UNIT - 3
MATHEMATICAL LOGIC

Propositions and logical operators – Truth table – Propositions generated by a set, Equivalence and implication – Basic laws – Some more connectives – Functionally complete set of connectives – Normal forms – ~~Proofs in propositional calculus – Predicate calculus.~~

3.1 INTRODUCTION

Logic is the basis of all mathematical reasoning and it provides rules and methods to check whether a given argument is valid or not. Logical reasoning is used in developing algorithms needed for computer programs. Logic is applied to decide whether one statement follows from, or is a logical consequence of one or more statements.

It has two parts – propositional calculus which deals with analysis of propositions and predicate calculus which deals with the analysis of predicates which are the propositions involving variables.

3.2 OBJECTIVES

- To enable students to identify the validity of statements or arguments.
- To make them understand the normal forms of statements which are used to compare statements.
- To help them to use quantifiers and identify the truth sets of propositions (i.e.) the set of values where the propositions are true.
- To highlight the usage of propositional calculus in predicate calculus which deals with propositions involving variables.

3.3 PROPOSITIONS

A proposition is a statement that is either true or false but not both. These values - true or false, are called truth values.

(i) True is denoted by T or 1.

(ii) False is denoted by F or 0.

The propositions are denoted using alphabets.(capitals or small)

Examples:

1. p: Chennai is the capital of Tamil Nadu (True)
2. q: 3 + 2 = 0 (False)

The following are not propositions.

1. Oh! How beautiful it is! (Exclamatory sentences are not propositions.)
2. Ring the bell. (Commands are not propositions.)
3. Where are you going? (Interrogative sentences are not propositions.)
4. This statement is false. (Self contradictory sentences are not propositions.)

They are not propositions because we cannot assign particular truth value – True or false as their truth values.

Examples

1) Today is Friday. (If this sentence is said on a Friday its truth value is true otherwise it is false. But is doesn't take both values at the same time. Hence it is a proposition.)

2) 101 + 1 = 110. (In decimal number system 101 + 1 = 102. Hence this will take truth value false. But in binary number system 101 + 1 = 110 and hence its value is true in binary number system. Anyway it takes only 1 value T or F & hence it is a proposition.)

3) Mrs. Christy is a teacher. (The truth value of this statement depends upon the profession of Mrs. Christy. If she is a teacher then this sentence will have truth value T or else it will take the truth value F. Since it cannot have both values at the same time it is a proposition.)

Propositions can either be

1. Primitive propositions or
2. Compound propositions.

3.3.1 PRIMITIVE PROPOSITIONS:

A proposition is said to be primitive if it cannot be broken down into simpler propositions. They are also called as atomic propositions.

Examples:

1. Sugar is sweet.
2. 6 + 4 = 10.

3.3.2 COMPOUND PROPOSITIONS:

A proposition which is composed of atomic propositions connected by 'and', 'or', etc., are called compound propositions. They are composite.

Examples:

Sam studies well and also plays keyboard.

Here 'Sam studies well' and 'Sam plays keyboard' are atomic statements connected by the connective 'and'.

3.3.3 TRUTH TABLE:

Let $P_1, P_2 \ldots P_n$ be atomic variables connected by the connecting word 'and', 'or' etc., Then the compound proposition consisting of these atomic statements $P_1, P_2 \ldots P_n$ will finally have a truth value which is either T or F. This final truth value depends upon the connectives which connect the atomic variables. If the given statement has 'n' atomic statements then the truth table will have 2^n rows. The truth table will be as follows

P_1	P_2	$P_3 \ldots\ldots\ldots P_n$	Compound proposition in terms of $P_1, P_2 \ldots P_n$
…………	……………	………………………………	……T or F
…………	……………	………………………………	……………
.	.	.	.
.	.	.	. 2^n rows
.	.	.	.
………..	…………..	………………………………	……………

HAVE YOU UNDERSTOOD THE CONCEPTS?

ANSWER THE FOLLOWING:

1. Are the following propositions or not:

(i) Shyam is rich — Ans: Yes

(ii) May God bless you. — Ans: No

(iii) Oh! What a wonderful party. — Ans: No

(iv) I am a liar. — Ans: No

(v) Close the door. — Ans: No

2. How many rows will be there in a truth table drawn to check the equivalence of a statement containing 4 variables?

(a) 2 (b) 4 (c) 8 (d) 16 Ans: $16 = 2^4$

3.4 LOGICAL OPERATORS:

The three basic logical operators are conjunction, disjunction and negation and they correspond to the English words 'and', 'or' and 'not'.

Conditional and biconditional operators are also logical operators. There are some more connectives NAND , NOR and exclusive-or. We define all these logical operators.

3.4.1 NEGATION:

The negation of a statement is generally formed by introducing the word 'not' at a proper place in the statement.

If 'P' denotes a statement then negation of P is denoted by $\neg$P or $\sqcup$ P and is read as 'not P'.

Example: If P : Shyam is clever then its negation is

$\neg$P : Shyam is not clever.

Truth Table for negation:

P	$\neg$P
T	F
F	T

3.4.2 CONJUNCTION:

Let p and q be two propositions. The proposition 'p and q' is called the conjunction of p and q and is denoted by $p \wedge q$.

$p \wedge q$ is true when both p and q are true and is false otherwise.

Examples:

1. If p : I will have idli for breakfast.

q : I will have curd rice for lunch.

Then the conjunction is $p \wedge q$: I will have idli for breakfast and I will have curd rice for lunch.

Truth Table for Conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3.4.3 DISJUNCTION:

Let p and q be two propositions. The proposition 'p or q' is called the disjunction of p and q and is denoted by '$p \vee q$'. $p \vee q$ is false when both p and q are false and is otherwise true.

Example:

If p : I will become a doctor.

q : I will become a software engineer.

Then the conjunction is $p \vee q$: I will become a doctor <u>or</u> a software engineer.

Truth Table for Disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3.4.4 CONDITIONAL:

If p and q are two propositions then the compound proposition 'if p then q' denoted by '$p \rightarrow q$' is called conditional proposition.

$p \rightarrow q$ is false only when p is true and q is false and is true otherwise.

here p - is called hypothesis or antecedent or premise.

q - is called conclusion or consequent.

Example:

Let p : Tomorrow is Sunday.

q : Today is Saturday.

Then the conditional $p \rightarrow q$ is

If tomorrow is Sunday then today is Saturday.

Truth Table for $p \rightarrow q$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

3.4.5 BICONDITIONAl:

If p and q are two propositions then the proposition 'p if and only if q' denoted by $p \leftrightarrow q$ is called biconditional proposition.

$p \leftrightarrow q$ is true when both p and q are true or both p and q are false.

Example :

Let p : You will score well.

q **:** You work hard.

Then the biconditional $p \leftrightarrow q$ is "You will score well if and only if you work hard".

Truth Table for $p \leftrightarrow q$:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

3.4.6 NAND:

'NAND' is the combination of 'NOT' and 'AND'. (i.e.) it is the combination of negation and conjunction. It is denoted by $\uparrow$. If p and q are two propositions then p NAND q is denoted by '$p \uparrow q$'.

$p \uparrow q$ is false only when both p and q are true.

Truth Table for NAND:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

3.4.7 NOR:

'NOR' is the combination of 'NOT' and 'OR'. (i.e.) it is the combination of negation and disjunction. It is denoted by $\downarrow$. If p and q are two variables then 'p NOR q' is denoted by $p \downarrow q$. It is also called as joint denial. It is true only when p and q are false.

Truth Table for NOR:

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

3.4.8 EXCLUSIVE – OR:

If p and q are two propositions then 'p exclusive – or q' is denoted by $p \oplus q$.

$p \oplus q$ is true when either p is true or q is true but not both. It is called exclusive – or because it excludes the possibility that both p and q are true.

Example:

Let p : I will earn an A in this course.

q : I will drop this course.

$p \oplus q$ is : I will either earn an A in this course or I will drop it. (but not both).

Truth table for exclusive – or :

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

BOOLEAN OPERATIONS SUMMARY:

Summarizing all the connectives we have studied, we have the following table:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \uparrow q$	$p \downarrow q$	$p \oplus q$
T	T	F	T	T	T	T	F	F	F
T	F	F	F	T	F	F	T	F	T
F	T	T	F	T	T	F	T	F	T
F	F	T	F	F	T	T	T	T	F

3.4.9 PRECEDENCE OF OPERATORS:

The connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ have the following order of precedence.

I	$\neg$
II	$\wedge$
III	$\vee$
IV	$\rightarrow, \leftrightarrow$

If the proposition does not have paranthesis then the order is selected according to the precedence of operators.

For example $\neg P \vee Q$ means $(\neg P) \vee Q$ because negation comes before $\vee$.

If there are paranthesis then the order of precedence is altered by paranthesis.

For example in $(P \vee Q) \wedge R$, $P \vee Q$ is evaluated first and then $(P \vee Q) \wedge R$ is evaluated.

When more than one set of paranthesis are used we have a nesting of paranthesis. Any proposition should contain even number of paranthesis.

i.e. Number of open brackets = number of close brackets.

Exercises:

1. If p is true and q is false then find the truth value of the following:

a) $\neg(p \to \neg q)$ b) $(p \wedge q) \to (p \vee q)$

c) $\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$ d) $(p \to q) \vee \neg(p \leftrightarrow \neg q)$

Solution: Given p - true; q - false.

(a) $\neg(p \to \neg q)$

p	q	$\neg q$	$p \to \neg q$	$\neg(p \to \neg q)$
T	F	T	T	F

Ans: $\neg(p \to \neg q)$ is false.

(b) $(p \wedge q) \to (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	F	F	T	T

Ans: $(p \wedge q) \to (p \vee q)$ is true.

(c) $\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$q \leftrightarrow p$	$\neg(q \leftrightarrow p)$	$\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$
T	F	F	T	T	F	T

Ans: $\neg(p \wedge q) \vee \neg(q \leftrightarrow p)$ is true.

(d) $(p \to q) \vee \neg(p \leftrightarrow \neg q)$

p	q	$p \to q$	$\neg q$	$p \leftrightarrow \neg q$	$\neg(p \leftrightarrow \neg q)$	$(p \to q) \vee \neg(p \leftrightarrow \neg q)$
T	F	F	T	T	F	F

Ans: $(p \to q) \vee \neg(p \leftrightarrow \neg q)$ is false.

2. Determine the truth value of

"If $30 \div 10 = 3$ then $7 \times 5 = 35$"

Solution: Let p : $30 \div 10 = 3$ which is true.

q : $7 \times 5 = 35$ which is true.

And it is a conditional statement $p \to q$

Therefore we have

p	q	$p \to q$
T	T	T

Ans: If $30 \div 10 = 3$ then $7 \times 5 = 35$ is true.

3: Consider the following

p : x is even.

q : x is divisible by 2.

state the following in words:

a) $\neg p$ b) $\neg p \wedge \neg q$ c) $p \vee q$ d) $p \to q$

e) $\neg p \to \neg q$ f) $p \leftrightarrow q$

Solution:

a) $\neg p$: x is not even.

b) $\neg p \wedge \neg q$: x is neither even nor divisible by 2.

c) $p \vee q$: x is even or divisible by 2.

d) $p \to q$: If x is even then it is divisible by 2.

e) $\neg p \to \neg q$: If x is not even then it is not divisible by 2.

f) $p \leftrightarrow q$: x is even if and only if it is divisible by 2.

4. If p : Anil is rich and q : Kanchan is poor. Write this in symbolic form

(1) Either Anil or Kanchan is rich.

(2) Anil is poor and Kanchan is rich.

(3) It is not true that Anil and Kanchan are both rich.

Solution: (1) Either Anil or Kanchan is rich is $p \vee \neg q$.

(2) Anil is poor and Kanchan is rich is $\neg p \wedge \neg q$.

(3) It is not true that Anil and Kanchan are both rich is $\neg(p \wedge \neg q)$.

5. Remove unnecessary paranthesis so that the meaning does not change.

(i) $\left(\left(A \wedge (B \vee C)\right) \vee \left(A \wedge (C \vee D)\right)\right)$

(ii) $\left(\left((\neg A) \wedge (\neg B)\right) \rightarrow \neg(A \vee B)\right)$

Solution: (i) The first and last brackets can be removed and hence we will have

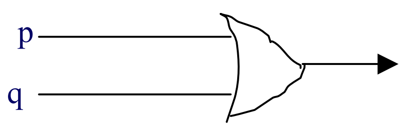
$\left(A \wedge (B \vee C)\right) \vee \left(A \wedge (C \vee D)\right)$

(ii) Since negation has the highest order of precedence the paranthesis are not needed there and hence $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$ has the same meaning as the given proposition.

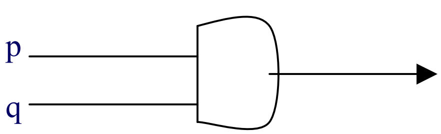
6. Draw the block diagram for $(a \vee b) \wedge (\overline{a} \vee \overline{b})$

Solution:

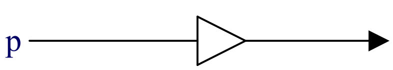
In block diagram



represents $p \vee q$

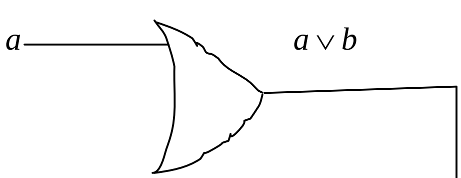


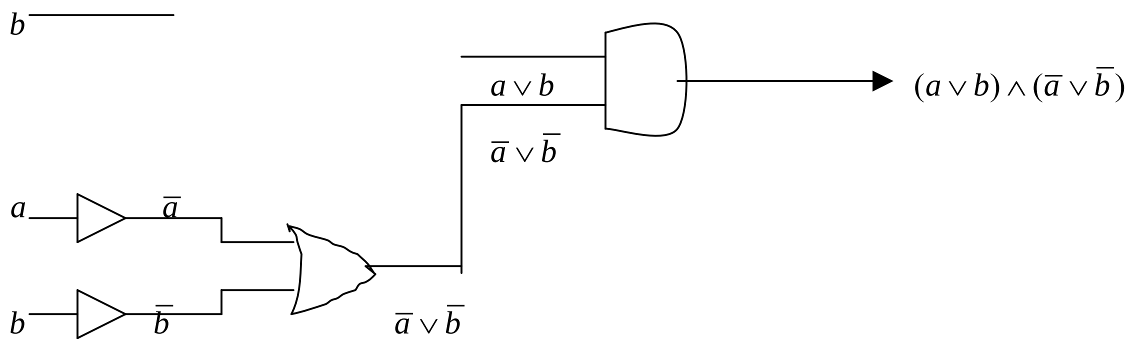
represents $p \wedge q$



represents $\neg p$ or $\overline{p}$

Therefore for the given proposition we have the following block diagram.





HAVE YOU UNDERSTOOD THE CONCEPTS ?
ANSWER THE FOLLOWING:

1. What is the truth value of 'If 6 + 6 = 12 then 4 – 2 = 6'

(a) T (b) F Ans : (b)

2. Negate 13<20.

(a) 13 >20 (b) $13 \geq 20$ (c) $13 \not\leq 20$ Ans: (b)

3. Negate ' n is odd or n is even'

(a) n is not odd and n is not even.

(b) n is not odd or n is not even.

(c) n is odd or n is not even. Ans : (a)

3.5 MORE ON CONNECTIVES:

3.5.1 TAUTOLOGY:

A proposition which is always true is called a tautology. (i.e.) it contains only 'TRUE' in the last column of its truth table irrespective of the truth values of its atomic propositions (or variables). A tautology is denoted by **T**.

For example: $p \vee \neg p$ is a tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

The last column is always true.

3.5.2 CONTRADICTION:

A proposition which is always false is called a contradiction. (i.e.) it contains only 'FALSE' in the last column of its truth table. A contradiction is denoted by **F**.

For example: $p \wedge \neg p$ is a contradiction.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

The last column is always false.

Exercises:

1: Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution:

P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg P$	$\neg P \wedge \neg Q$	$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	F	F	T
T	F	T	T	F	F	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T

Since the last column has only T irrespective of the value of P and Q the given proposition is a tautology.

2. Show that $(P \wedge Q) \wedge \neg(P \vee Q)$ is a contradiction.

Solution:

P	Q	$P \wedge Q$	$P \vee Q$	$\neg(P \vee Q)$	$(P \wedge Q) \wedge \neg(P \vee Q)$
T	T	T	T	F	F

T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the last column values are always false irrespective of the values of P and Q the given proposition is a contradiction.

3.5.3 LOGICAL EQUIVALENCE:

Two propositions are said to be logically equivalent if they have identical truth values.

If $P(p,q,...)$ and $Q(p,q,...)$ are two propositions then they are logically equivalent if P and Q columns in the truth table are identical. It is denoted by

$$P(p,q,...) \equiv Q(p,q,...)$$

Example: $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

P	Q	$(P \wedge Q)$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The columns of $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ have the same truth values F,T,T,T. $\therefore$ the two columns are identical. Hence $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

Hence $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$.

Exercise:

There are two restaurants next to each other. One has a sign that says 'Good food is not cheap' and the other has a sign that says, "Cheap food is not good". Are the signs saying the same thing?

Solution:

Let c : denote the proposition that the food is cheap.

g : denote the proposition that the food is good.

'Good food is not cheap' – can be symbolically be written as $g \to \overline{c}$. (The meaning of this statement is "if the food is good then it is not cheap" which is a conditional proposition)

'Cheap food is not good' – can be symbolically be written as $c \to \overline{g}$. (The meaning of this statement is "if the food is cheap then it is not good" which is a conditional proposition).

g	c	$\overline{g}$	$\overline{c}$	$g \to \overline{c}$	$c \to \overline{g}$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The columns of $g \to \overline{c}$ and $c \to \overline{g}$ are identical (i.e.) they have the same truth values F,T,T,T. Hence they are equivalent.

$\therefore g \to \overline{c} \quad \equiv c \to \overline{g}$

$\therefore$ the signs are saying the same thing.

3.5.4 DUALS AND DUALITY PRINCIPLE:

DUALS: Two statements A and B are duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

The connectives $\wedge$ and $\vee$ are called duals of each other.

Example: If X is $(p \vee q) \wedge r$ then the dual of X is $(p \wedge q) \vee r$.

DUALITY PRINCIPLE: Let s and t be two statements that contain no logical connectives other than $\wedge$ and $\vee$.

If $s \equiv t$ then

$s^d \equiv t^d$

where s^d - is the dual of s and

t^d - is the dual of t.

3.5.5 LAWS OF ALGEBRA OF PROPOSITIONS:

1. **Idempotent Laws:**

   $P \vee P \equiv P$ and $P \wedge P \equiv P$

2. **Associative Laws:**

   $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

   $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

3. **Commutative Laws:**

   $P \vee Q \equiv Q \vee P$

   $P \wedge Q \equiv Q \wedge P$

4. **Distributive Laws:**

   $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ and

   $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

5. **Identity Laws:**

   $P \vee T \equiv \mathbf{T}$; $P \wedge T \equiv P$

   $P \vee F \equiv P$; $P \wedge F \equiv \mathbf{F}$

where **T** represents tautology and

F represents contradiction.

6. **Complement Laws:**

   $P \vee \neg P \equiv \mathbf{T}$; $P \wedge \neg P \equiv$ **F**

   $\neg T \equiv F$; $\neg F \equiv T$

7. **Demorgans's Laws:**

   $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ and

   $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

8. **Involution Laws (or) Law of Double Negation:**

   $\neg\neg P \equiv P$

Note:

All these laws of algebra of propositions can be proved. The first law can be established by using truth table and the second law will follow by the duality principle as the second is the dual of the first law.

As an example let us prove distributive laws:

Example:

State and prove distributive laws:

Solution:

The distributive laws are

(i) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(ii) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

To prove (i) **:** Let $S_1 \equiv Q \wedge R;\ S_2 \equiv P \vee (Q \wedge R);$

$$S_3 \equiv P \vee Q;\ S_4 \equiv P \vee R;\ S_5 \equiv (P \vee Q) \wedge (P \vee R).$$

(Instead of writing big propositions in the truth table they can be replaced by some other small variables defined like this so that the table will have a neat appearance)

P	Q	R	S_1	S_2	S_3	S_4	S_5
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From the truth table $S_2 \equiv S_5$.

Hence $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R) - - - - - - - - \boxed{1}$

To prove (ii) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ contains the connectives $\vee$ and $\wedge$ only and it is of the form $s \equiv t$. Therefore by duality principle $s^d \equiv t^d$ where s^d and t^d are the duals of the left hand side and right hand side respectively.

Therefore taking the dulas of LHS and RHS in equation 1, we have

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R) - - - - - - - \boxed{2}$

Hence the second law follows by the duality principle. Hence the proof.

Note: Using the laws of algebra of propositions further laws can be obtained.

Exercise

Prove absorption laws: (i) $P \vee (P \wedge Q) \equiv P$

(ii) $P \wedge (P \vee Q) \equiv P$

Solution: (i) $P \vee (P \wedge Q) \equiv (P \wedge T) \vee (P \wedge Q)$ (using identity law)

$\equiv P \wedge (T \vee Q)$ (using distributive law)

$\equiv P \wedge T$ (using identity law)

$\equiv P$ (using identity law)

(ii) $P \wedge (P \vee Q) \equiv (P \vee F) \wedge (P \vee Q)$ (using identity law)

$\equiv P \vee (F \wedge Q)$ (using distribution law)

$\equiv P \vee F$ (using identity law)

$\equiv P$ (using identity law)

Note: To prove equivalence of propositions either truth tables or the laws of algebra of propositions can be used.

Exercise:

Show that $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P) \quad \equiv (P \vee Q) \to (P \wedge Q)$

Solution:

P	Q	$P \leftrightarrow Q$	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$	$(P \vee Q)$	$(P \wedge Q)$	$(P \vee Q) \to (P \wedge Q)$
T	T	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	F
F	T	F	T	F	F	T	F	F
F	F	T	T	T	T	F	F	T

In the above table the columns $P \leftrightarrow Q$, $(P \to Q) \wedge (Q \to P)$ and $(P \vee Q) \to (P \wedge Q)$ have the identical truth values.

Therefore they are equivalent.

Note:

In some cases, both truth tables and laws of algebra of propositions can be used to prove the equivalence as in this problem.

Exercise:

Show that $P \to (Q \to R) \equiv (P \wedge Q) \to R$

Solution:

Step 1: We first show that $Q \to R \equiv \neg Q \vee R$

Q	R	$Q \to R$	$\neg Q$	$\neg Q \vee R$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore$ from the table $Q \to R \equiv \neg Q \vee R --------\boxed{1}$

Step 2:

$P \to (Q \to R) \equiv P \to (\neg Q \vee R)$ from step $\boxed{1}$

$\equiv P \vee (\neg Q \vee R)$ using $\boxed{1}$.

$\equiv (\neg P \vee \neg Q) \vee R$ (by associative law)

$\equiv \neg(P \wedge Q) \vee R$ (by demorgan's law)

$\equiv (P \wedge Q) \to R$ by $\boxed{1}$.

$\therefore P \to (Q \to R) \equiv (P \wedge Q) \to R$

Hence proved.

Note: Hence we have proved the following conditional equivalence $P \to Q \equiv \neg P \vee Q$

and the biconditional equivalence $(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$

$\equiv (P \vee Q) \to (P \wedge Q)$

3.5.6 FUNCTIONALLY COMPLETE SET OF CONNECTIVES:

In any formula, we can replace the biconditionals first then the conditionals and finally all the conjunctions or all the disjunctions to obtain an equivalent formula.

This formula will either contain negation and disjunction or negation and conjunction. Thus the set of connectives $\{\neg,\wedge\}$ and $\{\neg,\vee\}$ are functionally complete set.

We have the equivalences which express $\neg,\wedge,\vee$ in terms of $\uparrow$ alone.

$$P\uparrow P\equiv\neg(P\wedge P)\equiv\neg P\vee\neg P\equiv\neg P$$
$$(P\uparrow Q)\uparrow(P\uparrow Q)\equiv\neg(P\uparrow Q)\equiv P\wedge Q$$
$$(P\uparrow P)\uparrow(Q\uparrow Q)\equiv\neg P\uparrow\neg Q\equiv\neg(\neg P\wedge\neg Q)\equiv P\vee Q$$

Similarly $\neg,\wedge,\vee$ can be expressed in terms of $\downarrow$ alone.

$$P\downarrow P\equiv\neg(P\vee P)\equiv\neg P\wedge\neg P\equiv\neg P$$
$$(P\downarrow Q)\downarrow(P\downarrow Q)\equiv\neg(P\downarrow Q)\equiv P\vee Q$$
$$(P\downarrow P)\downarrow(Q\downarrow Q)\equiv\neg P\downarrow\neg Q\equiv\neg(\neg P\vee\neg Q)\equiv P\wedge Q$$

NAND and NOR operators are functionally complete. The sets $\{\uparrow\}$ and $\{\downarrow\}$ are called minimal functionally complete set or a minimal set.

Exercise:

Write an equivalent for $p\wedge(q\leftrightarrow r)\vee(r\leftrightarrow p)$ which does not involve conditional and biconditional.

Solution:

$$p\wedge(q\leftrightarrow r)\vee(r\leftrightarrow p)$$
$$\equiv p\wedge((q\to r)\wedge(r\to q))\vee((r\to p)\wedge(p\to r))$$
$$\equiv p\wedge((\neg q\vee r)\wedge(\neg r\vee q))\vee((\neg r\vee p)\wedge(\neg p\vee r))$$

and it does not involve conditional and biconditionals.

3.5.7 CONVERSE, INVERSE, CONTRAPOSITIVE:

The converse, inverse and contrapositive can be associated with the conditional propositions.

Consider the conditional proposition $p\to q$.

Its converse is $q\to p$.

Its inverse is $\neg p\to\neg q$.

Its contrapositive is $\neg q\to\neg p$.

Example:

If a triangle ABC is a right angled triangle then $|AB|^2 + |BC|^2 = |AC|^2$. This is a conditional proposition.

Its converse is : If $|AB|^2 + |BC|^2 = |AC|^2$ then the triangle is a right angled triangle.

Its inverse is : If ABC is not a right angled triangle then $|AB|^2 + |BC|^2 \neq |AC|^2$.

Its contrapositive is : If $|AB|^2 + |BC|^2 \neq |AC|^2$ then the triangle ABC is not a right angled triangle.

TRUTH TABLE FOR CONVERSE, INVERSE AND CONTRAPOSITIVE:

p	q	$p \to q$ conditional	$q \to p$ converse	$\neg p \to \neg q$ inverse	$\neg q \to \neg p$ contrapositive
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Note: Since the columns of conditional and contrapositive are identical, conditional and contrapositive are logically equivalent.

HAVE YOU UNDERSTOOD THE CONCEPTS ?
ANSWER THE FOLLOWING:

1. $p \vee q$ is equivalent to

(a) $\neg p \vee \neg q$ (b) $\neg(\neg p \wedge \neg q)$ (c) $\neg p \wedge \neg q$ (d) $\neg p \wedge q$ Ans: (b)

2. $P \wedge P$ is equivalent to

(a) 1 (b) P (c) $-P$ (d) None of these Ans: (b)

3. The inverse of "If I run I will catch the train" is

(a) If I run I will not catch the train.

(b) If I don't run I will catch the train.

(c) If I don't run I will not catch the train. Ans: (c)

3.6 NORMAL FORMS:

A well formed formula is denoted by $A(P_1, P_2, ...P_n)$ where $P_1, P_2, ...P_n$ are atomic propositions. Hence a well formed formula is a compound proposition containing atomic propositions connected by connectives.

Any well formed formula can be reduced to any one of the following normal forms.

(i) Disjunctive Normal Form (DNF)

(ii) Conjunctive Normal Form (CNF)

(iii) Principal Disjunctive Normal Form (PDNF)

(iv) Principal Conjunctive Normal Form (PCNF)

We use product in the place of '$\wedge$' and the sum in the place of '$\vee$'.

3.6.1. DISJUNCTIVE NORMAL FORM (DNF)

Elementary Product:

A product of variables and their negations is called elementary product.

If p and q are two variables then $p \wedge q,\ p \wedge q \wedge p,\ p \wedge \neg q \wedge \neg p,\ q \wedge \neg p$ are some elementary products.

DNF: A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form.

Exercise:

Obtain DNF of $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$

Solution: $(p \wedge \neg(q \vee r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$

$\equiv (p \wedge (\neg q \wedge \neg r)) \vee (((p \wedge q) \vee \neg r) \wedge p)$ Demorgan's law.

$\equiv (p \wedge \neg q \wedge \neg r) \vee ((p \wedge q \wedge p) \vee (\neg r \wedge p))$ distributive law.

$\equiv (p \wedge \neg q \wedge \neg r) \vee (p \wedge q) \vee (\neg r \wedge p)$ is the required DNF.

3.6.2. CONJUNCTIVE NORMAL FORM: (CNF)

Elementary Sum: Sum of variables and their negations is called elementary sum.

$p \vee q,\ \neg p \vee q,\ p \vee \neg q \vee \neg p,\ p \vee \neg q \vee q$ are some examples of elementary sums in two variables p and q.

CNF: A formula which is equivalent to a given formula and which consists of a product of elementary sum is called as conjunctive normal form.

Exercises:

1. Obtain CNF of $p \to ((p \to q) \wedge \neg(\neg q \vee \neg p))$

Solution:

$p \to ((p \to q) \wedge \neg(\neg q \vee \neg p))$

$\equiv \neg p \vee ((p \to q) \wedge \neg(\neg q \vee \neg p))$

$\equiv \neg p \vee ((\neg p \vee q) \wedge (q \wedge p))$

$\equiv (\neg p \vee (\neg p \vee q)) \wedge (\neg p \vee (q \wedge p))$

$\equiv (\neg p \vee \neg p \vee q) \wedge ((\neg p \vee q) \wedge (\neg p \vee p))$

$\equiv (\neg p \vee q) \wedge (\neg p \vee q) \wedge (\mathbf{T})$

$\equiv (\neg p \vee q) \wedge \mathbf{T}$

$\equiv \neg p \vee q$ is the required CNF.

2. Obtain CNF of $q \vee (p \wedge r) \wedge \neg((p \vee r) \wedge q)$

Solution: $q \vee (p \wedge r) \wedge \neg((p \vee r) \wedge q)$

$\equiv (q \vee (p \wedge r)) \wedge (\neg(p \vee r) \vee \neg q)$

$\equiv (q \vee (p \wedge r)) \wedge ((\neg p \wedge \neg r) \vee \neg q)$

$\equiv (q \vee p) \wedge (q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg r \vee \neg q)$ is the required CNF.

Note: The disjunctive normal forms and the conjunctive normal forms of a given formula are not unique.

3.6.3 PRINCIPAL DISJUNCTIVE NORMAL FORM (PDNF):

Minterms:

Let p and q be two variables. The product of the variables and their negations such that none of the terms contain a variable and its negation are called minterms. For p and q the minterms are $p \wedge q,\ p \wedge \neg q,\ \neg p \wedge q,\ \neg p \wedge \neg q$.

Note: $p \wedge \neg q \wedge q$ is a elementary product but not a minterm because a minterm should not contain a variable and its negation.

If there are n variables then there are 2^n minterms.

PDNF:

A formula which is equivalent to a given formula consisting of sum of minterms only is called principal disjunctive normal form.

There are two ways of finding PDNF.

(i) using truth table

(ii) without using truth table.

PDNF is also called as sum of minterms form or sum of products canonical form.

(i) **Using truth table:**

1. Construct truth table for the given formula.
2. Identify the rows in which the formula has true – as truth value.
3. Construct minterms from each such rows by taking

   (i) the variable with true – as variable itself and

   (ii) the variable with false – as negated variable.
4. Sum of these minterms will be the required PDNF.

Exercise:

Find PDNF of $P \rightarrow Q$

Solution:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Here the 1st, 3rd and 4th rows are true for $P \rightarrow Q$. Select such rows.

In the 1st row since P and Q have true as truth value the corresponding minterm is $P \wedge Q$. In the third row since P has truth value false and Q has truth value true the corresponding minterms is $\neg P \wedge Q$.

In the fourth row since the variables P and Q have truth value false then corresponding minterm is $\neg P \wedge \neg Q$.

Hence the required PDNF is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

(ii) **Without the truth table**:

Exercise:

1. Find the minterm normal form of $\neg((P \vee Q) \wedge R) \wedge (P \vee R)$

Solution:

$\neg((P \vee Q) \wedge R) \wedge (P \vee R)$

$\equiv (\neg(P \vee Q) \vee \neg R) \wedge (P \vee R)$ (Using Demorgan's law)

$\equiv ((\neg P \wedge \neg Q) \vee \neg R) \wedge (P \vee R)$ (Using Demorgan's law)

$\equiv (\neg P \wedge \neg Q \wedge P) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R)$ (Using distributive law)

$\equiv \mathbf{F} \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee (\mathbf{F})$ $\quad [\because P \wedge \neg P \equiv \mathbf{F} \quad R \wedge \neg R \equiv \mathbf{F}]$

$\equiv (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P)$

$\equiv (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P \wedge \mathbf{T})$

$\equiv (\neg P \wedge \neg Q \wedge R) \vee ((\neg R \wedge P) \wedge (Q \vee \neg Q))$

$\equiv (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$ is the required PDNF.

3.6.4 PRINCIPAL CONJUNCTIVE NORMAL FORM: (PCNF)

Maxterms: Let P and Q be two variables. Maxterms are sums of P and Q and their negations such that none of the terms contain a variable and its negation. For P and Q the maxterms are $P \vee Q, P \vee \neg Q, \neg P \vee Q, \neg P \vee \neg Q$. If there are n variables then there are 2^n possible maxterms.

Note: $P \vee \neg P$ is not a maxterm because it contains both variable and its negation.

PCNF: A formula which is equivalent to the given formula and consisting of product of maxterms only is known as principal conjunctive normal form. It is also called as product of sums canonical form.

PCNF can be found in two ways:

(i) using truth tables.

(ii) without using truth tables.

(i) Using truth tables:

1. Construct truth table for the given formula.
2. Identify the rows in which the formula has false – as truth value.
3. Construct maxterm from each such row by taking
   (i) the variable with true -as negated variable and
   (ii) the variable with false- as the variable itself.
4. Product of these maxterms will be the required PCNF.

1. Find PCNF of $\neg(P \to Q)$ using truth table.

P	Q	$P \to Q$	$\neg(P \to Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Here 1st, 3rd and 4th row have false as the truth value of the given statement. Select such rows.

In the first row P and Q are true so the corresponding maxterm is $\neg P \vee \neg Q$.

In the third row P is false and Q is true and so the corresponding maxterm is $P \vee \neg Q$.

In the fourth row P is false and Q is false so the corresponding maxterm is $P \vee Q$.

And so the required PCNF is $(\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (P \vee Q)$

(ii) **Without using truth tables:**

1. Find PCNF of $\neg(P \to Q)$ without using truth table.

Solution:

$\neg(P \to Q) \equiv \neg(\neg P \vee Q)$

$\equiv (P \wedge \neg Q)$

$\equiv (P \wedge \neg Q) \vee \mathbf{F}$ (Identity law)

$\equiv (P \wedge \neg Q) \vee (P \wedge \neg P)$

$\equiv (P \vee P) \wedge (\neg Q \vee P) \wedge (P \vee \neg P) \wedge (\neg Q \vee \neg P)$

$\equiv (P \vee P) \wedge (P \vee \neg P) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\equiv [P \vee (P \wedge \neg P)] \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\equiv (P \vee \mathbf{F}) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\equiv (P \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\equiv (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\equiv (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg Q \vee \neg P)$ is the required PCNF.

3.6.5 CONVERSION OF PCNF TO PDNF:

Steps for converting PCNF to PDNF

(i) There are 2^n maxterms for n atomic variables.

(ii) Select the maxterms which are not used in PCNF

(iii) Conjunct those maxterms.

(iv) Take negation for previous step and use demorgan's law.

(v) The resultant formula is the required PDNF.

1. Find the PDNF of the PCNF $P \vee Q \vee R$

Solution: There are 3 variables and so there are $2^3 = 8$ maxterms.

$(P \vee Q \vee R), (P \vee Q \vee \neg R), (P \vee \neg Q \vee R), (P \vee \neg Q \vee \neg R), (\neg P \vee Q \vee R), (\neg P \vee Q \vee \neg R),$
$(\neg P \vee \neg Q \vee R), (\neg P \vee \neg Q \vee \neg R)$ are the 8 maxterms for P , Q and R.

Leaving the maxterm in the given PCNF we have 7 maxterms. Conjuncting them we have

$(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge$
$(\neg P \vee \neg Q \vee \neg R)$

Negation of the above is

$\neg\{(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R)$
$\wedge (\neg P \vee \neg Q \vee \neg R)\}$

$\neg(P \vee Q \vee \neg R) \vee \neg(P \vee \neg Q \vee R) \vee \neg(P \vee \neg Q \vee \neg R) \vee \neg(\neg P \vee Q \vee R) \vee \neg(\neg P \vee Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R)$
$\vee \neg(\neg P \vee \neg Q \vee \neg R)$

$(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$
is the required PDNF.

2. Obtain the PCNF and PDNF of $(\neg p \to R) \wedge (q \leftrightarrow p)$.

Solution:

Let S $\equiv (\neg p \to r) \wedge (q \leftrightarrow p)$

$\equiv (\neg(\neg p) \vee r) \wedge (q \to p) \wedge (p \to q)$ conditional and biconditional equivalence

$\equiv (p \vee r) \wedge ((\neg q \vee p) \wedge (\neg p \vee q))$ conditional equivalence & law of double negation

$\equiv ((p \vee r) \vee \mathbf{F}) \wedge ((\neg q \vee p) \vee \mathbf{F}) \wedge ((\neg p \vee q) \vee \mathbf{F})$ identity law

$\equiv ((p \vee r) \vee (q \wedge \neg q)) \wedge ((\neg q \vee p) \vee (r \wedge \neg r)) \wedge ((\neg p \vee q) \vee (r \wedge \neg r))$
complement law

$\equiv (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge$
$(\neg p \vee q \vee \neg r)$ Distributive law

$\equiv (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$
is the required PCNF.

To find PDNF:

The maxterms that are not present in PCNF are
$(p \vee q \vee \neg r), (\neg p \vee \neg q \vee r), (\neg p \vee \neg q \vee \neg r)$. Conjuncting them we have
$(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$
Taking Negation we have
$\neg\{(p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)\}$
$\equiv \neg(p \vee q \vee \neg r) \vee \neg(\neg p \vee \neg q \vee r) \vee \neg(\neg p \vee \neg q \vee \neg r)$
$\equiv (\neg p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$ is the required PDNF.

3.6.6 CONVERSION OF PDNF TO PCNF:

Steps for converting PDNF to PCNF

1. There are 2^n minterms for n atomic variables.
2. Choose the minterms which are not used in the PDNF.
3. Disjunct those minterms.
4. Take negation for previous step terms and apply Demorgan's Laws.
5. The resultant formula is the required PCNF.

1. Find PCNF of the PDNF $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

Solution:

This involves only 2 variables P and Q and has $2^2 = 4$ minterms.

This possible minterms are $P \wedge Q,\ P \wedge \neg Q, \neg P \wedge Q$ and $\neg P \wedge \neg Q$.

This minterms which is not present in the PDNF is $P \wedge \neg Q$. Since there is only 1 term we don't disjunct it with anything.

Taking negation of that we have

$\neg(P \wedge \neg Q) \equiv \neg P \vee Q$ and it is the required PCNF.

HAVE YOU UNDERSTOOD THE CONCEPTS ?

ANSWER THE FOLLOWING:

1. Which of the following is not a minterm.

(a) $p \wedge q \wedge r$ (b) $p \wedge \neg r \wedge r$ (c) $\neg p \wedge q \wedge \neg r$ (d) $\neg p \wedge \neg q \wedge \neg r$ Ans: (b)

2. Which of the following is a maxterm

(a) $p \wedge q$ (b) $p \vee q \vee r$ (c) $p \vee q \vee \neg q$ (d) $p \vee \neg p \vee q$

Ans: (b)

3. If a formula has 2 variables involved in it then it has __________ maxterms.

(a) 2 (b) 8 (c) 16 (d) 4 Ans: (d)

3.7 INFERENCE THEORY

In inference theory we check the validity of the arguments. There are two branches in it.

1. Inference theory for propositional calculus and
2. Inference theory for predicate calculus.

In the inference theory for propositional calculus the arguments are in terms of propositions (ordinary statements). But in inference theory for predicate calculus the arguments are in terms of predicates with quantifiers (which are defined later).

In this section we deal with the inference theory for propositional calculus.

3.7.1 ARGUMENTS:

An argument is an assertion that a given set of propositions yields another proposition called the conclusion. Such an argument is denoted by $P_1, P_2, \ldots P_n \vdash Q$

Here $P_1, P_2, \ldots P_n$ - are called premises.

Q – is called conclusion.

VALID ARGUMENTS:

An argument $P_1, P_2, ...P_n$ is said to be valid if Q is true whenever all the premises $P_1, P_2, ...P_n$ are true.

Eg. $p, p \to q \vdash q$

p	$p \to q$	q
T	T	T
T	F	F
F	T	T
F	T	F

Here the premises are p and $p \to q$. Both the premises are true only in the first row. So we check for validity only in the first row where all the premises are true. In that row the conclusion q is true.Therefore the conclusion is true when all the premises are true and hence it is a valid argument.

FALLACY:

An argument which is not valid is called a fallacy.

Eg: $p \to q, \neg p \vdash \neg q$

p	q	$p \to q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Here in the fourth row the conclusion is true when the premises are true. But in the third row when the premises $p \to q$ and $\neg p$ are true the conclusion is false. Hence it is not valid and is called as a fallacy.

The following three statements are equivalent.

(i) $P_1, P_2, ...P_n \vdash$ Q is a valid argument.

(ii) Q is true whenever $P_1 \wedge P_2 \wedge ... \wedge P_n$ is true.

(iii) $P_1 \wedge P_2 \wedge ... \wedge P_n \to$ Q is a tautology.

3.7.2 METHODS TO CHECK THE VALIDITY OF ARGUMENTS:

As we have done in the examples of valid argument and fallacy, the validity of an argument can be found out using truth tables. In such a case we first identify the premises and the conclusion and find the truth values of premises and conclusion in a truth table. We identify the rows where all the premises are true and check whether the conclusion is true there. If the conclusion is also true in all such rows then the argument is valid otherwise it is not valid.

As the truth table consists of 2^n rows for an argument with n variables, the construction of the truth table with a big value of n (i.e) with many variables is difficult and time consuming. Hence we use other methods to check the validity of arguments.

The following methods are used to check the validity of the arguments.

(i) Direct Proof

(ii) Indirect Proof

(i) **Direct Proof:** A direct proof is the one in which the truth of the premises are shown to directly imply the truth of the conclusion.

(ii) **Indirect Proof:** In indirect proof we show that the assumption of the negated conclusion results in a contradiction.

3.7.3 RULES OF INFERENCES:

In addition to the rules of algebra of proposition we use another set of rules called the rules of inference.

A proposition $P(p,q.....)$ is said to logically imply a proposition $Q(p,q.....)$ written as

$$P(p,q.....) \Rightarrow Q(p,q.....)$$

if $Q(p,q.....)$ is true whenever $P(p,q.....)$ is true. (i.e.) $P \rightarrow Q$ is a tautology.

Example:
Prove that $(p \wedge q) \Rightarrow (p \rightarrow q)$

Solution:
We have to prove that $(p \wedge q)$ logically implies $(p \rightarrow q)$.

(i.e.) we have to prove that $(p \wedge q) \rightarrow (p \rightarrow q) \equiv \mathbf{T}$

$(p \wedge q) \rightarrow (p \rightarrow q)$

$\equiv \neg(p \wedge q) \vee (\neg p \vee q)$ Conditional equivalence

$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q)$ Demorgans law

$\equiv \neg p \vee (\neg q \vee q)$ idempotent law

$\equiv \neg p \vee \mathbf{T}$ complement law

$\equiv \mathbf{T}$ identity law.

Hence proved.

SOME VALID INFERENCES:

1. $P \Rightarrow P \vee Q$ (disjunctive addition)

2. $P \wedge Q \Rightarrow P$ and $P \wedge Q \Rightarrow Q$ (simplification)

3. $P \wedge (P \rightarrow Q) \Rightarrow Q$ (modus ponens)

4. $\neg Q \wedge (P \to Q) \Rightarrow \neg P$ (modus tollens)

5. $[(P \to Q) \wedge (Q \to R)] \Rightarrow P \to R$ (chain rule)

6. $[\neg P \wedge (P \vee Q)] \Rightarrow Q$ (disjunctive syllogism)

7. $(P \to Q) \wedge (R \to S) \wedge (P \vee R) \Rightarrow Q \vee S$ (constructive dilemma)

8. $(P \to Q) \wedge (R \to S) \wedge (\neg Q \vee \neg S) \Rightarrow (\neg P \vee \neg R)$ (destructive dilemma)

9. $(P \to Q) \equiv \neg Q \to \neg P \equiv \neg P \vee Q$ (conditional equivalence)

10. $(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P) \equiv (P \vee Q) \vee (\neg P \wedge \neg Q)$ (biconditional equivalence)

To prove the arguments we use 3 rules.

(1) rule P

(2) rule T

(3) rule CP

(2) Rule P: A premise can be introduced in any stage of the derivation.

(3) Rule T : A formula S can be used in the derivation if it is tautologically implied by one or more of the preceeding steps in the derivation.

(4) Rule CP: If we can derive S from R and a set of premises then we can derive R→S from the set of premises alone.

Exercises: (with direct proof)

1. Show that $p \to r$, $\neg p \to q, q \to s \Rightarrow \neg r \to s$

Solution:

Step	Proposition	Justification
1.	$p \to r$	Rule P
2.	$\neg r \to \neg p$	Rule T & (1), conditional equivalence
3.	$\neg p \to q$	Rule P
4.	$\neg r \to q$	[(2), (3) and chain rule] Rule T
5.	$q \to s$	Rule P
6.	$\neg r \to s$	Rule T and [(4), (5) and Chain rule]

Note:

When we move along the rows of the table, at any intermediate step we mean that all the preceding steps are true and hence they can be connected by $\wedge$.

In the above problem in step 2 we have $\neg r \rightarrow \neg p$ in step 3 we have $\neg p \rightarrow q$.

Hence when we come to step 4 all the previous steps are taken to be true and hence combining step 2 and step 3 we have

$(\neg r \rightarrow \neg p) \wedge (\neg p \rightarrow q)$

By chain rule $(p \rightarrow q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R$ and hence we have

$(\neg r \rightarrow \neg p) \wedge (\neg p \rightarrow q) \Rightarrow \neg r \rightarrow q$ which is step 4.

2. Show that t is a valid conclusion from the premises

$p \rightarrow q, q \rightarrow r, r \rightarrow s, \neg s \ and \ p \vee t$.

Solution:

Step	Proposition	Justification
1.	$p \rightarrow q$	Rule P
2.	$q \rightarrow r$	Rule P
3.	$p \rightarrow r$	Rule T [(1) and (2) and chain rule]
4.	$r \rightarrow s$	Rule P
5.	$p \rightarrow s$	Rule T and (3) and (4) and chain rule
6.	$\neg s$	Rule P
7.	$\neg p$	Rule T and (5) and (6) and modus tollens
8.	$p \vee t$	Rule P
9.	t	Rule T and (7) , (8) and disjunctive syllogism

3. Verify the validity of the argument

$(\neg p \vee q) \rightarrow r, r \rightarrow s \vee t, \neg s \wedge \neg u, \neg u \rightarrow \neg t \Rightarrow p$

Solution:

Step	Proposition	Justification
1.	$\neg s \wedge \neg u$	Rule P

2.	$\neg u$	Rule T ((1) and simplification)
3.	$\neg u \to \neg t$	Rule P
4.	$\neg t$	Rule T & (2) and (3) and modus ponens
5.	$\neg s$	(1) and rule T and simplification
6.	$\neg s \wedge \neg t$	(4) and (5)
7.	$r \to (s \vee t)$	Rule P
8.	$\neg(s \vee t) \to \neg r$	Rule T and (7) and conditional equivalence
9.	$\neg s \wedge \neg t \to \neg r$	(8) and Demorgan's laws
10.	$\neg r$	(6), (7) and modus tollens and rule T (as $\neg s \wedge \neg t \equiv \neg(s \vee t)$ in (6))
11.	$(\neg p \vee q) \to r$	Rule P
12.	$\neg r \to \neg(\neg p \vee q)$	Rule T and (11) and conditional equivalence
13.	$\neg r \to p \wedge \neg q$	Demorgan's laws and law of double negation in RHS.
14.	$p \wedge \neg q$	Rule T, (10), (13) and modus ponens
15.	p	Rule T, (14) and simplification

4. If $p \to q$, $q \to r$, $\neg(p \wedge r)$ and $p \vee r$ then prove r.

Solution:

Step	**Proposition**	**Justification**
1.	$p \to q$	Rule P
2.	$q \to r$	Rule P
3.	$p \to r$	Rule T and (1), (2) and chain rule.
4.	$\neg p \vee r$	(3) and rule T and conditional equivalence
5.	$p \vee r$	Rule P
6.	$(p \wedge \neg p) \vee r$	(4) and (5) and distributive law $(\neg p \vee r) \wedge (p \vee r) \equiv (\neg p \wedge p) \vee r$
7.	$\mathbf{F} \vee r$	(6) and complement law
8.	r	(7) and identity law
9.	$\neg(p \wedge r)$	Rule P
10.	$\neg p \vee \neg r$	(9) and Demorgan's law

11.	$r \wedge (\neg p \vee \neg r)$	Combining (8) & (10)
12.	$(r \wedge \neg p) \vee (r \wedge \neg r)$	(11) and distributive law
13.	$(r \wedge \neg p) \vee (\mathbf{F})$	(12) & complement law for second term in bracket
14.	$r \wedge \neg p$	(13) & identity law
15.	r	(14) and simplification

5. Find the consistency of the following premises $p \to q, p \to r, q \to \neg r, p$.

Solution:

Step	Proposition	Justification
1.	$p \to q$	Rule P
2.	$q \to \neg r$	Rule P
3.	$p \to \neg r$	Rule T and (1), (2) (chain rule)
4.	p	Rule P
5.	$\neg r$	Rule P
6.	$p \to r$	Rule P
7.	$\neg p$	Rule T from (5) and (6)(modus tollens)
8.	p	Rule P
9.	$p \wedge \neg p$	$\Rightarrow\Leftarrow$ contradiction

Using the premises we arrive at a contradiction. Hence the given premises are inconsistent.

6. Check the validity of the argument:

If the band could not play rock music or the refreshments were not delivered on time then the New Year's party would have been cancelled and Alice would have been angry. If the party were cancelled, then refunds would have to be made. No refunds were made.

Therefore the band could play rock music.

Solution:

Let p : the band could play rock music.

q : the refreshments were made on time

r : The new Year's party was cancelled.

s : Alice was angry.

t : Refunds had to be made.

Premises are : $(\neg p \vee \neg q) \to (r \wedge s)$

$r \to t$

$\neg t$

Conclusion is p

This can also be represented as follows:

$(\neg p \vee \neg q) \to (r \wedge s)$

$r \to t$

$\neg t$

$\therefore p$

(The statements which are above the line are premises and the statement below the line is the conclusion.Since it is the conclusion it is written with a '$\therefore$'.)

Proof:

Step	Proposition	Justification
1.	$r \to t$	Rule P
2.	$\neg t$	Rule P
3.	$\neg r$	Rule T [(1) and (2) and modus tollens]
4.	$\neg r \vee \neg s$	(3) Rule T, rule of disjunctive addition
5.	$\neg(r \wedge s)$	Step 4 & Demorgan's laws
6.	$(\neg p \vee \neg q) \to (r \wedge s)$	Rule P
7.	$\neg(\neg p \vee \neg q)$	Rule T and (5) and (6) and modus tollens
8.	$p \wedge q$	(7), Demorgan's laws and law of double negation.
9.	p	Rule T , (8) and simplification

Using all the given premises the conclusion p is attained in direct proof method. Hence the given argument is a valid argument.

Problems with indirect proof:

1. Show that $a \to b, \neg(b \vee c) \Rightarrow \neg a$.

Solution:

Step	Proposition	Justification
1.	a	Negated conclusion (assumed) $\neg\neg a \equiv a$
2.	$a \to b$	Rule P
3.	b	Rule T, (1) & (2) & modus ponens
4.	$b \vee c$	Rule T, (3) & disjunctive addition
5.	$\neg(b \vee c)$	Rule P
6.	$(b \vee c) \wedge \neg(b \vee c)$	(4) & (5) Contradiction

By assuming the negated conclusion we get a contradiction. Hence the given premises imply the conclusion & the argument is valid.

2. Prove by the method of indirect proof that $\neg p \leftrightarrow q, q \to r, \neg r \Rightarrow p$.

Solution:

Step	Proposition	Justification
1.	$\neg p$	Negated conclusion (assume)
2.	$\neg p \leftrightarrow q$	Rule P
3.	$(\neg p \to q) \wedge (q \to \neg p)$	Rule T , (2) and biconditional equivalence
4.	$\neg p \to q$	Rule T, (3) & simplification
5.	$q \to r$	Rule P
6.	$\neg p \to r$	Rule T, (4), (5) & chain rule.
7.	r	Rule T, (6), (1) and modus ponens
8.	$\neg r$	Rule P
9.	$r \wedge \neg r$	(7) & (8) contradiction

Hence by the method of indirect proof the given argument is valid.

Problems with CP:

1. Show that $P, P \to (Q \to (R \wedge S)) \Rightarrow Q \to S$.

Solution: To prove using CP, we take Q as an assumed premise and prove S. This will imply $Q \to S$ as a valid conclusion.

Step	Proposition	Justification
1.	Q	Assumed premise
2.	$P \to (Q \to (R \wedge S))$	Rule P
3.	P	Rule P
4.	$Q \to (R \wedge S)$	(2), (3) & modus ponens & rule T.
5.	$R \wedge S$	Rule T, (1), (4) & modus ponens
6.	S	(5) and simplification
7.	$Q \to S$	Using CP & (1) & (6)

2. Derive $P \to (Q \to S)$ using the rule CP if necessary from

   $P \to (Q \to R), Q \to (R \to S)$

Solution:

Step	Proposition	Justification
1.	P	Assumed premise
2.	$P \to (Q \to R)$	Rule P
3.	$Q \to R$	Rule T & (1) & (2) & modus ponens
4.	$Q \to (R \to S)$	Rule P
5.	$\neg Q \vee R$	(3), Rule T, & conditional equivalence
6.	$\neg Q \vee (R \to S)$	(4), Rule T and conditional equivalence
7.	$\neg Q \vee (R \wedge (R \to S))$	(5), (6) & distributive law
8.	$\neg Q \vee S$	Rule T, (7) and modus ponens
9.	$Q \to S$	Rule T, (8) and conditional equivalence

10.	$P \to (Q \to S)$	Using CP, (1) & (10)

Hence proved.

HAVE YOU UNDERSTOOD THE CONCEPTS ?
ANSWER THE FOLLOWING:

1. $(P \vee Q) \wedge ((P \vee Q) \to R)$ implies

(a) P (b) Q (c) $P \vee Q$ (d) R Ans: (d)

2. $(P \to (Q \vee R)) \wedge (\neg Q \wedge \neg R)$ implies

(a) P (b) $\neg P$ (c) $Q \vee R$ (d) $\neg(Q \vee R)$ Ans: (b)

3. Rule CP can be used to prove only conditional statements.

(a) True (b) False Ans: (a)

4. If $(P \wedge T) \to S$ and $S \to R$ then

(a) $(P \wedge T) \to R$ (b) $P \to R$ (c) $T \to R$ (d) $\neg(P \wedge T) \to R$ Ans: (a)

3.8 PREDICATE CALCULUS

3.8.1 N-PLACE PREDICATES:

1-PLACE PREDICATES:

Consider the following sentences.

Ramu is a student of II CSE.

Somu is a student of II CSE

Latha is a student of II CSE.

Prema is a student of II CSE.

All these sentences are about the students of II CSE and it is the predicate.

If S – denotes a student of II CSE

r – Ramu, s – Somu, l – Latha and p – Prema then the above sentences can be symbolized as S(r), S(s), S(l), S(p) respectively.

Since only one variable is required to use this predicate it is called as 1 – place predicate.

2-PLACE PREDICATE:

Consider the following sentences

Tina is taller than Ramya.

Vasan is cleverer than Roshan.

In both the above sentences we require two variables (names) to define the predicates. hence they are 2-place predicates.

If T denotes : is taller than (predicate)

t : Tina and r : Ramya (Variables)

The first sentence can be denoted as T(t,r).

If C denotes : is cleverer than (predicate)

v : Vasan and r : Roshan (Variables)

The second sentence can be denoted as C(v,r).

3-PLACE PREDICATES:

If three variables are required to use a predicate then it is called 3 place predicate.

Example:

5 is in between 4 and 6 in the set of whole numbers.

Here 4,5,6 are variables.

……..is between …. and ……. is the predicate.

n-PLACE PREDICATES:

Similarly it can be extended for n variables and in such a case it is called a n-place predicates.

3.8.2 QUANTIFIERS:

There are two different quantifiers:

1. Universal quantifier
2. Existential quantifier

(i) UNIVERSAL QUANTIFIER:

Let $p(x)$ be a propositional function. "For every x" or "for all x" is called the universal quantifier and is denoted by $\forall x$ or (x).

$\forall x\ p(x)$ means that the proposition is true for all x.

$(\forall x \in A)\ p(x)$ means that the proposition is true for all x in the set A.

The set of values for which the statement is true is called truth set.

Examples:

1. Consider “all scents have pleasant fragrance”.

Let $S(x)$: x is a scent.

$F(x)$: x has pleasant fragrance.

$(x)\left(S(x) \to F(x)\right)$.

2. Let A = {7, 8, 9, 10}

Consider $(\forall x \in A)(x+5>9)$

Since $x + 5 > 9$

$x > 9 - 5$

$x > 4$

Here $x \in A$ and A = {7, 8, 9, 10} and all the values in this set are greater than 4 and satisfy the relation $x + 5 > 9$. Hence for every value x in the set the given statement is true.

ii) EXISTENTIAL QUANTIFIER:

Let p(x) be a propositional function. “There exists a x” or “There exists some x” is called as the existential quantifier and is denoted by $\exists x$.

$\exists x p(x)$ means that the proposition is true for some x.

$(\exists x \in A)\, p(x)$ means that there exists some $x \in A$ such that the proposition p(x) is true.

Examples:

1. Some students are intelligent.

   Let S(x): x is a student.

   I(x): x is intelligent.

   $(\exists x)\left(S(x) \to I(x)\right)$ is the symbolic representation of the given statement.

2. Let $A = \{5, 6, 7, 8, 9, 10\}$

   Consider $(\exists x \in A)(5x < 10)$

   Here $5x < 10 \Rightarrow x < 2$

   But the set A has values which are greater than 2. There are no elements in the set A that will satisfy this relation. Hence the statement is false and the truth set = ϕ (null set).

Note:

$\exists x$ is the negation of $\forall x$ and

$\forall x$ is the negation of $\exists x$.

NESTING OF QUANTIFIERS:

If there are more than one quantifier in front of a proposition then we have a nesting of quantifiers.

Example:

Consider "Everyone has someone whom they like".

Let $L(x, y): x$ likes y.

Hence the given sentence can be symbolized as

$\forall x\left[\exists y L(x, y)\right]$

The meaning is 'there is some y whom x likes and it is true for all x'. Therefore the universe for this proposition is the set of all people.

3.8.3 BOUND AND FREE VARIABLES:

Consider (x)A(x) and ($\exists y$)B(y). Here (x)A(x) is called x-bound part of the formula and ($\exists y$)B(y) is called an y-bound part of the formula. In the first case x is called bound variable and in the second case y is called as the bound variable.

Consider (x) P(x,y). Here x is a bound variable and y is a free variable.

The scope of the formula is the formula immediately following the quantifier. If the scope is an atomic formula no parenthesis are used to enclose the formula.

Example: (x)W(x)

If it is not an atomic formula then parenthesis are needed.

Example:

1. $(x)\left(A(x) \rightarrow B(x)\right)$
2. $\exists y\left(P(y) \rightarrow Q(y)\right)$
3. Consider the well known 'Socrates argument'

   "All men are mortal

   Socrates is a man

   Therefore Socrates is mortal"

If H(x) : x is a man.

M(x) : x is a mortal and

s : Socrates

Then the Socrates argument can be represented as

$(x)\big(H(x) \to M(x)\big)$

H(s)

$\therefore$ M(s).

(This can be proved using the inference theory for predicate calculus which will have methods similar to the methods which were used to prove the validity of arguments in propositional calculus).

Exercises:

1. Translate the statement, 'the sum of two positive integers is positive' into a logical expression.

Solution:

The meaning of the given statement is 'for every two positive integers x, y the sum $x + y$ is also positive'.

$\therefore \forall x \forall y \big((x > 0) \wedge (y > 0)\big) \to (x + y > 0)$

where the universe of discourse for both variables consists of all integers.

2. Express the statement, 'If a person is a female and is a parent then this person is someone's mother' as a logical expression.

Solution:

Let F(x): x is a female

P(x): x is a parent

M(x, y): x is a mother of y.

The meaning of the given statement is 'if x is a parent and a female then she is a mother of a child y'. It can be symbolized as $\forall x\big((F(x) \wedge P(x)) \to \exists y M(x, y)\big)$. Since y is not a bound variable in $F(x) \wedge P(x)$, y does not appear in $F(x) \wedge P(x)$. It can be brought to the front and in such a case an equivalent expression will be

$$\forall x \exists y\left((F(x) \wedge P(x)) \rightarrow M(x, y)\right)$$

3. Negate the statement, 'If x is odd then $x^2 - 1$ is even'.

Solution:

Let p(x): x is odd

q(x): $x^2 - 1$ is even.

The given statement is, 'If x is odd then $x^2 - 1$ is even' and it can be denoted as

$\forall x\left(p(x) \rightarrow q(x)\right)$.

The negation of this statement is determined as follows

$$\neg\left[\forall x\left(P(x) \rightarrow Q(x)\right)\right]$$

$\equiv \exists x\left[\neg\left(P(x) \rightarrow Q(x)\right)\right]$ Negation of universal quantifier $\forall x$ is the essential quantifier $\exists x$

$\equiv \exists x\left[\neg\left(\neg P(x) \vee Q(x)\right)\right]$ Conditional equivalence $P \rightarrow Q \equiv \neg P \vee Q$

$\equiv \exists x\left[\neg\left(\neg P(x)\right) \wedge \neg Q(x)\right]$ Demorgan's laws

$\equiv \exists x\left[P(x) \wedge \neg Q(x)\right]$ Law of double negation

Hence the negation is, "There exists an integer x such that x is odd and $x^2 - 1$ is not even (odd)".

(The truth value of this statement is false).

4. Let $p(x, y), q(x, y)$ and $r(x, y)$ represent three open statements. What is the negation of the following statement?

$$\forall x \exists y\left[\left(p(x, y) \wedge q(x, y)\right) \rightarrow r(x, y)\right]$$

Solution:

$$\neg\left\{\forall x \exists y\left[\left(p(x, y) \wedge q(x, y)\right) \rightarrow r(x, y)\right]\right\}$$

$$\equiv \exists x\left[\neg \exists y\left(p(x, y) \wedge q(x, y)\right) \rightarrow r(x, y)\right]$$

$$\equiv \exists x \forall y \left\{ \neg \left[\left(p(x,y) \wedge q(x,y) \right) \rightarrow r(x,y) \right] \right\}$$

$$\equiv \exists x \forall y \left\{ \neg \left[\neg \left(p(x,y) \wedge q(x,y) \right) \vee r(x,y) \right] \right\}$$ using conditional equivalence.

$$\equiv \exists x \forall y \left\{ \neg \left[\neg \left(p(x,y) \wedge q(x,y) \right) \right] \wedge \neg r(x,y) \right\}$$ using demorgan's law.

$$\equiv \exists x \forall y \left\{ p(x,y) \wedge q(x,y) \wedge \neg r(x,y) \right\}$$ using the rule of double negation.

5. Let the universe consist of integers 1, 2, 4, 8, 16, 32

Let P(x): x is an even integer.

Q(x): x is divisible by 8.

R(x , y): x is divisible by y .

Find the truth value of

(i) P(8) (ii) Q(4) (iii) R(1, 16) (iv) R(16, 1)

(v) P(8) $\wedge$ R(8, 4) (vi) R(1, 8) $\rightarrow$ Q(4)

Solution:

(i) P(8):

P(x): x is even integer.

Here x is 8. 8 is an even integer.

Therefore P(8) is true.

(ii) Q(4):

Q(x): x is divisible by 8.

Here x is 4. But 4 is not divisible by 8.

Therefore Q(4) is false.

(iii) R(1, 16):

R(x , y): x is divisible by y .

Here R(1, 16) is : 1 is divisible by 16, which is not true.

Hence its truth value is false

(iv) R(16, 1):

The meaning is '16 is divisible by 1'

And its truth value is true.

(v) P(8) $\wedge$ R(8, 4):

P(8) is true.

R(8, 4) is '8 is divisible by 4'

And it is true.

P(8)	R(8, 4)	P(8) $\wedge$ R(8, 4)
T	T	T

Hence true.

(vi) R(1, 8) $\rightarrow$ Q(4)

R(1, 8) is ' 1 is divisible by 8' and it is false

Q(4) is false (found in (ii))

Using definition of conditional statement

R(1, 8)	Q(4)	R(1, 8) $\rightarrow$ Q(4)
F	F	T

Hence true.

~~3.8.4 INFERENCE THEORY FOR PREDICATE CALCULUS~~

~~As discussed earlier, the validity of an argument is checked in inference theory. When the premises and the conclusion are propositions, the validity of the arguments are proved using the equivalence laws and the valid inferences.~~

~~When the propositions are quantified statements with predicates, the inferences and the equivalence laws are not sufficient to check their validity. Hence along with those laws and inferences we use the following for deriving the conclusion from the premises.~~

~~Some more Equivalence laws and inferences:~~

~~1. $(\exists x)(A(x) \vee B(x)) \equiv (\exists x)A(x) \vee (\exists x)B(x)$~~
~~2. $(x)(A(x) \wedge B(x)) \equiv (x)A(x) \wedge (x)B(x)$~~
~~3. $\neg(\exists x)A(x) \equiv (x)\neg A(x)$~~
~~4. $\neg(x)A(x) \equiv (\exists x)\neg A(x)$~~
~~5. $(x)A(x) \vee (x)B(x) \Rightarrow (x)(A(x) \vee B(x))$~~
~~6. $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$~~

~~As before the proof is either a direct proof or an indirect proof. Rule P, rule T and rule CP are also used here with the same meaning.~~

In the inference theory of predicate calculus the arguments contain either the universal quantifier $[(x) \text{ or } \forall x]$ or the existential quantifier $[\exists x]$. We either use these additional laws and inferences to derive the conclusion or we remove the quantifiers to get forms generalized to predicates. Then we use the ordinary equivalence laws and inferences to get the conclusion. If the conclusion is a quantified statement we introduce quantifiers to get the desired result. The additional laws and inferences involving quantifiers can also be used in any stage as required.

The elimination and addition of quantifiers can be done by using the following rules.

1. Rule US (Universal specification)
2. Rule UG (Universal generalization)
3. Rule ES (Existential specification) and
4. Rule EG (Existential generalization)

Rule US:

If a statement is true for every x in the universe then it is also true for some object "c" in the universe.

$$\frac{\forall x A(x)}{\therefore A(c)} \quad \text{or} \quad \frac{(x)A(x)}{\therefore A(c)}$$

(i.e.) from $(x)A(x)$ one can conclude $A(c)$.

Rule UG:

If a statement $A(c)$ is true for an arbitrary c of the universe then the universal quantifier may be prefixed to obtain $(\forall x)A(x)$ provided c is not free in any premise or in any prior step of the derivation.

In symbols,

$$\frac{A(c) \text{ for an arbitrary } c}{\therefore \forall x A(x)}$$

Rule ES:

If a statement is true for some x in the universe (i.e.) if $\exists x A(x)$ is true then we conclude that it will be true for some element 'c' in the universe. $\exists x A(x)$ means that there exists a 'x' for which $A(x)$ is true. We name it as 'c' and continue the proof provided 'c' is not free in any premise or in any prior step of the derivation.

In symbols,

$$\frac{\exists x A(x)}{\therefore A(c) \text{ for some element } 'c'}$$

Rule EG:

If $A(c)$ is true for some element 'c' in the universe then we conclude that $\exists x A(x)$ is true.

In symbols,

$$\frac{A(c) \text{ for some element } 'c'}{\therefore \exists x A(x)}$$

Problems:

1. Prove that $(\exists x)\big((p(x) \wedge q(x))\big) \Rightarrow (\exists x)\, p(x) \wedge (\exists x)(q(x))$.

Solution:

Step	Proposition	Justification
1.	$(\exists x)(p(x) \wedge q(x))$	Rule P
2.	$p(y) \wedge q(y)$	Rule ES
3.	$p(y)$	Rule T conjunctive simplification and step (3)
4.	$q(y)$	Rule T conjunctive simplification and step (3)
5.	$\exists x p(x)$	Rule EG and step (4)
6.	$\exists x q(x)$	Rule EG and step (5)
7.	$(\exists x)\, p(x) \wedge (\exists x) q(x)$	Rule T and step (6) and step (7)

2. Show that $(\exists x)M(x)$ follows from the premises

$$(x)(H(x) \rightarrow M(x)) \text{ and } (\exists x)H(x)$$

Proof:

Step	Proposition	Justification
1.	$(\exists x)H(x)$	Rule P
2.	$H(c)$	Rule ES & (1)
3.	$(x)(H(x) \rightarrow M(x))$	Rule P
4.	$H(c) \rightarrow M(c)$	Rule US & (3)
5.	$M(c)$	(2), (4) & modus ponens
6.	$(\exists x)M(x)$	Rule EG & (5)

Hence proved.

3. Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Proof:

We use the indirect method of proof.

Step	Proposition	Justification
1.	$\neg((x)P(x) \vee (\exists x)Q(x))$	Assumed premise. (Negated conclusion)
2.	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Demorgan's laws
3.	$\neg(x)P(x)$	Conjunctive simplification
4.	$(\exists x)\neg P(x)$	Negation of (x) is $\exists x$
5.	$\neg(\exists x)Q(x)$	(2) and conjunctive simplification
6.	$(x)\neg Q(x)$	Negation of $\exists x$ is (x)
7.	$\neg P(y)$	Rule ES & (4)
8.	$\neg Q(y)$	Rule US & (6)
9.	$\neg P(y) \wedge \neg Q(y)$	(7) & (8)

10.	$\neg\left(P(y)\vee Q(y)\right)$	(9) & Demorgan's laws
11.	$(x)\left(P(x)\vee Q(x)\right)$	Rule P
12.	$P(y)\vee Q(y)$	(1) & Rule US
13.	$\neg\left(P(y)\vee Q(y)\right)\wedge\left(P(y)\vee Q(y)\right.$	(10) & (12)
14.	0	contradiction

When we assume the negated conclusion we arrive at a contradiction. Hence by indirect method of proof the given argument is valid. Hence the proof.

4. Check the validity of the argument

$$(\forall x)\left(P(x)\rightarrow Q(x)\right),(\forall x)\left(R(x)\rightarrow\neg Q(x)\right)\Rightarrow(\forall x)\left(R(x)\rightarrow\neg P(x)\right)$$

Solution:

Since the conclusion involves the conditional connective we use the rule CP for proving it.

Step	Proposition	Justification
1.	$(\forall x)\left(P(x)\rightarrow Q(x)\right)$	Rule P
2.	$(\forall x)\left(R(x)\rightarrow\neg Q(x)\right)$	Rule P
3.	$R(x)\rightarrow\neg Q(x)$	Rule US & (2)
4.	$R(x)$	Rule P (assumed)
5.	$\neg Q(x)$	(3), (4) & modus polens
6.	$P(x)\rightarrow Q(x)$	Rule US & (1)
7.	$\neg P(x)$	(5), (6) and modus tollens
8.	$R(x)\rightarrow\neg P(x)$	Rule CP, (4) & (7)
9.	$(\forall x)\left(R(x)\rightarrow\neg P(x)\right)$	Rule UG and (9)

Hence by the rule CP the argument is valid. Hence the proof.

HAVE YOU UNDERSTOOD THE CONCEPTS ?
ANSWER THE FOLLOWING:

1. If the universe is the set of integers what is the truth value of $\exists x\left(x^2-5=0\right)$.

(a) T (b) F Ans: (b) $\left[\because x=\pm\sqrt{5}\notin z\right]$

~~2. What is the truth set for "$x+3>8$".~~

~~(a) $\{x : x \in N, x > 8\}$~~

~~(b) $\{x : x \in N, x > 3\}$~~

~~(c) $\{x : x \in N, x \geq 6\}$~~

~~(d) $\{x : x \in N, x \geq 5\}$~~ ~~Ans: (c)~~

~~3. Negate $\exists x \forall y\ p(x, y)$~~

~~(a) $\exists x \neg \forall y,\ p(x, y)$~~

~~(b) $\exists x \forall y \neg p(x, y)$~~

~~(c) $\forall x \exists y \neg p(x, y)$~~

~~(d) $\forall x \forall y \neg p(x, y)$~~ ~~Ans: (c)~~

~~4. Is this true or false?~~

~~$(x)(y)\,p(x, y) = (y)(x)\,p(x, y)$~~

~~(a) T~~ ~~(b)F~~ ~~Ans: (a)~~

~~5. $\neg((\forall x)(\neg P(x)))$ is~~

~~(a) $(\exists x)P(x)$~~ ~~(b) $\forall x P(x)$~~ ~~(c) $(\exists x)\neg P(x)$~~ ~~Ans: (a)~~

SUMMARY:

- Propositions are statements which are either true or false but not both.
- Primitive or atomic propositions are those which cannot be broken down into simpler propositions and they are represented by either capital letters or small letters.
- Compound propositions are composed of atomic propositions and are connected by connectives.
- The truth value of propositions can be found using truth tables and they contain 2^n rows if the proposition consists of n variables.
- Negation is the negative of a statement and it is true if the statement is false and vice versa and is represented by $\neg$.

- Conjunction is represented by $p \wedge q$ and is true if both the statements p and q are true otherwise it is false. Disjunction is represented by $p \vee q$ and is false only when both p and q are false otherwise it is true. If … then statement is called conditional and is denoted by $p \rightarrow q$ and is false only when p is true and q is false.
- If and only if statement is called biconditional and it is true only when both p and q are true or both are false and it is denoted by $p \leftrightarrow q$.
- NAND is 'not' and 'and' together and is denoted by $p \uparrow q$ and it is false only when both p and q are true. NOR is 'not' and 'or' together and is denoted by $p \downarrow q$ and it is true only when p and q are false. If p and q are two propositions then 'p exclusive – or q' is denoted by $p \oplus q$. $p \oplus q$ is true when either p is true or q is true but not both. It is called exclusive – or because it excludes the possibility that both p and q are true.
- Tautology is a statement which is always true. Contradiction is a statement which is always false. Two statements are equivalent when they have the identical truth values.
- Two statements are duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. If two statements are equivalent and are connected by $\vee$ and $\wedge$ only then their duals are also equivalent and this is called as duality principle. Any proposition can be written in terms of the connectives $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ and they are called as functionally complete set of connectives. $\{\uparrow\}, \{\downarrow\}$ are minimal sets. The laws of algebra of propositions and laws of equivalences are used for this.
- There are four Normal forms and any well formed formula can be reduced to any one of the normal forms. A formula consisting of a sum of elementary products and equivalent to a given formula is called DNF. A formula consisting of a product of elementary sum and equivalent to a given formula is CNF. A formula consisting of a sum of minterms and equivalent to a given formula is called PDNF. A formula consisting of a product of maxterms and equivalent to a given

formula is called PCNF. PCNF & PDNF can be obtained either using truth tables or without using them. PCNF to PDNF and PDNF to PCNF conversion is also possible.

- An assertion that a given set of premises yielding the conclusion is called an argument. An argument in which the conclusion is true when all the premises are true it is called as a valid argument. An argument which is not valid is called as a fallacy. The validity of arguments can be proved using direct proof or indirect proof. For these proofs a set of rules called rule T, rule P, rule CP are used along with some valid inferences.
- Predicates related by any number of variables constitute predicate calculus. Depending upon the number of variables they may be 1-place predicates, 2-place predicates and so on. Quantifiers are of two kinds-universal and existential. 'for every x' denoted by $\forall x$ or (x) is universal quantifier. 'there exists some x' denoted by $\exists x$ is called existential quantifier. Bound and free variables are also defined and they have applications in the inference theory of predicate calculus.
- Rules are defined for removal and addition of quantifiers. Equivalence laws and valid inferences which involve quantifiers are also charted. These concepts are used to check the validity of arguments in predicate calculus.

EXERCISES:

PART A

1. Negate the statement given below:

'If there is a will there is a way' Ans: $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

2. Obtain the disjunctive normal form for the formula $(p \wedge (q \to r)) \to r$

Ans: $\neg p \vee (q \wedge r) \vee r$

3. Symbolize the statement 'some rational number are powers of 2'.

Ans: $(\exists x)(R(x) \wedge P(x))$.

4. If $p \to (\neg p \vee q)$ is false the then truth value of p and q are respectively

(a) TT (b) TF (c) FT (d) FF Ans: TF $[p - True\ \neg p \vee q - False$;

$q - False]$

5. Determine whether the conclusion q follows logically from the premises $H_1 : p \rightarrow q$
$H_2 : p$. Ans: Yes $p \rightarrow q, p \Rightarrow q$ (modus ponens)

6. Express the following two statements symbolically using quantifiers.

(i) Some students in this examination hall know Java.

(ii) Every student in this examination hall knows C++ or Java.

Ans: $(\exists x)(S(x) \wedge J(x))$

$\forall x[S(x) \rightarrow (C(x) \vee J(x))]$

7. Express the following statement in symbolic form "Any integer is either positive or negative". Ans: $\forall x(I(x) \rightarrow (P(x) \vee N(x)))$

EXERCISES:

PART – B

1. Show that $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology without using truth tables.

   (show that it is equivalent to **T**).

2. Show that (i) $P \oplus Q \equiv (P \vee Q) \wedge \neg (P \wedge Q)$

   (ii) $P \oplus Q \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$

3. Show that $\rightarrow$ is distributive over $\wedge$.

   Hint: Show $P \rightarrow (Q \wedge R) \equiv (P \rightarrow Q) \wedge (P \rightarrow R)$

4. Show that $\rightarrow$ is distributive over $\vee$.

   Hint: Show $P \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$

5. Show that $\leftrightarrow$ is not distributive over $\wedge$.

   Hint: Show $P \leftrightarrow (Q \wedge R) \not\equiv (P \leftrightarrow Q) \wedge (P \leftrightarrow R)$

6. Show that $\rightarrow$ is not associative.

   Hint: Show $(P \rightarrow Q) \rightarrow R \not\equiv P \rightarrow (Q \rightarrow R)$

7. Show that $\leftrightarrow$ is associative.

   Hint: Show $(P \leftrightarrow Q) \leftrightarrow R \equiv P \leftrightarrow (Q \leftrightarrow R)$

(All these equivalences from problem 2 to 7 can be proved using the truth tables easily)

8. Find the converse, inverse and contrapositive of

   'If n is prime then n is odd or n is 2'

   Ans:

   Converse: If n is odd or n is 2 then n is prime.

   Inverse: If n is not prime, then n is not odd and n is not 2.

   Contrapositive: If n is not odd and n is not 2 then n is not prime.

9. Show that $R \to S$ can be derived from the premises $P \to (Q \to S), \neg R \vee P$ and Q.

   (Hint: Prove using CP)

10. Test the validity of the following argument. "If I get the job and work hard then I will get promoted. If I get promoted then I will be happy. Therefore either I will not get the job or I will not work hard.

    (Hint: Premises: $(p \wedge q) \to r$, $r \to s$, $\neg s$ Conclusion: $\neg p \vee \neg q$ valid argument)

11. Using indirect proof derive $P \to \neg S$ from $P \to (Q \vee R)$, $Q \to \neg P$, $S \to \neg R, P$.

12. Prove that $P \to Q, Q \to R, S \to \neg R$, $P \wedge S$ are inconsistent premises.

    (Hint: Using the premises arrive at a contradiction)

13. Find PDNF of $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$.

    Ans: $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

14. Translate the following predicate calculus formula into English sentence.

    $\forall x\left[C(x) \forall \exists y(C(y) \wedge F(x,y))\right]$.

    Here $C(x): x$ has a computer, $F(x,y): x$ *and* y are friends. The universe for both x *and* y is the set of all students of your college.

    (Ans: Every student of your college has a computer or has a friend who has a computer).

15. Symbolize the statement

    "Given any positive integer n, there is a greater positive integer."

    Ans: $(x)\left(p(x) \to (\exists y)\left(P(y) \wedge Q(y,x)\right)\right)$

where $P(x)$: x is a positive integer.

$Q(y,x)$: y is greater than x.

16. Show that from

a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows. (Hint: Use direct proof)

17. Prove the validity by indirect method

$(\forall x)(P(x) \rightarrow Q(x)), (\exists y) P(y) \Rightarrow (\exists z) Q(z)$

Books for References

1. Kenneth H.Rosen, " Discrete Mathematics and Its Applications", Tata McGraw Hill, Fourth Edition, 2002.
2. M.K.Venkataraman "Engineering Mathematics", Volume II, National Publishing Company, 2nd Edition,1989.
3. J.P.Tremblay and R.Manohar, "Discrete Mathematical Structures with Applications to Computer Science", Tata McGraw Hill, Third Edition, 2003