Mathematical Foundations for Computer Applications

Predicate Calculus

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- **A predicate is a proposition that is a function of one or more variables.**

E.g.: $P(x)$: x is an even number. So $P(1)$ is false, $P(2)$ is true,….

Examples of predicates:

- Domain ASCII characters - IsAlpha(x) : TRUE iff x is an alphabetical character.
- Domain floating point numbers - IsInt(x): TRUE iff x is an integer.

Quantifiers

- **Describes the values of a variable that make the predicate true.**

E.g. $\exists x \, P(x)$

- Domain or universe: range of values of a variable (sometimes implicit)
- The words *all, some*, *many*, *none*, and *few* are used in quantifications.

Quantifiers

- **Two types of quantification**

1.Universal quantification, which tells us that a predicate is true for every element under consideration,

2.Existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.

- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Two Popular Quantifiers

- **Universal**: $\forall x\, P(x)$ – "P(x) for all x in the domain"
- **Existential**: $\exists x\, P(x)$ – "P(x) for some x in the domain" or "there exists x such that P(x) is TRUE".
- Either is meaningless if the domain is not known/specified.
- Examples (domain real numbers)
  - $\forall x\, (x^2 >= 0)$
  - $\exists x\, (x > 1)$
  - $(\forall x > 1)\, (x^2 > x)$ – quantifier with restricted domain

The universal quantifier-Summary

Statement	When True?	When False?
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Problems

1.Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

- Solution: Because $P(x)$ is true for all real numbers x,

  the quantification $\forall x\, P(x)$ is true

Problems

2. Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false.

That is, $x = 3$ is a counter example for the statement $\forall x\, Q(x)$.

Thus $\forall x\, Q(x)$ is false.

Problems

3. What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x\, P(x)$ is the same as the conjunction

- $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$,

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false,

- it follows that **$\forall x\, P(x)$ is false**.

Problems

4. What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

- $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. { eg--$(1 /2)^2$ not $\geq 1/2$ }
- if the domain consists of the integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$.

The existential quantifier

- The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

- We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$.
- Here $\exists$ is called the *existential quantifier*.
- "for some," "for at least one," or "there is." ---words

Problems

1. Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x\, P(x)$, is true.

NOTE: The statement $\exists x\, P(x)$ is false if and only if there is no element x in the domain for which $P(x)$ is true.

That is, $\exists x\, P(x)$ is false if and only if $P(x)$ is false for every element of the domain.

2. Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$,

which is $\exists x\, Q(x)$, is false.

3. What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is {1, 2, 3, 4}, the proposition $\exists x\, P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

- Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true.

Logical Equivalences Involving Quantifiers

- **Statements** involving **predicates and quantifiers** are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
- We use the notation $\boldsymbol{S \equiv T}$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Logical Equivalences Involving Quantifiers

Eg--Show that $\forall x(P(x) \land Q(x))$ and $\forall xP(x) \land \forall xQ(x)$ are logically equivalent.

Suppose that $\forall x(P(x) \land Q(x))$ is true. This means that if a is in the domain, then $P(a) \land Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true.

- Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall xP(x)$ and $\forall xQ(x)$ are both true.
- This means that $\forall xP(x) \land \forall xQ(x)$ is true.
- $\forall x(P(x) \land Q(x)) \equiv \forall xP(x) \land \forall xQ(x)$.

THANK YOU

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