MATH 221 FINAL EXAM PASSMAN

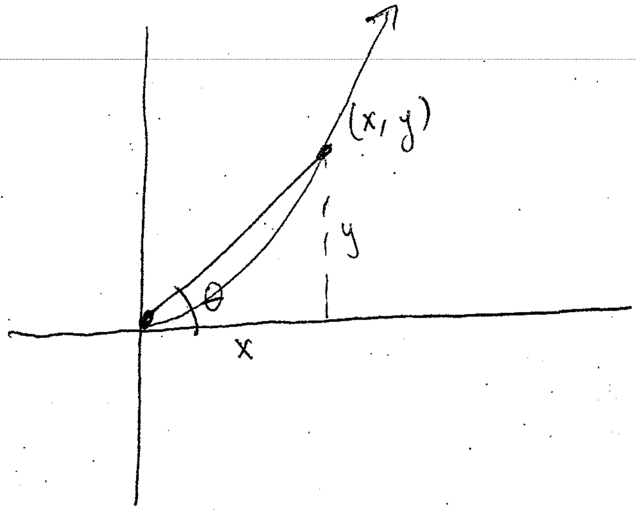
NAME X

T. A.'s NAME X

Do ALL 8 problems and show ALL work.
Each problem is worth 20 points.
Use only techniques that have been covered in class.

PROBLEM	GRADE
1 20 pts	
2 20 pts	
3 20 pts	
4 20 pts	
5 20 pts	
6 20 pts	
7 20 pts	
8 20 pts	
TOTAL	

1. A particle moves along the cubic curve $y = (1/3)x^3$ in the first quadrant in such a way that its x-coordinate increases at a steady rate of 25. How fast is the angle between the x-axis and the line joining the particle to the origin changing when $x = 2$.



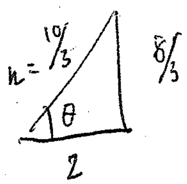
Need $\dfrac{d\theta}{dt}$. $\quad \dfrac{dx}{dt} = 25$

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{1}{3}x^3 \Big/ x = \frac{1}{3}x^2$$

$$\underbrace{\sec^2\theta}_{5/3}\,\frac{d\theta}{dt} = \frac{2}{3}\underbrace{x}_{2}\,\underbrace{\frac{dx}{dt}}_{25}$$

When $x = 2$, $y = \frac{1}{3}\cdot 2^3 = \frac{8}{3}$



$$x^2 + y^2 = h^2$$

$$4 + \frac{64}{9} = h^2$$

$$\frac{100}{9} = h^2$$

$$h = \frac{10}{3}$$

$$\Rightarrow \sec\theta = \frac{10}{3}\Big/2 = \frac{10}{6} = 5/3$$

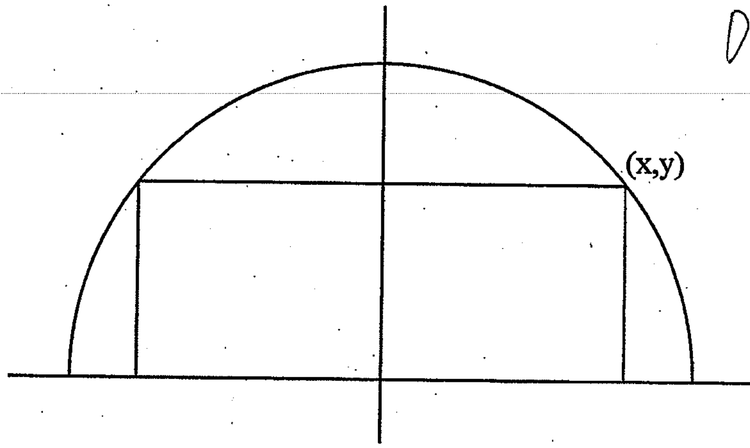
$$\left(\frac{5}{3}\right)^2 \frac{d\theta}{dt} = \frac{100}{3}$$

$$\frac{25}{9}\frac{d\theta}{dt} = \frac{100}{3}$$

$$\frac{d\theta}{dt} = 12.$$

Perimeter

2. A rectangle is inscribed in the semicircle $y = \sqrt{5 - x^2}$ as indicated below. Find the maximum value of the ~~circumference~~ c of the rectangle. Be sure to check the possible endpoints. Note that $2 < \sqrt{5} < 2.5$.



Domain of x:

$0 \le x \le \sqrt{5}$

radius of circle

$$P = 4x + 2y$$
$$= 4x + 2\sqrt{5-x^2}$$

$$\frac{dP}{dx} = 4 - \frac{2x}{\sqrt{5-x^2}} = 0$$

$\rightarrow \quad 4\sqrt{5-x^2} = 2x$ square both sides

$\rightarrow \quad 4(5-x^2) = x^2$

$\rightarrow \quad 20 - 4x^2 = x^2$

$5x^2 = 20 \quad \rightarrow \quad x = 2$

Test points

$x = 0, \quad P = 2\sqrt{5}$

$x = 2, \quad P = 8 + 2\sqrt{5-4} = 8 + 2 = 10$

$x = \sqrt{5}, \quad P = 4\sqrt{5} + 2\sqrt{5-5} = 4\sqrt{5}$

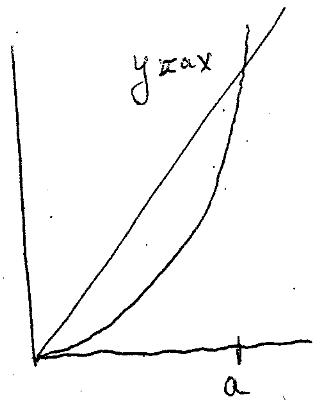
Now $\sqrt{5} < 2.5 \Rightarrow 4\sqrt{5} < 10$

So max is at $x = 2$, $P = 10$.

3. Let a be a positive constant. Find the two volumes if the region in the x, y-plane bounded above by the line $y = ax$ and below by the parabola $y = x^2$ is rotated about

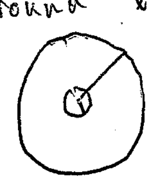
a) the x-axis,

b) the y-axis.



$ax = x^2 \Rightarrow x = a$

a) Rotate around x-axis.
Use Washers



O: ax

I: x^2

$$V = \int_0^a \left((ax)^2 - (x^2)^2\right) dx = \pi \int_0^a a^2x^2 - x^4 \, dx = \pi \left(\frac{a^2x^3}{3} - \frac{x^5}{5} \right) \Big|_0^a$$

$$= \pi \left(\frac{a^5}{3} - \frac{a^5}{5} \right) = \frac{2\pi a^5}{15}$$

b) Rotate around y-axis
Use Shells

$R = x$

$H = ax - x^2$

$$2\pi \int_0^a x(ax - x^2) \, dx = 2\pi \int_0^a ax^2 - x^3 dx = 2\pi \left(\frac{ax^3}{3} - \frac{x^4}{4} \right) \Big|_0^a$$

$$= 2\pi \left(\frac{a^4}{3} - \frac{a^4}{4} \right) = \frac{\pi a^4}{6}$$

4. Find the length of the curve given by the parametric equations

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad 0 \le t \le \ln 3$$

$$\text{Length} = \int_0^{\ln 3} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$x'(t) = -e^t \sin t + e^t \cos t = e^t(\cos t - \sin t)$$

$$y'(t) = e^t \cos t + e^t \sin t = e^t(\cos t + \sin t)$$

$$x'(t)^2 = e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t)$$

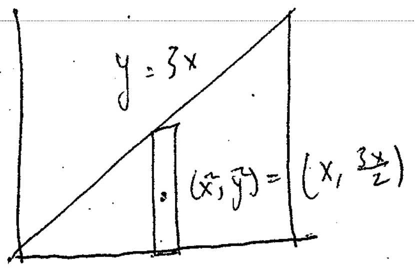
$$y'(t)^2 = e^{2t}(\cos^2 t + 2\sin t \cos t + \sin^2 t)$$

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{e^{2t}(\cos^2 t - \cancel{2\sin t\cos t} + \sin^2 t + \cos^2 t + \cancel{2\sin t\cos t} + \sin^2 t)}$$

$$= \sqrt{2}\, e^t$$

$$\int_0^{\ln 3} \sqrt{2}\, e^t = \sqrt{2}\, e^t \Big|_0^{\ln 3} = \sqrt{2}\,(3-1) = 2\sqrt{2}$$

5. A thin triangular plate in the x, y-plane is bounded above by the line $y = 3x$, below by the x-axis and on the right by the line $x = 2$. If the density at any point is given by $\delta = x$, find the center of mass of the plate.



$$dm = \delta \, dA$$
$$= 3x^2 \, dx$$

$$M = \int_0^2 dm = \int_0^2 3x^2 \, dx = x^3 \Big|_0^2 = 8$$

$$M_y = \int \tilde{x} \, dm = \int_0^2 3x^3 \, dx = \frac{3x^4}{4} \Big|_0^2 = 12$$

$$M_x = \int \tilde{y} \, dm = 9 \int_0^2 x^3 \, dx = \frac{9x^4}{4} \Big|_0^2 = 36$$

$$\bar{x} = \frac{M_y}{M} = \frac{12}{8} = \frac{3}{2} \qquad \bar{y} = \frac{36}{8} = \frac{9}{2}$$

6. Evaluate

a)

$$\int \frac{dx}{9x^2 - 6x + 5}$$

$$\int \frac{dx}{9x^2-6x+5} = \int \frac{dx}{(3x-1)^2+4} = \frac{1}{6} \tan^{-1} \frac{3x-1}{2} + C$$

b)

$$\int \frac{dx}{\sqrt{4 - \underbrace{9x^2}_{(3x)^2}}}$$

You can use u-sub with $u = 3x$

$$= \frac{1}{3} \sin^{-1} \frac{3x}{2} + C$$

c)

$$\int \frac{\cos x}{1 + \sin^2 x}\, dx = \int \frac{du}{1+u^2} = \tan^{-1}(\sin x) + C$$

$u = \sin x$

$du = \cos x \, dx$

7. a) The size $y(t)$ of a certain radioactive material satisfies the differential equation $dy/dt = -ay$ for some positive contant a. Assume that $y(0) = 120$ and that the material has a half-life of 10. At what time t will $y(t) = 40$. Leave your answer in terms of natural logarithms.

$$\frac{dy}{dt} = -ay$$

$$-\int \frac{dy}{ay} = \int dt \Rightarrow \frac{-1}{a} \ln y = t + c$$

$$\Rightarrow \ln y = -at + c \quad , \quad e^c \text{ is a constant } A$$

$$\boxed{y = Ae^{-at}}$$

- $y(0) = 120 \Rightarrow \quad 120 = A e^{-a \cdot 0} = A$
- Half-Life 10 $\Rightarrow$ in 10 minutes there will be $\frac{120}{2} = 60$ left

$$60 = 120 e^{-a \cdot 10} \Rightarrow \frac{1}{2} = e^{-a \cdot 10} \Rightarrow \frac{\ln 2}{10} = a$$

So $y = 120 e^{-\frac{\ln 2}{10} \cdot t}$

when $y(t) = 40$

$\Rightarrow$

$$40 = 120 e^{-\frac{\ln 2}{10} t}$$

$$\frac{1}{3} = e^{\frac{-\ln 2}{10} t}$$

$$\ln 3 = \frac{\ln 2}{10} t$$

$$\Rightarrow t = \frac{10 \ln 3}{\ln 2}$$

b) Compute

$$\int_0^{\sqrt{\ln 7}} x e^{x^2} \underbrace{dx}_{\frac{du}{2}}$$

$u = x^2$

$du = 2x \, dx$

$u(0) = 0, \quad u(\sqrt{\ln 7}) = \ln 7$

$$\int_0^{\sqrt{\ln 7}} x e^{x^2} dx = \int_0^{\ln 7} e^u \frac{du}{2} = \frac{e^u}{2} \Big|_0^{\ln 7} = \frac{7}{2} - \frac{1}{2} = 3$$

8. Evaluate

a)

$$\cancel{\int \frac{\sinh x}{3+\cosh x}\,dx}$$

b)

$$\frac{d}{dx}(x^{-x})$$

$x^{-x} = e^{-x \cdot \ln x}$

$\frac{d}{dx} x^{-x} = \frac{d}{dx}\left(e^{-x \ln x}\right) = e^{-x \ln x} \cdot (-x \ln x)'$

$= x^{-x}\left(-\frac{x}{x} + (-1)\ln x\right) = -x^{-x}(\ln x + 1)$

c)

$$\int \frac{(\ln x)^2}{x}\,dx$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}\,dx$

$\Rightarrow \int u^2\,du = \frac{u^3}{3} = \frac{(\ln x)^3}{3} + C$