Chapter 10

Euler Equations:

$\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \Gamma_1$, $\quad \Gamma$ is Torque
$\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 = \Gamma_2$
$\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 = \Gamma_3$

Rigid Body with Zero Torque

$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_2\omega_3$
$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_2)\omega_3\omega_1$
$\lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$

Knowing constant ω_3 (this case):

$\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3$, where b is for body.
$\mathbf{L} = (\lambda_1\omega_1, \lambda_1\omega_2, \lambda_3\omega_3) = (\lambda_1\omega_0 cos\Omega_b t, -\lambda_1\omega_0 sin\Omega_b t, \lambda_3\omega_3)$

Euler axis change:

- Step (a). Starting with the body axes aligned with the space axes, we first rotate the body through an angle about the axis i, as illustrated in the first frame of Figure 10.10. This rotates the first and second body axes in the xy plane. In particular, the second body axis now points in the direction labeled e2.
- Step (b). Next we rotate the body through an angle 9 about the new axis e2. This moves the body axis e 3 to the direction whose polar angles are 9 and O. Evidently our first two steps can bring the body axis e 3 to any assigned orientation, and, with e 3 in position, the only remaining freedom is a rotation about e 3 .
- Step (c). Finally, we'll rotate the body about e 3 through whatever angle * is needed to bring the body axes e 2 and e l into their assigned directions, as shown in the third frame of Figure 10.10.

Euler Angles:

$\omega = \omega_a + \omega_b + \omega_c = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\mathbf{e'}_2 + \dot{\psi}\mathbf{e}_3$ In terms of $\hat{x}, \hat{y}, \hat{z}$
$\mathbf{e'}_1 = \hat{x}cos\phi + \hat{y}sin\phi$
$\mathbf{e'}_2 = -\hat{x}sin\phi + \hat{y}cos\phi$
$\mathbf{e'}_3 = \hat{z}$

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} = \begin{bmatrix} cos\psi cos\phi - cos\theta sin\theta sin\psi & cos\psi sin\phi + cos\phi cos\phi sin\psi & sin\psi sin\theta \\ -sin\psi cos\phi - cos\theta sin\phi cos\psi & -sin\psi sin\phi + cos\theta cos\phi cos\psi & cos\psi sin\theta \\ sin\theta sin\phi & -sin\theta cos\phi & cos\theta \end{bmatrix} \begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$

Velocity of body in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$:
$\mathbf{w} = -(\dot{\theta}sin\phi - \dot{\psi}sin\theta sin\phi)\hat{x} + (\dot{\theta}cos\phi - \dot{\psi}sin\theta cos\phi)\hat{y} + (\dot{\phi} + \dot{\psi}cos\theta)\hat{z}$
Velocity of body in terms of $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$:
$\mathbf{w} = -(\dot{\phi}sin\theta cos\psi - \dot{\theta}sin\phi)\hat{e}_1 + (\dot{\phi}sin\theta sin\psi - \dot{\theta}cos\psi)\hat{e}_2 + (\dot{\psi} + \dot{\phi}cos\theta)\hat{e}_3$
$\mathbf{L} = (-\lambda_1\dot{\phi}sin\theta)\mathbf{e'}_1 + \lambda_2(\dot{\theta})\mathbf{e}'_2 + \lambda_3(\dot{\psi} + \dot{\phi}cos\theta)\mathbf{e}_3$

Motion of a spinning top:

$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2 sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}cos\theta)^2 - MgRcos\theta$
θ eqn: $\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2 sin\theta cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}cos\theta)cos\theta =$ const
ϕ eqn: $\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2 sin\theta cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}cos\theta)cos\theta =$ const
ψ eqn: $\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}cos\theta) =$ const

Notes:

- Swapping from longest to shortest axis is just swapping 1,2, or 3
$$\Omega^2 = \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1\lambda_2}\omega_3^2 \quad \rightarrow \quad \Omega^2 = \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3\lambda_2}\omega_1^2$$
- If the body is spinning about either the principal axis with largest moment or that with smallest moment, the motion is stable against small disturbances and ω is a sine or cos solution. Else, if λ lies between the other λs then ω is a real exponential which moves away from 0.
- Moment of inertia: $\mathbf{I} = mr^2$

Relations:

Problem 1 (Book) Rotating Wheel:
$KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \quad , \quad I = \frac{1}{2}MR^2, \quad v = R\omega$
so, $KE = \frac{3}{4}MR^2\omega^2$
Uniform motion about a wheel's center is $I = \frac{1}{2}MR^2$ and about the rim is $I' = \frac{3}{2}MR^2$

Formulas:

Center of Mass:
$$cm = \frac{\int_0^L x\frac{M}{L}dx}{M}$$
Center of Mass:
$$R = \frac{1}{M}\sum_\alpha m_\alpha r_\alpha, \quad \text{Where M is the total mass}$$

Used in Example #1
$\lambda_1 = I_{xx} = \int dV\rho(y^2 + z^2) \rightarrow dx, dy, dz$
$\lambda_2 = I_{yy} = \int dV\rho(x^2 + z^2)$
$\lambda_3 = I_{zz} = \int dV\rho(y^2 + x^2)$
$\rho = \frac{M}{V}$
$\mathbf{L} = \mathbf{Iw} = (\lambda_1 w_x, \lambda_2 w_y, \lambda_3 w_z)$

Kinetic energy of a rotating body
$\mathbf{T} = \frac{1}{2}\omega \cdot \mathbf{L} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$
$\mathbf{L} = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$

Chapter 13

Hamilton's Equations One Dimensional:

$p = \frac{\partial T}{\partial\dot{x}}$
$\dot{q} = \frac{\partial\mathcal{H}}{\partial p}$, $\dot{p} = -\frac{\partial\mathcal{H}}{\partial q}$
Check if q is a 'natural' coordinates (not dependent on t), if so we can use
$\mathcal{H} = T + U$
Otherwise,
$\mathcal{H} = pq - \mathcal{L}$
also note:
$\mathbf{F} = -\nabla U$

Hamiltonian general form from notes:

$$\frac{\partial\mathcal{H}}{\partial Q} = \frac{\partial\mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial Q}$$
and
$$\frac{\partial\mathcal{H}}{\partial P} = \frac{\partial\mathcal{H}}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial P}$$

Jacobian of H

$$J = \left(\frac{Q, P}{q, p}\right) = 1$$

Canonical transformation:

- Jacobian must be equal to 1
- To find the canonical transformation. It's the Original values of p,q or P,Q reversed, divided by the Jacobian solution (given arbitrary constant values, α, β,etc

Examples

Example #1 (Inertia tensor for a Solid Cone)
Find the moment of inertia tensor I for a spinning top that is a uniform solid cone (mass M, height h, and base radius R) spinning about its tip. Choose the z axis along the axis of symmetry of the cone. For an arbitrary angular velocity c_o, what is the top's angular momentum L?
i) The moment of Inertia about the $\hat{z}$ is the integral
$I_{zz} = \int dVQ(y^2 + x^2)$
$Q = \frac{M}{V} = \frac{3M}{\pi R^2 h}$
ii) Evaluated in cylindrical coordinates...
$x^2 + y^2 = \rho^2$
Thus, $I_{zz} = Q\int_0^h dz \int_0^{2\pi} d\phi \int_0^r \rho d\rho\rho^2 = \frac{3}{10}MR^2$
$I_{xx} = I_{yy}$, because of rotational symmetry of the top.
$I_{xx} = \int dVQ(y^2 + z^2) = \int dVQy^2 + dVQz^2 = \frac{3}{20}M(R^2 + 4h^2)$
$$\mathbf{I} = \frac{3}{20}M\begin{bmatrix} R^2 + 4h^2 & 0 & 0 \\ 0 & R^2 + 4h^2 & 0 \\ 0 & 0 & 2R^2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$
If ω points along the x axis, then $\omega_y = \omega_z = 0$ (unique property of this problem)

Example #2 (Atwood Machine with 2 hanging masses)
$\mathcal{H} = T + U = \frac{p^2}{2(m_1 + m_2)} - (m_1 - m_2)gx$
$p = \frac{\partial T}{\partial\dot{x}} = (m_1 + m_2)\dot{x}$
This example illustrates the general procedure to be followed in setting up Hamilton's equations for any given system: 1. Choose suitable generalized coordinates, q 1 , , qn . 2. Write down the kinetic and potential energies, T and U, in terms of the q's and 's. 3. Find the generalized momenta p i , , pn . (We are now assuming our system is conservative, so U is independent of qi and we can use pi = $\partial T/\partial\dot{q}_i$. In general, one must use $p_i = \partial\mathcal{L}/\partial\dot{q}_i$ 4. Solve for the $\dot{q}$'s in terms of the p's and q 's. 5. Write down the Hamiltonian $\mathcal{H}$ as a function of the p's and q 's. [Provided our coordinates are "natural" (relation between generalized coordinates and underlying Cartesians is independent of time), H is just the total energy H = T + U 6. Write down Hamilton's equations (13.25).

Example #3 (Mass on a cone)
$$T = \frac{1}{2}m\left[(c^2 + 1)\dot{z}^2 + (cz\dot{\phi})^2\right]$$
$$p_z = \partial T/\partial\dot{z} = m(c^2 + 1)\dot{z}, \quad p_\phi = \partial T/\partial\dot{\phi} = mc^2z^2\dot{\phi}$$
Then solve for $\dot{q}$ and $\dot{\psi}$

$$\mathcal{H} = T + U = \frac{1}{2m}\left[\frac{p_z^2}{(c^2+1)} + \frac{p_\phi^2}{c^2 z^2}\right] + mgz$$

now we can solve

$$\dot{z} = \partial\mathcal{H}/\partial p_z = \frac{p_z}{m(c^2+1)}, \quad \dot{p}_z = -\partial\mathcal{H}/\partial z = \frac{p_\phi^2}{mc^2 z^2} + mg$$

and do the same for $\dot{\phi}$ and $\dot{p}_\phi$

Coordinates: