2 3^0 5^1 = 20. \quad \blacktriangleleft$$

Prime factorizations can also be used to find the **least common multiple** of two integers.

DEFINITION 5 The *least common multiple* of the positive integers a and b is the smallest positive integer that is divisible by both a and b. The least common multiple of a and b is denoted by $\operatorname{lcm}(a, b)$.

The least common multiple exists because the set of integers divisible by both a and b is nonempty (as ab belongs to this set, for instance), and every nonempty set of positive integers has a least element (by the well-ordering property, which will be discussed in Section 5.2). Suppose that the prime factorizations of a and b are as before. Then the least common multiple of a and b is given by

$$\operatorname{lcm}(a, b) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_n^{\max(a_n, b_n)},$$

where $\max(x, y)$ denotes the maximum of the two numbers x and y. This formula is valid because a common multiple of a and b has at least $\max(a_i, b_i)$ factors of p_i in its prime factorization, and the least common multiple has no other prime factors besides those in a and b.

EXAMPLE 15 What is the least common multiple of $2^3 3^5 7^2$ and $2^4 3^3$?

Solution: We have

$$\operatorname{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3, 4)} 3^{\max(5, 3)} 7^{\max(2, 0)} = 2^4 3^5 7^2. \quad \blacktriangleleft$$

Theorem 5 gives the relationship between the greatest common divisor and least common multiple of two integers. It can be proved using the formulae we have derived for these quantities. The proof of this theorem is left as Exercise 31.

THEOREM 5 Let a and b be positive integers. Then

$$ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b).$$

The Euclidean Algorithm

Links

Computing the greatest common divisor of two integers directly from the prime factorizations of these integers is inefficient. The reason is that it is time-consuming to find prime factorizations. We will give a more efficient method of finding the greatest common divisor, called the **Euclidean algorithm**. This algorithm has been known since ancient times. It is named after the ancient Greek mathematician Euclid, who included a description of this algorithm in his book *The Elements*.

Before describing the Euclidean algorithm, we will show how it is used to find gcd(91, 287). First, divide 287, the larger of the two integers, by 91, the smaller, to obtain

$$287 = 91 \cdot 3 + 14.$$

Any divisor of 91 and 287 must also be a divisor of $287 - 91 \cdot 3 = 14$. Also, any divisor of 91 and 14 must also be a divisor of $287 = 91 \cdot 3 + 14$. Hence, the greatest common divisor of 91 and 287 is the same as the greatest common divisor of 91 and 14. This means that the problem of finding gcd(91, 287) has been reduced to the problem of finding gcd(91, 14).

Next, divide 91 by 14 to obtain

$$91 = 14 \cdot 6 + 7.$$

Because any common divisor of 91 and 14 also divides $91 - 14 \cdot 6 = 7$ and any common divisor of 14 and 7 divides 91, it follows that $\gcd(91, 14) = \gcd(14, 7)$.

Continue by dividing 14 by 7, to obtain

$$14 = 7 \cdot 2.$$

Because 7 divides 14, it follows that $\gcd(14, 7) = 7$. Furthermore, because $\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$, the original problem has been solved.

We now describe how the Euclidean algorithm works in generality. We will use successive divisions to reduce the problem of finding the greatest common divisor of two positive integers to the same problem with smaller integers, until one of the integers is zero.

The Euclidean algorithm is based on the following result about greatest common divisors and the division algorithm.

Links

EUCLID (325 B.C.E.–265 B.C.E.) Euclid was the author of the most successful mathematics book ever written, *The Elements*, which appeared in over 1000 different editions from ancient to modern times. Little is known about Euclid's life, other than that he taught at the famous academy at Alexandria in Egypt. Apparently, Euclid did not stress applications. When a student asked what he would get by learning geometry, Euclid explained that knowledge was worth acquiring for its own sake and told his servant to give the student a coin "because he must make a profit from what he learns."

LEMMA 1 Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

Proof: If we can show that the common divisors of a and b are the same as the common divisors of b and r, we will have shown that $\gcd(a, b) = \gcd(b, r)$, because both pairs must have the same *greatest* common divisor.

So suppose that d divides both a and b. Then it follows that d also divides $a - bq = r$ (from Theorem 1 of Section 4.1). Hence, any common divisor of a and b is also a common divisor of b and r.

Likewise, suppose that d divides both b and r. Then d also divides $bq + r = a$. Hence, any common divisor of b and r is also a common divisor of a and b.

Consequently, $\gcd(a, b) = \gcd(b, r)$. ◁

Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$. When we successively apply the division algorithm, we obtain

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \leq r_2 < r_1,\\
r_1 &= r_2 q_2 + r_3 && 0 \leq r_3 < r_2,\\
&\ \ \vdots\\
r_{n-2} &= r_{n-1} q_{n-1} + r_n && 0 \leq r_n < r_{n-1},\\
r_{n-1} &= r_n q_n.
\end{aligned}
$$

Eventually a remainder of zero occurs in this sequence of successive divisions, because the sequence of remainders $a = r_0 > r_1 > r_2 > \cdots \geq 0$ cannot contain more than a terms. Furthermore, it follows from Lemma 1 that

$$
\begin{aligned}
\gcd(a, b) &= \gcd(r_0, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{n-2}, r_{n-1})\\
&= \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n.
\end{aligned}
$$

Hence, the greatest common divisor is the last nonzero remainder in the sequence of divisions.

EXAMPLE 16 Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Solution: Successive uses of the division algorithm give:

$$
\begin{aligned}
662 &= 414 \cdot 1 + 248\\
414 &= 248 \cdot 1 + 166\\
248 &= 166 \cdot 1 + 82\\
166 &= 82 \cdot 2 + 2\\
82 &= 2 \cdot 41.
\end{aligned}
$$

Hence, $\gcd(414, 662) = 2$, because 2 is the last nonzero remainder. ◀

The Euclidean algorithm is expressed in pseudocode in Algorithm 1.

ALGORITHM 1 The Euclidean Algorithm.

procedure *gcd*(a, b: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x$ **mod** y
    $x := y$
    $y := r$
return x {gcd(a, b) is x}

In Algorithm 1, the initial values of x and y are a and b, respectively. At each stage of the procedure, x is replaced by y, and y is replaced by x **mod** y, which is the remainder when x is divided by y. This process is repeated as long as $y \neq 0$. The algorithm terminates when $y = 0$, and the value of x at that point, the last nonzero remainder in the procedure, is the greatest common divisor of a and b.

We will study the time complexity of the Euclidean algorithm in Section 5.3, where we will show that the number of divisions required to find the greatest common divisor of a and b, where $a \geq b$, is $O(\log b)$.

gcds as Linear Combinations

An important result we will use throughout the remainder of this section is that the greatest common divisor of two integers a and b can be expressed in the form

$$sa + tb,$$

where s and t are integers. In other words, gcd(a, b) can be expressed as a **linear combination** with integer coefficients of a and b. For example, gcd(6, 14) = 2, and $2 = (-2) \cdot 6 + 1 \cdot 14$. We state this fact as Theorem 6.

THEOREM 6 **BÉZOUT'S THEOREM** If a and b are positive integers, then there exist integers s and t such that gcd(a, b) = $sa + tb$.

Links

ÉTIENNE BÉZOUT (1730–1783) Bézout was born in Nemours, France, where his father was a magistrate. Reading the writings of the great mathematician Leonhard Euler enticed him to become a mathematician. In 1758 he was appointed to a position at the Académie des Sciences in Paris; in 1763 he was appointed examiner of the Gardes de la Marine, where he was assigned the task of writing mathematics textbooks. This assignment led to a four-volume textbook completed in 1767. Bézout is well known for his six-volume comprehensive textbook on mathematics. His textbooks were extremely popular and were studied by many generations of students hoping to enter the École Polytechnique, the famous engineering and science school. His books were translated into English and used in North America, including at Harvard.

His most important original work was published in 1779 in the book *Théorie générale des équations algébriques*, where he introduced important methods for solving simultaneous polynomial equations in many unknowns. The most well-known result in this book is now called *Bézout's theorem*, which in its general form tells us that the number of common points on two plane algebraic curves equals the product of the degrees of these curves. Bézout is also credited with inventing the determinant (which was called the Bézoutian by the great English mathematician James Joseph Sylvester). He was considered to be a kind person with a warm heart, although he had a reserved and somber personality. He was happily married and a father.

DEFINITION 6

If a and b are positive integers, then integers s and t such that $\gcd(a, b) = sa + tb$ are called *Bézout coefficients* of a and b (after Étienne Bézout, a French mathematician of the eighteenth century). Also, the equation $\gcd(a, b) = sa + tb$ is called *Bézout's identity*.

We will not give a formal proof of Theorem 6 here (see Exercise 36 in Section 5.2 and [Ro10] for proofs). We will provide an example of a general method that can be used to find a linear combination of two integers equal to their greatest common divisor. (In this section, we will assume that a linear combination has integer coefficients.) The method proceeds by working backward through the divisions of the Euclidean algorithm, so this method requires a forward pass and a backward pass through the steps of the Euclidean algorithm. (In the exercises we will describe an algorithm called the **extended Euclidean algorithm**, which can be used to express $\gcd(a, b)$ as a linear combination of a and b using a single pass through the steps of the Euclidean algorithm; see the preamble to Exercise 41.)

EXAMPLE 17 Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

Solution: To show that $\gcd(252, 198) = 18$, the Euclidean algorithm uses these divisions:

$$252 = 1 \cdot 198 + 54$$
$$198 = 3 \cdot 54 + 36$$
$$54 = 1 \cdot 36 + 18$$
$$36 = 2 \cdot 18.$$

Using the next-to-last division (the third division), we can express $\gcd(252, 198) = 18$ as a linear combination of 54 and 36. We find that

$$18 = 54 - 1 \cdot 36.$$

The second division tells us that

$$36 = 198 - 3 \cdot 54.$$

Substituting this expression for 36 into the previous equation, we can express 18 as a linear combination of 54 and 198. We have

$$18 = 54 - 1 \cdot 36 = 54 - 1 \cdot (198 - 3 \cdot 54) = 4 \cdot 54 - 1 \cdot 198.$$

The first division tells us that

$$54 = 252 - 1 \cdot 198.$$

Substituting this expression for 54 into the previous equation, we can express 18 as a linear combination of 252 and 198. We conclude that

$$18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198,$$

completing the solution. ◀

We will use Theorem 6 to develop several useful results. One of our goals will be to prove the part of the fundamental theorem of arithmetic asserting that a positive integer has at most one prime factorization. We will show that if a positive integer has a factorization into primes, where the primes are written in nondecreasing order, then this factorization is unique.

First, we need to develop some results about divisibility.

LEMMA 2 If a, b, and c are positive integers such that $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof: Because $\gcd(a, b) = 1$, by Bézout's theorem there are integers s and t such that

$$sa + tb = 1.$$

Multiplying both sides of this equation by c, we obtain

$$sac + tbc = c.$$

We can now use Theorem 1 of Section 4.1 to show that $a \mid c$. By part (*ii*) of that theorem, $a \mid tbc$. Because $a \mid sac$ and $a \mid tbc$, by part (*i*) of that theorem, we conclude that a divides $sac + tbc$. Because $sac + tbc = c$, we conclude that $a \mid c$, completing the proof. ◁

We will use the following generalization of Lemma 2 in the proof of uniqueness of prime factorizations. (The proof of Lemma 3 is left as Exercise 64 in Section 5.1, because it can be most easily carried out using the method of mathematical induction, covered in that section.)

LEMMA 3 If p is a prime and $p \mid a_1 a_2 \cdots a_n$, where each a_i is an integer, then $p \mid a_i$ for some i.

We can now show that a factorization of an integer into primes is unique. That is, we will show that every integer can be written as the product of primes in nondecreasing order in at most one way. This is part of the fundamental theorem of arithmetic. We will prove the other part, that every integer has a factorization into primes, in Section 5.2.

Proof (of the uniqueness of the prime factorization of a positive integer): We will use a proof by contradiction. Suppose that the positive integer n can be written as the product of primes in two different ways, say, $n = p_1 p_2 \cdots p_s$ and $n = q_1 q_2 \cdots q_t$, each p_i and q_j are primes such that $p_1 \le p_2 \le \cdots \le p_s$ and $q_1 \le q_2 \le \cdots \le q_t$.

When we remove all common primes from the two factorizations, we have

$$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v},$$

where no prime occurs on both sides of this equation and u and v are positive integers. By Lemma 3 it follows that p_{i_1} divides q_{j_k} for some k. Because no prime divides another prime, this is impossible. Consequently, there can be at most one factorization of n into primes in nondecreasing order. ◁

Lemma 2 can also be used to prove a result about dividing both sides of a congruence by the same integer. We have shown (Theorem 5 in Section 4.1) that we can multiply both sides of a congruence by the same integer. However, dividing both sides of a congruence by an integer does not always produce a valid congruence, as Example 18 shows.

EXAMPLE 18 The congruence $14 \equiv 8 \pmod 6$ holds, but both sides of this congruence cannot be divided by 2 to produce a valid congruence because $14/2 = 7$ and $8/2 = 4$, but $7 \not\equiv 4 \pmod 6$. ◀

Although we cannot divide both sides of a congruence by any integer to produce a valid congruence, we can if this integer is relatively prime to the modulus. Theorem 7 establishes this important fact. We use Lemma 2 in the proof.

THEOREM 7 Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then $a \equiv b \pmod{m}$.

Proof: Because $ac \equiv bc \pmod{m}$, $m \mid ac - bc = c(a - b)$. By Lemma 2, because $\gcd(c, m) = 1$, it follows that $m \mid a - b$. We conclude that $a \equiv b \pmod{m}$. ◁

Exercises

1. Determine whether each of these integers is prime.
a) 21 **b)** 29
c) 71 **d)** 97
e) 111 **f)** 143

2. Determine whether each of these integers is prime.
a) 19 **b)** 27
c) 93 **d)** 101
e) 107 **f)** 113

3. Find the prime factorization of each of these integers.
a) 88 **b)** 126 **c)** 729
d) 1001 **e)** 1111 **f)** 909,090

4. Find the prime factorization of each of these integers.
a) 39 **b)** 81 **c)** 101
d) 143 **e)** 289 **f)** 899

5. Find the prime factorization of 10!.

***6.** How many zeros are there at the end of 100!?

7. Express in pseudocode the trial division algorithm for determining whether an integer is prime.

8. Express in pseudocode the algorithm described in the text for finding the prime factorization of an integer.

9. Show that if $a^m + 1$ is composite if a and m are integers greater than 1 and m is odd. [*Hint:* Show that $x + 1$ is a factor of the polynomial $x^m + 1$ if m is odd.]

10. Show that if $2^m + 1$ is an odd prime, then $m = 2^n$ for some nonnegative integer n. [*Hint:* First show that the polynomial identity $x^m + 1 = (x^k + 1)(x^{k(t-1)} - x^{k(t-2)} + \cdots - x^k + 1)$ holds, where $m = kt$ and t is odd.]

***11.** Show that $\log_2 3$ is an irrational number. Recall that an irrational number is a real number x that cannot be written as the ratio of two integers.

12. Prove that for every positive integer n, there are n consecutive composite integers. [*Hint:* Consider the n consecutive integers starting with $(n + 1)! + 2$.]

***13.** Prove or disprove that there are three consecutive odd positive integers that are primes, that is, odd primes of the form p, $p + 2$, and $p + 4$.

14. Which positive integers less than 12 are relatively prime to 12?

15. Which positive integers less than 30 are relatively prime to 30?

16. Determine whether the integers in each of these sets are pairwise relatively prime.
a) 21, 34, 55 **b)** 14, 17, 85
c) 25, 41, 49, 64 **d)** 17, 18, 19, 23

17. Determine whether the integers in each of these sets are pairwise relatively prime.
a) 11, 15, 19 **b)** 14, 15, 21
c) 12, 17, 31, 37 **d)** 7, 8, 9, 11

18. We call a positive integer **perfect** if it equals the sum of its positive divisors other than itself.
a) Show that 6 and 28 are perfect.
b) Show that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is prime.

19. Show that if $2^n - 1$ is prime, then n is prime. [*Hint:* Use the identity $2^{ab} - 1 = (2^a - 1) \cdot (2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1)$.]

20. Determine whether each of these integers is prime, verifying some of Mersenne's claims.
a) $2^7 - 1$ **b)** $2^9 - 1$
c) $2^{11} - 1$ **d)** $2^{13} - 1$

The value of the **Euler ϕ-function** at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. [*Note:* ϕ is the Greek letter phi.]

21. Find these values of the Euler ϕ-function.
a) $\phi(4)$. **b)** $\phi(10)$. **c)** $\phi(13)$.

22. Show that n is prime if and only if $\phi(n) = n - 1$.

23. What is the value of $\phi(p^k)$ when p is prime and k is a positive integer?

24. What are the greatest common divisors of these pairs of integers?
a) $2^2 \cdot 3^3 \cdot 5^5$, $2^5 \cdot 3^3 \cdot 5^2$
b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, $2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

CHAPTER

5 Induction and Recursion

Many mathematical statements assert that a property is true for all positive integers. Examples of such statements are that for every positive integer n: $n! \leq n^n$, $n^3 - n$ is divisible by 3; a set with n elements has 2^n subsets; and the sum of the first n positive integers is $n(n+1)/2$. A major goal of this chapter, and the book, is to give the student a thorough understanding of mathematical induction, which is used to prove results of this kind.

Proofs using mathematical induction have two parts. First, they show that the statement holds for the positive integer 1. Second, they show that if the statement holds for a positive integer then it must also hold for the next larger integer. Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n\, P(n)$ is true. Mathematical induction can be used to prove a tremendous variety of results. Understanding how to read and construct proofs by mathematical induction is a key goal of learning discrete mathematics.

In Chapter 2 we explicitly defined sets and functions. That is, we described sets by listing their elements or by giving some property that characterizes these elements. We gave formulae for the values of functions. There is another important way to define such objects, based on mathematical induction. To define functions, some initial terms are specified, and a rule is given for finding subsequent values from values already known. (We briefly touched on this sort of definition in Chapter 2 when we showed how sequences can be defined using recurrence relations.) Sets can be defined by listing some of their elements and giving rules for constructing elements from those already known to be in the set. Such definitions, called *recursive definitions*, are used throughout discrete mathematics and computer science. Once we have defined a set recursively, we can use a proof method called structural induction to prove results about this set.

When a procedure is specified for solving a problem, this procedure must *always* solve the problem correctly. Just testing to see that the correct result is obtained for a set of input values does not show that the procedure always works correctly. The correctness of a procedure can be guaranteed only by proving that it always yields the correct result. The final section of this chapter contains an introduction to the techniques of program verification. This is a formal technique to verify that procedures are correct. Program verification serves as the basis for attempts under way to prove in a mechanical fashion that programs are correct.

5.1 Mathematical Induction

Introduction

Suppose that we have an infinite ladder, as shown in Figure 1, and we want to know whether we can reach every step on this ladder. We know two things:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying (2) again, because we can reach the second rung, we can also reach the third rung. Continuing in this way, we can show that we

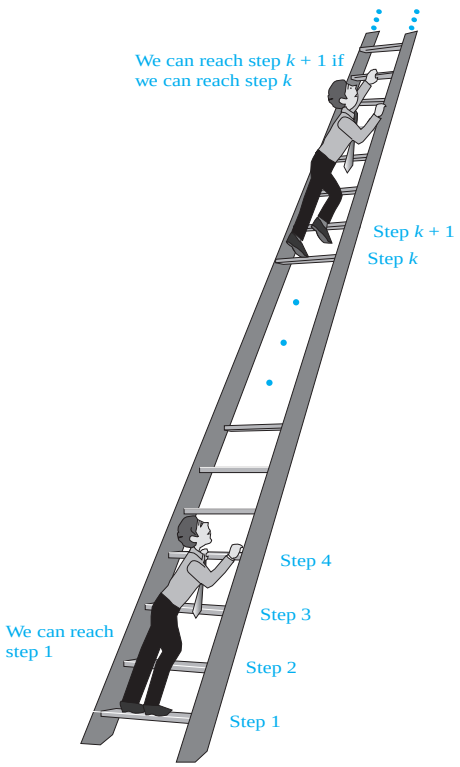

FIGURE 1 **Climbing an Infinite Ladder.**

can reach the fourth rung, the fifth rung, and so on. For example, after 100 uses of (2), we know that we can reach the 101st rung. But can we conclude that we are able to reach every rung of this infinite ladder? The answer is yes, something we can verify using an important proof technique called **mathematical induction**. That is, we can show that $P(n)$ is true for every positive integer n, where $P(n)$ is the statement that we can reach the nth rung of the ladder.

Mathematical induction is an extremely important proof technique that can be used to prove assertions of this type. As we will see in this section and in subsequent sections of this chapter and later chapters, mathematical induction is used extensively to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.

In this section, we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is *not* a tool for discovering formulae or theorems.

Mathematical Induction

Assessment

In general, mathematical induction * can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical

*Unfortunately, using the terminology "mathematical induction" clashes with the terminology used to describe different types of reasoning. In logic, **deductive reasoning** uses rules of inference to draw conclusions from premises, whereas **inductive reasoning** makes conclusions only supported, but not ensured, by evidence. Mathematical proofs, including arguments that use mathematical induction, are deductive, not inductive.

induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**. Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers, that is, we have shown that $\forall n\, P(n)$ is true where the quantification is over the set of positive integers. In the inductive step, we show that $\forall k(P(k) \rightarrow P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \rightarrow P(k+1)$ is true for every positive integer k. To prove that this conditional statement is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Remark: In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know that $P(2)$ is true, because $P(1)$ implies $P(2)$. Further, we know that $P(3)$ is true, because $P(2)$ implies $P(3)$. Continuing along these lines, we see that $P(n)$ is true for every positive integer n.

HISTORICAL NOTE The first known use of mathematical induction is in the work of the sixteenth-century mathematician Francesco Maurolico (1494–1575). Maurolico wrote extensively on the works of classical mathematics and made many contributions to geometry and optics. In his book *Arithmeticorum Libri Duo*, Maurolico presented a variety of properties of the integers together with proofs of these properties. To prove some of these properties, he devised the method of mathematical induction. His first use of mathematical induction in this book was to prove that the sum of the first n odd positive integers equals n^2. Augustus De Morgan is credited with the first presentation in 1838 of formal proofs using mathematical induction, as well as introducing the terminology "mathematical induction." Maurolico's proofs were informal and he never used the word "induction." See [Gu11] to learn more about the history of the method of mathematical induction.

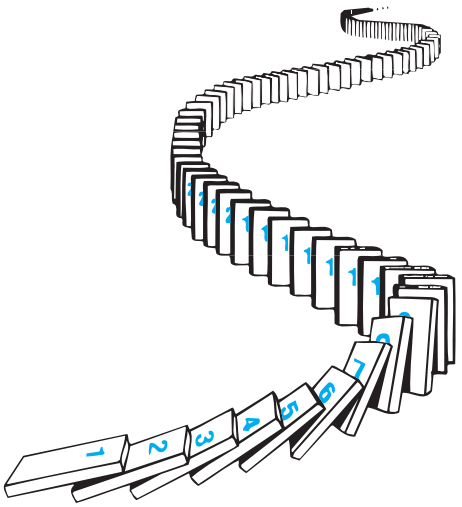

FIGURE 2 **Illustrating How Mathematical Induction Works Using Dominoes.**

WAYS TO REMEMBER HOW MATHEMATICAL INDUCTION WORKS Thinking of the infinite ladder and the rules for reaching steps can help you remember how mathematical induction works. Note that statements (1) and (2) for the infinite ladder are exactly the basis step and inductive step, respectively, of the proof that $P(n)$ is true for all positive integers n, where $P(n)$ is the statement that we can reach the nth rung of the ladder. Consequently, we can invoke mathematical induction to conclude that we can reach every rung.

Another way to illustrate the principle of mathematical induction is to consider an infinite row of dominoes, labeled $1, 2, 3, \ldots, n, \ldots$, where each domino is standing up. Let $P(n)$ be the proposition that domino n is knocked over. If the first domino is knocked over—i.e., if $P(1)$ is true—and if, whenever the kth domino is knocked over, it also knocks the $(k+1)$st domino over—i.e., if $P(k) \rightarrow P(k+1)$ is true for all positive integers k—then all the dominoes are knocked over. This is illustrated in Figure 2.

Why Mathematical Induction is Valid

Why is mathematical induction a valid proof technique? The reason comes from the well-ordering property, listed in Appendix 1, as an axiom for the set of positive integers, which states that every nonempty subset of the set of positive integers has a least element. So, suppose we know that $P(1)$ is true and that the proposition $P(k) \rightarrow P(k+1)$ is true for all positive integers k. To show that $P(n)$ must be true for all positive integers n, assume that there is at least one positive integer for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty. Thus, by the well-ordering property, S has a least element, which will be denoted by m. We know that m cannot be 1, because $P(1)$ is true. Because m is positive and greater than 1, $m-1$ is a positive integer. Furthermore, because $m-1$ is less than m, it is not in S, so $P(m-1)$ must be true. Because the conditional statement $P(m-1) \rightarrow P(m)$ is also true, it must be the case that $P(m)$ is true. This contradicts the choice of m. Hence, $P(n)$ must be true for every positive integer n.

The Good and the Bad of Mathematical Induction

An important point needs to be made about mathematical induction before we commence a study of its use. The good thing about mathematical induction is that it can be used to prove

You can prove a theorem by mathematical induction even if you do not have the slightest idea why it is true!

a conjecture once it is has been made (and is true). The bad thing about it is that it cannot be used to find new theorems. Mathematicians sometimes find proofs by mathematical induction unsatisfying because they do not provide insights as to why theorems are true. Many theorems can be proved in many ways, including by mathematical induction. Proofs of these theorems by methods other than mathematical induction are often preferred because of the insights they bring.

Examples of Proofs by Mathematical Induction

Many theorems assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. Mathematical induction is a technique for proving theorems of this kind. In other words, mathematical induction can be used to prove statements of the form $\forall n\, P(n)$, where the domain is the set of positive integers. Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form. (Remember, many mathematical assertions include an implicit universal quantifier. The statement "if n is a positive integer, then $n^3 - n$ is divisible by 3" is an example of this. Making the implicit universal quantifier explicit yields the statement "for every positive integer n, $n^3 - n$ is divisible by 3.)

We will use how theorems are proved using mathematical induction. The theorems we will prove include summation formulae, inequalities, identities for combinations of sets, divisibility results, theorems about algorithms, and some other creative results. In this section and in later sections, we will employ mathematical induction to prove many other types of results, including the correctness of computer programs and algorithms. Mathematical induction can be used to prove a wide variety of theorems, not just summation formulae, inequalities, and other types of examples we illustrate here. (For proofs by mathematical induction of many more interesting and diverse results, see the *Handbook of Mathematical Induction* by David Gunderson [Gu11]. This book is part of the extensive CRC Series in Discrete Mathematics, many of which may be of interest to readers. The author is the Series Editor of these books).

Note that there are many opportunities for errors in induction proofs. We will describe some incorrect proofs by mathematical induction at the end of this section and in the exercises. To avoid making errors in proofs by mathematical induction, try to follow the guidelines for such proofs given at the end of this section.

Look for the $\overset{\text{IH}}{=}$ symbol to see where the inductive hypothesis is used.

SEEING WHERE THE INDUCTIVE HYPOTHESIS IS USED To help the reader understand each of the mathematical induction proofs in this section, we will note where the inductive hypothesis is used. We indicate this use in three different ways: by explicit mention in the text, by inserting the acronym IH (for inductive hypothesis) over an equals sign or a sign for an inequality, or by specifying the inductive hypothesis as the reason for a step in a multi-line display.

PROVING SUMMATION FORMULAE We begin by using mathematical induction to prove several summation formulae. As we will see, mathematical induction is particularly well suited for proving that such formulae are valid. However, summation formulae can be proven in other ways. This is not surprising because there are often different ways to prove a theorem. The major disadvantage of using mathematical induction to prove a summation formula is that you cannot use it to derive this formula. That is, you must already have the formula before you attempt to prove it by mathematical induction.

Examples 1–4 illustrate how to use mathematical induction to prove summation formulae. The first summation formula we will prove by mathematical induction, in Example 1, is a closed formula for the sum of the smallest n positive integers.

EXAMPLE 1 Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Extra Examples

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

If you are rusty simplifying algebraic expressions, this is the time to do some reviewing!

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n. ◀

As we noted, mathematical induction is not a tool for finding theorems about all positive integers. Rather, it is a proof method for proving such results once they are conjectured. In Example 2, using mathematical induction to prove a summation formula, we will both formulate and then prove a conjecture.

EXAMPLE 2 Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution: The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$$1 = 1, \qquad 1 + 3 = 4, \qquad 1 + 3 + 5 = 9,$$
$$1 + 3 + 5 + 7 = 16, \qquad 1 + 3 + 5 + 7 + 9 = 25.$$

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let $P(n)$ denote the proposition that the sum of the first n odd positive integers is n^2. Our conjecture is that $P(n)$ is true for all positive integers. To use mathematical induction to prove this conjecture, we must first complete the basis step; that is, we must show that $P(1)$ is true. Then we must carry out the inductive step; that is, we must show that $P(k + 1)$ is true when $P(k)$ is assumed to be true. We now attempt to complete these two steps.

BASIS STEP: $P(1)$ states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1. The basis step is complete.

INDUCTIVE STEP: To complete the inductive step we must show that the proposition $P(k) \rightarrow P(k + 1)$ is true for every positive integer k. To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(k)$ is true for an arbitrary positive integer k, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

(Note that the kth odd positive integer is $(2k - 1)$, because this integer is obtained by adding 2 a total of $k - 1$ times to 1.) To show that $\forall k(P(k) \rightarrow P(k + 1))$ is true, we must show that if $P(k)$ is true (the inductive hypothesis), then $P(k + 1)$ is true. Note that $P(k + 1)$ is the statement that

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.$$

So, assuming that $P(k)$ is true, it follows that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + \cdots + (2k - 1)] + (2k + 1) \\ &\overset{\text{IH}}{=} k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

This shows that $P(k + 1)$ follows from $P(k)$. Note that we used the inductive hypothesis $P(k)$ in the second equality to replace the sum of the first k odd positive integers by k^2.

We have now completed both the basis step and the inductive step. That is, we have shown that $P(1)$ is true and the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. Consequently, by the principle of mathematical induction we can conclude that $P(n)$ is true for all positive integers n. That is, we know that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all positive integers n. ◀

Often, we will need to show that $P(n)$ is true for $n = b,\ b + 1,\ b + 2, \ldots$, where b is an integer other than 1. We can use mathematical induction to accomplish this, as long as we change the basis step by replacing $P(1)$ with $P(b)$. In other words, to use mathematical induction to show that $P(n)$ is true for $n = b, b + 1, b + 2, \ldots$, where b is an integer other than 1, we show that $P(b)$ is true in the basis step. In the inductive step, we show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for $k = b, b + 1, b + 2, \ldots$. Note that b can be negative, zero, or positive. Following the domino analogy we used earlier, imagine that we begin by knocking down the bth domino (the basis step), and as each domino falls, it knocks down the next domino (the inductive step). We leave it to the reader to show that this form of induction is valid (see Exercise 83).

We illustrate this notion in Example 3, which states that a summation formula is valid for all nonnegative integers. In this example, we need to prove that $P(n)$ is true for $n = 0, 1, 2, \ldots$. So, the basis step in Example 3 shows that $P(0)$ is true.

EXAMPLE 3 Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k + 1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n. ◀

The formula given in Example 3 is a special case of a general result for the sum of terms of a geometric progression (Theorem 1 in Section 2.4). We will use mathematical induction to provide an alternative proof of this formula.

EXAMPLE 4 **Sums of Geometric Progressions** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n + 1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k + 1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\begin{aligned}\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} &= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1} \\ &= \frac{ar^{k+2} - a}{r - 1}.\end{aligned}$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$ must also be true. This completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction $P(n)$ is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct. ◀

As previously mentioned, the formula in Example 3 is the case of the formula in Example 4 with $a = 1$ and $r = 2$. The reader should verify that putting these values for a and r into the general formula gives the same formula as in Example 3.

PROVING INEQUALITIES Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer, as Examples 5–7 illustrate.

EXAMPLE 5 Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers n.

Extra Examples

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

INDUCTIVE STEP: We first assume the inductive hypothesis that $P(k)$ is true for anarbitrary positive integer k. That is, the inductive hypothesis $P(k)$ is the statement that $k < 2^k$. To complete the inductive step, we need to show that if $P(k)$ is true, then $P(k + 1)$, which is the statement that $k + 1 < 2^{k+1}$, is true. That is, we need to show that if $k < 2^k$, then $k + 1 < 2^{k+1}$. To show

that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k < 2^k$, and then note that $1 \leq 2^k$. This tells us that

$$k + 1 \overset{\text{IH}}{<} 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

This shows that $P(k + 1)$ is true, namely, that $k + 1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n. ◀

EXAMPLE 6 Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1$, 2, and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k + 1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k + 1)!$. We have

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\
&< 2 \cdot k! && \text{by the inductive hypothesis} \\
&< (k + 1)k! && \text{because } 2 < k + 1 \\
&= (k + 1)! && \text{by definition of factorial function.}
\end{aligned}$$

This shows that $P(k + 1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

An important inequality for the sum of the reciprocals of a set of positive integers will be proved in Example 7.

EXAMPLE 7 **An Inequality for Harmonic Numbers** The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true for all nonnegative integers n. That is, the inequality $H_{2^n} \geq 1 + \frac{n}{2}$ for the harmonic numbers holds for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

PROVING DIVISIBILITY RESULTS Mathematical induction can be used to prove divisibility results about integers. Although such results are often easier to prove using basic results in number theory, it is instructive to see how to prove such results using mathematical induction, as Examples 8 and 9 illustrate.

EXAMPLE 8 Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer. (Note that this is the statement with $p = 3$ of Fermat's little theorem, which is Theorem 3 of Section 4.4.)

Extra Examples

Solution: To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k. To complete the inductive

step, we must show that when we assume the inductive hypothesis, it follows that $P(k+1)$, the statement that $(k+1)^3-(k+1)$ is divisible by 3, is also true. That is, we must show that $(k+1)^3-(k+1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3-(k+1) &= (k^3+3k^2+3k+1)-(k+1)\\ &= (k^3-k)+3(k^2+k).\end{aligned}$$

Using the inductive hypothesis, we conclude that the first term k^3-k is divisible by 3. The second term is divisible by 3 because it is 3 times an integer. So, by part (i) of Theorem 1 in Section 4.1, we know that $(k+1)^3-(k+1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that n^3-n is divisible by 3 whenever n is a positive integer. ◀

The next example presents a more challenging proof by mathematical induction of a divisibility result.

EXAMPLE 9 Use mathematical induction to prove that $7^{n+2}+8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Extra Examples

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2}+8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer. We see that $P(0)$ is true because $7^{0+2}+8^{2\cdot 0+1}=7^2+8^1=57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2}+8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2}+8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}7^{(k+1)+2}+8^{2(k+1)+1} &= 7^{k+3}+8^{2k+3}\\ &= 7\cdot 7^{k+2}+8^2\cdot 8^{2k+1}\\ &= 7\cdot 7^{k+2}+64\cdot 8^{2k+1}\\ &= 7(7^{k+2}+8^{2k+1})+57\cdot 8^{2k+1}.\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2}+8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2}+8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57\cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2}+8^{2k+1})+57\cdot 8^{2k+1}=7^{k+3}+8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2}+8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

PROVING RESULTS ABOUT SETS Mathematical induction can be used to prove many results about sets. In particular, in Example 10 we prove a formula for the number of subsets of a finite set and in Example 11 we establish a set identity.

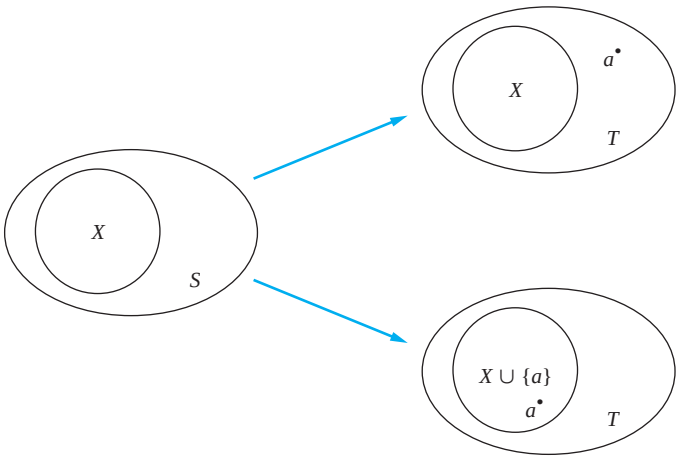

FIGURE 3 **Generating Subsets of a Set with $k + 1$ Elements. Here $T = S \cup \{a\}$.**

EXAMPLE 10 **The Number of Subsets of a Finite Set** Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets. (We will prove this result directly in several ways in Chapter 6.)

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

BASIS STEP: $P(0)$ is true, because a set with zero elements, the empty set, has exactly $2^0 = 1$ subset, namely, itself.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k, that is, we assume that every set with k elements has 2^k subsets. It must be shown that under this assumption, $P(k + 1)$, which is the statement that every set with $k + 1$ elements has 2^{k+1} subsets, must also be true. To show this, let T be a set with $k + 1$ elements. Then, it is possible to write $T = S \cup \{a\}$, where a is one of the elements of T and $S = T - \{a\}$ (and hence $|S| = k$). The subsets of T can be obtained in the following way. For each subset X of S there are exactly two subsets of T, namely, X and $X \cup \{a\}$. (This is illustrated in Figure 3.) These constitute all the subsets of T and are all distinct. We now use the inductive hypothesis to conclude that S has 2^k subsets, because it has k elements. We also know that there are two subsets of T for each subset of S. Therefore, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. This finishes the inductive argument.

Because we have completed the basis step and the inductive step, by mathematical induction it follows that $P(n)$ is true for all nonnegative integers n. That is, we have proved that a set with n elements has 2^n subsets whenever n is a nonnegative integer. ◀

EXAMPLE 11 Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$.

Solution: Let $P(n)$ be the identity for n sets.

BASIS STEP: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's laws; it was proved in Example 11 of Section 2.2.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary integer with $k \geq 2$; that is, it is the statement that

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_k$ are subsets of the universal set U. To carry out the inductive step, we need to show that this assumption implies that $P(k+1)$ is true. That is, we need to show that if this equality holds for every collection of k subsets of U, then it must also hold for every collection of $k+1$ subsets of U. Suppose that $A_1, A_2, \ldots, A_k, A_{k+1}$ are subsets of U. When the inductive hypothesis is assumed to hold, it follows that

$$\begin{aligned}
\overline{\bigcap_{j=1}^{k+1} A_j} &= \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}} && \text{by the definition of intersection} \\
&= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} && \text{by De Morgan's law (where the two sets are } \textstyle\bigcap_{j=1}^{k} A_j \text{ and } A_{k+1}) \\
&= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} && \text{by the inductive hypothesis} \\
&= \bigcup_{j=1}^{k+1} \overline{A_j} && \text{by the definition of union.}
\end{aligned}$$

This completes the inductive step.

Because we have completed both the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true whenever n is a positive integer, $n \geq 2$. That is, we know that

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2, \ldots, A_n$ are subsets of a universal set U and $n \geq 2$. ◀

PROVING RESULTS ABOUT ALGORITHMS Next, we provide an example (somewhat more difficult than previous examples) that illustrates one of many ways mathematical induction is used in the study of algorithms. We will show how mathematical induction can be used to prove that a greedy algorithm we introduced in Section 3.1 always yields an optimal solution.

EXAMPLE 12 Recall the algorithm for scheduling talks discussed in Example 7 of Section 3.1. The input to this algorithm is a group of m proposed talks with preset starting and ending times. The goal is to schedule as many of these lectures as possible in the main lecture hall so that no two talks overlap. Suppose that talk t_j begins at time s_j and ends at time e_j. (No two lectures can proceed in the main lecture hall at the same time, but a lecture in this hall can begin at the same time another one ends.)

Without loss of generality, we assume that the talks are listed in order of nondecreasing ending time, so that $e_1 \leq e_2 \leq \cdots \leq e_m$. The greedy algorithm proceeds by selecting at each stage a talk with the earliest ending time among all those talks that begin no sooner than when

the last talk scheduled in the main lecture hall has ended. Note that a talk with the earliest end time is always selected first by the algorithm. We will show that this greedy algorithm is optimal in the sense that it always schedules the most talks possible in the main lecture hall. To prove the optimality of this algorithm we use mathematical induction on the variable n, the number of talks scheduled by the algorithm. We let $P(n)$ be the proposition that if the greedy algorithm schedules n talks in the main lecture hall, then it is not possible to schedule more than n talks in this hall.

BASIS STEP: Suppose that the greedy algorithm managed to schedule just one talk, t_1, in the main lecture hall. This means that no other talk can start at or after e_1, the end time of t_1. Otherwise, the first such talk we come to as we go through the talks in order of nondecreasing end times could be added. Hence, at time e_1 each of the remaining talks needs to use the main lecture hall because they all start before e_1 and end after e_1. It follows that no two talks can be scheduled because both need to use the main lecture hall at time e_1. This shows that $P(1)$ is true and completes the basis step.

INDUCTIVE STEP: The inductive hypothesis is that $P(k)$ is true, where k is an arbitrary positive integer, that is, that the greedy algorithm always schedules the most possible talks when it selects k talks, where k is a positive integer, given any set of talks, no matter how many. We must show that $P(k+1)$ follows from the assumption that $P(k)$ is true, that is, we must show that under the assumption of $P(k)$, the greedy algorithm always schedules the most possible talks when it selects $k+1$ talks.

Now suppose that the greedy algorithm has selected $k+1$ talks. Our first step in completing the inductive step is to show there is a schedule including the most talks possible that contains talk t_1, a talk with the earliest end time. This is easy to see because a schedule that begins with the talk t_i in the list, where $i > 1$, can be changed so that talk t_1 replaces talk t_i. To see this, note that because $e_1 \leq e_i$, all talks that were scheduled to follow talk t_i can still be scheduled.

Once we included talk t_1, scheduling the talks so that as many as possible are scheduled is reduced to scheduling as many talks as possible that begin at or after time e_1. So, if we have scheduled as many talks as possible, the schedule of talks other than talk t_1 is an optimal schedule of the original talks that begin once talk t_1 has ended. Because the greedy algorithm schedules k talks when it creates this schedule, we can apply the inductive hypothesis to conclude that it has scheduled the most possible talks. It follows that the greedy algorithm has scheduled the most possible talks, $k+1$, when it produced a schedule with $k+1$ talks, so $P(k+1)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step. So, by mathematical induction we know that $P(n)$ is true for all positive integers n. This completes the proof of optimality. That is, we have proved that when the greedy algorithm schedules n talks, when n is a positive integer, then it is not possible to schedule more than n talks. ◀

CREATIVE USES OF MATHEMATICAL INDUCTION Mathematical induction can often be used in unexpected ways. We will illustrate two particularly clever uses of mathematical induction here, the first relating to survivors in a pie fight and the second relating to tilings with regular triominoes of checkerboards with one square missing.

EXAMPLE 13 **Odd Pie Fights** An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Use mathematical induction to show that there is at least one survivor, that is, at least one person who is not hit by a pie. (This problem was introduced by Carmony [Ca79]. Note that this result is false when there are an even number of people; see Exercise 75.)

Solution: Let $P(n)$ be the statement that there is a survivor whenever $2n+1$ people stand in a yard at distinct mutual distances and each person throws a pie at their nearest neighbor. To prove this result, we will show that $P(n)$ is true for all positive integers n. This follows because as n runs through all positive integers, $2n+1$ runs through all odd integers greater than or equal

to 3. Note that one person cannot engage in a pie fight because there is no one else to throw the pie at.

BASIS STEP: When $n = 1$, there are $2n + 1 = 3$ people in the pie fight. Of the three people, suppose that the closest pair are A and B, and C is the third person. Because distances between pairs of people are different, the distance between A and C and the distance between B and C are both different from, and greater than, the distance between A and B. It follows that A and B throw pies at each other, while C throws a pie at either A or B, whichever is closer. Hence, C is not hit by a pie. This shows that at least one of the three people is not hit by a pie, completing the basis step.

INDUCTIVE STEP: For the inductive step, assume that $P(k)$ is true for an arbitrary odd integer k with $k \geq 3$. That is, assume that there is at least one survivor whenever $2k + 1$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor. We must show that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that there is at least one survivor whenever $2(k + 1) + 1 = 2k + 3$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor, is also true.

So suppose that we have $2(k + 1) + 1 = 2k + 3$ people in a yard with distinct distances between pairs of people. Let A and B be the closest pair of people in this group of $2k + 3$ people. When each person throws a pie at the nearest person, A and B throw pies at each other. We have two cases to consider, *(i)* when someone else throws a pie at either A or B and *(ii)* when no one else throws a pie at either A or B.

Case (i): Because A and B throw pies at each other and someone else throws a pie at either A and B, at least three pies are thrown at A and B, and at most $(2k + 3) - 3 = 2k$ pies are thrown at the remaining $2k + 1$ people. This guarantees that at least one person is a survivor, for if each of these $2k + 1$ people was hit by at least one pie, a total of at least $2k + 1$ pies would have to be thrown at them. (The reasoning used in this last step is an example of the pigeonhole principle discussed further in Section 6.2.)

Case (ii): No one else throws a pie at either A and B. Besides A and B, there are $2k + 1$ people. Because the distances between pairs of these people are all different, we can use the inductive hypothesis to conclude that there is at least one survivor S when these $2k + 1$ people each throws a pie at their nearest neighbor. Furthermore, S is also not hit by either the pie thrown by A or the pie thrown by B because A and B throw their pies at each other, so S is a survivor because S is not hit by any of the pies thrown by these $2k + 3$ people.

We have completed both the basis step and the inductive step, using a proof by cases. So by mathematical induction it follows that $P(n)$ is true for all positive integers n. We conclude that whenever an odd number of people located in a yard at distinct mutual distances each throws a pie at their nearest neighbor, there is at least one survivor. ◀

Links

In Section 1.8 we discussed the tiling of checkerboards by polyominoes. Example 14 illustrates how mathematical induction can be used to prove a result about covering checkerboards with right triominoes, pieces shaped like the letter "L."

EXAMPLE 14 Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes, where these pieces cover three squares at a time, as shown in Figure 4.

FIGURE 4 A Right Triomino.

Solution: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. We can use mathematical induction to prove that $P(n)$ is true for all positive integers n.

BASIS STEP: $P(1)$ is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino, as shown in Figure 5.

FIGURE 5 Tiling 2 × 2 Checkerboards with One Square Removed.

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(k)$ is true for the positive integer k; that is, it is the assumption that every $2^k \times 2^k$ checkerboard with one square removed can be tiled using right triominoes. It must be shown that under the assumption of the inductive hypothesis, $P(k+1)$ must also be true; that is, any $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

To see this, consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions. This is illustrated in Figure 6. No square has been removed from three of these four checkerboards. The fourth $2^k \times 2^k$ checkerboard has one square removed, so we now use the inductive hypothesis to conclude that it can be covered by right triominoes. Now temporarily remove the square from each of the other three $2^k \times 2^k$ checkerboards that has the center of the original, larger checkerboard as one of its corners, as shown in Figure 7. By the inductive hypothesis, each of these three $2^k \times 2^k$ checkerboards with a square removed can be tiled by right triominoes. Furthermore, the three squares that were temporarily removed can be covered by one right triomino. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard can be tiled with right triominoes.

We have completed the basis step and the inductive step. Therefore, by mathematical induction $P(n)$ is true for all positive integers n. This shows that we can tile every $2^n \times 2^n$ checkerboard, where n is a positive integer, with one square removed, using right triominoes. ◀

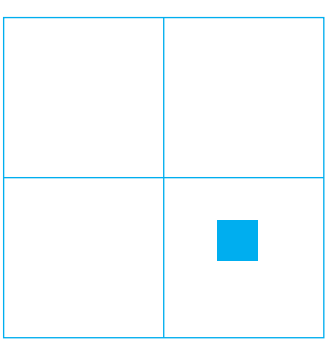

FIGURE 6 Dividing a $2^{k+1} \times 2^{k+1}$ Checkerboard into Four $2^k \times 2^k$ Checkerboards.

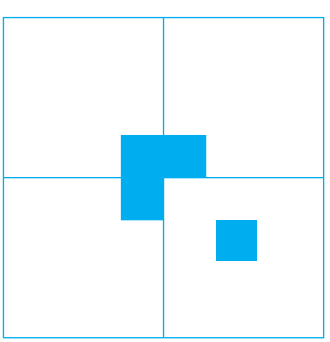

FIGURE 7 Tiling the $2^{k+1} \times 2^{k+1}$ Checkerboard with One Square Removed.

Mistaken Proofs By Mathematical Induction

As with every proof method, there are many opportunities for making errors when using mathematical induction. Many well-known mistaken, and often entertaining, proofs by mathematical induction of clearly false statements have been devised, as exemplified by Example 15 and Exercises 49–51. Often, it is not easy to find where the error in reasoning occurs in such mistaken proofs.

Consult *Common Errors in Discrete Mathematics* on this book's website for more basic mistakes.

To uncover errors in proofs by mathematical induction, remember that in every such proof, both the basis step and the inductive step must be done correctly. Not completing the basis step in a supposed proof by mathematical induction can lead to mistaken proofs of clearly ridiculous statements such as "$n = n + 1$ whenever n is a positive integer." (We leave it to the reader to show that it is easy to construct a correct inductive step in an attempted proof of this statement.) Locating the error in a faulty proof by mathematical induction, as Example 15 illustrates, can be quite tricky, especially when the error is hidden in the basis step.

EXAMPLE 15 Find the error in this "proof" of the clearly false claim that every set of lines in the plane, no two of which are parallel, meet in a common point.

"Proof:" Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. We will attempt to prove that $P(n)$ is true for all positive integers $n \geq 2$.

BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point (by the definition of parallel lines).

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer k, that is, it is the assumption that every set of k lines in the plane, no two of which are parallel, meet in a common point. To complete the inductive step, we must show that if $P(k)$ is true, then $P(k + 1)$ must also be true. That is, we must show that if every set of k lines in the plane, no two of which are parallel, meet in a common point, then every set of $k + 1$ lines in the plane, no two of which are parallel, meet in a common point. So, consider a set of $k + 1$ distinct lines in the plane. By the inductive hypothesis, the first k of these lines meet in a common point p_1. Moreover, by the inductive hypothesis, the last k of these lines meet in a common point p_2. We will show that p_1 and p_2 must be the same point. If p_1 and p_2 were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, p_1 and p_2 are the same point. We conclude that the point $p_1 = p_2$ lies on all $k + 1$ lines. We have shown that $P(k + 1)$ is true assuming that $P(k)$ is true. That is, we have shown that if we assume that every k, $k \geq 2$, distinct lines meet in a common point, then every $k + 1$ distinct lines meet in a common point. This completes the inductive step.

We have completed the basis step and the inductive step, and supposedly we have a correct proof by mathematical induction.

Solution: Examining this supposed proof by mathematical induction it appears that everything is in order. However, there is an error, as there must be. The error is rather subtle. Carefully looking at the inductive step shows that this step requires that $k \geq 3$. We cannot show that $P(2)$ implies $P(3)$. When $k = 2$, our goal is to show that every three distinct lines meet in a common point. The first two lines must meet in a common point p_1 and the last two lines must meet in a common point p_2. But in this case, p_1 and p_2 do not have to be the same, because only the second line is common to both sets of lines. Here is where the inductive step fails. ◀

Guidelines for Proofs by Mathematical Induction

Examples 1–14 illustrate proofs by mathematical induction of a diverse collection of theorems. Each of these examples includes all the elements needed in a proof by mathematical induction. We have provided an example of an invalid proof by mathematical induction. Summarizing what we have learned from these examples, we can provide some useful guidelines for constructing correct proofs by mathematical induction. We now present these guidelines.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

It is worthwhile to revisit each of the mathematical induction proofs in Examples 1–14 to see how these steps are completed. It will be helpful to follow these guidelines in the solutions of the exercises that ask for proofs by mathematical induction. The guidelines that we presented can be adapted for each of the variants of mathematical induction that we introduce in the exercises and later in this chapter.

Exercises

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

2. Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Prove that this golfer plays every hole on the course.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

9. a) Find a formula for the sum of the first n even positive integers.
b) Prove the formula that you conjectured in part (a).

10. a) Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.
b) Prove the formula you conjectured in part (a).

11. a) Find a formula for

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$$

by examining the values of this expression for small values of n.
b) Prove the formula you conjectured in part (a).

12. Prove that

$$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j=\frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$$

whenever n is a nonnegative integer.

13. Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}n(n+1)/2$ whenever n is a positive integer.

14. Prove that for every positive integer n, $\sum_{k=1}^{n}k2^k=(n-1)2^{n+1}+2$.

15. Prove that for every positive integer n,

$$1\cdot 2+2\cdot 3+\cdots+n(n+1)=n(n+1)(n+2)/3.$$

16. Prove that for every positive integer n,

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=n(n+1)(n+2)(n+3)/4.$$

17. Prove that $\sum_{j=1}^{n}j^4=n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

Use mathematical induction to prove the inequalities in Exercises 18–30.

18. Let $P(n)$ be the statement that $n!<n^n$, where n is an integer greater than 1.
a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of the proof.
c) What is the inductive hypothesis?
d) What do you need to prove in the inductive step?
e) Complete the inductive step.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

19. Let $P(n)$ be the statement that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n},$$

where n is an integer greater than 1.
a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of the proof.
c) What is the inductive hypothesis?
d) What do you need to prove in the inductive step?
e) Complete the inductive step.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

20. Prove that $3^n<n!$ if n is an integer greater than 6.

21. Prove that $2^n>n^2$ if n is an integer greater than 4.

22. For which nonnegative integers n is $n^2\le n!$? Prove your answer.

23. For which nonnegative integers n is $2n+3\le 2^n$? Prove your answer.

24. Prove that $1/(2n)\le[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]/(2\cdot 4\cdot\cdots\cdot 2n)$ whenever n is a positive integer.

***25.** Prove that if $h>-1$, then $1+nh\le(1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

***26.** Suppose that a and b are real numbers with $0<b<a$. Prove that if n is a positive integer, then $a^n-b^n\le na^{n-1}(a-b)$.

***27.** Prove that for every positive integer n,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2(\sqrt{n+1}-1).$$

28. Prove that $n^2-7n+12$ is nonnegative whenever n is an integer with $n\ge 3$.

In Exercises 29 and 30, H_n denotes the nth harmonic number.

***29.** Prove that $H_{2^n}\le 1+n$ whenever n is a nonnegative integer.

***30.** Prove that

$$H_1+H_2+\cdots+H_n=(n+1)H_n-n.$$

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

31. Prove that 2 divides n^2+n whenever n is a positive integer.

32. Prove that 3 divides n^3+2n whenever n is a positive integer.

33. Prove that 5 divides n^5-n whenever n is a nonnegative integer.

34. Prove that 6 divides n^3-n whenever n is a nonnegative integer.

***35.** Prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

***36.** Prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever n is a positive integer.

***37.** Prove that if n is a positive integer, then 133 divides $11^{n+1}+12^{2n-1}$.

Use mathematical induction in Exercises 38–46 to prove results about sets.

38. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j\subseteq B_j$ for $j=1,2,\ldots,n$, then

$$\bigcup_{j=1}^{n}A_j\subseteq\bigcup_{j=1}^{n}B_j.$$

39. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

41. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

42. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

43. Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}.$$

44. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

49. What is wrong with this "proof" that all horses are the same color?

Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

50. What is wrong with this "proof"?

"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = (n + \frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n+1) + \frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof"?

"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers. Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

52. Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

***53.** Use mathematical induction to show that n people can divide a cake (where each person gets one or more separate pieces of the cake) so that the cake is divided fairly, that is, in the sense that each person thinks he or she got at least $(1/n)$th of the cake. [*Hint:* For the inductive step, take a fair division of the cake among the first k people, have each person divide their share into what this person thinks are $k+1$ equal portions, and then have the $(k+1)$st person select a portion from each of the k people. When showing this produces a fair division for $k+1$ people, suppose that person $k+1$ thinks that person i got p_i of the cake where $\sum_{i=1}^{k} p_i = 1$.]

54. Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***55.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose that we have an infinite chessboard, made up

of all squares (m, n) where m and n are nonnegative integers that denote the row number and the column number of the square, respectively. Use mathematical induction to show that a knight starting at (0, 0) can visit every square using a finite sequence of moves. [*Hint:* Use induction on the variable $s = m + n$.]

56. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

57. (*Requires calculus*) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

58. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

59. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

60. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***61.** Show that

$$[(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge \cdots \wedge (p_{n-1} \rightarrow p_n)] \rightarrow [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \rightarrow p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions, where $n \geq 2$.

***62.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

****63.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean** of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

64. Use mathematical induction to prove Lemma 3 of Section 4.3, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

65. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

***66.** Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k + 1)$ are both true, then $P(k + 2)$ is true.

67. Show that if $A_1, A_2, \ldots, A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$ either A_i is a subset of A_j or A_j is a subset of A_i, then there is an integer i, $1 \leq i \leq n$ such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

***68.** A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question—asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n - 1)$ questions. [*Hint:* First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69–71 deal with the gossip problem.

69. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

70. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [*Hint:* In the inductive step, have a new person call a particular person at the start and at the end.]

****71.** Prove that $G(n) = 2n - 4$ for $n \geq 4$.

***72.** Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

***73.** Show that if $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, $n \geq 2$, and every pair of these intervals has a nonempty intersection, that is, $I_i \cap I_j \neq \emptyset$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n$, then the intersection of all these sets is nonempty, that is, $I_1 \cap I_2 \cap \cdots \cap I_n \neq \emptyset$. (Recall that an **open interval** is