ECE 411 Tutorial 1 Jan 17, 2020 ①

# Difference Equations & $\mathbb{Z}$-Transforms

## Difference Equations:

General form: $y(k) + a_1 y(k-1) + \dots + a_n y_n(k-n) = b_0 u(k) + b_1 u(k-1) + \dots + b_m u(k-m)$

$k \in \mathbb{Z}$, the discrete variable. think of $u$ as an inp. seq. and $y$ as an output sequence. The $a_i, b_i$ are real and constant.

Some Remarks: • What if we have $y(k+\bar{k})$ terms, with $\bar{k} > 0$?
No problem! simply subtract $\bar{k}$ from each term.
• What if $a_0 \neq 1$? Also, no problem! Simply divide through by $a_0$.

**Solution Method 1:** Since the DE is linear, exploit superposition!

- As with ODE's, sol'ns are lin. combos of homog. and particular sol'ns (natural & forced responses).
- In CT ODE's, we use $e^{\lambda t}, \lambda \in \mathbb{C}$ as a candidate homog. sol'n. In DT DE's, use $\lambda^k, \lambda \in \mathbb{C}$.
- Subbing into the homogeneous DE, we have:

$$\lambda^k + a_1 \lambda^{k-1} + \dots + a_n \lambda^{k-n} = 0 \Rightarrow \text{multiply through by } \lambda^{n-k}$$
$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$$

this is an $n^{th}$-order polynomial $\Rightarrow n$ roots. Assume they are all distinct
$\Rightarrow$ We have $n$ l.i. solutions. By superposition, any lin. combo is also a solution. $\Rightarrow$

$$y_h(k) = c_1 \lambda_1^k + \dots + c_n \lambda_n^k$$

(complex roots? Take Re{ }, Im{ } as basis fxns and use polar form + Euler to get polyn. envelope trig)

- For particular sol'n, "guess" form and use method of undet. coeffs. (see pg. 50 of notes for ex. + details.

**Solution Method 2:** Use $\mathbb{Z}$-Transform.

- If $x(k)$ is a ~~real-valued~~ real-valued DT signal, its $\mathbb{Z}$-Tsfm is

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$ (one-sided) (details on ROC in text).

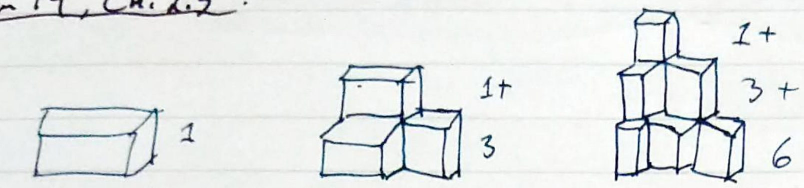
- Key property for solving DE: Time-shift!

let $m > 0$. Then,

$$\mathcal{Z}\{x(k-m)\} = z^{-m} X(z) + x(-m) + z^{-1}x(-m+1) + \dots + z^{-m+1} x(-1)$$
$$\mathcal{Z}\{x(k+m)\} = z^{m} X(z) - [z^{m} x(0) + z^{m-1} x(1) + \dots + z\, x(m-1)]$$

- Inverting: $\mathcal{Z}^{-1}\{X(z)\} = x(k) = \sum \operatorname{Res}[X(z) z^{k-1}]$ for each pole of $X(z)$. (Def'n of inverse uses contour integral, but residue method easier).

## Problem 14, Ch. 2.5:




Stage 1: 1 Block     Stage 2: ~~3~~ 4 Blocks     Stage 3: 10 Blocks

How many cubes at stage 9?

Let $y(K-1)$ be number of blocks at stage $K = 1, 2, 3, \dots$

Why $K-1$? Because i.c. is for stage $K=1$ so that $y(1-1) = y(0)$.

What does the ~~form~~ recursion on $y$ look like?

$$y(0) = 1, \quad y(1) = y(0) + \underset{\text{top}}{1} + \underset{\text{sides}}{2}, \quad y(2) = y(1) + \underset{\text{top}}{1} + \underset{\substack{\text{side}\\ \text{layer 2}}}{2} + \underset{\substack{\text{side}\\ \text{layer 1}}}{3}$$

So we see that in general:

$$y(K+1) = y(K) + 1 + 2 + \dots + K + 2$$

Now we have 2 approaches:

(i) Treat $1+2+\dots+K+2$ as input $v(K)$ which satisfies ~~$v(K) = 1+2+\dots+K+2$~~

$v(K+1) = v(K) + K+3 \longrightarrow$ to see why: $v(0) = 1+2$

$v(0) = 3.$

$v(1) = 1+2+3$
$= v(0) + (0+3)$

$v(2) = 1+2+3+4$
$= v(1) + (1+3)$

Then take $u(K) = K+3$ as another input and find $u(K+1) = K+3+1 = u(K)+1$

$u(0) = 3$

Now have system:

$$\begin{aligned} y(K+1) &= y(K) + v(K), & y(0) &= 1 \\ v(K+1) &= v(K) + u(K), & v(0) &= 3 \\ u(K+1) &= u(K) + 1, & u(0) &= 3 \end{aligned}$$

So we kept defining inputs until.... The ~~remaing~~ remaining input became ~~...~~ constant! Started quadratic since $\sum^n i = \frac{n(n+1)}{2}$, so we needed 2 steps.

Apply a Z-Transform:

$$\left.\begin{aligned} zY(z) - zy(0) &= Y(z) + V(z) \\ zV(z) - zv(0) &= V(z) + U(z) \\ zU(z) - zu(0) &= U(z) + \frac{z}{z-1} \end{aligned}\right\} \rightarrow \begin{aligned} Y(z) &= \frac{1}{z-1}\left(z + V(z)\right) \\ V(z) &= \frac{1}{z-1}\left(3z + U(z)\right) \\ U(z) &= \frac{1}{z-1}\left(3z + \frac{z}{z-1}\right) \end{aligned}$$

$$\Rightarrow V(z) = \frac{1}{z-1}\left(3z + \frac{3z}{z-1} + \frac{z}{(z-1)^2}\right) \Rightarrow Y(z) = \frac{1}{z-1}\left(z + \frac{3z}{z-1} + \frac{3z}{(z-1)^2} + \frac{z}{(z-1)^3}\right)$$

factor out

$$\Rightarrow Y(z) = \frac{z}{(z-1)^4}\left((z-1)^3 + 3(z-1)^2 + 3(z-1) + 1\right)$$

$$= \frac{z}{(z-1)^4}\left[1 + (z-1)(z^2 - 2z + 1 + 3z - 3 + 3)\right] = \frac{z}{(z-1)^4}\left[1 + z^3 - 1\right] = \frac{z^4}{(z-1)^4}.$$

Then use residues: $y(k) = \frac{1}{3!}\frac{d^3}{dz^3}\Big|_{z=1} (z-1)^4 \cdot \frac{z^4 \cdot z^{k-1}}{(z-1)^4} = \frac{1}{6}\frac{d^3}{dz^3}\Big|_{z=1} z^{k+3} = \frac{(k+3)(k+2)(k+1)}{6}$

Num blocks at stage $K=9$ is $y(9-1) = y(8) \Rightarrow \frac{(11)(10)(9)}{6} = 165.$

(ii) Alternatively, notice that:

$$y(k+1) = y(k) + \sum_{i=0}^{k+2} i = y(k) + \frac{(k+2)(k+3)}{2} \quad \left(\text{recall } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$$

$$\Rightarrow y(k+1) = y(k) + \underbrace{\frac{k^2}{2}} + \underbrace{\frac{5k}{2}} + \underbrace{3}$$

can find transforms!

$$\Rightarrow zY(z) - zy(0) = Y(z) + \frac{z(z+1)}{2(z-1)^3} + \frac{5z}{2(z-1)^2} + \frac{3z}{z-1}$$

$$\Rightarrow \cancel{Y(z) = \frac{1}{(z-1)^2}\left[z(z+1)\right.} \quad Y(z) = \frac{1}{z-1}\left[z + \frac{3z}{z-1} + \frac{5z}{2(z-1)^2} + \frac{z(z+1)}{2(z-1)^3}\right]$$

Then invert:

$$y(k) = 1 + 3k + \frac{5}{2\cdot 2!}\left[\frac{d^2}{dz^2}\bigg|_{z=1} z^k\right] + \frac{1}{2\cdot 3!}\left[\frac{d^3}{dz^3}\bigg|_{z=1} z^{k+1} + z^k\right]$$

$$= 1 + 3k + \frac{5}{4}k(k-1) + \frac{1}{12}\cdot(k)(k-1)\left[k+1 + k-2\right]$$

$$= 1 + 3k + \frac{5}{4}k(k-1) + \frac{1}{12}k(k-1)(2k-1)$$

Again, we get $y(8) = 165$.