Fourier uncertainty and exact signal recovery

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Finite Signals and Discrete Fourier transform

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- Fourier Inversion says that we can recover the signal by using the Fourier inversion:

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- The basic question is, can we recover f **exactly** from its discrete Fourier transforms if

$$\left\{ \widehat{f}(m) : m \in S \right\}$$

are unobserved (or missing due to noise, other interference, or security), for some $S \subset \mathbb{Z}_N^d$?

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are unobserved (or missing due to noise, other interference, or security), for some $S \subset \mathbb{Z}_N^d$?

- The answer turns out to be *YES* if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Fourier Uncertainty Principle.

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- (Plancherel)

$$\sum_{m \in \mathbb{Z}_N^d} \left|\widehat{f}(m)\right|^2 = \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

An elementary point of view: setup

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- $$= N^{-\frac{d}{2}} \sum_{m \notin S} \chi(x \cdot m) \widehat{1}_E(m) + N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \widehat{1}_E(m)$$

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- Since we know nothing about S, the best we can do is assume that the quantity above is small.

An elementary point of view: rounding

- If
$$N^{-d}|E||S| < \frac{1}{2},$$
we can take the modulus of $I(x)$ and round it up to 1 if it is $\geq \frac{1}{2}$, and round it down to 0 otherwise.

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- This gives us **exact recovery** using a simple and direct algorithm (to be henceforth referred to as the Direct Rounding Algorithm (DRA)) if
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- But what happens if we consider general signals?

Matolcsi-Szucks/ Donoho-Stark point of view

- Let $h : \mathbb{Z}_N^d \to \mathbb{C}$. Then the classical Uncertainty Principle says that

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- If f cannot be recovered uniquely, then there exists a signal $g : \mathbb{Z}_N^d \to \mathbb{C}$ such that g also has $|E|$ non-zero entries,

$$\widehat{f}(m) = \widehat{g}(m) \text{ for } m \notin S,$$

and f is not identically equal to g.

Uncertainty Principle $\rightarrow$ Unique Recovery

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- Therefore, if we assume that

$$|E| \cdot |S| < \frac{N^d}{2},$$

we must have $h = 0$, and hence the recovery is *unique*.

A general recovery method

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- Donoho and Stark showed, using a beautiful idea due to Benjamin Logan, that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in E, and the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, then if

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- then f can be recovered as

$$\mathit{arg\ min}_u \ ||u||_{L^1(\mathbb{Z}_N^d)} \text{ subject to } \widehat{f}(m) = \widehat{u}(m) \text{ for } m \notin S.$$

Benjamin Franklin Logan

- Logan was an accomplished bluegrass musician in addition to his groundbreaking work in signal processing.

Proof of the L^1 recovery method

- Let $f = g + h$, where g is the solution to the L^1 minimization problem above, and note that $\widehat{h}$ is supported in S. We have

$$||g||_{L^1(\mathbb{Z}_N^d)} = ||f - h||_{L^1(\mathbb{Z}_N^d)}$$

$$= ||f - h||_{L^1(E)} + ||h||_{L^1(E^c)} \geq ||f||_{L^1(\mathbb{Z}_N^d)} + \left[||h||_{L^1(E^c)} - ||h||_{L^1(E)} \right].$$

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- If we can show that $||h||_{L^1(E^c)} > ||h||_{L^1(E)}$, then

$$||f||_{L^1(\mathbb{Z}_N^d)} < ||g||_{L^1(\mathbb{Z}_N^d)},$$

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- The resulting contradiction will prove that $h \equiv 0$.

The uncertainty principle strikes again

- We have

$$|h(x)| = N^{-\frac{d}{2}} \cdot \left| \sum_{m \in S} \chi(x \cdot m) \widehat{h}(m) \right| \leq N^{-d} \cdot |S| \cdot ||h||_{L^1(\mathbb{Z}_N^d)}.$$

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- It follows that

$$||h||_{L^1(E)} \leq N^{-d} \cdot |E| \cdot |S| \cdot ||h||_{L^1(\mathbb{Z}_N^d)} < \frac{1}{2} \cdot ||h||_{L^1(\mathbb{Z}_N^d)}.$$

We conclude that

$$||h||_{L^1(E)} < ||h||_{L^1(E^c)},$$

as desired.

The L^2-minimization algorithm

- The L^2 algorithm works in a very different way. If we try to run the same algorithm with L^1-minimization replaced by L^2-minimization, we run into an annoying complication, which is that

  $$||f-h||_{L^2(\mathbb{Z}_N^d)} \geq ||f-h||_{L^2(E)} + ||f-h||_{L^2(E^c)},$$

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- There is a workaround, but it is computationally costly.

- We must consider

$$arg\ min_u||\widehat{u}-\widehat{f}||_{L^2(S^c)} \text{ subject to the constraint } |spt(f)| = |spt(u)|.$$

Proof of the L^2-minimization algorithm

- As before, we have $f = g + h$ where $\widehat{h}$ is supported on S, the set of missing frequencies. Also, g is supported on a set of size $|E|$, hence if h is supported on $T \subset \mathbb{Z}_N^d$, then $|T| \leq 2|E|$.

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- We have

$$|h(x)| = N^{-\frac{d}{2}} \cdot \left| \sum_{m \in S} \chi(x \cdot m) \widehat{h}(m) \right| \leq N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \cdot ||h||_{L^2(T)}.$$

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- It follows that

$$||h||_{L^2(T)} \leq \sqrt{|T||S|N^{-d}} \cdot ||h||_{L^2(T)}.$$

Proof of the L^2-minimization algorithm (conclusion)

- This leads to an immediate contradiction if

$$|T| \cdot |S| < N^d,$$

which amounts to

$$|E| \cdot |S| < \frac{N^d}{2}.$$

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- This leads to an immediate contradiction if

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- We conclude that the L^2-minimization works under this assumption, though the necessity to consider candidate signals supported in a set of size $|E|$ makes the practical value of this algorithm quite limited.

Can we loosen the $|E| \cdot |S| < \frac{N^d}{2}$ assumption?

- In general, the answer is no. Suppose that $d = 1$, N **is not** prime, and E is a subgroup of $\mathbb{Z}_N$.

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- Then if f is supported on E, $\widehat{f}$ is supported on

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- However, we shall that for a generic set S of missing frequencies, the situation is much better.

The prime case

- If N is prime and $d \geq 2$, it is not difficult to check that if f is supported on a k-dimension plane H, $\widehat{f}$ is supported on the orthogonal subspace $H^{\perp}$.

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- If N is prime and $d = 1$, a beautiful result due to Terry Tao says that if f is supported on E and $\widehat{f}$ is supported on S, then $|E| + |S| \geq N + 1$, with the corresponding improvement for the exact signal recovery condition.

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

where $C(q)$ depends only on q.

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  where $C(q)$ depends only on q.

- As we shall see, this result has a simple and effective built-in uncertainty principle.

Jean Bourgain

- My first conversation with Bourgain was during my first year out of grad school. In less than 20 minutes, Jean worked out on a napkin three out of the four results I had up to that point :).

The meaning of generic

- The notion of **generic** above means the following. Let $0 < \delta < 1$ and let $\{\xi_j\}_{1 \leq j \leq n}$ denote independent $0, 1$ random variables of mean $\int \xi_j(\omega) d\omega = \delta$, $1 \leq j \leq n$.

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- Choosing $\delta = n^{\frac{2}{q}-1}$ generates a random subset

$$S_\omega = \{1 \le j \le n : \xi_j(\omega) = 1\} \text{ of } \{1, 2, \dots n\}$$

of expected size $\lceil n^{\frac{2}{q}} \rceil$. Bourgain's theorem holding for a **generic** set S means that the result holds for the set S_ω with probability $1 - o_N(1)$.

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- In a simpler language, if we randomly choose a subset of $\{1, 2, \dots, n\}$ by choosing each element with probability $p = n^{\frac{2}{q}-1}$, then Bourgain's theorem holds for such a set with probability close to 1.

Bourgain's Λ_q theorem

- It is a consequence of Bourgain's celebrated Λ_p theorem in locally compact abelian groups that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $\widehat{f}$ **is supported in** S, then for a "generic" set of size $\lceil N^{\frac{2d}{q}} \rceil$, $2 < q < \infty$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}} \leq C(q) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}},$$

with $C(q)$ independent of N.

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with $C(q)$ independent of N.

- It is not difficult to see that this inequality implies that the support of f must be a positive proportion of $\mathbb{Z}_N^d$.

Signal recovery in the presence of the Λ_q inequality

Theorem

(A. Iosevich and A. Mayeli (2024)) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ *be a signal supported in* $E \subset \mathbb{Z}_N^d$. *Suppose that the frequencies* $\{\widehat{f}(m)\}_{m \in S}$ *are unobserved, where* S *satisfies the* Λ_q *inequality with constant* $C(q)$, *i.e whenever* $\widehat{g}$ *is supported in* S, $|S| = \lceil N^{\frac{2d}{q}} \rceil$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |g(x)|^q \right)^{\frac{1}{q}} \leq C(q) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |g(x)|^2 \right)^{\frac{1}{2}},$$

with K_q *independent of* N. *Then* f *can be recovered exactly provided that*

$$|E| < \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}},$$

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A direct consequence of Bourgain's Λ_q theorem

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- Suppose that f is supported in $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported in S. Bourgain's theorem implies that
- $$N^{-\frac{d}{q}} \cdot |E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^q \right)^{\frac{1}{q}}$$
  $$\leq K_q N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}.$$

A direct consequence of Bourgain's Λ_q theorem

- It follows that
$$|E| \geq \frac{N^d}{(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}.$$

A direct consequence of Bourgain's Λ_q theorem

- It follows that

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- We conclude that if we send the Fourier transform of a signal f supported on a set of size

$$< \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}},$$

and the frequencies in $S \subset \mathbb{Z}_N^d$, $|S| = \lceil N^{\frac{2d}{q}} \rceil$, satisfying the Λ_q, $q > 2$, inequality with constant $C(q)$ are missing, we can recover f exactly with very high probability using L^2-minimization.

A very large set of missing frequencies

- The following result due Guedon, Mendelson, Pajor and Tomczak-Jaegermann (2008).

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Theorem

There exist two positive constants c and C such that for any even integer n and any orthonormal system $\{\phi_j\}_{j=1}^n$ in L^2 with $||\phi_j||_{L^\infty} \leq L$, $1 \leq j \leq n$, we can find a subset $I \subset \{1, \dots, n\}$ with $\frac{n}{2} - c\sqrt{n} \leq |I| \leq \frac{n}{2} + c\sqrt{n}$ such that for every $a = (a_i) \in \mathbb{C}^n$,

$$\left\| \sum_{i \in I} a_i \phi_i \right\|_{L^2} \leq CL \log n (\log \log n)^{\frac{5}{2}} \left\| \sum_{i \in I} a_i \phi_i \right\|_{L^1}.$$

-

A very large set of missing frequencies (continued)

- Moreover, the proof of this result shows that the result holds for a generic set I with

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- We can apply this result to the setting of $\mathbb{Z}_N^d$, where $\{e_1, \dots, e_n\}$, $n = N^d$ denotes the set of all characters $\{\chi(x \cdot m)\}_{m \in \mathbb{Z}_N^d}$.

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- We can show that there exists $C, c > 0$ such that for a generic set $S \subset \mathbb{Z}_N^d$, with

$$|S| \sim \frac{N^d}{\log(N)(\log(\log(N)))^{\frac{5}{2}}},$$

and $f(x) = N^{-\frac{d}{2}} \sum_m \chi(x \cdot m)\widehat{f}(m)$, we have

Another uncertainty principle

- $$\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|f(x)|^2\right)^{\frac{1}{2}} \leq C(d)\left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|f(x)|\right).$$

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- Using the same ideas as before, we can check that if f is supported in E, then
$$|E| \geq \frac{N^d}{C^2(d)}.$$

Exact recovery with a large set of missing frequencies

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Theorem

(A. Iosevich, B. Kashin, I. Limonova, and A. Mayeli (2024)) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a signal supported in $E \subset \mathbb{Z}_N^d$, and suppose that the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved. There exists constants $C, c > 0$ such that if S is generic set of size

$$\sim_{C,c} \frac{N^d}{\log(N)(\log(\log(N)))^{\frac{5}{2}}},$$

and

$$|E| < \frac{N^d}{2C^2(d)},$$

then f can be recovered exactly using L^1-minimization.

The key calculation in the proof

- We have
$$||h||_{L^1(E)} \leq |E|^{\frac{1}{2}} \cdot ||h||_{L^2(\mathbb{Z}_N^d)}$$

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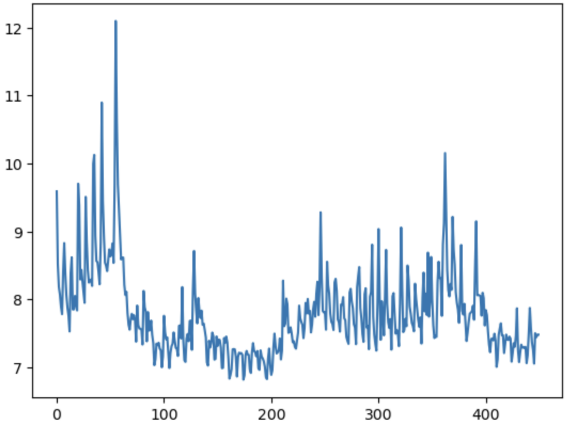
- It follows that if
$$|E| < \frac{N^d}{2C^2(d)}, \text{ then } ||h||_{L^1(E)} < \frac{1}{2} ||h||_{L^1(\mathbb{Z}_N^d)},$$
- hence
$$||h||_{L^1(E)} < ||h||_{L^1(E^c)}.$$

A real-life application

- The following data set describes the number of daily hits on Peyton Manning's website.



Filling in missing values

- The time series you just saw has length 450. We are going to take out the values from 200 to 250 and fill them in using the L^1-minimization algorithm.

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- More precisely, we are going to encode the time series above by a function $f : \mathbb{Z}_{450} \to \mathbb{Z}^+$, and consider all possible functions $g : \mathbb{Z}_{450} \to \mathbb{Z}^+$ such that $f(x) = g(x)$ away from the set of missing values. We are then going to find such a g with the smallest possible $||\widehat{g}||_{L^1(\mathbb{Z}_N)}$ norm, thus approximating f.

Filling in missing values - diagram

- The original time series is in **red**. The imputed values are in **blue**.

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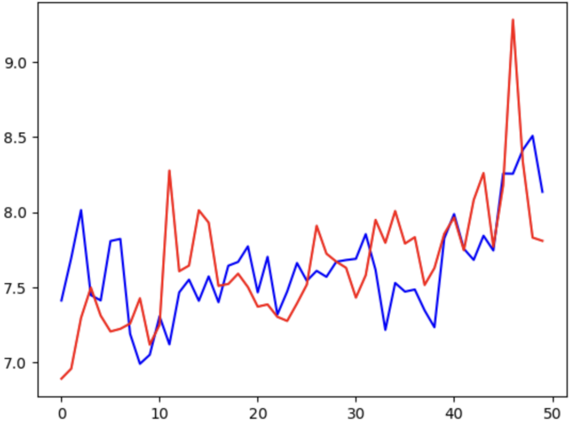

A more advanced imputation

- We are now going to encode the time series above by a function $f : \mathbb{Z}_{450} \to \mathbb{Z}^+$, and consider all possible functions $g : \mathbb{Z}_{450} \to \mathbb{Z}^+$ such that

$$||f - g||_{L^2(M^c)} < \epsilon,$$

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- This approach accounts for the noise in the data set.

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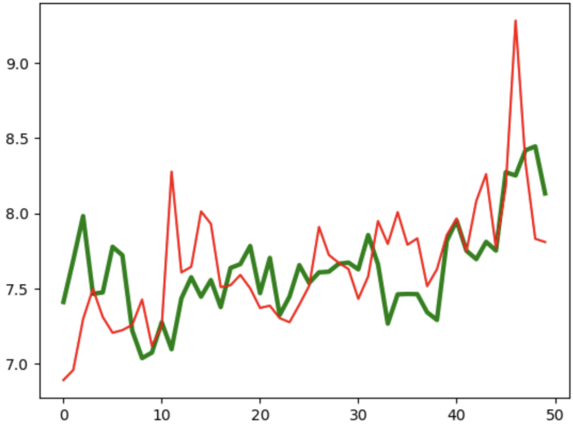


Eh...
thanks
a lot!