University of Toronto
Department of Electrical and Computer Engineering
ECE411S Real-Time Computer Control
Solution of Homework 1

Problem 1 Solve the initial-value problem

$$\begin{aligned} y(k) - 0.5y(k-1) &= 0.2^k, \quad k \geq 1 \\ y(0) &= 1. \end{aligned}$$

Solution This method uses time-domain theory. The homogeneous equation is

$$y(k) - 0.5y(k-1) = 0.$$

The general solution of this is

$$y_h(k) = c_1(0.5)^k.$$

For a particular solution of the nonhomogeneous equation, let's try the form

$$y_p(k) = c_2(0.2)^k.$$

Subbing into the equation, we get

$$c_2(0.2)^k - 0.5c_2(0.2)^{k-1} = 0.2^k.$$

Divide by 0.2^k:

$$c_2 - 0.5c_2(0.2)^{-1} = 1.$$

So $c_2 = -2/3$ and the general solution of the nonhomogeneous equation is

$$y(k) = c_1(0.5)^k - \frac{2}{3}(0.2)^k.$$

Finally, get c_1 from the initial condition:

$$y(k) = \frac{5}{3}(0.5)^k - \frac{2}{3}(0.2)^k.$$

You could also use z-transforms. Then

$$Y(z) = \frac{z^2}{(z-0.5)(z-0.2)}.$$

Application of the residue approach gives the same result as above.

Problem 3 Find two real-valued, linearly independent solutions to the homogeneous equation

$$y(k) + 4y(k-2) = 0.$$

Solution Try the solution $y(k) = \lambda^k$. Then

$$\lambda^2 + 4 = 0,$$

so $\lambda_1 = 2j$, $\lambda_2 = \overline{\lambda_1}$. The function $y(k) = (2j)^k$ is a complex-valued solution. Its real and imaginary parts are real, linearly independent solutions:

$$(2j)^k = \left(2e^{j\pi/2}\right)^k = 2^k e^{jk\pi/2}.$$

The real and imaginary parts are

$$y_1(k) = 2^k \cos(k\pi/2), \quad y_2(k) = 2^k \sin(k\pi/2).$$

Problem 7 Find the final value of $x(k)$, if it exists, where $X(z) = \frac{z^2}{(z+1)(z-0.2)}$. Repeat for $X(z) = \frac{z^2}{(z-1)(z-0.2)}$.

Solution First case: doesn't exist. Second case: $x(k) \to \frac{1}{0.8}$.

Problem 8 Solve the initial-value problem

$$\begin{aligned} y(k) - y(k-1) &= 4k, \quad k \geq 1 \\ y(0) &= 1 \end{aligned}$$

using z-transforms and the method of undetermined coefficients.

Solution First, we solve the problem using z-transforms. Noting that the initial condition for y is given at time 0, we write the equation as

$$y(k+1) - y(k) = 4k + 4, \quad k \geq 0.$$

Then

$$zY(z) - zy(0) - Y(z) = 4\frac{z}{(z-1)^2} + 4\frac{z}{z-1},$$

so

$$zY(z) - z - Y(z) = 4\frac{z}{(z-1)^2} + 4\frac{z}{z-1},$$

and hence

$$Y(z) = \frac{z}{z-1} + \frac{4z}{(z-1)^2} + \frac{4z}{(z-1)^3}.$$

By the method of residues, the inverse transforms of the three terms are, in order,

$$\text{residue } \frac{z^k}{z-1} = z^k\Big|_{z=1} = 1$$

$$\text{residue } \frac{4z^k}{(z-1)^2} = \frac{d}{dz}4z^k\bigg|_{z=1} = 4k$$

$$\text{residue } \frac{4z^k}{(z-1)^3} = \frac{1}{2}\frac{d^2}{dz^2}4z^k\bigg|_{z=1} = \frac{4k(k-1)}{2}.$$

So $y(k)$ equals the sum:

$$y(k) = 2k^2 + 2k + 1.$$

If we use the method of undetermined coefficients, we will quickly see that a trial solution for $y_p(k)$ in the form $y_p(k) = Ak + B$ does not work. The reason is that part of the particular solution is also the solution of the homogeneous equation. We need to try a particular solution of the form

$$y_p(k) = \alpha k^2 + \beta k$$

On matching coefficients, we get

$$2\alpha = 4$$

$$-\alpha + \beta = 0$$

This results in

$$\alpha = 2 \quad \beta = 2$$

The homogeneous solutions is given by $y_h(k) = c$, a constant. Applying the initial condition gives $c = 1$. The solution for $y(k)$ is given by

$$y(k) = 2k^2 + 2k + 1$$

Problem 14 Stages 1, 2 and 3 of a pattern of cubes are shown (see problem in course notes). How many cubes will there be at stage 9 of this pattern? solve the problem by z-transforms.

Solution Let $y(k-1)$ be the number of blocks at stage $k = 1, 2, \ldots$ (We take the argument of y to be $k-1$ so that the initial condition is on $y(0)$.) Thus

$$y(0) = 1$$

$$y(1) = y(0) + 1 + 2$$

$$y(2) = y(1) + 1 + 2 + 3$$

and in general

$$y(k+1) = y(k) + 1 + 2 + \cdots + (k+2), \quad k \geq 0.$$

Define the input to this equation

$$v(k) = 1 + 2 + \cdots + (k+2).$$

Then

$$v(k+1) = v(k) + (k+3), \quad v(0) = 3.$$

We can also generate the input to this equation, $u(k) = k + 3$, via

$$u(k+1) = u(k) + 1, \quad u(0) = 3.$$

Thus, we have the coupled equations

$$\begin{aligned} y(k+1) &= y(k) + v(k), \quad y(0) = 1 \\ v(k+1) &= v(k) + u(k), \quad v(0) = 3 \\ u(k+1) &= u(k) + 1, \quad u(0) = 3. \end{aligned}$$

Take z-transforms

$$\begin{aligned} zY(z) - z &= Y(z) + V(z) \\ zV(z) - 3z &= V(z) + U(z) \\ U(z) - 3z &= U(z) + \frac{z}{z-1}. \end{aligned}$$

Thus,

$$\begin{aligned} Y(z) &= \frac{1}{z-1}[z + V(z)] \\ V(z) &= \frac{1}{z-1}[3z + U(z)] \\ U(z) &= \frac{1}{z-1}\left[3z + \frac{z}{z-1}\right]. \end{aligned}$$

Solving in reverse order gives

$$\begin{aligned} U(z) &= \frac{z(3z-2)}{(z-1)^2} \\ V(z) &= \frac{z(3z^2 - 3z + 1)}{(z-1)^3} \\ Y(z) &= \frac{z^4}{(z-1)^4}. \end{aligned}$$

We want $y(8)$:

$$y(8) = \sum_{p_i} \mathrm{Res}[Y(z)z^7, z = p_i] = \mathrm{Res}\Big[\frac{z^4}{(z-1)^4} z^7, z = 1\Big]$$

Thus,

$$y(8) = \frac{1}{3!} \frac{d^3}{dz^3} z^{11}\Big|_{z=1} = \frac{11 \cdot 10 \cdot 9}{3!} = 165$$

Extra problem Find the transfer function of the digital differentiator using backward and forward differences.

Solution Using forward differences, we have

$$y(k) = \frac{u(k+1) - u(k)}{T}.$$

The z-transform of the above equation with zero initial condition is

$$Y(z) = \frac{zU(z) - U(z)}{T},$$

so that the transfer function is

$$\frac{Y(z)}{U(z)} = \frac{1}{T}(z-1).$$

Note that the transfer function is not proper, i.e., the degree of the numerator is higher than that of the denominator. This is a manifestation of the fact that the differentiator using forward differences is non-causal. Its implementation requires the knowledge of $u(k+1)$, one sample ahead in the future.

Using backward differences, we have

$$y(k) = \frac{u(k) - u(k-1)}{T}.$$

The z-transform of the above equation with zero initial condition is

$$Y(z) = \frac{U(z) - z^{-1}U(z)}{T},$$

so that the transfer function is

$$\frac{Y(z)}{U(z)} = \frac{1}{T}\frac{z-1}{z}.$$

This transfer function is proper, and indeed the system is now causal.