Stochastic Non-Cooperative Games

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Computation Issues in Stochastic Processes
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Outline

- Introduction
- Bounded Rationality: The Minority Game. Results
- Non-rationality: the Thermal Minority Game.
- Numerical Methods
- Conclusions

Introduction

Game

- N players
- s strategies
- Payoff/utility function u.

Game theory

- Mathematical Theory (John von Neumann, 1944)
- Market design, microeconomy, evolutionary biology, etc.

Standard (Equilibrium) theory

Game Theory	**Statistical Mechanics**
Nash Equilibrium	Thermodynamical Equilibrium
Replicator dynamics	Model A, Model B, etc.

J. Berg and A. Engel, Phys. Rev. Lett **81**, 1998

Agent-based (non-equilibrium) models like:

- The **Santa Fe Institute Stock Market Model**
  W. B. Arthur *et al.*
- The **Doyne Farmer Model** J. D. Farmer, 1998
- The **Minority Game** D. Challet and Y.-C. Zhang, 1997

Game Theory	**Statistical Mechanics**
Bounded Rationality Learning, Evolution	Non-equilibrium Absence of detailed Balance

AIM:
Apply techniques/ideas of statistical physics to Game Theory

Introduction (II)

Markets are complex systems because:

1. Strategies: Each agent plays in the market with a different set of strategies (quenched disorder)
2. Non-cooperative behaviour: each agent wants to maximize its own utility function (frustration)

Markets are non-equilibrium problems because:

3. Learning process gives non-Markovian effects.
4. Non-rational players give stochasticity.

Minority Game = 1. + 2. + 3.
Thermal Minority Game = Minority Game + 4.

Related problems:

- Spin-glasses (1. + 2.)
- Neural networks (1. + 2. + 3.).

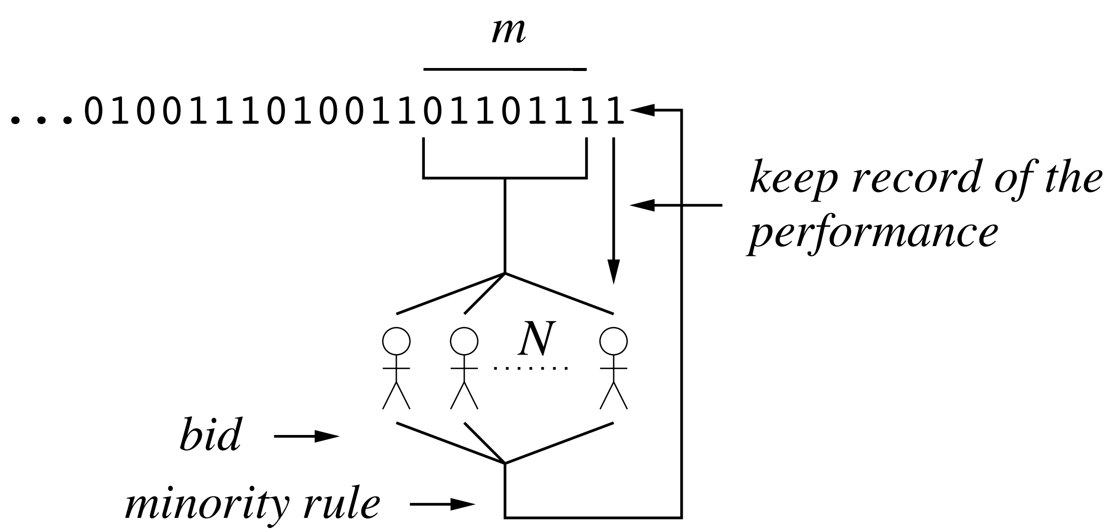
Main goals of agent-based models

- Test rationality
- Understand why/when markets are efficient

Bounded Rationality: Minority Game (I)

Arthur's "El Farol" bar problem '94

Challet and Zhang '97,



1. **Agents**: $i = 1, \ldots, N$ agents able to go to room 0 or room 1 (buy/sell) at each time step.
2. **External information**, I, available to all the agents.
3. **Strategies**:
   - s strategies $\{R_i^1, R_i^2, \ldots, R_i^s\}$
   - Random functions of I, $R_i^\alpha : I \to \{0, 1\}$.
   - In general $R_i^\alpha \neq R_j^\beta$.
   - At each time step each agent chooses one of its strategies to decide which room to go, R_i^*.
4. **Non-cooperative**.
   - **Minority rule**: at each time step the winning room is the less crowded.
   - The public information is the sequence of winning rooms.

Bounded Rationality: Minority Game (II)

5. **Learning**:
   - Agents can only analyze the last m winning groups (brain size m).
   - Each agent only chooses one of its strategies to decide which room to go: the **best strategy** i.e. the one with the maximum number of virtual points.
   - After each time step, each agent keeps record of its strategies performance: it gives a **virtual point** to each of its strategies if they have been able to predict the next winning group.

- Last m winning rooms: $I(t)$.
- Action of agent i, $R_i^*(I(t)) \in \{0, 1\}$
- Total action

$$A(t) = \sum_{i=1}^{N} R_i^*(I(t))$$

- Minority rule:

$$W(t+1) = \Theta(N/2 - A(t))$$

- Virtual points:

$$P_i^\alpha(t+1) = P_i^\alpha(t) + \delta_{W(t+1), R_i^\alpha(I(t))}$$

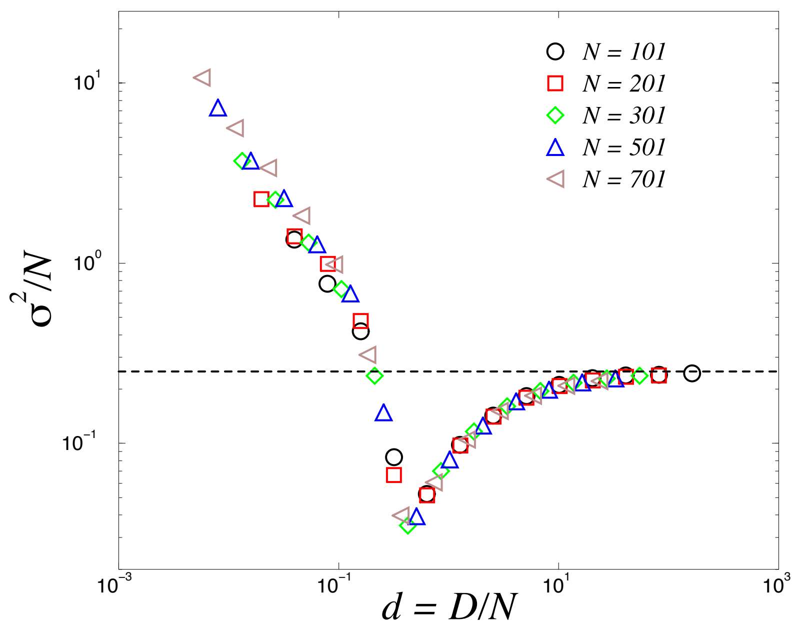
- Best strategy

$$R_i^* = \{R_i^\alpha | P_i^\alpha \geq P_i^\beta \ \forall \beta = 1, \ldots, s\}$$

- We have 2^{2^m} possible random boolean functions R_i^α.

Results

1. Attendance to room 1: $A(t) = \sum_{i=1}^{N} R_i^*(I(t))$
   We get that $\langle A(t) \rangle = N/2$
2. Variance of the attendance, $\sigma^2 = \langle [A(t) - \langle A(t) \rangle]^2 \rangle$.
   If the agents chose their strategies randomly, then $\sigma_r^2 = N/4$



Savit, Manuca and Riolo, PRL '99

The relevant parameter is $d = D/N$ where $D = 2^m$ (*the reduced dimension*) (scaling)

Two phases:

- $d < d_c$ efficient phase. (non-cooperative)
- $d > d_c$ inefficient phase. (cooperative)

The system does not reach full rationality, Nash equilibrium.

Non-rationality: Thermal Minority Game (I)

A. Cavagna, J. P. Garrahan, I. Giardina and D. Sherrington, Phys. Rev. Lett. **83**, 4429 (1999)

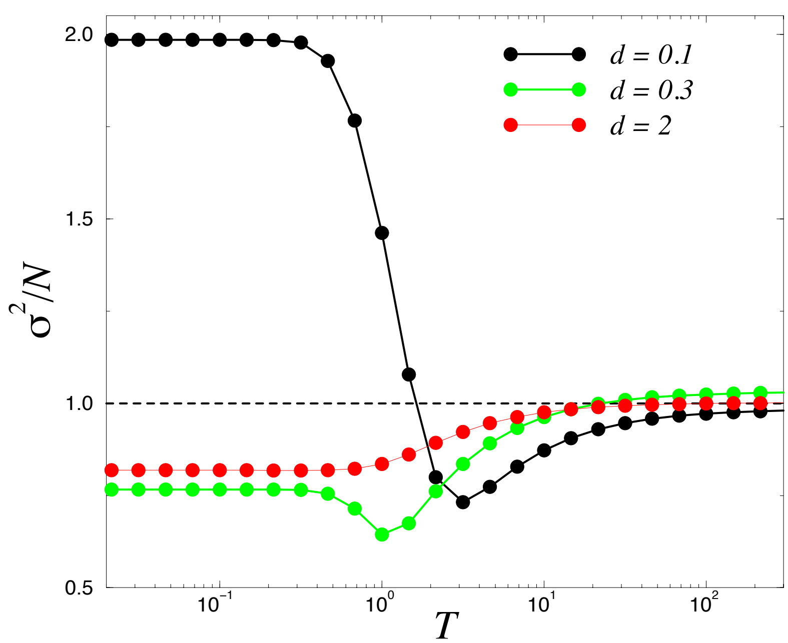
Mixed strategies

- At each time step, each agent chooses its best strategy according to some probability π_i^α.

$$\pi_i^\alpha = \exp(-\beta P_i^\alpha)$$

- $\beta = 1/T$, where T is a parameter that controls the stochasticity (temperature)

Towards cooperation:



Non-rationality: Thermal Minority Game (II)

J. P. Garrahan, E. Moro and D. Sherrington,
Phys. Rev. E (Rapid Comm.) **62**, (2000)

- Only two strategies $s = 2$.
- $p_i(t) = P_i^1(t) - P_i^2(t)$.
- $\sigma_i(t) = \mathrm{sgn}(p_i(t))$

Diffusion approximation + continuous time

$$\boxed{dp_i = -\frac{\delta \mathcal{H}}{\delta s_i} dt + \sum_{j=1}^{N} M_{ij} \circ dW_j(t)}$$

where

$$s_i(t) = \tanh(\beta \sigma_k(t))$$

$$M_{ij} = \sum_{k=1}^{N} J_{ik} J_{jk} [1 - s_k^2(t)]$$

$$\mathcal{H} = \sum_{i=1}^{N} h_i s_i + \frac{1}{2} \sum_{i,j} J_{ij} s_i s_j$$

- Although the system reaches a stationary state in the long run, in general is not in equilibrium (even for $T = 0!$).
- But for some values of $d = D/N$ the stationary state is well described by $\mathcal{H}$.
- Thus, the dynamics of learning are not just relaxational.
- But, how is the dynamics in the space of the strategies? Neural Networks

Numerical Methods

In General

- **Equilibrium**: Calculating the Nash equilibrium is a hard problem like
  - spin-glass minima.
  - optimization problems.

  But in general, there is not a Hamiltonian to minimize (J. Berg and A. Engel).
- **Non-equilibrium**: Agent-based simulations. Simulations by brute force, mainly

For the Minority Game

- Standard stochastic algorithms.
- Typical numbers are
  - $N = 500$,
  - $m = 1, \ldots, 10$, so $2^{2^m} = 4, \ldots, 10^{300}$
- But, once we know that the system tends to minimize a function we could improve the algorithms using
  - Simulated annealing
  - Entropic Sampling method
- We have seen that the stochasticity helps the system to cooperate

Conclusions

General

- Agent-based models could help to understand some features of real markets.
- And to test ideas like rationality and efficiency.
- Tools and concepts from statistical mechanics could be useful to describe some features of these games.
- We have studied one of these models: the Minority Game
  - We have been able to derive the simplest set of equations for the dynamics using stochastic processes tools.
  - The system is not in equilibrium (not even for $T = 0$), at least not the one described by $\mathcal{H}$.
  - We have also check the dependence on the initial conditions, the character of the phase transition, etc.

Further work

- **Analytical**: Is it there any other Lyapunov function $\mathcal{H}'$? The dynamics in terms of the strategies are like parallel artificial neural networks with memory effects.
- **Analytical**: Like in statistical mechanics which are the macroscopic parameters that describe completely the system? ($\sigma +$?).
- **Numerical**: Temperature/non-rationality helps agents to cooperate up to some value of T. Could we increase this cooperation by using simulated-annealing techniques?
- **Theoretical**: Apply tools from statistical mechanics (replica trick) to calculate Nash equilibrium in general matrix games.