TODO: [A] Kurven

[N]: PM, IPM, Shift, Rayleigh-Coeff.

[O]: Klausuranmeldung, Pü3 Testat Moodle!

Globalübung 09
Mathe 3 WS20

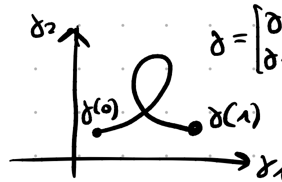
06.01.2021 ①

Lambert Theisen

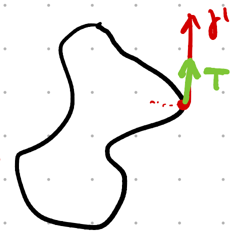
[A]

Kurven: → stetige Abbildung $\gamma : [a,b] \to \mathbb{R}^n$

$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}(t)$



- **Weg**: $\hat{=}$ Bild $\Gamma = \gamma([a,b])$ Γ [γ = Parametrisierung von Γ]
- **Einfachheit**: Falls γ injektiv ("doppelpunktfrei") ↯ nicht einfach
- **Geschlossenheit**: Falls $\gamma(a) = \gamma(b)$ ↯ ✓
- **Einfachheit** ⇐ s.o. Falls γ geschlossen & $\gamma|_{[a,b)}$ injektiv ↯ ✓
- **C^1-Kurve**: Falls γ stetig diff'bar ✓
- **Tangentialvektor im Pkt. $\gamma(t)$**: $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t))^T$



- **Glattheit**: Falls γ eine C^1-Kurve & $\|\gamma'(t)\| \neq 0 \ \forall t \in [a,b]$

Vektor!

- **Tangential $\boxed{\text{einheits}}$ vektor für glatte Kurven**: $T : [a,b] \to \mathbb{R}^n$: $T(t) := \dfrac{\gamma'(t)}{\|\gamma'(t)\|} \Rightarrow \|T(t)\| = 1$ ✓

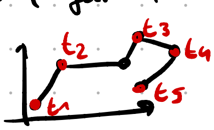
Skalar!

- **Umparametrisierung** $\xi := \gamma \circ \phi : [c,d] \to [a,b] \to \mathbb{R}^n$, $t \mapsto \gamma(\phi(t))$

  → Falls ϕ Diffeomorphismus ("gute Trafo") & $\gamma \in C^1 \Rightarrow \xi \in C^1$

geht nur für glatte Kurven!

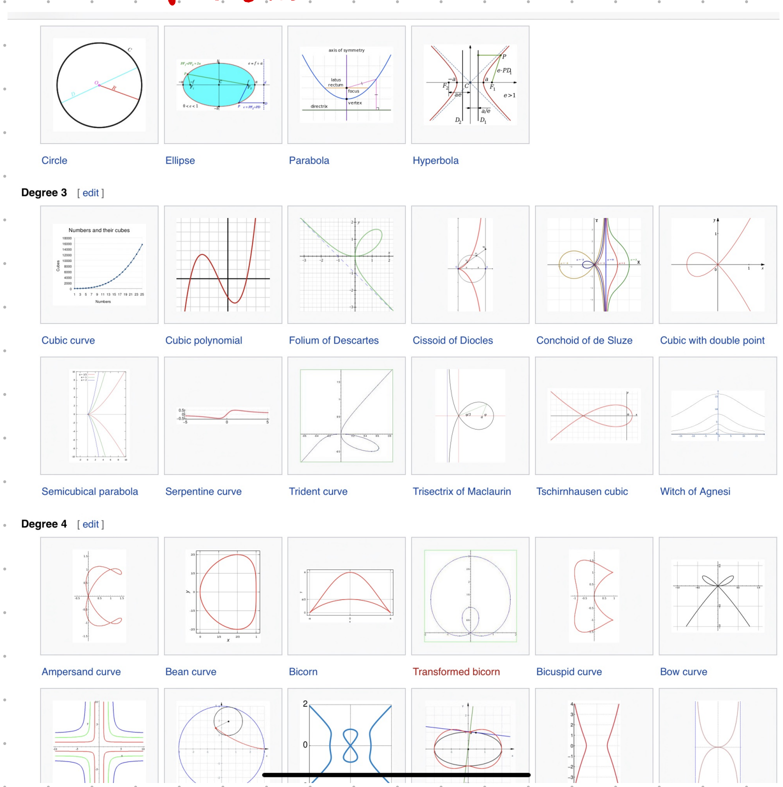
- **Stückweise Eigenschaften**:



Wenn $\tilde{T} := \{t_1, \dots, t_n\}$ & Menge endlich

$|\tilde{T}| = 5$ endlich

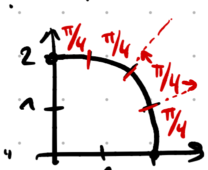
  → zB Polygonzug stückweise glatt



- **Bogenlänge für C^1-Kurven**: $\frac{1}{4} \, 2 \cdot \pi \cdot 2$

  → $L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$

  "Quasi Maßband an Kurve anlegen"



- **Invarianz Bogenlänge unter Umparametrisierung**:

Sei $\phi : [c,d] \to [a,b]$ Diffeomorphismus und $\xi := \gamma \circ \phi$, dann

$$L(\xi) = \int_c^d \|\xi'(s)\| \, ds = \int_a^b \|\gamma'\| \, dt = L(\gamma)$$

ϕ'

$\|\gamma'\| = 1$ wäre schön.... 🤔

Natürliche Parametrisierung (HA/SELF): $\xi : [0, L] \to \mathbb{R}^n$ (γ) ist nat. Param. eines Weges Γ falls $\|\xi'(t)\| = 1$. [Einfaches Ablesen der Bogenlänge]

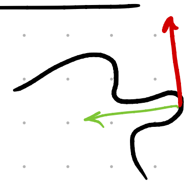
→ Konstruktion: Benutze Diffeomorphismus $\varphi : [a,b] \to [0,L]$

mit $t \mapsto \varphi(t) := \int_a^t \|\gamma'(s)\| \, ds$

$\gamma : [a,b] \to \mathbb{R}$, $\gamma'(s) = 2$

$\varphi(t) = \int_a^t \cancel{t}\; 2 \, ds = 2t - 2a$

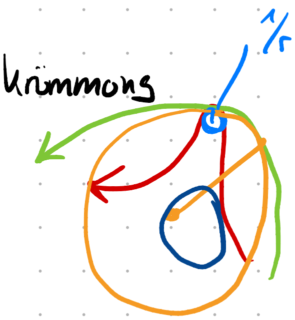
Normaleneinheitsvektor: Sei $T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$ Tang'einh'vektor, dann



$$N(t) := \frac{T'(t)}{\|T'(t)\|}$$ Norm'einh'vektor.

Krümmung: Sei ξ eine natürliche Param., dann ist die Krümmung von ξ in γ:

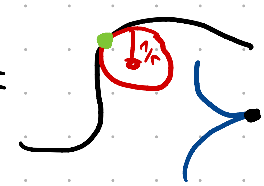
$$\kappa(\gamma) := \|T'(\gamma)\|$$



Kreiskrümmung: $\xi : [0, 2\pi r] \to \mathbb{R}$, $s \mapsto \xi(s) = \begin{bmatrix} r\cos(\frac{s}{r}) \\ r\sin(\frac{s}{r}) \end{bmatrix}$

$$\xi'(s) = \begin{bmatrix} -\sin(\frac{s}{r}) \\ \cos(\frac{s}{r}) \end{bmatrix} \Rightarrow T(t) = \frac{\xi'(t)}{\|\xi'(t)\|_2} = \xi'(t) \text{ weil } \|\xi'(t)\|_2 = \sqrt{\sin^2 + \cos^2} = 1$$

$$\Rightarrow T'(\gamma) = \begin{bmatrix} -\frac{1}{r}\cos(s/r) \\ -\frac{1}{r}\sin(s/r) \end{bmatrix} \Rightarrow \kappa(\gamma) = \sqrt{\tfrac{1}{r^2}(\sin^2 + \cos^2)} = \frac{1}{r}$$



Beispiel: a) Check:

Bdg. muss erst parametrisiert werden!

$x^{2/3} + y^{2/3} = a^{2/3}$ mit $\gamma : [0, 2\pi] \to \mathbb{R}^2$

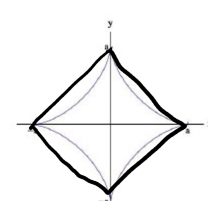
$$[a\cos^3(t)]^{2/3} + [a\sin^3(t)]^{2/3} = a^{2/3}$$

$$\Leftrightarrow a^{2/3}\cos^2(t) + a^{2/3}\sin^2(t) \overset{\Leftrightarrow}{=} a^{2/3} \quad ✓$$

b) Bogenlänge $L(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|_2 \, dt$

Daher $$\gamma'(t) = \begin{bmatrix} \partial/\partial t (a\cos^3(t)) \\ \partial/\partial t(a\sin^3(t)) \end{bmatrix} = \begin{bmatrix} -3a\cos^2(t)\sin(t) \\ 3a\sin^2(t)\cos(t) \end{bmatrix}$$

Aufgabe 68. (Bogenlänge einer Kurve)



The so-called astroid (Sternkurve) is described by the Cartesian equation

$$x^{2/3} + y^{2/3} = a^{2/3}$$

for some $a > 0$.

a) Verify that the astroid can be expressed in parametric form $\gamma : [0, 2\pi] \to \mathbb{R}^2$ for

$$x(t) = a\cos^3 t \quad ; \quad y(t) = a\sin^3 t$$

b) Find the length of the astroid.

c) Find the area of the astroid using the change of variables $F : (x,y) \mapsto (r, \varphi)$ defined as

$$x = r\cos^3\varphi \quad , \quad y = r\sin^3\varphi$$

Begin by verifying that

$$J(r,\varphi) = \left\| \begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{matrix} \right\| = \frac{8}{3} r (1 - \cos 4\varphi)$$

and notice that we can describe the astroid in terms of the coordinates (r, φ) as $0 \le r \le a$ and $0 \le \varphi \le 2\pi$.

$$\Rightarrow \|\gamma'(t)\|_2 \overset{\boxed{a>0}}{=} 3a\sqrt{(\cos^2\sin)^2+(\sin^2\cos)^2} = 3a\sqrt{\underbrace{(\cos^2+\sin^2)}_{=1}(\sin^2\cos^2)}$$

$$\Rightarrow L(\gamma) = \int_0^{2\pi} \|\gamma'(t)\| dt = 3a\int_0^{2\pi} \sqrt{\sin^2(t)\cos^2(t)}\, dt$$

$$= 3a\int_0^{2\pi} \Big(\big[\underbrace{\sin(t)\cos(t)}_{=\frac{1}{2}\sin(2t)}\big]^2\Big)^{1/2} dt = 3a\int_0^{2\pi} \left|\tfrac{1}{2}\sin(2t)\right| dt$$

$$= 12a\int_0^{\pi/2} \tfrac{1}{2}\sin(2t)\, dt = 12a \cdot \tfrac{1}{2}\pi = 6a\pi$$

Check?!

c) **Fläche** (nicht so wichtig) siehe Tips

$$A = \int_0^a \int_0^{2\pi} 1 \cdot J(r,\varphi)$$

$$= \int_0^a \int_0^{2\pi} \tfrac{3}{8} r(1-\cos(4\varphi))\, dr d\varphi$$

$$= \ldots = \tfrac{3}{8}\pi a^2$$

N **Poweriteration (Vektoriteration):**

$\lambda_1 = \max(\sigma(A))$

- Let $A \in \mathbb{R}^{n\times n}$
- Choose $x^0 \in \mathbb{R}^n \setminus \{0\}$
- While ($k \le k_{max} \wedge |\rho_A(x^i) - \rho_A(x^{i+1})| > \varepsilon$)
  - $\tilde{x}^k = Ax^{k-1}$ // iterate
  - $x^k = \tilde{x}^k / \|\tilde{x}^k\|$ // normalize
  - $\lambda^k = \rho_A(x^k)$ // eval estimate
- Return (λ^k, x^k)

"Mehrmaliges Anwenden der Matrix"

$x_1 = Ax_0$

$x_2 = A\cdot x_1$

(Rayleigh Quotient s.b.)

Aus x Näherung $\Rightarrow \lambda$ Näherung

$$\boxed{q = \left|\frac{\lambda_{r+1}}{\lambda_1}\right|}$$

→ Convergence (usually):

I) $\lim_{k\to\infty} \lambda^k \to \lambda_1$ Fehler: $e \in O(q^{2k})$

II) $\lim_{k\to\infty} x^k \to x^1 \subset E(\lambda_1)$ $e \in O(q^k)$

Rayleigh-Quotient: $[\rho_A(x) =]\ R(x) := \dfrac{x^TAx}{x^Tx}$ für A symmetrisch

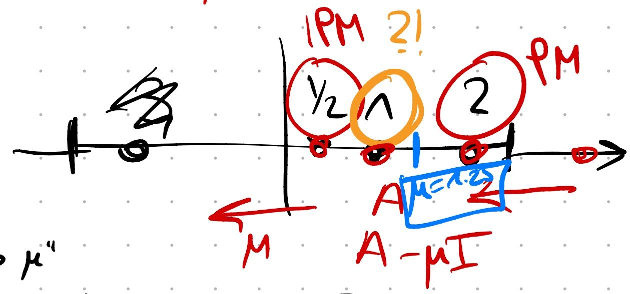
$$\lambda_n = \inf_{x\in\mathbb{R}^n/\{0\}} R(x) \le R(x) \le \max_{x\in\mathbb{R}^n/\{0\}} R(x) = \lambda_1$$ (Rayleigh Prinzip)

Finde Eigenwert zu gegebenem Eigenvektor: $\boxed{R(v_j) = \lambda_j \quad \forall j}$

(Shifted) Inverse Power Iteration: ("PM für A^{-1} findet kleinsten EWert")

One Variant:

Let $A \in \mathbb{R}^{n \times n}$ and Shift μ
Choose $x^0 \in \mathbb{R}^n \setminus \{0\}$
While ("not finished") do
  Solve $(A - \mu I) x^{k+1} = x^k$ ("$\Leftrightarrow$" $x^{k+1} = (A - \mu I)^{-1} x^k$)
  Normalize x^{k+1}
  $\lambda^{k+1} = \rho_A(x^{k+1})$
Return (λ^k, x^k)



→ Shifted IPM konvergiert zu $\lambda_{\text{"closest to } \mu\text{"}}$

→ Nützlich wenn Näherung bekannt oder für Nicht-extreme EWerte.

Demo:

→ Analysis: Kurven → Numerik: PM, IPM, SIPM

Define a Curve

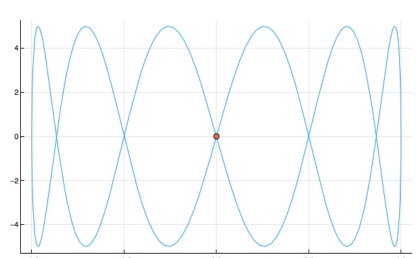
Define the curve γ:

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, t \mapsto \gamma(t) = (\sin(t/3), 5\sin(2t))^T$$

```
γ (generic function with 1 method)
γ(t) = [sin(t/3), 5sin(2t)]
```

0.0

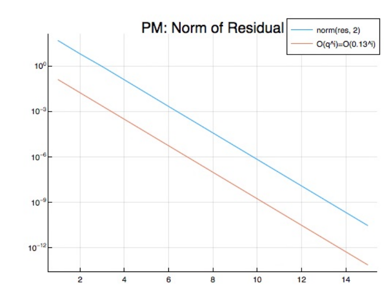
```
@bind t Slider(0:π/20:2π+π/20, show_value=true)
```



```
begin
    plot(t->γ(t)[1], t->γ(t)[2], 0, 6π, leg=false)
    scatter!([γ(t)[1]], [γ(t)[2]])
end
```

Residual Error

The normed error of the residual $e = ||x_k - x^*||_2$ is in $\mathcal{O}(q^k)$ with the eigenvalue ratio $q = \lambda_1/\lambda_2$ of the considered matrix.

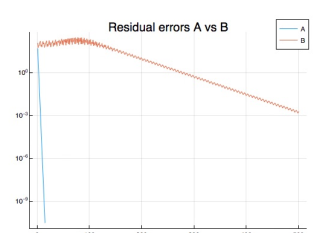



```
plot([pm[3].errors, [1 * abs(eigvals(A)[end-1] / eigvals(A)[end])^i for i = 1 :
size(pm[3].errors)[1]]], yaxis=:log, title="PM: Norm of Residual", label=["norm(res,
2)" "O(q^i)=O($(@sprintf("%.2f", abs(eigvals(A)[end-1] / eigvals(A)[end])))^i)"])
```

Comparison of Matrices with Different Fundamental Ratios

- Matrix A has ratio $q =$ 0.13333333333333336
- Matrix B has ratio $q =$ 0.9718253158075516

Therefore, much faster converge for A.

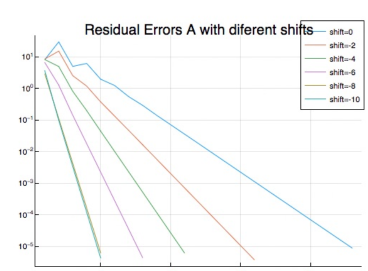



```
plot(map(X -> PM(X, ones(3), maxit=500)[3].errors, [A, B]), yaxis=:log, label=["A"
"B"], title="Residual errors A vs B")
```

Check the Convergence Behavior for Differen Shifts

Notice that better shifts significantly improve the performance of the algorithm. Shift eight and t are the same because they have the same absolute distance to the real eigenvalue.

```
shifts = 0:-2:-10
# Compare zeros shift with good estimation
shifts = 0:-2:-10 # real lowest eval is -9
```




```
plot(map(X -> IPM(A, ones(3), tol=1E-5, shift=X)[3].errors, shifts), yaxis=:log,
label=reshape(map(x -> string("shift=", x), shifts), 1, :), title="Residual Error
with diferent shifts")
```