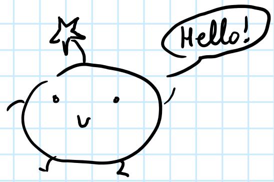
Tutorial 2



Topics (cont'd from last week)
- Difference eq'ns
- z-transforms
- FVT

REVIEW

a) Difference Eq'ns

$$y(k) + a_1 y(k-1) + \dots + a_n y(k-n) = b_0 u(k) + \dots + b_m u(k-m)$$

You have seen two ways of solving this:

i) $y = y_h + y_p$

y_h = sol'n to homogeneous eq'n

$$y(k) + a_1 y(k-1) + \dots + a_n y(k-n) = 0$$

→ Use λ^k, $\lambda \in \mathbb{C}$ as candidate sol'n
→ You saw the cases for real & complex roots
→ There are n roots in total.
→ If the roots are distinct, there are n lin. indep sol'ns
→ $y_h = c_1 \lambda_1^k + \dots + c_n \lambda_n^k$
Lin. combo of sol'ns is also a sol'n

y_p = a particular sol'n
→ In general need to guess form

ii) z-transforms

- Like Laplace Transforms

$$\mathcal{Z}(x(k)) = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \qquad z \in \mathbb{C}$$

b) Properties of z-transform

- Convolution & multiplication of signals (see class notes)
- Multiplication by a^k

$$\mathcal{Z}(a^k x(k)) = X(z/a)$$

- Backward shift by m samples

$$\mathcal{Z}(x(k-m)) = z^{-m} X(z) + x(-m) + z^{-1} x(-m+1) + \dots + x(-1) z^{-m+1}$$

- Forward shift by m samples

$$\mathcal{Z}(x(k+m)) = z^m X(z) - [z^m x(0) + z^{m-1} x(1) + \dots + z\, x(m-1)]$$

- Forward shift by m samples

$$\mathcal{Z}(x(k+m)) = z^m X(z) - [z^m x(0) + z^{m-1}x(1) + \ldots + z\,x(m-1)]$$

c) Inverse z-transform

Find using residues:

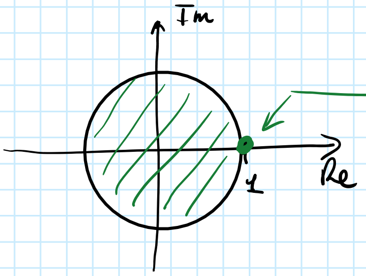
$$x(k) = \sum \text{Residues of } X(z)z^{k-1} \text{ at all poles of } X(z)z^{k-1}$$

→ See table of z-tfs for common cases

d) FVT

$$\boxed{\begin{array}{l} \text{Suppose } \lim_{k\to\infty} x(k) \text{ exists, finite.} \\ \text{Then } \lim_{k\to\infty} x(k) = \lim_{z\to 1} (z-1)X(z) \end{array}}$$

** $\lim_{k\to\infty} x(k)$ exists, finite iff
- $X(z)$ has no poles in $|z|>1$, and
- At most one pole at $z=1$



Important: One pole at $z=+1$ is ok

Any other value of z s.t. $|z|=1$ ✗ NOT ok

e.g. $z=-1$ ✗ Limit DNE

$z=j$ ✗ Limit DNE

PROBLEMS

Problem 1

Solve the IVP

$$y(k) - 4y(k-2) = 3^k \qquad y(-2)=0, \quad y(-1)=1$$

We will solve this using z-transforms

1) Take z-tf

$$Y(z) - 4z^{-2}Y(z) - 4y(-2) - 4z^{-1}y(-1) = \frac{z}{z-3}$$

↑ backwards shift property

Plug in I.C. & isolate $Y(z)$:

$$Y(z)[1-4z^{-2}] - 0 - 4z^{-1} = \frac{z}{z-3}$$

$$Y(z)[1-4z^{-2}] = \frac{z}{z-3} + 4z^{-1}$$

$$Y(z)[1 - 4z^{-2}] = \frac{z}{z-3} + 4z^{-1}$$

$$Y(z) = \left[\frac{z}{z-3} + 4z^{-1}\right] \underbrace{\frac{1}{1-4z^{-2}}}_{\text{Multiply by } \frac{z^2}{z^2}}$$

$$\frac{z^2}{z^2-4}$$

$$Y(z) = \frac{z^3}{(z-3)(z^2-4)} + \frac{4z}{z^2-4}$$

2) Take inverse z-tf using residues

1st term

$$\sum \text{Res}\left(\frac{z^3}{(z-3)(z^2-4)} \cdot z^{k-1}\right) \text{ at all poles}$$

The poles are $z=3$, $z=2$, $z=-2$

Then:

$$= \frac{z^{k+2}}{\cancel{(z-3)}(z^2-4)}\cancel{(z-3)}\Bigg|_{z=3} + \frac{z^{k+2}}{(z-3)(z-2)}\Bigg|_{z=-2} + \frac{z^{k+2}}{(z-3)(z+3)}\Bigg|_{z=2}$$

$$= \frac{3^{k+2}}{5} + \frac{(-2)^{k+2}}{20} - \frac{2^{k+2}}{4}$$

2nd term

$$\sum \text{Res}\left(\frac{4z \cdot z^{k-1}}{z^2-4}\right) \text{ at all poles}$$

The poles are $z=2$, $z=-2$

$$= \frac{4z^k}{(z+2)}\Bigg|_{z=2} + \frac{4z^k}{(z-2)}\Bigg|_{z=-2}$$

$$= \frac{4 \cdot 2^k}{4} - \frac{4(-2)^k}{4} = 2^k - (-2)^k$$

Putting terms together:

$$y(k) = \frac{1}{5}(3^{k+2}) - \frac{1}{4}(2^{k+2}) + \frac{1}{20}(-2)^{k+2} + 2^k - (-2)^k$$

Problem 2

For each of the following, does the final value of $x(k)$ exist?

If so, what is it?

a) $X(z) = \dfrac{3z+2}{z^2+1}$

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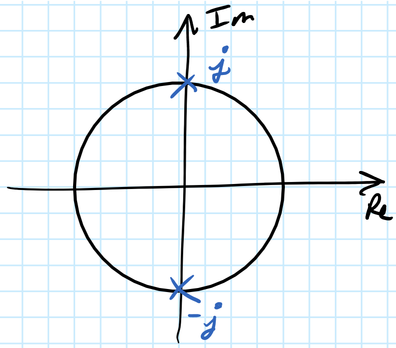
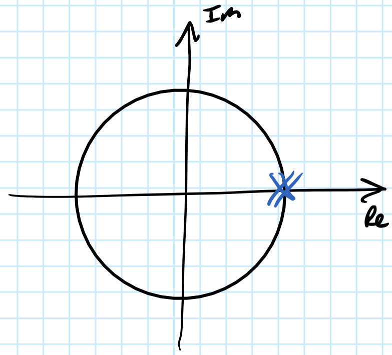


$X(z) = \dfrac{3z+2}{(z+j)(z-j)}$ $\rightarrow$ Roots at $\pm j$

- We cannot have poles w̄ magnitude 1 anywhere <u>except</u> at <u>$z=1$</u>
- Poles at $\pm j$ ✗ Limit DNE

b) $X(z) = \dfrac{z}{(z-1)^2}$

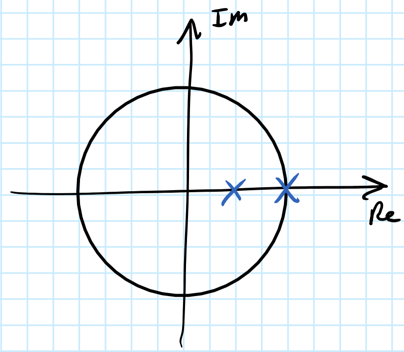


↳ Repeated poles at $z=1$

- Can have <u>max</u> <u>one</u> pole at $z=1$

✗ Limit DNE

c) $X(z) = \dfrac{2z}{(z-1)(z-0.5)}$



Roots at $z=1$, $z=0.5$

✓ Limit exists

Can apply FVT

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} \frac{2z}{(z-1)(z-0.5)}(z-1)$$

$$= \lim_{z\to 1} \frac{2z}{(z-0.5)} = \frac{2}{0.5} = 4$$