[G]: Dinner
[E]: ST

Exercise 06
MACC – SS22

4.7.22 ①

[O]: → Dinner fixed, see Moodle

→ Last sheet till 19.07.22 14:30

→ Other questions? → Next week: Question round: 11. July → Questions via Mail

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[HW6] **Operator Theory**

[E1] **Boundedness:** Op $A: \mathcal{D}(A) \to L^2(\mathbb{R}^3)$ bounded iff

$$\|A\|_{op} := \sup_{\psi \in \mathcal{D}(A)} \frac{\|A\psi\|}{\|\psi\|} < \infty$$

Operator norm

(Total) Momentum operator: $p := p_1^2 + p_2^2 + p_3^2$, $p: \mathcal{D}(p) \to L^2(\mathbb{R}^3)$

with $p_\alpha = -i \frac{\partial}{\partial x_\alpha}$

→ Show unboundedness for p_1 by construction a counterexample.

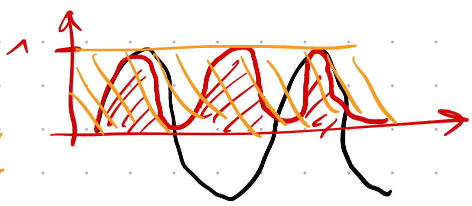
$p_1 = -i\,\partial_x$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad \psi_n := \sin(\pi \cdot x \cdot n)$

$$p_1 \cdot \psi_n = -i\,\partial_x \psi_n(x,y,z) = -i\,\pi \cdot n \; \cos(\pi \cdot x \cdot n)$$

$$\|p_1 \psi_n\|_{L^2} = \left[\int_{\mathbb{R}^3} \overline{\cancel{+}i\,\pi n \cos(\pi x n)}\,(-i\,\pi \cdot n \; \cos(\pi \cdot x \cdot n))\, dx\right]^{0.5}$$

$i^2 = -1$

$$= \left[\int_{\mathbb{R}^3} \pi^2 n^2 \; \underbrace{\cos^2(\pi \cdot x \cdot n)}_{\leq 1}\right]^{0.5}$$

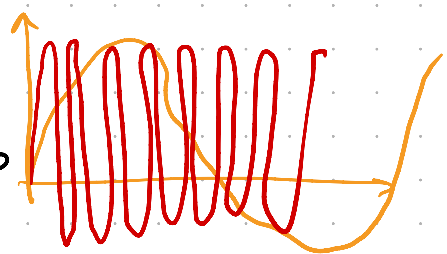


Upper bound is a bad idea

$$\pi \cdot n \int_{\mathbb{R}^3} \cdot 1 \quad dx$$

$$\frac{\|p_1 \psi_n\|_{L^2}}{\|\psi_n\|_{L^2}} = \boxed{\pi \cdot n}\left(\frac{\int_{\mathbb{R}^3} \overbrace{\cos^2(\pi \cdot x \cdot n)}^{u^2}}{\underbrace{\int_{\mathbb{R}^3} \sin^2(\pi \cdot x \cdot n)}_{=1}}\right)^{0.5} = \pi \cdot n \to \infty$$

(*)

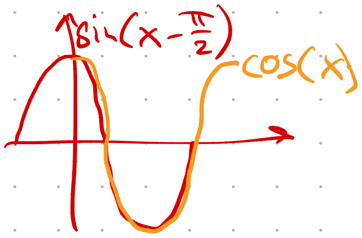


$$\frac{\int_{\mathbb{R}^3} \cos^2(\pi\cdot x\cdot 1)\, dx}{\int_{\mathbb{R}^3} \sin^2(\pi\cdot x\cdot 1)\, dx} = 1$$

$$\boxed{x_n := n\cdot x}$$

$$\frac{dx_n}{dx} = n \Rightarrow \frac{dx_n}{n} \qquad \approx \frac{n}{n}$$

$$\frac{\int_{\mathbb{R}^3} \cos^2(\pi\cdot \overbrace{x\cdot n}^{x_n})}{\int_{\mathbb{R}^3} \sin^2(\pi\cdot x\cdot n)}\, dx = \boxed{\frac{\frac{1}{n}\int_{\mathbb{R}^3} \cos^2(\pi\cdot x_n)\, dx_n}{\frac{1}{n}\int_{\mathbb{R}^3} \sin^2(\pi\cdot x_n)\, dx_n}}^{=1} = 1$$



$$\cos^2(\pi\cdot x\cdot n) = \sin^2\left(\pi\cdot\left(x-\frac{\pi}{2}\right)\cdot n\right)$$

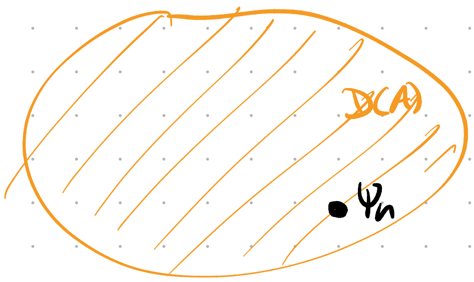
$$\left(\frac{\int \cos^2(\pi\cdot x\cdot n)}{\int \sin^2(\pi\cdot x\cdot n)}\right)^{0.5} = \left(\frac{\int \cos^2(\pi\cdot x\cdot n)}{\int \cos^2(\pi\cdot (x-\frac{\pi}{2})\cdot n)}\right)^{0.5} = \left(\frac{\int_{\mathbb{R}^3} \cos^2(\pi x n)}{\int_{\mathbb{R}^3} \cos^2(\pi\cdot x\cdot n)}\right)^{0.5} = 1^{0.5} = 1$$

doesn't matter when integrate $\mathbb{R}$

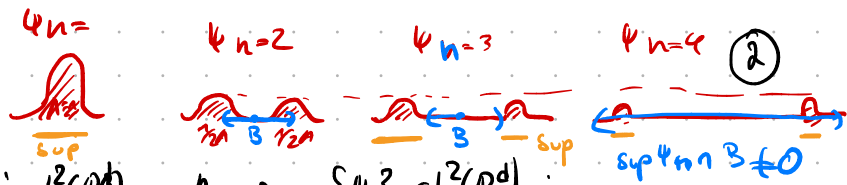
(*) shows that at least one seq. $\{\psi_n\}$ exists. s.t.

$$\frac{\| p_1 \psi_n \|}{\| \psi_n \|} \to \infty$$

$$\Rightarrow \sup_{\psi \in D(A)} \geq \frac{\| p_1 \psi_n \|}{\| \psi_n \|} \to \infty \Rightarrow p_1 \text{ is unbounded.}$$

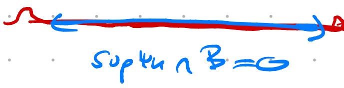


$\boxed{E3}$ **Spectra Schrödinger Op.** ②



Spreading Sequence: Let A be op. in $L^2(\mathbb{R}^d)$. A seq. $\{\varphi_n\} \subset L^2(\mathbb{R}^d)$ is called spreading sequence for A and λ if

$\{x \mid \varphi_n(x) \neq 0\}$ w.r.t.



1) $\|\varphi_n\| = 1 \quad \forall n$

2) for any bounded set $B \subset \mathbb{R}^d$, $\operatorname{supp}\{\varphi_n\} \cap B = \{\}$ for sufficiently large n

3) $\|(A-\lambda)\varphi_n\| \to 0$ as $n \to \infty$

Theorem 1: If $H = -\frac{\hbar^2}{2m}\Delta + V$ with real potential $V(x) \in C^0(\mathbb{R}^d)$, $V(x) \geq L \;\forall x \in \mathbb{R}^d$, then

x^2

$$\sigma_{ess}(H) = \{\lambda \in \mathbb{C} \mid \text{there exists a spreading seq. for } H \text{ and } \lambda\}$$

?!

$\sigma_{ess}(H) = \{\}$

$\boxed{E3}$: → Use the theorem to show that $\sigma_{ess} = \{\}$ via contradiction.

Proof by contradiction:

E3 Let us assume that φ_n is Sp. Seq. w.r.t. $H = -\Delta + V$ and λ

$$0 \leftarrow \langle \varphi_n, \underbrace{(H-\lambda)\varphi_n}_{\to 0 \text{ by our assumption}} \rangle = \ldots \quad \geq \ldots \quad = \ldots \longrightarrow \infty$$

⇒ Assumption is wrong

$$\langle \varphi_n, (H-\lambda)\varphi_n \rangle = -\underbrace{\int \bar{\varphi}_n \Delta\varphi_n}_{\langle \varphi_n, \Delta\varphi_n \rangle} + \int \bar{\varphi}_n \cdot V(x) \cdot \varphi_n - \lambda \underbrace{\int \bar{\varphi}_n \cdot \varphi_n}_{1 \text{ by Sp. Seq prop I)}}$$

⇒ $\nexists$ Sp. Seq.

$$= \int \underbrace{(\nabla\varphi_n)^2}_{\geq 0} + \int V(\varphi_n)^2 - \lambda$$

$$\geq \left(\int_{\mathbb{R}} V(\varphi_n)^2\right) - \lambda$$

$$\geq \inf_{x \in \operatorname{supp}\varphi_n} V(x)\,\varphi_n - \lambda$$

.....

$\longrightarrow \infty$

Don't forget to show that $\lambda \to \infty$ and the rest of Ex

Show that there are ∞ many λ's

they have to accumulate somewhere