Line Search Stepsize Control and Trust-Region Methods

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Trust-Region Methods

1. Given $x^{(k)}$
2. Replace f by (e.g 2nd order) approximation $\hat{f}$
3. Solve $\hat{x} = \operatorname{argmin}_{x \in D_k} \hat{f}(x)$ for a given thrust region $D_k = \{x \in \mathbb{R}^n \mid \|x - x^{(k)}\|_p \leq \delta\}$
4. Test improvement $\rho = \frac{\text{actual improvement}}{\text{predicted improvement}} = \frac{f(x^k) - f(\hat{x})}{f(x^k) - \hat{f}(\hat{x})}$
5. If $\rho > \rho_{\min}$, set $x^{(k+1)} = \hat{x}$, else decrease thrust region radius $\delta \leftarrow \sigma\delta$

```
trust_region (generic function with 1 method)
```

```
function trust_region(
    f, fhat, x0, solve_subproblem, kmax, rhomin, delta0, sigma
)
    println("START")
    hist = []
    x = x0
    push!(hist, [x0, 0])
    for k=1:kmax
        @show k
        delta = delta0
        @show delta
        xhatval = nothing
        xhatval = solve_subproblem(x, delta)
        @show xhatval
        rho = (f(x) - f(xhatval)) / (f(x) - fhat(xhatval, x))
        @show rho
        i = 0
        while abs(rho-1) > 0.005 && i<10
            delta *= sigma
            @show delta
            xhatval = solve_subproblem(x, delta)
            @show xhatval
            rho = (f(x) - f(xhatval)) / (f(x) - fhat(xhatval, x))
            @show rho
            i += 1
        end
        @show delta
        x = xhatval
        @show x
        push!(hist, [x, delta])
    end
    return x, hist
end
```

Define Problem

- Define objective: $f(x, y) = x^2 + y^2(y^2 - 1)$
- Derive quadratic approximation $\hat{f} = \hat{f}(x) := f(x^{(k)}) + (x - x^{(k)})^T \nabla f(x^{(k)}) + \frac{1}{2}(x - x^{(k)})^T \nabla^2 f(x^{(k)})(x - x^{(k)})$
- Minima are at $(0, \pm 1/\sqrt{6})$, saddle point at $(0, 0)$

```
f (generic function with 1 method)
```

```
# objective
f(x) = x[1]^2 + x[2]^2 * (x[2]^2 - 1)
```

fhat (generic function with 1 method)

```
# quadratic approximation
fhat(x, x0) = (
    f(x0) + (x-x0)' * derivative(f, x0)
    + 1/2 * (x-x0)' * hessian(f, x0) * (x-x0)
)
```

Define Solution to Subproblem

- Either analytically (see below)
- Or use approximate solutions (Cauchy point, ...)

solve_subproblem (generic function with 1 method)

```
solve_subproblem(x, delta) = [
    if (abs(x[1]) <= delta)
        0
    else
        x[1] - sign(x[1])*delta
    end,
    if (x[2] == 0)
        if (abs(x[2]) <= delta)
            delta
        else
            x[2] + sign(x[2]) * delta
        end
    elseif (x[2]^2 >= 1/6)
        if (abs(x[2] - (4*x[2]^3)/(6*x[2]^2-1)) <= delta)
            (4*x[2]^3)/(6*x[2]^2-1)
        else
            x[2] - sign(x[2] - (4*x[2]^3)/(6*x[2]^2-1)) * delta
        end
    else
        nothing
        # TOOD: Fix this
    end
]
```

Test Thrust-Region Method with Saddle Point

- We can escape the saddle point $x^{(0)} = (0, 0)$ 👏

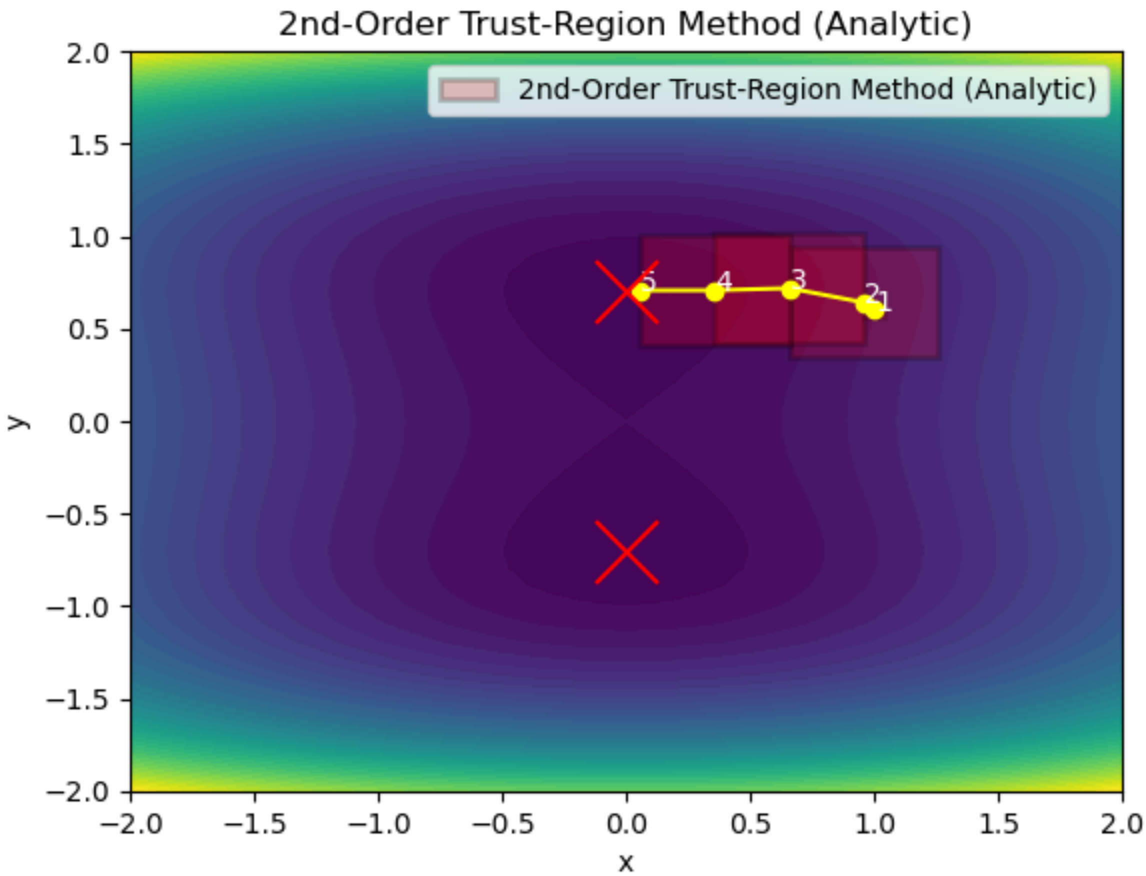
([0.0, 0.707107], [[[more], 0], [[more], 0.523018], [[more], 0.523018], [[more],

```
trust_region(f, fhat, [0.,0.], solve_subproblem, 10, 0.5, 1.5, 0.9) # TODO, fixme <=> implement solve_subproblem better
```

```
START
k = 1
delta = 1.5
xhatval = [0.0, 1.5]
rho = -1.2500000001833427
delta = 1.35
xhatval = [0.0, 1.35]
rho = -0.8225000001206395
delta = 1.215
xhatval = [0.0, 1.215]
rho = -0.47622500006984997
delta = 1.0935000000000001
xhatval = [0.0, 1.0935000000000001]
rho = -0.1957422500287106
delta = 0.9841500000000002
xhatval = [0.0, 0.9841500000000002]
rho = 0.031448775046123556
delta = 0.8857350000000002
xhatval = [0.0, 0.8857350000000002]
rho = 0.21547350980660407
delta = 0.7971615000000002
xhatval = [0.0, 0.7971615000000002]
rho = 0.3645335429712174
delta = 0.7174453500000002
xhatval = [0.0, 0.7174453500000002]
rho = 0.48527216983455407
delta = 0.6457008150000002
xhatval = [0.0, 0.6457008150000002]
rho = 0.5830704575938568
```

Trust-Region Method in Action 😎

x01 = 1 x02 = 0.6 k = 4 rhomin = 0.91 deltao = 0.3 sigma = 0.61


2nd-Order Trust-Region Method (Analytic)
2nd-Order Trust-Region Method (Analytic)
5
4
3
2
1
y
x
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

```
let
    # Perform Optimization
    tr = trust_region(f, fhat, [Float64(x01),Float64(x02)], solve_subproblem, k,
    rhomin, delta0, sigma)
    tr_x = [
        tr[2][i][1][1] for i=1:length(tr[2])
    ]
    tr_y = [
        tr[2][i][1][2] for i=1:length(tr[2])
    ]
    deltas = [
        tr[2][i][2][1] for i=1:length(tr[2])
    ]

    # Plot annotations
    clf()
    ax = gca()
    Δ = 0.1
    X=collect(-2:Δ:2)
    Y=collect(-2:Δ:2)
    F=[f([X[j],Y[i]]) for i=1:length(Y), j=1:length(X)]
    contourf(X,Y,F, levels=50)
    PyPlot.title("2nd-Order Trust-Region Method (Analytic)")

    # Trust Regions
    for i=2:length(tr_x)
        ax.add_patch(PyPlot.matplotlib.pyplot.Rectangle((tr_x[i-1]-deltas[i],
        tr_y[i-1]-deltas[i]), 2deltas[i], 2deltas[i], facecolor="red", alpha=0.2,
        edgecolor="black", linewidth=2.))
    end

    # Trajectory
    PyPlot.plot(tr_x, tr_y, color="yellow", zorder=2)
    scatter(tr_x, tr_y, color="yellow", zorder=2)
    for i=1:length(tr_x)
        annotate(string(i), [tr_x[i], tr_y[i]], color="w", zorder=3)
    end

    # Plot annotations
    legend(["2nd-Order Trust-Region Method (Analytic)"])
    xlabel("x")
    ylabel("y")

    # Mark minima
    scatter(0, 1/sqrt(2), color="r", s=500, zorder=3, marker="x")
    scatter(0, -1/sqrt(2), color="r", s=500, zorder=3, marker="x")

    gcf()
end
```

```
START
k = 1
delta = 0.3
```

```
xhatval = [0.7, 0.7448275862068965]
rho = 0.9855320796996289
delta = 0.183
xhatval = [0.817, 0.7448275862068965]
rho = 0.9783360454267123
delta = 0.11163
xhatval = [0.88837, 0.71163]
rho = 0.9850596830522886
delta = 0.0680943
xhatval = [0.9319057, 0.6680942999999999]
rho = 0.9947717508804039
delta = 0.041537523
xhatval = [0.958462477, 0.641537523]
rho = 0.9981246406560301
delta = 0.041537523
x = [0.958462477, 0.641537523]
k = 2
delta = 0.3
xhatval = [0.658462477, 0.7187527775000918]
rho = 0.9975357747347572
delta = 0.3
x = [0.658462477, 0.7187527775000918]
k = 3
delta = 0.3
xhatval = [0.35846247700000006, 0.7073838506308193]
```

See you next week ✌️

Questions?