All the distributions that we will discuss, can be categorised into two broad groups : `Discrete` and `Continuous` .

- Discrete distributions pertain to data which contains discrete outcomes [Integers] as their values. E.g. Number of people on the train , Count of bees in an honeycomb . [Imagine the horror of seeing three and quarter of a person getting off the train!]
- Continuous distributions on the other hand have values which can take values in decimals . E.g. Weight of students in a class, Length of trees in a forest [Number of people in a zombie movie?]

We'll start our discussion with a very simple and intuitive distribution known as uniform distribution . In that discussion we will also learn about defining and calculating mean and variance as `Expectation` .

Uniform Distribution [Discrete]

Uniform distribution is where each value in the distribution has equal probability of occurrence . If we had to come up with a general expression for probability of occurrence of each value in the uniform distribution, we can write .

$$P(X = x) = \frac{1}{n}$$

Where n is the number of values in the range of the distribution. A classic example which produces uniform distribution is throwing a fair dice. 🎲

Each throw might result in any of the $\{1, 2, 3, 4, 5, 6\}$ on the top face with equal probability . So in this case

$$P(X = x) = \frac{1}{6} \ \ \forall \, x \in \{1, 2, 3, 4, 5, 6\}$$

Expectation

Consider a gambling party where the host has gone bonkers and is asking guests to throw a dice and collect as much money as the number which shows on the face of the dice. However if you get 1 , you have to give $ 10 to the host.On an Average how much money the host is going to make [or lose] on each player.

We can start by assuming there are 72 [the number doesn't really matter] people in the party . Since the dice is fair; we can assume that each of $\{1, 2, 3, 4, 5, 6\}$ will be obtained by 12 guests.

- So the money lost by host $= 12 * 6 + 12 * 5 + 12 * 4 + 12 * 3 + 12 * 2 = \$240$
- money gained by host $= 12 * 10 = \$120$
- Overall money lost by host $= 240 - 120 = \$120$
- Money lost per player $\frac{\$120}{72} = \$1.67$

A simpler way of doing this will be to multiply probability of each outcome with the outcome and sum them.

Average gain $= -6 * \frac{1}{6} - 5 * \frac{1}{6} - 4 * \frac{1}{6} - 3 * * \frac{1}{6} - 2 * \frac{1}{6} + 10 * \frac{1}{6} = \$1.67$

This is also known as expected value of a distribution [or mean]. A formal expression for the same will be

$$\mu = E[X] = \sum xP(X = x)$$

Let's calculate mean of uniform distribution like this . Say distribution takes integer values in the interval $[a, b]$ with n values in it.

$[b = a + (n - 1)]$, Since b is the nth element .

$$\mu = E[X]$$
$$= [a + (a + 1) + (a + 2).....+b] * \frac{1}{n}$$
$$= [n * a + (1 + 2 + 3....n - 1)] * \frac{1}{n}$$
$$= [n * a + (n - 1) * \frac{n}{2}] * \frac{1}{n}$$
$$= \frac{2a + n - 1}{2}$$
$$= \frac{a + a + n - 1}{2}$$
$$= \frac{a + b}{2}$$

Before we proceed further I will mention here few algebraic properties of expectation which are easy to derive; so I am going to put them here without extensive proof.

- $E[g(X)] = \sum g(x)P(X = x)$
- $E[const] = const$
- $E(aX + b) = aE(X) + b$

Variance of a distribution is defined as expectation of $(X - \mu)^2$. [remember variance was mean of squared deviations from the mean]

$$Var(X) = E[(X - \mu)^2]$$
$$= E[X^2 + \mu^2 - 2\mu X]$$
$$= E[X^2] + \mu^2 - 2\mu E[X]$$
$$= E[X^2] - \mu^2$$
$$= E[X^2] - (E[X])^2$$

Lets calculate variance of uniform distribution with this

$$E(X^2) = [a^2 + (a+1)^2 + \ldots \ldots (a+n-1)^2] * \frac{1}{n}$$

$$= [n * a^2 + (1 + 2^2 + 3^2 + \ldots \ldots (n-1)^2) - 2 * a(1 + 2 + 3 \ldots n - 1)] * \frac{1}{n}$$

$$= [n * a^2 + \frac{(n-1)n(2n-1)}{6} - 2 * a * \frac{(n-1)(n)}{2}] * \frac{1}{n}$$

$$= a^2 + \frac{(b-a)(b-a+1)(2b-2a+1)}{6} - a(b-a)(b-a+1)$$

$$\vdots$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{(b-a)^2}{12}$$

I have skipped some algebraic gymnastic there. Do it yourself if it rubbed your marbles the wrong way.

For continuous distributions ; if the pdf is given as $f(x)$ then
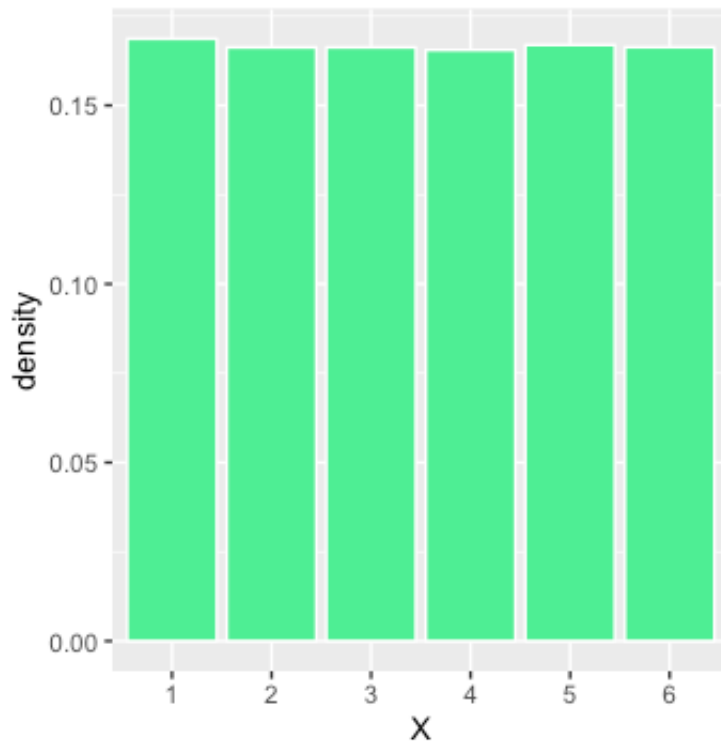
- $E[X] = \int x f(x) dx$

Key Takeaways

About expectations :

- $E[X] = \sum x P(X = x) = \int x f(x) dx$
- $Var(X) = E[X^2] - (E[X])^2$

About discrete uniform distribution :

- $P(X = x) = \frac{1}{n} \forall x \in \{a, a+1, \ldots b\} \; with \; n \; elements$
- $\mu = E(X) = \frac{(a+b)}{2}$
- $Var(X) = \frac{(b-a)^2}{12}$

What does it look like

Bernoulli's Distribution [Discrete]

We get a call on our phone , we receive it or we don't . We are driving on the road , we reach home safely or we die [well that went from 0 to dead pretty fast] . Life is full of transactions which have two outcomes . 0/1 , failure/success, accept/decline and so on.

And each of these outcomes have some associated probability . Like there is 90% chance of an advice being solid gold if given by an old dude while peeling an apple with a pocket knife and eating pieces right off the blade . Oddly specific but relatable.

Formally we write :

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = 0, \text{ where q } = 1\text{-p} \end{cases}$$
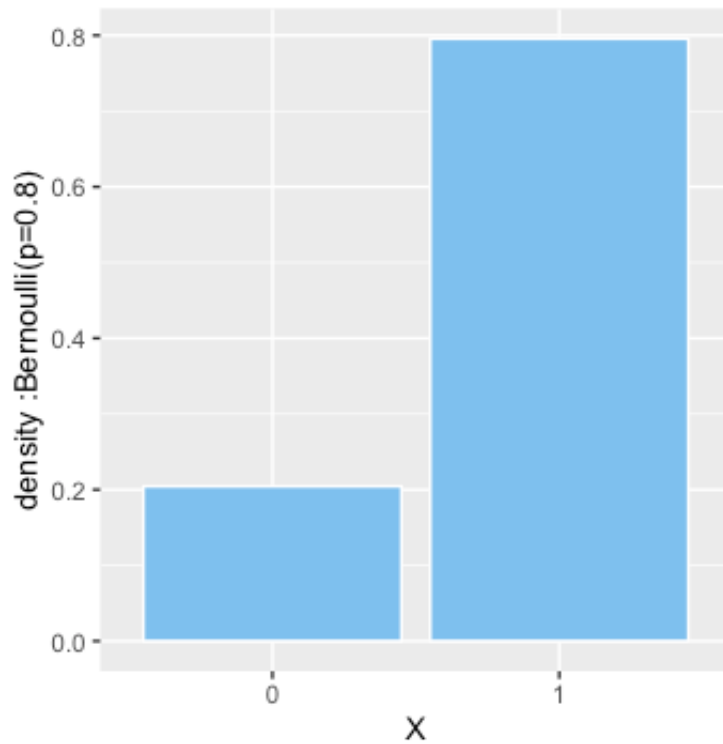
Mean

$$\begin{aligned} \mu &= E[X] \\ &= 1 * p + 0 * q \\ &= p \end{aligned}$$

Variance

$$\begin{aligned} E[X^2] &= 1^2 * p + 0^2 * q \\ &= p \\ Var(X) &= E[X^2] - [E[X]]^2 \\ &= p - p^2 = p * (1 - p) \\ &= pq \end{aligned}$$

What does it look like

Life is nothing but a sequence of countless Bernoulli Trials .

Binomial Distribution [Discrete]

During the pandemic lockdown, you have learnt to make absolutely killer cocktails. Probability of a person getting shitfaced drunk on the cocktail is p . Let's say n people attended your party once the lockdown opens , what are the chances that x of them will have no memory of what happened that night ?

x is just number of guests . It can be made up of any x guests out of n . We can chose x guests out of n in ^nC_x ways.

$$^nC_x = \frac{!n}{!x!(n-x)}$$

$$\text{given } !n = n * n(n-1)\cdots2*1$$

formally we can write [considering $q = 1 - p$]

$$P(X = x) = {^nC_x} * p^x * q^{n-x}$$

Before we get down to calculating mean and variance , let's get done with some mean looking maths involving binomial coefficients .

We know that $(a + b)^n$ can be expanded using following expression involving binomial coefficients .

$$(a+b)^n = \sum_{r=0}^{n} {^nC_r} a^r b^{n-r}$$

Now to the mean maths; we are going to derive some results which will come in handy later.

$$\sum_{r=0}^{n} r * {}^nC_r * a^r * b^{n-r} = a * \sum_{r=0}^{n} r * {}^nC_r * a^{r-1} * b^{n-r}$$

$$= a * \frac{d}{da}\left[\sum_{r=0}^{n} {}^nC_r * a^r * b^{n-r}\right]$$

$$= a * \frac{d}{da}\left[(a+b)^n\right]$$

$$= na(a+b)^{n-1}$$

$$\sum_{r=0}^{n} r^2 * {}^nC_r * a^r * b^{n-r} = \sum_{r=0}^{n} r(r-1) * {}^nC_r * a^r * b^{n-r} + \sum_{r=0}^{n} r * {}^nC_r * a^r * b^{n-r}$$

$$= a^2 \sum_{r=0}^{n} r(r-1) * {}^nC_r * a^{r-2} * b^{n-r} + na(a+b)^{n-1}$$

$$= a^2 * \frac{d}{da}\left[\sum_{r=0}^{n} r * {}^nC_r * a^{r-1} * b^{n-r}\right] + na(a+b)^{n-1}$$

$$= a^2 * \frac{d}{da}\left[\frac{d}{da}\left[\sum_{r=0}^{n} {}^nC_r * a^r * b^{n-r}\right]\right] + na(a+b)^{n-1}$$

$$= a^2 * \frac{d}{da}\left[\frac{d}{da}\left[(a+b)^n\right]\right] + na(a+b)^{n-1}$$
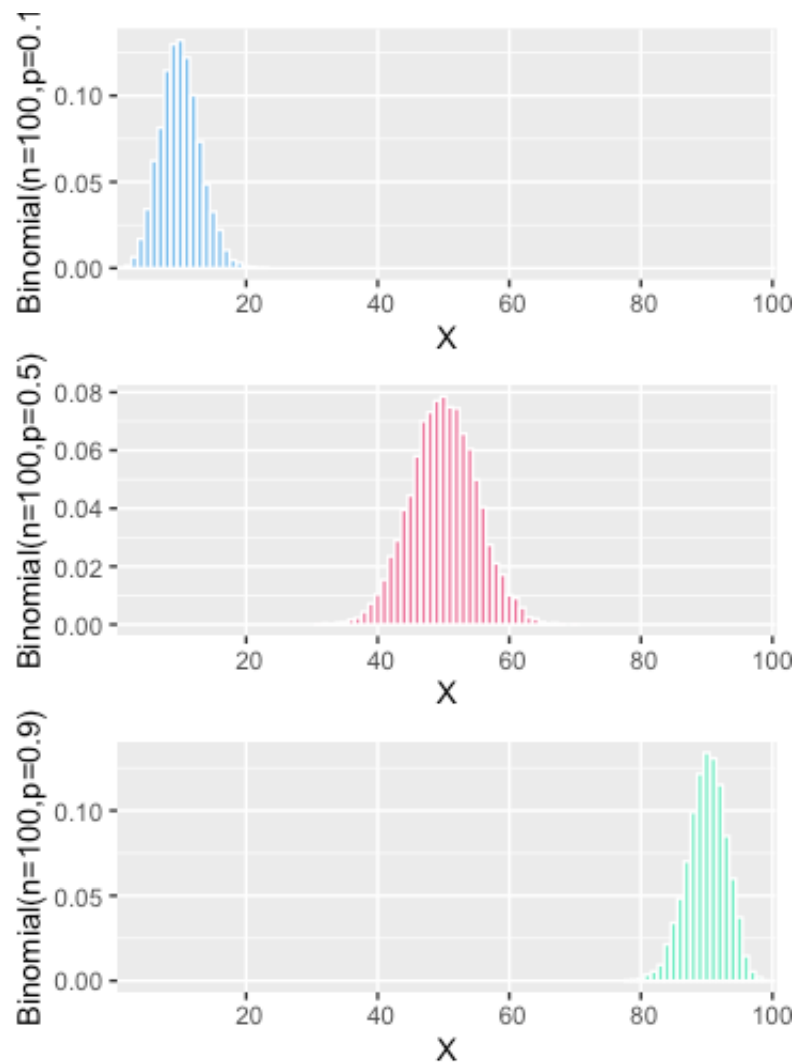
$$= n(n-1)a^2(a+b)^{n-2} + na(a+b)^{n-1}$$

Mean

$$\mu = E[X]$$

$$= \sum_{x=0}^{n} x * {}^nC_x * p^x * q^{n-x}$$

$$= np(p+q)^{n-1}$$

$$= np \quad [\text{since } p+q=1]$$

Variance

$$E[X^2] = \sum_{r=0}^{n} x^2 * {}^nC_x * p^x * q^{n-x}$$

$$= n(n-1)p^2(p+q)^{n-2} + np(p+q)^{n-1}$$

$$= n(n-1)p^2 + np$$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p)$$

$$= npq$$

What does it look like

CDF [cumulative density function]

Many at times , probability of an instance is not very useful. And it's better to have a look at probability of an interval instead . This can be calculated as difference between cumulative probabilities which are defined as follows [given that X takes values in the interval $[a, b]$]

- for discrete distribution $CDF = P(X \leq x) = \sum\limits_{x=a}^{x} P(X = x)$

- for continuous distributions $= \int\limits_{a}^{x} f(x)dx$

An Example

for a large clinical trial where there were 1000 patients and probability of success was 20% . answer following questions [I have added R codes, wherever needed]

- whats the expected number of cured patients = np = 1000*0.2=200
- whats the expected standard deviation across multiple such clinical trials
  $= \sqrt{npq} = \sqrt{1000 * 0.2 * 0.8} = 12.6$

```
# whats the probability that 200 of the patients will be cured
dbinom(200,1000,0.2)

# whats the probability that upto 200 patients will be cured [use functions for
cdf]
pbinom(200,1000,0.2)

# whats the probability that number of patients cured
# will be between 190 and 210 [use function for cdf]
pbinom(210,1000,0.2)-pbinom(190,1000,0.2)
```

I did not explicitly mention, but I hope you realised that Binomial Distribution is collection of Bernoulli trials . Where each individual outcome is from Bernoulli distribution. Binomial Distribution's outcome is nothing but sum of the outcomes of those Bernoulli trials.

Poisson Distribution [Discrete]

You hear the unsavoury news that someone found an iron nail in their sausage and suddenly everyone in the group starts recounting similar experiences; and makes you feel as if this has become a very common phenomenon and world has gone to dogs and its a mistake to become a parent in these times and blah blah blah .

But if you take a step back and for a moment consider that; millions of sausages are consumed everyday and you rarely get to hear about someone really finding a nail in theirs. Mostly because, **not finding a nail in your sausage** is not NEWS, but finding a nail means; hell has finally frozen over. Same as a thousands of cars mundanely passing a town centre do not become talk of the town, but a single accident does.

I did not take this crazy sounding; sudden tangent in our discussion to play down the risk of nails or accident. The strange take aways that i want you to have from following news bite : `there were 5 accidents last month in this traffic junction` are :

- We don't know whats the probability of an individual vehicle passing the junction getting into an accident , **but we do know that its pretty small** despite the hullabaloo about the accidents
- We also know that thousands of vehicle must have passed through the junction which eventually resulted in these 5 accidents. But we don't know the exact value of n either.

So this month of 5 accidents was essentially a huge collection of Bernoulli trials with a very small probability of success. Basically this value 5 is coming from a Binomial distribution with very large n and very small p

We'll call this 5 accidents/month to be rate of events and denote it with symbol λ

$$\lambda = np$$
$$p = \frac{\lambda}{n}$$

Now with this information, someone asks you, can you tell me whats the probability that we'll see x accidents next month [given that flow of vehicles through the junction remains similar and other traffic conditions also remain unchanged]. Let's try to answer that question in absence of information of both n and p . All that we have, is the rate λ and hunch that n is large and p is small.

But it still is a binomial distribution . [well for now at-least , but we are going in for the kill boys!]

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty} {}^nC_x * p^x * q^{n-x}$$

$$= \lim_{n\to\infty} \frac{!n}{!x!(n-x)} * \left(\frac{\lambda}{n}\right)^x * \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{!x} \lim_{n\to\infty} \frac{!n}{!(n-x)} * \frac{1}{n^x} * \left(1-\frac{\lambda}{n}\right)^n * \left(1-\frac{\lambda}{n}\right)^{-x}$$

Ah i guess we are stuck now. We'll have to kill this beast in parts and then come back and put together it all.

$$\lim_{n\to\infty} \frac{!n}{!(n-x)} * \frac{1}{n^x} = \frac{n*(n-1)*\cdots*(n-x+1)*(n-x)*\cdots 2*1}{(n-x)*\cdots*2*1} * \frac{1}{n*n*\cdots*n}$$

$$= 1*\left(1-\frac{1}{n}\right)*\left(1-\frac{2}{n}\right)*\cdots*\left(1-\frac{x-1}{n}\right)$$

$$= 1$$

thats one down

$$e = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty}\left[\left(1-\frac{1}{n/\lambda}\right)^{-n/\lambda}\right]^{-\lambda}$$

$$= e^{-\lambda}$$

the remaining guy is just a formality now

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1^{-x} = 1$$

Putting all this together we get

$$\lim_{n\to\infty} P(X=x) = \frac{\lambda^x * e^{-\lambda}}{!x}$$

Thats the density function for Poisson Distribution

Mean

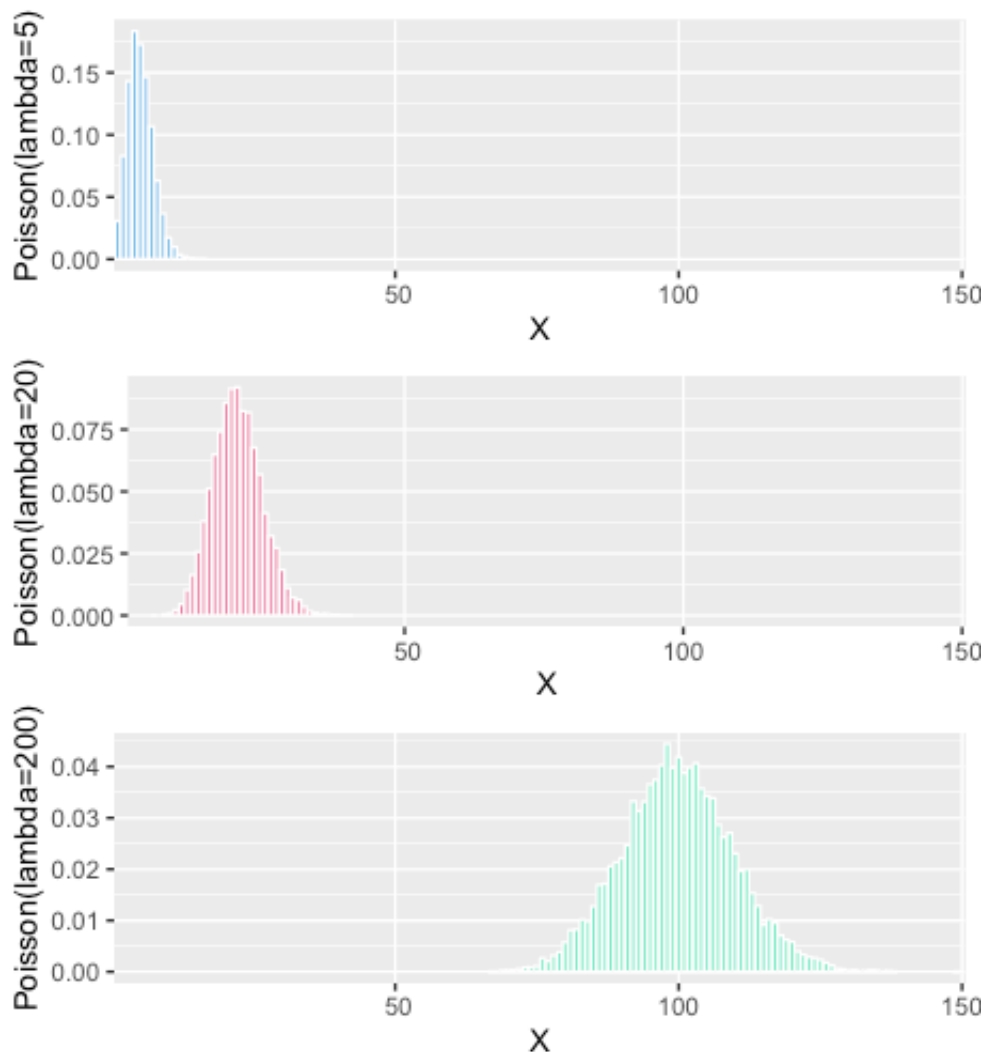$$\mu = E[X]$$
$$= \sum_{x=1}^{\infty} \frac{x * \lambda^x * e^{-\lambda}}{!x}$$
$$= \lambda * e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{!(x-1)}$$
$$= \lambda * e^{-\lambda} \left[1 + \frac{\lambda}{!1} + \frac{\lambda^2}{!2} + \frac{\lambda^3}{!3} \cdots \right]$$
$$= \lambda * e^{-\lambda} e^{\lambda}$$
$$= \lambda$$

Variance

$$E[X^2] = \sum_{x=1}^{\infty} \frac{x^2 * \lambda^x * e^{-\lambda}}{!x}$$
$$= \sum_{x=1}^{\infty} \frac{x * (x-1) * \lambda^x * e^{-\lambda}}{!x} + \sum_{x=1}^{\infty} \frac{x * \lambda^x * e^{-\lambda}}{!x}$$
$$= \lambda^2 * e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{!(x-2)} + \lambda$$
$$= \lambda^2 + \lambda$$
$$Var(X) = E[X^2] - [E[X]]^2$$
$$= \lambda^2 + \lambda - \lambda^2$$
$$= \lambda$$

What does it look like

Notice, since variance is λ, for higher values of λ; the distribution is becoming more spread out.

An Example

A McDonald's on an highway receives 20 orders per hour . If that exceeds 25, they'd need to hire another person. Whats the probability that they'd need to do so.

- the cdf with x=25 here will give us probability of number of orders being *upto* 25
- we need probability of $x \geq 25$

```
1-ppois(25,20)
```

Exponential Distribution [Continuous]

You know; the fear [father of all the questions, doubts and drugs] is not always about how many. It's about, more often than not , when the first/next would happen. As in the first heart attack , the next kiss, [an odd couple indeed], the first covid positive patient and of-course; for the dinosaurs, the next big meteor.

Continuing with the example that we took above; what if the question that we want to answer is :

`whats the probability that they will receive their next order within 30 mins`

Of course the answer isn't that straight forward to come by.

We know that, events are happening at the rate λ per unit time. So, in t time units , events will happen at the rate λt

Using Poisson distribution, probability of x events happening in next t time units

$$P(X = x) = \frac{e^{-\lambda t} * (\lambda t)^x}{!x}$$

Whats the probability that no event will happen in next t time units?

$$P(X = 0) = e^{-\lambda t}$$

Whats the probability that at-least 1 event will happen in t time units ?

$$1 - P(X = 0) = 1 - e^{-\lambda t}$$

Hey, but thats cumulative probability or in other words

$$P(T \le t) = 1 - e^{-\lambda t}$$

and if the distribution of t was defined by function $f(t)$, we could write

$$\int_{t=0}^{t} f(t)dt = 1 - e^{-\lambda t}$$

$$f(t) = \lambda e^{-\lambda t}$$

And thats the exponential distribution.

Intuitively , by the looks of it , we can infer that if some events are happening at a fast rate [large λ], its going to be exponentially rare that you will get to see the next event after a very long time [high value of t will have exponentially diminishing probabilities]

Mean

$$\mu = E[X]$$

$$= \int_{0}^{\infty} t\lambda e^{-\lambda t} dt$$

Remember integration by parts ? Good for you if you said yes. I don't . So here is a reminder for all of us

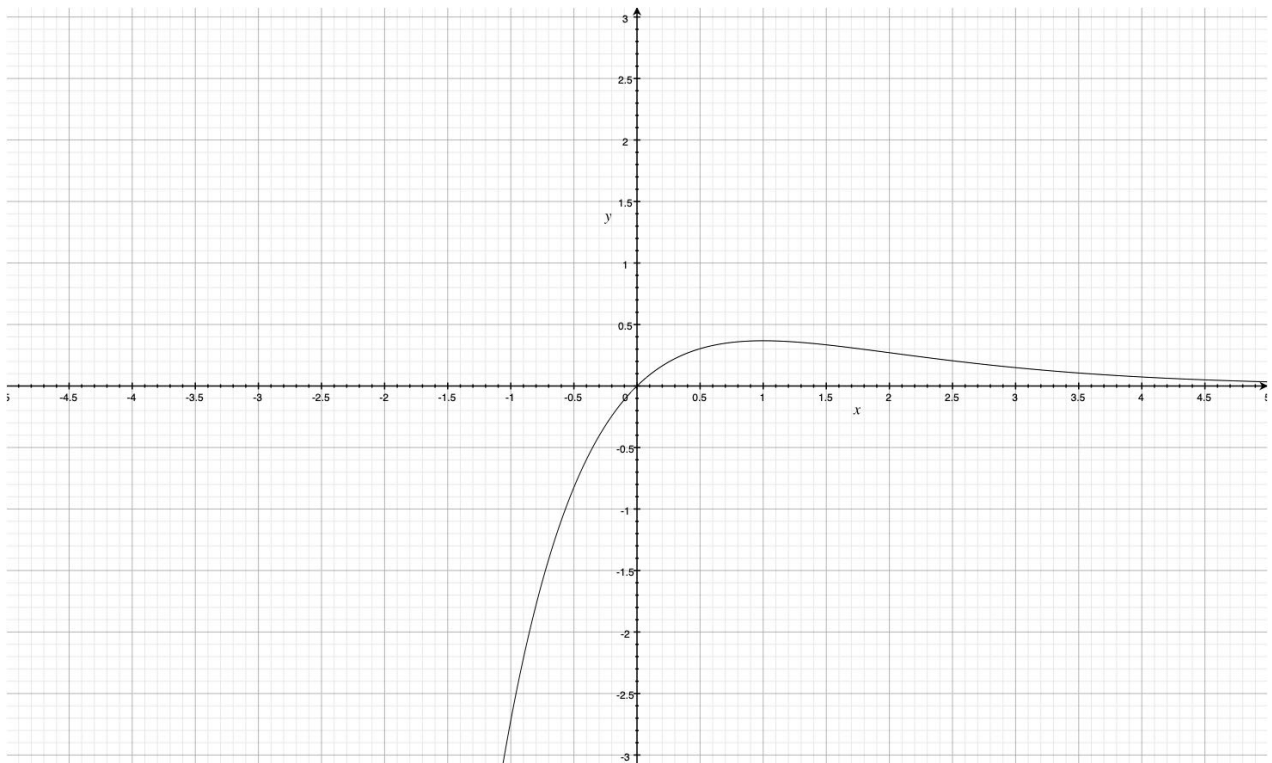$$\int u(\frac{dv}{dt})dt = uv - \int v\frac{du}{dt}dt$$

In our expression earlier

$$u = t$$

$$\&\ v = -\frac{e^{-\lambda t}}{\lambda}$$

so we can rewrite that expression using integration by parts formula

$$\mu = \lambda\left[-\left.\frac{te^{-\lambda t}}{\lambda}\right|_0^\infty + \int_0^\infty \frac{e^{-\lambda t}}{\lambda}dt\right]$$

Easiest and very intuitive to understand that expression $te^{-\lambda t}$ goes to 0 as $t \to \infty$ is to look at its plot

so ,

$$\mu = \lambda\int_0^\infty \frac{e^{-\lambda t}}{\lambda}dt$$

$$= \int_0^\infty e^{-\lambda t}dt$$

$$= \frac{1}{\lambda}$$

Variance
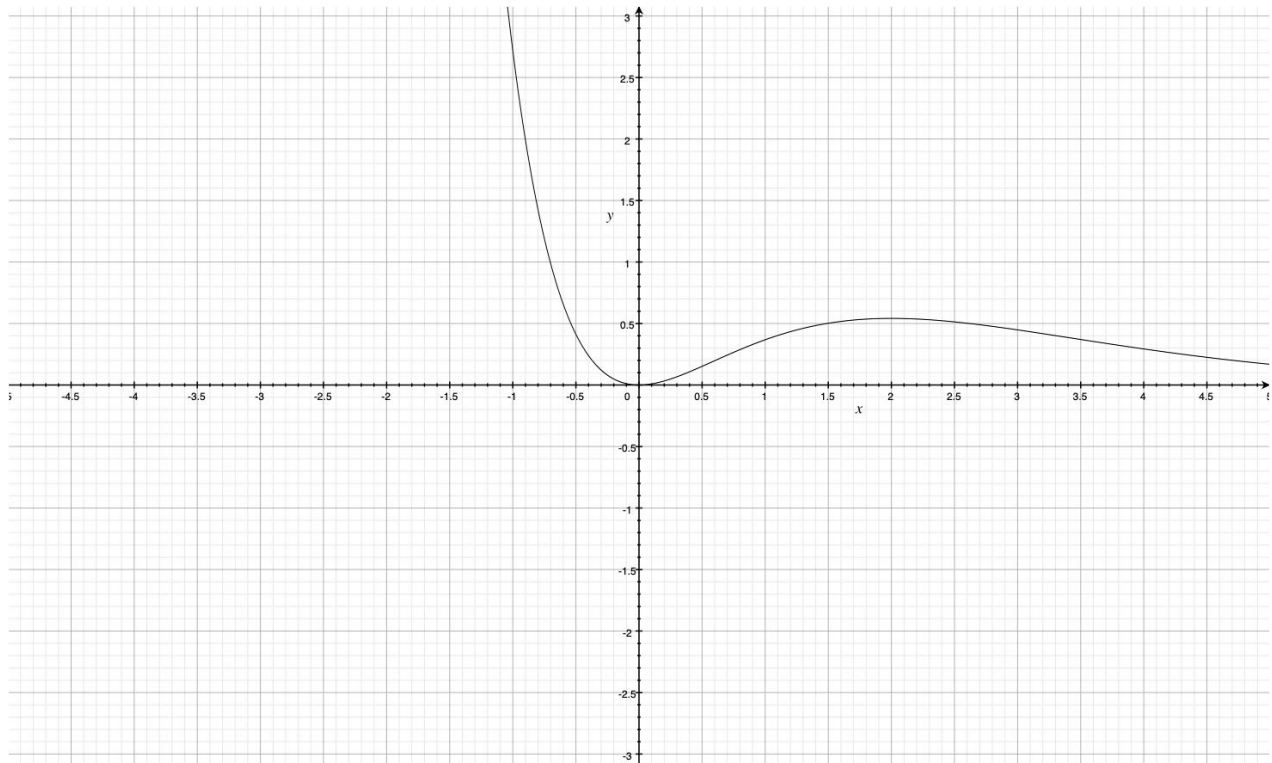
$$E[X^2] = \int_0^\infty t^2 \lambda e^{-\lambda t}dt$$

now here

$$u = t^2$$

$$\&\ v = -\frac{e^{-\lambda t}}{\lambda}$$

so,

$$E[X^2] = \int_0^\infty t^2 \lambda e^{-\lambda t}\,dt$$

$$= \lambda\left[-\frac{t^2 e^{-\lambda t}}{\lambda}\Big|_0^\infty + \int_0^\infty \frac{te^{-\lambda t}}{\lambda}\,dt\right]$$

Graph for $t^2 e^{-\lambda t}$ looks like this
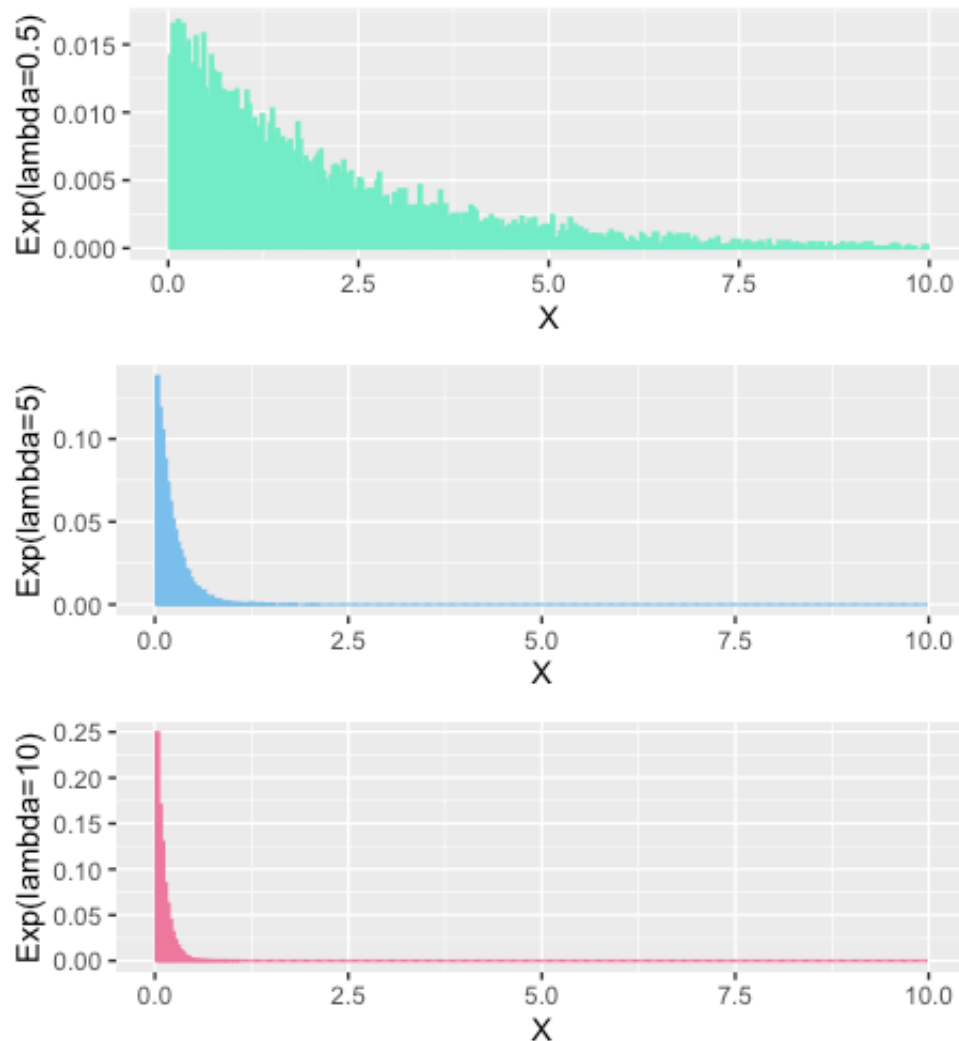

Clearly the first term in the expression above is going to be simply 0 again.

$$E[X^2] = \int_0^\infty t^2 \lambda e^{-\lambda t}\,dt$$

$$= \lambda\left[-\frac{t^2 e^{-\lambda t}}{\lambda}\Big|_0^\infty + 2\int_0^\infty \frac{te^{-\lambda t}}{\lambda}\,dt\right]$$

$$= 2\int_0^\infty te^{-\lambda t}\,dt$$

$$= \frac{2}{\lambda^2}$$

using this for variance

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

What does it look like

An example

Chef takes 5 minutes to make a burger . Whats the probability that chef will take 3 to 4 minutes to cook the next burger . You have all the tools at your disposal . Do it yourself .

What Next?

- Geometric, Negative Binomial , Hypergeometric
- Beta, Gamma, Weibull, Double Exponential
- Normal, Cauchy, Chi-Squared, t, F

Part 2 coming soon!

-- Lalit Sachan [https://www.linkedin.com/in/lalitsachan/]