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CBCS PATTERN

B.B.A.
(Computer Applications)
Semester-II

CBCS
3 CREDITS

BUSINESS MATHEMATICS

A. V. RAYARIKAR   Dr. P. G. DIXIT

NIRALI PRAKASHAN
ADVANCEMENT OF KNOWLEDGE

SPPU New Syllabus

Course Code 203

BUSINESS MATHEMATICS

Credit-3

For B.B.A. (Computer Application) Semester-II

As Per New Syllabus (CBCS Pattern) Effective from June 2019

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Statistical Thinking will one day be necessary for effective citizenship as the ability to read and write

H.G. Wells

Preface ...

We are very happy to place this book in the hands of **first year 'B.B.A. (C.A.)' Semester-II** students. This book is written according to new prescribed syllabus (CBCS Pattern) by Pune University which comes into force from the academic year 2019.

The main purpose of the book is to provide foundation as well a comprehensive background of Business Mathematics' to beginners in simple and intersecting manner. In order to make the contents of the book easier to comprehend, We have included a requisite number of illustrations, remarks, figures, diagrams etc. To elucidate Mathematical concepts, Applications of Mathematics in real life situations is emphasized through illustrative examples. Ample number of graded problems, are provided at the end of each chapter along with hints and answers. A Model Question paper is set for student's self assessment.

While writing the book we have borne in mind that many students have not offered mathematics at XIth and XIIth std.

This book will also serve the purpose of reference book for M.B.A., C.A., B.C.A., I.C.W.A., M.P.M., classes.

We are thankful to Mr. D. K. Furia, Mr. Jignesh Furia and the staff of Nirali Prakashan for bringing out this book in short time. Mrs. Anagha Medhekar, Mr. Santosh Bare, Mrs. Anjali Muley and Mr. Pandya deserve special thanks for the co-operation they have extended to us. Finally, our families deserve special thanks for their support, encouragement and tolerance.

We request our colleagues, teaching Mathematics to offer their criticisms and suggestions, for further improvement of the book.

November 2019 **Authors**

Syllabus ...

1. Ratio, Properties and Percentage **(08)**

Ratio - Definition, Continued ratio, Inverse ration, Proportion, Continued proportion, Direct proportion, Inverse proportion, Variation, Inverse variation, Joint variation, Percentage, Computation of percentage.

2. Profit and Loss **(06)**

Terms and formulae, Trade discount, Cash discount, Problems involving cost price, Selling price, Trade discount and cash discount. Introduction to commission and brokerage, Problems on commission and brokerage.

3. Interest and Annuity : **(07)**

Simple interest, Compound interest, Equated monthly instalments (EMI) by interest of reducing balance and flat interest methods and problems.

Ordinary annuity, Sinker fund, Annuity due, Present value and future value of annuity.

Shares and Mutual Funds : **(07)**

Shares, Face value, Market value, Dividend, Brokerage, Equity shares, Preferential shares, Bonus shares, Examples and problems, Concept of mutual funds, Change in Net Asset Value (NAV), Systematic Investment Plan (SIP), Examples and Problems.

4. Matrices and Determinant : **(10)**

Matrices, Types of matrices, Algebra of matrices, Determinant, Ajoint of matrix, Inverse of matrix, System of linear equations, Solution of system of linear equation by adjoint method (upto 3 variables only).

5. Linear Programming Problem (L.P.P.) **(05)**

Concept of L.P.P., Formulation of LPP and solution of L.P.P. by graphical method.

Transportation Problem (T.P.) **(05)**

Concept of transportation problem, Initial basic feasible solution, North-West Corner Method (NWCM), Least Cost Method (LCM), Vogal's Approximation Method (VAM).

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Contents ...

Chapter 1 ...

Ratio, Proportion and Percentage

Contents ...

Key Words :

Ratio, Proportion, Direct Proportion, Continued proportion, Inverse proportion, Continued ratio variation, Joint variation, Inverse variation, Percentage.

Objectives :

To understand the concept of ratio proportion and percentage in business.

1.1 Ratio

In day-to-day life, we come across several situations in which we have to make comparisons among quantities.

For example, salaries of persons, prices of commodities, sales of different firms etc.

In mathematics, the operations of subtraction and division are mainly used for comparison. When two quantities are compared by division, we use *ratio*.

Suppose that a firm having sale of ₹ 40 crores in a year made a profit of ₹ 13 crores and another firm having sale of ₹ 60 crores in that year made a profit of ₹ 16 crores. Therefore, do we infer that the performance of second firm is better ? Taking the quotient profit/sales we can compare the performance of the two firms. (Note that this is one of the criteria for judging the performance of a firm). In other words, we are using ratio (of profit to sales) to compare the performance of the two companies, and then our conclusion is that the performance of first firm is better than that of the second $\left(\because \frac{13}{40} > \frac{16}{60}\right)$.

From the above illustration, it is clear that, we can find the ratio of two quantities of the same type. Moreover, the unit for measurement must be same.

Thus, we have a formal definition.

Definition : *If 'a' and 'b' are magnitudes of same kind, expressed in same units, then the quotient $\frac{a}{b}$ is called the ratio of 'a' to 'b' and it is denoted by a : b.*

Note :

(i) Ratio is a pure number i.e. it has no units.

(ii) In the ratio a : b, a is called antecedent and b is called consequent.

(iii) If we multiply the numerator and denominator in any ratio by the same (non-zero) number, the ratio remains the same.

i.e. $$\frac{a}{b} = \frac{ma}{mb} \qquad (m \neq 0)$$

From this it is clear that, the antecedent and consequent in a ratio may not be actual quantities. It also indicates that, if the ratio of two quantities is a : b, the actual quantities should be taken as xa and xb $(x \neq 0)$.

Continued Ratio : *It is the relation between the magnitudes of three or more quantities of the same kind.*

The continued ratio of three similar quantities a, b, c is denoted by a : b : c.

Solved Examples

Example 1.1 : *Two numbers are in the ratio 7 : 8 and their sum is 195. Find the numbers.*

Solution : Let the numbers be 7x and 8x.

$$\therefore \quad 7x + 8x = 195$$

$$\text{i.e} \quad 15x = 195$$

$$\therefore \quad x = 13$$

$\therefore$ Required numbers are 91 and 104.

Example 1.2 : *If a : b = 4 : 7 and b : c = 9 : 5, find a : c.*

Solution : $$\frac{a}{b} = \frac{4}{7}$$

$$\therefore \quad 7a = 4b$$

$$\therefore \quad a = \frac{4b}{7}$$

Again, $$\frac{b}{c} = \frac{9}{5}$$

$$\therefore \quad 5b = 9c$$

$$\therefore \quad c = \frac{5b}{9}$$

$$\therefore \quad \frac{a}{c} = \frac{\frac{4}{7}b}{\frac{5}{9}b} = \frac{36}{35}$$ i.e. a : c = 36 : 35.

Example 1.3 : *The sum of present ages of 3 persons is 66 years. Five years ago, their ages were in the ratio 4 : 6 : 7. Find their present ages.*

Solution : Let the ages of three persons, five years ago be 4x, 6x and 7x years respectively.

$\therefore$ Their present ages are 4x + 5, 6x + 5 and 7x + 5.

From the information given,

$$(4x + 5) + (6x + 5) + (7x + 5) = 66$$

$$\therefore \quad 17x + 15 = 66$$

$$\therefore \quad 17x = 51$$

$$\therefore \quad x = 3$$

$\therefore$ Present ages are 4x + 5, 6x + 5 and 7x + 5.

i.e. 17, 23 and 26 years respectively.

Example 1.4 : *The monthly salaries of two persons are in the ratio 3 : 5. If each receives an increase of ₹200 in monthly salary, the new ratio is 13 : 21. Find their original salaries.*

(Oct. 2013)

Solution : Let the original salaries be ₹ 3x and ₹ 5x.

Due to increase in salaries, the revised salaries are ₹ (3x + 200) and ₹ (5x + 200). It is given that

$$\frac{3x + 200}{5x + 200} = \frac{13}{21}$$

$$\therefore \quad 63x + 4200 = 65x + 2600$$

$$\therefore \quad 2x = 1600$$

$$\therefore \quad x = 800$$

$\therefore$ Original salaries were ₹ 2,400 and ₹ 4,000.

Example 1.5 : *The ratio of prices of two houses was 4 : 5. Two years later, when the price of first had risen by 10% and that of the second by ₹6,000, the ratio became 11 : 15. Find the new prices of the houses.*

Solution : Let the original prices be ₹ 4x and ₹ 5x. Two years later, the price of first house increased to ₹ $4x + \frac{4x}{10}$ and that of the second to ₹ (5x + 6,000).

$\therefore$ Ratio of new prices,

$$\frac{4x + \frac{4x}{10}}{5x + 6000} = \frac{11}{15} \qquad \text{(given)}$$

$$\therefore \quad 15\left(4x + \frac{4x}{10}\right) = 11\,(5x + 6000)$$

$$\therefore \quad 60x + 6x = 55x + 66000$$

$$\therefore \quad 11x = 66000$$

$$\therefore \quad x = 6000$$

$\therefore$ New prices of the houses are $4x + \frac{4x}{10}$ and $5x + 6000$

i.e. $24000 + 2400$ and $30000 + 6000$

i.e. ₹ 26400 and ₹ 36000.

Example 1.6 : *Incomes of P, Q, R are in the ratio 2 : 3 : 4 and their expenditures are in the ratio 5 : 7 : 9. If P saves (1/5)th of his income, find the ratio of their savings.*

(April 2016, 2017, 2018)

Solution : Let the incomes of P, Q, R be ₹ 2x, 3x, 4x respectively. (Note that, we cannot take 5x, 7x, 9x. Why ?)

$\therefore$ Their savings will be ₹ $2x - 5y$, $3x - 7y$, $4x - 9y$ respectively.

But it is given that P saves $1/5^{th}$ of his income i.e. ₹ $\frac{2x}{5}$.

$$\text{Thus,} \quad 2x - 5y = \frac{2x}{5}$$

$$\therefore \quad 10x - 25y = 2x$$

$$\therefore \quad 8x = 25y$$

$$\therefore \quad x = \frac{25}{8}y$$

$\therefore$ Savings of P, Q, R will be $\frac{50}{8}y - 5y$, $\frac{75}{8}y - 7y$, and $\frac{100}{8}y - 9y$ respectively.

i.e. $\frac{10}{8}y$, $\frac{19}{8}y$ and $\frac{28}{8}y$ respectively.

$\therefore$ Ratio of savings is 10 : 19 : 28.

1.2 Proportion (April 2015, 2018)

If two ratios are equal, then the four quantities given by them are said to be in *proportion* i.e. if the ratios a : b and c : d are equal, then a, b, c, d are said to be in proportion and we write a : b :: c : d. Here, b and c are called *means* while a and d are called *extremes*, further d is called 4^{th} proportional to a, b and c.

Note : If a, b, c, d are in proportion, then,

$$\frac{a}{b} = \frac{c}{d}$$

$$\therefore \quad ad = bc$$

i.e. Product of extremes = Product of means

Continued Proportion (Oct. 2014, April 2015, 2018) : If a, b, c are three quantities of the same kind and if $a/b = b/c$, then a, b, c are said to be in *continued proportion*.

In this case, b is called mean proportional to a and c.

Note that $b^2 = ac$.

The concept of continued proportion can also be extended to more than three quantities of the same kind.

Direct Proportion : Petrol costs ₹ 11 per litre. If a person buys 3 litres of petrol, clearly he has to pay ₹ 33. Thus, as the consumption of petrol increases, expenditure on it also increases. Similarly, if the consumption is less, expenditure is also less.

Thus, we have a relation between two variables viz., consumption of petrol and expenditure on it. They are said to be in *direct proportion*.

Definition : Direct Proportion (Oct. 2014, April 2015, 2018) : *When two variables are so related that an increase (or reduction) in one causes an increase (or reduction) in the other in same ratio then the proportion is called direct proportion.*

Inverse Proportion (Oct. 2014, April 2015, 2018) : Suppose that a man completes a job in 15 days working 4 hours per day. Then we know that if the job is to be completed in 10 days, he will have to work 6 hours per day. Thus, if the job is to be completed in lesser days, the man has to work more everyday.

In this illustration, number of days and working hours are two variables such that if number of hours is increased, number of days is decreased in the same ratio. Also if number of hours is decreased, the number of days is increased in the same ratio. This type of variation is called inverse variation and two variables are said to be in **inverse proportion**, since in this case, one ratio is reciprocal of the other as shown below :

No. of days	Working hours
15	4
10	6
$\frac{15}{10} = \left(\frac{3}{2}\right) = \frac{1}{\frac{4}{6}}$	

Definition : *If two variables are so related that, an increase (or reduction) in one causes a reduction (or increase) in the same ratio in the other, then they are said to be in inverse proportion.*

1.3 Variation (Oct. 2013)

If two variables x and y are in direct proportion, we write it as $x \propto y$, then, $x = ky$, where k is called constant of proportionality.

If a value of x and corresponding value of y are known, then this constant can be obtained at once.

For a circle, circumference $\propto$ radius is an illustration of direct variation.

Inverse Variation (Oct. 2013, 2014) : If x and y are two variables such that x varies directly as $\frac{1}{y}$, then we say that x varies inversely as y and write,

$$x \propto \frac{1}{y}$$

then, $$x = \frac{k}{y}$$ where, k is constant of proportionality.

If a value of x and corresponding value of y are known, then this constant can be obtained at once.

Joint Variation (Oct. 2013, 2014) :

1. A variable x is said to vary jointly with respect to the variables y and z, if it varies as their product i.e. if

$$x \propto yz$$

then, $$x = kyz$$

For example, we know that area of a triangle varies jointly as its base and altitude.

2. A variable x is said to vary directly as y and inversely as z, if it varies as $\frac{y}{z}$.

i.e. $$x \propto \frac{y}{z}$$

then, $$x = \frac{ky}{z}$$

Solved Examples

Type 1 :

Example 1.7 : *Find x, if (i) 6 : 15 :: 2 : x, (ii) 15 : 27 :: x : 45.*

Solution : (i) 6 : 15 :: 2 : x

i.e. $$\frac{6}{15} = \frac{2}{x}$$

i.e. $$6x = 30$$

$\therefore$ $$x = 5$$

(ii) 15 : 27 :: x : 45

i.e. $$\frac{15}{27} = \frac{x}{45}$$

i.e. $$\frac{5}{9} = \frac{x}{45}$$

$\therefore$ $$\frac{45 \times 5}{9} = x$$

$\therefore$ $$x = 25$$

Example 1.8 : *Find fourth proportional to 6, 8, 10.*

Solution : Let x be the fourth proportional.

$\therefore \quad 6 : 8 :: 10 : x$

i.e. $$\frac{6}{8} = \frac{10}{x}$$

i.e. $$6x = 80$$

$$x = \frac{40}{3} = 13.3333$$

Example 1.9 : *Ages of Madhav, Ajit, and Dilip are in continued proportion. If Madhav is 4 years old and Dilip is 9 years old, what is the age of Ajit ?*

Solution : Let Ajit be x years old.

$\therefore \quad 4 : x :: x : 9$ (since they are in continued proportion)

i.e. $$\frac{4}{x} = \frac{x}{9}$$

i.e. $$x^2 = 36$$

$\therefore$ $$x = 6$$

Hence, the age of Ajit is 6 years.

Type 2 :

Example 1.10 : *If sugar costs ₹8 per kg, how many tonnes can be bought for ₹48,000 ?*

Solution : The price of sugar and quantity purchased are in direct proportion.

$\therefore$ If x kg sugar can be bought for ₹ 48,000.

$$\frac{8}{1} = \frac{48000}{x}$$

$\therefore$ $$x = 6{,}000 \text{ kg}$$

i.e. $$x = 6 \text{ tonnes}$$

Type 3 :

Example 1.11 : *A student finishes a book by reading 30 pages per day in 16 days. If he wants to finish the book in 12 days, how many pages should be read everyday ?*

Solution : We know that the number of pages read and the number of days required are in inverse proportion.

Let x be the number of pages that he has to read everyday to finish the book in 12 days.

	No. of pages	No. of days
Original data	30	16
New data	x	12

Because of inverse proportion,

$$\frac{x}{30} = \frac{1}{12/16}$$

$$\therefore \quad \frac{x}{30} = \frac{4}{3}$$

$$\therefore \quad x = 40$$

∴ He has to read 40 pages per day.

Type 4 :

Example 1.12 : *If A ∝ B and A = 4 when B = 6, find the value of A when B = 27.*

Solution : $A \propto B;$ ... (1)

$$\therefore \quad A = kB$$

When A = 4, and B = 6,

$$\therefore \quad 4 = 6k \qquad \therefore \quad k = \frac{2}{3}\ B$$

$$\therefore \text{ from (1)} \quad A = \frac{2}{3}\ B$$

$$\therefore \text{ when} \quad B = 27$$

$$A = \frac{2}{3} \times 27 \qquad \therefore \quad A = 18$$

Example 1.13 : *If x varies directly as y and inversely as z and x = 12 when y = 9 and z = 16, find y when x = 9 and z = 24.*

Solution : $x \propto y \quad \text{and} \quad x \propto \frac{1}{z}$

$$\therefore \quad x \propto \frac{y}{z} \qquad \therefore \quad x = \frac{ky}{z} \qquad \text{... (1)}$$

When x = 12, y = 9, z = 16,

$$12 = \frac{9k}{16} \qquad \therefore \quad k = \frac{64}{3}$$

$$\therefore \text{ from (1),} \quad x = 64\frac{y}{3}$$

When x = 9, z = 24,

$$\therefore \quad 9 = \frac{64\ y}{3} \times 24 \qquad \therefore \quad y = \frac{81}{8}$$

Example 1.14 : *A diamond worth ₹ 25,600 is accidentally broken into two pieces whose weights are in the ratio 3 : 5. The value of the diamond varies as the square of the weight. Calculate the loss due to the breakage.*

Solution : Let c denote the cost and w denote the weight of diamond. Then,

$$c \propto w^2$$

$$\therefore \quad c = kw^2 \qquad \text{... (1)}$$

where, k is constant of proportionality.

Let weight of the diamond be 8 gm.

$$25{,}600 = k\,(64)$$

$\therefore$ $k = 400$

$\therefore$ (1) gives $c = 400\,w^2$ ... (2)

Due to breakage, the two pieces now weigh 3 gm and 5 gm respectively.

The cost of the diamond weighing 3 gm is

$$c = 400\,(9) = 3{,}600$$

Similarly, the cost of the diamond weighing 5 gm is

$$c = 400\,(25) \qquad \therefore\ c = 10{,}000$$

$\therefore$ The total cost of the two pieces

$$= 3{,}600 + 10{,}000 = 13{,}600$$

$\therefore$ Loss $= 25{,}600 - 13{,}600$

i.e. Loss $=$ ₹ 12,000

Exercise (1.1)

1. Fill in the blanks :

(i) A person was getting a salary of ₹ 37500 in 2005. His salary increased to ₹ 41,500 in 2007 ..... Ratio of salary in 2007 to the salary in 2005 is ......

(ii) Arvind has ₹ 8 while Sameer has 40 paise.

$\therefore$ Ratio of amount with Arvind to that of Sameer is ......

(iii) If $x : y = 5 : 7$ and $x = 30$, then $y =$ ......

(iv) A has ₹ 3,200 and B has ₹ 2,600

$\therefore$ B has ...... times the amount with A.

(v) Manish has 3 bananas and Hari has 7 dozen bananas.

$\therefore$ Ratio of Manish's bananas to that of Hari is ......

(vi) Ratio of two numbers is 4 : 7. The bigger number is 147. Hence the smaller number is ......

(vii) Ratio of two numbers is 3 : 5 and the sum of the numbers is 232. Hence, the bigger number is ......

(viii) The ratio of length to breadth of a rectangle is 8 : 3. If the perimeter of the rectangle is 352 cm, then the sides of the rectangle are ......

2. Ratio of two numbers is 4 : 7. The smaller is 24, find the bigger. **(April 2016)**

3. Monthly incomes of A and B are in the ratio 9 : 11 and those of B and C are in the ratio 13 : 10. If monthly income of C is ₹ 1,430, then find the incomes of A and B. **(April 2018)**

4. If 8, x and 50 are in continued proportion, then find x.
5. If numerator and denominator in a ratio are each increased by same non-zero constant, the ratio remains the same. What can you say about the ratio ?
6. An alloy of gold and copper weighing 100 gm contains 97 gm of gold. How much gold should be added to the alloy to increase the percentage of gold to 98 ?
7. If an article is sold at 25% profit, find the ratio of cost price to selling price.
8. Sand and cement are in the ratio 6 : 5 in a mixture weighing 671 kg. How much sand must be added to the mixture so as to make the ratio 8 : 5 ?
9. A and B are two alloys of gold and copper prepared by mixing them in the ratio 3 : 2 and 5 : 8 respectively. If equal quantities of the alloys are melted to form a third alloy, then find the ratio of the gold to copper in the new alloy.

Answers (1.1)

(1) (i) 83 : 75, (ii) 20 : 1, (iii) 42, (iv) $\frac{13}{16}$, (v) 1 : 28, (vi) 84, (vii) 145, (viii) 128 cm, 48 cm.

(3) Income of A ₹ 1521, Income of B ₹ 1,859.

(4) x = 20	(5) 1 : 1	(6) 50 gm
(7) 4 : 5	(8) 122 kg	(9) 32 : 33

1.4 Percentage

It is a special type of ratio, in which the consequent (denominator) is 100. When a ratio is expressed in this form, the numerator is said to express the percentage. Thus $\frac{x}{100}$ denotes x%. Consider the illustration studied in earlier section, in which profits of two firms were compared.

The ratio of profit to sales of the first firm is $\frac{13}{40} = \frac{32.5}{100}$ and the ratio for other firm is $\frac{16}{60} = \frac{\frac{80}{3}}{100}$.

In other words, the percentage profits of the two firms are 32.5 and $\frac{\frac{80}{3}}{100}$. Hence, according to this criterion, the performance of first firm is better than that of the second.

Note that with the help of percentage, we could compare the performance of the two companies very easily.

Remarks :

1. As percentage is a ratio, it has no units.
2. Percentages are very useful in profit and loss, commission and brokerage, simple interest, compound interest etc.

3. If we want to find x% of a quantity, we should multiply the quantity by $\frac{x}{100}$, in other words, x% of Y = $Y \times \frac{x}{100}$

   For example, 3% of 58 = $58 \times \frac{3}{100} = 1.74$

4. If we want to write a given fraction in percentages, we should multiply the fraction by 100.

   For example, $\frac{3}{4} = \frac{3}{4} \times 100\% = 75\%$.

Solved Examples

Example 1.15 : *The population of a city according to 2001 census was 84,500 and it rose to 1,11,200 in 2011. Find the percentage increase in the population.*

Solution : The increase in population

$$= 1,11,200 - 84,500 = 26,700.$$

This growth is w.r.t. original population 84,500.

on 84,500, increase is 26,700

∴ on 100, the increase is 31.6 (approx.)

$$\frac{100}{84500} \times 26700 = 31.6$$

∴ The percentage increase in population over the decade is 31.6 (approx.)

Example 1.16 : *A salesman gets 5% commission in his sales. If he gets a commission of ₹8700 in a month, find his sales in that month.*

Solution : For a commission of ₹ 5, sales are ₹ 100.

∴ For a commission of ₹ 870, sales are ₹ 17,400

$$\frac{8700}{5} \times 100 = 17,4000.$$

∴ His sales in that month are ₹ 17,4000.

Example 1.17 : *The price ₹ 50 of an article was increased by 12%. As a result, consumption decreased by 10%. Find the percentage increase or decrease in the original income.*

Solution : Suppose that the original consumption is 100 articles.

Due to 12% rise, the article now costs ₹ 56.

Original price 100, new price ₹ 112

∴ Original price ₹ 50, new price ₹ 56

$$\left(\frac{80}{100} \times 112\right) = 89.6$$

Due to 10% reduction in number of articles, now the consumption is 90 articles.

∴ New income = 90×56 = ₹ 5,040

∴ There is an increase of ₹ (5,040 – 5,000) = ₹ 40.

For original income of ₹ 5,000, increase ₹ 40.

∴ For original income of ₹ 100, increase ₹ $\frac{8}{10}$.

Thus, there is an increase of 0.8% in the original income.

Example 1.18 : *In a school, there are 12% girls. If 5 boys and 15 girls are newly admitted to the school, the percentage of girls becomes 15. What is the total strength of the school ?*

Solution : Let the original strength of the school be x.

$$\text{Number of girls} = \frac{12\,x}{100}$$

$$\text{After new admissions, number of girls} = \frac{12\,x}{100} + 15$$

$$\text{Revised strength of the school} = x + 20$$

$$\therefore \quad \text{Revised percentage of girls} = \frac{\frac{12\,x}{100} + 15}{x + 20} \times 100$$

$$= \frac{12\,x + 1500}{x + 20}$$

But the percentage is given to be 15.

$$\therefore \quad \frac{12\,x + 1500}{x + 20} = 15$$

$$\therefore \quad 12\,x + 1500 = 15\,x + 300$$

$$\therefore \quad 3\,x = 1200$$

$$\therefore \quad x = 400$$

∴ Original strength of the school is 400.

Example 1.19 : *A sold a car to B at 15% profit. B sold the car to C at 5% profit for ₹48,300. Find the price at which A has purchased the car.*

Solution : Suppose that A purchased the car for ₹ x. Since his profit is 15% i.e. $\frac{15\,x}{100}$ he has sold the car to B for

$$₹\left(x + \frac{15\,x}{100}\right) = ₹\,\frac{115\,x}{100}$$

B takes a profit of 5% on this.

For C.P. ₹ 100, S.P. is ₹ 105

For C.P. ₹ $\frac{115\,x}{100}$, S.P. is ₹ $\frac{483\,x}{400}$

$$\frac{115\,x}{100} \times \frac{105}{100} = \frac{483\,x}{400}$$

$$\text{Thus,} \quad \frac{483}{400}\,x = 48{,}300$$

$$\therefore \quad \frac{x}{400} = 100$$

$$\therefore \quad x = 40{,}000$$

∴ A had purchased the car for ₹ 40,000.

Example 1.20 : *Rates of electricity charges increased by 25%. In order to keep expenses on electricity at the same level, by what per cent a family should reduce its consumption of electricity ?*

Solution : Let the original consumption of electricity be 100 units and rate be Re. 1 per unit.

$$\therefore \quad \text{Original bill} = 100 \times 1 = ₹\, 100$$

Due to the increase in the rates, new rate is ₹ 1.25 per unit.

Let the new consumption be x units.

$$\therefore \quad \text{Revised expenditure} = 1.25 \times x = ₹\, \frac{5}{4}\, x$$

As the family wants to maintain expenditure at original level,

$$\frac{5}{4}\, x = 100$$

$$\therefore \quad x = 80$$

Thus, the family should reduce its consumption by 20%.

Example 1.21 : *Price of sugar increased by 10% as a result of which a person gets 1 kg. less in ₹ 88. Find the original rate.*

Solution : Let the original rate be ₹ x per kg. In ₹ 88, the person would get $\frac{88}{x}$ kg.

Due to increase, the new rate is $x + \frac{x}{10} = \frac{11\,x}{10}$

$\therefore$ In ₹ 88, the person would get $\dfrac{88}{\frac{11\,x}{10}} = \dfrac{80}{x}$

By given information, $\dfrac{88}{x} - \dfrac{80}{x} = 1$

$$\therefore \quad x = 8$$

Thus, the original rate of sugar is ₹ 8 per kg.

Exercise (1.2)

1. Fill in the blanks :

   (i) $\left(\frac{2}{5}\right)^{th}$ of an amount = ... % of that amount.

   (ii) A person having income of ₹ 1,640 spends ₹ 1,230, therefore, his expenses are ...... % of his income.

   (iii) A student scores 42 marks out of 60. $\therefore$ His percentage score is ......

   (iv) A clerk writes 9 in place of 90. $\therefore$ He committees an error of ...... %

2. Population of a city in a specific year was 80,000. There was rise of 10% for each of the next 3 years. Find the population at the end of 3 years.

3. A company declares a dividend of 20% on its equity shares. If a share whose face value is ₹ 10, is purchased for ₹ 30, what is the percentage return on the investment ?
4. Electricity rates increased by 5%, as a result of which there was a fall of 8% in the electricity consumption. If the total revenue of electricity board decreased by ₹ 6,80,000, find the revenue of the board.
5. In an examination, a student got 28% of total marks and failed as he got 80 marks less than the total required. Another student passed by securing 38% of the total marks, when he had scored 20 marks more than the required total. How many marks were required for passing ?
6. A person gave 20% of his total amount to his son. Of the remainder, he gave 50% to his wife and 30% of the remainder to each of three daughters. The remaining amount of ₹ 3,416 was donated to a charity trust. Find the total amount and share each one got.
7. An alloy of gold and silver weighs 50 gm. It contains 80% gold. How much gold should be added to the alloy so that percentage of gold is increased to 90 ?
8. Which investment gives a better return : 5% stock at 75, subject to 30% income-tax or 4% stock at 90, tax free ?

   (Assume the face value of share to be ₹ 100.)
9. Monthly incomes of A and B are in the ratio 9 : 7 and those of B and C in the ratio 3 : 2. If 10% of A's income exceeds 15% of C's income by ₹ 90, find the incomes of A, B and C.
10. A person invests some amount and loses 10% in the first year but in the next year, he gains 20% of what he had at the end of the first year. If there is an increase of ₹ 1,440 in his capital at the end of two years, find his original capital.

Answers (1.2)

(1) (i) 40, (ii) 75, (iii) 70, (iv) 90.

(2) 1,06,480.

(3) 6.66

(4) 2 crores

(5) 360

(6) ₹ 85,400, ₹ 17,080, ₹ 34,160, ₹ 10,248

(7) 50 gm

(8) 5% stock at 75

(9) ₹ 4,050, ₹ 3,150, ₹ 2,100

(10) ₹ 18,000.

Miscellaneous Exercise (1)

1. Find the value of x, if (i) $x : 3 = (x + 2) : 5$, (ii) $20 : x = 4 : 5$.
2. The ratio of cost of a pencil to that of a pen is 2 : 3 while that of cost of a pen to a note book is 4 : 5. If the cost of a note book is ₹ 15, find the cost of a pencil.

3. In 135 litres of milk mixed with water, the ratio of milk to water is 7 : 2. How much water should be added so that the ratio of milk to water becomes 5 : 2.

4. Monthly incomes of A and B are in the ratio 6 : 7 and those of B and C are in the ratio 7 : 9. If A earns ₹ 36000, find the incomes of B and C.

5. The ages of two friends are in the ratio 16 : 27 and the difference in their age is 5.5 years. Find their ages.

6. The monthly incomes of A and B are in the ratio 9 : 7 and those of B and C are in the ratio 7 : 5. If 10% of A's income exceeds 16% of C's income by ₹ 120. Find monthly incomes of A, B and C.

7. An ornament of gold weighing 28 gm contains gold and copper in the ratio 13 : 1. How much of pure gold must be added to it, so as to make the ratio of gold to copper 15 : 1 ?

8. Incomes of A, B and C are in the ratio 1 : 2 : 3 while their expenditures are in the ratio 2 : 3 : 4. If A saves one-third of his income, find the ratio of their savings.

9. The total salary bill of a company employing 25 men and 35 women is ₹ 7,80,000 per month. A man earns 20% more than a woman, on an average. What is monthly salary of each man and each woman ?

10. A student declared "Passed" if he secures 40% of the total marks. In an examination a student secured 24 marks and hence scored 4 marks more than that are necessary for passing. Find the maximum marks at the examination.

11. In a constituency of 80500 voters 60% cast their votes and in a straight contest between two candidates, the winner obtained 2600 votes more than his opponent. What is percentage of votes cast in favour of each candidate ?

12. The market share of an organisation in the year 2006 was 50%. After the takeover of another company, it increased by 40% in the year 2007. Due to competition by other organisation it reduced by 10% in the 2007 year 2008 as compared to the year 2007. What is the share in the year 2008 as compared to the year 2006 ?

13. The sales in 2 shopes were in the ratio 4 : 7. After advertising, sales in shop A increased by 10% and in B by 8%. The difference in new sales is 79 articles. Find the original sales.

14. A customer paid ₹ 330 for 220 telephone calls. The telephone charges increased by 10% what should be the percentage reduction in the number of calls in order to keep the expenditure over telephone sanu ?

15. A man spends 30% on house rent, 30% on food, 12% on clothing, 10% on fuel and the remaining on miscellaneous items from his salary. If he spends ₹ 1350 on miscellaneous items, what is his salary ? What is the amount spent under each heading ?

16. Population of a city in a specific year was 80,000. There was a rise of 10% for each of next 3 years. Find the population at the end of 3 years.

Answers

(1) (i) 3, (ii) 25 (2) ₹ 8 (3) 12 litres

(4) ₹ 4,2000, ₹ 5,4000 (5) 8 years, 13.5 years.

(6) A's ₹ 10800, B's ₹ 8400, C's ₹ 6000

(7) 4 gm (8) 1 : 3 : 5 (9) ₹ 14,400, ₹ 12,000

(10) 50 (11) loser 47.31%, winner 52.69%

(12) 26% increase in the year 2008 (13) 100, 175

(14) reduction of 9.09% in number of telephone calls

(15) ₹ 2250, ₹ 2250, ₹ 900, ₹ 750.

❑❑❑

Chapter 2...

Profit and Loss

Contents ...

Key Words :

Market price, Selling Price, Trade Discount, Cash Discount, Commission, Brokerage.

Objectives :

To understand the concept and application of profit and loss in business.

Although you have studied this topic earlier in school, we give below certain terms and formulae :

2.1 Terms and Formulae

Cost Price (C.P.) : The total amount paid for purchasing an article, transport charges, octroi etc. is called *cost price* of that article.

Marked Price (M.P.) (April 2016) : The price of an article which is printed in the price list or catalogue so as to ensure that the consumer gets the article at that price is called *marked price* or *list price* or *catalogue price.*

Selling Price (S.P.) (April 2016) : Total amount realised by selling an article is called *selling price* of that article.

2.2 Trade Discount and Cash Discount

Traders who buy goods in large quantities from manufacturers or wholesalers are generally given an allowance on the list price of the goods. This is known as *trade discount.*

The discount given by retailer to a purchaser is called *cash discount.*

Sometimes, a buyer may get benefit of both discounts. In such a case, cash discount should be calculated on the net value of the goods after deducting trade discount.

Thus, net selling price is the amount realised by the seller after giving discount. If no discount is given, then S.P. will be same as the M.P.

If S.P. is more than the C.P. then we say that a profit is realised in the transaction.

$$\boxed{\text{Profit} = \text{S.P.} - \text{C.P.}}$$

If C.P. is more than the S.P. then we say that a loss is incurred the transaction.

$$\boxed{\text{Loss} = \text{C.P.} - \text{S.P.}}$$

While solving the problems, students are advised to remember the following points :

1. *Percentage profit or loss is always calculated on the C.P.*
2. *Commission or discount is always given on S.P.*
3. *Percentage profit or loss does not depend on number of articles sold (or purchased).*

Note : If an article costing ₹ x is sold at y% profit, its

$$\text{S.P.} = \left(\frac{100+y}{100}\right)x \text{ and if it is sold at } y\% \text{ loss,}$$

$$\text{then its S.P.} = \left(\frac{100-y}{100}\right)x.$$

Solved Examples

Type 1 : To find percent profit/loss when S.P. and C.P. are given.

Example 2.1 : *A scooter costing ₹ 12,000 was sold for ₹ 10,400 after two years. Find the percentage loss.*

Solution : Loss = ₹ 12,000 – 10,400 = ₹ 1,600

On C.P. ₹ 12,000 loss ₹ 1,600 $\frac{100}{12,000} \times 1,600$

∴ On C.P. ₹ 100, loss ₹ $\frac{40}{3}$ $= \frac{40}{3}$

∴ There is $\frac{40}{3}$ = 13.33% loss.

Type 2 : To find S.P. when C.P. and per cent profit/loss is given.

Example 2.2 : *Goods worth ₹6,000 are purchased. What should be the S.P. to earn a profit of 12% ?*

Solution : To earn 12% profit, an article costing ₹ 100 should be sold for ₹ 112. Thus,

	when C.P. is ₹ 100, S.P. is ₹ 112	$\frac{6000}{100} \times 112$
∴	when C.P. is ₹ 6,000, S.P. is ₹ 6,720	= 6,720
∴	Goods should be sold for **₹ 6,720.**	

Type 3 : To find C.P. when S.P. and percent profit/loss is given

Example 2.3 : *A camera when sold for ₹1,674 resulted in a loss of 7%. What was the C.P. of the camera ?*

Solution : A loss of 7% means if the camera was purchased for ₹ 100, it is sold for ₹ 93. Thus,

	when S.P. is ₹ 93, C.P. is ₹ 100	$\frac{1,674}{93} \times 100$
∴	when S.P. is ₹ 1,674, C.P. is ₹ 1,800	= 1,800
∴	C.P. of the camera was ₹ 1,800	

Type 4 : To find per cent profit/loss when some goods are sold at profit and remaining at loss.

Example 2.4 : *A book-seller purchased 800 copies of a book for ₹ 4,400. He sold 600 at a profit of 20% and remaining copies at a loss of 25%. Find per cent profit/loss in the total transaction.*

Solution : Since C.P. of 800 copies is ₹ 4,400 each copy costs ₹ 5.50.

$$\text{C.P. of 600 copies is ₹ } (600 \times 5.50) = ₹\ 3{,}300$$

$$\text{S.P. of these copies at 20\% profit} = ₹\left(\frac{100+20}{100}\right) \times 3{,}300$$

$$= ₹\ 1.2 \times 3300$$

$$= ₹\ 3{,}960$$

$$\text{C.P. of remaining 200 copies} = ₹\ (200 \times 5.50)$$

$$= ₹\ 1{,}100$$

These copies are sold at a loss of 25%

$$\therefore \quad \text{S.P.} = ₹\left(\frac{100-25}{100}\right) 1{,}100$$

$$= ₹\ \frac{3}{4} \times 1{,}100 = ₹\ 825$$

$$\therefore \quad \text{Total S.P.} = ₹\ (3{,}960 + 825) = ₹\ 4{,}785$$

$$\therefore \quad \text{Profit} = \text{S.P.} - \text{C.P.}$$

$$= ₹\ (4{,}785 - 4{,}400) = ₹\ 385$$

when C.P. is ₹ 4,400, profit ₹ 385 | $\frac{100}{4{,}400} \times 385 = 8.75$

∴ when C.P. is ₹ 100, profit ₹ 8.75

∴ There is 8.75% profit in the total transaction.

Type 5 : To decide the marked price of an article when C.P., per cent profit and per cent commission is known.

Example 2.5 : *A dealer in furniture buys chairs at ₹ 340 each. At what price should he mark them for sale, so that he may earn a profit of 25% after giving 15% discount ?*

Solution : Let the marked price of a chair be ₹ x.

$$\therefore \quad \text{Discount at 15\%} = \frac{15\,x}{100}$$

$$\therefore \quad \text{Net S.P.} = x - \frac{15\,x}{100} = \frac{85\,x}{100} = \frac{17\,x}{20}$$

The chair costs ₹ 340 and 25% profit is to be realised.

$$\therefore \quad \text{Net S.P.} = ₹\left(\frac{100+25}{100}\right) \times 340 = ₹\frac{5}{4} \times 340 = ₹\ 425$$

Thus, $$\frac{17x}{20} = 425$$

$$\therefore \quad x = 425 \times \frac{20}{17} = 500$$

∴ Marked price of the chair should be ₹ 500.

Note : In this problem, we have seen how to find the M.P. of an article when the following are known :

(a) C.P. of the article.

(b) % profit to be realised.

(c) % discount to be given.

However, we give below an elegant formula to find M.P. at once :

$$\text{M.P.} = \left[\frac{100 + (\%\text{ profit})}{100 - (\%\text{ discount})}\right] \times \text{C.P.}$$

Thus in the above problem,

$$\text{M.P.} = \left(\frac{100+25}{100-15}\right) \times 340 = \frac{125}{85} \times 340 = 125 \times 4 = 500 \text{ rupees.}$$

Type 6 : Miscellaneous

Example 2.6 : *The profit realised by selling an article at ₹ 49.50 is 7/4 times the profit realised by selling it at ₹45. Find the C.P. of the article.*

Solution : Let the C.P. of the article be ₹ x.

∴ Profit when it is sold at ₹ 45. = ₹ (45 – x)

∴ Profit when it is sold at ₹ 49.50 = ₹ (49.50 – x)

By given information,

$$49.50 - x = \frac{7}{4}(45 - x)$$

$$\therefore \quad 198 - 4x = 315 - 7x$$

$$\therefore \quad 3x = 315 - 198 = 117$$

$$\therefore \quad x = 39$$

∴ C.P. of the article is ₹ 39.

Example 2.7 : $\frac{1}{17}$ *th of the cost price is* $\frac{1}{22}$ *of S.P. 10% of the C.P. and 5% of the S.P. differ by 12. Find the C.P. and S.P.* **(April 2018)**

Solution : Let x be the C.P. and y be the S.P. of article.

$$\therefore \quad \frac{x}{17} = \frac{y}{22}$$

i.e. $$22x = 17y \quad \ldots (i)$$

Also, $\frac{x}{10} = \frac{y}{20} + 12$ (why ?)

$\therefore \quad 2x = y + 240 \quad \dots \text{(ii)}$

From (i) and (ii)

$$22x - 17y = 0$$

and $22x - 11y = 240 \times 11$

Subtracting, $6y = 240 \times 11$

$\therefore \quad y = 440$

Substituting in equation (ii)

$$x = 340$$

Example 2.8 : *A man sold two machines at ₹990 each. On one, he gained 10 % and on the other, he lost 10%. Find percentage profit or loss in the total transaction.* **(April 2018)**

Solution : Let the C.P. of the machine on which he gained 10% be ₹ x.

By formula, $\text{S.P.} = \frac{100+10}{100}(\text{C.P.}) = \frac{11x}{10}$

$\therefore \quad \frac{11x}{10} = 990$

$\therefore \quad x = 900$

Let the C.P. of the machine on which he lost 10 % be ₹ y.

By formula, $\text{S.P.} = \frac{100-10}{100}(\text{C.P.}) = \frac{9y}{10}$

$\therefore \quad \frac{9y}{10} = 990$

$$9y = 9900$$

$$y = \frac{9900}{9} = 1100$$

$\therefore$ Total C.P. = ₹ (900 + 1100) = ₹ 2000

Total S.P. = 2 × 990 = ₹ 1,980

$\therefore$ Loss = ₹ 20

For C.P. of ₹ 2000, loss ₹ 20

$\therefore$ For C.P. of ₹ 100, loss ₹ 1

Thus there is 1% loss in the total transaction.

Example 2.9 : *Niraj sold a car to Sarthak at 15% profit. Sarthak sold the car to Yash at 5% loss for ₹43,700. Find the price at which Niraj has purchased the car.*

Solution : Let the C.P. of the car for Niraj be ₹ x. Niraj sold the car to Sarthak at 15% profit.

$\therefore$ S.P. of Niraj $= \frac{115x}{100}$

This is C.P. for Sarthak.

Sarthak sold the car at 5% loss.

Thus, when C.P. is 100, S.P. is 95 | $\frac{115x}{100} \times 95 \times \frac{1}{100}$

$\therefore$ when C.P. is $\frac{115x}{100}$ S.P. is $\frac{10925}{10000}x$

This is given as 43700

$$\therefore \quad \frac{10925}{10,000}x = 43700$$

$$\therefore \quad \frac{437 \times 25}{10,000}x = 43700$$

$$\therefore \quad \frac{x}{400} = 100$$

$$\therefore \quad x = 40,000$$

$\therefore$ Niraj had purchased the car for ₹ 40,000.

Example 2.10 : *Ravi sold two machines at ₹960 each. On one he gained 20% and on the other, he lost 20%. Find percentage profit or loss in his total transaction.*

Solution : The S.P. of each of the machines is ₹ 960.

On one of the machines, he gained 20%.

When S.P. is 120, C.P. is 100 | $\frac{960}{120} \times 100 = 800$

$\therefore$ When S.P. is 960, C.P. is 800.

On the other machine, he lost 20%.

When S.P. is 80, C.P. is 100 $\frac{960}{80} \times 100 = 1200$

$\therefore$ When S.P. is 960, C.P. is 1200.

$$\text{Thus, Total C.P.} = 800 + 1200 = 2000$$

$$\text{Total S.P.} = 2 \times 960 = 1920$$

$$\therefore \quad \text{Loss} = \text{C.P.} - \text{S.P.} = 2000 - 1920 = 80$$

For C.P. of ₹ 2000, loss is ₹ 80 $\Rightarrow \frac{100 \times 80}{2000}$

$\therefore$ For C.P. of ₹ 100, loss is ₹ 4.

$\therefore$ % loss in total transaction is 4.

Example 2.11 : *A trader offers 25% discount on the catalogue price of a bicycle and yet makes 20% profit. If he gains ₹200 per bicycle, what must be the catalogue price of the bicycle ? (Take C.P. = ₹1,000).*

Solution : Let the catalogue price of the bicycle be ₹ x.

$$\therefore \quad \text{Discount} = \frac{25x}{100} = \frac{x}{4}$$

$$\therefore \quad \text{Net S.P.} = x - \frac{x}{4} = \frac{3x}{4}$$

In this he gains ₹ 200.

$$\therefore \quad \text{C.P.} = \frac{3x}{4} - 200$$

$$\therefore \quad 1000 = \frac{3x}{4} - 2000$$

$$\therefore \quad 4000 = 3x - 8000$$

$$\therefore \quad 3x = 12000$$

$$\therefore \quad x = 4000$$

∴ Catalogue price of bicycle is ₹ 4000.

Example 2.12 : *A car having cost ₹ 1,00,000 was sold for ₹ 80,000 after 4 years. Find percentage of loss.*

Solution :

$$\begin{aligned} \text{Loss} &= \text{C.P.} - \text{S.P.} \\ &= 1,00,000 - 80,000 \\ &= 20,000 \end{aligned}$$

For a C.P. ₹ 1,00,000 loss ₹ 20,000.

∴ For C.P. ₹ 100, loss is 20.

Hence % loss is 20.

Example 2.13 : *Mr. Prakash Paranjape gets 15% commission upto the sale of ₹ 40,000 and 20% on the sale exceeding ₹40,000. In a year his sales are ₹85,000. Find his commission.*

Solution : Since sales are of ₹ 85,000, Excess sales over ₹ 40,000 is ₹ 45,000.

For sales of first ₹ 40,000, commission @ 15% = ₹ 6000.

For sales of ₹ 45,000, commission @ 20% = ₹ 9,000.

∴ Total commission = ₹ 15,000

Example 2.14 : *An article is sold for ₹ 380 after allowing a trade discount of 20% and subsequently a cash discount of 5%. Find percentage of his sales.*

Solution : (There is printing mistake in last sentence. However, we take it as "find catalogue price of the article").

Let x be the catalogue price of the article.

$\therefore$ Trade discount @ 20% $= \frac{20x}{100} = \frac{x}{5}$.

$\therefore$ Price after trade discount $= x - \frac{x}{5} = \frac{4x}{5}$.

Cash discount 5% is given on this.

$$\therefore \quad \text{Cash discount} = \frac{4x}{5} \times \frac{5}{100} = \frac{4x}{100} = \frac{x}{25}$$

$$\therefore \quad \text{Net S.P.} = \frac{4x}{5} - \frac{x}{25} = \frac{19x}{25}$$

But net S.P. is given as ₹ 380.

$$\therefore \quad \frac{19x}{25} = 380 \quad \therefore \frac{x}{25} = 20 \quad \therefore x = 500$$

Hence catalogue price is ₹ 500.

Example 2.15 : *A car costing ₹ 3,50,000 was sold for ₹ 2,85,000 after two years. Find the percentage of loss.*

Solution : Given :

$$\text{C.P.} = ₹\,3{,}50{,}000$$

$$\text{S.P.} = ₹\,2{,}85{,}000$$

$$\because \quad \text{S.P.} < \text{C.P.}$$

$$\therefore \quad \text{Loss} = \text{C.P.} - \text{S.P.}$$

$$\text{Loss} = 3{,}50{,}000 - 2{,}85{,}000$$

$$\text{Loss} = 65{,}000$$

$\therefore$ On C.P. of ₹ 3,50,000 loss ₹ 65,000.

$$\text{Percent loss} = \frac{\text{Loss}}{\text{C.P.}} \times 100$$

$$= \frac{65000}{350000} \times 100$$

$$= 18.57\% \text{ loss}$$

$\therefore$ There is 18.57% loss.

Example 2.16 : *A T.V. set costing ₹ 12,000 was sold after one year to ₹ 13500. Find the percentage profit.*

Solution : Given :

$$\text{C.P.} = ₹\,12{,}000$$

$$\text{S.P.} = ₹\,13{,}500$$

Here, S.P. > C.P.

$$\therefore \quad \text{Profit} = \text{S.P.} - \text{C.P.}$$

$$\text{Profit} = 13500 - 12000$$

$$\text{Profit} = 1500$$

$\therefore$ On C.P. ₹ 12,000 there is profit of ₹ 1500.

$$\therefore \quad \text{Percent profit} = \frac{\text{Profit}}{\text{C.P.}} \times 100$$

$$= \frac{1500}{12000} \times 100$$

$$= 12.5\%$$

$\therefore$ There is 12.5% profit.

Example 2.17 : *A goods worth ₹10000 are purchased. What should be S.P. to earn a profit of 10% ?*

Solution : Given : C.P. = ₹ 10,000,

Profit = 10%

The ern 10% profit on article costing ₹ 100 should be sold for ₹ 110.

$\therefore$ When C.P. is ₹ 100, S.P. is ₹ 110.

$\therefore$ When C.P. is ₹ 10,000 S.P. is x.

$$\therefore \quad \text{S.P.} = \left(\frac{100 + y}{100}\right) \times 10000$$

$$= \frac{110}{100} \times 10000$$

$$= 11000$$

$\therefore$ When C.P. is ₹ 10,000, S.P. is ₹ 11000.

$\therefore$ Goods should be sold for ₹ 11000.

Example 2.18 : *A person buys a motor cycle for ₹50000 and sells it at a loss of 20%. What is the selling price of the motor cycle ?* **(April 2016)**

Solution : A loss of 20% means if the motor cycle was purchased for ₹ 100, it is sold for ₹ 80.

$\therefore$ To find the selling price of 20% loss.

When C.P. is ₹ 100, S.P. is ₹ 80.

$\because$ C.P. is ₹ 50000, S.P. is ₹ x.

$$\therefore \quad \text{S.P. at 20\% loss} = \left(\frac{100 - y\%}{100}\right) x$$

$$= ₹\left(\frac{100 - 20}{100}\right) \times 50000$$

$$= ₹\left(\frac{80}{100} \times 50000\right)$$

$$= ₹\ 80 \times 500$$

$$= ₹\ 40000$$

$\therefore$ Selling price of the motor cycle is ₹ 40,000/-.

2.3 Introduction to Commission and Brokerage (April 2015)

Sale and purchase of various commodities form an important part of business and commerce. However, it is not possible for the manufacturer of goods to approach customers directly. The goods are generally sold through middlemen called agents or brokers. Our Government encourages industrialization in rural and backward areas. But then how to arrange the markets of such goods ? The middlemen look after this process. They are called wholesale dealers, retailers, agents etc. This arrangement suits the producer also since he can concentrate on the production of his goods.

The remuneration paid by manufacturer to the agent is called commission. The rate of commission can be fixed on the entire sale or it may vary after certain amount of sale (as an incentive). Sometimes, apart from commission, a fixed amount is also paid to the agent per month or per year.

In modern times, we come across various types of agents, involved in activities such as selling tea, arranging accommodation, arranging sale and purchase of shares etc.

A commission agent involved in selling of goods, normally gets commission from manufacturer only, but commission agents involved in selling shares or vehicles charge commission to both the parties, viz buyer and seller. This type of commission is called brokerage. **Note that commission is always paid on selling price.**

Solved Examples

Example 2.19 : *A commission agent gets 12 % commission upto a sale of ₹ 30,000/- and 15 % on the sales exceeding ₹ 30,000/-. In a month, his sales are ₹ 67,000/- find his commission.*

Solution : The amount of commission on ₹ 30,000/- @ 12 % is ₹ 3600. (April 2016)

Since the total sale is ₹ 67,000/- the sale exceeding ₹ 30,000/- is worth ₹ 37,000/-.

The amount of commission on ₹ 37000/- @ 15 % is ₹ 4050.

Hence, the total commission earned by the agent is ₹ (3600 + 4050) = ₹ 7650/-.

Example 2.20 : *The rate of commission is increased from 5 % to 8 %; still the income of an agent remains the same. Find the percentage change in his sales.*

Solution : Suppose that originally the sale of the agent was ₹ 100/-.

∴ His commission, then @ 5 % was ₹ 5.

Let the sale of the agent, now be ₹ x.

∴ His commission, @ 8 % would be $\frac{8x}{100}$.

This is same as his earlier commission.

$$\therefore \quad \frac{8x}{100} = 5$$

$$\therefore \quad x = \frac{500}{8} = 62.5$$

Thus now his sale is ₹ 62.50.

∴ There is a reduction of (100 − 62.50) = 37.5 % in his sales.

Example 2.21 : *A commission agent is paid a fixed monthly income plus commission on the sales at a fixed percent of sales. If in two successive months, his incomes are ₹4406 and ₹5555 respectively and the sales in these two months are ₹46350 and ₹65,500, find his monthly salary and rate of commission.*

Solution : Let the monthly salary of the agent be ₹ x and rate of commission be y %.

∴ On the sale of ₹ 46350, his total income would be

$$x + 463.5\,y = 4406 \qquad \text{(given)} \ldots (1)$$

Similarly, his income on sale of ₹ 65,500 would be

$$x + 655\,y = 5555 \qquad \ldots (2)$$

Subtracting (1) from (2)

$$191.5\,y = 1149$$

$$\therefore \quad y = \frac{1149}{191.5} = 6$$

Putting this value in equation (2),

$$x + (655)(6) = 5555$$

$$\therefore \quad x + 3930 = 5555$$

$$\therefore \quad x = 5555 - 3930 = 1625$$

Thus, the monthly salary is 1625 and rate of commission is 6 %.

Another Method : Since monthly salary is fixed, the rise in income has taken place due to the rise in sales only.

Rise in income = ₹ (5555 – 4406) = ₹ 1149

Rise in sales = ₹ (65500 – 46350) = ₹ 19150

Thus, for a sale of ₹ 19150, the commission is ₹ 1149

∴ for a sale of ₹ 100 the rise is 6.

∴ Rate of commission is 6 %.

∴ On the sale of ₹ 65,500, the commission is ₹ 3930.

∴ Salary = Total income – Commission

= 5555 – 3930

= ₹ 1625

Example 2.22 : *The salary of a salesman was reduced from ₹3000 to ₹2500 but his rate of commission increased from $2\frac{1}{2}$ % to 3 %. Due to this, his income increased by ₹100 than in previous month. Find his sales in the month.*

Solution : The salary of the salesman is decreased by ₹ 500/- still his income increases by ₹ 100. This means there is net rise of ₹ 600/- in his income. This has taken place due to rise in rate of commission which is $\frac{1}{2}$ %.

Thus $\frac{1}{2}$ % commission corresponds to sales of ₹ 600.

∴ 100 % commission corresponds to sales of ₹ 1,20,000. Thus his sales in the month is ₹ 1,20,000.

Example 2.23 : *Three scooters were sold through an agent for ₹ 20,000/- ₹ 16,800 and ₹ 15,000 respectively. The rate of commission were 15 % on the first and 12 % on the second. If, on the whole, the agent received a commission of 14 %, find the commission received by him on the third scooter.*

Solution : The commission on first scooter @ 15 % = 3000 and the commission on second scooter @ 12 % $= 16800 \times \frac{12}{100} = 2016$.

∴ Total of commission on two scooters = 5016.

The total S.P. of three scooters

$$= ₹\,(20000 + 16800 + 15000)$$

$$= ₹\,51800$$

The agent receives a commission on this at 14 %.

∴ Total commission

$$= 51800 \times \frac{14}{100} = ₹\,7252$$

∴ Commission on third scooter

$$= ₹\,(7252 - 5016) = ₹\,2236.$$

The third scooter is sold for ₹ 15000.

∴ Rate of commission is $100 \times \frac{2236}{15000}$

$$= 14.91\ \%\ \text{(approx.)}$$

Example 2.24 : *A commission agent gets 12%. Commission upto sales ₹25,000 and 15% on the sales exceeding ₹25,000. In a month, his sales are ₹67,000. Find the commission.*

(April 2015)

Solution : Given :

(i) Commission % = 12%.

(ii) Sales upto ₹ 25000/-.

(iii) 15% on the sales exceeding ₹ 25,000.

(iv) Sale in a month is ₹ 67000/-.

Find : Commission = ?

The amount of commission on ₹ 30,000 @ 12% is $25000 \times \frac{12}{100} = ₹\,3000$.

Since the total sale is ₹ 67000 the sale exceeding ₹ 25000 is worth

$$67000 - 25000 = 42000/-$$

The amount of commission on ₹ 42000 @ 15% is

$$42000 \times \frac{15}{100} = ₹\ 6300$$

Hence the total commission earned by the agent is

$$₹\ 3000 + ₹\ 6300 = ₹\ 9300/-$$

The total commission earned by the agent is ₹ 9300/-.

Example 2.25 : *A commission agent gets 10%. Commission upto sale of ₹ 50,000 and 13% on the sale exceeding ₹ 50,000. In a month his sale are ₹ 75,000. Find his commission.*

(April 2016)

Solution : The amount of commission ₹ 50,000 @ 10% is

$$50000 \times \frac{10}{100} = ₹\ 5000$$

Since the total sale is ₹ 75000/- the sale exceeding ₹ 50,000 is worth

$$75000 - 50000 = ₹\ 25000/-$$

The amount of commission on ₹ 25000 @ 13% is

$$\frac{25000}{100} \times 13 = ₹\ 3250/-$$

Hence the total commission earned by the agent is

$$₹\ 5000 + ₹\ 3250 = ₹\ 8250$$

∴ The total commission earned by the agent is ₹ 8250/-.

Example 2.26 : *An agent was paid ₹ 30000 as commission on the sale of vehicles. If the commission was 10% and price of each vehicle is ₹ 15000/-. Find how much vehicles did he sell.*

(Oct. 2016)

Solution : Given : Commission = ₹ 30000, Commission % = 10%,

Price of each vehicles = ₹ 15000/-

Find : Number of vehicles sold = ?

Commission paid to the agent @ 10% is ₹ 30000/-

$$\therefore \quad \text{Total sales price} = ₹\ 30000 \times \frac{100}{10}$$

$$= ₹\ 300000$$

$$\therefore \quad \text{Number of vehicle sold by agent} = \frac{\text{Total sales price}}{\text{Price of each vehicle}} = \frac{300000}{15000} = 20$$

∴ Number of vehicle sold by the agent = 20.

Example 2.27 : *The rate of commission is increased from 5% to 9% still the income of an agent remains the same. Find the percentage change in his sales.* (May 2007)

Solution : Given : The rate of commission is increased from 5% to 9%.

Find : Parentage change in the sale

Suppose that originally the sale of agent was 100/-.

∴ His commission is @ 5 is ₹ 5

Let the sale of agent now be ₹ x

∴ The commission is @ 9% would be $\frac{9x}{100}$.

This commission is same as its earlier commission.

$$\frac{9x}{100} = 5$$

$$x = \frac{5 \times 100}{9} = 55.56$$

Thus now his sale is ₹ 55.56.

∴ There is reduction of 100 – 55.56 = 44.44 in his sale.

∴ Percentage change in agent's sale is 44.44%.

Exercise (2.1)

Theory Questions :

1. Define the following terms :
   (i) Cost price. (ii) Market price. (iii) Selling price.
2. Explain : Trade discount and Cash discount.
3. Explain the terms : Commission and Brokerage.

Exercise (2.2)

Numerical Problems :

Discount :

1. Fill in the blanks :
   (i) The C.P. of an article should be .... times S.P. if there is 25% profit.
   (ii) The S.P. of an article is $\frac{6}{5}$ times the C.P.
   ∴ The profit is .... %.
   (iii) An article when sold for ₹ 20, yields 25% profit. If it is sold for ₹ 24, the profit would be .... %
   (iv) A man sold 12 pens for the C.P. of 15 pens.
   ∴ Profit is .... %.

(v) A dealer sells 25 chairs for the cost price of 20 chairs.

∴ He losses .... %.

(vi) If an article is marked 10% above C.P. and if 10% discount is given, there is ...... (profit, loss, neither profit nor loss).

2. A piano is sold for ₹ 42,500 at a loss of 15%. For how much should it have been sold to earn a profit of 15% ?

3. 60 litres of diesel is bought at ₹ 8 per litre. If 10% is lost in transit, at what rate should the remainder be sold to earn 10% on the whole ?

4. A dealer bought a T.V. set and music system for ₹ 50,000. He sold T.V. set at a gain of 20% and music system at a loss of 10%. He gained 2% on the whole. Find the C.P. of T.V. set.

5. The cost of printing 2000 copies of a book is as follows :

Paper : ₹ 6,400

Printing : ₹ 4,000 per thousand copies

Binding : ₹ 280 per thousand copies.

If the publisher allows 15% discount to the bookseller and realizes a profit of $27\frac{1}{2}$ %, what is the S.P. of each copy ?

6. A trader allows trade discount at 25% off the list price and further 5% for cash. How much an article costing ₹ 100, he should mark, so as to realise net profit of 14% on the cost ?

7. 10 kg tea of quality 'A' costing ₹ 70 per kg and 20 kg tea of quality B at ₹ 65 per kg are mixed. What should be the rate of selling the mixture to earn a profit of 20% ?

8. A manufacturer sells his article at 20% profit to the wholesaler. The wholesaler sells it to the retailer at 25% and the retailer sells it to the customer at 40% for ₹ 175. Find the C.P. to the manufacturer.

9. Material worth ₹ 5,000 was required for making 10 chairs. Labour charges were ₹ 2,500. What should be the S.P. of each chair to realise a profit of $33\frac{1}{3}$ % ?

10. A, B, C are partners in a business with capitals ₹ 50,000, ₹ 40,000 and ₹ 30,000 respectively gets 20% of the profit for managing the business and rest is divided in the ratio of their capitals. At the end of the year, C gets ₹ 4,000 more than B. Find the profit and share of each.

11. A dealer purchased two machines for a total of ₹ 22,000. He sold one of them at a gain of 6% and the other at a loss of 5%, but then he finds that he has neither gain nor loss in the total transaction. Find C.P. of the machines.

12. An article when sold for ₹ 875 resulted in a loss of 30% ₹ 150 were paid towards transport and octroi. What was the original C.P. of the article ?

13. A merchant purchased rice worth ₹ 16,500. 1/3 of the rice was partly damaged and had to sold at 10% loss. Find the percentage of profit at which he should sell the remaining stock so that he may make 20% on the whole.

14. A VCR is sold at a profit of 20%. If the C.P. and S.P. would each be less by ₹ 1,000, there would be an increase in profit by $\frac{5}{3}$ %. Find the C.P. of the VCR.

15. A retailer is given 15% trade discount and further 5% cash discount. What is the list price of an article for which retailer pays ₹ 1,615 in cash ?

16. By selling a fruit juice at 15 the trade losses 10%. At what is should be sold to again 10% ?

(April 2016)

17. Cost price of an article is ₹ 20,000. What should be the market price in orders to earn 20% profit after allowing 4% discount.

(April 2016)

18. Mr. Sunil sold his scooter at a loss of 10%. If he had sold it for ₹ 1200 more, he would have gained 2%. Find the cost price of the scooter.

(April 2016)

Commission and Brokerage :

19. A merchant asked his agent to sell 350 shirts on 6 % commission and to invest the balance in purchasing sportswear. The agent charged 9 % commission on the purchase and earned ₹ 10,122/- on the two transactions. At what price was each shirt sold ?

20. A car was bought for ₹ 86,000/- and sold for ₹ 92,000/- through a broker who charges commission of 2 % on purchase and 3 % on sales. Find the total gain on the transactions.

21. An agent is paid commission at 8 % on cash sales and 6 % on credit sales made by him. If 35 % of his sales are for cash and rest on the credit, find the average rate of commission earned by him.

22. An agent was paid ₹ 19,440/- as commission on the sale of T.V. sets. If the rate of commission was 12 % and the price of each set, ₹ 10,800/-, how many sets did he sell ?

23. An agent buys 500 pens and sells them at 20 % profit. The agent charges commission of 10 % on the purchase and 20 % on the sales. If his total commission is ₹ 2040/-, find the S.P. of each pen.

24. A salesman is paid a fixed monthly salary plus a commission based as a percentage on the sales made by him. If on the sales of ₹ 64000/- and 72000/-, in two successive months, he receives in all ₹ 10650/- and 11450/- respectively find his monthly salary and rate of commission paid to him.

25. A house is sold at 25 % profit. The amount of brokerage at 3/4 % comes to ₹ 5250/-. Find the cost of the house.

26. An insurance agent gets a commission of 25 % on the first years, 20 % on second years and 10 % on each of the subsequent years premia on a life insurance policy. If the rate of premium is ₹ 62.5 per thousand and amount of policy is 20,000/- find his total income for which 5 premia have been paid.

27. By selling a plot of land through an agent, the owner received. ₹ 7,12,500 net. If the agent charged commission at 5 %, what was the cost of the land ?

28. The income of a broker remains unchanged though rate of commission is increased from 8 % to 12 %. Find the percentage reduction in his business.

29. A trader offers 25 % discount on the catalogue price of a bicycle and yet makes 20 % profit. If he gains ₹ 200 per bicycle, what must be the catalogue price of the cycle ?

30. After deducting commission at $7\frac{1}{2}$ % on first ₹ 10,000 and 5 % on the balance of sales made by him, an agent remits ₹ 33950 to his principal. Find the value of goods sold by him.

31. An article was sold at a loss of 3%. Had it been sold for ₹ 35 more, there would be a gain of 4%. Find the cost price of the article.

32. A man sells two plots for ₹ 98,560 each. On one, he gets 12% profit and on the other he loses 12%. Find his gain/loss percent.

33. At what price should an article costing ₹ 510 be sold so that after giving 15% cash discount, a profit of 20% is made ?

34. A man sold his radio-set at a loss of 20%. If he would have sold it for ₹ 90 more, his loss would have been only 10%. Find C.P. of the radio-set.

35. A dealer sells two qualities of washing powders at ₹ 18 and ₹ 14 per kg respectively making a profit of 8% and 5% respectively. How, should he mix the two if he desires to sell the mixture at ₹ 17 per kg to earn a profit of $10\frac{1}{2}$ % ?

36. At what price should goods costing ₹ 48,250 be sold though an agent so that after paying him a commission of $3\frac{1}{2}$ % on sales, a net gain of 20% be made ?

37. An agent is paid a commission of 12% on cash sales and 8% on credit sales. If on the sales of ₹ 1,75,000, the agent receives a total commission of ₹ 16,880, find sales made by him in cash and on credit.

38. The cost price of a book to the book seller is ₹ 75.00. What should be the marked price of the book, so that after allowing a discount of $7\frac{1}{2}$ % to the customer, he realizes a profit of 20%.

39. A salesman gets fixed monthly salary plus commission based on the sales. In two successive months he received ₹ 16000 and ₹ 16500. On the sales of ₹ 3,00,000 and ₹ 3,50,000 respectively. Find his monthly salary and the rate of commission.

40. An agent is paid commission at following rates : 16% on sand, 10% on wood and 8% on cement. If he sold these goods in the ratio 8 : 5 : 7, what is the average rate of commission ?

41. A salesman receives 4% commission on the sales upto ₹ 10,000 and 5% commission on the sales over ₹ 10,000. Find his total income on a sale of ₹ 35,000. (April 2018)

42. Mr. Sunil gets a commission of 8% on cash sales and 6% on credit sales. If he receives ₹ 1500 as commission on total sales of ₹ 24,000/-, find the sales made by him in cash or credit. (April 2017)

Answers (2.2)

(1) (i) $\frac{4}{5}$ (ii) 20 (iii) 50 (iv) 25 (v) 20 (vi) loss.

(2) ₹ 57,500 (3) ₹ 9.78 (approx.) (4) ₹ 20,000 (5) ₹ 15 (6) ₹ 160

(7) ₹ 80 per kg (8) ₹ 83.33 (9) ₹ 1,000 (10) ₹ 30,000, ₹ 10,000, ₹ 8,000, ₹ 12,000

(11) ₹ 12,000, ₹ 10,000 (12) ₹ 1,100 (13) 35% (14) ₹ 13,000 (15) ₹ 2,000

(19) ₹ 200	(20) ₹ 1520	(21) 6.7 %
(22) 15	(23) ₹ 14.40	(24) ₹ 4250 and 10 %
(25) ₹ 5,60,000	(26) ₹ 937.50	(27) ₹ 7,50,000
(28) 33.33	(29) ₹ 1600	(30) ₹ 36,000
(31) ₹ 500	(32) 1.46% loss	(33) ₹ 720
(34) ₹ 900	(35) 8 : 5	(36) ₹ 60,000

(37) ₹ 72,000, ₹ 1,04,000 (38) ₹ 97.30 (approx.) (39) ₹ 13,000, 1% (40) 11.7%.

Chapter 3...

Interest and Annuity

Contents ...

Key Words :

Simple Interest, Compound Interest, Nominal and Effective Rates of Interest, Annuity, EMI.

Objectives :

To understand difference between effective and nominal rate of interest. To enable to calculate EMI.

3.1 Simple Interest

When a person borrows some amount of money from other person or institution, the borrower has to pay some *charge* to the lender for the use of the money. This charge is called *interest.* The interest depends on two things viz., the period for which the money is borrowed and secondly the rate of interest.

The sum borrowed is called *principal,* time for which it is borrowed is called *term.* The total sum returned by the borrower i.e. principal together with interest is called *amount.*

When interest is calculated on the original principal, whatever the term may be, it is called as *simple interest.*

If P denotes the principal,
n denotes term in years,
r denotes rate of interest % per annum (p.c.p.a.),
I denotes simple interest,

then we have a simple formula $\boxed{I = \frac{P.r.n.}{100}}$

amount A is found by adding interest to the principal.

Thus, $A = P + I = P + \frac{P.r.n.}{100} = P\left(1 + \frac{r.n.}{100}\right)$

Solved Examples

Example 3.1 : *Find the simple interest on ₹1,250 for $2\frac{1}{2}$ years at 12% p.a.*

Solution : Here P = ₹ 1,250 ; $n = \frac{5}{2}$; r = 12

Now, $$I = \frac{P.r.n.}{100} = \frac{1250 \times \frac{5}{2} \times 12}{100} = 375$$

∴ Simple interest is ₹ 375.

Example 3.2 : *A sum of money borrowed on 26th March is repaid on 7th June in the same year. If the simple interest paid is ₹75.75 at 7.5% p.a., find the sum borrowed.*

Solution : Let the sum borrowed be ₹ P.

Given I = ₹ 75.75, r = 7.5

Let us calculate the period for which interest is charged.

Month	March	April	May	June	Total
Number of days	5	30	31	7	73

∴ Interest is charged for 73 days.

$$n = \frac{73}{365} = \frac{1}{5} \text{ year} \qquad (\because 1 \text{ year} = 365 \text{ days})$$

Now, $$I = \frac{P.r.n.}{100}$$

∴ $$75.75 = \frac{P \times \frac{1}{5} \times 7.5}{100}$$

∴ $$P = 5050$$

i.e. Sum borrowed is ₹ 5,050.

Example 3.3 : *A sum of ₹ 4,800 amounted to ₹ 6,240 in a certain period. If the rate of simple interest is 12% p.a., find the period.*

Solution : Here P = ₹ 4,800; A = ₹ 6,240 ; r = 12% p.a. To find n.

Now, $$A = P\left(1+\frac{r.n.}{100}\right)$$

$$\therefore \quad 6240 = 4800\left(1+\frac{12\,n}{100}\right)$$

$$\frac{6240}{4800} = 1+\frac{12\,n}{100}$$

$$\therefore \quad \frac{13}{10} = 1+\frac{12\,n}{100}$$

$$\therefore \quad \frac{3}{10} = \frac{12\,n}{100}$$

$$\therefore \quad n = \frac{5}{2}$$

Thus, the period is 2.5 year.

Example 3.4 : *A sum of money doubles itself in 10 years. Find the rate of simple interest.* **(April 2005)**

Solution : Let P be the principal.

$\therefore$ Amount $A = 2P$; $n = 6$. To find r.

Now, $$A = P\left(1+\frac{r\,n}{100}\right)$$

$$\therefore \quad 2P = P\left(1+\frac{10\,r}{100}\right)$$

$$\therefore \quad 2 = 1+\frac{10\,r}{100}$$

$$\therefore \quad 1 = \frac{10\,r}{100}$$

$$\therefore \quad 10r = 100$$

$$r = 10$$

Thus the rate of interest is 10% p.a.

Example 3.5 : *What sum will amount to ₹3,296 in 4 months at 9% p.a. simple interest ?*

Solution : Here $A = ₹\,3{,}296$; $r = 9$; $\left(n = \frac{4}{12} = \frac{1}{3}\right)$. To find P.

$$\therefore \quad A = P\left(1+\frac{r\,n}{100}\right)$$

$$\therefore \quad 3296 = P\left(1+\frac{9\times\frac{1}{3}}{100}\right)$$

$$\therefore \quad 3296 = P\left(\frac{103}{100}\right)$$

$$\therefore \quad P = \frac{3296\times 100}{103} = 3200$$

$\therefore$ Principal is ₹ 3,200.

Example 3.6 : *A person borrows ₹ 15,000, partly at 10% and remaining at 12%. If at the end of $2\frac{1}{2}$ years, he pays a total simple interest ₹ 4,050, how much did he borrow at each rate ?*

Solution : Suppose he borrows ₹ x at 10%.

$\therefore$ He borrows ₹ (15,000 – x) at a 12%

$$\therefore \quad \text{Total S.I. for } 2\tfrac{1}{2} \text{ years} = \frac{x \times \frac{5}{2} \times 10}{100} + \frac{(15000 - x) \times \frac{5}{2} \times 12}{100}$$

$$= \frac{25\,x + 30\,(15000 - x)}{100}$$

$$= \frac{25\,x + 30 \times 15000 - 30\,x}{100}$$

$$= 4500 - \frac{5\,x}{100}$$

But total S.I. is 4,050.

$$\therefore \quad 4500 - \frac{5\,x}{100} = 4050$$

$$\therefore \quad \frac{5\,x}{100} = 450$$

$$\therefore \quad x = 9{,}000$$

Thus he has borrowed ₹ 9,000 at 10% and hence ₹ 6,000 at 12%.

Example 3.7 : *S.I. on a sum at 4% p.a. for $1\frac{1}{2}$ years is less than that on the same sum at rate of $3\frac{1}{4}$ % p.a. for $2\frac{1}{2}$ years by ₹ 38.25. Find the sum.*

Solution : Let P be the sum. S.I. on P at 4% for $\frac{3}{2}$ years is

$$\frac{\text{P.n.r.}}{100} = \frac{P \times \frac{3}{2} \times 4}{100} = \frac{6\,P}{100} = I_1 \text{ (say)}$$

S.I. on P at $3\frac{1}{4}$ % for $\frac{5}{2}$ years is

$$\frac{\text{P.n.r.}}{100} = \frac{P \times \frac{5}{2} \times \frac{13}{4}}{100} = \frac{65\,P}{800} = I_2 \text{ (say)}$$

It is given that $I_1 < I_2$.

Moreover, $$I_2 - I_1 = 38.25$$

$$\therefore \quad \frac{65\,P}{800} - \frac{6\,P}{100} = 38.25$$

$$\therefore \quad \frac{65\,P - 48\,P}{800} = 38.25$$

$$\therefore \quad 17\,P = 38.25 \times 800$$

$$\therefore \quad P = 1800$$

Example 3.8 : *A certain sum of money is deposited in a bank annually for 5 years at 8% p.a. S.I. If at the end of 5 years, the amount standing to the credit of the depositor is ₹ 49,600, then find the amount deposited each year.*

Solution : Suppose the person deposits ₹ x every year. Then he will get interest on ₹ x deposited in the first year for 5 years, interest on ₹ x deposited in the second year for 4 years and so on. In other words, he will get interest on ₹ x for (5 + 4 + 3 + 2 + 1) years i.e. for 15 years.

$$\therefore \quad S.I. = \frac{P.n.r.}{100} = \frac{x \times 15 \times 8}{100} = \frac{12\,x}{10}$$

$$\text{Now, Amount} = P + I$$

$$\therefore \quad 49,600 = 5\,x + \frac{12\,x}{10}$$

$$\therefore \quad 49,600 = \frac{62\,x}{10}$$

$$\therefore \quad x = 8,000$$

∴ The person deposits ₹ 8,000 each year.

Example 3.9 : *What will be the simple interest on ₹ 1200 at 10% p.a. for 2 years ?* **(April 2009)**

Solution : $P = 1200, \; n = 2, \; r = 10$

$$S.I. = ?$$

$$S.I. = \frac{P.n.r.}{100}$$

$$= \frac{1200 \times 2 \times 10}{100}$$

$$S.I. = ₹\,240$$

Example 3.10 : *Find difference between compound interest and simple interest on ₹ 5000 for 3 years at 5% p.a. compounded half yearly.* **(April 2009, 2018)**

Solution : Let us first calculate the amount A, when interest is compounded half yearly.

$$P = 5000, \quad n = \text{No. of periods} = 3 \times 2 = 6$$

$$r = \frac{5}{2} \text{ (interest for 6 months)}$$

$$\text{Now,} \quad A = P\left(1 + \frac{r}{100}\right)^n$$

$$\therefore \quad A = 5000\left(1 + \frac{5/2}{100}\right)^6$$

$$A = 5000\left(\frac{205}{200}\right)^6 = 5000\,(1.025)^6 = 5798.47$$

$$\text{Now,} \quad C.I. = A - P$$

$$= 5798.47 - 5000$$

$$= ₹\,798.47$$

Now, let us calculate simple interest

$$S.I. = \frac{P.r.n.}{100} = \frac{5000 \times 5 \times 3}{100}$$

$$= ₹\ 750$$

$\therefore$ Difference between C.I. and S.I.

$$= C.I. - S.I. = 798.47 - 750$$

$$= ₹\ 48.47$$

Example 3.11 : *In what period a sum of ₹ 10,000 will earn a simple interest of ₹ 3200 8% p.a. ?* **(April 201**

Solution : P = 10,000, r = 8, S.I. = 3200, n = ?

$$S.I. = \frac{P.r.n.}{100}$$

$$\therefore \quad 3200 = \frac{10,000 \times 8 \times n}{100}$$

$$\therefore \quad 32 = 8n$$

$$\therefore \quad n = 4$$

$\therefore$ 4 years is required period.

Example 3.12 : *What sum of money will amount to ₹ 43,200 at 5% p.a. simple interest f years ?* **(April 20**

Solution : Here A = ₹ 43,200, r = 5, n = 4. To find P.

$$A = P\left(1 + \frac{rn}{100}\right)$$

$$\therefore \quad 43200 = P\left(1 + \frac{5 \times 4}{100}\right)$$

$$\therefore \quad 43200 = P\left(1 + \frac{1}{5}\right)$$

$$\therefore \quad 43200 = P\left(\frac{6}{5}\right)$$

$$\therefore \quad P = \frac{5}{6} \times 43200 = 36000$$

$\therefore$ ₹ 36000 will amount to ₹ 43200 in 4 years.

Example 3.13 : *Find the simple interest on ₹ 6000 at 7 p.c.p.a. for 10 months.* **(April 20**

Solution : Given : $P = 6000, \; r = 7, \; n = \frac{10}{12}$

$$S.I. = \frac{P.r.n.}{100} = \frac{6000 \times 7 \times 10}{100 \times 12}$$

$$= ₹\ 350$$

Example 3.14 : *Find the simple interest on ₹40,000 for 5 years at 12% p.a.*

Solution : Here P = ₹ 4000, n = 5 years, r = 12.

$$\text{Simple interest} = I = \frac{n \cdot P \cdot r}{100}$$

$$I = \frac{40000 \times 5 \times 12}{100} = 24000$$

Example 3.15 : *A sum of ₹3000 amount of ₹3960 at 8% p.a. simple interest in a certain period. Find period.* **(Oct. 2009)**

Solution : Here P = 3000, r = 8, I = 3960, n = ?

$$\text{S.I.} = \frac{n \cdot P \cdot r}{100}$$

$$\therefore \quad 3960 = \frac{3000 \times n \times 8}{100} = 240 \times n$$

$$\therefore \quad n = \frac{3960}{240} = 16.5$$

$$\therefore \quad \text{Period} = 16.5 \text{ years} = 16\frac{1}{2} \text{ years}$$

Example 3.16 : *In what period of sum of ₹10000 will earn a simple interest of ₹3200 at 8% p.a. ?* **(April 2010)**

Solution : Given : P = 10000, S.I. = ₹ 3200, r = 8%, n = ?

$$\text{S.I.} = \frac{P \cdot n \cdot r}{100}$$

$$3200 = \frac{10000 \times n \times 8}{100}$$

$$3200 = 800 \times n$$

$$\therefore \quad n = \frac{3200}{800} = 4$$

$$\therefore \quad \text{Period } n = 4 \text{ years.}$$

Example 3.17 : *Find the rate of simple interest at which principal of ₹ 3000 earns on interest of ₹2600 in 5 years.* **(April 2011)**

Solution : Given : P = 3000, n = 5, S.I. = 2600. To find r.

$$\text{S.I.} = \frac{P \cdot n \cdot r}{100}$$

$$\therefore \quad 2600 = \frac{3000 \times r \times 5}{100} = 150r$$

$$\therefore \quad r = \frac{2600}{150} = 17.33$$

$\therefore$ Rate of simple interest = 17.33%.

3.2 Compound Interest

When interest is added to the principal at the end of each period, and the total so obtained is treated as the principal for the next period, then the interest so obtained is called *compound interest.*

For example, suppose Rasiklal starts an industry by taking a loan of ₹ 50,000 from a bank at 16% p.a. Bank charges interest quarterly. However, Rasiklal is not able to pay interest at the end of each quarter. Naturally, the bank will add the interest at the end of each quarter to the principal. Let us find how much Rasiklal has to pay towards interest at the end of 1st year.

Interest on ₹ 50,000 for 1st quarter = ₹ 2,000

Principal for 2nd quarter = 50,000 + 2,000

= 52,000

∴ Interest on ₹ 52,000 for 2nd quarter = 2,080

∴ Principal for 3rd quarter = 52,000 + 2,080

= 54,080

∴ Interest on ₹ 54,080 for 3rd quarter = 2,163.20

∴ Principal for 4th quarter = 54,080 + 2163.20

= 56,243.20

∴ Interest on ₹ 56,243.20 for 4th quarter = 2249.73

∴ Total interest that Rasiklal has to pay

= ₹ (2,000 + 2,080 + 2,163.20 + 2,249.73) = ₹ 8,492.93

However, it is not necessary to find principal for each period while solving problems.

If P is the principal, r the rate of interest p.a. and n, the period in years then amount A at the end of n years is given by;

$$\boxed{A = P\left(1 + \frac{r}{100}\right)^n}$$

Then the compound interest is given by;

$$\begin{aligned} C.I. &= A - P \\ &= P\left(1 + \frac{r}{100}\right)^n - P \\ &= P\left[\left(1 + \frac{r}{100}\right)^n - 1\right] \end{aligned}$$

Thus,

$$\boxed{C.I. = P\left\{\left(1 + \frac{r}{100}\right)^n - 1\right\}}$$

Note :

1. In general, the compound interest on a given sum will be greater than the simple interest on the same sum for the same period. (rate of interest being same)
2. In case of calculation of compound interest, it is convenient to find amount first and then C.I. is given by subtracting principal from the amount.
3. As we have to find powers of $\left(1 + \frac{r}{100}\right)^n$, use of log tables/calculators will be helpful.
4. The S.I. and C.I. for 1st year are same when interest is calculated annually.

Compound interest when period is different from a year :

Sometimes the period at the end of which interest is calculated is different from a year. It may be a month, a quarter, half year. In such a case, the same formula for amount viz.

$$A = P\left(1 + \frac{r}{100}\right)^n \text{ is valid.}$$

However, then n = Number of periods.

r = Rate of compound interest % *per period.*

For example, if we want to find compound interest on ₹ 1000 at 10 % p.a. for 3 years when interest is compounded half yearly,

$$n = \text{number of periods} = 6$$

$$r = \frac{10}{2} = 5$$

then,
$$A = P\left(1 + \frac{r}{100}\right)^n = 1000\left(1 + \frac{5}{100}\right)^6$$
$$= 1341$$

Compound Interest on a given sum in the n^{th} year :

We know that the compound interest for the n^{th} year will be given by

$$I_n = \text{C.I. in } n \text{ years} - \text{C.I. in } (n-1) \text{ years}$$

$$= P\left\{\left(1 + \frac{r}{100}\right)^n - 1\right\} - P\left\{\left(1 + \frac{r}{100}\right)^{n-1} - 1\right\}$$

$$= P\left\{\left(1 + \frac{r}{100}\right)^n - \left(1 - \frac{r}{100}\right)^{n-1}\right\}$$

$$= P\left(1 + \frac{r}{100}\right)^{n-1}\left\{\left(1 + \frac{r}{100}\right) - 1\right\} \qquad \left\{\begin{array}{l}\because x^n - x^{n-1} \\ = x^{n-1}(x-1)\end{array}\right\}$$

$$= P\left(1 + \frac{r}{100}\right)^{n-1}\left(\frac{r}{100}\right)$$

Thus,
$$\boxed{I_n = \frac{P.r.}{100}\left(1 + \frac{r}{100}\right)^{n-1}}$$

3.3 Problems on Growth and Decay

The formula for C.I. can also be used in the cases of uniform periodical increase or decrease at a constant rate. They are known as problems on growth and decay respectively.

For the problems on growth, the formulae for amount viz.

$A = P\left(1+\frac{r}{100}\right)^n$ gives the total quantity at the end of n periods when the rate of growth is r % per period.

For example, Consider the problem. "The population of a town is 40,000 and increase every year 1.8 % of the population at the beginning of that year. Find the population after 15 year".

Since increase in population in each year is taken as the base for next year, this is an illustration of compounding.

Here, $P = 40{,}000, \quad r = 1.8, \quad n = 15$

Now, $A = P\left(1+\frac{r}{100}\right)^n$

$$= 40{,}000\left(1+\frac{1.8}{100}\right)^{15} = 40{,}000\,(1.018)^{15}$$

$$= 40{,}000 \times 1.3 = 52{,}000$$

In the case of problems on decay, we replace $\frac{r}{100}$ by $-\frac{r}{100}$ in the formula for compound interest.

Thus for depreciated value at the end of n periods, we use the formula

$$V = P\left(1-\frac{r}{100}\right)^n$$

For example, a machine depreciates at rate of 20 % on the reducing balance. The original cost was ₹ 1,00,000, find its cost after $7\frac{1}{2}$ years.

Here initial value $P = 1{,}00{,}000, \quad r = 20, \quad n = \frac{15}{2}$

Now, $V = P\left(1-\frac{r}{100}\right)^n$

$$= 1{,}00{,}000\left(1-\frac{20}{100}\right)^{15/2}$$

$$= 1{,}00{,}000\,(0.8)^{15/2} = 1{,}00{,}000\,(0.1876)$$

$$= ₹\,18{,}760$$

Solved Examples

Example 3.18 : *Find the compound interest on ₹ 10,000 for 4 years at 5 % p.a.*

Solution : Here P = 10,000, r = 5, n = 4, To find C.I.

The amount of the end of 4 years is given by,

$$A = P\left(1+\frac{r}{100}\right)^n = 10{,}000\left(1+\frac{5}{100}\right)^4$$

$$= 10{,}000\,(1.05)^4 = ₹\,12{,}155$$

$\therefore$ C.I. = A − P = ₹ 2,155

Example 3.19 : *Find the rate % p.a. at which a sum of money trebles itself in 12 years.*

Solution : Let P be the sum which trebles itself in 12 years, at r % p.a.

We have
$$A = P\left(1+\frac{r}{100}\right)^n$$
$$\therefore \quad 3P = P\left(1+\frac{r}{100}\right)^{12}$$
$$\therefore \quad 3 = \left(1+\frac{r}{100}\right)^{12}$$
$$\therefore \quad \log 3 = 12\log\left(1+\frac{r}{100}\right)$$
$$\therefore \quad 0.4771 = 12\log\left(1+\frac{r}{100}\right)$$
$$\therefore \quad \log\left(1+\frac{r}{100}\right) = \frac{0.4771}{12} = 0.0398$$
$$\therefore \quad 1+\frac{r}{100} = \text{Antilog}\ (0.0398) = 1.096$$
$$\therefore \quad \frac{r}{100} = 0.096$$
$$\therefore \quad r = 9.6$$

$\therefore$ Required rate is 9.6 % p.a.

Example 3.20 : *The difference between the simple and compound interest on a certain sum of money for 4 years at 6 % p.a. is ₹168.75. What is the sum ?*

Solution : Let the sum be ₹ 100

$$\therefore \quad \text{S.I.} = \frac{100\times 6\times 4}{100} = ₹\ 24$$

By compound interest,
$$A = P\left(1+\frac{r}{100}\right)^n$$
$$= 100\left(1+\frac{6}{100}\right)^4 = 100\,(1.06)^4 = 126.25$$

$\therefore$ Interest = ₹ 26.25

$\therefore$ Difference between C.I. and S.I.
$$= ₹\ 26.25 - ₹\ 24 = ₹\ 2.25$$

For difference ₹ 2.25, the sum = ₹ 100

$\therefore$ For difference ₹ 168.75, the sum
$$= \left(\frac{100\times 168.75}{2.25}\right) = ₹\ 7{,}500$$

Example 3.21 : *A sum of money amounts to ₹ 2812.16 in 3 years and to ₹ 3041.50 in 5 years, find the sum and rate of interest.*

Solution : Given

$$A_3 = P\left(1+\frac{r}{100}\right)^3 = 2812.16 \quad \text{... (1)}$$

$$A_5 = P\left(1+\frac{r}{100}\right)^5 = 3041.50 \quad \text{... (2)}$$

$$\therefore \quad \frac{A_5}{A_3} = \left(1+\frac{r}{100}\right)^2 = \frac{3041.50}{2812.16} = 1.08155$$

$$\therefore \quad 1+\frac{r}{100} = \sqrt{1.08155} = 1.04$$

$$\therefore \quad 1+\frac{r}{100} = 1.04$$

$$\therefore \quad \frac{r}{100} = 0.04$$

$$\therefore \quad r = 4$$

Now

$$P\left(1+\frac{r}{100}\right)^3 = 2812.16 \quad \text{[From equation (1)]}$$

$$\therefore \quad P\left(1+\frac{4}{100}\right)^3 = 2812.16$$

$$\therefore \quad P = \frac{2812.16}{(1.04)^3} = 2500$$

The principal is ₹ 2500 and rate of interest is 4 % p.a.

Example 3.22 : *The depreciation on a machine is charged at 15 % p.a. on the diminishing balance method. In how many years will the original value of ₹ 1,50,000 be reduced to its scrap value estimated at 10 % of the cost ?*

Solution : Since the original value of the machine is ₹ 1,50,000, scrap value is ₹ 15,000 (10 %).

Suppose that it takes n years for the machine to reduce it to scrap value.

Now,

$$V = P\left(1-\frac{r}{100}\right)^n$$

i.e.

$$15{,}000 = 1{,}50{,}000\left(1-\frac{15}{100}\right)^n$$

$$\therefore \quad \frac{1}{10} = (0.85)^n$$

$$\therefore \quad \log\left(\frac{1}{10}\right) = n \log (0.85)$$

$$\bar{1}.0000 = n(\bar{1}.9294) = n(-0.0706)$$

$$\therefore \quad -1 = -0.0706\, n$$

$$\therefore \quad n = \frac{10000}{706} = 14.16 \approx 14 \text{ years and 2 months.}$$

Example 3.23 : *Find the difference between compound interest and simple interest on ₹500 for 2 years at 10% p.a. (Compounded yearly).* **(April 2010, 2015)**

Solution : Let us first find S.I.

$$P = 500, \quad n = ?, \quad r = 10$$

$$\text{S.I.} = \frac{\text{P.n.r.}}{100}$$

$$= \frac{500 \times 2 \times 10}{100}$$

$$\text{S.I.} = ₹\ 100$$

$$\text{Compound interest} = A - P$$

where, $$A = P\left(1 + \frac{r}{100}\right)^n$$

$$A = 500\left(1 + \frac{10}{100}\right)^2 = 500\,(1.1)^2 = 500\,(1.21)$$

$$A = ₹\ 605$$

$$\text{Compound interest} = A - P = 105$$

∴ The difference between C.I. and S.I. is

$$105 - 100 = ₹\ 5$$

Example 3.24 : *What sum will amount to ₹4000 in 3 years at 6 p.c.p.a. compound interest ?* **(April 2015)**

Solution : A = 4000, n = 3, r = 6.

To find : P.

$$A = P\left(\frac{1 + r}{100}\right)^n$$

∴ $$4000 = P\,(1.06)^3 = 1.1910\ P$$

$$P = \frac{4000}{1.1910}$$

∴ $$P = \frac{4000}{1.1910} = ₹\ 3358.5$$

Example 3.25 : *Find the amount of ₹4500 at 12% p.a. in 4 years, compounded half yearly.* **(April 2018)**

Solution : Given : P = 4500, r = 6 (since half yearly), no. of periods n = 8.

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 4500\left(1 + \frac{6}{100}\right)^8$$

$$= 4500\,(1.06)^8$$

$$\approx 4500 \times 1.6$$

$$\approx 7200 \text{ (approx)}$$

Example 3.26 : *Find compound interest on ₹ 4000 for the 5^{th} year when the rate of interest is 8% p.a.* **(April 2011)**

Solution : Given : P = 4000, r = 8.

Compound interest for 5^{th} year.

$= A_5 - A_4$ (A_n : amount of the end of n^{th} year)

$$= P\left\{\left(1+\frac{r}{100}\right)^5 - 1\right\} - P\left\{\left(1+\frac{4}{100}\right)^4 - 1\right\}$$

$$= P\left\{\left(1+\frac{r}{100}\right)^5 - 1\left(1+\frac{r}{100}\right)^4\right\}$$

$$= 4000\left\{\left(1+\frac{8}{100}\right)^5 - \left(1+\frac{8}{100}\right)^4\right\}$$

$$= 4000\,\{(1.085)^5 - (1.085)^4\}$$

$$= 4000\,\{(1.4694) - (1.3605)\}$$

$$= 4000 \times 0.10884 = 435.37$$

$\therefore$ Compound interest ₹ 435.37

Example 3.27 : *Find difference between compound interest and simple interest on ₹ 500 for 2 years at 10% p.a. (compounded yearly).* **(April 2010, 2013)**

Solution : Given : P = 500, n = 2, r = 10.

$$S.I. = \frac{P \cdot n\, r}{100} = \frac{500 \times 2 \times 10}{100} = 100$$

Now,

$$A = P\left(1+\frac{r}{100}\right)^n = 500\left(1+\frac{10}{100}\right)^2$$

$$= 500\left(\frac{11}{10}\right)^2 = 500\,(1.1)^2 = 500 \times 121 = 605$$

$\therefore$ C.I. = A − P = 605 − 500 = 105

No difference between C.I. and S.I.

$$= C.I. - S.I. = 105 - 100 = 5$$

$\therefore$ No difference between C.I. and S.I. is 5.

Example 3.28 : *Find interest on ₹ 15000 for 5 years at 8% p.a. compounded half yearly.* **(April 2009)**

Solution : Here, P = 15,000, $n = \frac{1}{2}$ years = 6 months, r = 8.

$$A = P\left(1+\frac{r}{100}\right)^n = 15000\left(1+\frac{8}{100}\right)^6$$

$$= 15000\left(\frac{108}{100}\right)^7 = 15000\,(1.08)^6$$

$$= 15000\,(1.5869) = 23803.1148$$

$$C.I. = A - P = 23803.1148 - 15000 = 8803.1148$$

$\therefore$ Compound Interest = ₹ 880.

Example 3.29 : *Find compound interest on ₹ 5,000 for three years at 5% p.a. compounded yearly.* **(April 2009)**

Solution : Here $n = \frac{1}{2}$ years = 6 months, P = 5000, r = 5%

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 5000\left(1 + \frac{5}{100}\right)^6 = 5000\left(\frac{105}{100}\right)^6$$

$$= 5000\,(1.05)^6 = 5000\,(1.3400)$$

$$= 6700.4782$$

$$C.I. = A - P = 6700.4782 - 5000 = 1700.4782$$

∴ Compound Interest 1700.4782.

Example 3.30 : *Find the compound interest on ₹5000 at 4% p.a. for 5 years.*

Solution : $A = P\left(1 + \frac{r}{100}\right)^n$

Here, P = ₹ 5,000, r = 4%, N = 5 years

∴ $A = 5000\left(1 + \frac{4}{100}\right)^5$

$$= 5000\left(\frac{104}{100}\right)^5 = 5000\,(1.04)^4$$

$$= 5000\,(1.2166529) = ₹\ 6083.264$$

$$C.I. = ₹\ 6083.264\ (-)\ ₹\ 5000$$

$$= ₹\ 1083.264$$

∴ Compound Interest = ₹ 1083.264

3.4 Nominal and Effective Rate of Interest

A reputed financial institution advertised its fixed deposit scheme as follows :

Tenure / Fixed deposit	12 months		24 months		36 months	
	Annual	Quarterly	Annual	Quarterly	Annual	Quarterly
Regular	8.70%	8.45%	9.10%	8.85%	9.40%	9.10%
For senior citizens	8.95%	8.70%	9.35%	9.10%	9.65%	9.35%

What do we observe ? Apparently annual rate of interest appears to be more. However, calculations only will help us in deciding which rate of interest is more beneficial.

Consider the case of regular fixed deposit for 12 months. For comparison, suppose a sum of ₹ 100 is deposited in annual as well as quarterly interest scheme. By annual interest scheme is 100 will amount to ₹ 108.70 (∵ rate of interest is 8.70% p.a.).

For quarterly interest scheme, the interest for the first quarter is ₹ $\frac{8.45}{4}$ = 2.1125 so t amount of the end of first quarter is ₹ 102.1125. On this amount, interest will be calculated the second quarter and so on. Thus, the amount at the end of one year, by compound inter formula is

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= 100\left(1+\frac{2.1125}{100}\right)^4 \qquad (\because \text{ 1 year } = 4 \text{ quart}$$

$$= 108.72$$

Thus, the amount by way of quarterly calculations exceeds (slightly in this case) tha annual interest. Note that the quarterly rate of interest appears to be less but the yield is m For bigger principal the difference in interest will be significant.

The quarterly rate mentioned (8.45%) is called nominal rate of interest and annualised re of 8.72% is called **effective rate of interest.**

Example 3.31 : *If the rate of interest is 10.5% p.a., what is the effective rate if compoun is done (i) quarterly ? (ii) half yearly ?*

Solution : Let principal be ₹ 100 and period be one year.

(i) When compounding is done quarterly,

$$r = \frac{10.5}{4} = 2.625 \text{ and } n = 4$$

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= 100\left(1+\frac{2.625}{100}\right)^4$$

$$= 110.92 \text{ (approximately)}$$

∴ Effective rate of interest is 10.92%.

(ii) When compounding is done half yearly $r = \frac{10.5}{2} = 5.25$ and $n = 2$.

$$A = P\left(1+\frac{r}{100}\right)^n$$

$$= 100\left(1+\frac{5.25}{100}\right)^2$$

$$= 110.78 \text{ (approximately)}$$

∴ Effective rate of interest is 10.78%.

3.5 Simple Annuities

Introduction : We know that loans are made available by banks as well as other finan companies for the purchase of household items like furniture, T.V. sets, refrigerator to cos items like flats, luxury cars etc.

In such cases the repayment of loan is made by the borrower in installments.

Suppose that an ownership flat whose cash down price is ₹ 50 lakhs can be purchased by paying ₹ 10 lakh cash down and remaining amount in monthly installments of ₹ 50000 for 10 years.

Thus here is a series of payments made at equal intervals. Such a series is called an *annuity*.

Common examples of annuity are insurance premia, recurring deposit etc.

Some definitions and terms

We have just learnt that an annuity is a series of payments made at equal intervals. The payments may be equal or different. When payments are equal, the annuity is called "Simple Annuity". (We shall study simple annuities only).

The time interval between two successive payments is called *period* of the annuity. The period may be a month, a quarter, six months or even an year.

In the above mentioned example of purchase of flat, the period is one month. The payments are to be made for 10 years. This span is called *status* of the annuity. When this span is fixed, as in the present case, the annuity is called certain annuity. Sometimes this span is uncertain.

For example, pension of a retired employee. Here the paying authority (or institution) does not know how long it will have to pay pension to the employee. Such an annuity is called *annuity contingent.*

The amount of loan ₹ 40 lakhs is called *present value* of the annuity.

Since the purchaser does not have this amount ready with him, he has to take loan. This loan is repaid alongwith compound interest. In the present case this amount is ₹ 60 lakhs (₹ 50000 × 120). This amount is called *amount of the annuity*.

Perpetuity : In some annuities, the payments are made forever.

For example, academic prizes etc. such an annuity is called **perpetuity**.

Sinking Fund : A sinking fund is an account that is used to deposit and save money to repay a debt or replace a an asset which gets worn out over period of time in the future. In other words, it's like a saving account that you deposit money in regularly and can only be used for a set purpose.

3.6 Immediate Annuity and Annuity Due

We have already stated that some common examples of annuity are recurring deposits, insurance premia, repayment of loan etc. In some of these cases like insurance premia, recurring deposits the payment is made at the beginning of each period while in case of repayment of loan, installment is paid at the end of each period.

If payments of an annuity are made at the beginning of each period then the annuity is called *annuity due* and when payments are made at the end of each period, then the annuity is called *immediate annuity*. Thus, recurring deposit is an example of annuity due and house rent is example of immediate annuity if rent is paid at the end of every month.

We shall study problems of immediate annuity only, however formulae for annuity due are also given at the end.

Students are advised to use calculators for solving problems in annuities.

3.7 Formulae for Amount & Present Value of Simple Immediate Annuity

Let, P : Present value of immediate annuity

x : Periodic installment

n : Number of installments

i : Rate of (compound) interest per rupee per period

Then
$$\boxed{P = \frac{x}{i}\{1-(1+i)^{-n}\}}$$

If A denotes the amount of an immediate annuity then,

$$\boxed{A = \frac{x}{i}\{(1+i)^{n}-1\}}$$

3.8 Relation between Amount and Present Value of Immediate Annuity

$$\boxed{\frac{1}{P}-\frac{1}{A} = \frac{i}{x}}$$

3.9 Present Value of a Perpetuity

$$\boxed{P = \frac{x}{i}}$$

Remark : Since in most cases, amount of loan P, rate of interest i and number of installments through which loan is to be repaid are known the following formula helps us deciding x, the installment.

$$P = \frac{x}{i}[1-(1+i)^{-n}]$$

Solved Examples

Example 3.32 : *Find the amount of an immediate annuity of ₹ 15000 per year payable for 12 years at 10 % p.a.*

Solution : Here,

$$x = \text{Periodic installment} = 15000,\ n = 12, i = 0.1$$

∴ Amount of the annuity

$$A = \frac{x}{i}\{(1+i)^{n}-1\} = \frac{1500}{0.1}\{(1.1)^{12}-1\}$$
$$= 15000\{3.1384-1\} = 320764.26$$

Example 3.33 : *A company wishes to set aside a certain sum at the end of each year to create a sinking fund. If it should amount to 10 lakhs in 10 years at 12 % p.a., find the sum to be set aside each year.*

Solution : Here,

Amount of annuity A = 10,00,000

No. of installments n = 10

Rate of interest per rupee i = 0.12 (per year)

To find x.

Now $$A = \frac{x}{i}[(1+i)^n - 1]$$

$$\therefore \quad 10,00,000 = \frac{x}{0.12}[(1.12)^{10} - 1]$$

$$\therefore \quad 10,00,000 \times \frac{12}{100} = x[3.1058 - 1]$$

$$\therefore \quad 1,20,000 = 2.1058\,x$$

$$\therefore \quad x = \frac{1,20,000}{2.1058} = 56985.47$$

Hence the company should set aside ₹ 56985 per year for the requisite sinking fund.

Example 3.34 : *A sewing machine worth ₹ 5000 is purchased on installment basis under five equal annual installments including compound interest at 10 % p.a. Find the amount of installment.*

Solution : It is clear that ₹ 5000/- is the present value of the annuity under consideration.

Thus $P = 5000,\ n = 5,\ i = 0.1$

To find x.

Now, $$P = \frac{x}{i}\{1 - (1+i)^{-n}\}$$

$$\therefore \quad 5000 = \frac{x}{0.1}\{1 - (1.1)^{-5}\}$$

$$\therefore \quad 500 = x\{1 - 0.6209\}$$

$$\therefore \quad 0.3790\,x = 500$$

$$\therefore \quad x = 1319.26$$

∴ Amount of installment is ₹ 1319 / -.

Example 3.35 : *What is the purchase price of a perpetuity of ₹ 1500 per year at 10 % p.a ?*

Solution : The purchase price of a perpetuity means the present value of the perpetuity P.

Here $x = 1500$ and $r = 10$ % p.a.

$\therefore \quad i = 0.1$

For a perpetuity, present value

$$P = \frac{x}{i} = \frac{1500}{0.1} = 15000$$

Example 3.36 : *ULIP is a scheme of Unit Trust of India under which a person can deposit upto ₹ 10,000/- per year. The status of ULIP is 10 years or 15 years. A person takes membership of ULIP by paying ₹ 10,000/- for 10 years. Assuming the rate of compound interest to be 12 %, find the amount he will receive at the end of 10 years.*

Solution : Here

$$x = 10,000$$
$$n = 10$$
$$i = 0.12$$

To find amount A

Now,

$$A = \frac{x}{i}[(1+i)^n - 1] = \frac{10000}{0.12}[(1.12)^{10} - 1]$$
$$= 83333.33\,[3.1058 - 1] = ₹\ 175483$$

Example 3.37 : *The present value of an annuity of ₹ 3600 per year is ₹ 12,000 and amount at the same rate for the same period is ₹ 15000. Find the rate of interest.*

Solution : Here, we are given that

$$\text{Amount A} = 15,000$$
$$\text{Present value} = 12,000$$
$$\text{Installment x} = 3,600$$

To find r, rate % p.a. we have,

$$\frac{1}{P} - \frac{1}{A} = \frac{i}{x}$$

$$\therefore \quad \frac{1}{12000} - \frac{1}{15000} = \frac{i}{3600}$$

$$\therefore \quad 1000\left[\frac{1}{12000} - \frac{1}{15000}\right] = 1000 \times \frac{i}{3600}$$

$$\therefore \quad \frac{1}{12} - \frac{1}{15} = \frac{10}{36}\,i$$

$$\therefore \quad \frac{5-4}{60} = \frac{10}{36}\,i$$

$$\therefore \quad \frac{1}{60} = \frac{10}{36}\,i$$

$$\therefore \quad i = \frac{36}{600} = \frac{6}{100}$$

This is rate per rupee per period.

$$\therefore \quad \text{Rate \% per period} = \frac{6}{100} \times 100 = 6$$

Example 3.38 : *A person deposits ₹ 20,000 in a bank at 15 % p.a. to give scholarship to needy students every year. Find the amount of yearly scholarship.*

Solution : Since the amount of ₹ 20,000 is deposited forever, to give scholarship perpetual this is an example of perpetuity in which present value P is ₹ 20,000.

Now, rate % per year = 15. ∴ Rate per rupee per year = $i = 0.15$.

Let x be the amount of annual scholarship.

Now, for a perpetuity,

$$P = \frac{x}{i}$$

$$\therefore \quad 20{,}000 = \frac{x}{0.5}$$

$$\therefore \quad 20{,}000 \times \frac{15}{100} = x$$

i.e. $\quad x = 3000$

Example 3.39 : *Find the amount of an annuity of ₹ 400 payable quarterly for 3 years at 16 % p.a.*

Solution : Here installment x = 400

Period is 1 quarter.

16 % p.a. means 4 % per quarter i.e. 4 paise per rupee.

Thus, i = 0.04

n : number of installments = 3 × 4 = 12

To find amount A.

We have
$$A = \frac{x}{i}[(1+i)^n - 1] = \frac{400}{0.04}[(1.04)^{12} - 1]$$
$$= 10{,}000\,[1.60103 - 1]$$
$$= 10{,}000 \times 0.60103 = ₹\ 6010 \text{ approx.}$$

Example 3.40 : *A man deposits his provident fund of ₹ 2,00,000 in a charity trust at 5 % p.a. and settles to withdraw at ₹ 15000/- per month for his expenses. If he begins to spend from first year and goes on spending at this rate, show that he will not be able to withdraw same amount in 23rd year.*

Solution : Suppose that the person can withdraw ₹ 15000 for n years, after which his balance will be less than ₹ 15000. Here, present value P = 2,00,000

Annual installment x = 15000

Rate per rupee p.a. i = 0.05

To find n.

We have
$$P = \frac{x}{i}[1 - (1+i)^{-n}]$$

$$\therefore \quad 2{,}00{,}000 = \frac{15000}{0.05}[1 - (1.05)^{-n}]$$

$$\therefore \quad \frac{200}{15} \times \frac{5}{100} = 1 - (1.05)^{-n}$$

$$\therefore \quad \frac{2}{3} = 1 - (1.05)^{-n}$$

$$\therefore \quad (1.05)^{-n} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \quad -n \log(1.05) = \log \frac{1}{3}$$

$$\therefore \quad -n(0.021) = -0.4771$$

$$\therefore \quad n = \frac{0.4771}{0.021} = 22.61$$

i.e. approximately 23 years.

$\therefore$ He will be bankrupt before the end of 23rd year.

3.10 Equated Monthly Instalments (E.M.I.)

Of late we find more and more people purchasing vehicles and homes by taking loans from the bank. The repayment is generally made in monthly instalments over a period of one year, two years, five years etc. This monthly instalment of repayment is called **equated monthly instalment (EMI)**. There are two ways by which banks or housing finance companies charge interest.

Interest on reducing balance :

Suppose, Mr. Desai takes a loan of ₹ 15 lacs from a bank for a period of 20 years at 7.5% reducing balance. Every month the E.M.I. consists of principal component and interest component. The principal component is deducted from Mr. Desai's outstanding loan amount and interest is charged on the lesser outstanding principal. For remaining months The equation (composition) between interest and principal keeps changing every month. Since interest is calculated on the outstanding principal, a higher share goes towards interest during the initial part of the term. The composition of interest and principal for first three months is shown in following table. (Calculation of EMI is shown below).

	Loan	EMI	Interest	Principal
1st month	15,00,000	12090	9375	2715
2nd month	14,97,285	12090	9358	2731
3rd month	14,94,554	12090	9341	2749

The EMI is calculated using formula already given.

i.e. $$P = \frac{x}{i}\{1-(1+i)^{-n}\}$$

If r is rate of interest % per year then i = rate per rupee per month $= \frac{r}{1200}$.

$\therefore$ Formula (1) becomes

$$P = \frac{x}{r/1200}\left\{1-\left(1+\frac{r}{1200}\right)^{-n}\right\}$$

where, n = Total number of months in the term.

Thus, in Mr. Desai's case

$$1500000 = \frac{x}{7.5/1200}\left\{1-\left(1+\frac{7.5}{1200}\right)^{-240}\right\}$$

$$= 160 \times \{1-(1.00625)^{-240}\}$$

$$\frac{1500000}{160} = x\,\{1 - 0.224175\}$$

$$9375 = x\,(0.77582)\,58$$

$$\therefore \quad x = 12090$$

From this it is clear that, for calculation of EMI, usual formulae for annuity are to be employed.

Flat interest rate :

In this method, the lending agency (banks, housing finance companies etc.) calculates the amount (principal + interest) for the given term at specified rate of interest. Then the EMI is arrived at by dividing this amount by total number of months in the term. Thus, the method consists of

(i) Calculate amount by using formula

$$A = P\left(1 + \frac{rn}{100}\right)$$

(ii) $$\text{EMI} = \frac{A}{K}, \qquad \text{where, } K = \text{Number of months.}$$

For example, calculate EMI for ₹ 1 lacks at 5% flat rate over 20 years.

Solution : Here, $P = 1,00,000$

$r = 5$

$n = 20$

$K = 240$

The amount is given by

$$A = P\left(1 + \frac{r.n}{100}\right) = 100000\left(1 + \frac{5 \times 20}{100}\right)$$

$$= 2,00,000$$

$$\text{EMI} = \frac{A}{K} = \frac{2,00,000}{240} = ₹\ 833.33$$

Comparison between flat interest rate and rate of interest on reducing balance :

The following example illustrates that flat interest rate is equivalent to a significantly lower reducing balance interest rate for the same period.

Let amount of loan i.e. P be ₹ 1,00,000 and flat rate of interest be 5%.

Let the term be 20 years then EMI turns out to be ₹ 833.33 as seen above.

Let us calculate EMI on reducing balance, at the same rate and for same term.

$$P = \frac{x}{i}\,\{1 - (1 + i)^{-n}\}$$

$$\therefore \quad 1,00,000 = \frac{x}{5/1200}\left\{1 - \left(1 + \frac{5}{1200}\right)^{-240}\right\}$$

$$416.67 = x\,(1 - 0.3686)$$

$$\therefore \quad x = \frac{416.67}{0.6314} = ₹\ 660 \text{ (approximately)}$$

Thus borrowing money on reducing balance (at the same rate) is highly beneficial.

Note : In view of the difference between flat interest rate and rate of reducing balanc people prefer reducing balance scheme.

Now-a-days, flat interest scheme is rarely used.

Solved Examples

Example 3.41 : *A two wheeler manufacturing company sells a motor-cycle costing ₹44,0 on installment basis by charging EMI ₹4500 for 1 year. Find flat rate of interest.*

Solution : Here, $A = 4500 \times 12 = 54000$, $P = 44,000$, $r = ?$, $n = 1$.

$$A = P\left(1 + \frac{rn}{100}\right)$$

$$\therefore \quad 54,000 = 44,000\left(1 + \frac{r}{100}\right)$$

$$\therefore \quad \frac{54}{44} = 1 + \frac{r}{100}$$

$$\therefore \quad \frac{r}{100} = \frac{54}{44} - 1 = \frac{10}{44}$$

$$\therefore \quad r = \frac{1000}{44} = 22.7$$

Example 3.42 : *A person borrows ₹5,00,000 from HDFC for purchase of the flat at 8% p annum reducing balance interest rate. Find EMI for a period of 10 years. Calcula corresponding flat interest rate.*

Solution : Here, $P = 5,00,000$, $r = 8\%$

$$\therefore \quad i = \frac{8}{1200}, \quad n = \text{number of months} = 120$$

$$P = \frac{x}{i}\{1 - (1 + i)^{-n}\}$$

$$\therefore \quad 5,00,000 = \frac{x}{8/1200}\left\{1 - \left(1 + \frac{8}{1200}\right)^{-120}\right\}$$

$$\therefore \quad 5,00,000 \times \frac{8}{1200} = x\{1 - (0.4503)\}$$

$$\therefore \quad 3333.33 = x(0.5497)$$

$$\therefore \quad x = 6064 \text{ (approximately)}$$

The total amount payed by the person

$$A = 6064 \times 120$$

$$= 727680$$

Now,

$$A = P\left(1 + \frac{rn}{100}\right)$$

$$\therefore \quad 727680 = 5,00,000\left(1 + \frac{10r}{100}\right)$$

$$\therefore \quad 1.4554 = \left(1 + \frac{r}{10}\right)$$

$$\therefore \quad r = 10 \times 0.4554$$

$$r = 4.554$$

Example 3.43 : *What is the EMI of loan of ₹ 25000 if repaid in 4 years, at the rate of interest 5% p.a. on the outstanding amount at the beginning of each year ?* **(April 2005)**

Solution : P = 25000, r = 5, n = 4 years = 48 months

$$i = \text{Interest per rupee per month}$$

$$= \frac{12}{1200} = \frac{1}{100} = 0.01$$

Now,
$$P = \frac{x}{i}\left[1 - (1+i)^{-n}\right]$$

$$25000 = \frac{x}{0.01}\left[1 - (1+0.01)^{-48}\right]$$

$$25000 = \frac{x}{0.01}\left[1 - (1.01)^{-48}\right]$$

$$250 = x\,[1 - 0.6203]$$

$$250 = x\,[0.3797]$$

$$\therefore \quad x = \frac{250}{0.3797}$$

$$x = 658.3459$$

$\therefore$ EMI = ₹ 658

Example 3.44 : *Find the EMI on a loan of ₹ 3,00,000 to be paid in 4 years at 12% p.a. on the outstanding amount at the beginning of each month.* **(April 2007, Oct. 2006)**

Solution : Given : P = 3,00,000, r = 12%, n = 4 years = 48 months

$$i = \text{Interest per rupee per month}$$

$$= \frac{12}{1200} = \frac{1}{100} = 0.01$$

Now,
$$P = \frac{x}{i}\left[1 - (1+i)^{-n}\right]$$

$$\therefore \quad 300000 = \frac{x}{0.01}\left[1 - (1+0.01)^{-48}\right]$$

$$\therefore \quad 3000 = x\left[1 - (1.01)^{-48}\right]$$

$$3000 = x\,[1 - 0.6203]$$

$$3000 = (0.3797)\,x$$

$$x = \frac{3000}{0.3797}$$

$$x = 7900.9745$$

Example 3.45 : *Find EMI on a loan of ₹ 1,00,000 to be repaid in equal ... installments. Interest is charged at 12% p.a. on the loan outstanding at the beginning of ... month and the time span in 5 years $[(1.01)^{60} = 1.8199]$.* (April ...)

Solution : $P = 1,00,000,\ r = 12,\ n = 5 = 60$ months

$$i = \frac{12}{1200} = 0.01$$

$$P = \frac{x}{i}\,[1-(1+i)^{-n}]$$

$$\therefore \quad 1,00,000 = \frac{x}{0.01}\,[1-(1+0.01)^{-60}] = 100 \times [1-(0.1)^{-60}]$$

$$\therefore \quad 1000 = x\,[1-(1.01)^{-60}]$$

$$\therefore \quad 1000 = x\,[1-0.5504]$$

$$\therefore \quad 1000 = x\,[0.44955]$$

$$\therefore \quad x = \frac{1000}{0.44955}$$

$$\therefore \quad x = 2224.4448$$

$\therefore$ EMI is ₹ 2224/- approx.

Example 3.46 : *Find EMI on a loan of ₹ 1,00,000 to be repaid in equal ... installments. Interest is charged at 12% p.a. on the loan outstanding at the beginning of ... month and the time span in 2 years $[(1.01)^{60} = 1.8199]$* (April ...)

Solution : $P = 1,00,000,\ r = 12,\ n = 2 = 24$ months

$$i = \frac{12}{1200} = 0.01$$

$$P = \frac{x}{i}\,[1-(1+i)^{-n}]$$

$$\therefore \quad 1,00,000 = \frac{x}{0.01}\,[1-(1+0.01)^{-24}] = 100 \times [1-(0.1)^{-24}]$$

$$\therefore \quad 1000 = x\,[1-(1.01)^{-24}]$$

$$\therefore \quad 1000 = x\,[1-0.7876]$$

$$\therefore \quad 1000 = x\,[0.2124]$$

$$\therefore \quad x = \frac{1000}{0.2124}$$

$$\therefore \quad x = 4707.3472$$

$\therefore$ EMI is ₹ 4707/- approx.

Example 3.47 : *Find the EMI on a loan of ₹ 1,00,000/- to be repaid in 4 years at 12% on the balance outstanding at the beginning of each month.* (April 200...)

Solution : Given : $P = 1,00,000,\ r = 12\%,\ n = 4$ years $= 48$ months

$$i = \frac{12}{1200}$$

$$P = \frac{x}{i}[1-(1+i)^{-n}]$$

$$1,00,000 = \frac{x}{12/1200}\left[1-\left(1+\frac{12}{1200}\right)^{-48}\right]$$

$$1,00,000 = \frac{x}{0.01}[1-(1+0.01)^{-48}]$$

$$1,00,000 \times 0.01 = x[1-(1.01)^{-48}]$$

$$1000 = x[1-0.6203]$$

$$1000 = x[0.3797]$$

$$x = \frac{1000}{0.3797} = 2633.6582$$

$\therefore$ E.M.I. = 2633.6582

Example 3.48 : *What is the EMI of loan of ₹ 25000 is repaired in 4 years, if the rate of interest is 5% p.a. on the outstanding amount at the beginning of each year.* **(April 2005, 09, 13)**

Solution : Given : Total loan amount $= \frac{₹\ 25,000}{48 \text{ months}} =$ ₹ 520.83

5% interest on ₹ 25,000 = ₹ 1250 p.a.

$\therefore$ For 1 month = ₹ $\frac{1250}{12}$ = ₹ 104.17

$\therefore$ ₹ 520.83 (+) ₹ 104.17 = ₹ 625/-

$\therefore$ E.M.I. of loan = ₹ 625.

Example 3.49 : *A washing machine worth ₹ 20000 is purchased on instalment basis under 20 monthly instalments, including interest at 18% p.a. Find equated monthly instalment EMI by monthly reducing balance method.* **(Oct. 2009)**

Solution : Given : P = 20000, n = 20 months

$$\text{Interest} = \frac{r}{1200} = \frac{18}{1200}$$

$$P = \frac{x}{r/1200}\left\{1-\left(1+\frac{r}{1200}\right)^{-n}\right\}$$

$$20000 = \frac{x}{18/1200}\left\{1-\left(\frac{18}{1200}\right)^{-20}\right\}$$

$$20000 = 66.67 \times \{1-(1.015)^{-20}\}$$

$$20000 = 66.67 \times (1-0.7425)$$

$$20000 = 66.67 \times 0.2575$$

$$\therefore \quad x = \frac{20000}{66.67 \times 0.2575}$$

$$x = 1164.8565$$

$\therefore$ E.M.I. is ₹ 1164.8565

3.11 Formulae for Present Value and Amount of an Annuity Due

Let x : Periodic installment

n : Number of installments

i : Rate of interest per rupee per period

P' : Present value of annuity due

A' : Amount of annuity due

Then

$$P' = \frac{x(1+i)}{i}\{1-(1+i)^{-n}\}$$

$$A' = \frac{x(1+i)}{i}\{(1+i)^{n}-1\}$$

$$\frac{1}{P'} - \frac{1}{A'} = \frac{1}{x(1+i)}$$

Present value of a perpetuity :

$$P' = \frac{x(1+i)}{i}$$

Exercise (3.1)

Theory Questions :

1. Give meaning of the following terms :
   (i) Period of annuity (ii) Status of annuity
   (iii) Annuity contingent (iv) Perpetuity
   (v) Present value (vi) Immediate annuity
2. Define the following terms :
   (i) Principal, (ii) Term, (iii) Amount, (iv) EMI.
3. Define the term 'Simple Annuity'.

Exercise (3.2)

Numerical Problems :

1. Find the simple interest on ₹ 1,000 at 6% p.a. for 5 months.
2. What sum of money put at simple interest for two years at 8% will amount to ₹ 1,276 ?
3. In how many years will ₹ 35,000 will amount to ₹ 87,500 at 10% p.a. simple interest ?
4. Suresh borrowed ₹ 1,750 from Dinesh for 5 months. Dinesh charged him ₹ 43.75 as simple interest. At what rate was the interest reckoned ?
5. Mahesh invested ₹ 2,000 for 3 years and ₹ 3,550 for 5 years at the same rate of simple interest. If he received a total simple interest of ₹ 893.75, find the rate of interest.
6. A sum of money amounts to ₹ 10,400 in 6 years and ₹ 11,200 in 8 years. Find the sum and rate of interest.

7. A sum of money doubles itself in 6 years at certain rate of interest. In how many years would it triple if the rate of interest is increased by $3\frac{1}{3}$ p.a. ?
8. How should a sum of ₹ 25,000 be divided between two boys aged 15 years and 10 years now, so that at the age of 25 they would get equal amounts, interest being charged at 8% p.a. ?
9. A certain sum amounted to ₹ 5,750 in $2\frac{1}{2}$ years at 6% p.a. Find after how many more years it will amount to ₹ 6,200.
10. A person deposits ₹ 100 on the 1st day of each month throughout the year. If the bank pays 8% p.a. rate of interest, what amount will be get at the end of year ?
11. A bank increased the rate of interest on saving bank accounts from 3.5% p.a. to 4% p.a. Sumit withdrew ₹ 1570 from his account and as a result, in that half of the year his interest was reduced by ₹ 20.40. How much amount had he in his account at the beginning ?
12. A person borrowed ₹ 20,000 from a bank, part of it was borrowed at 8% p.a. and part of it was borrowed at 12% p.a. simple interest. At the end of 2.5 years he paid an interest of ₹ 4240 on the whole sum. Find the sum borrowed at 12%.
13. The simple interest on ₹ 1900 for 4 years exceeds the simple interest on ₹ 1200 for the same period by ₹ 308. Find the rate of interest.
14. A person brought a shares of ₹ 100 for ₹ 250. 20% dividend was declared on that share, what is the rate of interest realised on his investment ?
15. Find the C.I. on ₹ 2750 at $4\frac{1}{2}$ % p.a. for 6 years.
16. A person borrows ₹ 6,000 for $3\frac{1}{2}$ years at $7\frac{1}{2}$ % p.a. C.I. How much does he replay ?
17. What sum will amount to ₹ 12167 in 5 years at 4% p.a. C.I. ?
18. Find the amount at the end of 5 years of a sum of ₹ 2,000 at 6% p.a. payable half yearly.
19. How long would a sum of money take to double itself, if allowed to accumulated at $7\frac{1}{2}$ % p.a. ?
20. Find the rate of C.I. at which a sum of money triples itself in 8 years.
21. If the population of a town increased every year by 2.1% of the population at the beginning of that year, in how many years will the total increase of population be 20% ?
22. The difference between S.I. and C.I. on a sum of money for 2 years is ₹ 15. The S.I. on the same sum for 4 years is ₹ 1,200. Find the sum and rate of interest.
23. A man left for his 2 sons aged 10, 12 years ₹ 15,000 and ₹ 12,000 respectively. The money is invested in 6% and 9% compound interest respectively. The sons will receive the amounts when each of them attains the age of 20 years. Which one will receive larger amount ?
24. A sum of money amounts to ₹ 5681.15 in 5 years and ₹ 6766.33 in 8 years.
25. Find the C.I. on ₹ 6,200 at $4\frac{1}{2}$ % p.a. in the 3rd year.

26. A machine is depreciated at rate of 20% on the reducing balance. The original cost was ₹ 1,00,000 and the ultimate scrap value was ₹ 30,000. Find the effective life of machine.
27. A machine depreciated 10% p.a. for the first two years and then 7% p.a. for the next 3 years, depreciation being calculated on the diminishing value. If the value of the machine be ₹ 10,000 initially, then find the average rate of depreciation and the depreciated value of the machine at the end of the 5th year.
28. A person invested ₹ 4 lac in a business. His money grew at the compound rate of 10% for first 5 years and then there was a decline at the compound rate of 10%. What shall be the amount at the end of 10 years ? Is it same as the invested amount ?
29. The rate of compound interest is 12% p.a. compounded quarterly. What is the effective rate of interest ?
30. A bank payes interest quarterly at 8.85%. What is effective rate of interest ?
31. A person wants to deposit ₹ 1 lac quarterly rate of interest is 9.10% and annual rate of interest is 9.40%. Which scheme will give him better yield ?
32. A bank payes 6.25% interest on monthly recurring deposit scheme. What is effective rate of interest ?
33. Find the present value of an annuity of ₹ 2500 per year for 20 years at 10 % p.a.
34. A loan is repaid in 4 equal quarterly installments of ₹ 4000 including principal and interest at 16 % p.a. what was the amount of the loan ?
35. What is the amount of an immediate annuity of ₹ 2000 per year payable for 12 years at 10 % ?
36. A man deposits ₹ 500 at the end of every month in a bank for 3 years. If the rate of interest is 12 % p.a. find the amount he will get at the end of 3 years.
37. A company sets aside ₹ 75000 at the end of every year to create a sinking fund. What will it amount to at the end of 25 years at 10 % ?
38. The present value and amount of an immediate annuity are ₹ 20,000 and ₹ 30,000 respectively (for the same period). If the installment is ₹ 4500 p.a., find the rate of interest.
39. The cash price of a flat is ₹ 4,80,000. A person paid 25 % of this in cash and borrowed 75 % from HDFC at 15 % p.a. repayable in monthly equal installments spread over 15 years. Find the amount of installment.
40. A loan of ₹ 1,00,000 is to be repaid in 30 equal annual installments payable at the end of each year to cover principal and compound interest at 12 %. Find the amount of installment.
41. Find the present value of an annuity of ₹ 500 payable at the end of every half year for 10 years at 10 % p.a. compound interest.
42. Find the amount of an annuity of ₹ 200 payable quarterly for 5 years at 12 % p.a.
43. What sum set aside every year will amount to ₹ 5 lakhs at the end of 10 years at 12 % ?
44. For what period should a man mortgage his property yielding. ₹ 30000 per year to clear a debt of ₹ 2 lakhs at 10 % p.a. ?
45. A sum was borrowed at 10 % compound interest and was repaid in 3 equal annual installments of ₹ 6655 each, at the end of each year. Find the sum borrowed.

46. A man deposits ₹ 3000 at the end of every year in a bank and at the end of 3 years receives a sum of ₹ 10,000. Find the rate of compound interest. (**Given :** $\sqrt{93} = 9.6436$.)
47. How much amount a person should deposit in a bank to ensure yearly scholarship of ₹ 500 for ever ? Rate of interest is 16 %.
48. A man pays ₹ 1200 as house rent to the owner of the house per month. If he goes abroad for 2 years, how much rent he should pay in advance if the owner is to receive 10 % p.a.
49. A person wants to borrow ₹ 5 lacs for purchase of a house. Bank A lends at the rate of 5% p.a. flat while bank B lends at 8% on monthly reducing balance for the same period of 10 years. Compare the EMI's. Which bank the person should opt for ?
50. A motor-cycle manufacturing company gives partial loan of ₹ 33000 and charges ₹ 3018 per month for one year. The company claims it charges 5% rate of interest. Is it flat or on monthly reducing balance ?
51. Mr. Choudhary wants to take a loan of ₹ 10 lacks on monthly reducing balance. Find the interest paid by for a period of 15 years if rate of interest is 8%.

    $\left[\textbf{Hint : } \text{Find } A = \frac{x}{i}\,[(1+i)^n - 1],\ \text{Interest} = A - P.\right]$
52. What is the simple interest on ₹ 4980 at 12% p.a. for 8 months ? What is the amount ?
53. The simple interest on a certain sum of the end of 4 years is $\frac{1}{4}^{th}$ the sum itself. Find the rate of interest.
54. Further wanted to gift his son on his 18th birthday a sum of ₹ 6000. If the present age of the son is 12 years and rate of interest is 10% p.a. simple interest, what amount should be invest now ?
55. Madhu invested ₹ 10,000 at 16% p.a., ₹ 8000 at 14% p.a., ₹ 5000 at 10% p.a. simple interest. Find the average rate of return on his investment.
56. The simple interest on a sum at 7% p.a. for $3\frac{1}{3}$ years exceeds that on the same sum at 10% p.a. for 2 years by ₹ 200. Find the sum.
57. The simple interest on a certain sum for 2 years at 12% p.a. exceeds the compound interest on it for the same period at 10% p.a. by ₹ 240. Find the sum.
58. ₹ 16000 invested at 10% p.a. compounded semi-annually amounts to ₹ 18522. Find the time period of investment.
59. Find C.I. on ₹ 5500 for 6 months at 12% p.a. compounded quarterly.
60. Find the rate of interest p.a. if ₹ 2,00,000 amount to ₹ 231525 in $1\frac{1}{2}$ years interest compounded half yearly.
61. ₹ 15000 is invested in a Term Deposit Scheme that hatches interest 6% p.a. compound quarterly. What will be the interest after one year ? What is the effective rate of interest ?
62. Find effective rate of interest corresponding to a nominal rate 3% p.a. payable (i) half yearly, (ii) quarterly.

63. Find EMI if a loan of ₹ 2,00,000 at the rate of 12% p.a. (reducing balance) is to be repaid equal monthly instalments in 10 years.
64. Amol borrows ₹ 8 lacs from a finance company at 10% flat rate of interest for a period of years. Compute EMI.
65. Find the difference between, EMI's by flat rate of interest and reducing balance of interest ₹ 6 lacks at 9% p.a. for a period of 10 years.
66. Find the EMI on a loan of ₹ 16,00,000 for a period of 20 years at 12% p.a. reducing balance
67. Find the simple interest on ₹ 17500 at 6% p.a. for 5 years. (April
68. Find the compound interest on ₹ 11000 at 7% p.a. for 2 years. (April 201
69. What is the difference between simple interest and compound interest on ₹ 500 for 2 years 10% p.a. (compounded yearly). (April 201
70. What is the EMI of a loan of ₹ 1,00,000 if repaid in 5 years ? If the rate of interest 10% p.a. on the outstanding amount at the (n single) n of each year.
71. Find the simple interest on ₹ 6000 at 7% p.a. for 10 months. (April 201
72. What sum will amount to ₹ 4764 in 3 years at 6% p.a. compounded interest ?
73. Which is the better investment ?
    (a) 12% at ₹ 296, (b) 15% at ₹ 120. (April 201
74. Find the amount of ₹ 4500 at 12% p.a. in 4 years compounded yearly. (April 201
75. What is the EMI of loan of ₹ 2,00,000 repaid is 5 years if the rate of interest is 12% of p on the outstanding amount at the beginning of each year ?
76. Find the amount of EMI of ₹ 1000 per year payable for 10 years at 12% p.a. (April 201
77. Find the amount at the end of 5 years of a sum of ₹ 2000 at 8% p.a. compounded quarterly.

Answers (3.2)

(1) 25	(2) ₹ 1,100	(3) 5/4 years
(4) 6% p.a.	(5) 3.76% p.a.	
(6) ₹ 8,000, 5% p.a.	(7) 5/6	
(8) ₹ 13,750 , ₹ 11,250		(9) $1\frac{1}{2}$
(10) ₹ 1,252	(11) ₹ 3,840	(12) ₹ 2,400
(13) 11% p.a.	(14) 8%	(15) ₹ 831.22
(16) ₹ 7728.24	(17) ₹ 10,000	(18) ₹ 2687.83
(19) 9.55 years	(20) 14.8% p.a.	(21) 8.7 years
(22) ₹ 6,000, 5% p.a.	(23) Younger son will receive larger amount.	
(24) ₹ 4245.29, 6% p.a.		(25) ₹ 304.68
(26) 5.4 years	(27) Rate 8.2%, Value ₹ 6515	
(28) ₹ 380396.02 less than the original.		(29) 12.55%
(30) 9.15%		

(31) Quarterly rate of interest gives him ₹ 15 more (32) 6.43%

(33) ₹ 21284 (34) ₹ 14520

(35) ₹ 42769 (36) ₹ 21538

(37) ₹ 73,76,029 (38) $7\frac{1}{2}$ % p.a.

(39) ₹ 5040 (40) ₹ 12414

(41) ₹ 6231 (42) ₹ 5374

(43) ₹ 28492 (44) 11.5 years

(45) ₹ 16550 (46) 10.7 %

(47) ₹ 3125 (48) ₹ 25065

(52) ₹ 398.40, ₹ 5378.4

(53) 6.25% (54) ₹ 3750

(55) ₹ 6000 (56) 14% p.a. (57) ₹ 8000

(58) 3 (59) ₹ 334.95 (60) 0.5%

(61) ₹ 920.45, 6.13% (62) 3.0225% p.a., 3.033%

(63) ₹ 2869.42

(64) ₹ 15000 (65) ₹ 1899.45 (66) ₹ 17617.38

(Almost all answers are approximate)

Multiple Choice Questions

Choose the correct alternative from the following questions :

1. The simple interest on some principal at some rate for 3 years is ₹ 525. At the same rate, the compound interest on ₹ 1600 for 2 years is ₹ 164. What is the principal ?
   (a) ₹ 4500 (b) ₹ 3500
   (c) ₹ 3000 (d) ₹ 4000
2. The compound interest for 2 years and 3 years on certain principal at certain rate of interest is ₹ 8820 and ₹ 9261 respectively. Find the rate of interest and principal.
   (a) 5% ₹ 5000 (c) 6% ₹ 8000
   (b) 5% ₹ 9000 (d) 5% ₹ 8000
3. What amount should be invested at 4% for 3 years to get ₹ 19,510 as compared interest ?
   (a) ₹ 1,86,250 (c) ₹ 1,56,250
   (b) ₹ 1,06,250 (d) ₹ 1,76,250
4. Nisha deposited ₹ 12,000 in a bank for 3 years at a rate of 10 p.c.p.a. compound interest. What amount will she receive from the bank at the end of the term ?
   (a) ₹ 15,972 (c) ₹ 15,790
   (b) ₹ 15,792 (d) ₹ 15,970
5. In how many years simple interest on ₹ 700 at the rate of 12.5 p.c.p.a. be ₹ 525 ?
   (a) 3 (c) 6
   (b) 5 (d) 7

6. Ananya deposited ₹ 15,000 in a bank for 3 years at simple interest. If she received ₹ 24 at the end of the period find the rate of interest.
   (a) 20 p.c.p.a. (c) 22 p.c.p.a.
   (b) 15 p.c.p.a. (d) 25 p.c.p.a.
7. What is the principal, if it amounts to ₹ 3528 at 5 p.c.p.a. compound interest for 2 years ?
   (a) ₹ 3200 (c) ₹ 2800
   (b) ₹ 3000 (d) ₹ 3100
8. The population of a village increases by 10% every year. Its population at the end of 2 was 7986. What was its population at the beginning of 2013 ?
   (a) 4000 (c) 5000
   (b) 4500 (d) 600
9. How much is the simple interest earned an ₹ 75 for 3 months at 8 p.c.p.a. ?
   (a) ₹ 1.05 (c) ₹ 15
   (b) ₹ 18 (d) ₹ 1.50
10. If the principal at ₹ 1,200 doubles in 6 years by simple interest, find the rate of interest.
   (a) $16\frac{2}{3}$ % p.c.p.a. (c) $16\frac{3}{1}$ % p.c.p.a.
   (b) $16\frac{1}{3}$ % p.c.p.a. (d) $16\frac{3}{2}$ % p.c.p.a.
11. In how many years will the principal be 2.25 times, at the rate of 10% p.a. simple interest ?
   (a) $12\frac{1}{2}$ years (c) $8\frac{1}{2}$ years
   (b) $10\frac{1}{2}$ years (d) $14\frac{1}{2}$ years
12. What will be the simple interest earned on ₹ 10,000 in 2.5 years at the rate of 8 p.c.p.a. ?
   (a) ₹ 2000 (c) ₹ 2060
   (b) ₹ 2050 (d) 2080

Answers

1. (b)	2. (d)	3. (c)	4. (a)	5. (c)	6. (a)
7. (a)	8. (d)	9. (d)	10. (a)	11. (a)	12. (a)

□□

Chapter 4...

Shares and Mutual Funds

Contents ...

Key Words :

Equaity Share, Preference Share, Debentures, Stock Market.

Mutual fund, Net asset value, Risks, Entry load, Exit load.

Objectives :

To understand dealings in stock market.

To understand the concept of mutual fund, Benefits of mutual funds and different concepts related to mutual fund.

(A) SHARES AND DIVIDENDS

4.1 Introduction

We know that a person or a small group of persons can start a business upto medium size. However, when a big industry is to be launched, several persons come together to raise required capital. These are called *promoters* of the company. The capital is divided into small parts called *shares*. The people who purchase shares are called *shareholders* of the company and in a way they are owners of the company. The company is managed by a body of persons known as *Board of Directors* of the company.

4.2 Share Capital

The total capital of the company is divided into a number of small unit of equal value called *'shares'*. Thus a capital of ₹ 1,00,000 may be divided into 1000 shares of ₹ 100 each, or 2000 shares of ₹ 50 each. The rights of the holders of each class of shares are governed by the companies act and also by the Articles and Association of the company.

4.3 Kinds of Shares

(i) Preference shares : The holders of these shares enjoy a preferential rights as regards the payment of dividend, and the repayment of capital in the event of winding-up. The rate of profit or dividend is fixed but it is paid before the profit is distributed on equity shares.

The preference shares are of the following kinds :

(a) Cumulative preference shares : The holders of these shares are entitled to a fixed dividend each year. But the amount of the dividend not paid in any year stands as arrears and is payable out of the profits of subsequent years.

(b) Non-cumulative preference shares : The holders of these shares have a preferential right for a fixed dividend out of the profits before the same are distributed to other classes of shares, but such dividend is payable only out of the profits of each particular year. Thus each year it lapses and cannot be claimed out of the future profits.

(ii) Equity shares : These shares ware formerly called 'ordinary' shares. They have no special rights attached to them. The holders of these shares are paid dividend after the claims of the preference shares holders are satisfied. The rate of the dividend is not fixed; it varies from year to year depending on the profits of the company. In some years they may have to go without dividend while in others they may get a very high rate of dividend. An equaity share is also called a "scrip".

4.4 Dividend

(April 2007, 2009)

The net profit made by the company every year is ascertained from its Profit and Loss Account prepared at the end of the year. Out of the net profits, dividend at a specified rate is paid on preference shares. Arrears of dividend, if any, on cumulative preference shares are also paid, if the amount of profit permits. The balance is then utilised for payment of dividend on equity shares. Dividend may be declared as fixed amount per share or as a percentage of the capital of the company.

4.5 Debentures

A company may require additional long-term capital for extension and development schemes. One of the methods of raising such finance is by means of debentures. Debentures are long-term loans taken by the company from the public. The total amount to be borrowed is divided like share capital into small units of equal amounts; and the members of the public are invited to lend such amounts to the company at a specified rate of interest.

4.6 Bonus Shares (April 2007, 2015)

Sometimes a company rewards its share holders by issuing free shares to them in proportion of the shares held by them. These (free) shares are called **bonus shares**. They are entitled for all rights, that an ordinary share has. In this way, the holding of a person increases and the amount corresponding to bonus shares is capitalised. The company can use this amount for capital expenditure. Bonus shares are issued in some ratio. The ratio **a** : b means **a** free shares for b shares held.

4.7 Stock Exchange, Face Value, Market Value of Shares (April 2011, 2017)

Shares and debentures are transferable assets. They are bought and sold in "Stock Exchanges". A stock exchange is a form of exchange, which provides services for stock brokers and traders to trade stocks, bonds and other securities. Stock exchanges also provide facilities for issue and redemption of securities and other financial instruments and capital events including payment of income and dividends.

In India there are two prominent stock exchanges, the Bombay Stock Exchange (BSE) and National Stock Exchange (NSE).

Bombay Stock Exchange, known as BSE limited is the oldest stock exchange in entire Asea. It is located at Jeejee bhoy towers, Dalal street in fort, Mumbai. It has largest number of companies of the world listed on it. As per March 2012 there are more than 5000 Indian companies listed on BSE. The BSE sensex which is also known as BSE-30 (weighed average of 30 leading companies) is most commonly used term while referring to trading volume in India and Asea. The total capital of all shares listed on BSE in 2012 was approximately ₹ 50000 crores. In term of share volume NSE is almost twice that of BSE.

Now-a-days shares in physical form have given way to shares in demat (dematerialize) form. In short, an investor has a list of shares he possesses and not physical share certificates. This has simplified several procedures and reduced lot of paper work.

The price stated on the body of share or debenture is called its **face value** (F.V.) or **nominal value.**

The price at which a debenture or share is actually bought or sold is called **market value** or **cash value** of the share.

If the face value and market value of a share are equal, the share is said to be **"at par"**.

Solved Examples

Example 4.1 : *A sum of money got by selling shares of ₹ 1,500 in 10% at 135 was deposited in a bank at 8% p.a. which investment gives a better return ?*

Solution : As there is no mention of brokerage, we need not think of it.

By 10% at 135 we mean a share with face – value ₹ 100 is having a market price ₹ 135 and fetches dividend ₹ 10. Let us find the amount realised by selling shares of face-value ₹ 1,500.

When face-value is ₹ 100, market price is 135.

∴ When face-value is ₹ 1,500, market price

$$= \frac{135 \times 1500}{100}$$

$$= ₹\ 2{,}025$$

However, the income by way of dividend on these shares is ₹ 15 × 10 = 150 (since there 15 shares of face-value ₹ 100 each).

The annual interest received from the bank at 8% p.a.

$$= \frac{P.n.r.}{100}$$

$$= \frac{2025 \times 1 \times 8}{100}$$

$$= ₹\ 162$$

∴ Investment in bank is better as it gives ₹ 12 more annually.

Example 4.2 : *A sum of ₹ 1,350 was invested in 6% stock at 87. When it rose to 91 all shares were sold. In the meanwhile dividend was received. For purchasing the brokerage w 3%, while selling it was 2%. What is the total gain or loss in the total transaction ?*

Solution : Since rate of commission is 3% for purchase, each share costs ₹ (87 + 3) = ₹ 90

∴ In ₹ 1,350, a person will get $\frac{1350}{90}$ = 15 shares.

The dividend received on it will be ₹ (15 × 6) = ₹ 90.

While selling the shares, amount received per share will be ₹ (91 – 2) = ₹ 89.

∴ Amount received by selling all 15 shares = ₹ (15 × 89) = ₹ 1,335.

∴ Total gain = ₹ (1,335 + 90 – 1,350) = ₹ 75.

Example 4.3 : *Two companies have shares of 12% at 124 and 16% at 145. In which of shares would the investment be more profitable ?*

Solution : Clearly, the face-value of the share must be ₹ 100 in each case.

Let us find percentage return in each case.

For ₹ 124, the return is ₹ 12

For ₹ 100, the return is ₹ 9.677 $\left(\frac{100}{124} \times 12 = 9.677\right)$

For the second company.

For ₹ 145, the return is ₹ 16

∴ For ₹ 100, the return is ₹ 11.03 $\left(\frac{100}{145} \times 16 = 11.03\right)$

As the percentage return in second case is more, the investment in the second company more profitable.

Example 4.4 : *The capital of a company consists of ₹6,00,000, 10% preference shares and ₹24,00,000 equity shares. What percentage dividend can be declared out of a total profit of ₹3,75,000 after making a tax provision of 20% on the profit ?*

Solution : Tax @ 20% on 3,75,000 = 75,000.

Net profit = ₹ 3,00,000

less 10% dividend on ₹ 6 lacs. (preference shares) = ₹ 60,000.

∴ Profit available for equity dividend = ₹ 2,40,000.

Equity share capital is ₹ 24 lacks.

∴ Rate of dividend on equity shares = 10%.

Example 4.5 : *The capital of a company consists of 1 lac, 8% cumulative preference shares of ₹ 100 each and 5 lack equity shares of ₹ 10 each. In a year there was no profit, in the next year company decided to pay 15% on equity shares. What was the total dividend distribution ?*

Solution : Cumulative preference share capital

$$= ₹\ 1{,}00{,}000 \times 100$$
$$= ₹\ 1 \text{ crore}$$

The company has to pay dividend for 2 years at 8% (each year).

i.e. 16% in all.

∴ dividend outgo = ₹ 16,00,000

equity capital = ₹ 5,00,000 × 10

= ₹ 50,00,000

dividend at the rate of 15% = ₹ 7,50,000

∴ Total dividend outgo = ₹ 16,00,000 + 7,50,000

= ₹ 23,50,000

Example 4.6 : *A person holds 400, 8% preference shares of ₹ 100 each, ₹ 50 paid-up and 300 equity shares of ₹ 10 each, 5 paid-up. If the company declares a dividend of 20% on equity shares, find the total dividend received by him.*

Solution : Since the company declares dividend on equity shares, it has to pay dividend on preference shares.

Preference capital of the person = 400 × 100

= 40,000

But it is 50% paid up.

∴ Paid-up preference capital = 20,000

$$\text{Dividend on preference shares} = 20{,}000 \times \frac{8}{100} = ₹\ 1600$$

Equity capital = 300 × 10 = 3000

It is also 50% paid up.

∴ Paid-up equity capital = 1500

Dividend @ 20% = 300

∴ Total dividend received by him = 1600 + 300 = ₹ 1900.

Example 4.7 : *A persons finds that if he invests his money in 15% stock at 225, his income will be ₹270 greater than if he invest it in 22% stock at 375. Find the sum invested.*

Solution : Suppose that the person invests ₹ x. On ₹ 225, earn ₹ 15.

$\therefore$ On ₹ x, he earn $\frac{15x}{225}$

In the other case, on ₹ 375, he earns ₹ 24.

$\therefore$ On ₹ x, he earns $\frac{22x}{375} = 270$

$$\frac{15x}{225} - \frac{22x}{375} = 270$$

$$x = 33750$$

$\therefore$

$\therefore$ His investment is ₹ 33750.

Example 4.8 : *Ashok purchased 10 share of Infosys at ₹2000 per share cum-bonus. Bonus was declared at 1 : 1. Ashok sold 15 share ex-bonus at ₹1250. He had to pay 1% brokerage each time on the market value. What is the cost price of remaining 5 shares held by him ?*

Solution : Cost price of 10 shares @ ₹ 2000 = ₹ 20,000.

Selling price of 15 shares @ ₹ 1250 = ₹ 18,750.

Total brokerage paid on ₹ 38750 @ 1% = ₹ 387.50.

$\therefore$ His net outgo = ₹ 20,000 + 387.50 = ₹ 20387.50

His earning by selling shares = ₹ 18750.

$\therefore$ His net cost of 5 shares = ₹ 1637.50.

$\therefore$ Cost price per share ₹ 327.50.

Example 4.9 : Mr. *Pradeep invested ₹3,100 in 6% shares at ₹124. How much divided will he get ? (Face value = ₹100)* **(April 2010, 2016)**

Solution :

Market value of the share = ₹ 124

Amount invested = ₹ 3100

Number of shares purchased = $\frac{₹\,3100}{124} = 25$

Face value of each share = ₹ 100

Face value of 25 shares = 25 × 100

= ₹ 2500

$\because$ Mr. A received dividend at 6%

$\therefore$ Dividend received by him = ₹ 2500 × $\frac{6}{100}$

= ₹ 150

Example 4.10 : *A man invested ₹2000 in 10% shares at 125 of company 'A' and ₹2400 in 15% shares at 120 of company 'B'. Which investment is more profitable ? Why ?*

Solution : The man invested ₹ 2000 in 10% shares at 125 in company 'A'.

Number of shares = $\frac{2000}{125} = 16$

So. Dividend on 16 shares of company A $= 16 \times 10$

$= ₹\ 160$

$\therefore$ % return is $\frac{160}{20} = 8$

The man also invested ₹ 2400 in 15% shares at 120 in company 'B'.

$\therefore$ Number of shares $= \frac{2400}{120} = 20$

So, Dividend on 20 shares of company B $= 20 \times 15 = ₹\ 300$ $\therefore$ % return is $\frac{300}{24} = 14.5$

Therefore, the investment in company B is more profitable than company A.

Example 4.11 : *Pragat invested ₹ 13,568/- in 7% shares at ₹ 106/-. Find his profit at the end of the year. [F.V. 100].* **(April 2008)**

Solution : Pragat invested ₹ 13568/- in 7% shares at 106/-.

$$\text{Number of shares} = \frac{13568}{106} = 128$$

$$\text{Profit on 128 shares} = 128 \times ₹\ 7\ (7\%\ \text{of}\ ₹\ 100) = 896$$

Example 4.12 : *Which of the following is the better investment ?*

(i) 8% at ₹ 80/-.

(ii) 15% at ₹ 120/- [F.V. = ₹ 100] **(April 2008, 2011, 2016, 2017)**

Solution : Two companies have shares 8% at 80 and 15% at 120. The face value in each case is 100. Let us find percentage return in each case.

For the first company : For ₹ 80 the return is 8 for ₹ 100 the return is

$$\frac{100}{80} \times 8 = 10$$

For the second company : For ₹ 120 the return is 15 for ₹ 100 the return is

$$\frac{100}{120} \times 15 = 14.5$$

As the percentage return in second case is more than first case, investment in the second company is more profitable.

Example 4.13 : *Mrs. 'A' buys 100 shares of ₹ 100 each at ₹ 125 of a company. If company pays dividend at 12% what is the percentage return on her investment ?*

Solution : The dividend is declared on the face value ₹ 100 at the rate of 12%. Mrs. A will get $100 \times \frac{12}{100} = ₹\ 12$ divided on an investment of ₹ 125 on each share.

On ₹ 125, return is ₹ 14.

$\therefore$ On ₹ 100, return is ₹ 9.6

$$\left(\frac{1200}{125} = 9.6\right)$$

$\therefore$ % return is 9.6.

Example 4.14 : *A person invested ₹7000 in 8% shares at ₹140. How much dividend will get ?* (April 20

Solution : Since the price of a share is ₹ 140, he will get 50 shares in ₹ 7000.

∴ At ₹ 8 per share, he will get $50 \times 8 =$ ₹ 400 as dividend.

Example 4.15 : Sahib holds 300 equity shares of ₹ 10 each of a company. The comp issues bonus shares in the ratio of 5 : 3. Company declared dividend of 15% on enlarged cap How much dividend will Sahib receive ? (April 20

Solution : 5 bonus shares are issued for 3.

∴ For 300 equity shares, Sahib will receive 500 bonus shares.

∴ Now he holds 800 shares.

At 15%, the divided on each share (of ₹ 10) is ₹ 1.5.

∴ His income by way of dividend = $1.5 \times 8000 =$ ₹ 1200/-

Example 4.16 : Two companies have shares of 6% at 84 and 12% at 140. In which of shares would the investment be more profitable ?

(April 2011, 20

Solution : For 1st company : Market value (price) of the share = ₹ 84. Rate dividend = 6%.

The share-holder will get ₹ 6 (6% of ₹ 100) as income on ₹ 84.

∴ % Return on investment $= \frac{6}{84} \times 100 = \frac{600}{84} = 7.1429\%$

For 2nd company : Market value (price) of the share = ₹ 140, Rate of dividend = 12%.

The share-holder will get ₹ (12% of ₹ 100) as income on ₹ 140.

% Return on investment $= \frac{12}{140} \times 100 = \frac{1200}{140} = 8.5714\%$

Since in 2nd company % rate of return on investment is higher than in 1st company investm in 2nd company would be more profitable.

∴ Invest in 2nd company (i.e. shares of 12% at 140) would be more profitable.

Exercise (4.1)

Theory Questions :

1. Explain the term 'shares'.
4. Explain the term dividend. State its types. (Nov. 20
3. Explain the term "Bonus Shares". (April 2007, 2009, Nov. 20
4. Define the following terms : (April 20

   (i) Share. (ii) Dividend. (iii) Bonus shares.

Exercise (4.2)

Numerical Problems :

1. A man purchased shares of face-value ₹ 3,200 by investing ₹ 4,000. What was the market price of a share ? If the shares fetched 6% dividend, what percentage of dividend did he get on his investment ?
2. Suresh invested ₹ 1,080 in shares of face value ₹ 50 at ₹ 54. After receiving dividend on them at 8% he sold them at 54. In each of the transactions he paid 2% brokerage. How much did he gain or lose in the overall transaction ?
3. A man invested ₹ 6,200 in 6% shares at 124. How much dividend will he get ? How much per cent of dividend does he get on his investment ?
4. Shah spent ₹ 7,560 in purchasing 5% shares at 126. After getting the dividend on them, he sold them at the same price. In each transaction he had to pay 2% brokerage. Did he gain or lose in the total transaction ? By how much ?
5. A man invested ₹ 13,568 in 7% shares at 106 and ₹ 12,648 in 11% shares at 124. How much income would he get in all ?
6. Hussen and Altaf each invested ₹ 4,550 in $5\frac{1}{2}$ at 91 and $7\frac{1}{2}$ at 130 respectively. Whose investment is more profitable and by how much ?
7. A man purchased shares worth ₹ 18,900 when the market price was ₹ 94.50. Out of those shares he sold shares of face value ₹ 12,600 when the market rate was 104, and sold the remaining shares at 98. He had to pay 1.5% brokerage each time. What was his gain or loss on the whole ?
8. The amount realised by selling 8% shares at 144 of the face value of ₹ 2,400 was invested in 6% shares at 96. What would be the difference in annual income ?
9. Two companies have shares of 13% at 122 and 17% at 150 respectively. In which of the shares would the investment be more profitable ?
10. The capital of a company consists of 10 lack, 8% cumulative preference shares and of ₹ 10 each and 50 lac equity shares of ₹ 10 each. The company could not declared dividend on preference shares. In the third year company decided to pay 12% dividend on equity shares. Find the total amount paid by the company, by way of dividend.
11. The ordinary share capital of a company is thrice preference capital. Preference shares carry 8% dividend. When the distributable profit amounted to ₹ 22 lacs, equity holders received dividend at 12%. Find amount of each kind of capital.

14. Mahesh purchased 400 shares of a company of ₹ 10 at ₹ 80 each through a broker. Mahesh paid 1.5% brokerage and 0.5% of the market price towards transfer charges. The company declared a dividend of 25% and declared bonus in the ratio 1 : 4. Mahesh sold all the shares at ₹ 60 each. Find his net gain/loss.

13. Mr. X invested ₹ 12,400 in 14.4% shares at ₹ 124. How much dividend will he get ?

14. Aditya holds 300 shares of ₹ 10 each. The company issues bonus shares in the ratio 3 : 5. The company declared a dividend of 25% on the charged capital. What is the average rate of return on his investment. **(April 2015)**

15. Which investment is better 12% at ₹ 120 or 10% at ₹ 140. Justify. (April 20..)
16. Mohnish invested 5000 in 8% at ₹ 120. How much dividend will he get ? (April 20..)
17. Ravindra invested 4500 in 9% at ₹ 1.25. How much dividend will he get ? (April 20..)

Answers (4.2)

1. ₹ 125, 4.8%.
2. No gain, no loss.
3. ₹ 300, 4.84%.
4. Gain of ₹ 60.
5. ₹ 2,108.
6. Hussen's investment is profitable by ₹ 14.50.
7. Gain of ₹ 856.
8. Gain ₹ 24.
9. 17% at 150
10. ₹ 84 lacs.
11. Preference capital ₹ 50 lacs.
14. Loss : ₹ 1640.

Multiple Choice Questions

Choose the correct alternative from the following questions :

1. Find the number of shares that can be bought for ₹ 8200, if the market value is ₹ 20 ea... with brokerage being 4.5%.
   (a) 450
   (b) 500
   (c) 400
   (d) 410
2. Company pays 14.5% dividend to its investors. If an investor buys ₹ 50 shares and gets 2... on investment, at what price did the investor buy the shares ?
   (a) 6.26
   (b) 25
   (c) 50
   (d) 14.5
3. Major industries has issued 50,000 ordinary shares of face value ₹ 10 each. The comp... declared a total dividend of ₹ 55,000. Find the rate of dividend paid by the company
   (a) 11%
   (b) 12%
   (c) 13%
   (d) 14%
4. Find the number of shares if the total dividend at 8% on the shares with face value ₹ 10 ... ₹ 240.
   (a) ..0
   (c) 350
   (d) 250
5. Find the total dividend at 10% on 340 shares of face value ₹ 100 each.
   (a) ₹ 4003
   (b) ₹ 3004
   (c) ₹ 3040
   (d) ₹ 3400
6. Find the market price of share at the time of purchase if an investment of ₹ 1,09,350 in ... ₹ 100 shares gave a total dividend of ₹ 9000.
   (a) 243
   (b) 234
   (c) 244
   (d) 233

7. Find the face value of a share if share purchased at a market price of ₹ 180 each by investing ₹ 4,41,000 gave a total dividend of ₹ 1470 at 6% rate of dividend.
   (a) ₹ 200
   (b) ₹ 102
   (c) ₹ 100
   (d) ₹ 101

8. What is the rate of return on investment if Mr. X bought 350 shares of nominal value ₹ 10 at ₹ 50 each and received 8% dividend ?
   (a) 1.5%
   (b) 1.6%
   (c) 1.4%
   (d) 1.8%

9. Find the market price of a 20% ₹ 100 share giving 8% yield.
   (a) ₹ 250
   (b) ₹ 205
   (c) ₹ 520
   (d) ₹ 502

10. Find the number of shares purchased by Mr. Shantanu who invested ₹ 2,79,837 in ₹ 5 shares quote at ₹ 62, paying 0.3% brokerage.
   (a) 4050
   (b) 4005
   (c) 4500
   (d) 5400

Answers

1. (c)	4. (b)	3. (a)	4. (b)	5. (d)	6. (a)
7. (c)	8. (b)	9. (a)	10. (c)		

(B) MUTUAL FUND

4.8 Introduction

Whenever a person has money to invest, he weighs various options. In olden days, people had only two choices namely gold and land. After the introduction formal banking system, people started earning by way of interest on their bank deposits. Due to corporate world, a new avenue was opened to the people, namely **shares**.

Mutual fund is a step forward which tries to minimize the risk, one faces in the stock market.

The mutual fund industry in India began in 1963 with the formation of unit trust of India (UTI). Much later in 1987, SBI mutual fund was the first non-UTI mutual fund in India.

Today, there are 43 assets under management (AUMs) of mutual funds in India and the total number of mutual fund schemes is around 2800.

4.9 Types of Mutual Funds

Broadly, mutual funds can be classified into 3 categories :

1. **Equity Schemes :** These schemes invest a large portion of the funds in equity shares or equity linked instruments.

4. **Debt Schemes :** These schemes invest in debt instruments, namely, bonds and cash.

3. **Hybrid Schemes :** These schemes invest in both, equity and debt instruments.

4.10 Definition and NAV

Definition : Mutual fund is a trust that collects money from number of investors with common investment objectives. The trust then invests the money in equities, bonds, money market instruments and other securities. Each investor owns units which is a portion of holdings of the fund. The income/gains generated by this collective investment is distributed proportionately amongst the investors after deducting the expenses.

The combined holdings of a mutual fund are known as its portfolio.

Net Asset Value (N.A.V.) :

The N.A.V. represents market value of a unit of the fund. This is the price at which investors buy units from a fund or sell it back to the fund. It is calculated by dividing the total value of assets in a portfolio, minus all its liabilities.

4.11 Advantages of Investment in Mutual Funds

1. **Professional Management :** The fund managers do the research and select securities. They monitor the performance of the securities regularly.
4. **Diversification :** Since mutual funds invest in various companies/industries, it helps lower risk of the investor.
3. **Affordability :** Most mutual funds set a relatively low amount for initial investment and subsequent purchases. This helps a person with marginal savings to invest in a mutual fund.
4. **Liquidity :** Mutual fund investors can easily redeem their units at any time.

4.12 Benefits of Mutual Funds

1. **Dividend Payment :** If a fund earns income from dividends on stocks or interests on bonds, then the fund pays divided to unit holders, after deducting expenses.
4. **Capital Gains Distribution :** When a fund sells a security whose price is increased, the fund has capital gain. At the end of the year the fund distributes this capital gain to investors (after considering capital loses, if any).
3. **Increased N.A.V. :** If the market value of a fund's portfolio increases after deducting expenses, the value of the fund also increases. The higher N.A.V. reflects higher value of one's investment.

4.13 Risks

1. All funds carry some level of risk. If the value of securities held by a fund goes down investors may lose some money.
4. Dividends or interest payments also change as market conditions change.
3. The past performance of a fund may not predict future returns. However, past performance helps in determining volatility of a fund. The more volatile a fund, the higher is the investment risk.
4. Integrity of fund managers.
5. Political interference.
6. Huge commission./fees of fund managers.

4.14 Entry Load and Exit Load

Running a mutual fund incurs cost. For this, mutual funds charge (levy) entry load/exit load or both on various schemes.

Entry Load : This is the charge, a fund levies on the investor at the time of purchase. This is generally in percentage. If the face value of a unit is ₹ 100 and entry load is 2%, then an investor has to pay ₹ 102 to purchase a unit. Now-a-days very few mutual funds charge entry load.

Exit Load : This is the charge, an investor has to pay to the fund while selling units of a scheme. It is also in percentage.

It is levied on the market value of the unit (N.A.V.) : For example, if the market value of a unit is ₹ 140 and exit load is 2%, then the mutual fund company charges ₹ 4.80 on purchase (redeem) of a unit. Thus the investor will receive ₹ 137.20 per unit.

Solved Examples

Example 4.1 : *Amit invests ₹ 10,000 in a mutual fund. Entry load is 2% and exit load is 1%, before 2 years and nil after 3 years. He receives dividend of ₹ 1,000, ₹ 1,100 and ₹ 1,200 for 3 years in a cumulaiive dividend scheme. What is his*

(a) total rate of return.

(b) annualised rate of return, if he redeems all units after 3 years ?

Solution : Due to entry load of 2%, his outgo is

₹ 10000 + ₹ 200 = ₹ 10,200.

He has received a total amount of

₹ 10,000 + 1000 + 1100 + 1200 = 13,300.

Since he sells the unit after 3 years, he does not have to pay any exit load.

$$\therefore \quad \text{Total return} = 13{,}300 - 10{,}200 = 3{,}100$$

$$\therefore \quad \text{Rate of return} = \frac{3100 \times 100}{10200} = 30.39$$

$$\text{Annualised return} = \frac{30.39}{3} = 10.13$$

Example 4.2 : *Salmaan invested ₹ 50,000 in a mutual fund scheme with nil entry load and 2% exit load. He received a dividend of ₹ 3000, 4000 and 4500 for first, second and third year. He redeemed all his units after 3 years. What is the rate of return ?*

Solution : Since the rate of exit load is 2%, total exit load on units of ₹ 50,000 is ₹ 1,000.

∴ At the end of 3 years, Salmaan receives

₹ 50,000 + 3000 + 4000 + 4500 − 1000 = 60,500.

$$\therefore \quad \text{His net gain} = ₹\ 10{,}500$$

$$\therefore \quad \text{Total rate of return} = \frac{10{,}500 \times 100}{50{,}000} = 21\%$$

$$\text{Annualised rate of return} = 7\%.$$

Example 4.3 : *Prakash invests ₹ 1,00,000 in a mutual fund scheme in which there is no entry load and no exit load. He received dividend of ₹ 10,000 for first year, nil for the second year and 15,000 for third year. At the end of third year, he redeems all units, but N.A.V. has dropped to 95000. What is his annualised return ?*

Solution : In all, he has received a dividend of ₹ 25,000. But due to a drop in NAV, he will receive ₹ 95000 for his units.

∴ At the end of 3 years, he has received ₹ 1,20,000.

∴ His gain is 3 years is ₹ 20,000 i.e. 20%

$$\therefore \text{ Annualised rate of return} = \frac{20}{3} = 6.66\%$$

Example 4.4 : Anita invests ₹ 2,00,000 in a debt fund for 4 years. She received a dividend of ₹ 16,000 for first year, 12000 for second year, 14000 for third year and 15000 for fourth year. What is the rate of return if NAV remains unchanged ?

Solution : By way of dividend, Anita received ₹ 57,000.

$$\therefore \quad \text{Rate of return} = \frac{57000 \times 100}{200000} = 28.5\%$$

$$\text{Annualised rate of return} = \frac{28.5}{4} = 7.125\%$$

Example 4.5 : *Suresh invested ₹ 2,50,000 in a mutual fund Scheme with entry load of 1% and exit load 2% (for 3 years). He had to redeem all the units after 2 years when NAV has dropped to 2,30,000, in the mean while, he had received a dividend of ₹ 15,000. What is his gain or loss percent ?*

Solution : By 1%, entry load = ₹ 2,500.

At 2%, exit load = ₹ 4600 (on NAV at time of exist)

∴ His outgo = 250000 + 2500 + 4600 = 2,57,100

At the time of sale, he will receive ₹ 2,30,000.

Adding dividend, he has received ₹ 2,45,000.

∴ Loss = 2,57,100 − 2,45,000 = ₹ 12,100/-

$$\therefore \quad \text{Rate of loss} = \frac{12100 \times 100}{2,50,000} = 4.84 \text{ (for 2 years)}$$

∴ Annualised rate to loss 4.42%.

Exercise (4.3)

1. Satish invests ₹ 1,50,000 in a mutual fund scheme with no entry load and exit load of 3% upto 3 years. He receives dividend at 5% per year for 7 years. At the end of 7 years, the NAV is ₹ 1,80,000. What is the rate of return ?
4. Geeta invested ₹ 80,000 in a cumulative (no dividend payout) mutual fund scheme, in which there is no entry load or exit load. The NAV after 7 years is 2,10,000. What is the rate of return ?
3. Thomas invested ₹ 1,25,000 in a mutual fund scheme with no entry load and no exit load. He received a dividend of 6%, 8% and 10% for first three years. At the end of three years the NAV dropped to ₹ 1,00,000. What is the rate of return ?

4. Mohan had a fixed deposit of ₹ 5 lakhs in a Nationalised Bank, which fetched him 7% p.a. interest. He decided to park his funds in a mutual fund scheme (NAV 100) with no entry load but 4.5% exit load, if one exits the scheme before completion of 5 years. The scheme declared dividend of 8%, 7%, 10% and 5% for first four years. Mohan redeemed all units at the end of 4 years. When NAV was 90. Was his decision to shift to mutual fund beneficial ?
5. Mr. X invest ₹ 80,000/- in a mutual fund scheme in which dividend is cumulative. The entry load is 1% and exist load is 2% upto 3 years. Company declares a dividend of 8%, nil, 6% and 10% for four years. What is annualised return at the end of 4 years ?
6. Ganesh invested ₹ 40,000/- is a mutual fund scheme. Entry load is 1% and exit load 2% upto 3 years. NAV at initial point is ₹ 40,000/-. If Ganesh redeems all units after 2 years, when NAV is 60,000/-, what is rate of return ?
7. A person invests ₹ 25,000/- in a mutual fund scheme. There is no entry load but 2% exist load if units are redeemed within 3 years of 8%, 12%, 10%, 5% for four successive years. He sells all units after 4 years. When NAV is dropped to ₹ 21,000. Does he gain or lose ?

   Find annualised returns of gain/loss.
8. Mr. Choudhari invested ₹ 1,00,000 in a mutual fund for which entry load was 1% and exit load 2% if redeemed in 3 years. After 2 years Mr. Choudhari was in need of money so he sold units of ₹ 40,000. When NAV remained same. Company declared dividend of 8%, 5%, 6% and 9% for 4 years. At the end of 4 years Mr. Chaudhari sold remaining units when NAV was ₹ 65000. Find the gain/loss in entire transaction.

Answers (4.3)

1. Annual rate of return = 7.85%
4. Annual rate of return = 23.21%
3. Annual rate of return = 1.33%
4. No, loss of ₹ 51250
5. 6%
6. 15.38, annualised : 7.7
7. ₹ 4750 gain is 4 years. Annualised gain : 4.75%
8. Gain ₹ 20,200

Multiple Choice Questions

Choose the correct alternative from the following questions :

1. Sheela sold 125.6 units of mutual fund when its NAV was ₹ 15.35. If the exit load is 0.6% then in this deal how much money has she received ?
   (a) ₹ 1916.39
   (b) ₹ 1916.93
   (c) ₹ 1619.39
   (d) ₹ 1619.93
2. Vijay invested ₹ 9000 in Tata Mutual Fund, when NAV was ₹ 150. He sold all his units when NAV touched ₹ 165. Find his gain in the transaction.
   (a) ₹ 700
   (b) ₹ 500
   (c) ₹ 900
   (d) ₹ 800

3. Rekha invested ₹ 7000 in a mutual fund when its NAV was ₹ 14.50 if 20% dividend is declared by the mutual fund, then how much amount Rekha gets ?
   (a) ₹ 1102
   (c) ₹ 1201
   (b) ₹ 1120
   (d) ₹ 1210
4. Closed-end funds have lower cash requirements than open-end funds because ....
   (a) closed end funds have limited life
   (b) open-end funds allow investors to redeem their shares at any time
   (c) both (a) and (b)
   (d) none
5. Which type of fund invests in debt securities with very short maturities ?
   (a) money market mutual fund
   (b) bond fund
   (c) index fund
   (d) none
6. Which type of fund is most likely to have the lowest management fee ?
   (a) bond fund
   (c) management fund
   (b) index fund
   (d) none
7. Most mutual funds are members of a group of mutual funds known as ......
   (a) a family of funds
   (b) a no-load funds
   (c) a dual purpose funds
   (d) none
8. Shares of closed-end funds often sell ......
   (a) exactly at the net asset value
   (b) at a discount to the net asset value
   (c) at the end asset value
   (d) at a premium to the net asset value
9. Which of the following is not a benefit of investing in mutual funds ?
   (a) transaction cost
   (b) diversification
   (c) professional management
   (d) none
10. Shares of open-end funds often trade :
    (a) at a discount to the net asset value
    (b) at a price premium to the net asset value
    (c) exactly at the net asset value
    (d) none

Answers

1. (a)	4. (c)	3. (b)	4. (b)	5. (a)	6. (b)
7. (a)	8. (b)	9. (a)	10. (c)		

Chapter 5...

Matrices and Determinants

(Up to Order 3 Only)

Contents ...

Key Words :

- Determinant, matrix, minors and cofactors, adjoint and inverse of a square matrix, system of equations, homogeneous and non-homogeneous system.

Objectives :

- To understand applications of matrices in business.

5.1 Introduction

Although the title of this chapter is "Matrices and Determinants", we begin with Determinants for two reasons. Firstly you are already familiar with 2×2 determinants in your school and secondly, the knowledge of determinant is a pre-requisite for learning definitions of some special matrices like singular and non-singular.

5.2 Determinants

In standard X, you have studied the use of determinants of order 2 in the solution of two linear equations in two unknowns. A determinant D of order 2 may be written as

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, where a, b, c, d are certain numbers. They are called the elements of the determinant. The expression ad − bc is the expansion or the value of the determinant and we write $D = ad - bc$.

For example, $\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (-1 \times 3) = 11,$

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = (1 \times 6) - (3 \times 2) = 0,$$

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \text{ etc.}$$

5.3 Determinant of Order 3

A determinant of order 3 contains $3 \times 3 = 9$ elements in all. It is convenient to take nine elements as $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$. These are arranged below in the form of a square array and enclosed between two vertical lines.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This is a determinant of order 3. Here a_1, b_1, c_1 form the first row, a_2, b_2, c_2 form the second row and a_3, b_3, c_3 form the third row. These rows may be denoted by R_1, R_2, R_3 respectively. Similarly, a_1, a_2, a_3 form the first column b_1, b_2, b_3 form the second column and c_1, c_2, c_3 form the third column. These three column may be denoted by C_1, C_2, C_3 respectively. The determinant may be denoted by D.

The value of the expansion of this determinant is given by the following expression :

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad \text{... (5.1)}$$

$$= a_1 (b_2 c_3 - c_2 b_3) - b_1 (a_2 c_3 - c_2 a_3) + c_1 (a_2 b_3 - b_2 a_3)$$

$$= a_1 b_2 c_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3 \quad \text{... (5.2)}$$

Expression (5.1) is written down as follows. Elements a_1, b_1, c_1 of R_1 appear in this expression with alternately positive and negative signs and each is multiplied by a certain determinant of order 2. The element a_1 is in the first row and the first column. If we imagine all the elements in this row and column as deleted, there remains the array $b_2\ c_2$, a_1 is multiplied by the determinant of $b_3\ c_3$ order 2 formed by this array. Similarly, b_1 and c_1 are multiplied by the determinants of order 2 formed by the arrays $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$. The determinants of order 2 are then expanded and finally expression (5.2) results after simplification. This expansion is called the expansion by the first row.

Solved Examples

Example 5.1 : *Let* $D = \begin{vmatrix} 4 & -3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$

Solution : Following the procedure explained above, we get

$$D = 4\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 4(-4-1) + 3(-2-3) + 2(1-6)$$

$$= -20 - 15 - 10 = -45$$

Example 5.2 : *Let* $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

$$= a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix}$$

Solution : $= a(a^2 - bc) - b(ac - b^2) + c(c^2 - ab)$

$$= a^3 + b^3 + c^3 - 3abc$$

Example 5.3 : *Expand the determinant* $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Solution : We have

$$D = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$$

$$= abc - af^2 - ch^2 + fgh + fgh - bg^2 = abc + 2fgh - af^2 - bg^2 - ch^2$$

Example 5.4 : *Find the value of x if* $\begin{vmatrix} 3+x & 4+x & 5+x \\ 1 & -1 & 2 \\ 1 & 4 & 2 \end{vmatrix} = 0$

Solution : Expanding the determinant on the L.H.S, we get

$$(3+x)(-2-8) - (4+x)(2-2) + (5+x)(4+1) = 0$$

$-30 - 10x - 0 + 25 + 5x = 0 \therefore -5x - 5 = 0$ $\therefore x = -1$

Example 5.5 : *Find the value of x if* $\begin{vmatrix} 5 & 5 & x \\ x & 5 & 5 \\ 5 & 5 & 4 \end{vmatrix} = 0$

Solution : Expanding the determinant on the L.H.S. we get

$5\,(20-25) - 5\,(4x-25) + x\,(5x-25) = 0$

$\therefore -5 - 4x + 25 + x^2 - 5x = 0 \quad \therefore \quad x^2 - 9x + 20 = 0$

$\therefore (x-4)(x-5) = 0 \qquad \therefore \quad x = 4$ or $x = 5$.

Example 5.6 : *Evaluate :* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$. **(Oct. 2008, April 2017 BBA II)**

Solution : $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$= a\,(bc - f^2) - h\,(hc - gf) + g\,(hf - gb)$

$= abc - af^2 - h^2c - hgf + hgf + g^2b$

$= abc - af^2 - bg^2 - ch^2$

Exercise (5.1)

Evaluate the following determinants. (Problem 1 to 6)

1. $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 4 & 3 \\ -10 & -2 & 4 \\ 0 & 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 10 & 20 & 30 \\ 30 & 40 & 50 \\ 50 & 60 & 70 \end{vmatrix}$

4. $\begin{vmatrix} -b & c & 1 \\ a & 1 & -c \\ -1 & a & -b \end{vmatrix}$

5. $\begin{vmatrix} a & -b & 1 \\ 1 & c & -a \\ -c & 1 & b \end{vmatrix}$

6. $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ 1 & 1 & 1 \end{vmatrix}$

Solve the following equations. (Problem 7, 8)

7. $\begin{vmatrix} x-1 & x+1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$

8. $\begin{vmatrix} 2 & x+1 & 4 \\ 4 & x+2 & x+1 \\ x+2 & 1 & x+1 \end{vmatrix} = 0$

9. Prove that $\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix} = a(b-c)(a-b)$

Hence find the value of $\begin{vmatrix} 5 & 5 & 5 \\ 5 & 4 & 4 \\ 5 & 4 & 2 \end{vmatrix}$

Answers (5.1)

1. 3	2. 42	3. 0
4. $a^2+b^2+c^2+1$	5. $a^2+b^2+c^2+1$	6. 0
7. $x = 3$	8. $x = 0$ or $x = -3$ or $x = 5$	9. 10

5.4 Properties of Determinants

1. *The value of a determinant remains unchanged if its rows and columns are interchanged.*

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note that first row becomes first column, second row becomes second column and third row becomes third column.

Remark : In view of this property, any result applicable to rows is also applicable to columns of a determinant.

2. *The value of determinant is changed in sign only, if any two rows (or columns) in it are interchanged.*

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$

Column 1 is interchanged with column 3. This is denoted by C_{13}.

3. *The value of a determinant is zero if any two rows (or columns) in it are identical.*

For example, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$ $(\because R_1 \equiv R_3)$

4. *If all the elements of a row (or column) are multiplied by a constant, then it is equivalent to multiplying the determinant by that constant.*

For example, $\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note : The property, in other words, states that if all elements of a row (or column) has factor k it can be taken out as coefficient of determinant.

For example, $\begin{vmatrix} 2 & 3 & 10 \\ -1 & 4 & -15 \\ 7 & 2 & 25 \end{vmatrix} = 5 \begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 7 & 2 & 5 \end{vmatrix}$

(5 is taken common from C_3)

5. *Two determinants can be added if they have two identical rows (or columns).*

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ l & m & n \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+l & b_2+m & c_2+n \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note that identical rows (columns) remain as they are after addition. Only the unequal elements get added.

For example, $\begin{vmatrix} 2 & -3 & 4 \\ 10 & 8 & 5 \\ 5 & 6 & -7 \end{vmatrix} + \begin{vmatrix} 5 & 1 & -10 \\ 10 & 8 & 5 \\ 5 & 6 & -7 \end{vmatrix} = \begin{vmatrix} 7 & -2 & -6 \\ 10 & 8 & 5 \\ 5 & 6 & -7 \end{vmatrix}$

(Corresponding elements of first row are added. R_2 and R_3 remain the same.)

6. *The value of a determinant remains the same if k multiples of any row (or column) are added to corresponding elements of any other row (or column).*

For example, Consider the determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D \text{ (say)}$$

if we add k multiples of first row and m multiples of third row to second row then we get the determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+ka_1+ma_3 & b_2+kb_1+mb_3 & c_2+kc_1+mc_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = D' \text{ (say)}$$

Then $D = D'$

As a simple case, consider the determinant

$$D = \begin{vmatrix} 5 & 4 \\ +1 & 6 \end{vmatrix} = 6\times 5 - (1)(4) = 30 - 4 = 26$$

Suppose elements of second row are multiplied by 100 and added to first row, then

$$\begin{vmatrix} 5+100 & 4+600 \\ 1 & 6 \end{vmatrix} = \begin{vmatrix} 105 & 604 \\ 1 & 6 \end{vmatrix}$$

$$= (105\times 6) - (604\times 1)$$

$$= 630 - 604$$

$$= 26$$

Note : By "without expanding the determinant" we mean by using properties of determinant mentioned above.

Notation : We know that R_1, R_2, R_3 denote first, second and third row. Also C_1, C_2, C_3 denote first, second, third column. For different operations on rows/columns following notations are used.

(1) kR_i : Multiplying every element of i^{th} row by k.

(2) R_{ij} : Interchange of i^{th} and j^{th} row.

(3) $R_i + kR_j$: Adding k multiples of j^{th} row to corresponding elements of i^{th} row.

Similarly, kC_i, C_{ij}, $C_i + kC_j$ are used for column operations.

Solved Examples

Example 5.7 : *Find the value of x if* $\begin{vmatrix} 2+x & 3+x & 4+x \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 0.$

(April 2011, April 2018)

Solution : $\begin{vmatrix} 2+x & 3+x & 4+x \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 0.$

$\therefore$ $(2+x)[(2\times 3)-(-1\times 1)] - (3+x)[1\times 3 - (2\times -1)] + (4+x)[1\times 1 - 2\times 2] = 0.$

$\therefore$ $(2+x)(6+1) - (3+x)(3+2) + (4+x)(1-x) = 0$

$\therefore$ $7(2+x) - 5(3+x) - 3(4+x) = 0$

$\therefore$ $14 + 7x - 15 - 5x - 12 - 3x = 0$

$\therefore$ $-x - 13 = 0$

$\therefore$ $x + 13 = 0$

$\therefore$ $x = -13$

Example 5.8 : *Evaluate* $\begin{vmatrix} 47 & 78 & 25 \\ 111 & 173 & 58 \\ 64 & 95 & 33 \end{vmatrix}$.

Solution : $\begin{vmatrix} 47 & 78 & 25 \\ 111 & 173 & 58 \\ 64 & 95 & 33 \end{vmatrix} = \begin{vmatrix} 47 & 78 & 25 \\ 0 & 0 & 0 \\ 64 & 95 & 33 \end{vmatrix}$ [(by $R_2 - (R_1 +$

$= 0$

Example 5.9 : *Evaluate* $D = \begin{vmatrix} 28 & 45 & 63 \\ 20 & 34 & 48 \\ 21 & 36 & 51 \end{vmatrix}$.

Solution : $D = \begin{vmatrix} 28 & 45 & 63 \\ 20 & 34 & 48 \\ 21 & 36 & 51 \end{vmatrix} = \begin{vmatrix} 8 & 11 & 15 \\ 20 & 34 & 48 \\ 1 & 2 & 3 \end{vmatrix}$ (by $R_1 - R_2, R_3 -$

$= \begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}$ (by $R_2 - 16R_3, R_1 - 5$

$= 3(6-0) - 1(12-0) + 0$

$= 18 - 12$

$= 6$

Example 5.10 : *Without expanding prove that*

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$$

Solution : $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$ (Interchanging rows and colum

$= -\begin{vmatrix} b & y & q \\ a & x & p \\ c & z & r \end{vmatrix}$ (by R

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$ (by C

$=$ R.H.S.

Exercise (5.2)

1. Without expanding the determinants, show that

(i) $$\begin{vmatrix} 7 & 2 & 5 \\ -a & -3 & 1 \\ 15 & 4 & 8 \end{vmatrix} + \begin{vmatrix} -1 & 2 & 5 \\ 5 & -3 & 1 \\ -3 & 4 & 8 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 & 5 \\ -2 & -3 & 1 \\ 6 & 4 & 8 \end{vmatrix}.$$

(ii) $$\begin{vmatrix} 1 & bc & bc\,(b+c) \\ 1 & ca & ca\,(c+a) \\ 1 & ab & ab\,(a+b) \end{vmatrix} = 0$$

2. Show that : $$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

3. Show that : $$\begin{vmatrix} a & b-c & b+c \\ c+a & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)\,(a^2+b^2+c^2)$$

4. Prove that : $$\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = (x-y)\,(y-z)\,(z-x).$$

5. Show that : $$\begin{vmatrix} a & b & c \\ b & c & a \\ b+c & c+a & a+b \end{vmatrix} = 3\,abc - a^3 - b^3 - c^3.$$

6. Prove that : $$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc.$$

7. Show that : $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$

5.5 Matrices

Introduction : Early development of matrix theory is due to European mathematicians Sylvester, Cayley and Hamilton. This branch of mathematics has applications in many other subjects such as Statistics, Physics, Economics, Psychology etc.

Many a times, it is convenient to present numerical data in the form of rectangular array of rows and columns. For example, (i) At recently concluded IPL matches various teams fared as follows :

Team	P	W	L	Points
Chennai Superkings	16	11	5	22
Mumbai Indian	16	11	5	22
Rajasthan Royals	16	10	6	20
Sunrisers Hydrabad	16	10	6	20
Royal Challengers Bangalore	16	9	7	18
Kings XI Punjab	16	8	8	16
Kolkata Knight Riders	16	6	10	12
Pune Warriers India	16	4	12	08
Delhi Daredevils	16	3	13	06

P : Played, W : Won, L : Lost

(ii) The daily sales of hot drinks from a shop for 5 days of a week are as follows :

$$\begin{array}{c} \\ \text{Mon.} \\ \text{Tue.} \\ \text{Wed.} \\ \text{Thu.} \\ \text{Fri.} \end{array} \begin{bmatrix} \text{Tea} & \text{Coffee} & \text{Milk} \\ 37 & 56 & 49 \\ 50 & 58 & 70 \\ 80 & 45 & 79 \\ 85 & 79 & 43 \\ 30 & 24 & 13 \end{bmatrix}$$

Such a system is called a **matrix**.

Definition of matrix, Notation for writing a matrix :

Definition : An arrangement of mn numbers in the form of a rectangular block of m rows and n columns, enclosed in rectangular brackets is called a matrix of the order m × n.

For example, $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 1 \end{bmatrix}$

Any element in a matrix may be located by stating the number of row and number of column in which the element occurs. Thus in matrix A above, element 2 is in place (1, 1), 3 is in place (1, 2), 1 is in place (2, 1) etc.

A useful notation for writing matrices in theoretical work is as follows :

We use letters such as a, b, c, d, ... with proper suffixes to denote the elements of a matrix.

For example, we may denote the elements in the places (1, 1), (1, 2), (1, 3), (2, 1), ... etc. by a_{11}, a_{12}, a_{13}, a_{21} ... etc. With this notation a matrix of the order 2 × 2 may be written as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. This matrix is then denoted by the corresponding capital letter A. Thus we write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ or just } A = [a_{ij}]\ i = 1, 2;\ j = 1, 2.$$

Similarly, a matrix of the order 3×2 may be written as

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \text{ or } B = [b_{ij}] \quad i = 1, 2, 3; j = 1, 2$$

Note :

(1) There should be no confusion between the ideas of a determinant and a matrix. Unlike a determinant, (a) the number of rows and the number of columns in a matrix need not be equal and (b) there is no such thing as the value or expansion of matrix.

(2) A matrix of the order 1×1 is regarded as a scalar. (i.e. just a real number)

Solved Examples

Example 5.11 : *Write down the matrices of the coefficients of the following sets of equations. State their orders.*

(i) $x + 2y - z = 2$
$2x - 5y + 6z = 3$

(ii) $x + z = 4$
$x + y + z = 6$
$2y + z = 5$

Solution : (i) There are 3 unknowns x, y, z and 2 equations. The required matrix is $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -5 & 6 \end{bmatrix}$. It is of the order 2×3.

(ii) There are 3 unknowns x, y, z and 3 equations. In the first equation, y is absent. We can say that coefficient of y is zero. Thus this equation may be written as $x + 0y + z = 6$. Similarly, taking the coefficient of x in the third equation as zero, it can be written as $0x + 2y + z = 5$.

Therefore the required matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. It is of the order 3×3.

Example 5.12 : *Find the elements a_{12}, a_{21}, a_{23} in the matrix.*

$$A = \begin{bmatrix} 5 & 4 & -3 \\ 2 & 1 & 6 \\ 7 & -2 & -1 \end{bmatrix}$$

Solution : a_{12} is the element in the first row and second column. It is 4. Thus $a_{12} = 4$. Similarly $a_{21} = 2$, $a_{23} = 6$.

Exercise (5.3)

1. Write down two matrices of each of the following orders :
(i) 2×2 (ii) 2×3 (iii) 3×2 (iv) 1×3 (v) 3×1

2. Write down the matrices of the coefficients of the following sets of equations. State their orders.
(i) $x + 2y = 5$
$5x - 3y = -1$
(ii) $x + y = 3$
$y + z = 5$
$2z + x = 7$
(iii) $x + y + z = 12$
$x + z = 8$
$y - z = -1$

3. Find the elements a_{22}, a_{32}, a_{33} in the following matrix :

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & 6 \\ -1 & 1 & -4 \end{bmatrix}$$

4. State the orders of the following matrices :

(i) $[1, 0, 0]$ (ii) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (vi) $\begin{bmatrix} 5 & 0 & 1 \\ 4 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}$

5. Write down the matrix A given that $a_{11} = a_{22} = a_{33} = 0$,

$a_{12} = -a_{21} = 2, a_{13} = -a_{31} = -4, a_{23} = -a_{32} = -1$.

Answers (5.3)

2. (i) $\begin{bmatrix} 1 & 2 \\ 5 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

3. $a_{22} = -5, a_{32} = 1, a_{33} = -4$

4. (i) 1×3 (ii) 2×1 (iii) 2×3 (iv) 3×1 (v) 3×2 (vi) 3×3

5. $A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -1 \\ 4 & 1 & 0 \end{bmatrix}$

5.6 Types of Matrices

Depending on the order and the nature of the elements, matrices are classified into various categories.

(1) Zero matrix : A matrix whose all elements are zero is called a *zero matrix* (or a null matrix).

For example, $[0, 0]$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all zero matrices. They are of the orders 1×2, 3×1, 2×2 and 3×3 respectively. A zero matrix is denoted by O. (Capital letter O).

(2) Row matrix : A matrix having only one row is called a *row matrix*. A row matrix containing n elements is a matrix of the order $1 \times n$.

For example, $[1 \;\; 2]$, $[2 \;\; 3 \;\; 4]$, $[0 \;\; 1 \;\; 2]$, $[p \;\; q \;\; r]$

are all row matrices. They are of the orders 1×2, 1×3, 1×3, 1×3 respectively.

(3) Column matrix (April 2015) : A matrix having only one column is called a *column matrix*. A column matrix containing n elements is a matrix of the order n × 1.

For example, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} k \\ l \\ m \end{bmatrix}$

are all column matrices. They are of the orders 2 × 1, 3 × 1, 3 × 1, 3 × 1 respectively.

(4) Square matrix (April 2015) : A matrix in which the number of rows is equal to the number of columns is called a square matrix. A square matrix having n rows and n columns is a matrix of the order n × n. It is usually called a *square matrix of order n* (instead of order n × n).

For example, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

are all square matrices. The first two are of order 2 and the next two are of order 3.

In a square matrix, the elements in the places (1, 1), (2, 2), (3, 3), ... are called the diagonal elements. (In the four matrices above, the diagonal elements are 1, 4; 1, 1; 0, 0, 0; a, b, c respectively.) They are said to form the principal diagonal or the leading diagonal of the matrix. In a square matrix, the elements other than the diagonal elements are called the *non-diagonal elements*. Thus in the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ b, c are the non-diagonal elements.

(5) Diagonal matrix (April 2015) : A square matrix in which all the non-diagonal elements are zero is called *a diagonal matrix*.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$

are diagonal matrices. Note that in a diagonal matrix, one or more of the diagonal elements may be zero while all the non-diagonal elements must be zero.

(6) Scalar matrix : A diagonal matrix in which all the diagonal elements are equal, is called a *scalar matrix*.

For example,

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

are scalar matrices.

(7) Unit matrix : A scalar matrix in which all the diagonal elements are 1 is called a *unit matrix* (or an identity matrix).

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are unit matrices of order 2 and 3 respectively. A unit matrix is denoted by I. If we want to mention the order of a unit matrix, we write I_2, for unit matrix of order 2, I_3 for unit matrix of order 3 etc.

(8) Upper triangular matrix (Oct. 2008) : A square matrix in which all the elements below the principal diagonal are zero is called an *upper triangular matrix*.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{vmatrix} p & q & r \\ 0 & 0 & m \\ 0 & 0 & k \end{vmatrix}$

are upper triangular matrices.

(9) Lower triangular matrix (Oct. 2008) : A square matrix in which all the elements above the principal diagonal are zero is called a *lower triangular matrix*.

For example, $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$

are lower triangular matrices.

A matrix which is both, upper triangular and lower triangular is clearly a diagonal matrix.

(10) Symmetric matrix : A square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is called } \textit{symmetric matrix}.$$

if $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$. This means the elements which are symmetrically situated w.r.t. the principal diagonal are equal.

For example, $\begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

are symmetric matrices.

(11) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called a skew-symmetric matrix. if $a_{ij} = -a_{ji}$ for all i and j. This means that the $(ij)^{th}$ element and the $(ji)^{th}$ element are equal in magnitude but have opposite signs. Further, all its diagonal elements are zero.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & p & q \\ -p & 0 & -r \\ -q & r & 0 \end{bmatrix}$$

are all skew-symmetric matrices.

5.7 Transpose of a Matrix

Consider the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$. It is of the order 2×3. Let us write another matrix B from A by writing the rows of A as columns of B. We have

$$B = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ 4 & 3 \end{bmatrix}$$

The matrix B is called the transpose of A and it is denoted by A' (read A prime).

Definition : The matrix obtained from a given matrix A by interchanging its rows and columns is called the transpose of A. Transpose of A is denoted by A'.

If A is of the order $m \times n$, then A' is of the order $n \times m$.

Clearly, the transpose of the transpose of A is the matrix A itself. i.e. $(A')' = A$.

Determinant of a square matrix

With any square matrix A we can associate a determinant which is formed by the same array of elements as the matrix A. This determinant is called the *determinant of the matrix A* and it is denoted by |A| (or det A).

For example, if

(i) $$A = \begin{bmatrix} 10 & 3 \\ 2 & 1 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 10 & 3 \\ 2 & 1 \end{vmatrix} = 4$$

(ii) $$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$$

(iii) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 8 \text{ etc.}$$

5.8 Singular and Non-singular Matrices (April 2010)

A square matrix A is called a singular matrix if $|A| = 0$ and a non-singular matrix if $|A| \neq 0$.

For example, let $$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}.$$

Then $$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{vmatrix} = 2 + 3 = 5$$

$\therefore$ $|A| \neq 0$ $\therefore$ the matrix A is a non-singular matrix.

Let $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}.$$

Then $$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1 - 2 + 1 = 0$$

$\therefore$ the matrix A is a singular matrix.

Exercise (5.4)

1. State the types of the following matrices :

(i) $[1, 2, 3]$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (iv) $\begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$

(v) $\begin{bmatrix} p & s & t \\ 0 & 0 & 0 \\ 0 & 0 & r \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (ix) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (xi) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (xii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(xiii) $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ (xiv) $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$

(xv) $\begin{bmatrix} 1 & -2 & 5 \\ 2 & 4 & 3 \\ -5 & -3 & 2 \end{bmatrix}$

2. Write down the transposes of the following matrices :

(i) $[1\ 3\ 5]$ (ii) $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3. Consider the symmetric matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -4 \\ 2 & -4 & 5 \end{bmatrix}$

Write down A'. What do you observe ?

4. Determine whether the following matrices are singular or non-singular :

(i) $\begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$ (vii) $\begin{bmatrix} 50 & 51 & 52 \\ 52 & 51 & 50 \\ 102 & 102 & 102 \end{bmatrix}$

Answers (5.4)

1. (i) row matrix (ii) zero matrix
(iii) column matrix
(iv) square matrix (v) diagonal matrix
(vi) upper triangular matrix
(vii) unit matrix (viii) scalar matrix
(ix) symmetric matrix
(x) diagonal matrix (xi) scalar matrix
(xii) unit matrix
(xiii) symmetric matrix (xiv) skew-symmetric matrix
(xv) square matrix

2. (i) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 3 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. (i) singular matrix (ii) non-singular matrix
(iii) non-singular matrix (iv) non-singular matrix
(v) singular matrix (vi) singular matrix
(vii) singular matrix (viii) singular matrix

5.9 Algebra of Matrices

Equality of matrices

Definition : Two matrices A and B are said to be equal if (i) they are of the same order and (ii) each element of one is equal to the element in the corresponding place of the other. If A and B are equal, we write A = B. If A and B are not equal, we write $A \neq B$.

(1) Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$

A is of the order 2×2 and B is the of the order 2×3. Since the orders of A and B are not the same, A and B are not equal. (i.e. $A \neq B$).

(2) Consider the matrices $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 0 \end{bmatrix}$

They are of the same order viz. 3×2 and their corresponding elements are equal. Therefore A and B are equal. (i.e. A = B)

We now proceed to define some basic operations on matrices. These are (i) Scalar multiplication (ii) Addition (iii) Subtraction (iv) Multiplication. You will see how these operations on given matrices gives rise to new matrices.

(1) Scalar multiplication

Let A be any matrix and c a scalar (i.e. a real number). Then the matrix obtained by multiplying every element of A by c is called the scalar multiple of A by c. It is denoted by cA. Clearly, cA is a matrix of same order as A.

Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ and let c = 2. Then

$$cA = 2\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 2 & 2\times 3 & 2\times(-1) \\ 2\times 1 & 2\times 0 & 2\times 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 & -2 \\ 2 & 0 & 8 \end{bmatrix}$$

Similarly, $-3A = \begin{bmatrix} -6 & -9 & 3 \\ -3 & 0 & -12 \end{bmatrix}$.

$$\frac{1}{2}A = \begin{bmatrix} 1 & 3/2 & -1/2 \\ 1/2 & 0 & 2 \end{bmatrix}$$

(Note that c need not be a positive integer.)

In particular, if we take c = − 1, then cA = − 1A. We write − A for − 1A. The matrix − A is called the negative of the matrix A and it is obtained by changing the sign of every element of A.

For example, if $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 0 & -1 \end{bmatrix}$, then $-A = \begin{bmatrix} -1 & -2 \\ 3 & -4 \\ 0 & 1 \end{bmatrix}$

(2) Addition of matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & -3 \end{bmatrix}$$

Matrices A and B are of the same order (viz. 2 × 3). The sum of A and B written as A + B is a matrix C (say), which is of the same order as A and B and whose elements are obtained by adding the corresponding elements of A and B. Thus

$$C = A + B = \begin{bmatrix} 1+2 & 2+(-1) & 2+1 \\ -2+3 & -1+(-2) & 4+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 1 & -3 & 1 \end{bmatrix}$$

Similarly, (i) if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ then $A + B = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

(ii) if $A = [3 \;\; 2 \;\; -1]$, $B = [-1 \;\; 1 \;\; 1]$ then $A + B = [2 \;\; 3 \;\; 0]$

We now state the following definition.

Definition : Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order m × n (say). Then their sum, denoted by A + B, is a matrix $C = [c_{ij}]$ of the order m × n and such that $c_{ij} = a_{ij} + b_{ij}$ for i = 1 to m, j = 1 to n.

Two matrices which are of the same order are said to be conformable for addition. If two matrices are not of the same order then their sum is not defined.

Commutative and Associative laws of addition of matrices

(i) $A + B = B + A$. This is known as the *commutative law of addition of matrices.*

(ii) $(A + B) + C = A + (B + C)$. This is known as the *associative law of addition of matrices.*

Let us verify these laws for three particular matrices A, B, C.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix}, C = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 4 & 3 \\ 1 & -3 \end{bmatrix}$,

$B + A = \begin{bmatrix} 4 & 3 \\ 1 & -3 \end{bmatrix}$ showing that $A + B = B + A$.

Further, $(A + B) + C = \begin{bmatrix} 4 & 3 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -5 \end{bmatrix}$

and $A + (B + C) = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & -5 \end{bmatrix}$

showing that $(A + B) + C = A + (B + C)$

In view of the associative law, we may just write $A + B + C$ to mean $(A + B) + C$ or $A + (B + C)$. (i.e. the parentheses may be dropped)

Also $A + A = [a_{ij} + a_{ij}] = [2a_{ij}]$ and $2A = [2a_{ij}]$

Therefore we can write $A + A = 2A$.

Similarly, we can write $A + A + A = 3A$ etc.

Solved Examples

Example 5.13 : Find the matrix X if $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + X = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$ **(April 2017)**

Solution : Clearly X is a square matrix of order 2. Let

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 + a & 3 + b \\ -1 + c & 5 + d \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$

$\therefore$ by the definition of equality of matrices,

$\therefore 2 + a = 5, 3 + b = 2, -1 + c = 3, 5 + d = 3$

$\therefore a = 3, b = -1, c = 4, d = -2$

$\therefore X = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

Example 5.14 : If $A = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$

find the matrix C such that A + B + C is a zero matrix.

Solution : Clearly C is a matrix of the order 2×3. Let

$$C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}. \text{ Then}$$

$$A + B + C = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 1 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$= \begin{bmatrix} 3+a & -1+b & 5+c \\ -1+d & 3+e & 1+f \end{bmatrix}$$

$A + B + C = O$ gives $3 + a = 0, -1 + b = 0, 5 + c = 0, -1 + d = 0,$
$3 + e = 0, 1 + f = 0 \quad \therefore \quad a = -3, b = 1, c = -5, d = 1, e = -3, f = -1$

$$\therefore \quad C = \begin{bmatrix} -3 & 1 & -5 \\ 1 & -3 & -1 \end{bmatrix}$$

(3) Subtraction of matrices

Let A and B be two matrices of the same order ($m \times n$, say). Let $A = [a_{ij}]$, $B = [b_{ij}]$. Then $A - B$ is a matrix of the same order as A and B and its elements are obtained by subtracting the elements of B from the corresponding elements of A. Thus if $C = [c_{ij}] = A - B$, then $c_{ij} = a_{ij} - b_{ij}$ for $i = 1$ to m, $j = 1$ to n.

For example, let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$. Then

$$A - B = \begin{bmatrix} 2-1 & 3-4 \\ -1-2 & 4-1 \\ 1-(-1) & 2-(-2) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\text{Similarly, } B - A = \begin{bmatrix} 1-2 & 4-3 \\ 2-(-1) & 1-4 \\ -1-1 & -2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \\ -2 & -4 \end{bmatrix}$$

In view of our definition of the negative of a matrix, we can say that $A - B = A + (-B)$. Also, for any matrix A, $A - A = O$ (zero matrix). If two matrices are not of the same order, then subtraction is not defined.

Note : If A, B, C are matrices of the same order and if $A + B = C$, then we have $A = C - B$, $B = C - A$. In particular, if $A + B = O$, then $A = -B$, $B = -A$.

Ex. 1 of § 3. 13 may now be worked out alternatively, as follows :

$$\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + X = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

Solved Examples

Example 5.15 : If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 6 \\ 4 & 4 \end{bmatrix}$

Find $2A + 3B - \frac{1}{2}C$.

Solution : We have

$$2A + 3B - \frac{1}{2}C = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ -12 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 13 \\ -6 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ -8 & 12 \end{bmatrix}$$

Example 5.16 : If $A = \begin{bmatrix} 2 & 4 & 3 \\ -3 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$, find a matrix X such that $2X + A - 2B = O$.

Solution : $2X + A - 2B = O \quad \therefore \; 2X = O + 2B - A = 2B - A$

$$\therefore \quad 2X = 2\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 3 \\ -3 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 6 \\ 4 & 8 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 3 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 & 3 \\ 7 & 9 & 10 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{2}\begin{bmatrix} 0 & -8 & 3 \\ 7 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 3/2 \\ 7/2 & 9/2 & 5 \end{bmatrix}$$

Example 5.17 : Find the values of x, y, z if

$$\begin{bmatrix} 2x-1 & 3 \\ 4 & 2 \\ 3z-1 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ 1 & y+3 \\ z & -4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 9 \\ 11 & 1 \end{bmatrix}$$

Solution : Adding the two matrices on the L.H.S. we get

$$\begin{bmatrix} 2x+6 & 5 \\ 5 & y+5 \\ 4z-1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 9 \\ 11 & 1 \end{bmatrix}$$

$\therefore$ by the definition of equality of matrices,

$2x + 6 = 10,\ y + 5 = 9,\ 4z - 1 = 11. \quad \therefore\ x = 2, y = 4, z = 3.$

Exercise (5.5)

1. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$. Write down 3A and A + A + A. Are they equal ? **(April 2017)**

2. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

   (i) Write down $A + B$, $A - B$, $B - A$, $2A + 3B$, $3A - 2B$.

   (ii) Find a matrix X such that $A - 2B + X = O$ (iii) Find a matrix X such that $2A - B + X = I$

3. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -4 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$, find a matrix C such that $A + B + C$ is a zero matrix.

4. If $A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$,

   $B = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

   find (i) $2A + B - C$, $A - B + C$, $A + 2B + 3C$ (ii) A matrix X such that

   $A - 2B + C + X = O$ (iii) Verify that $A - (B - C) = A - B + C$

5. If $A = \begin{bmatrix} 4 & 5 \\ 3 & 7 \end{bmatrix}$, find a matrix X such that $A - 2X = \begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix}$

Answers (5.5)

1. $3A = \begin{bmatrix} 6 & 3 \\ 3 & 6 \\ 9 & 3 \end{bmatrix} = A + A + A$

2. (i) $A + B = \begin{bmatrix} 3 & 4 \\ 7 & 3 \end{bmatrix}$, $A - B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$,

   $B - A = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$,

   $2A + 3B = \begin{bmatrix} 7 & 9 \\ 17 & 8 \end{bmatrix}$, $3A - 2B = \begin{bmatrix} 4 & 7 \\ 6 & -1 \end{bmatrix}$

   (ii) $X = 2B - A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

   (iii) $X = I + B - 2A = \begin{bmatrix} -2 & -5 \\ -5 & 1 \end{bmatrix}$

3. $C = \begin{bmatrix} 1 & 2 \\ -3 & -1 \\ 3 & -2 \end{bmatrix}$

4. (i) $2A + B - C = \begin{bmatrix} 6 & 3 & 1 \\ -3 & 6 & 2 \end{bmatrix}, A - B + C = \begin{bmatrix} 0 & 6 & -4 \\ 0 & 0 & -2 \end{bmatrix},$

$A + 2B + 3C = \begin{bmatrix} 1 & 7 & 10 \\ 12 & 1 & 4 \end{bmatrix}$

(ii) $X = 2B - A - C = \begin{bmatrix} 1 & -7 & 8 \\ 2 & 1 & 4 \end{bmatrix}$

(iii) $A - (B - C) = A - B + C = \begin{bmatrix} 0 & 6 & -4 \\ 0 & 0 & -2 \end{bmatrix}$

5. $X = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

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(4) Multiplication of matrices

Let $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ 4 & -2 & 5 \end{bmatrix}$

A is of the order 2×2 and B is of the order 2×3. We have taken the number of columns of A and the number of rows of B equal. The product of the matrices A and B, taken in this order, is defined. It is denoted by AB. It is a matrix which has the same number of rows as A and the same number of columns as B. Thus, here AB is of the order 2×3. Let us denote AB by $C = [c_{ij}]$. Then $i = 1, 2$ and $j = 1, 2, 3$.

The elements c_{ij} are obtained as follows :

To find c_{11}, we take the first row of A viz. [2 1], and the first column of B viz. $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and form the sum of the products of the corresponding elements. This sum gives c_{11}. Thus $c_{11} = (2 \times 1) + (1 \times 4) = 2 + 4 = 6$

To find c_{12}, we take the first row of A viz. [2 1] and the second cloumn of B viz. $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and form the sum of the products of the corresponding elements. This sum gives c_{12}. Thus $c_{12} = (2 \times 2) + [1 \times (-2))] = 2$

Similarly, $c_{13} = [2 \times (-1)] + (1 \times 5) = 3$

$c_{21} = (2 \times 1) + (3 \times 4) = 14$

$c_{22} = (2 \times 2) + [3 \times (-2)] = -2$

$c_{23} = [2 \times (-1)] + (3 \times 5) = 13$

$$\therefore \qquad AB = C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} 6 & 2 & 3 \\ 14 & -2 & 13 \end{bmatrix}$$

You will observe that in forming the product AB, the rows of A and the columns of B come in the picture. In the above illustration, A has two rows.

Let us denote them by R_1 and R_2 and write $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. B has three columns.

Let us denote them by C_1, C_2, C_3 and write $B = [C_1\ C_2\ C_3]$. Then AB may be written symbolically as;

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$

Note :

(1) Two matrices A and B such that the number of columns of A is equal to the number of rows of B are said to be *comformable for the product* AB.

For example, if A is of the order $m \times n$ and B is of the order $n \times p$ then A and B are conformable for the product AB which is of the order $m \times p$. In the product AB, A is called the *prefactor* and B is called the *postfactor*. A is said to be postmultiplied by B and B is said to be premultiplied by A.

(2) Two matrices A and B which are conformable for the product AB may not be conformable for the product BA.

For example, if A is a 3×3 matrix and B is a 3×2 matrix then we can form the product AB but not the product BA.

(3) Even if the products AB and BA are both defined, they may not be equal.

For example, if A is a 2×3 matrix and B is a 3×2 matrix then AB is a square matrix of order 2 and BA is a square matrix of order 3. Thus the orders of AB and BA are not the same. Hence the two products are not equal. In general, if A is matrix of the order $m \times n$ and B is a matrix of the order $n \times m$, then AB and BA are both defined and are square matrices of order m and n respectively. If $m \neq n$ then they are of different orders and hence cannot be equal. If A and B are both square matrices of the same order n, then AB and BA are also square matrices of order n. Even in this case, they are, in general, not equal, because their elements are formed in different ways. Thus matrix multiplication is not commutative.

We explain this point by following examples.

(1) Let $\qquad A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$

Then $\qquad AB = \begin{bmatrix} 3 & 6 \\ 3 & 7 \end{bmatrix}, \; BA = \begin{bmatrix} 0 & 7 & 5 \\ 3 & 5 & 4 \\ -5 & 8 & 5 \end{bmatrix}$

AB is a square matrix of order 2 while BA is a square matrix of order 3. They are of course not equal.

(2) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 8 & 2 \\ 5 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

Here AB and BA are of the same order but their corresponding elements are not equal. Hence $AB \neq BA$.

(3) If A and B are square matrices of the same order, then, in some special cases, we may have $AB = BA$.

For example,

Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$\therefore \quad AB = BA$ (Observe that here $B = I_2$.)

Let $A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 4 & 11 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 9 \\ 18 & 37 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 9 \\ 18 & 37 \end{bmatrix}$

$\therefore \quad AB = BA$

(**Note :** Two matrices A and B such that $AB = BA$ then they are said *to commute*.)

Associative law of matrix multiplication

Let A, B, C be three matrices of the orders $m \times n$, $n \times p$, $p \times q$ respectively. Then the product AB is defined and it is of the order $m \times p$. The product (AB) C is also defined and it is of the order $m \times q$. Similarly, the product BC is defined and it is of the order $n \times q$. The product A (BC) is also defined and it is of the order $m \times q$. Thus (AB) C and A(BC) are matrices of the same order viz. $m \times q$. Will these products be always equal ? We state (without proof) that the answers is yes. The two products are always equal. We do have

$$(AB)C = A(BC)$$

This result is known as the *associative law of matrix multiplication*.

Let us verify this result by an example.

Let $A_{3\times 2} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B_{2\times 2} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $C_{2\times 2} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Then $\quad AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \ (AB)\,C = \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$

$BC = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad A(BC) = \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$

Thus $\quad (AB)\,C = A\,(BC)$

In view of this law, each of the products (AB) C and A (BC) is denoted by ABC i.e. brackets are removed. One may find ABC as (AB) C or A (BC).

(5) Positive integral powers of a square matrix

Let A be a square matrix. Then the product AA is defined. We write $AA = A^2$. Then A^2A and AA^2 are defined.

We write $A^2A = AA^2 = A^3$.

Similarly, $A^3A = AA^3 = A^4$ and so on. In this way we can find A^n where n is any positive integer.

For example,

(1) if $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$

$A^3 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 10 & 7 \end{bmatrix}$ etc.

(2) if $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

then $\quad I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$I^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ etc.

We can show that $I^n = I$ where, n is any positive integer and I is unit matrix of any order.

Solved Examples

Example 5.18 : $A = \begin{bmatrix} -1 & 2 \\ 5 & 1 \end{bmatrix}$, find matrix X such that

$$2A + 3X = \begin{bmatrix} 4 & 16 \\ -5 & 17 \end{bmatrix}.$$

(April 201

Solution : Since A is of order 2×2, X must be of the order 2×2.

Let $\quad X = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$

$$\therefore \quad 2\begin{bmatrix} -1 & 2 \\ 5 & 1 \end{bmatrix} + 3\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ -5 & 17 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2 & 4 \\ 10 & 2 \end{bmatrix} + \begin{bmatrix} 3x & 3y \\ 3z & 3u \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ -5 & 17 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2+3x & 4+3y \\ 10+3z & 2+3u \end{bmatrix} = \begin{bmatrix} 4 & 16 \\ -5 & 17 \end{bmatrix}$$

By equality of matrices,

$$\begin{array}{llll} -2+3x=4, & 4+3y=16, & 10+3z=-5, & 2+3u=17 \\ 3x=6, & 3y=12, & 3z=-15, & 3u=15 \\ x=2, & y=4, & z=-3, & u=5. \end{array}$$

$$\therefore \quad X = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$$

Example 5.19 : If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 5 \\ 4 & -3 & 7 \\ 3 & 6 & 0 \end{bmatrix}$, find matrix X such that $AB + X = \begin{bmatrix} 5 & 10 & 15 \\ 0 & -17 & 30 \end{bmatrix}$. **(Oct. 2008)**

Solution : Since A is 2×3 matrix and B is 3×3 matrix AB is a matrix of order 2×3.

$\therefore$ X must be matrix of order 2×3

$$\text{Let} \quad X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\text{Now, } AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}\begin{bmatrix} -1 & 2 & 5 \\ 4 & -3 & 7 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times-1+3\times4+4\times3 & 2\times2+3\times-3+4\times6 & 2\times5+3\times7+4\times0 \\ 1\times-1+5\times4+7\times3 & 1\times2+5\times-3+7\times6 & 1\times5+5\times7+7\times0 \end{bmatrix}$$

$$= \begin{bmatrix} -2+12+12 & 4-9+24 & 10+21+0 \\ -1+20+21 & 2-15+42 & 5+35+0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 19 & 31 \\ 40 & 29 & 40 \end{bmatrix}$$

$$AB + X = \begin{bmatrix} 5 & 10 & 15 \\ 0 & -17 & 30 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 22 & 19 & 31 \\ 40 & 29 & 40 \end{bmatrix} + \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 0 & -17 & 30 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 22+a & 19+b & 31+c \\ 40+d & 29+e & 40+f \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 0 & -17 & 30 \end{bmatrix}$$

By equality of matrices

$22 + a = 5, \quad 19 + b = 10, \quad 31 + c = 15$

$40 + d = 0, \quad 29 + e = -17, \quad 40 + f = 30$

$\therefore \quad a = -17, \quad b = -9, \quad c = -16, \; d = -40, \; e = -46, \; f = -10$

$$\therefore \quad X = \begin{bmatrix} -17 & -9 & -16 \\ -40 & -46 & -10 \end{bmatrix}$$

Example 5.20 : If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, show that $A^2 = 4A - I$ (April 2018)

Solution : $$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} \quad \ldots (1)$$

$$4A - I = \begin{bmatrix} 8 & 4 \\ 12 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $A^2 = 4A - I$

Example 5.21 : If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & -1 \\ 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ find x, y, z if $(2A + B)\,C = X$

Solution : $$2A + B = \begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 3 & 3 \end{bmatrix}$$

$$\therefore \quad (2A + B)\,C = \begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$(2A + B)\,C = X \text{ gives} \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore \; x = 2, \; y = -3, \; z = 3$$

Example 5.22 : If $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, find a matrix B such that $AB = I_2$.

Verify that $BA = I_2$.

Solution : Clearly, B must be a square matrix of order 2. Let

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $$AB = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix}$$

$$AB = I_2 \text{ gives} \begin{bmatrix} 3a + c & 3b + d \\ 5a + 2c & 5b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $3a + c = 1,\ 5a + 2c = 0,\ 3b + d = 0,\ 5b + 2d = 1$. First two equations gives $a = 2,\ c = -5$. Next two equations gives $b = -1,\ d = 3$.

$$\therefore \quad B = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}.$$

$$\text{Also} \quad BA = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Example 5.23 : Determine x, y, z if

$$\left\{5\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution :

$$\text{L.H.S} = \left\{\begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ -6 & 9 \\ 9 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 \\ 6 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -6 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -2 \\ 8 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ $x = -2,\ y = 8,\ z = -6$

Example 5.24 : Find the values of a and b if $\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 7 & 11 \end{bmatrix}$

Solution : We have, forming the product of the matrices on the L.H.S.

$$\begin{bmatrix} 3a + 2b & 7 \\ a + 5b & 11 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 7 & 11 \end{bmatrix} \quad \therefore \quad 3a + 2b = 8,\ a + 5b = 7$$

Solving these equations we get $a = 2,\ b = 1$.

Example 5.25 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$ then show that $(AB)' = B'A'$.

Solution :

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 4 + 2 \times -5 + 3 \times 7 \\ 2 \times 4 + (-1)(-5) + 0 \times 7 \end{bmatrix} = \begin{bmatrix} 4 - 10 + 21 \\ 8 + 5 + 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \end{bmatrix} \quad \ldots (1)$$

$$\therefore \quad (AB)' = [15 \quad 13]$$

$$B' = \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} = [4 \ -5 \ \ 7]$$

$$A' = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$\therefore \quad B'A' = [\ 4 \ \ -5 \ \ 7\ ] \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= [4 \times 1 + (-5)(2) + 7 \times 3 \quad 4 \times 2 + (-5)(-1) + 7 \times 0]$$

$$= [4 - 10 + 21 \quad 8 + 5 + 0]$$

$$= [15 \quad 13] \qquad \ldots (2)$$

From equation (1) and (2),

$$(AB)' = B'A'$$

Example 5.26 : Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ satisfies the equation

$$A^2 - 4A + I = 0$$ **(April 2011)**

Solution :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 2 \times 1 & 1 \times 2 + 2 \times 3 \\ 1 \times 1 + 3 \times 1 & 1 \times 2 + 3 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$$

$$A^2 - 4A + I = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 4 + 1 & 8 - 8 + 0 \\ 4 - 4 + 0 & 11 - 12 + 1 \end{bmatrix}$$

$$= 0$$

Exercise (5.6)

1. Given the following pairs of matrices, form the products whichever are defined :

(i) $A = [1 \;\; 2]$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (ii) $A = [2 \;\; 1 \;\; 1]$ $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

(iv) $A = [p \;\; q \;\; r]$ $B = \begin{bmatrix} a & b & c \\ l & m & n \\ u & u & w \end{bmatrix}$

2. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, find AB and BA. What do you observe ?

3. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, find a matrix X such that $AX = I_3$.

4. If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, find a matrix X such that $AX = I_2$.

5. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \end{bmatrix}$. Show that $(AB)' = B'A'$

(April 2011)

6. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $B = \begin{bmatrix} l & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & n \end{bmatrix}$. Form the products AB and BA. Comment on the result.

7. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 2 \end{bmatrix}$

Show that (i) $A(B + C) = AB + AC$ (ii) $A(B - C) = AB - AC$

8. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Find AB and BA. What do you observe ?

9. Let A h = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Find AB and AC. What do you observe ?

10. If $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $C=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, verify that $(AB)C = A(BC)$.

11. Compute $\left\{3\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}-\begin{bmatrix} 1 & 5 & -2 \\ -3 & -4 & 4 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

12. If $A=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, determine a matrix X such that $AX = B$.

13. If $A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, show that $A^2 = 2A$

14. If $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A$ is a scalar matrix.

15. If $A=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, show that A satisfied $A^2 - 3A + I_2 = O$

Answers (5.6)

1. (i) $AB = [4 \;\; 3]$ (ii) $AB = 7$, $BA=\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{bmatrix}$ (iii) $AB=\begin{bmatrix} -4 \\ 6 \end{bmatrix}$

   (iv) $BA = [pa + ql + ru \quad pb + qm + rv \quad pc + qn + rw]$

2. $AB=\begin{bmatrix} -4 & 7 \\ -7 & -4 \end{bmatrix}$, $BA=\begin{bmatrix} -4 & 7 \\ -7 & 4 \end{bmatrix}$ $AB = BA$.

3. $X=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

4. $X=\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ 5. $(AB)' = B'A' = \begin{bmatrix} 4 & 7 \\ 3 & -1 \\ -2 & -1 \end{bmatrix}$

6. $AB = BA = \begin{bmatrix} al & 0 & 0 \\ 0 & bm & 0 \\ 0 & 0 & cn \end{bmatrix}$

   We observe that two diagonal matrices of the same order commute and that the product is again a diagonal matrix.

7. (i) $A(B+C) = AB + AC = \begin{bmatrix} 5 & -3 & 8 \\ 8 & -4 & 13 \end{bmatrix}$,

$A(B-C) = AB - AC = \begin{bmatrix} 3 & 9 & -6 \\ 6 & 16 & -11 \end{bmatrix}$

8. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ Neither A nor B is a zero matrix but AB is a zero matrix. At the same time, BA is not a zero matrix.

9. $AB = AC = \begin{bmatrix} 4 & 6 \\ 8 & 12 \end{bmatrix}$. A is not a zero matrix and $AB = AC$ but $B \neq C$. Note that A is a singular matrix.

10. $(AB)C = A(BC) = \begin{bmatrix} 9 & 3 \\ 3 & 3 \\ 6 & 0 \end{bmatrix}$ 11. $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$

12. Let $X = \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix}$. Then $AX = B$ gives

$\therefore \begin{bmatrix} p & q & r \\ s-p & t-q & u-r \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$\therefore p = 1, q = 2, r = 3, s = 5, t = 7, u = 9.$

$\therefore X = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$

13. $A^2 = 2A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

14. $A^2 - 4A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

5.10 Minors and Co-factors

Given a square matrix A, by minor of an element a_{ij} we mean the value of the determinant obtained by deleting i^{th} row and j^{th} column of A. It is denoted by M_{ij}.

Consider the square matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

We first calculate minor of element 2. Since it is (1, 1) element of A, we delete first row and first column, so that determinant of remaining array is;

$$\begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} = (4 \times -2) - (2 \times -1) = -8 + 2 = -6 = M_{11}$$

Since –1 is (1, 2) element, we delete first row and second column. The determinant of remaining array is

$$\begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} = 0 \times -2 - (2 \times 1)$$

$$= -2 = M_{12}$$

The minor of 3 is $\begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} = 0 - 4 = -4 = M_{13}$

The minor of 0 is $\begin{vmatrix} -1 & 3 \\ -1 & -2 \end{vmatrix} = (-1)(-2) - (3)(-1) = 2 + 3 = 5 = M_{21}$

The minor of 4 is $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = (2)(-2) - (3)(1) = -4 - 3 = -7 = M_{22}$

The minor of 2 (in (2, 3) place is

$$\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (-1)(1)$$

$$= -2 + 1$$

$$= -1 = M_{23}$$

The minor of 1 is $\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} = (-1)(2) - (3)(4)$

$$= -2 - 12$$

$$= -14 = M_{31}$$

The minor of (–1) is $\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = (4) - 0 = 4 = M_{32}$

The minor of (–2) is $\begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix} = (2)(4) - 0 = 8 = M_{33}$

For a 2 × 2 matrix, calculation of minors is very simple.

Consider the matrix $P = \begin{bmatrix} 2 & 6 \\ -4 & 7 \end{bmatrix}$

For finding minor of 2 we delete first row and first column.

For example, $\begin{bmatrix} -2 & 6 \\ -4 & 7 \end{bmatrix}$

So that remaining array is $|7| = 7 = M_{11}$.

Similarly, minors of 6, – 4 and 7 will be – 4, 6, 2 respectively.

Co-factor : Let A be a square matrix. By cofactor C_{ij} of an element a_{ij} of A, we mean minor of a_{ij} with a positive or negative sign depending on i and j. For a 2 × 2 matrix, negative sign is to be given the minors of elements a_{12} and a_{21}.

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\therefore \quad C_{11} = M_{11}, \quad C_{12} = -M_{12}$$
$$C_{21} = -M_{21}, \quad C_{22} = M_{22}$$

For example, Consider the matrix $A = \begin{bmatrix} 5 & -3 \\ -2 & 0 \end{bmatrix}$.

The minor of 5 is 0 i.e. $M_{11} = 0$.

$\therefore$ Cofactor of 5 is 0 (no change in sign)

The minor of − 3 is − 2 i.e. $M_{12} = -2$.

$\therefore$ Cofactor of − 3 is + 2 (change in sign)

The minor of − 2 is − 3 i.e. $M_{21} = -3$.

$\therefore$ Cofactor of − 2 is + 3 (change in sign)

The minor of 0 is 5 i.e. $M_{22} = 5$.

$\therefore$ Cofactor of 0 is 5 (no change in sign).

For a 3 × 3 matrix, negative sign is to be given to minors of elements encircled below :

$$\begin{bmatrix} + & \bigcirc & + \\ \bigcirc & + & \bigcirc \\ + & \bigcirc & + \end{bmatrix}$$

Thus for five positions, co-factor is same as minor while for four positions (encircled) cofactor is obtained by changing sign of the minor.

i.e.
$$C_{11} = M_{11}, \quad C_{12} = -M_{12}, \quad C_{13} = M_{13}$$
$$C_{21} = -M_{21}, \quad C_{22} = M_{22}, \quad C_{23} = -M_{23}$$
$$C_{31} = M_{31}, \quad C_{32} = -M_{32}, \quad C_{33} = M_{33}$$

Note : For a 3 × 3 matrix, cofactors are minors with alternate positive and negative signs, starting from positive sign for C_{11}.

Consider the third order matrix $M = \begin{bmatrix} 2 & -3 & -1 \\ 6 & 4 & 1 \\ 0 & 5 & 3 \end{bmatrix}$.

Minor of 2 is $\begin{vmatrix} 4 & 1 \\ 5 & 3 \end{vmatrix} = 12 - 5 = 7$.

$\therefore$ Cofactor of 2 is 7.

Minor of -3 is $\begin{vmatrix} 6 & 1 \\ 0 & 3 \end{vmatrix} = 18 - 0 = 18$

$\therefore$ Cofactor of -3 is -18 (change in sign)

Minor of -1 is $\begin{vmatrix} 6 & 4 \\ 0 & 5 \end{vmatrix} = 30 - 0 = 30$

$\therefore$ Cofactor of -1 is 30

Minor of 6 is $\begin{vmatrix} -3 & -1 \\ 4 & 1 \end{vmatrix} = -3 - (4)(-1) = -3 + 4 = 1$

$\therefore$ Cofactor of 6 is -1 (change in sign)

Minor of 4 is $\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = 2 \times 3 - 0 = 6$

$\therefore$ Cofactor of 4 is 6

Minor of 1 is $\begin{vmatrix} 2 & -3 \\ 0 & 5 \end{vmatrix} = 10 - 0 = 10.$

$\therefore$ Cofactor of 1 is -10 (change in sign)

Minor of 0 is $\begin{vmatrix} -3 & -1 \\ 4 & 1 \end{vmatrix} = (-3)(1) - (4)(-1) = -3 + 4 = 1$

$\therefore$ Cofactor 0 is 1.

Minor of 6 is $\begin{vmatrix} 2 & -1 \\ 6 & 1 \end{vmatrix} = (2)(1) - (6)(-1) = 2 + 6 = 8.$

$\therefore$ Cofactor of 5 is -8 (change in sign)

Minor of 3 is $\begin{vmatrix} 2 & -3 \\ 6 & 4 \end{vmatrix} = (2)(4) - (6)(-3) = 8 + 18 = 26.$

$\therefore$ Cofactor of 3 is 26.

5.11 Adjoint of a square matrix (Oct. 2008)

Definition : Given a square matrix A, the transpose of matrix of cofactors of A is called *adjoint of A* and is denoted by adj A.

For example, Consider the matrix $A = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}$.

$M_{11} = 2,\quad M_{12} = 2,\quad M_{21} = -1,\quad M_{22} = 5$

$\therefore\quad C_{11} = 2,\quad C_{12} = -2,\quad C_{21} = +1,\quad C_{22} = 5.$

$\therefore$ Cofactor matrix $= \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$ (transpose of above matrix)

(2) Find adj A, where, $A = \begin{bmatrix} 13 & -9 \\ 0 & 8 \end{bmatrix}$

$M_{11} = 8, \; M_{12} = 0, \; M_{21} = -9, \; M_{22} = 13$

$\therefore$ $C_{11} = 8, \; C_{12} = 0, \; C_{21} = 9, \; C_{22} = 13.$

$\therefore$ Cofactor matrix $= \begin{bmatrix} 8 & 0 \\ 9 & 13 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} 8 & 9 \\ 0 & 13 \end{bmatrix}$

(3) Consider the matrix

$$P = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 1 & 5 \\ 0 & 6 & 4 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} 1 & 5 \\ 6 & 4 \end{vmatrix} = 4 - 30 = -26$ $\therefore$ $C_{11} = -26$

$M_{12} = \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix} = 8 - 0 = 18$ $\therefore$ $C_{12} = -8$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} = 12 - 0 = 12$ $\therefore$ $C_{13} = 12$

$M_{21} = \begin{vmatrix} -2 & -1 \\ 6 & 4 \end{vmatrix} = (-2)(4) - (6)(-1) = -8 + 6 = -2$ $\therefore$ $C_{21} = 2$

$M_{22} = \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} = 12 - 0 = 12$ $\therefore$ $C_{22} = 12$

$M_{23} = \begin{vmatrix} 3 & -2 \\ 0 & 6 \end{vmatrix} = 18 - 0 = 18$ $\therefore$ $C_{23} = -18$

$M_{31} = \begin{vmatrix} -2 & -1 \\ 1 & 5 \end{vmatrix} = (-2)(5) - (1)(-1) = -10 + 1 = -9$ $\therefore$ $C_{31} = -9$

$$M_{32} = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = (3)(5) - (2)(-1) = 15 + 2 = 17$$

$\therefore \quad C_{32} = -17$

$$M_{33} = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (2)(-2) = 3 + 4 = 7$$

$\therefore \quad C_{33} = 7$

$$\therefore \quad \text{Cofactor matrix of P} = \begin{bmatrix} -26 & -8 & 12 \\ 2 & 12 & -18 \\ -9 & -17 & 7 \end{bmatrix}$$

$$\therefore \quad \text{adj A} = \begin{bmatrix} -26 & 2 & -9 \\ -8 & 12 & -17 \\ 12 & -18 & 7 \end{bmatrix}$$

(4) Show that adjoint of the following matrix is itself.

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0 - 4 = -4 \qquad \therefore \quad C_{11} = -4$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 3 - 4 = -1 \qquad \therefore \quad C_{12} = +1$$

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} 4 - 0 = 4 \qquad \therefore \quad C_{13} = 4$$

$$M_{21} = \begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = (-3)(3) - (4)(-3) = -9 + 12 = 3 \qquad \therefore \quad C_{21} = -3$$

$$M_{22} = \begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = (-4)(3) - (4)(-3) = 0 \qquad \therefore \quad C_{22} = 0$$

$$M_{23} = \begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = (-4)(4) - (4)(-3) = -16 + 12 = -4 \qquad \therefore \quad C_{23} = +4$$

$$M_{31} = \begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = (-3)(1) - 0 = -3 \qquad \therefore \quad C_{31} = -3$$

$$M_{32} = \begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = (-4)(1) - (-3)(1) = -4 + 3 = -1 \qquad \therefore \quad C_{32} = +1$$

$$M_{33} = \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = 0 - (-3)(1) = +3 \qquad \therefore \quad C_{33} = 3$$

$$\therefore \quad \text{Cofactor matrix} = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = A.$$

5.12 Inverse of a Matrix

In school algebra you have studied the concept of multiplicative inverse. Given a non-zero real number a, we know that its multiplicative inverse is $\frac{1}{a}$. Can we extend the same idea to Matrices ? The multiplicative inverse of a is $\frac{1}{a}$ since $a \times \frac{1}{a} = 1$, which is multiplicative identity in the set of reals. In case of matrices, identity matrix is the multiplicative identity. Hence, given a matrix A, if there exists a matrix B such that AB = I, then B should be taken as inverse of A.

This is precisely taken as definition of inverse.

Definition : Given a matrix A, if there exists a matrix B such that AB = BA = I, then B is called *inverse of A*.

Inverse of A is denoted by A^{-1}.

The following results are extremely important.

(1) Only a non-singular matrix can possess inverse i.e. a square matrix A possesses inverse if and only if $|A| \neq 0$. Then A is said to be *invertible*.

(2) Inverse of a matrix, when exists, is unique. i.e. a non-singular matrix A cannot possess different inverses, say B and C.

Result : If A is a non-singular matrix, then

$$A^{-1} = \frac{1}{|A|} \text{ adj } A.$$

Algorithm of finding inverse : Suppose a square matrix A is given whose inverse is to be obtained.

(1) Find |A|. If |A| = 0, write "inverse does not exist". If $|A| \neq 0$ write "inverse exists" and proceed to step 2.

(2) Find cofactors of all elements of A.

(3) Write matrix of cofactors of A.

(4) Write adj A.

(5) $A^{-1} = \frac{1}{|A|}$ adj A.

(6) We advise the students to check whether the inverse is correct by verifying $AA^{-1} = I$.

Solved Examples

Example 5.27 : If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ satisfy the matrix equation $A^2 - kA + 2I = 0$, find k.

Solution :

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 3 + (-2)\times 4 & 3\times -2 + (-2)(-2) \\ 4\times 3 + (-2)\times 4 & 4\times -2 + (-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$A^2 - kA + 2I = 0$

(given)

$$\therefore \quad \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\therefore \quad \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} -3k & 2k \\ -4k & 2k \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

$$\therefore \quad \begin{bmatrix} 1-3k+2 & -2+2k+0 \\ 4-4k+0 & -4+2k+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 3-3k & 2k-2 \\ 4-4k & 2k-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$3 - 3k = 0 \Rightarrow k = 1.$

Example 5.28 : *Find the inverse of the matrix.* **(April 2018)**

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

Solution :

$$|A| = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = -4-(-3) = -1$$

$\therefore \quad |A| \neq 0$

$\therefore \quad A^{-1}$ exists.

The minors of elements are given by

$$M_{11} = -2, \; M_{12} = 1, \; M_{21} = -3, \; M_{22} = 2$$

$$\therefore \quad C_{11} = -2, \; C_{12} = 1, \; C_{21} = +3, \; C_{22} = 2$$

$\therefore$ Cofactor matrix $= \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}$ adj A

$= \frac{1}{-1}\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

Verification : $AA^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 2\times 2+(-3)(1) & 2\times(-3)+(-3)(-2) \\ 1\times 2+(-2)(1) & 1\times(-3)+(-2)(-2) \end{bmatrix}$

$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I_2$

Example 5.29 : *Find the inverse of the following matrix :*

$P = \begin{bmatrix} 5 & -8 \\ -10 & 16 \end{bmatrix}$

Solution : $|P| = \begin{vmatrix} 5 & -8 \\ -10 & 16 \end{vmatrix}$

$= 16\times 5-(-10)(-8)$

$= 80-80$

$= 0$

$\therefore$ P^{-1} does not exist.

Example 5.30 : *Find inverse of the matrix* **(April 2018)**

$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solution : $|A| = 1(16-9)-3(4-3)+3(3-4)$

$= 7-3(1)+3(-1)$

$= 7-3-3$

$= 1$

$\therefore$ $|A| \neq 0$, A^{-1} exists.

The minors of various elements are

$$M_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16 - 9 = 7 \qquad \therefore C_{11} = 7$$

$$M_{12} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1 \qquad \therefore C_{12} = -1$$

$$M_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1 \qquad \therefore C_{13} = -1$$

$$M_{21} = \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 12 - 9 = 3 \qquad \therefore C_{21} = -3$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1 \qquad \therefore C_{22} = 1$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0 \qquad \therefore C_{23} = 0$$

$$M_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 9 - 12 = -3 \qquad \therefore C_{31} = -3$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 3 = 0 \qquad \therefore C_{32} = 0$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1 \qquad \therefore C_{33} = 1$$

$$\therefore \text{Cofactor matrix of A} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad \text{adj. A} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now, $$A^{-1} = \frac{1}{|A|} \text{adj A} = \text{adj A} \qquad (\because |A| =$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Verification $AA^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Example 5.31 : Computer $\left\{(-2)\begin{bmatrix} 1 & -3 \\ 7 & 9 \\ 8 & 0 \end{bmatrix} + (3)\begin{bmatrix} 6 & 0 \\ 9 & 5 \\ 1 & 2 \end{bmatrix}\right\}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ **(April 2013)**

Solution : $\left\{(-2)\begin{bmatrix} 1 & -3 \\ 7 & 9 \\ 8 & 0 \end{bmatrix} + (3)\begin{bmatrix} 6 & 0 \\ 9 & 5 \\ 1 & 2 \end{bmatrix}\right\}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$= \left\{\begin{bmatrix} -2 & 6 \\ -14 & 18 \\ -16 & 0 \end{bmatrix} + \begin{bmatrix} 18 & 0 \\ 27 & 15 \\ 3 & 6 \end{bmatrix}\right\}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+18 & 6+0 \\ -14+27 & 18+15 \\ -16+3 & 0+6 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 6 \\ 13 & 33 \\ -13 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \times 3 + 6 \times -2 \\ 13 \times 3 + 33 \times -2 \\ -13 \times 3 + 6 \times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 48-12 \\ 39-66 \\ -39-12 \end{bmatrix} = \begin{bmatrix} 36 \\ -27 \\ -51 \end{bmatrix}$$

Example 5.32 : Find adjoint of the matrix A, where $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ and show that

$$A\,(\text{adj } A) = |A|\, I$$ (April 2011)

Solution :

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$\therefore$ $|A| = 2 \times 7 - 5 \times 3 = 14 - 15 = -1$

Minor of 2 = 7

$\therefore$ Cofactor of 2 = 7 (no change in sign)

Minor of 5 = 3

$\therefore$ Cofactor of 5 = − 3 (change in sign)

Minor of 3 = 5

$\therefore$ Cofactor of 3 = − 5 (change in sign)

Minor of 7 = 2

$\therefore$ Cofactor of 7 = 2 (no change in sign)

$\therefore$ Matrix of cofactor of A $= \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$

$\therefore$ $\text{adj } A = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}$

$$A\,(\text{adj } A) = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 7 + 5 \times -3 & 2 \times -5 + 5 \times 2 \\ 3 \times 7 + 7 \times -3 & 3 \times -5 + 7 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 - 15 & -10 + 10 \\ 21 - 21 & -15 + 14 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= |A|\, I$$

Exercise (5.7)

1. Find adjoint of each of following matrices :

   (i) $\begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. Find adjoint of each of following matrices :

   (i) $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ (ii) $P = \begin{bmatrix} 3 & -4 & 1 \\ -3 & 6 & -1 \\ 4 & -8 & 2 \end{bmatrix}$

3. Find the inverse of the following matrices :

(i) $A = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 4 & -5 \\ 2 & 1 \end{bmatrix}$ (iii) $C = \begin{bmatrix} -20 & -43 \\ 40 & 36 \end{bmatrix}$

4. Find the inverse of the following matrices :

(i) $P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, (ii) $Q = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (Oct. 08),

(iii) $R = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(iv) $S = \begin{bmatrix} 4 & 2 & 3 \\ 4 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Answers (5.7)

1. (i) $\begin{bmatrix} 4 & -5 \\ 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. (i) $\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 0 & -2 \\ 2 & 2 & 0 \\ 0 & 8 & 6 \end{bmatrix}$.

3. (i) $-\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, (ii) $\frac{1}{14}\begin{bmatrix} 1 & 5 \\ -2 & 4 \end{bmatrix}$, (iii) $\frac{1}{1000}\begin{bmatrix} 36 & 43 \\ -40 & -20 \end{bmatrix}$.

4. (i) $\begin{bmatrix} 2 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iii) $\frac{1}{11}\begin{bmatrix} -1 & 7 & -24 \\ 2 & -3 & 4 \\ 2 & -3 & 15 \end{bmatrix}$, (iv) $\frac{1}{10}\begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 8 \\ 4 & -2 & -8 \end{bmatrix}$

5.13 System of Linear Equations

You are familiar with equations of the type $2x - y = 1$, $3x + 2y = 12$

This is called a system of two linear equation in two unknowns x and y.

Let us generalize this idea and then extend it to three equations in three unknowns.

A system of two linear equations in two unknowns x and y is as follows :

$$\left.\begin{aligned} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{aligned}\right\} \quad \ldots (5.3)$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Then system (5.3) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

i.e. $AX = B$

If the R.H.S., namely B, is O then the system is called **homogeneous**, otherwise **non-homogeneous**.

Thus
$$2x + 3y = 0,$$
$$x - 4y = 0$$

is a **homogeneous** system of two equations in two unknowns x and y and

$$3x - 4y = 7$$
$$2x + 9y = 0$$

is a **non-homogeneous** system of equations.

A system of three linear equations in three unknowns x, y, z is as follows :

$$\left.\begin{array}{c} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array}\right\} \quad \ldots (5.4)$$

Which can also be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Then the matrix form of given system is

$$AX = B$$

This system is, as before, homogeneous if $B = 0$ and non-homogeneous if $B \neq 0$.

Solution of a System of Equations :

Consider the system $2x - y = 1$ and $3x + 2y = 12$. We observe that $x = 2$ and $y = 3$ satisfy both the equations. (i.e. the system of equations.) Hence $x = 2, y = 3$ (or (2, 3)) is a **solution** of a given system.

A set of values of unknowns (x, y, z etc.) which satisfy all the equations in the system simultaneously is called a **solution** of given system.

Consistency of Equations :

Definition : A system of equations is said to be **consistent** if it has a solution.

Thus the system $2x - y = 1$ and $3x + 2y = 12$ is consistent.

Consider the equation $2x - 3y = 4$ we observe that $x = 8, y = 4$ is a solution of the equation. $x = 2, y = 0$ is also a solution of the given equation. In fact, it has infinitely many solutions. Hence this system (of only one equation) is consistent.

The system $2x - y = 4$ and $6x - 3y = 7$ is not consistent. (Why ? Try to find a solution).

This discussion leads us to find some criteria about existence of solution for a given system. In other words, we are interested in finding some condition/conditions which determines the consistency of equations.

Consistency of a Homogeneous System :

Consider the homogeneous system

$$a_{11} x + a_{12} y = 0$$
$$a_{21} x + a_{22} y = 0$$

of two equations in two unknowns x and y. We observe that, whatever be the values of $a_{11}, a_{12}, a_{21}, a_{22}$, $x = 0, y = 0$ always satisfy given system. Thus (0, 0) is always a solution of the given system and hence the system is consistent.

Similarly, the following system

$$a_{11} x + a_{12} y + a_{13} z = 0$$
$$a_{21} x + a_{22} y + a_{23} z = 0$$
$$a_{31} x + a_{32} y + a_{33} z = 0$$

is satisfied by $x = 0, y = 0, z = 0$. **Thus a homogeneous system is always consistent.**

In fact, it has a unique solution, namely $x = 0, y = 0, z = 0$ etc.

Consistency of Non-homogeneous System :

Consider the non-homogeneous system

$$a_{11} x + a_{12} y = b_1,$$
$$a_{21} x + a_{22} y = b_2$$

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then the system can be written in matrix form as $AX = B$. $(B \neq 0)$.

Pre-multiplying both sides by A^{-1}.

$$A^{-1}(AX) = A^{-1}B$$

$$\therefore \quad X = A^{-1}B$$

If A^{-1} exists the R.H.S. exists i.e. solution exists. Thus, existence of A^{-1} is necessary and sufficient condition for the system to be consistent.

But we know that A^{-1} exists if and only if $|A| \neq 0$.

Thus we have following result

Result : A system of equations $AX = B$ $(B \neq 0)$ has unique solution if and only if A is non-singular i.e. $|A| \neq 0$. Since A^{-1} is unique, the solution $A^{-1}B$ is also unique.

Note that this result is true for a system of n linear equations in n unknowns. (However we shall confine ourselves to a system containing atmost three unknowns.)

Solution of a system of Non-homogeneous Equations :

Let $AX = B$ be a given non-homogeneous system of, n linear equations in n unknowns. Assuming existence of A^{-1}, $X = A^{-1}B$ is the solution.

Algorithm :

(1) Write the given system in the form of matrix equation as $AX = B$.

(2) If $|A| = 0$, A^{-1} does not exist so that solution does not exist. Write "System is not consistent".

(3) If A^{-1} exists find A^{-1}.

(4) Find $X = A^{-1}B$.

(5) Write values of x, y, z.

(6) Students are advised to verify the solution. i.e. whether the values of x, y, z so obtained satisfy given equations.

Solved Examples

Example 5.33 : *Solve the equations*

$$4x + 7y - 9 = 0$$
$$5x - 8y + 15 = 0$$

Solution : Given equations can be written as

$$4x + 7y = 9$$
$$5x - 8y = -15$$

Let, $$A = \begin{bmatrix} 4 & 7 \\ 5 & -8 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ -15 \end{bmatrix}$$

$\therefore$ Given system can be written as

$$AX = B$$

$$\therefore \quad X = A^{-1}B \qquad \ldots(1)$$

Let us find A^{-1}.

$$|A| = (4 \times -8) - (5 \times 7)$$
$$= -32 - 35$$
$$= -67$$

Minors and co-factors of matrix A are

$M_{11} = -8$, $\therefore C_{11} = -8$

$M_{12} = 5$ $\therefore C_{12} = -5$

$M_{21} = 7$ $\therefore C_{21} = -7$

$M_{22} = 4$ $\therefore C_{22} = 4$.

$\therefore$ Co-factor matrix of A $= \begin{bmatrix} -8 & -5 \\ -7 & 4 \end{bmatrix}$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} -8 & -7 \\ -5 & 4 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

$$= \frac{-1}{67}\begin{bmatrix} -8 & -7 \\ -5 & 4 \end{bmatrix}$$

From equation (1), $X = A^{-1}B$

$$= -\frac{1}{67}\begin{bmatrix} -8 & -7 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ -15 \end{bmatrix}$$

$$= -\frac{1}{67}\begin{bmatrix} -72+105 \\ -45-60 \end{bmatrix}$$

$$= -\frac{1}{67}\begin{bmatrix} 33 \\ -105 \end{bmatrix}$$

$$= \begin{bmatrix} -33/67 \\ 105/67 \end{bmatrix}$$

$$\therefore \quad x = -\frac{33}{67}, \quad y = \frac{105}{67}.$$

Example 5.34 : *Solve the system of linear equations by matrix method*

$$2x + y + 3z = 1$$

$$x + z = 2$$

$$2x + y + z = 3$$

Solution : Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ Given equations can be written as

$$AX = B$$

$$\therefore \quad X = A^{-1}B \qquad \ldots (1)$$

Let us find A^{-1}

$$|A| = 2(0-1) - 1(1-2) + 3(1-0)$$

$$= -2 + 1 + 3 = 2 \ (\neq 0)$$

$\therefore$ A^{-1} exists.

Minors and cofactors of elements of A are

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 - 1 = -1 \qquad \therefore C_{11} = -1$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1 \qquad \therefore C_{12} = 1$$

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1 - 0 = 1 \qquad \therefore C_{13} = 1$$

$$M_{21} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2 \qquad \therefore C_{21} = 2$$

$$M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 - 6 = -4 \qquad \therefore C_{22} = -4$$

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 2 - 2 = 0 \qquad \therefore C_{23} = 0$$

$$M_{31} = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \qquad \therefore C_{31} = 1$$

$$M_{32} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1 \qquad \therefore C_{32} = -1$$

$$M_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \qquad \therefore C_{33} = 1$$

$$\therefore \quad \text{Cofactor matrix of A} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -4 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj (A)} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj (A)}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

From equation (1),

$$X = A^{-1}B$$

$$X = \frac{1}{2}\begin{bmatrix} -1 & 2 & 1 \\ 1 & -4 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} -1+4+3 \\ 1-8+3 \\ 1+0-3 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore$ $x = 3,\ y = -2,\ z = -1.$

Example 5.35 : *Solve the equations*

$$2x + y + z = 2,\ x + y + z = 0,\ 4x - y - 3z = 20$$

Solution : Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 4 & -1 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 20 \end{bmatrix}$

$\therefore$ Given equations can be written as

$$AX = B$$

$\therefore$ $$X = A^{-1} B \qquad \dots (1)$$

Let us find A^{-1}.

$$|A| = 2(-3+1) - 1(-3-4) + 1(-1-4)$$
$$= -4 + 7 - 5 = -2\ (\neq 0)$$

$\therefore$ A^{-1} exists

Minors and cofactors of elements of A are

$$M_{11} = \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} = -3 + 1 = -2 \quad \therefore \quad C_{11} = -2$$

$$M_{12} = \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} = -3 - 4 = -7 \quad \therefore \quad C_{12} = 7$$

$$M_{13} = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = 5 \quad \therefore \quad C_{13} = -5$$

$$M_{21} = \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} = -3+1 = -2 \quad \therefore \quad C_{21} = 2$$

$$M_{22} = \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} = -6-4 = -10 \quad \therefore \quad C_{22} = -10$$

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -2-4 = -6 \quad \therefore \quad C_{23} = 6$$

$$M_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0 \quad \therefore \quad C_{31} = 0$$

$$M_{32} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1 = 1 \quad \therefore \quad C_{32} = -1$$

$$M_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1 = 1 \quad \therefore \quad C_{33} = 1$$

$$\therefore \quad \text{Cofactor matrix of A} = \begin{bmatrix} -2 & 7 & -5 \\ 2 & -10 & +6 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad \text{adj (A)} = \begin{bmatrix} -2 & 2 & 0 \\ 7 & -10 & -1 \\ -5 & 6 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{ adj (A)} = \frac{1}{-2}\begin{bmatrix} -2 & 2 & 0 \\ 7 & -10 & -1 \\ -5 & 6 & 1 \end{bmatrix}$$

From equation (1),

$$X = A^{-1}B$$

$$X = \frac{-1}{2}\begin{bmatrix} -2 & 2 & 0 \\ 7 & -10 & -1 \\ -5 & 6 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 20 \end{bmatrix}$$

$$X = \frac{-1}{2}\begin{bmatrix} -4+0+0 \\ 14+0-20 \\ -10+0+20 \end{bmatrix}$$

$$X = \frac{-1}{2}\begin{bmatrix} -4 \\ -6 \\ 10 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$$

$\therefore$ $x = 2,\ y = 3,\ z = -5.$

Example 5.36 : *Solve the system of linear equations*

$$5x + y = 8$$
$$2x + 3y = 11$$

using inverse of the coefficient matrix. **(Oct. 2008)**

Solution : The given system of equations can be written as

$$\begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

i.e. $AX = B$ (say)

where, $A = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}.$

$\therefore$ $AX = B$

$\therefore$ $X = A^{-1}B$ ... (1)

To find A^{-1}. $|A| = 5 \times 3 - 2 \times 1 = 13$

We shall find A^{-1} by adjont method.

$$C_{11} = (-1)^{1+1}(3) = 3$$
$$C_{12} = (-1)^{1+2}(2) = -2$$
$$C_{21} = (-1)^{2+1}(1) = -1$$
$$C_{22} = (-1)^{2+2}(5) = 5$$

$\therefore$ Matrix of cofactor of $A = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix}$

$\therefore$ $\text{adj } A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{13}\begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$$

From equation (1) $X = A^{-1}B = \frac{1}{13}\begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix}$

$$= \frac{1}{13}\begin{bmatrix} 3 \times 8 - 1 \times 11 \\ -2 \times 8 + 5 \times 11 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 24-11 \\ -16+55 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 13 \\ 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\therefore$ $x = 1, \quad y = 3.$

Example 5.37 : *To control a crop disease, it is necessary to use 8 units of chemical A, 14 units of chemical B and 13 units of chemical C. One barrel of P contains 1 unit of A, 2 units of B and 3 units of C. One barrel of Q contains 2, 3, 2 units of chemicals A, B, C respectively, 1 barrel of R contains 1, 2, 2 units of A, B, C respectively. Find how many barrels of each type of spray be used to just meet the requirements.*

Let x, y, z be number of barrels of P, Q, R be used. Therefore the matrix form of the given problem is

$$\begin{array}{c c} & \begin{array}{ccc} P & Q & R \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

i.e. $AX = B$ (say) ... (I)

We find A^{-1} by adjoint method.

$$|A| = 1(6-4) - 2(4-6) + 1(4-9)$$
$$= 1(2) - 2(-2) + 1(1-5)$$
$$= 2 + 4 - 5$$
$$= 6 - 5$$
$$= 1$$

The co-factors are $A_{11} = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 6 - 4 = 2$

$$A_{12} = -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -(4-6) = 2$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$$

$$A_{21} = -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -(4-2) = -2$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = 2 - 3 = -1$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -(2-6) = 4$$

$$A_{31} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = -(2-2) = 0$$

$$A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$\therefore \quad \text{Matrix of cofactors} = \begin{bmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \text{adj } A \qquad (\because |A| = 1)$$

From (1), $X = A^{-1}B$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 14 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 16 - 28 + 13 \\ 16 - 14 + 0 \\ -40 + 56 - 13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 3.$

Example 5.38 : *Solved the following system of equations $x - 2y = 5$, $2x + 3y = 2$ by using inverse of coefficient matrix.* **(April 2011)**

Solution : The given system can be written in matrix form as follows :

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

i.e. $AX = B$ (say)

$\therefore \quad X = A^{-1}B \qquad \ldots (1)$

To find A^{-1}, $|A| = 1 \times 3 - (-2 \times 2) = 3 + 4 = 7$

Cofactor of 1 = 3

Cofactor of −2 = −2 (change in sign)

Cofactor of 2 = 2 (change in sign)

Cofactor of 3 = 1

$\therefore$ Matrix of cofactors of A $= \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore$ $\text{adj } A = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{7}\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$$

From equation (1), $X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 \times 5 + 2 \times 2 \\ -2 \times 5 + 1 \times 2 \end{bmatrix}$

$$= \frac{1}{7}\begin{bmatrix} 19 \\ -8 \end{bmatrix} = [\ 19/7 \quad -8/7\ ]$$

$\therefore$ $x = \frac{19}{7}, y = \frac{-8}{7}$

Points to Remember

- A determinant D, i.e. $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{2 \times 2}$ where a, b, c, d are certain numbers, they are called the element of the determinant.
- Value of determinant $D = ad - bc$.
- An arrangement of mn numbers in the form of a rectangular block of m rows and n columns enclosed in rectangular brackets is called a matrix of the order $m \times n$.

  Types of matrices : Zero matrix, Row matrix, Column matrix, Diagonal matrix, Null matrix, Scalar matrix, Square matrix.
- Associative law of matrix multiplication $(AB)C = A(BC)$.
- Inverse of matrix A denoted by A^{-1}.
- Inverse by adjoint method : $A^{-1} = \frac{1}{|A|} \text{adj } A$.

Exercise (5.8)

Solve the following equations by matrix method.

1. $2x + 3y = 9$, $\quad -x + y = -2$
2. $x + 3y = -2$, $\quad 3x + 5y = 4$ (April 2015)
3. $x + y = 1$, $\quad 3y + 3z = 5$, $3z + 3x = 4$ (April 2010)
4. $x + y + z = 1$, $\quad 2x + y + 2z = 3$, $3x + 3y + 4z = 4$
5. $x + y + z = 6$, $\quad 3x - y + 3z = 10$, $5x + 5y - 4z = 3$ (April 2013)

6. Find the values of following determinants.

(i) $\begin{vmatrix} 1 & 2 & 3 \\ 8 & 4 & 6 \\ 4 & 2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 5 & -2 & 0 \\ -1 & 4 & 8 \\ 0 & -9 & 6 \end{vmatrix}$ (iv) $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ -3 & 7 & -6 \end{vmatrix}$ (vi) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

(vii) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

7. Solve the equations: (i) $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x & 2 & x+3 \\ 3 & 5 & 8 \\ x+1 & 7-x & 12 \end{vmatrix} = 0$

8. Find x and y if $\begin{bmatrix} 2x+y & 4 \\ 5 & x+2y \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 5 \end{bmatrix}$

9. If $A = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}$, $B = \begin{bmatrix} +1 & -2 \\ 1 & 2 \end{bmatrix}$ Find a matrix X satisfying the equation $A + 2X = B$.

10. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$, Find $2A - 3B$.

11. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I_2 = 0$

12. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ satisfies the matrix equation $A^2 - KA + 2I = 0$, find k.

13. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 6 \end{bmatrix}$, find $|A - 2I|$

14. Solve for matrices X and Y if

(i) $2X + 3Y = \begin{bmatrix} 3 & -2 \\ 5 & 6 \end{bmatrix}$ and $2X - 3Y = \begin{bmatrix} 4 & 2 \\ -5 & -6 \end{bmatrix}$

(ii) $X - 2Y = \begin{bmatrix} 5 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} -1 & -1 & 2 \\ 5 & 3 & 4 \end{bmatrix}$

15. If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 3 & 6 \end{bmatrix}$

    Find AB and BA. Are they equal ?

16. If $A = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$, show that $AA' = A'A = I_2$

17. If $A = \begin{bmatrix} 0 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 1 \end{bmatrix}$

    Show that $BA' = (AB')'$.

18. If $A = \begin{bmatrix} 3 & 0 \\ -4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & -7 \\ 0 & -1 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$

    Verify that $A(BC) = (AB)C$.

19. Find x, y, z if $\left\{4\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -2 & 4 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

20. Find the matrix X such that

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} X = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}$$

21. If $\begin{bmatrix} x & 4 \\ 2 & 8 \end{bmatrix}$ is a singular matrix, find value of x.

22. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$. Verify that $|AB| = |A|\,|B|$.

23. Show that $A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$ satisfies the equation $A^2 - 12A + I = 0$. Hence find A^{-1}

24. Find the adjoint of each of the following matrices.

    (i) $\begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

    (iv) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

25. Find the adjoint of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$ and verify that I_3. A (adj. A) = |A| I_3. **(April 2018 BBA IB)**

26. Find the inverse of the following matrices :

(i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

27. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$. Verify $(AB)^{-1} = B^{-1}A^{-1}$.

28. Solve the following equations. **(April 2018 BBA)**

(i) $x + y + z = 6,\quad 2x + y + 2z = 10,\quad 3x + 3y + 4z = 21$

(ii) $x + y + z = 1,\quad 2x - y + 7z = 7\quad 3x + y + 2z = 2$

(iii) $x + y + z = 6,\quad x - y + z = 2,\quad 2x + y - z = 1$

(iv) $2x - y + z = 1,\quad x + 2y + 3z = 8,\quad 3x + y - 4z = 1$

(v) $2x - 4y + 3z = 1,\quad x - 2y + 4z = 3,\quad 3x - y + 5z = 2$

(vi) $-3x + 2y + z = 3,\quad 4x - y + 3z = 11,\quad x + y + 4z = 8$

Answers (5.8)

1. $x = 3, y = 1$

2. $x = \frac{1}{2}, y = \frac{1}{2}$

3. $x = \frac{1}{3}, y = \frac{2}{3}, z = 1$

4. $x = 1, y = -1, z = 1$

5. $x = 1, y = 2, z = 3.$

6. (i) 0 (ii) 27 (iii) 468 (iv) $(x-1)^2 + (x+2)$

   (v) 0 (vi) 2 (vii) 0 (viii) xy

7. (i) $x = -9$ (ii) $x = 2/3$

8. $x = -\frac{1}{3}, y = \frac{8}{3}$

9. $X = \begin{bmatrix} -1 & -3 \\ 0 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 9 & -7 \\ -3 & -8 & 7 \end{bmatrix}$

12. $K = 1$

13. -2

14. (i) $X = \begin{bmatrix} \frac{7}{4} & 0 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} \frac{1}{6} & \frac{-2}{3} \\ \frac{5}{3} & 2 \end{bmatrix}$

(ii) $X = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ \frac{9}{5} & \frac{6}{5} & \frac{9}{5} \end{bmatrix}, Y = \begin{bmatrix} \frac{-4}{5} & -1 & \frac{3}{5} \\ \frac{7}{5} & \frac{3}{5} & \frac{2}{5} \end{bmatrix}$

15. $AB = \begin{bmatrix} 1 & 22 \\ 2 & 12 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & 6 & -1 \\ -8 & -10 & -4 \\ 30 & 27 & 27 \end{bmatrix}$, $AB \neq BA$.

19. $x = 12, y = 16, z = 8$

20. $[3 \ 1 \ 2]$

21. $x = 1$

23. $A^{-1} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$

24. (i) $\begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -2 \\ -7 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & -1 & 1 \\ -12 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

25. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

26. (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$ (iii) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (v) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

28. (i) $x = 1, y = 2, z = 3$ (ii) $x = 0, y = 0, z = 1$

(iii) $x = 1, y = 2, z = 3$ (iv) $x = 1, y = 2, z = 1$

(v) $x = -1, y = 0, z = 1$ (vi) System has no solution.

□□□

Chapter 6...

Linear Programming Problems
(For Two Variables Only)

Contents ...

Key Words :

Objective function, constraints, convex set of region, solution, feasible region, initial solution, feasible solution, optimal solution, unbounded solution, degenerate solution, alternative solution.

Objectives :

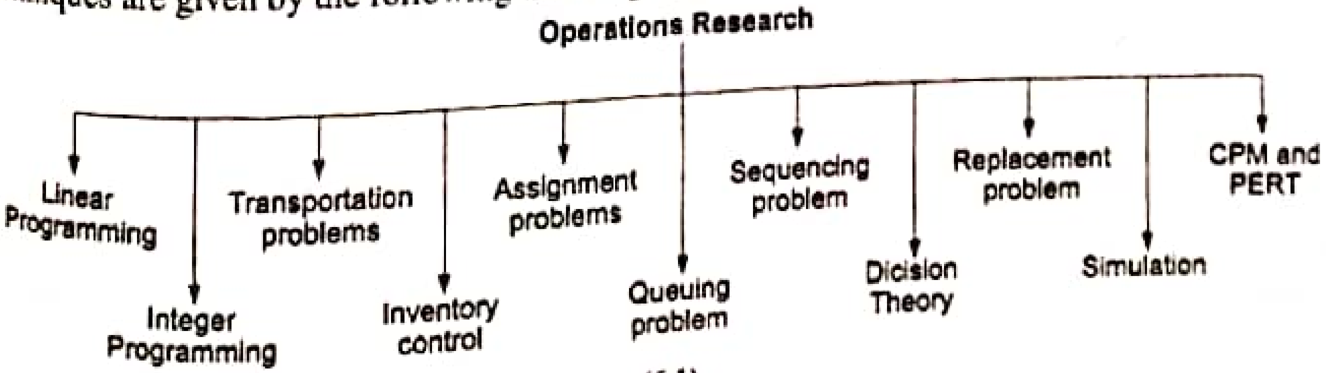
To formulate the problem as L.P.P. To obtain the best solution using graphical method.

6.1 Introduction

In business management there are many activities such as production, inventory, marketing, maintenance etc. Many times the situations are complex. The variables involved behave adversely and they have conflicting effects. In this situation one has to find a golden mean or a break-even point. A set of several techniques to achieve such a solution is developed around 1940, it is now referred as '**Operations Research**'.

Churchman, Ackoff and Arnoff defined operations research as the application of scientific methods, techniques and tools, to problems involving the operations of system so as to provide those in control of the system with optimum solutions to the problem.

The techniques in operations research has its applications in wide areas. It helps decision-making by giving a solution to problem which optimises resources. The various techniques are given by the following tree diagram.



In order to solve the problems using methods in operations research, we need to go through the following steps :

1. Decide the model to be used to solve the problem.
2. Formulation of the problem.
3. Use of appropriate technique to get the best or optimal solution.

In general the resources such as money, raw material, man hours, machine hours, electric supply etc. are limited. These resources are the **decision variables**. The limitations are represented in the form of equations or inequalities using the **decision variables**. These equations are called as **constraints**. The whole exercise is to use the resources to give best results. By best results we mean in most economic way or which minimises the cost of production or minimise the total time required to complete the task or to maximise the profit or to maximise the efficiency. Such criterion is called as an **objective function**. The objective function is also written in terms of mathematical expression. This function is to be maximised or minimised we call it as to be **optimised**.

Thus in short the problem reduces to the three ingredients (i) the decision variables, (ii) constraints, (iii) objective function or optimising function.

There may be many solutions to the problem. We are interested in those solutions which are confined to the constraints. Such solutions are called as **feasible solutions**. Among all feasible solutions we search for the optimum solution, which maximises (or minimises as the case may be) the objective function.

6.2 Meaning of Linear Programming Problem (L.P.P.) (April 2015)

The meaning of word programme is the plan of action. Particularly a linear programming is a problem of minimising or maximising the objective function, which is linear in nature, subject to the constraints which are also linear in nature.

The general L.P.P., after formulation, with x and y as decision variable is of the type :

$$\text{Maximize/Minimize } Z = c_1 x + c_2 y \quad \textbf{Objective function}$$

$$\text{Subject to} \left.\begin{array}{l} a_{11} x + a_{12} y \ (\leq, =, \geq)\ b_1 \\ a_{21} x + a_{22} y \ (\leq, =, \geq)\ b_2 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1} x + a_{m2} y \ (\leq, =, \geq)\ b_m \end{array}\right\} \quad \textbf{Constraints}$$

$$x, y \geq 0 \text{ (or unrestricted)} \quad \textbf{Non-negativity Restrictions}$$

The decision variables x and y may be non-negative or unrestricted. The linear constraints may be equations, less than type inequality or greater than type inequality. The linear objective function involves the coefficients c_1 and c_2 may be costs or profits or time required or efficiencies etc. with respect to the variables x and y respectively.

Illustration :

$$\begin{aligned} \text{Maximise} \quad & z = 5x + 4y \\ \text{Subject to} \quad & 2x + 3y \leq 100 \\ & 4x - 5y \geq 30 \\ & 3x + y = 75 \\ & x \geq 0 \quad y \geq 0 \end{aligned}$$

6.3 Mathematical Formulation of L.P.P. (April 2016)

Decision variables (Oct. 2010) : In the formulation of L.P.P. or any model in operations research one has to decide the unknowns to be found out. These unknowns are called as decision variables. The decision variables may take non-negative values or they are unrestricted in sign.

Constraints (April 2015) : The conditions, limitations in the given problem are expressed in terms of equations or inequalities in terms of decision variables are called as **constraints.** In L.P.P. the cosntraints are linear in nature.

Objective function : The goal or objective to be achieved in the model is called as objective function. It is to be maximised or minimised (optimised). In L.P.P. the objective function is linear in nature.

Thus in the formulation of L.P.P. one has to :

(i) identify the decision variables

(ii) identify the constraints

(iii) identify the objective function.

Solved Examples

Example 6.1 : ***Product Mix Problem :*** *A company has to produce A and B the two types of products. Each product has to be processed by three machines (i) moulding (M_1), (ii) grinding (M_2), (iii) finishing (M_3). Suppose machine M_1 can be operated for total time of 2700 minutes. It takes 12 minutes for an item of type A and 5 minutes for an item of type B.*

Machine M_2 is available for 2000 minutes and it takes 5 minutes for processing an item of type A and 10 minutes for an item of type B.

Machine M_3 is available for 450 minutes and it takes 2 minutes for processing an item of type A and 2 minutes for an item of type B.

The profit per item of type A is ₹10 and that of per item of type B is ₹15. Find the number of items of type A and B to be produced so as maximise the profit. Formulate the L.P.P.

Solution : There are two types of items to be manufactured. Thus, there will be two **decision variables,**

Let x = The number of items of type A

y = The number of items of type B

The objective function (z) is a total profit if x items of type A and y items of type B are manufactured hence $z = 10x + 15y$. We need to maximise the profit, hence objective is to

$$\text{Maximize } z = 10x + 15y$$

The constraints will be time required to process x items of A and y items of B on machine M_1, which can run for maximum 2700 minutes.

$$\begin{pmatrix}\text{Sum of time required} \\ \text{for x items of A and} \\ \text{y items of B}\end{pmatrix} \leq 2700$$

$$\therefore \qquad 12x + 5y \leq 2700 \qquad \ldots (1)$$

Similarly, for machine M_2 we get the constraint

$$5x + 10y \leq 2000 \qquad \ldots (2)$$

and for M_3 we get

$$2x + 2y \leq 450 \qquad \ldots (3)$$

Thus the L.P.P. is

Maximise $z = 10x + 5y$ (Objective function)

Subject to
$$\left.\begin{aligned} 12x + 5y &\leq 2700 \\ 5x + 10y &\leq 2000 \\ 2x + 2y &\leq 450 \\ x \geq 0, \quad y &\geq 0 \end{aligned}\right\}$$
(Constraints)

(Non-negativity restrictions)

Example 6.2 : ***Budget Allocation problem :*** *Two different kinds of foods A and B are being considered to form a weekly diet. The minimum weekly requirements of fats, Carbohydrates and Proteins are 18, 24 and 24 units respectively. One kg of food A contains 4, 16 and 8 units of fats, carbohydrates and proteins respectively. One kg. of food B contains 12, 4 and 6 units of fats, carbohydrates and proteins respectively. The prices of food A is ₹ 4 per kg. and that of B is ₹ 3 per kg. Find the quantity food A and B to be purchased so that the total cost is minimum and the requirement is fulfilled.*

Formulate the problem as L.P.P.

Solution : Let x : No. of kg. of food A to be purchased

y : No. of kg. of food B to be purchased

Then we require to find x and y such that

Minimize $z = 4x + 3y$ (Objective function)

Subject to $4x + 12y \geq 18$ (Fats constraint)

$16x + 4y \geq 24$ (Carbohydrates constraints)

$8x + 6y \geq 24$ (Proteins constraints)

$x, y, \geq 0$ (non-negativity constraint)

6.4 Solution of L.P.P. by Graphical Method

We need to solve the formulated L.P.P. It involves to find the values of decision variable which will satisfy the constraints and optimize the value of objective function.

Some definitions :

(a) Solution : A set of values of decision variables which satisfy the constraints of L.P.P. is called as a solution to corresponding L.P.P.

(b) **Feasible solution (April 2018, 2017) :** A solution which satisfies the non-negativity restrictions of L.P.P. is called as feasible solution to corresponding L.P.P.

(c) **Optimal solution (B.B.A. April 2018, 2017, 2010, Oct. 2010) :** A feasible solution of L.P.P. which optimizes (minimises or maximises as the case may be) the objective function is called as a optimal solution of the corresponding L.P.P.

(d) **Convex set :** A region in X-Y plane is called as a convex region or convex set if the line segment joining any two points in the set lies completely within the region. The following are convex sets, since the line segment AB completely lies in the set.

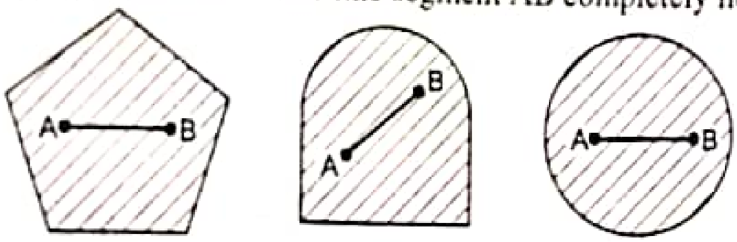


Fig. 6.1

The following are not convex sets.

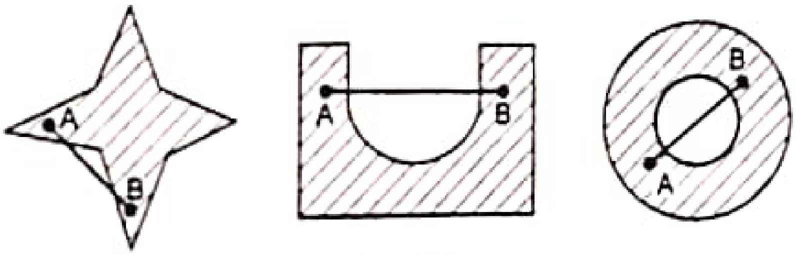


Fig. 6.2

The line segment AB having points A and B within the set does not lie completely in the set, hence those are non-convex sets.

Result (1) : The region formed by the constraints of L.P.P. is a convex set.

Result (2) : Every point in the convex set formed by the constraints of L.P.P. is a feasible solution. Such a region is also called as feasible region.

Result (3) : The extreme points (maxima or minima) of objective function, if exist, are found at the vertices (or boundary) of the convex region.

Hence in order to find the optimal solution, we should plot the linear inequations given by the constraints and find the region common to all the constraints. To find the optimal solution we evaluate the objective function at the vertices of convex set. We find the maximum (or minimum) value of the objective function. We read the co-ordinates of vertex at which optimal solution is found. The co-ordinates are values of decision variable which give optimal solution to L.P.P.

Note that the line $ax + by = c$ divides the x-y plane in two parts. One of the side of the line we observe $ax + by < c$, the other side gives $ax + by > c$ and for the points on line we get $ax + by = c$. Thus, the constraints in the form of linear inequalities define the region in X-Y plane.

Example 6.3 : *Plot the inequality $12x + 5y \leq 2700$.*

Solution : To plot the line $12x + 5y = 2700$ we take the intercepts on the axes (or any two points on the line).

If $x = 0$, $5y = 2700$, $y = \frac{2700}{5} = 540$ hence A(0, 540) is a point in the line. If $y = 0$, $12x = 2700$, $x = \frac{2700}{12} = 225$ hence B(225, 0) is a point on the line. We plot the points A and B and the line AB is denoted by $12x + 5y = 2700$.

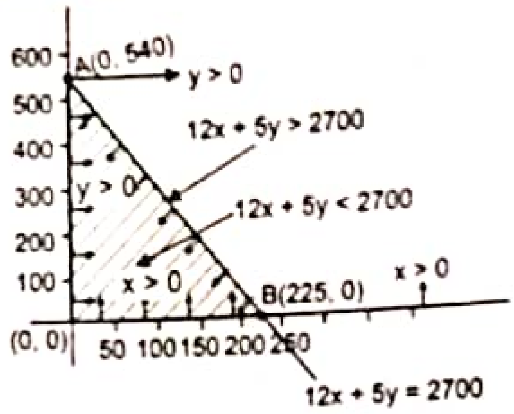


Fig. 6.3

The line divides the plane in two parts on one side of the origin (0, 0) lies. We substitute $x = 0$, $y = 0$ in the equation and evaluate.

$$(12 \times 0) + (5 \times 0) = 0 < 2700$$

Thus, the side containing origin is denoted by $12x + 5y < 2700$.

Clearly the side $12x + 5y > 2700$ corresponds to non-origin side.

Note : If the line passes through origin then any other point can be used to determine which part satisfies the inequality.

Example 6.4 : *Solve the following L.P.P. graphically :*

$$\begin{aligned} \textit{Maximize} \quad & z = 10x + 15y \\ \textit{Subject to} \quad 12x + 5y & \leq 2700 \\ 5x + 10y & \leq 2000 \\ x \geq 0,\ & y \geq 0 \end{aligned}$$

Solution : We need to plot the lines given by the inequalities in the constraints.

The line $12x + 5y = 2700$ passes through (0, 540) and (225, 0).

To plot $5x + 10y = 2000$, let us take $x = 0$ then we get $y = 200$ and for $y = 0$, $x = 400$. Therefore, the line joining the points (0, 200) and (400, 0) give the required line.

Substituting the (0, 0) in first equation $12x + 5y$ we get $(12 \times 0) + (5 \times 0) = 0 < 2700$, hence the required region is origin side region (See Fig. 6.4 (a)).

Substituting the origin in second line we get $(5 \times 0) + (10 \times 0) = 0 < 2000$. Hence the region containing origin gives $5x + 10y < 2000$ (See Fig. 6.4 (b)).

Plotting both the region we get a common region, where both the constraints are satisfied. Non-negativity restrictions $x \geq 0$, $y \geq 0$ confine the region to the first quadrant. (See Fig. 6.4 (c)).

Thus we get

Line	Points on the line
$12x + 5y = 2700$	(0, 540), (225, 0)
$5x + 10y = 2000$	(0, 200), (400, 0)

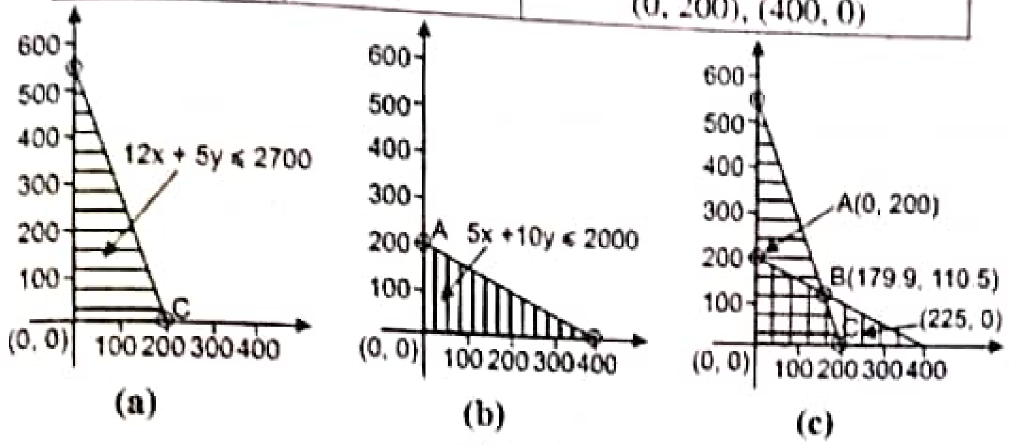


Fig. 6.4

The feasible region or solution space is given by a polygon OABC. We evaluate the objective function at every vertex.

Vertex	$z = 10x + 15y$
O (0, 0)	0
A (0, 200)	3000.0
B (178.9, 110.5)	3446.5 Maximum
D (225, 0)	2250. 0

Thus the solution of L.P.P. is $x = 178.9$, $y = 110.5$ and $z =$ ₹ 3446.5 by rounding-off we get $x \approx 179$, $y \approx 111$, $z =$ ₹ 3455.

Solutions of L.P.P. : There are two types of L.P.P. : (i) minimisation, (ii) maximisation.

L.P.P. has following types of solutions :

(a) **Unique optimal solution.**

(b) **Multiple optimal solutions or alternate optimal solution :** This situation arises when the line given by objective function is parallel to any of the boundary of the feasible region. The two vertices by such constraint will give the optimal solutions with the same value of z. In fact every point joining these two points will be a solution to L.P.P. Thus, there will be multiple or infinitely many solutions.

(c) **Unbounded solution :** The feasible region may be unbounded with unbounded value of z.

(d) **No solution or infeasible solution :** If there is no point in the feasible region which satisfies all the constraints then L.P.P. has no optimal solution.

(e) **Degenerate solution :** If any of the decision variable is 0 in the optimal solution then the solution is degenerate. If the vertex at which optimal solution exists is on any of the axis then degenerate solution is possible. This situation will arise if (i) there is only one constraint (ii) the objective function has only one variable in it, (iii) the cost coefficients c_1 and c_2 differ too much relatively etc.

Example 6.5 : *Solve the following L.P.P. (Formulated in example 3)*

Minimize $z = 4x + 3y$

Subject to

$$4x + 12y \geq 18$$
$$16x + 4y \geq 24$$
$$8x + 6y \geq 24$$
$$x, y \geq 0$$

Is the optimal solution unique ? If not then find the alternate solution. **(April 2018)**

Solution : To plot the lines we find two points on line

The line corresponding to the constraint	The points on the line
$4x + 12y = 18$ ... (1)	(0, 1.5), (4.5, 0)
$16x + 4y = 24$ ... (2)	(0, 6), (1.5, 0)
$8x + 6y = 24$ ... (3)	(0, 4), (3, 0)

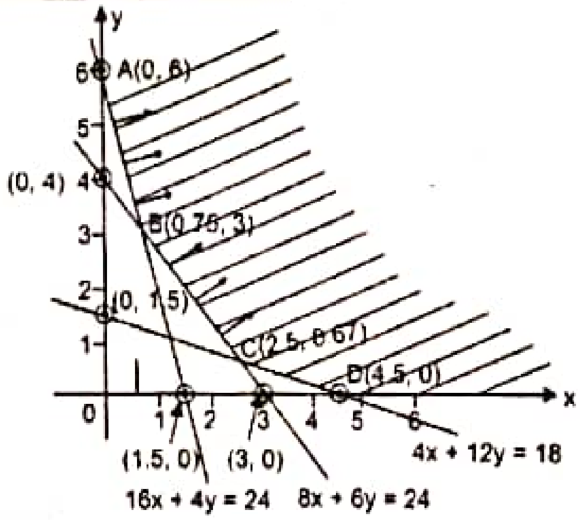


Fig. 6.5

Note that the line $4x + 12y = 18$ divides the x-y plane in two regions. Since origin (0, 0) gives $(4 \times 0) + (12 \times 0) = 0 < 18$, the region containing (0, 0) is $4x + 12y < 18$. The complementary part is $4x + 12y \geq 18$. Similarly, the regions given by $16x + 4y \geq 24$ and $8x + 6y \geq 24$ are also the regions in which (0, 0) is not included. The non-negativity restrictions give the region as the first quadrant. Thus, the shaded region with vertices A(0, 6), B(0.75, 3), C(2.5, 0.67), D(4.5, 0) is a feasible region. We search optimal solution at the vertices.

Vertex	The value of objective function $z = 4x + 3y$
A (0, 6)	$z = (4 \times 0) + (3 \times 6) = 18$
B (0.75, 3)	$z = (4 \times 0.75) + (3 \times 3) = 12$ Minimum
C (2.5, 0.67)	$z = (4 \times 2.5) + (3 \times 0.67) = 12.01 \doteq 12$ Minimum
D (4.5, 0)	$z = (4 \times 4.5) + (3 \times 0) = 18$

The optimising function has minima at B and C. Thus, there are two solutions of L.P.P.

Solution 1 : At B (0.75, 3), z = ₹ 12

$\therefore$ x = 0.75 kg, y = 3 kg , z = ₹12

Solution 2 : At C (2.5, 0.67)

$x = 2.5$ kg, $y = 0.67$ kg, $z = ₹\ 12.01 \doteqdot ₹\ 12$

Thus, the optimum solution is not unique the alternate solution is at C.

Note : 1. The objective function $z = 4x + 3y$ is parallel the constraint $4x + 3y = 24$, hence there are multiple solutions. Infact every point on the line joining B and C is a optimal solution to L.P.P. with common value of z as ₹ 12.

2. If (x_1, y_1) and (x_2, y_2) are the two solutions then $(ax_1 + (1 - a)x_2, ay_1 + (1 - a)y_2)$ is also a solution for $0 < a < 1$. Choosing any value of a between 0 and 1 we get a solution. Hence there are infinitely many optimum solutions to L.P.P.

Example 6.6 : *Solve the following L.P.P. using graphical method*

Maximise $z = x_1 + x_2$

Subject to

$$x_1 + x_2 \le 1$$
$$2x_1 + 3x_2 \ge 6$$
$$x_1 \ge 0, \quad x_2 \ge 0$$

Solution : $x_1 + x_2 = 1$ passes through (1, 0), (0, 1), $2x_1 + 3x_2 = 6$ passes through (3, 0) and (0, 2). The region $x_1 + x_2 \le 1$ is the origin side region and the region $2x_1 + 3x_2 \ge 6$ is the non-origin side region.

From the figure we get there is no region which satisfies both the constraints. Hence there is no solution to L.P.P.

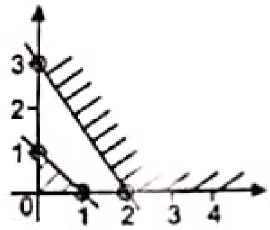


Fig. 6.6

Example 6.7 : *Show that the following L.P.P. has unbounded solution*

Maximize $z = 2x + y$

Subject to

$$x - y \le 10$$
$$2x - y \le 40$$
$$x, y \ge 0$$

Solution :

The line corresponding to the constraint	The points on the line
$x - y = 10$	(0, – 10), (10, 0)
$2x - y = 40$	(0, – 40), (20, 0)

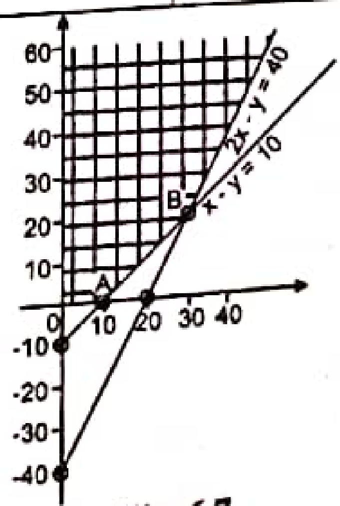


Fig. 6.7

Note that feasible region is unbounded hence solution is unbounded.

Example 6.8 : *The vertices of feasible region of a L.P.P. are given by A(0, 1), B(0, 4), C(4, 6), D(7, 5), E(10, 2). Solve the L.P.P. for the objective functions and state the nature of solutions.*

(a) Maximize $z_1 = x + y$

(b) Minimize $z_2 = x - y$

(c) Maximize $z_3 = 2x + y$

Solution :

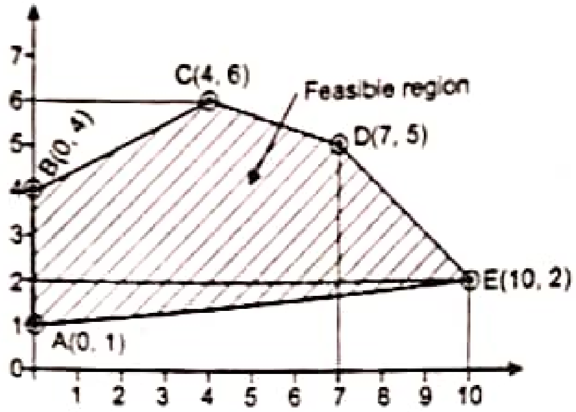


Fig. 6.8

Vertex	Max. $z_1 = x + y$	Min. $z_2 = x - y$	Max. $z_3 = 2x + y$
A (0, 1)	1	− 1	1
B (0, 4)	4	− 4*	4
C (4, 6)	10	− 2	14
D (7, 5)	12*	2	19
E (10, 2)	12*	8	22*

Nature of solutions :

(a) $z_1 = x + y$ has two solutions

Solution (1) : $x = 7, y = 5, z = 12$

Solution (2) : $x = 10, y = 2, z = 12$

Hence there are infinite solutions

General solution
$$x = 7a + 10(1 - a) = -3a + 10$$
$$y = 5a + 2(1 - a) = 3a + 2$$
$$z = 12$$

We get solution for every a, $(0 < a < 1)$

(b) $z_2 = x - y$ has unique solution at $(x = 0, y = 4)$ and $z = -4$.

It is a degenerate solution since only one variable is non-zero.

(c) $z_3 = 2x + y$ has unique solution at $(x = 10, y = 2)$ and $z = 22$.

Points to Remember

- The general L.P.P., after formulation, with x and y as decision variable is of the type :

Maximize/Minimize $Z = c_1 x + c_2 y$ **Objective function**

Subject to $a_{11} x + a_{12} y \ (\leq, =, \geq) \ b_1$
$a_{21} x + a_{22} y \ (\leq, =, \geq) \ b_2$
....
....
$a_{m1} x + a_{m2} y \ (\leq, =, \geq) \ b_m$ **Constraints**

$x, y \geq 0$ (or unrestricted) **Non-negativity Restrictions**

- Term used in L.P.P. are : Decision variables, Constraint, Objective function, Feasible solution, Optimal solution etc.

Exercise (6.1)

Theory Questions :

1. Explain the term 'Linear Programming'.
2. Explain the following terms used in L.P.P. Decision variable, constraint, objective function, feasible solution, optimal solution, alternate solution, convex set.

Exercise (6.2)

Numerical Problems :

1. A manufacturing company produces two types of batteries low volt and medium volt. A low volt battery requires 1 hour processing time on machine and 2 hours of labour time. A medium volt battery requires 2 hours of processing times on machine and 1.5 hours of labour time. In a week, processing machine is available for 70 hours and labour time available is 60 hours. The profit due to each of the low volt battery is ₹ 60 where as due to medium volt battery it is ₹ 75.

   Formulate the problem as L.P.P. with a view to maximise the total profit and solve it graphically. **(April 2018)**

2. A and B are two types of fertilizers available at ₹ 30 per kg and ₹ 50 per kg respectively. Fertilizer A contains 20 units of potash, 10 units of nitrogen and 40 units of phosphorus. Fertilizer B contains 15 units of potash, 20 units of nitrogen and 10 units of phosphorus. The requirement of potash, nitrogen and phosphorus is atleast 1800, 1700, 1600 units. **(April 2018)**

   Formulate the problems as L.P.P. in order to minimize the total purchasing cost. Also obtain the solution using graphical method.

3. A farmer has 100 hectares of land for cultivation. He grows potatoes and tomatoes. He is expected to get profit of ₹ 5000 per hectare for potatoes and ₹ 6000 per hectare for tomatoes. He needs 100 kg fertilizer per hectare for potatoes and 300 kg fertilizer per hectare for tomatoes. The cost of fertilizer is Re. 1 per kg and he can spent maximum ₹ 15,000 on fertilizer. The labour required for sowing, cultivation and harvesting for potatoes and tomatoes is 3 man days and 1 man day respectively. Total man days of labour available is 150.

   Formulate and solve the problem as L.P.P. to maximise the profit.

4. A cold drink company has two plants one at each place Mumbai and Pune. It produces three brands of cold drinks A, B and C. The number of bottles produced per day are as follows :

Brand	A	B	C
At Mumbai	900	1800	1200
At Pune	900	600	3000

A market survey reveals that in the month of April the demand for the types of cold drinks A, B, C is respectively 12,000, 24,000, 26,400 bottles. The operating cost per day for plants at Mumbai and at Pune are 500 and 380 monetary units. Formulate and solve the problem as L.P.P. so as minimise the total operating cost if the plant at Mumbai is run for x days and at Pune is run for y days in the month of April in order to meet the demand.

5. Suppose there are two types of food items First type of food contains 3 units of vitamin A and 4 units of vitamin B per kg. The second type of food item contains 1 unit of vitamin A and 3 units of vitamin B per kg. The minimum requirement of vitamin A is 12 units and that of vitamin B is 24 units. The costs per gram of food of type 1 and type 2 are respectively ₹ 50 per kg and ₹ 60 per kg. The minimum quantity to be ordered for each type of food is 1 kg. Formulate and solve the L.P.P. to find the quantity to be purchased of each type of food so as minimise to total cost.

6. A small farmer builds two types of garden shed. Type A requires 2 hours of machine time and 5 hours of craftsman time. Type B requires 3 hours of machine time and 5 hours of craftsman time. Each day there are 60 hours of machine time and 80 hours craftsman time available. The profit on each type of A shed is ₹ 160 and each type of B shed is ₹ 184. Formulate L.P.P. assuming that all garden sheds are sold. **(April 2017)**

7. Solve the following L.P.P. graphically

Maximize $z = 20x_1 + 17x_2$

Subject to

$$2x_1 + 2x_2 \leq 22$$
$$12x_1 + 10x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

8. Solve the following L.P.P. graphically

Minimise $z = 2x_1 + 4x_2$

Subject to

$$6x_1 + x_2 \geq 18$$
$$x_1 + 4x_2 \geq 12$$
$$2x_1 + x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

9. Solve the following L.P.P. using graphical method

Maximize $z = x + 2y$

Subject to

$$x + y \leq 100$$
$$0 \leq x \leq 75$$
$$0 \leq y \leq 60$$

10. Solve the following L.P.P. using graphical method

Maximize $z = x + 3y$

Subject to
$$\begin{aligned} 2x + 3y &\leq 20 \\ x + y &\geq 12 \\ x, y &\geq 0 \end{aligned}$$

11. Solve the following L.P.P. using graphical method

Maximize $z = 4x + 3y$

Subject to
$$\begin{aligned} 2x + 3y &\leq 6 \\ 2x + 2y &\geq 20 \\ x \geq 0,\ & y \geq 0 \end{aligned}$$

Show that there is no solution to L.P.P.

12. Solve the following L.P.P. using graphical method

Minimize $z = x + 5y$

Subject to
$$\begin{aligned} x + 3y &\geq 12 \\ x + y &\geq 9 \\ 4x + 2y &\geq 20 \\ x \geq 0,\ & y \geq 0 \end{aligned}$$

Show that the solution is degenerate.

13. Solve the following L.P.P. using graphical method

Minimize $z = x + y$

Subject to
$$\begin{aligned} x + 3y &\geq 12 \\ x + y &\geq 8 \\ 4x + 2y &\geq 20 \\ x \geq 0,\ & y \geq 0 \end{aligned}$$

Show that there is a alternate solution. Hence find the general solution.

14. Solve the following L.P.P. graphically

Maximise $z = 3x + 5y$

Subject to
$$\begin{aligned} x + y &\leq 30 \\ x \geq y &\geq 0 \\ 0 \leq y &\leq 10 \end{aligned}$$

15. A manufacturer makes two types of lamps shades A and B which require treatment by a cutter and a finisher. Lamp shade A requires 2 hours of cutter's time and 1 hour of finisher's time. Lamp shade B requires 1 hour of cutter's and 2 hours of finishing time. The cutter has 104 hours and finisher has 70 hours of available time each month. Profit on one lamp shade A is ₹ 6 and on one lamp shade of B is ₹ 11. How many of each type of lamps shades should be manufactured to obtain the best returns. Formulate and solve L.P.P. graphically. **(April 2015, 2016)**

16. A small manufacturing firm produces two types of Gadgets A and B. They have to undergo two processes. First in foundry and then in machine shop.

The following table gives the men hours of labour required for each of the processes and for each type of Gadgets.

Gadget	Foundry	Machine shop
A	10	5
B	6	4
Capacity	1000	600

The profit on the sale of A is ₹ 30 per unit and on that of B it is ₹ 20 per unit. How many units of A and B should be produced in order to maximise the profit ?

17. A company makes two kinds of leather belts. Belt A is of high quality and B is of lower quantity. The respective profits are 4 ₹ and 3 ₹ per belt. Each belt of type A requires twice as much time as a belt of type B. If all the belts were of type B the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (Both A and B combined). Belt A requires a fancy buckle and only 400 such buckles are available per day. There are only 700 buckles a day available for type B. Determine the number of belts to be produced for each type so as to make maximum profit.

18. Solve the following L.P.P. by graphical method : **(April 2019)**

Maximize $Z = 5x + 10y$

$$2x + 3y \leq 10$$
$$6x + 10y \geq 30$$
$$x, y \geq 0$$

19. Solve the following L.P.P. by graphical method : **(April 2016)**

Subject to $10x + 6y \leq 1000$

$$5x + 4y \leq 500$$
$$x, y \geq 0$$

20. Maximize $Z = 4x + 3y$ **(April 2017)**

Subject to constraints $x + 2y \leq 6$

$$x \geq 2$$
$$y \geq 3$$
$$y \geq 0$$

21. Solve the following Linear Programming problem (L.P.P.) by graphical method :

Maximize $Z = 3x + 5y$

Subject to constraints : $2x + 3y \leq 12$

$$3x + 2y \leq 18$$
$$x, y \geq 0$$

22. In a cattle breeding farm two types of fodder F_1 and F_2 are available. They cost ₹ 50 and ₹ 30 per unit. Each fadder contains two types of nutrients N_1 and N_2 in different quantities. Each unit of F_1 contains 2 units of N_1 and 4 units of N_2. Each unit of F_2 contains 3 units of N_1 and 2 units of N_2. The minimum daily requirement of N_1 and N_2 of an animal is 6 units and 8 units respectively. Formulate this L.P.P. problem to minimize the cost. **(April 2017)**

23. A firm manufacturing two types of electrical items A and B, can make a profit of ₹ 200 per unit of A and ₹ 300 per unit B. make use of two essential components, a motor and a transformer. Each unit of A requires 3 motors and 2 transformers while each unit of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 240 motors and 280 transformers. Formulate the above problem as L.P.P. to maximise the profit. Obtain the solution by graphical method. **(April 2017)**

Answers (6.2)

1. Maximize $z = 60x + 75y$

Subject to
$$x + 2y \le 70$$
$$2x + 1.5y \le 60$$
$$x \ge 0 \quad y \ge 0$$

x = No. of low volt batteries,

y = No. of medium volt batteries.

$x = 6, y = 32, z = ₹\ 2760.$

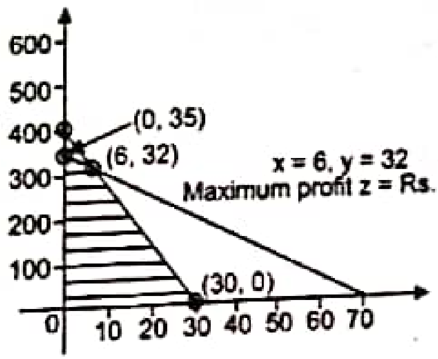


Fig. 6.9

2. Minimize $z = 30x + 50y$

Subject to
$$20x + 15y \ge 1800$$
$$10x + 20y \ge 1700$$
$$40x + 10y \ge 1600$$
$$x, y > 0$$

x = Amount of fertilizer of type A

y = Amount of fertilizer of type B.

$x = 42$ kg, $y = 64$ kg, $z = ₹\ 4460.$

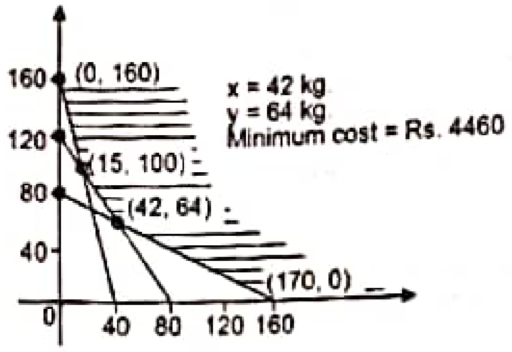


Fig. 6.10

3. Maximize $z = 5000x + 6000y$

Subject to

$$x + y \le 100$$
$$100x + 300y \le 15000$$
$$3x + y \le 150$$
$$x \ge 0 \quad y \ge 0$$

x = No. of hectares for potato.

y = No. of hectares for tomato.

$x = y = 37.5$ hectares, z = ₹ 4,12,500.

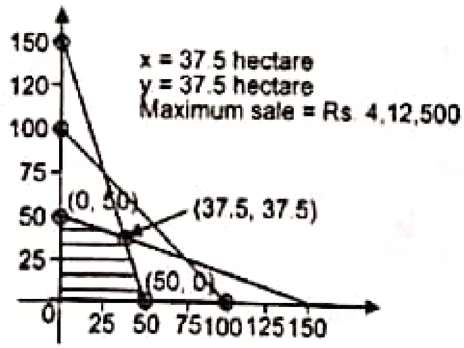


Fig. 6.11

4. Minimize $z = 500x + 380y$

Subject to

$$900x + 900y \ge 12000$$
$$1800x + 600y \ge 24000$$
$$1200x + 3000y \ge 26400$$
$$0 \le x \le 30$$
$$0 \le y \le 30$$

x = No. of days to run the Mumbai plant.

y = No. of days to run the Pune plant.

$x = 12$ days, $y = 4$ days, z = ₹ 7520.

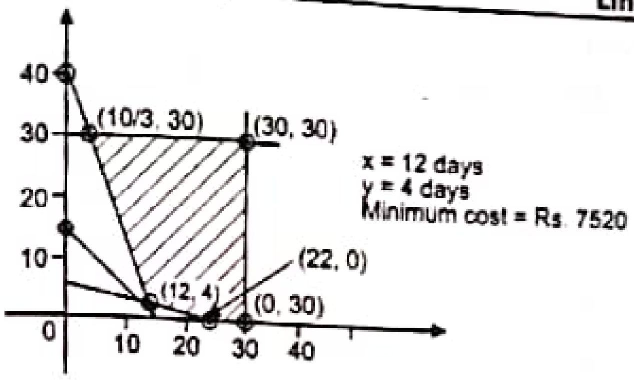


Fig. 6.12

5. Minimize $z = 50x + 60y$

Subject to
$$3x + y \geq 12$$
$$4x + 3y \geq 24$$
$$x \geq 1 \quad y \geq 1$$

x = Food of type 1 in kg.

y = Food of types 2 in kg.

$x = 5.25$ kg, $y = 1$ kg, $z =$ ₹ 322.5.

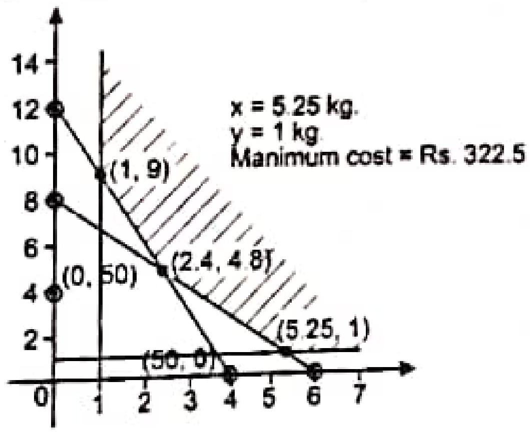


Fig. 6.13

7. $x_1 = 5, x_2 = 6, z = 202$

8. $x_1 = 4, x_2 = 2, z = 16$

9. $x = 40, y = 60, z = 160$

10. No solution

12. $x = 12, y = 0, z = 12$, solution is degenerate since, $y = 0$.

13. There are two solutions :

(a) $x = 2, y = 6, z = 8$ (b) $x = 6, y = 2, z = 8$

General solution $x = 2a + 6(1 - a), y = 6a + 2(1 - a), z = 8 \; 0 < a < 1$.

14. $x = 20, y = 10, z = 110$

15. x : No. of lamp shades of type A.

    y : No. of lamp shades of type B.

    Max. : $z = 6x + 11y$

    Subject to $2x + y \leq 104$

    $x + 2y \leq 70$

    $x \geq 0, \quad y \geq 0$

    Solution : $x = 44, y = 16, z =$ ₹ 440.

16. x : No. of gadgets of type A, y = No. of gadgets of type B

    Max. : $z = 30x + 20y$,

    Subject to $10x + 6y \leq 1000$

    $5x + 4y \leq 100$

    $x, y \geq 0$

    Solution : $x = 40, y = 100, z =$ ₹ 1000

17. x = No. of belts of type A

    y = No. of belts of type B

    Max. $z = 4x + 3y$

    Subject to $2x + y \leq 1000$

    $x + y \leq 800$

    $0 \leq x \leq 400$

    $0 \leq y \leq 700$

    Solution : $x = 200, y = 600, z =$ ₹ 2600.

❑❑❑

Chapter **7...**

Transportation problem

Contents ...

Key Words :

Sources, destination, cost of transportation, demand, availability, transportation problem, balanced problem, initial basic feasible solution, Unbalanced transportation problem, North-West corner rule, matrix-minima method, Vogel's approximation method.

Objectives :

To give a solution to transportation problem which satisfies the total availability and total requirements at various sources and destinations, so as to minimise the total cost of transportation.

7.1 Introduction

Often we need to transport material from one place to the other place.

For example, milk dairy collects milk from various centres and then distributes to several agents. So also fruits, vegetables. Manufacturing companies have warehouses at various places and shops at various places. Requirement of shops is to be fulfilled by transporting material from various warehouses. The transportation is required in business and marketing. Definitely we would be interested in transportation schedule so that the total transportation cost, time required is minimum. It also may be with a view to maximum profit, efficiency etc.

Sources : Sources are the centres where the material is available for transportation. Those are also called as origins, depots, dispatch centres, supply centres, godowns warehouses etc.

We denote the sources by $S_1, S_2, \ldots, S_m$.

Destinations : Destinations are the centres where the material is required to be transported. Those are also called as demand centres, requirement centres, reception centres, showrooms etc. We denote the destinations by $D_1, D_2, \ldots, D_n$.

7.2 Notation and Mathematical Formulation

Sources : $S_1, S_2, \ldots, S_m$

Destinations : $D_1, D_2, \ldots, D_n$

a_i = Number of units of material available at source i $(i = 1, 2, \ldots, m)$

b_j = Number of units of material required at destination j $(j = 1, 2, \ldots, n)$

c_{ij} = Cost of transportation per unit from S_i to D_j

(i.e. from i^{th} source to j^{th} destination)

x_{ij} = The number of units of material to be transported from i^{th} source (S_i) to j^{th} destination (D_j) ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, thus there are $m \times n$ variables)

Clearly, the problem is mathematically as follows (similar to L.P.P.)

The cost of transportation of x_{ij} units from S_i to D_j is $c_{ij}\, x_{ij}$. Thus the total transportation cost is $\sum\sum c_{ij}\, x_{ij}$. It is required to be minimum. Hence, the objective function is

Minimize $Z = \sum\sum c_{ij}\, x_{ij}$.

Suppose at source S_i the quantity available is a_i. Hence, the total quantity to be transported to various destination has to be a_i, thus we get

$$\sum_{j=1}^{m} x_{ij} = a_i \qquad \text{for } i = 1, 2, \ldots, m.$$

Similarly at destination D_j the requirement is b_j. Hence, the total quantity to be received at D_j from various sources is b_j, thus we get

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad \text{for } j = 1, 2, \ldots, n.$$

The mathematical formulation is as follows :

Minimize $Z = \sum\sum c_{ij}\, x_{ij}$

Subject to constraints

$$\sum_{j=1}^{m} x_{ij} = a_i \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \qquad \text{for all i and j.}$$

It can be noticed that the problem is like **Linear Programming Problem (L.P.P.)** where there are **m + n constraints** and **mn variables**.

Note :

(1) The feasible solution of the above type of transportation problem exists if and only if.

Total demand = Total availability

$$\sum_{j=1}^{n} b_j = \sum_{i=1}^{m} a_i \quad \textbf{(Rim condition)}$$

It is called as **balanced transportation problem.**

(2) If $\sum_{1}^{n} b_j \neq \sum_{1}^{m} a_i$ the problem is unbalanced transportation problem. First of all we need to make the problem balanced and then solve it. It is explained latter.

(3) The objective function may be to maximize the total profit or efficiency etc. also.

(4) There are $m \times n$ variables.

(5) The (m + n) constraints are interms of m row sums and n columns sums.

7.3 Unbalanced Transportation Problem

As long as the total demand ($\sum b_j$) is same as the total availability ($\sum a_i$), the transportation problem is called as balanced. However every problem is not balanced.

Unbalanced Transportation Problem : The transportation is said to be unbalanced if the total availability $\sum a_i \neq \sum b_j$ where $\sum b_j$ is the total demand.

In order to solve the unbalanced transportation problem first of all we have to convert it a balanced one. It gives rise to the following two cases.

Case (i) : $\sum a_i > \sum b_j$:

In this case the total availability is more than the total requirement or demand. The surplus availability is ($\sum a_i - \sum b_j$) units. To convert the unbalanced transportation problem to balanced one we consider a dummy destination with requirement ($\sum a_i - \sum b_j$) units. The cost of transportation to the dummy destination from each source is set to be zero.

Illustration : Examine whether the following transportation problem is balanced. If it is unbalanced convert it to balance transportation problem.

Sources \ Destinations	D_1	D_2	D_3	D_4	Availability (a_i)
S_1	4	2	2	3	9
S_2	5	4	5	2	13
S_3	6	5	4	7	15
S_4	3	2	1	1	3
Requirement b_j	8	10	7	9	-

Solution : Note that, the total availability $= \sum a_i = 9 + 13 + 15 + 3 = 40$ and the total requirement $= \sum b_j = 8 + 10 + 7 + 9 = 34$. Since $\sum a_i \neq \sum b_j$ the problem is unbalanced. To convert it to a balanced one, we consider a dummy destination D_5. The cost of transportation from each source to D_5 is set to be zero. The requirement at this dummy destination D_5 is ($\sum a_i - \sum b_j$) $= 40 - 34 = 6$ units. The balanced transportation problem is as follows.

Sources \ Destinations	D_1	D_2	D_3	D_4	D_5	b_j
S_1	4	2	2	3	0	9
S_2	5	4	5	2	0	13
S_3	6	5	4	7	0	15
S_4	3	2	1	1	0	3
b_j	8	10	7	9	$\sum a_i - \sum b_j = 6$	40

Case (ii) : $\sum a_i < \sum b_j$:

In this case the total demand or requirement ($\sum b_j$) is more than the total supply or availability ($\sum a_i$). The surplus requirement is ($\sum b_j - \sum a_i$) units. The surplus requirement is to be transported from dummy source. The cost such transportation is considered zero. Thus the transportation problem will be reduced to balanced one.

The following example illustrates the procedure.

Illustration : Is the following transportation problem balanced ? If not convert it to balanced one.

Sources \ Destinations	D_1	D_2	D_3	D_4	Availability a_i
S_1	5	4	2	2	10
S_2	3	6	3	8	12
S_3	2	1	5	3	15
Requirement b_j	12	10	8	13	–

Solution : In this problem the total availability $= \sum a_i = 10 + 12 + 15 = 37$ and the total requirement $= \sum b_j = 12 + 10 + 8 + 13 = 43$. Clearly the problem is unbalanced because $\sum a_i \neq \sum b_j$. There is surplus requirement of $(\sum b_j - \sum a_i) = 43 - 37 = 6$ units. Here we introduce a dummy source S_4. The surplus requirement of 6 units can be obtained from S_4 with transportation cost zero and the problem can be balanced. The modified transportation in the balanced form will look as follows.

Sources \ Destinations	D_1	D_2	D_3	D_4	a_i
S_1	5	4	2	2	10
S_2	3	6	3	8	12
S_3	2	1	5	3	15
S_4	0	0	0	0	$\sum b_j - \sum a_i = 6$
b_j	12	10	8	13	43

Initial Basic Feasible Solution (IBFS) : Among $m \times n$ variables (x_{ij}) selecting $m + n - 1$ basic variables we get initial basic feasible solution. So that it will satisfy all the $(m + n)$ constraints.

There are several solutions of which some will be basic feasible solutions with $(m + n - 1)$ variables non-zero. We have to select the basic feasible solution, so that it is **optimal** (minimum or maximum as the case may be).

There are some standard methods of (i) obtaining initial basic feasible solution (ii) from initial basic feasible solution the optimal solution.

Here, we are discussing the methods of finding IBFS. We study the following three methods to solve balanced transportation problem :

(A) North-West corner method.

(B) Matrix minima or Least cost method.

(C) Vogel's approximation method (VAM).

The transportation problem can be described using table of the following type :

Table 7.1 : Transportation Table

Sources	Destinations D_1	D_2	......	D_j	......	D_n	Availability
S_1	x_{11} c_{11}	x_{12} c_{12}		x_{1j} c_{1j}		x_{1n} c_{1n}	a_1
S_2	x_{21} c_{21}	x_{22} c_{22}		x_{2j} c_{2j}		x_{2n} c_{2n}	a_2
:							
S_i	x_{i1} c_{i1}	x_{i2} c_{i2}		x_{ij} c_{ij}		x_{in} c_{in}	a_i
:							
S_m	x_{m1} c_{m1}	x_{m2} c_{m2}		x_{mj} c_{mj}		x_{mn} c_{mn}	a_m
Requirement	b_1	b_2		b_j		b_n	

Clearly the sum of x elements in the 1st row, 2nd row, ..., mth row are respectively $a_1, a_2, \ldots a_m$. The column sum of x elements in the 1st column, 2nd column, ... nth column are respectively $b_1, b_2, \ldots, b_n$.

The entries x_{ij} in the box given at top left hand corner of each rectangle in the above table denote the quantity transported at j^{th} destination from i^{th} source. The values c_{ij} are the respective costs of transportation. Thus, finding the sum of product of cost and number of units in every rectangle gives the total cost of transportation.

7.4 Methods of Finding IBFS

(A) North West Corner Rule : (April 2018)

Step 1 : Start allocation as large as possible from North-West corner, i.e. top left hand corner, in other words first row and first column rectangle of the transportation table. We can allocate either the entire available quantity a_1 at origin or the entire required quantity b_1. Thus, we can allocate min (a_1, b_1). Denote $x_{11} = \min(a_1, b_1)$. After the allocation we get one of three cases.

Step 2 : Case (i) : If $b_1 > a_1$, then $x_{11} = \min(a_1, b_1) = a_1$ then the first row availability is exhausted. Cross or discard the row. The first column requirement will be reduced by a_1. Hence it is $b_1 - a_1$.

Case (ii) : If $b_1 < a_1$, then $x_{11} = \min(a_1, b_1) = b_1$ then the first column requirement is exhausted. Cross or discarded the column. The first row availability will be reduced by b_1 hence it is $a_1 - b_1$.

Case (iii) : If $a_1 = b$ then $x_{11} = a_1 = b_1$. Here first column and first row both get exhausted simultaneously. Cross out both the row as well as column.

Step 3 : We will be left with table with (i) m – 1 rows and n columns or (ii) m rows and n – 1 columns or (iii) m – 1 rows and n-1 columns.

With this table of reduced dimension or size we allocate at North-West corner using step 1. We repeat step 1 and 2 until all rows and columns are exhausted. In other words we continue allocation till all the demand and supply values are exhausted.

Note : The North-West corner method does not take in to account the costs, thus it is not a optimal solution. Moreover it is not very close to optimal solution as compared to other methods. However, the method is very simple to operate.

Example 7.1 : *The following table gives the unit transportation cost in ₹ using North-West corner rule find the initial basic feasible solution and total cost of transportation.*

Origins	Destination				Availability
	D_1	D_2	D_3	D_4	
O_1	20	22	17	04	120
O_2	24	37	09	07	70
O_3	32	37	20	15	50
Demand	60	40	30	110	240

Note that $\sum a_i = \sum b_j = 240$, hence the problem is a balanced transportation problem. We proceed further as follow.

Solution : There are 3 rows and 4 columns, in all 12 cells for allocation. Using North-West corner rule we allocate at x_{11} units at cell (O_1, D_1).

$$x_{11} = \min(a_1, b_1) = \min(120, 60) = 60$$

Thus, the first column D_1 is fulfilled and the first row availability is now reduced by b_1 hence it is $a_1 - b_1 = 120 - 60 = 60$. The new table 3 × 3 looks like (unshaded) as follows.

	D_1	D_2	D_3	D_4	
O_1	60				60
O_2					70
O_3					50
	–	40	30	110	180

In the new table North West cell is (O_1, D_2).

$$\therefore \quad x_{12} = \min(60, 40) = 40$$

The second column D_2 is fulfilled, it can be crossed and table remains 3 × 2 as follows.

	D_1	D_2	D_3	D_4	
O_1	60	40			20
O_2					70
O_3					50
	–	–	30	110	140

The availability at first row changes to $60 - 40 = 20$.

The new table (unshaded) has North West corner cell (O_1, D_3).

$$x_{13} = \min(20, 30) = 20$$

The first row O_1 gets exhausted. The requirement of column 3 changes to 30 – 20 = 10. The new table looks like as follows :

	D_1	D_2	D_3	D_4	
O_1	60	40	20		–
O_2					70
O_3					50
	–	–	10	110	120

The table has North West corner at (O_2, D_3)

$$x_{23} = \min(70, 10) = 10$$

Column D_3 gets exhausted. The available at O_2 becomes 70 – 10 = 60.

The new table will be (unshaded).

	D_1	D_2	D_3	D_4	
O_1	60	40	20		–
O_2			10		60
O_3					50
	–	–	–	110	110

The North-West corner is (O_2, D_4).

$$x_{24} = \min(60, 110) = 60$$

We cross row O_2 since the availability becomes 0, there is now only one cell remaining i.e. (O_3, O_4) with availability 50 and demand also 50. Hence, we allocate $x_{34} = 50$. The complete transportation schedule looks like as follows :

	D_1	D_2	D_3	D_4	
O_1	$\boxed{60}$ 20	$\boxed{40}$ 22	$\boxed{20}$ 17	04	120
O_2	24	37	$\boxed{10}$ 09	$\boxed{60}$ 07	70
O_3	32	37	20	$\boxed{50}$ 15	50
	60	40	30	110	240

The left hand top corner box in each cell is the allocation. The total cost of transportation is $(20 \times 60) + (22 \times 40) + (17 \times 20) + (10 \times 9) + (7 \times 60) + (15 \times 50) =$ ₹ 3680.

(B) Matrix Minima Method or Least Cost Method (April 2018) : In this method we find the cell with minimum cost and try to allocate as large quantity as possible. It gives a better solution than North-West corner rule. The allocation procedure is slightly complicated.

The procedural steps are as follows :

Step (1) : Locate the cell with the smallest cost in the transportation table. Suppose it is c_{ij}. Allocate $x_{ij} = \min(a_i, b_j)$. There will be one of the three cases.

Step (2) : Case (i) : $x_{ij} = a_i$, then cross the i^{th} row since it is exhausted. The requirement at j^{th} column now will be reduced to $b_j - a_i$

Case (ii) : $x_{ij} = b_j$, then cross the j^{th} column since the requirement at that column is exhausted. The availability at i^{th} row will reduce by b_j. The new availability is $a_i - b_j$

Case (iii) : $x_{ij} = a_i = b_j$. Cross out either i^{th} row or j^{th} column and recompute the availability at i^{th} row is now 0 and column requirement at j^{th} column is also 0.

Step (3) : Repeat steps (1) and (2) for newly reduced table till all the rows and columns are exhausted.

Note : If at any point the minimum cost c_{ij} is not unique choose any cost arbitrarily and start allocations.

Example 7.2 : *Find the initial basic feasible solution to the transportation problem in Example 7.1 using matrix minima method.*

Solution : Note that the smallest cost in the transportation cost table is 4. It is found at (O_1, D_4). Allocate x_{14} = min (120, 110) = 110. It will exhaust column D_4 and the availability at D_1 is 120 − 110 = 10. The new table is follows.

	D_1	D_2	D_3	D_4	
O_1	20	22	17	$\boxed{110}$ 04	10
O_2	24	37	09	07	70
O_3	32	37	20	15	50
	60	40	30	–	130

In the new table (unshaded) the smallest cost is 9 at (O_2, D_3).

Allocate x_{23} = min (70, 30) = 30.

The D_3 column gets exhausted. It reduces the requirement at O_2 as 70 − 30 = 40. The new table (unshaded) is given below.

	D_1	D_2	D_3	D_4	
O_1	20	22	17	$\boxed{110}$ 4	10
O_2	24	37	$\boxed{30}$ 9	7	40
O_3	32	37	20	15	50
	60	40	–	–	100

The minimum cost among uncovered elements is at (O_1, D_1). It is 20.

Allocate x_{11} = min (10, 60) = 10.

Note the row O_1 gets exhausted.

The new table (unshaded) is as follows.

	D_1	D_2	D_3	D_4	
O_1	[10] 20	22	17	[110] 4	–
O_2	24	37	[30] 9	7	40
O_3	32	37	20	15	50
	50	40	–	–	90

The minimum cost among unconverted elements is 24 at (O_2, D_1). Allocate $x_{21} = \min(40, 50) = 40$. It exhausts row O_2 gives, new table as follows.

	D_1	D_2	D_3	D_4	
O_1	[10] 20	22	17	[110] 4	–
O_2	[40] 24	37	[30] 9	7	–
O_3	32	37	20	15	50
	10	40	–	–	50

Now the minimum cost is 32 at (O_3, D_1). Allocate $x_{32} = \min(50, 10) = 10$. Ultimately $x_{32} = 40$.

The complete transportation schedule is as follows.

	D_1	D_2	D_3	D_4	
O_1	[10] 20	22	17	[110] 04	120
O_2	[40] 24	37	[30] 09	07	70
O_3	[10] 32	[40] 37	20	15	50
	60	40	30	110	240

The cost of transportation is Z.

$= (20 \times 10) + (4 \times 110) + (24 \times 40) + (9 \times 30) + (32 \times 10) + (37 \times 40) = ₹\ 3670$

Note : Total cost of transportation by North-West rule was ₹ 3680 where as it ₹ 3670 by matrix minima method. Thus, the matrix minima method gives better solution is verified.

(C) Vogel's Approximation Method (VAM) : In this method allocation is made by considering the smallest cost and next to the smallest cost in the same row or column. The difference in the smallest and next to the smallest is taken to be penalty cost in case not allocated at the cell with smallest cost. Naturally one has to choose the cost with largest penalty of non-allocation. This method gives solution to transportation problem which very close to optimal solution. Many times solution by VAM turns out to be the optimal solution.

Procedure :

Step (1) : Determine the smallest and next the smallest cost for each row and each column. Compute the difference between the smallest cost and next to the smallest cost. The difference is displayed near the respective rows and columns as the penalties of non-allocation. If the smallest and the next cost are equal the penalty (difference is) 0.

Step (2) : Locate the largest difference. Choose the respective row or column for allocation. Use the cell with smallest cost in the chosen row or column (If the largest penalties are same or tie occurs choose any difference arbitrarily or choose the cell with minimum cost among both penalties or choose the cell where maximum allocation is possible). Suppose $(i, j)^{th}$ cell is selected for allocation. Allocate $x_{ij} = \min(a_i, b_j)$. This gives rise to any one of three cases.

Step (3) : Case (i) : If $x_{ij} = a_i$, then cross out the i^{th} row since it is exhausted. Reduce the requirement at j^{th} column by a_i. Thus, the revised requirement is $b_j - a_i$.

Case (ii) : If $x_{ij} = b_j$, then cross out the j^{th} column since the requirement get exhausted. Reduce the availability at i^{th} now by b_j. The availability at i^{th} row is now $a_i - b_j$.

Case (iii) : If $x_{ij} = a_i = b_j$ then cross out either i^{th} row or j^{th} column.

Step (4) : For newly reduced table repeat the steps (1), (2) and (3) till all the rows and columns get exhausted.

Note : Due to arbitrary breaking of ties different allocations are possible, however the total cost of transportation is same in each of the such cases.

Example 7.3 : *Obtain the initial basic feasible solution to the transportation problem in Example 7.1 using VAM.*

Solution :

	D_1	D_2	D_3	D_4	a_i	Minimum	Next to minimum	Diff.
O_1	20	22	17	4	120	4	17	13
O_2	24	37	9	7	70	7	9	2
O_3	32	37	20	15	50	15	20	5
b_j	60	40	30	110				
Min.	20	22	9	4	4			
Next min.	24	37	17	7	7			
Difference	4	15	8	3	3			

The largest difference is 15 at column D_2. The smallest cost in D_2 is 22 at (O_1, D_2). We allocate $x_{12} = \min(120, 40) = 40$. The column D_2 gets exhausted. The requirement at O_1 reduces by 40 hence it is $120 - 40 = 80$. Dropping column D_2 we get new table as follows :

	D_1	D_3	D_4	Availability	Penalties
O_1	20	17	4	80	$17 - 4 = 13$
O_2	24	9	7	70	$9 - 7 = 2$
O_3	32	20	15	50	$20 - 15 = 5$
Requirement	60	30	110	200	–
Penalties	$24 - 20 = 4$	$17 - 9 = 8$	$7 - 4 = 3$		

The largest penalty is 13 at O_1. The smallest cost in O_1 row is 4 at D_4. Allocate x_{14} = min (80, 110) = 80. The row O_1 gets exhausted. The new requirement at D_4 is revised to 110 − 80 = 30. The new table is obtained by crossing row 1.

	D_1	D_3	D_4	Availability	Penalties
O_2	24	9	7	70	9 − 7 = 2
O_3	32	20	15	50	20 − 15 = 5
Requirement	60	30	30	120	–
Penalty	32 − 24 = 8	20 − 9 = 11	15 − 7 = 8		

The highest penalty is 11 at column D_3. The smallest cost in D_3 columns is 9 at (O_2, D_3). We allocate x_{23} = min (70, 30) = 30. The column D_3 gets exhausted. The availability at O_2 changes to 70 − 30 = 40. The revised table is given below by dropping column D_3.

	D_1	D_4	Availability	Penalties
O_2	24	7	40	24 − 7 = 17
O_3	32	15	50	32 − 15 = 17
Requirement	60	30		
Penalties	32 − 24 = 8	15 − 7 = 8		

The largest penalty is 17. There is a tie at O_2 and O_3. We select O_2, it has the smallest cost 7 at D_4 (rather than selecting O_3 with smallest cost 15 at D_4). Allocate x_{24} = min (30, 40) = 30. The column D_4 gets exhausted. We are left with D_1 column having two cells.

	D_1	a_i
O_2	24	10
O_3	32	50
b_j	60	60

Clearly we have to allocate x_{21} = 10 and x_{31} = 50. The complete allocation is of the type.

	D_1	D_2	D_3	D_4	
O_1	20	[40] 22	17	[80] 4	120
O_2	[10] 24	37	[30] 9	[30] 7	70
O_3	[50] 32	37	20	15	50
	60	40	30	110	240

The total cost of transportation is

$$Z = (22 \times 40) + (4 \times 80) + (24 \times 10) + (9 \times 30) + (7 \times 30) + (32 \times 50) = ₹\ 3520$$

Note : The solution by VAM has the smallest total transportation cost. Thus, VAM gives a solution better than the other two methods. It can be seen from following data :

	Method	**Cost of transportation**
1.	N-W corner	3680
2.	Matrix minimum	3670
3.	VAM	3520

7.5 Maximization in Transportation Problem

In a typical transportation problem we consider the cost of transportation and obviously we minimise the total cost. In stead of cost we may consider profit due to transportation. In this case the objective function will be the total profit and we try maximise the same. Thus a transportation problem can be viewed as maximisation problem. To solve such problem we need some modification and use the same method to get optimum solution.

Note that : Maximum {a, b, c, ...} = – Minimum {– a, – b, – c, ...}.

Using the above technique we convert a transportation problem of maximisation to minimisation and find the solution. The above technique changes the sign of cost (C_{ij}); however handling negative values $-C_{ij}$ is inconvenient hence we add a number to each of $-C_{ij}$ so that all values will be positive.

It can be achieved as follows :

Steps to solve maximisation transportation problem :

1. Make the problem balanced if it is not.
2. Find the maximum value of C_{ij} let it be 'C'.
3. Replace C_{ij} by $C - C_{ij}$ (This will convert the original problem of maximisation to minimisation).
4. Use the desired method of obtaining IBFS.

Illustration : The following table gives profit (in suitable units) earned by sending the items from manufacturing plants A, B, C to the market centres in cities X and Y. Obtain the solution using VAM in order to maximise the total profit.

Plants \ Cities	X	Y	Production capacity
A	19	17	20
B	8	1	30
C	12	5	40
Requirement	35	36	–

Solution : Note that $\sum a_i = 20 + 30 + 40 = 90 \neq \sum b_i = 35 + 36 = 71$ hence the problem is unbalanced. We create a dummy city Z (destination) and assign the requirement of $90 - 71 = 19$ units. The corresponding cost will be 0. Thus the balanced transportation problem of maximisation will be as follows.

	X	Y	Z	a_i
A	19	17	0	20
B	8	1	0	30
C	12	5	0	40
b_j	35	36	19	90

We convert the above problem to minimisation.

Maximum $C_{ij} = 19$, we replace C_{ij} by $19 - C_{ij}$.

	X	Y	Z	a_i
A	$19 - 19 = 0$	$19 - 17 = 2$	$19 - 0 = 19$	20
B	$19 - 8 = 11$	$19 - 1 = 18$	$19 - 0 = 19$	30
C	$19 - 12 = 7$	$19 - 5 = 14$	$19 - 0 = 19$	40
b_j	35	36	19	90

	X	Y	Z	a_i	Penalty
A	0	$\boxed{20}$ 2	19	20	2 –
B	11	$\boxed{11}$ 18	$\boxed{19}$ 19	30	7 7 1
C	$\boxed{35}$ 7	$\boxed{5}$ 14	19	40	7 7* 5*
b_j	35	36	19	90	
Penalty	7 4 –	12* 4 4	0 0 0		

There are apparently 5 allocations, however Z city is dummy so really there will be only 4 allocations as follows with original profit values. We use original profit values rather than modified C_{ij}'s.

	X	Y	a_i
A	19	$\boxed{20}$ 17	20
B	8	$\boxed{11}$ 1	11
C	$\boxed{35}$ 12	$\boxed{5}$ 5	40
b_j	35	36	71

$$\text{Total profit} = (12 \times 35) + (17 \times 20) + (1 \times 11) + (5 \times 5)$$
$$= 420 + 340 + 11 + 25 = 796$$

Exercise (7.1)

1. Explain the transportation problem.
2. What is the transportation problem ? What is an unbalanced case in transportation problem ? **(April 2017)**
3. Explain the similarity between L.P.P. and transportation problem.
4. Give the mathematical formulation of transportation problem.
5. State and explain the various methods of obtaining initial basic feasible solution.
6. What is unbalanced Transportation Problem ? How will you convert unbalance T.P. to balanced T.P. ? **(April 2017)**
7. Write the algorithm to obtain an initial basic feasible solution for the Transportation Problem by (i) North West Corner Rule, (ii) Matrix Minima Method, (iii) Vogel's Approximation Method. **(April 2018)**

Exercise (7.2)

1. Using North-West corner rule find the initial basic feasible solution for the following transportation problems. Where a_i's are availability and b_j's are requirements.

(a) $a_1 = 10$ $a_2 = 5$ $a_3 = 4$ $a_4 = 6$

$b_1 = 10$ $b_2 = 5$ $b_3 = 7$ $b_4 = 3$

(b) $a_1 = 1$ $a_2 = 16$ $a_3 = 7$ $a_4 = 8$

$b_1 = 3$ $b_2 = 4$ $b_3 = 5$ $b_4 = 2,$ $b_5 = 8$

(c) $a_1 = 10$ $a_2 = 3$ $a_3 = 7$

$b_1 = 14$ $b_2 = 3$ $b_3 = 4$ $b_4 = 9$

2. Use North-West corner rule to determine an initial basic feasible solution to the transportation problem.

	To			Supply
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Demand	7	9	18	–

Also obtain the solution using (i) matrix minima method (ii) VAM (iii) North-West corners method. State which solution is better.

3. Obtain the initial basic feasible solution using (i) North-West corner rule and (ii) VAM for the following transportation problem with given cost per unit of transportation. Also find which of two solution is better. **(April 2018)**

From wavehouse	To Store				Total supply
	S_1	S_2	S_3	S_4	
W_1	50	150	70	60	50
W_2	80	70	90	10	60
W_3	15	90	80	80	40
house					
Total demand	20	70	50	10	150

4. Obtain the initial basic feasible solution to the following transportation problem using matrix minima method with a view to minimise the total transportation time. The following table gives time required in hours for transportation.

Plant	Ware houses				Supply
	1	2	3	4	
P_1	3	4	9	2	23
P_2	6	5	8	8	27
Demand	12	13	15	10	50

5. Find the solution for the following transportation problem using VAM. The matrix given below is of travelling time required in minutes.

Men required at	Men available at		No. of men required
	Station A	Station B	
Restaurant	30	15	3
Plant	20	30	5
Garage	40	40	8
Other	45	60	2
No. of men available	10	8	–

6. Obtain an initial basic feasible solution to the following T.V. by Voge's Approximation Method (VAM) : **(April 2018)**

Sources	Destinations				Supply
	D_1	D_2	D_3	D_4	
S_1	1	2	1	4	30
S_2	3	3	2	1	50
S_3	4	2	5	9	20
	20	40	30	10	50

7. Obtain initial basic feasible solution using North West corner method for following transportation problem : **(April 2017)**

Markets → Sources ↓	D_1	D_2	D_3	D_4	Supply
O_1	5	2	4	3	22
O_2	4	8	1	6	15
O_3	4	6	7	5	8
Demand	7	12	17	9	45

8. Determine an initial basic feasible solution to the following transportation problem by using Matrix Minima Method : **(April 2017)**

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Supply
F_1	21	16	25	13	21
F_2	17	18	14	23	23
F_3	32	27	18	41	19
Demand	11	15	17	20	63

Unbalanced Transportation Problems :

9. Obtain initial basic feasible solution using Vogel's Approximation method for following transportation problem : **(April 2017)**

Markets → Sources ↓	D_1	D_2	D_3	D_4	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	275	

10. Obtain the solution to the following transportation problem using VAM to find the minimum cost of transportation.

	D_1	D_2	D_3	a_i
O_1	7	3	6	5
O_2	4	6	8	7
O_3	5	8	4	7
O_4	5	4	3	3
b_j	5	8	10	-

11. Obtain the solution using VAM to minimize the total cost of transportation.

	D_1	D_2	D_3	Supply
S_1	5	1	7	10
S_2	6	4	6	80
S_3	3	2	5	15
Demand	75	20	50	

12. Obtain the optimum schedule of transportation using :
   (i) North-West corner method.
   (ii) Matrix-minima method.
   (iii) VAM.

   So that the total cost of transportation is minimum.
   Show that the obtaining using VAM is the best.

Factories	Stores				Production capacity
	1	2	3	4	
A	4	6	8	13	50
B	13	11	10	8	70
C	14	4	10	13	30
D	9	11	13	8	50
Demand	25	35	105	20	-

Maximisation in Transportation Problems :

13. The following table gives profit per unit due to transportation from factories to sales agencies. Find the allocation so that total profit is maximum use VAM.

Factories	Sales agencies					Capacity
	A	B	C	D	E	
1	15	17	12	11	11	140
2	5	9	7	15	7	190
3	14	15	16	20	10	115
Demand	74	94	69	39	119	-

14. Solve the following transportation problem with a view to maximise the profit. Use matrix minima method.

From \ To	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90	90	100	110	200
Kolkatta	50	70	130	85	100
Demand	75	100	100	30	–

Answers (7.2)

1. (a)

10				10
	5			5
		4		4
		3	3	6
10	5	7	3	25

(b)

1						1
2	4	5	2	3		16
				5	2	7
					8	8
3	4	5	2	8	10	32

(c)

10				10
3				3
1	3	3		7
		1	9	10
14	3	4	9	30

2. N-W corner

5			5
2	6		8
	3	4	7
		14	14
7	3	18	34

Total Cost = 102

Matrix Minima

	2	3	5
		8	8
	7		7
7		7	14
7	9	18	34

Total Cost = 83

VAM

5			5
		8	8
	7		7
2	2	10	14
7	9	18	34

Total Cost = 80

3. N-W corner

20	30			50
	40	20		60
		30	10	40
20	70	50	10	150

Total Cost = 13300

VAM

		50		50
	50		10	60
20	20			40
20	70	50	10	150

Total Cost = 9200

4.

12	1		10	23
	12	15		27
12	13	15	10	50

Total Cost = 240

5.

	3	3
5	12	5
3	5	8
2		2
10	8	18

Total Cost = 915

6.

20		10		30
	20	20	10	50
	20			20
20	40	30	10	100

Total Cost = 180

7.

7	12	3		22
		14	1	15
			8	8
7	12	17	9	45

Total Cost = 108

8.

	1		20	21
6		17		23
5	14			19
11	15	17	20	63

Total Cost = 1154

9.

200	50			250
	150		150	300
		275	125	400
200	200*	275	275	950

Total Cost = 11875

* Problem is unbalanced, hence from D_2 out of 225 only 200 items will be transported.

10.

	D_1	D_2	D_3	
O_1		5		5
O_2	5		2	7
O_3			7	7
O_4		2	1	3
	5	7	10	

Total Cost = 90

11.

	D_1	D_2	D_3	
S_1		10		10
S_2	60	10	10	80
S_3	15			15
	75	20	10	

Total Cost = 515

12. (i) N-W corner.

	1	2	3	4	
A	25	25			50
B		10	60		70
C			30		30
D			15	20	35
	25	35	105	20	

Total Cost = 1615

(ii) Matrix-Minima :

	1	2	3	4	
A	25	5	5		35
B			70		70
C		30			30
D			30	20	50
	25	35	105	20	

Total Cost 1540

(iii) VAM

	1	2	3	4	
A	25	5	20		50
B			70		70
C		30			30
D			15	20	35
	25	35	105	20	

Total Cost = 1465

13.

46	94				140
			39	101	140
28		69		18	115
74	94	69	39	119	

Total Profit = 5256

14.

70	100		30	200
		100		100
70	100	100	30	200

Total Profit = 31600

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MODEL QUESTION PAPER

Time : 3 Hours **Maximum Marks : 70**

Instructions to the candidates :

1. All questions are compulsory.
2. Figures to right indicate full marks.
3. Use of calculator is allowed.

1. Attempt any two of the following : **(7 each)**

(a) Explain the following terms :

(i) Proportion, (ii) Continued proportion, (iii) Direct proportion, (iv) Inverse proportion.

(b) A house is sold at 25% profit. The amount of brokerage at $\frac{3}{4}$ % comes to ₹ 5250. Find the cost of house.

(c) The sum of present ages of 3 persons is 66 years. Five years ago their ages where in the proportion 4 : 6 : 7. Find their present age.

2. Attempt any two of the following : **(7 each)**

(a) (i) What are equity shares and preference shares ?

(ii) Pranave invested ₹ 13568/- in 7% shares at ₹ 106/-. Find his profit at the end of the year [F.V. 100].

(b) For producing 8 tables, material worth ₹ 4000 is required. Labour charges are 2,400. What should be the sales price (S.P.) of each table to realise a profit of 25%.

(c) A sum of ₹ 10,000 at 8% p.a. in five years is compounded half yearly. Find the amount.

3. Attempt any two of the following : **(7 each)**

(a) Which of the following is the better investment ?

(i) 8% at ₹ 80.

(ii) 15% of ₹ 120 (F.V. = ₹ 100).

(b) A sum of ₹ 2314 was invest in 7% at 85. When it rose to 90 all the shares were sold. In the meanwhile dividend was received. For purchasing the brokerage was 1% while selling it was $\frac{1}{2}$ %. What is the total gain or loss in the total transaction ?

(c) Obtain an initial basic feasible solution using Matrix Minima method to the following Transportation Problem (T.P.) :

	To			Supply
From	2	7	4	5
	3	3	1	8
	1	6	2	22
	5	4	7	10
Demand	10	20	15	45

4. **Attempt any two of the following :** **(7 each)**

(a) If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, then show that the matrix A satisfies $A^2 - 4A + I = 0$.

(b) Solve the following Linear Programming Problem (L.P.P.) by graphical method :

Maximize $Z = 3x + 5y$

Subject to constraints :

$$2x + 3y \leq 12$$
$$3x + 2y \leq 18$$
$$x, y \geq 0$$

(c) Solve the following system of linear equations by matrix inverse method.

$$x + y + z = 6$$
$$2x + y + 2z = 10$$
$$3x + 3y + 4z = 21$$

5. **Attempt any two of the following :** **(7 each)**

(a) Obtain an initial basic feasible solution to the following T.P. by Vogel's Approximation Method (VAM).

		Destinations				Supply
		D_1	D_2	D_3	D_4	
Sources	S_1	1	2	1	4	30
	S_2	3	3	2	1	50
	S_3	4	2	5	9	20
Demand		20	40	30	10	

(b) A and B are two types of fertilizers available at ₹ 30 per kg and ₹ 50 per kg respectively. Fertilizers A contains 20 units of potash, 10 units of nitrogen and 40 units of phosphorus. Fertilizer B contains 15 units of potash, 20 units of nitrogen and 10 units of phosphorous. The requirement of potash, nitrogen and phosphorus is at least 1800, 1700, 1600 units.

(c) Write an algorithm to obtain an initial basic feasible solution for the Transportation Problem by North West Corner Rule and Matrix Minima Method.

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