Chapter 5: Initial-Value Problems for Ordinary Differential Equations.

Numerical methods for solving IVPs
- euler
- midpoint
- modified Euler method
- Runge Kutta

5.2 Euler's Method

IVP $\frac{dy}{dt} = f(t,y)$ $\quad a \le t \le b, y(a)=\alpha, \alpha \in \mathbb{R}$

The object of Euler's method is to obtain approximations to the initial-value problem:

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

A continuous approximation to the solution $y(t)$ will not be obtained; instead, approximation to y will be generated at various values, called mesh points, in the interval $[a,b]$. Once the approximate solution is obtained at the points, the approximate solution at other points in the interval can be found by interpolation.

We first make the condition that the mesh points are equally distributed throughout the interval $[a,b]$. This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0,1,2,\dots,N.$$

The common distance between the points $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ is called the step size.

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We will use Taylor's Theorem to derive Euler's method:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i) \, ; \qquad \xi_i \in (t_i, t_{i+1})$$

one indep variable

dep → y indep → t

$\frac{dy}{dt} = t$, $y^2 y' = e^t$, $y'' + 2y' + y = 0$

1st order ode

Ex $\frac{dy}{dt} = 2t$

$dy = 2t\,dt$

$y = \frac{2t^2}{2} + c$

The analytic sol.

$y = t^2 + c$

approximation sol.

Initial value problems for odes

① $\frac{dy}{dt} = f(t,y)$, ② $a \le t \le b$ ③ $y(a) = \alpha$

$y(5) = 0.67853$

$t_i = a + ih$
$i = 0, 1, \ldots, N$ "mesh point"

* N
* $h = \frac{b-a}{N}$ "step size"

Euler's method constructs $w_i \approx y(t_i)$, for each $i = 1, 2, \ldots, N$, by deleting the remainder term. Thus Euler's method is

$w_0 \approx y(t_0)$, ⋮, $w_N \approx y(t_N)$ $\quad h = \frac{b-a}{N}$ $\quad \frac{dy}{dt} = f(t, y)$

$$(5.8) \quad \begin{cases} w_0 = \alpha, & (= y(t_0)) \ ① \\ \underbrace{w_{i+1}}_{y(t_{i+1})} = w_i + h f(t_i, w_i), & \text{for each } i = 0, 1, \ldots, N-1. \end{cases}$$

Illustration:

In Example 1 we will use an algorithm for Euler's method to approximate the solution to

euler's meth

$$① \; y' = y - t^2 + 1, \qquad ② \; 0 \le t \le 2, \qquad ③ \; y(0) = 0.5,$$

$y(2) = ?$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

$w_0 = \alpha$

$w_{i+1} = w_i + h f(t_i, w_i), \; i = 0, \ldots, N-1$

to find mesh points $t_i = a + ih, \; i = 0, \ldots, N$

$t_0 = 0 + 0h = 0$

$t_1 = 0 + 1h = 0.5$

$t_2 = 0 + 2h = 1$

$t_3 = 0 + 3h = 1.5$

① $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$, $t_3 = 1.5$, $t_4 = 2$

② $f(t_i, w_i) = w_i - t_i^2 + 1$

③ Euler method

$\rightarrow i = 0 \quad t_0 = 0 \quad w_0 = 0.5$

we need to find w_1, $i = 1$, $t_1 = 0.5$

$w_1 = w_0 + h f(t_0, w_0)$

$= 0.5 + 0.5(w_0 + t_0^2 + 1)$

$= 0.5 - 0^2 + 1$

$w_1 = 1.25 \approx y(t_1) = y(0.5)$

to find w_2 $\; i = 2 \;$ $t_2 = 1$

$1.25 + (0.5)(w_1 - t_1^2 + 1) = 2.25 \approx y(t_2)$

To find w_3, $i = 3$, $t_3 = 1.5$ $\qquad i+1 \to 3$, $i \to 2$

$w_3 = w_2 + h f(t_2, w_2)$

$= 2.25 + 0.5(w_2 - t_2^2 + 1)$

$= 2.25 + 0.5(2.25 - 1 + 1)$

$w_3 = 3.375 \approx y(t_3) = y(1.5)$

$i \to 4$, $\to 3$

To find w_4, $i = 4$, $t_4 = 2$

$w_4 = w_3 + h f(t_3, w_3)$

$= 3.375 + 0.5(w_3 - t_3^2 + 1)$

$= 3.375 + 0.5(3.375 - 1.5^2 + 1)$

$w_4 = 4.4375 \approx y(t_4) = y(2)$

∴ The approximation solution $y(2) \approx 4.4375$

$f(t_i, w_i) = w_i - t_i^2 + 1$

Algorithm 5.1 implements Euler's method. p 267.

Example 1:

Use Algorithm 5.1 with $N = 10$ to determine approximations to the problem $y' = y - t^2 + 1$, $0 \le t \le 2$, $y(0) = 0.5$.

and compare these approximations with the exact values given by $y(t) = (t+1)^2 - 0.5e^t$.

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$y' = y - t^2 + 1$

① $t_1 \; t_2 \; t_3$, we need to find the

$t_0 = 0 \quad 0.2, \; 0.4 \quad 0.6 \qquad t_N = 2$

mesh points: $N = 10$, $h = \frac{2-0}{10} = \frac{2}{10} = 0.2$,

$t_i = a + ih = 0 + ih$

$t_0 = 0$

$t_1 = 0 + h = 0.2$

$t_2 = 0 + 2h = 2(0.2) = 0.4$.

$t_3 = 3(0.2) = 0.6$

$t_4 = 0.8$

$t_5 = 1.0$

$t_6 = 1.2$

$t_7 = 1.4$

$t_8 = 1.6$

$t_9 = 1.8$

$t_{10} = 2$

② $f(t_i, w_i) = w_i - t_i^2 + 1$

③ To find w_0, $i = 0$, $t_0 = 0$, $\overset{y(0)=}{w_0} = 0.5$

To find w_1, $i = 1$, $t_1 = 0.2$

$w_1 = w_0 + h f(t_0, w_0)$

$= 0.5 + 0.2\,(w_0 - t_0^2 + 1)$

$= 0.5 + 0.2\,(0.5 - 0^2 + 1)$

$w_1 = 0.5 + 0.2(1.5) = 0.8 \approx y(t_1) \approx y(0.2)$

$w_1 \rightarrow$ app solution

$y(0.2) \rightarrow$ exact solution

$y_{exact}(0.2) = (0.2+1)^2 - 0.5e^{0.2} = 0.8292986$

$\Rightarrow |w_1 - y_{exact}(0.2)| = |0.8 - 0.8292986| = 0.0292986$

To find w_2, $i = 2$, $t_2 = 0.4$

$w_2 = w_1 + h f(t_1, w_1)$

$= 0.8 + 0.2\,(w_1 - t_1^2 + 1)$

$= 0.8 + 0.2\,(0.8 - 0.2^2 + 1)$

$w_2 = 1.152 \approx y(t_2) = y(0.4)$

$y_{exact}(0.4) = (0.4+1)^2 - 0.5e^{0.4} = 1.2140877$

$\Rightarrow |w_2 - y_{exact}(0.4)| = |1.152 - 1.2140877|$

Table 5.1

t_i	w_i	$y_i = y(t_i)$	$\|y_i - w_i\|$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

5.4 Runge-Kutta Methods

t_i $\quad t+\frac{h}{2}$ $\quad t_{2+1}$

w_i $\quad\quad\quad t_{i+1} = ?$

Midpoint Method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2},\ w_i + \frac{h}{2} f(t_i, w_i)\right); \text{ for } i = 0, 1, \ldots, N-1.$$

① ②

Modified Euler Method

$$w_0 = \alpha,$$

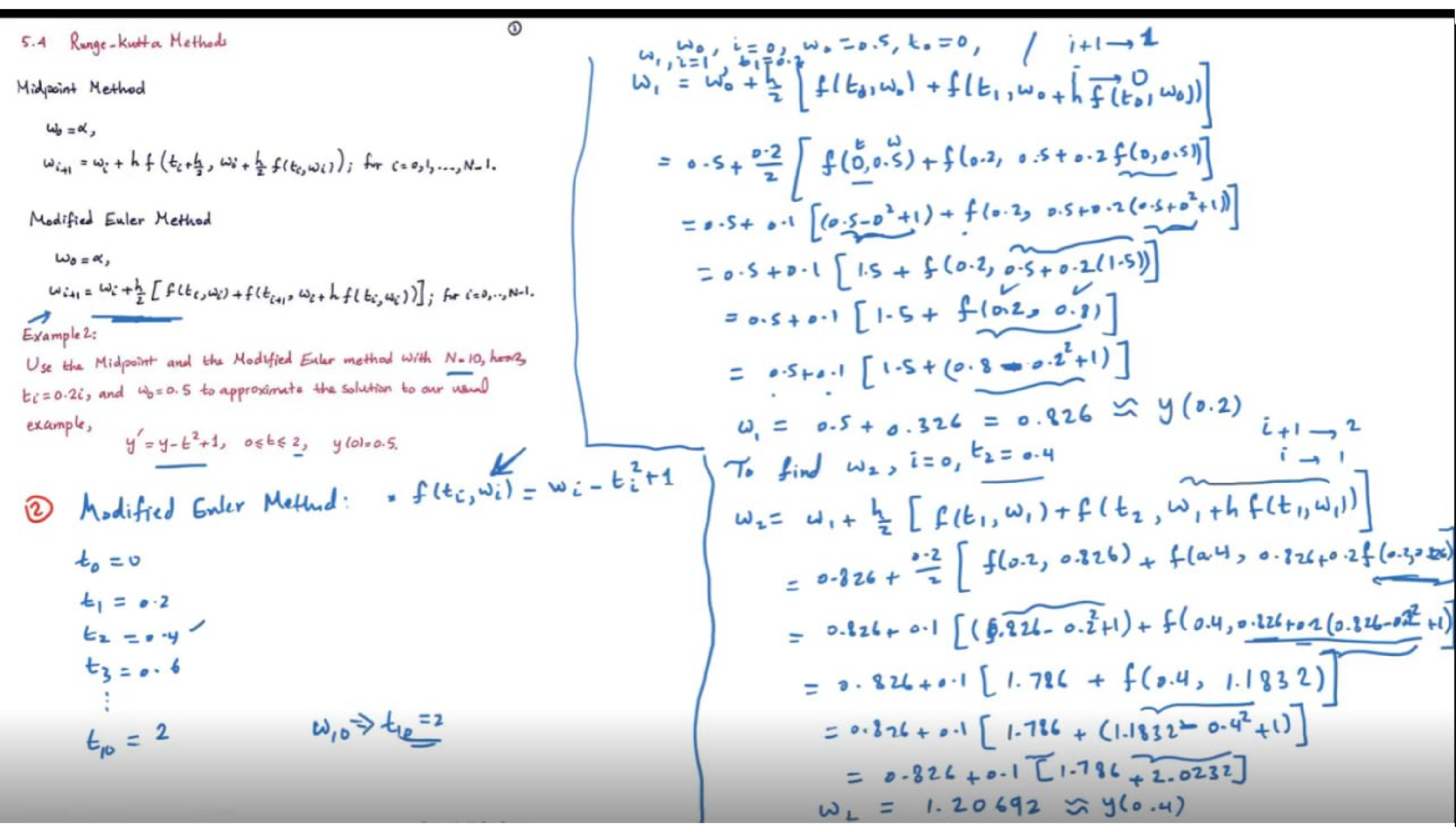
$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right]; \text{ for } i = 0, \ldots, N-1.$$

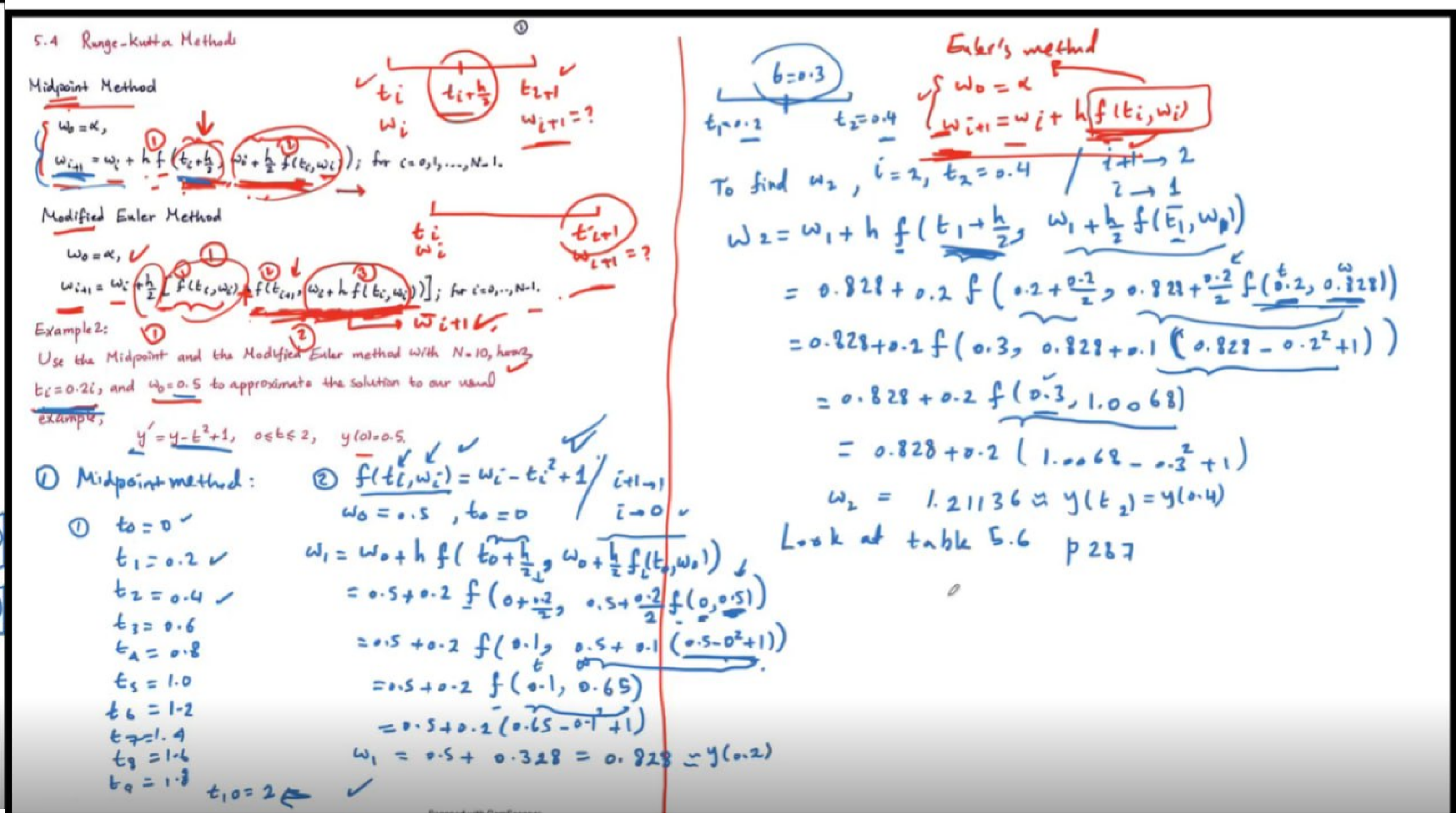
① ② ③

Example 2:

Use the Midpoint and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$





t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

Runge-kutta Order Four

$w_{i+1} \simeq y(t_{i+1})$

$w_i \simeq y(t_i)$

$$w_0 = \alpha, \quad y(a) = \alpha$$

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$

$$k_4 = hf(t_{i+1}, w_i + k_3),$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

for each $i = 0, 1, \ldots, N-1$. Algorithm 5.2 implements the Runge-kutta method of order four.

t_i $t_i + h/2$ t_{i+1}

w_i $w_i + \frac{1}{2}K_1$ $w_{i+1} \approx ?$

$w_i + 1/2\, K_2$ $w_i + K_3$

Example3:

Use Runge-kutta method of order Four with $h = 0.2$, $N = 10$, and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

① $t_0 = 0$ —— $t_{10} = 2$

$t_i = a + ih$

$t_0 = a + 0h = 0$

- $t_1 = 0 + 0.2 = 0.2$
- $t_2 = 0.2 + 0.2 = 0.4$
- $t_3 = 0.4 + 0.2 = 0.6$
- $t_4 = 0.6 + 0.2 = 0.8$
- $t_5 = 1.0$
- $t_6 = 1.2$
- $t_7 = 1.4$
- $t_8 = 1.6$
- $t_9 = 1.8$
- $t_{10} = 2$

② $f(t_i, w_i) = w_i - t_i^2 + 1$

③ $i = 0 \to t_0 = 0, w_0 = 0.5$

To find w_1, $i = 1$, $t_1 = 0.2$

$K_1 = hf(t_0, w_0) = 0.2 f(0, 0.5) = 0.2[0.5 - 0 + 1] = 0.3$

$K_2 = hf(t_0 + h/2, w_0 + 1/2 K_1) = 0.2 f(0.1, 0.65)$
$= 0.2[0.65 - 0.1^2 + 1] = 0.328$

$K_3 = hf(t_0 + \frac{h}{2}, w_0 + \frac{1}{2}K_2) = 0.2 f(0.1, 0.5 + \frac{1}{2}\, 0.328)$

$= 0.3308$

$= 0.2 f(0.1, 0.664) = 0.3308$

$k_4 = hf(t_1, w_0 + k_3) = 0.2 f(0.2, 0.5 + 0.3308)$
$= 0.2 f(0.2, 0.8308)$
$= 0.2[0.8308 - 0.2^2 + 1] = 0.35816$

$i \to 1$, $i+1 \to 2$; $t_1 = 0.2$, w_1; 0.3; $t_2 = 0.4$, $w_2 = ?$

$\omega_1 = \omega_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816)$

$\boxed{\omega_1 = 0.8292933 \simeq y(t_1)}$ $y(0.2)$

To find w_2, $i = 2$, $t_2 = 0.4$

$k_1 = hf(t_1, w_1) = 0.2 f(0.2, 0.8292933)$
$= 0.2[0.8292933 - 0.2^2 + 1] = 0.35785866$

$k_2 = hf(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_1) = 0.2 f(0.2 + \frac{0.2}{2}, 0.8292933 + \frac{1}{2}(0.35785866))$
$= 0.2 f(0.3, 1.00822263) = 0.2[1.00822263 - 0.3^2 + 1]$
$= 0.38364526$

$k_3 = hf(t_1 + \frac{h}{2}, w_1 + \frac{1}{2}k_2)$
$= 0.2 f(0.3, 0.8292933 + \frac{1}{2}(0.38364526)) = 0.38622311226$

$k_4 = hf(t_2, w_1 + k_3) = 0.2 f(0.4, 0.8292933 + 0.38622311226)$
$= 0.41110032825$

$\omega_2 = \omega_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8292933 + \frac{1}{6}(\text{---}) = 1.2140767$
$\simeq y(t_2) \simeq y(0.4)$

Table 5.8

t_i	Exact $y_i = y(t_i)$	Runge-Kutta Order Four w_i	Error $\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324000	3.7323401	0.0000599
1.6	4.2834838	4.2834095	0.0000743
1.8	4.8151763	4.8150857	0.0000906
2.0	5.3054720	5.3053630	0.0001089