Note: Multiple poverty lines

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1 This Note

Comparing economic poverty across regions (or time) requires the consistent estimation of relative prices to account for differences in purchasing power. More expensive places need more wealth for the same amount of welfare. Quantifying this difference across regions led to the development of a significant number of relative price measures called price indexes. Diewert (1976) provided second-order approximations to the true *cost-of-living* index when consumer preferences are homothetic[1]. I summarize the results in this note and provide a numerical example using quasi-linear preferences ($u(c, l) = \log c + l$) highlighting the differences across three commonly used indexes: Paasche, Laspeyres, and Fisher along with the true cost-of-living index.

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[1]Can be represented by a utility function that is linearly homogeneous: $f(\lambda x) = \lambda f(x)$

In section 2, we briefly review the relevance and challenges in specifying poverty lines[2]. In section 3 we introduce a few definitions that are utilized in the following sections. In section 4, we recall from Deaton and Muellbauer (1980) first-order approximations of the true cost-of-living index under general consumer preferences and certain restrictions on prices. In section 5, we summarize the results from Diewert (1976) highlighting its implication for homothetic preferences. Finally, in section 6 we look at the difference in the indexes using a class of non-homothetic preferences (quasi-linear). A summary follows.

2 Poverty Lines

Poverty Lines (PLs) serve as a monetary divide to categorize populations into poor and non-poor. Individuals with expenditures below the PL are labeled poor or *below poverty line* and thus can benefit from policies specifically targeting the poor. As a result, specifying and measuring PLs for different populations[3] is of substantial importance to policymakers around the world.

How does one go about specifying and measuring PLs? Broadly, an authority responsible for the classification of some population (say, of a state) into poor and non-poor must (i) identify a level of welfare that it believes *should* be afforded to all, and (ii) evaluate the lowest monetary cost of ob-

[2] For a more detailed review of consumption-based poverty measurement, see Deaton and Zaidi (2002)

[3] For example, urban v. rural, region A v. region B, time t v. time t', etc.

taining that welfare. This cost is defined as the PL. The choice of welfare as the fundamental variable in specifying the PL ensures that differences in individual decision problems do not alter the experience at the PL. PLs that deliver the same welfare (or *utility*) across regions or time are called *utility-consistent* since they do not alter the maximum utility feasible. Formally, if the optimal expenditure of an individual in different regions (or times) equals the appropriate PL, then each bundle she consumes belongs to the same indifference curve. However, the impossibility of measuring welfare objectively restricts its use beyond theoretically defining PL(s). To overcome this, a PL is generally defined as the cost of a consumption bundle at prices prevailing in a reference region and time.

The authority identifies all possible sources of welfare (consumption goods) in the population such as food, housing, durables, health, schooling, safety, etc. Then it decides on the composition of the *cheapest* bundle of goods that deliver the least acceptable welfare. Since the bundle is the cheapest bundle to deliver the targeted welfare, households with lower expenditure are identified as poor. The rationale is that households maximize their welfare subject to their income, and if their expenditure is below the set PL, the cost of the cheapest bundle to deliver the required welfare, the households must not be able to afford the bundle and so experience lower welfare. Analysts face challenges in specifying the composition of the bundle as well as measuring the PL once the bundle is specified.

In practice the bundles are determined by, a) the food energy intake (FEI), and b) the cost of basic needs (CBN) approach. The former generates the bundle by answering the following question: what is the lowest cost of delivering a threshold amount of calories in a day? Due to significant price variation across regions, it does not account for other necessities such as housing, healthcare, social needs, etc. which are usually accounted for in the CBN approach.[4].

The composition of the bundle used to specify the PL is affected by a) what can be reliably quantified and, b) relevance of the item in poverty measurement. For example, access to basic public healthcare while an important variable for individual welfare poses a difficult measurement problem and is typically omitted from the bundle. On the other hand, it can be argued that television can be excluded from it. Excluding goods essential for the functioning of an individual understates poverty. It also makes a comparison between regions (or times) who have a varying degree of access to the excluded items, biased. Proxy measures for such variables could improve the measurement, however, if the proxy is noisy across regions (or time), it is recommended to exclude it to restrict the bias across regions (even if the measurement itself is biased) to maintain consistency (Deaton and Zaidi (2002)). The relevance of certain goods in the measure of PL may vary over regions primarily due to differences in tastes, but also due to societal norms or regional needs. For

[4] More details on the two approaches can be found in Marivoet and De Herdt (2015)

example, heating in cold regions should be included in the basic consumption bundle but not in tropical regions. For such differences in consumption patterns, using different bundles to define PL yields more accurate measures of poverty than using a restricted common bundle. This is at the heart of the argument for multiple PLs across regions using region-specific consumption bundles.

If the population is homogeneous in their tastes over the goods, the authority can indeed identify and quantify all consumption goods, and the prices faced across the population are the same, then measuring the PL is the simple exercise of calculating the cost of a specified bundle at existing price levels. However, each of these assumptions comes at a price in the form of reduced accuracy of the desired PL, which delivers utility consistency.

To see how heterogeneous tastes over consumption goods alters the specification and measurement of PL, consider an economy with two regions. Urban (U) and Rural (R). The populations in both these regions consume only two goods: food and schooling[5]. Suppose the authority identifies that the welfare attained, $\bar{u}$[6], by consuming one unit of each food, and schooling in U is the upper bound to be classified as poor. For now, assume that prices are the same across goods and regions, and equal to \$1. Therefore the official PL is \$2 in the two regions. However, if the population of U has a stronger

[5] In this example, we ignore the investment good properties of schooling. As a result, the returns from schooling channel which add consumption overtime is not utilized.

[6] By quantifying welfare here, we implicitly assume a scale of welfare for the entire population.

affinity for schooling relative to food than the population of R, then the \$2 PL in R overestimates poverty in the region. First, recollect that \$2 is the smallest expenditure in U to obtain $\bar{u}$ and the bundle chosen is one unit of each good. The population of R at the PL can afford the bundle that their counterparts in U chose. However, since they find food more delightful, they trade some of the schooling for food and improve their well being over at \$2. The assumption that tastes are similar across populations is relatively harmless for small regions or across short to medium time horizons. For large differences in tastes such as observed across countries or diverse regions or long time horizons, defining a different PL according to the different tastes yields more reliable estimates of the size of the poor.

The most common difficulty in consistently measuring poverty is the price difference across regions (and time). Consider the above example of two regions U and R with one change: schooling refers to quality-adjusted schooling. Suppose that each costs the same in U and that is \$1 / unit. However, to obtain the same set of instructions in R, the instructor must travel farther from her home and so costs \$2 per unit of schooling. If the authority ignored the difference in the relative prices across U and R, and set, for example, \$2 as the universal poverty, then it would a) identify the true poor population in U, and b) underestimates the poor population in R. The difference in relative price across regions hinders our objectives of correctly identifying the poor in two ways. First, it excludes all the individuals who spend between \$2 and

\$3 in R but obtain lower welfare than -a downward bias on poverty measurement. Second, the higher relative price of schooling in R shifts budget share away from schooling and towards food (*substitution effect*)-an upward bias on poverty measurement. Two solutions are proposed to address this bias: (i) Single PL with price deflators and, (ii) multiple PLs method. We discuss them below.

In the single PL approach, a reference region is chosen and the consumption bundle is specified which corresponds to the lowest acceptable welfare to the authority in the region. The PL is the monetary cost of this consumption bundle at the prices prevailing at the reference region[7]. Expenditures in other regions are deflated using price index which measures the relative costliness (/ differences in *cost of living*) between regions and the reference region. Individuals with deflated expenditure below the specified PL are classified poor.

The approach is simple to implement and the data requirements are mild as compared to MPL. It is significantly cheaper than MPL approach since it does not involve estimating multiple demand systems to recover preferences which require detailed consumption data across all regions. However, it does not account for differences in nutritional and social needs, and public goods provision across groups. In other words, to make comparable expenditure profiles across regions, the approach trades off on specific needs of the regions

[7] It is implicitly assumed that the PL is the smallest expenditure to be incurred to obtain the targeted welfare

should they vary.

Additional criticism of using the single poverty line approach is the reliability of the price indexes in deflating expenditures. It has been shown that for small differences in prices across regions, various price indexes provide first-order approximations to the true difference in cost of living (4). However, prices have been known to move significantly across the region, which is reflected in varied consumption patterns. This is particularly true for non-food expenditure where prices are dispersed. Since the actual substitution pattern is not known, large differences in prices can push the price index away from the true cost of living. Marivoet and De Herdt (2015) argue that near the poverty line there is limited substitutability across goods. They recommend a restrictive food threshold and non-food allowance in defining the poverty bundle to limit this substitution. This permits them to estimate a food-based PL and scale it according to the share of non-food expenditure in the individual budget to obtain the overall PL.

The multiple poverty lines approach addresses some of the concerns raised by the critics of the single poverty line approach. To improve the accuracy of poverty measurement, the specific needs of the region are taken into consideration. Consumption bundles are specified carefully for each region keeping the local needs at the forefront of the construction. The cost of each bundle is computed at the domestically prevailing prices and this cost is set as the poverty line for the region.

Specifying separate bundles to define the poverty line comes at the cost of consistency. Bundle used in one region need not correspond to the same welfare as the bundle in another region. As a result, the non-poor in one region, may, in fact, experience lower welfare than poor in another region, biasing the measures of relative poverty across regions. Solutions proposed in literature to address this concern include, revealed preferences tests (Arndt and Simler (2010)), *functioning(s)* approach (Marivoet and De Herdt (2015)), etc. We do not discuss these resolutions here and refer interested readers corresponding papers. Specifying different consumption bundles across regions also is harder to justify to policymakers because it could lead to different poverty lines in different regions even without a price differential.

The consumption-based approach of setting utility consistent poverty lines is influenced by the preferences of the population. Lack of knowledge of substitution pattern along with the requirement of utility consistency severely restricts the objective accuracy of poverty measurement. While the notion of utility consistency finds support across the literature, the impossibility of directly observing and quantifying welfare forces us to adopt consumption bundles as proxies for welfare. Since a firm link between the consumption bundles and welfare cannot be established, the use of price index as a substitute for the true cost of living cannot be tested and is a rather important requirement to assume. Region-specific bundles, on the other hand, weaken the link between poverty across regions and as a result, make it harder to prioritize the allocation of limited public resources. However, if the key intent is to identify the weakest sections of the society, taking into account their specific basic needs appear acceptable.

3 Definitions

Notation[8]: Unless noted otherwise, $p \in \mathbb{R}^N_{++}$ (the positive orthant) represents the prices and $q \in \mathbb{R}^N_+$ (non-negative orthant) represents quantities consumed of N consumption goods.

3.1 Homothetic preferences

If individual preferences over consumption bundles can be represented by a homogeneous function then the preferences are said to be homothetic. Formally, individual preferences are homothetic iff there exists a mapping $u : \mathbb{R}^N_+ \longrightarrow R_+$ such that if q^1 is preferred to q^2 then

$$u(\lambda.q) = \lambda.u(q) \ \text{ and } \ u(q^1) \geq u(q^2) \tag{1}$$

Homothetic preferences can also be represented by an expenditure function, $e(u, p)$, which can be decomposed as $e(u, p) = ub(p)$ where $b(\cdot)$ is homogeneous of degree one[9]. We use this latter representation of Homothetic preference in our analysis to follow since the poverty line is defined as the

[8]Notation borrowed from Diewert (1976) and Deaton and Muellbauer (1980).

[9]Problems 3.C.5 and 3.E.5, Mas-Colell et al. (1995)

lowest expenditure of the household below which the household is identified as poor.

Homothetic preferences exhibit linear wealth expansion paths, that is, the expenditure shares are independent of wealth. The linearity of the wealth expansion path is not observed in the data. Food exhibits declining shares in expenditure, a result known as Engel's law. Nonetheless, they serve as a useful benchmark and are convenient to work with.

3.2 True Cost-of-living index

The true cost-of-living index is the relative expenditure needed to maintain the same level of welfare across two regions. If $e(u, p)$ represented the expenditure to experience welfare level u, at price p, then the cost-of-living index is defined by:

$$P_{CoL}(p^R, p^D, u) = \frac{e(u, p^D)}{e(u, p^R)} \tag{2}$$

The *true* cost-of-living index delivers *utility consistent* poverty lines across regions and hence is an ideal price index. That is, if the poverty line in R is given by $w^{R,PL}$ then the corresponding poverty line in D is given by $P_{CoL}(p^R, p^D, u) * w^{R,PL}$. Note that the true cost-of-living index requires that the expenditures be evaluated at the same welfare, a requirement that is impossible to test and is an assumption implicit in all price indexes used in specifying consumption-based poverty lines.

3.2.1 Laspeyres Price Index

The Laspeyres price index measures the relative cost of obtaining a *reference* consumption bundle (q^R) at the domestic prices (p^D) with respect to *reference* region prices (p^R). That is:

$$L_p(p^R, p^D, q^R) = \frac{q^R.p^D}{q^R.p^R} \tag{3}$$

The weights in the index are fixed to quantities in the reference region. Substitution arising from differences in prices are not accounted for and so Laspeyres overestimates poverty. However, the Laspeyres is simpler to calculate as it can be calculated relatively well even with sparse price data since the weights are fixed to the reference region's consumption shares.

3.2.2 Paasche Price Index

The Paasche price index measures the relative cost of obtaining a *domestic* consumption bundle (q^D) at the domestic prices (p^D) with respect to *reference* region prices (p^R). That is:

$$P_p(p^R, p^D, q^D) = \frac{q^D.p^D}{q^D.p^R} \tag{4}$$

If the price differential is small, Paasche and Laspeyres are approximately equal. However, with larger differences in prices and therefore consumption patterns, the two indexes provide different poverty lines. The weights used in determining the Paasche price index are specific to the region in consideration, D. As a result, based on the consumption shares of goods, the corresponding price index varies more than the Laspeyres price index. Deaton and Zaidi (2002) argue that this variation in weights in Paasche is relevant for consistent poverty measurement and ignoring substitution effects of price differences makes Laspeyres less informative.

3.2.3 Fisher Price Index

The Fisher price index is the geometric mean of the Paasche and Laspeyres price indices are given by:

$$F_p(p^R, p^D, q^R, q^D) = \left[\frac{q^R.p^D}{q^R.p^R}.\frac{q^D.p^D}{q^D.p^R}\right]^{1/2} \tag{5}$$

4 First-order approximation

Deaton and Muellbauer (1980) show that when price differences across regions are small, or if prices are proportional across regions, or if there is limited substitution, then the indexes provide first-order approximation to the true cost-of-living index. In addition to the above assumptions, the result uses a Taylor approximation of the expenditure function which requires a differentiable expenditure representation and Sheperd's lemma (9). If $e(u, p)$ represents the expenditure function, then a Taylor approximation around price p^R is given by:

$$e(u,p) \approx e(u,p^R) + \nabla_p e(u,p)|_{p^R}.(p-p^R) + \frac{1}{2!}(p-p^R)^T \nabla_p^2 e(u,p)|_{p^R}.(p-p^R)$$

Under the above restrictions on price and substitution behavior, the last term on the right hand side is small. Using $e(u,p) = p.q(u,p)$ and Sheperd's lemma to reduce the above expression to:

$$\begin{aligned} e(u,p) &\approx p^R q(u,p^R) + q(u,p^R)(p-p^R) \\ &= p.q(u,p^R) \end{aligned}$$

To see how this approximation improves the informational quality of the indexes, consider the cost-of-living index between R, D:

$$\begin{aligned} \frac{e(u,p^D)}{e(u,p^R)} &\approx \frac{p^D q(u,p^D)}{p^R q(u,p^D) + q(u,p^D)(p^R - p^D)} \\ &= \frac{p^D.q(u,p^D)}{p^R.q(u,p^D)} \\ &= P_p(p^R, p^D, q(u,p^D)) \end{aligned}$$

Similarly, if the approximation is carried out on the numerator, the resulting measure is the Laspeyres index.

This approximation is useful since it doesn't restrict the results to a set of preferences. However, this generality comes at the cost of comparability across large price differences, which are generally observed in non-food related expenses. Also, notice that tastes are assumed to be the same across regions, which is a key criticism of using global poverty lines.

5 Cost-of-living indexes under Homothetic preferences:

This section summarizes the results from Diewert (1976) and discusses them briefly in the context of *utility consistent* poverty lines.

Theorem 1 *Diewert (1976): If preferences are homothetic, represented by the unit expenditure function $b(p)$ such that $b(p) = \left(\sum_i \sum_j a_{ij} p_i^{r/2} p_j^{r/2}\right)^{1/r}$ where $a_{ij} = a_{ji}$ and prices in two regions are given by p^R, p^D, then, the true cost-of-living index is given by*

$$P_{CoL}(p^R, p^D, u) = \left\{ \frac{\sum_j \left(\frac{p_j^D}{p_j^R}\right)^{r/2} w_j^R}{\sum_j \left(\frac{p_j^R}{p_j^D}\right)^{r/2} w_j^D} \right\}^{1/r} \tag{6}$$

where w^R, w^D are the expenditure weights under p^R and p^D respectively. Proof in section 7. Under the assumptions of theorem 1, when preferences belong to a special class of homothetic preferences, the true cost-of-living index can be pinned down using prices across regions and expenditure weights which are generally available in consumer expenditure surveys. However, as noted before, the index assumes that the optimal bundle for the two regions under their respective price regimes lie on the same indifference curve.

Corollary 1.1 *When r=2, $P_{CoL}(p^R, p^D, u)$ in (6) can be re-written as the Fisher price index as in (5) i.e, Fisher price index coincides with the cost-of-living index when the preferences can be represented by a quadratic unit expenditure function.*

However, to extend the above results to all homothetic preferences, we use the following approximation result[10].

Theorem 2 *Diewert (1976): If $f : \mathbb{R}^N_{++} \longrightarrow \mathbb{R}_+$ be twice differentiable and homogeneous of degree 1, then for any $r \neq 0$, $f_r = [x^{T^{r/2}} Ax]^{1/r}$ provides a second order approximation to f.*

Theorem 2 extends the results from theorem 1 to all homothetic functions up to a second-order approximation of the expenditure function.

I briefly discuss the above results. In theorem 1, the true cost-of-living index is approximated by equation 6 which can be evaluated based on expenditure data. This allows us to estimate utility consistent poverty line as long as the consumption weights used to belong to the same indifference curve. The assumption that only a certain substitution patterns satisfied by equation $b(p) = \left(\sum_i \sum_j a_{ij} p_i^{r/2} p_j^{r/2}\right)^{1/r}$ is weakened using theorem 2 which expands the set of preferences to all homothetic preferences at the cost of small bias.

The significance of the result lies in the wide class of preferences where the poverty line can be consistently estimated across regions using the consump-

[10] The proof can be found in Diewert (1976).

tion data which is generally available. Separate consumption bundles need not be specified as long as similar households are compared across regions. In contrast to the results in 4, the second-order approximation is improved on the previous approximation. The index is independent of the welfare of the households as long as they are the same, i.e, consumption data from comparable households across regions suffices to obtain the price index. Linear wealth expansion curves provide us with utility independent price indexes because it allows the expenditure function to scale equivalently with the welfare in consideration. In section 6.1, we show how the price index is dependent on the welfare and biases the price index.

6 Non-Homothetic example: Quasilinear preferences

While the above results are useful benchmarks, the assumptions needed to satisfy them are not observed in the data. Engel's law states that as one grows wealthier, the proportion of wealth spent on food decreases. This observation makes the preferences between food and non-food items non-homothetic. To illustrate the differences, we provide the expression for the cost-of-living index of quasi-linear preferences over food and land. We then use a numerical example to see the variation in the magnitude of differences across the price indexes.

We restrict ourselves to a 2-good case, where the preferences over them are represented by $u(c, l) = \log c + l$. We assume the prices are such that the solutions to the optimization problems lie in the interior of the budget sets. We discuss later what happens when a corner solution is reached. We derive the true cost-of-living index for a welfare level and the corresponding Paasche, Laspeyres and Fisher indexes. Let the prices for the consumption bundle in region i be (p_c^i, p_l^i), then for a given welfare level, $\bar{u}$, the Hicksian demand functions are given by $c(\bar{u}, p^i) = \frac{p_l^i}{p_c^i}$ and $l = \bar{u} - \log \frac{p_l^i}{p_c^i}$. The corresponding expenditure minimizing bundle costs:

$$c(\bar{u}, p^i) = \left[\bar{u} + 1 - \log \frac{p_l^i}{p_c^i}\right] p_l^i \tag{7}$$

Hence the relative cost of obtaining $\bar{u}$ between two regions R and D is given by:

$$P_{CoL}(p^R, p^D, \bar{u}) = \frac{c(\bar{u}, p^D)}{c(\bar{u}, p^R)} = \frac{\left[\bar{u} + 1 - \log \frac{p_l^D}{p_c^D}\right] p_l^D}{\left[\bar{u} + 1 - \log \frac{p_l^R}{p_c^R}\right] p_l^R} \tag{8}$$

When $\frac{p_l^i}{p_c^i} < 1$, the consumer prefers to only consume food, since the marginal loss of giving up consumption then is higher than the marginal gain of consuming land, l, which is always 1. For the numerical example in 6.1 the Fisher index is given by:

$$F_p(p^R, p^D, q^R, q^D) = \Bigg[\underbrace{\frac{p_c^D \frac{p_l^R}{p_c^R} + p_l^D\left(\bar{u} - \log\frac{p_l^R}{p_c^R}\right)}{c(\bar{u}, p^R)}}_{Laspeyres}\underbrace{\frac{c(\bar{u}, p^D)}{p_c^R \frac{p_l^D}{p_c^D} + p_l^R\left(\bar{u} - \log\frac{p_l^D}{p_c^D}\right)}}_{Paasche}\Bigg]^{0.5}$$

$$= \left[P_{CoL}(p^R, p^D, \bar{u}) * \frac{p_c^D \frac{p_l^R}{p_c^R} + p_l^D\left(\bar{u} - \log\frac{p_l^R}{p_c^R}\right)}{p_c^R \frac{p_l^D}{p_c^D} + p_l^R\left(\bar{u} - \log\frac{p_l^D}{p_c^D}\right)}\right]^{0.5}$$

The Fisher index differs from the cost-of-living index and the difference depends on the welfare level chosen and the reference price. In fact, when the corner solution is reached, l is not consumed making the ratio only dependent on the level of food c which is consumed. Thus the indexes ignore l altogether which in turn leave Paasche and Laspeyres equal to the price ratio of food. The equality of the three indexes at the corner can be seen in 6.1 in the first four regions. Note that this result does not extend to multiple goods with multiple sources of non-homotheticity because in these cases, the solution can be in the interior for more than one good but in the corner for others making the prices of goods that are in the interior relevant.

6.1 Numerical Example

The equality of the three price indexes in the first four regions is a result of being at a corner solution to the optimizing problem. Since p_c was normalized

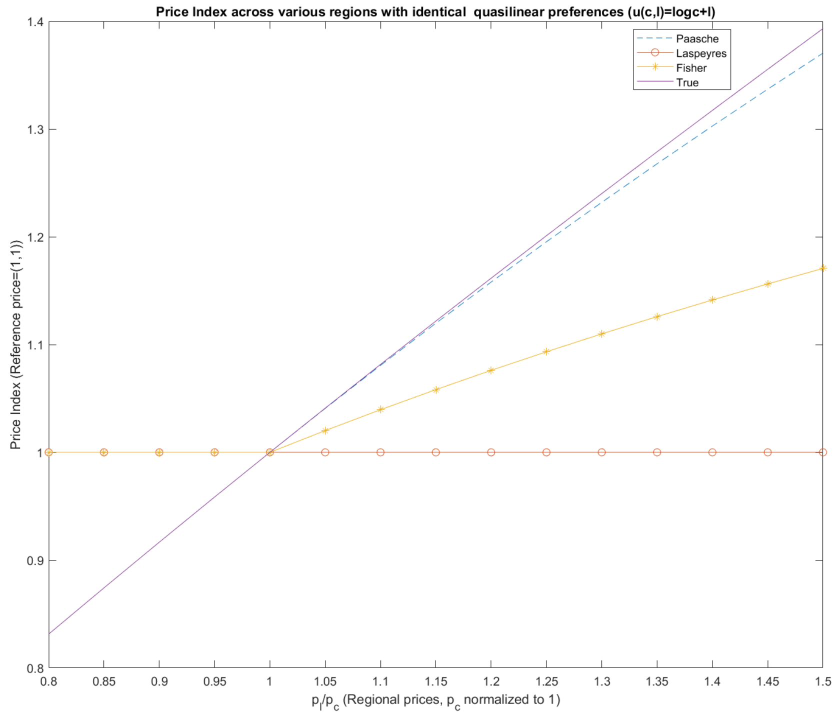

Figure 1: Poverty line set such that the welfare obtained at the poverty line is equal to the welfare from consuming $(c, l) = (1, 2)$

to 1, the price indexes also suggest that these regions are equally expensive to live in. However, this is an artifact of the normalization and not a result. The three indexes are the same when at a corner but can vary across regions who are also in the corner depending on the relative price of the food across regions. The Paasche tracks the true cost-of-living index relatively well when the solution is away from the corner.

7 Summary

In this note, we first review the literature and methodologies on specifying the poverty line. We then show that first-order approximations to the relative prices in two regions can help us pin down utility consistent poverty line with generally available consumption data. The key result of this note implies that when preferences are homothetic, and price differences are small, then the true cost-of-living index can be approximated to the second-order thus relaxing the need to specify region-specific poverty bundles. This provides strength to the argument for a single poverty line with price indexes. In addition to this, we also show that the index obtained reduces to Fisher price index when the preferences are quadratic as seen in theorem 1. Finally, as a first pass, we explore the evolution of the indexes under non-homothetic (quasi-linear) preferences.

Proofs of theorem 1 and corollary 1.1

Proof of 1: Given $b(p)$, the corresponding expenditure function can be written as $e(u,p) = ub(p)$. By Sheperd's lemma,

$$q(u,p) = \nabla_p e(u,p) = u.\nabla_p b(p) \tag{9}$$

Hence,

$$q_i(u,p) = u.(\sum_i \sum_j a_{ij} p_i^{r/2} p_j^{r/2})^{1/r-1} (\sum_j a_{ij} p_j^{r/2}) p_i^{r/2-1} \tag{10}$$

Dividing both sides by $e(u,p)$ yields

$$\frac{q_i(u,p)}{e(u,p)} = \frac{p_i^{r/2-1} \sum_j a_{ij} p_j^{r/2}}{\sum_i \sum_j a_{ij} p_i^{r/2} p_j^{r/2}} \tag{11}$$

So, the above expression when computed for p^R is:

$$\frac{q_i(u,p^R)}{e(u,p^R)} = \frac{p_i^{R^{r/2-1}} \sum_j a_{ij} p_j^{R^{r/2}}}{\sum_i \sum_j a_{ij} p_i^{R^{r/2}} p_j^{R^{r/2}}} \tag{12}$$

Multiplying both sides by $p_i^{D^{r/2}}$, rearranging the terms and summing over all the consumption goods i, we get:

$$\sum_i \frac{q_i(u,p^R)}{e(u,p^R)} p_i^{R^{1-r/2}} p_i^{D^{r/2}} = \frac{\sum_i \sum_j p_i^{D^{r/2}} a_{ij} p_j^{R^{r/2}}}{\sum_i \sum_j a_{ij} p_i^{R^{r/2}} p_j^{R^{r/2}}} \tag{13}$$

Since $\frac{p_i^R q_i(u,p^R)}{e(u,p^R)} = w_i^R$, we obtain;

$$\sum_i w_i^R \frac{p_i^{D^{r/2}}}{p_i^{R^{r/2}}} = \frac{\sum_i \sum_j p_i^{D^{r/2}} a_{ij} p_j^{R^{r/2}}}{\sum_i \sum_j a_{ij} p_i^{R^{r/2}} p_j^{R^{r/2}}} \tag{14}$$

Similarly, we can write the expression for p^D. This is given by:

$$\sum_i w_i^D \frac{p_i^{R^{r/2}}}{p_i^{D^{r/2}}} = \frac{\sum_i \sum_j p_i^{R^{r/2}} a_{ij} p_j^{D^{r/2}}}{\sum_i \sum_j a_{ij} p_i^{D^{r/2}} p_j^{D^{r/2}}} \tag{15}$$

Using the symmetry of a_{ij}s it can be shown that the numerator on the right hand side are equal for p^R and p^D. Dividing (14) by (15) gives:

$$\frac{\sum_i w_i^R \frac{p_i^{D^{r/2}}}{p_i^{R^{r/2}}}}{\sum_i w_i^D \frac{p_i^{R^{r/2}}}{p_i^{D^{r/2}}}} = \frac{\sum_i \sum_j a_{ij} p_i^{D^{r/2}} p_j^{D^{r/2}}}{\sum_i \sum_j a_{ij} p_i^{R^{r/2}} p_j^{R^{r/2}}} = \left[\frac{e(u,p^D)}{e(u,p^R)}\right]^r \tag{16}$$

where the second equality stems from using $e(u,p) = ub(p) = u.(\sum_i \sum_j a_{ij} p_i^{r/2} p_j^{r/2})^{1/r}$. Take the rth root on both sides to get (6).

Proof of 1.1: By plugging $r = 2$, (6) simplifies to

$$P_{CoL}(p^R, p^D, u) = \left[\frac{\sum_j \frac{p_j^D.q_j^R}{p^R.q^R}}{\sum_j \frac{p_j^R.q_j^D}{p^D.q^D}}\right]^{0.5}$$

$$= \left[\frac{\sum_j p_j^D.q_j^R}{\sum_j p_j^R.q_j^R}\frac{\sum_j p_j^D.q_j^D}{\sum_j p_j^R.q_j^D}\right]^{0.5}$$

$$= [L_p(p^R, p^D, q^R) P_p(p^R, p^D, q^D)]^{0.5}$$

$$= F_p(p^R, p^D, q^R, q^D)$$

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