From finite to infinite populations

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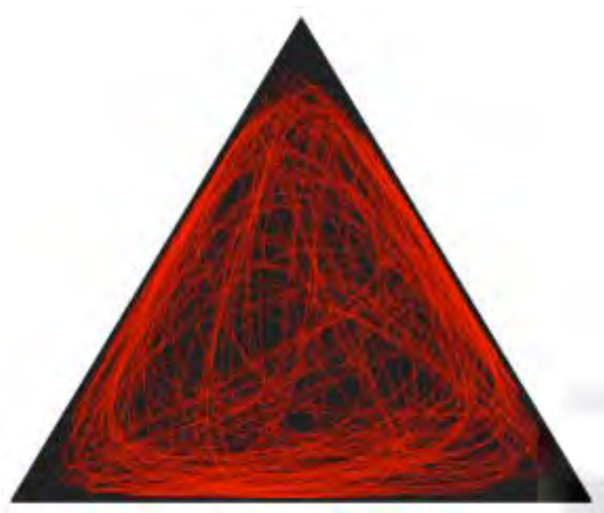


1. Replicator dynamics
2. Moran process
3. From finite to infinite populations
4. (Fermi process)

Games

Player 2

Player 1

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

Maynard Smith, Evolution and the Theory of games (1981)

Evolutionary game dynamics

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

Populations

Payoff=Reproductive fitness

Evolutionary dynamics

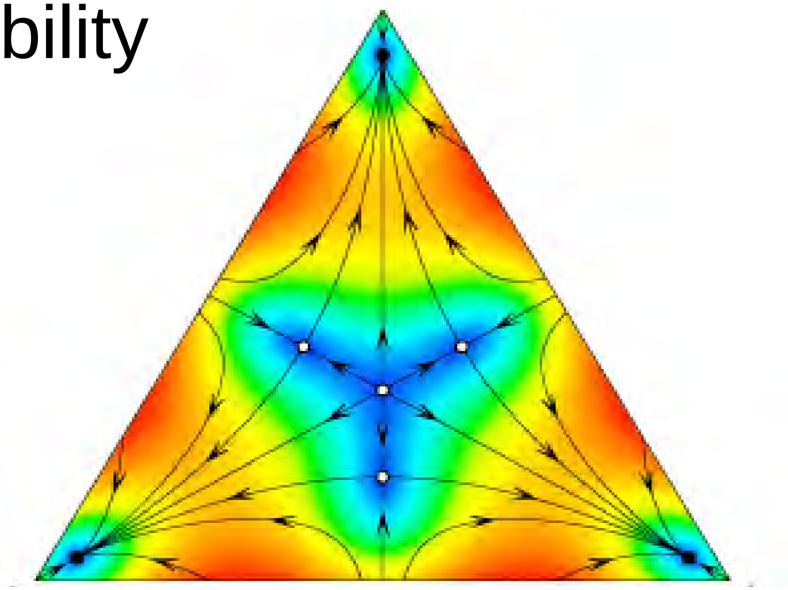
Fitter individuals produce more offspring

Maynard Smith, Evolution and the Theory of games (1981)

1. Replicator dynamics

$$\dot{x}_i = x_i \left(\pi_i - \langle \pi \rangle\right)$$

- Fixed points & their stability

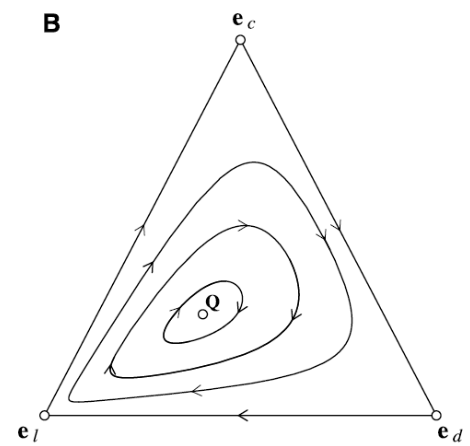


Taylor & Jonker, Math. Biosci. (1978); Hofbauer & Sigmund, Evolutionary Games and Population Dynamics (1998)

Replicator dynamics

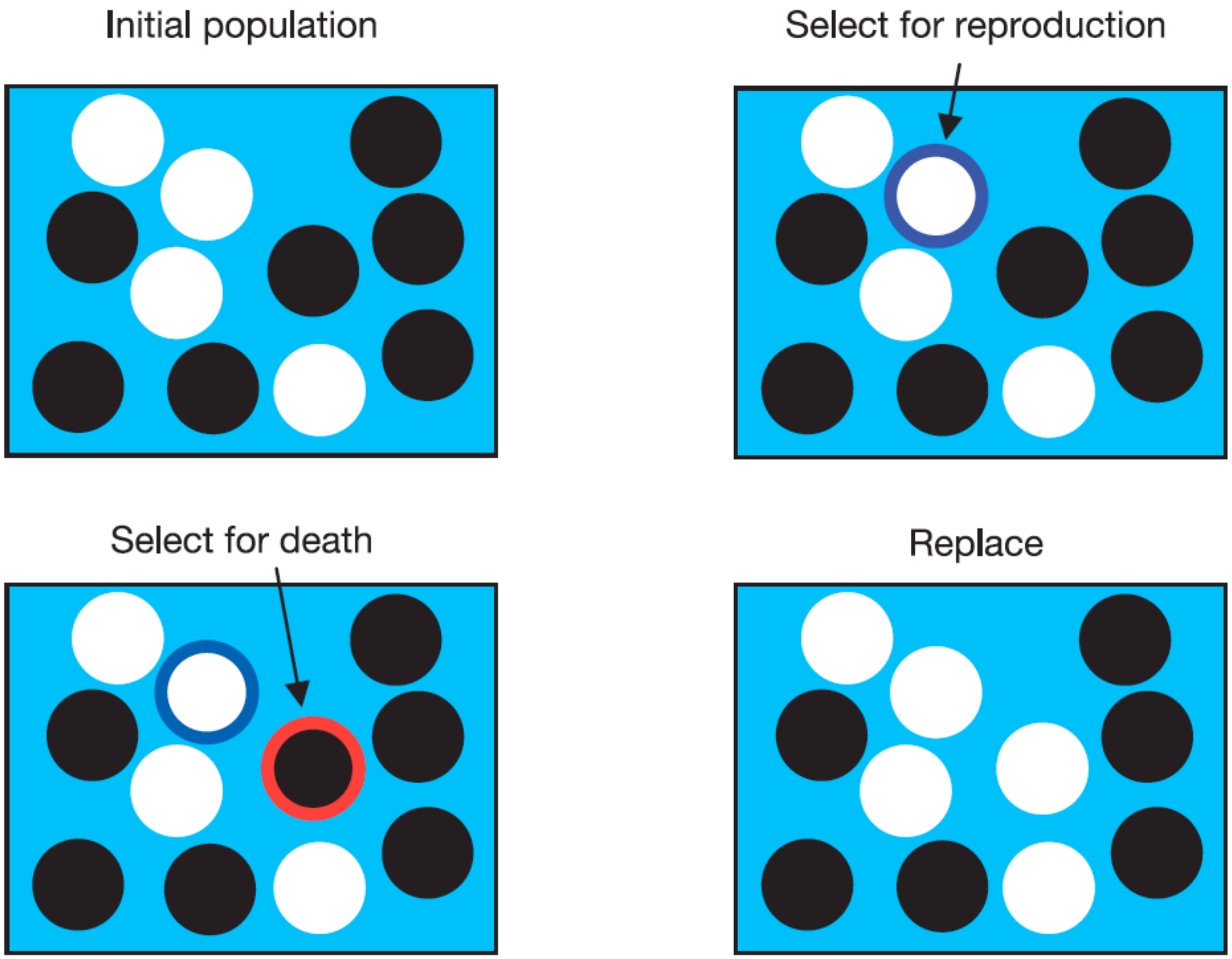
$$\dot{x}_i = x_i\left(\pi_i - \langle\pi\rangle\right)$$

- Constants of motion



- Deterministic Chaos

2. Moran process in finite populations

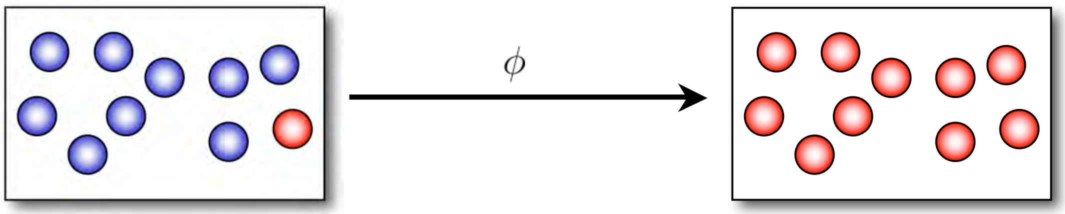


Nowak, Sasaki, Taylor, and Fudenberg, Nature 428, 646 (2004).

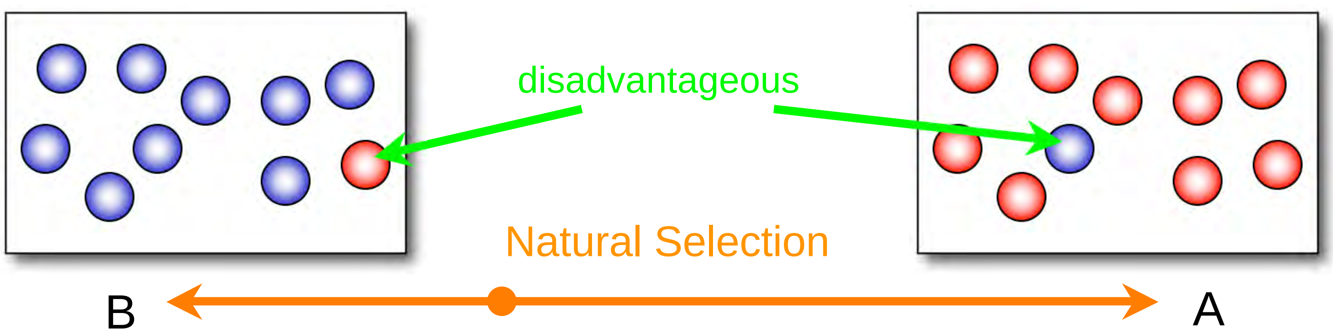
Results for the Moran process

- Fixation probabilities

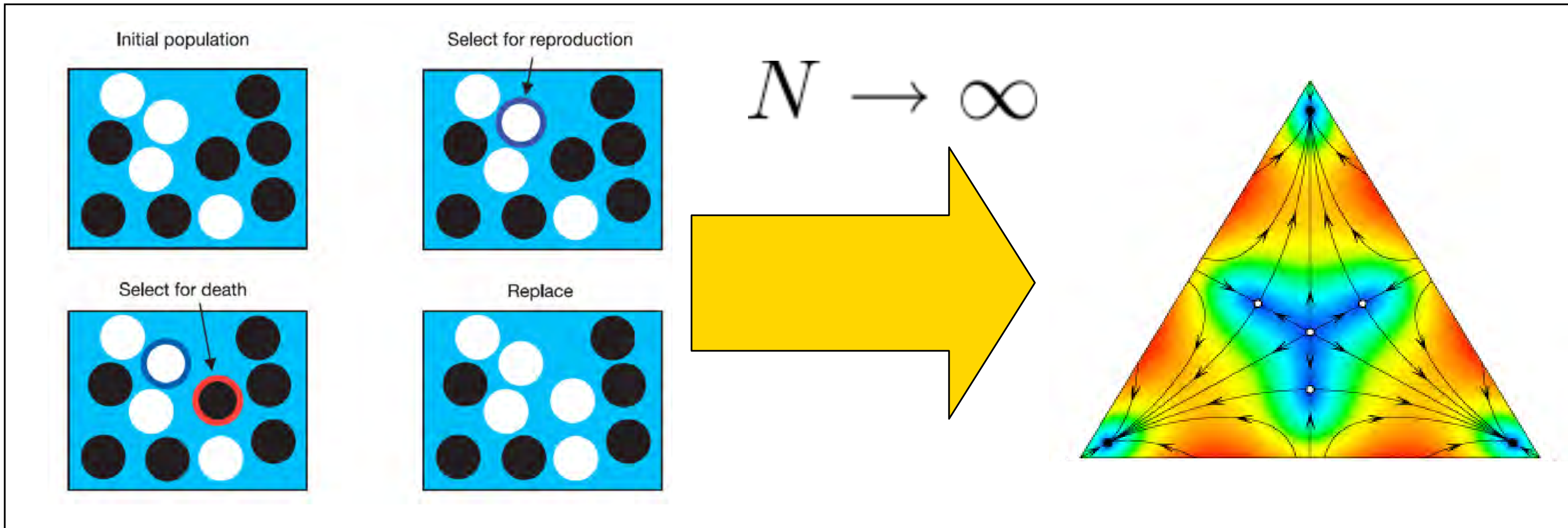
$$\phi = \frac{1}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \frac{T_{j\to j-1}}{T_{j\to j+1}}}$$



- 1/3 rule for weak selection

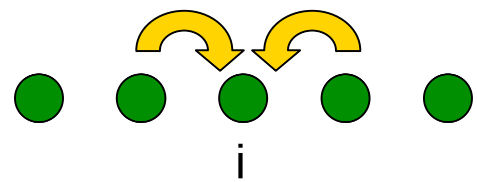


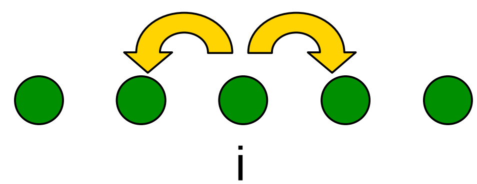
3. From finite to infinite populations



Temporal dynamics of the Moran process

$$
\begin{aligned}
P_i^{\tau+1} &= P_i^{\tau} \\
&+ P_{i-1}^{\tau} T_{i-1}^{+} + P_{i+1}^{\tau} T_{i+1}^{-} \\
&- P_i^{\tau} T_i^{+} - P_i^{\tau} T_i^{-}
\end{aligned}
$$





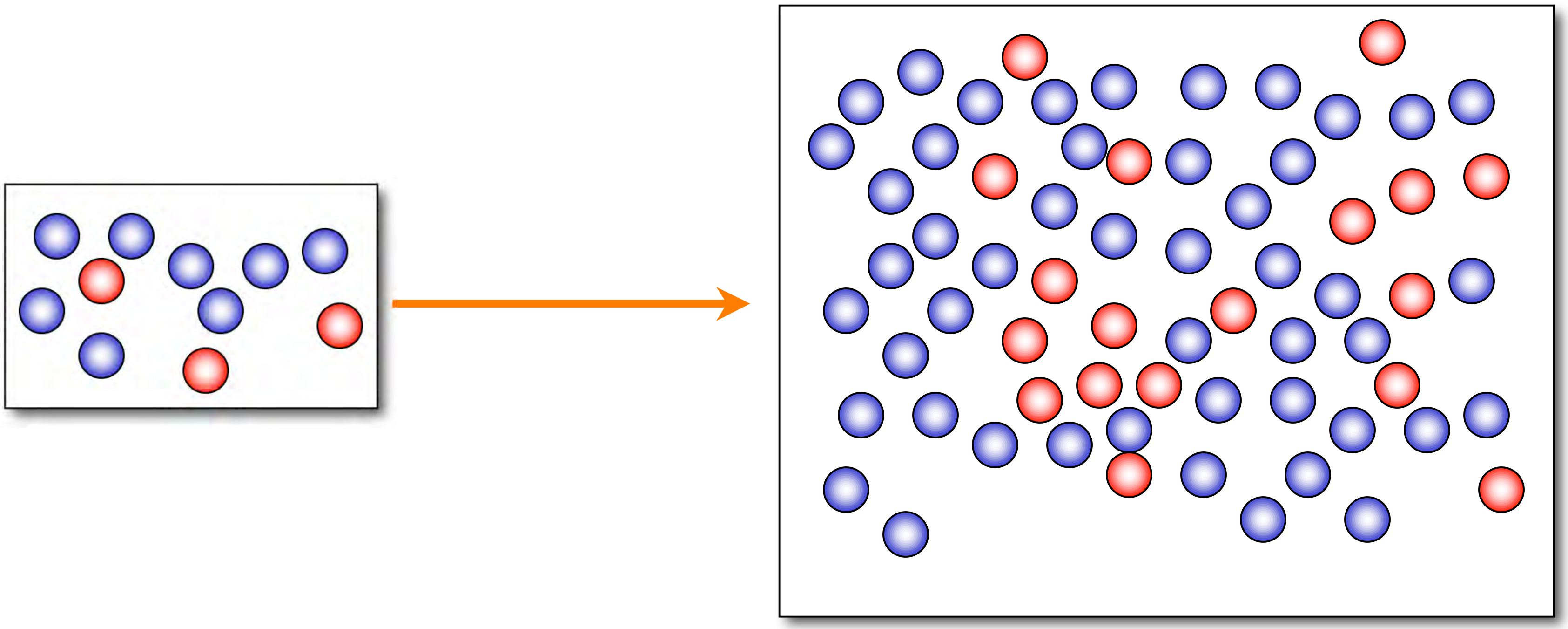
$$x = \frac{i}{N} \qquad t = \frac{\tau}{N} \qquad \rho(x,t) = N P_i^{\tau}$$

Temporal dynamics of the Moran process

$$x = \frac{i}{N} \qquad t = \frac{\tau}{N} \qquad \rho(x,t) = N P_i^\tau$$

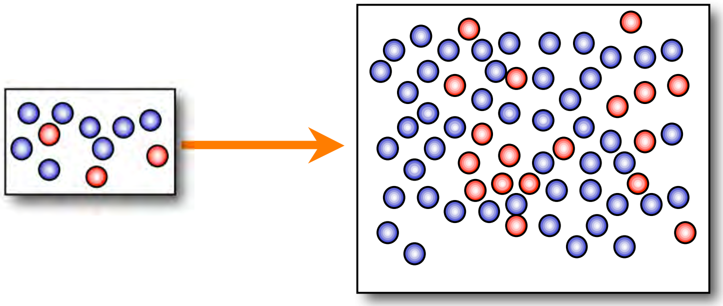
$$\begin{aligned}
\rho(x, t + 1/N) &= \rho(x,t) \\
&+ \rho(x - 1/N, t) T^+(x - 1/N) \\
&+ \rho(x + 1/N, t) T^-(x + 1/N) \\
&- \rho(x,t) T^+(x) \\
&- \rho(x,t) T^-(x)
\end{aligned}$$

The limit of large populations



Temporal dynamics of the Moran process

$$\begin{aligned}
\rho(x, t+1/N) &= \rho(x,t) \\
&+ \rho(x-1/N, t)T^{+}(x-1/N) \\
&+ \rho(x+1/N, t)T^{-}(x+1/N) \\
&- \rho(x,t)T^{+}(x) \\
&- \rho(x,t)T^{-}(x)
\end{aligned}$$



$$\begin{aligned}
\rho(x, t+1/N) &\approx \rho(x,t) + \frac{1}{N}\frac{\partial}{\partial t}\rho(x,t) \\
\rho(x \pm 1/N, t) &\approx \rho(x,t) \pm \frac{1}{N}\frac{\partial}{\partial x}\rho(x,t) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}\rho(x,t) \\
T^{\pm}(x \pm 1/N) &\approx T^{\pm}(x) \pm \frac{1}{N}\frac{\partial}{\partial x}T^{\pm}(x) + \frac{1}{2N^2}\frac{\partial^2}{\partial x^2}T^{\pm}(x)
\end{aligned}$$

Fokker-Planck equation

$$\dot{\rho}(x,t) \approx -\frac{\partial}{\partial x}\left[a(x)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[b^2(x)\rho(x,t)\right]$$

Drift

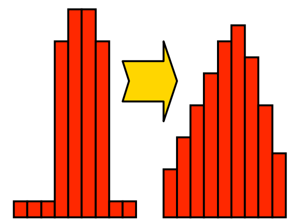
$$a(x) = T^+(x) - T^-(x)$$

Diffusion

$$b(x) = \sqrt{\frac{T^+(x) + T^-(x)}{N}}$$

From Fokker-Planck to Langevin

$$\dot{\rho}(x,t) \approx -\frac{\partial}{\partial x}\left[a(x)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[b^2(x)\rho(x,t)\right]$$



Stochastic Calculus

$$\dot{x} = a(x) + b(x)\xi$$

Gardiner, Handbook of Stochastic Methods (1981).

Back to the replicator equation

$$\dot{x} = a(x) + \cancel{b(x)\xi}$$

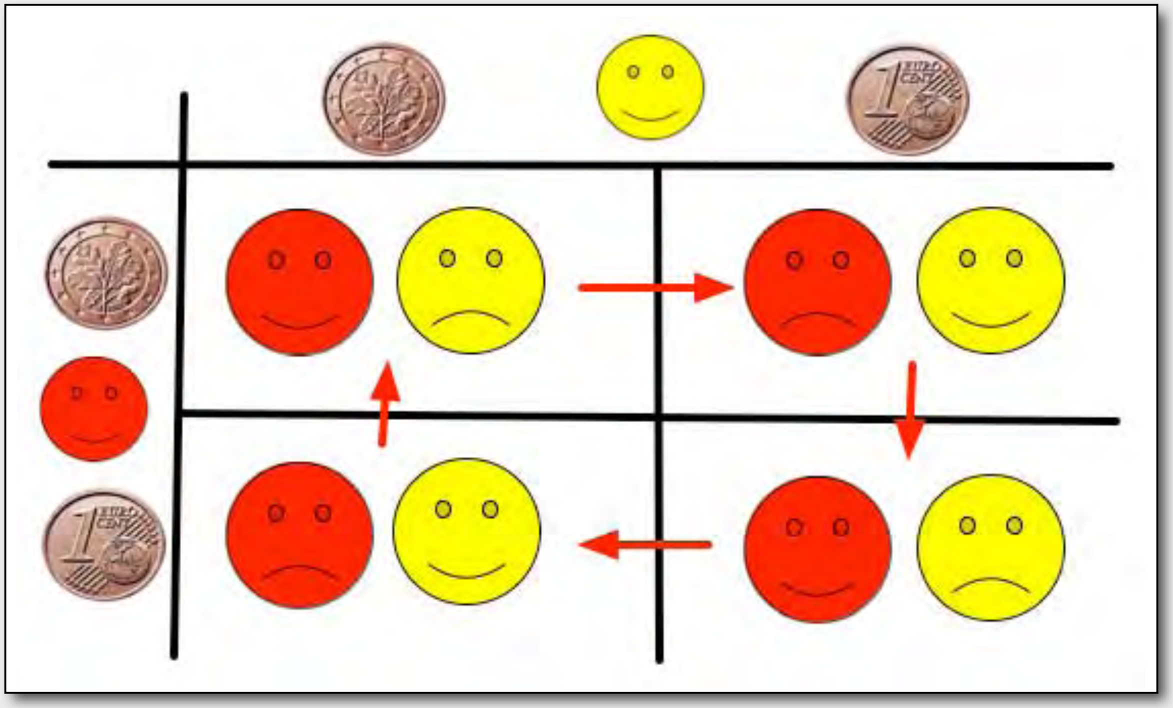
$$b(x) = \sqrt{\frac{T^+(x) + T^-(x)}{N}}$$

$$N \to \infty$$

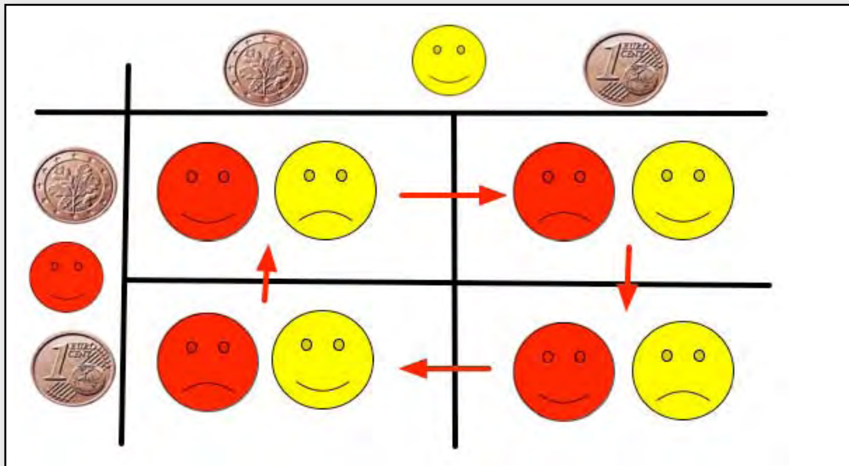
$$\begin{aligned} \dot{x} &= a(x) \\ &= T^+(x) - T^-(x) \\ &= x\frac{\pi_A}{\langle\pi\rangle}(1-x) - (1-x)\frac{\pi_B}{\langle\pi\rangle}x \\ &= x\frac{\pi_A - \langle\pi\rangle}{\langle\pi\rangle} \end{aligned}$$

Does this matter?

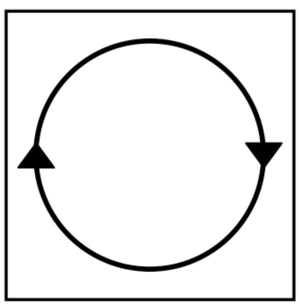
Matching pennies



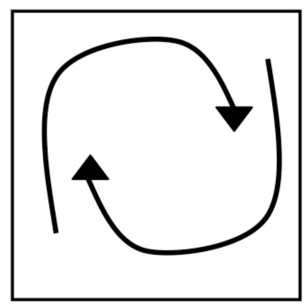
Matching pennies



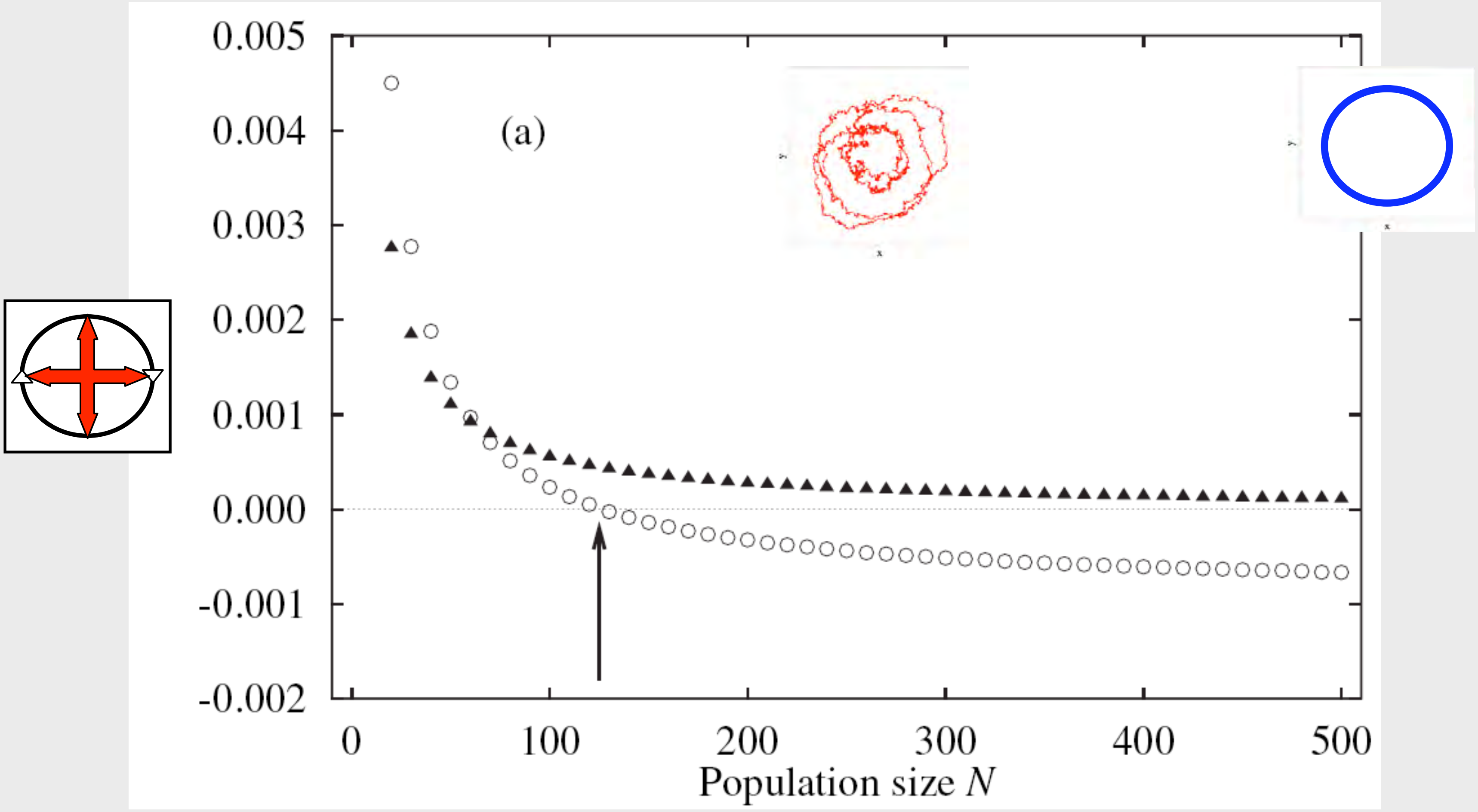
$$\dot{x} = x(\pi_A - \langle \pi \rangle)$$



$$\dot{x} = x\frac{\pi_A - \langle \pi \rangle}{\langle \pi \rangle}$$

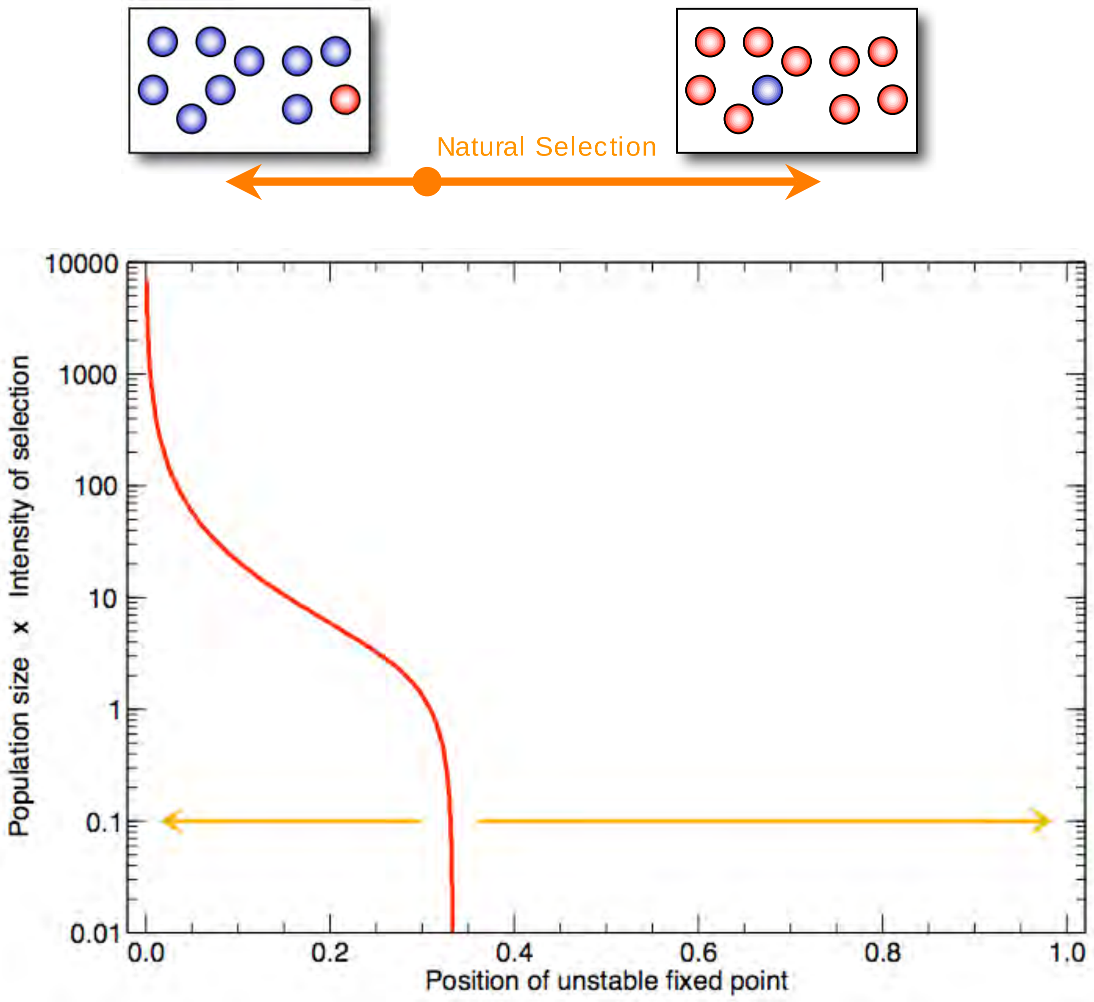


Matching pennies in finite populations



Traulsen, Claussen, and Hauert, Phys. Rev. Lett. 95, 238701 (2005).

The 1/3 rule for large populations



Traulsen, Pacheco & Imhof, PRE 74:021905 (2006)

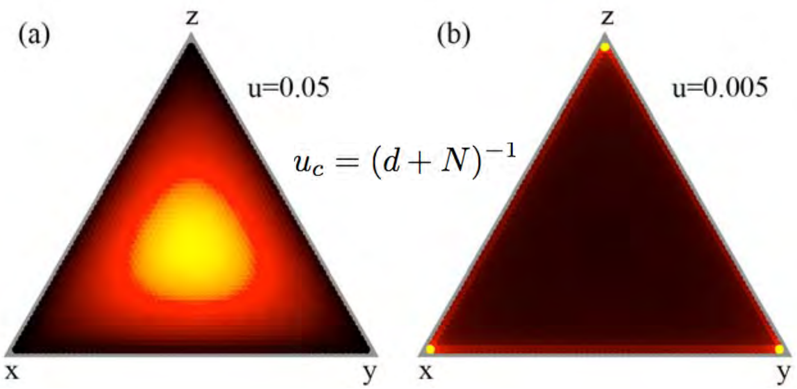
Generalization to more strategies

- Use d instead of 2 strategies

$$\dot{\rho}(\boldsymbol{x}) = -\sum_{k=1}^{d-1} \frac{\partial}{\partial x_k} \rho(\boldsymbol{x}) a_k(\boldsymbol{x}) + \frac{1}{2} \sum_{j,k=1}^{d-1} \frac{\partial^2}{\partial x_k \partial x_j} \rho(\boldsymbol{x}) b_{jk}(\boldsymbol{x})$$

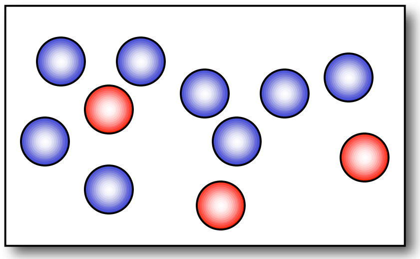
$$b_{jk}(\boldsymbol{x}) = \frac{1}{N}\left[-T_{jk}(\boldsymbol{x}) - T_{kj}(\boldsymbol{x}) + \delta_{jk} \sum_{l=1}^{d} T_{jl}(\boldsymbol{x}) + T_{lj}(\boldsymbol{x})\right]$$

$$a_k(\boldsymbol{x}) = \sum_{j=1}^{d} T_{jk}(\boldsymbol{x}) - T_{kj}(\boldsymbol{x})$$



Traulsen, Claussen and Hauert, Phys. Rev. E 74, 011901 (2006)

4. The Fermi process



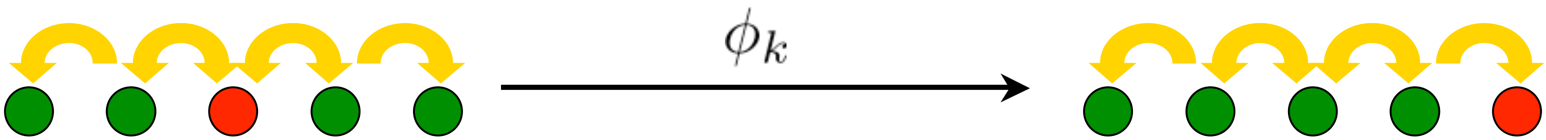
$$T_{i,i+1} = \frac{N-i}{N}\frac{i}{N}\underbrace{\frac{1}{1+e^{-\beta(\pi_A-\pi_B)}}}_{\text{Probability to switch to }\mathbf{A}}$$

A **B** player updates its strategy

The role model is **A**

$$T_{i,i-1} = \frac{i}{N}\frac{N-i}{N}\frac{1}{1+e^{+\beta(\pi_A-\pi_B)}}$$

Fixation probabilities



$$\phi_k = T_{k,k-1}\phi_{k-1} + T_{k,k+1}\phi_{k+1} + T_{k,k}\phi_k$$

$$\phi_k = \frac{\sum_{i=0}^{k-1}\prod_{j=1}^{i} T_{j,j-1}/T_{j,j+1}}{\sum_{i=0}^{N-1}\prod_{j=1}^{i} T_{j,j-1}/T_{j,j+1}}$$

Ratio of transition probabilities

$$\phi_k = \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} \boxed{T_{j,j-1}/T_{j,j+1}}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \boxed{T_{j,j-1}/T_{j,j+1}}}$$

Fixation probability for the Fermi process

$$\phi_k = \frac{\sum_{i=0}^{k-1} \prod_{j=1}^{i} \frac{T_{j,j-1}}{T_{j,j+1}}}{\sum_{i=0}^{N-1} \prod_{j=1}^{i} \frac{T_{j,j-1}}{T_{j,j+1}}}$$

General equation

Fixation probabilities for arbitrary N and β

$$\pi_A - \pi_B = u \cdot i + v$$

Frequency dependence

1. Frequency independent, *u=0*

$$\phi_k = \frac{1 - r^{-k}}{1 - r^{-N}} \qquad r = e^{2\beta v}$$

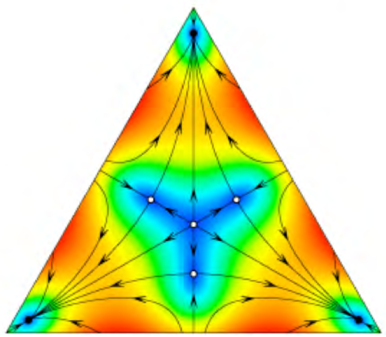
2. Frequency dependent

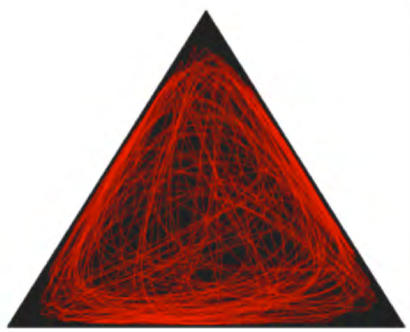
$$\phi_k = \frac{\operatorname{erf}[\xi_k] - \operatorname{erf}[\xi_0]}{\operatorname{erf}[\xi_N] - \operatorname{erf}[\xi_0]} \qquad \xi_k = \sqrt{\frac{\beta}{u}}(ku + v)$$

Take home message

- Large populations
  deterministic

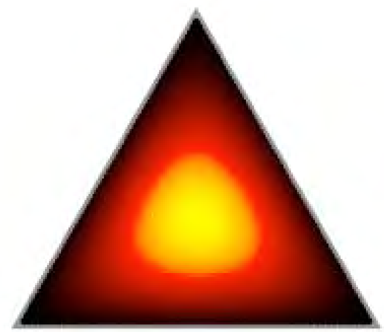
$$\dot{x}_i = x_i \left(\pi_i - \langle \pi \rangle\right)$$

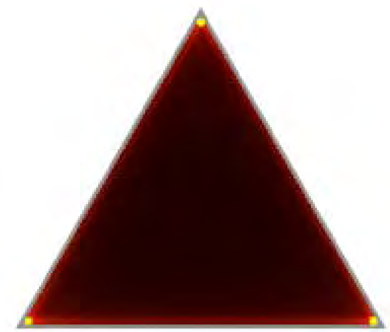




- Small populations
  stochastic

$$P_i^{t+1} = \sum_j P_j^t T_{j \to i}$$





Unified framework

$$\dot{x}_i = x_i \left(\pi_i - \langle \pi \rangle\right) + \frac{b_i(x_j)}{\sqrt{N}} \xi$$

Convenient process

$$T_{i,i+1} = \frac{N-i}{N} \frac{i}{N} \frac{1}{1 + e^{-\beta(\pi_A - \pi_B)}}$$

Thanks

- Collaborators
  J.C. Claussen (Kiel)
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  M.A. Nowak (Harvard)
  J.M. Pacheco (Lisbon)

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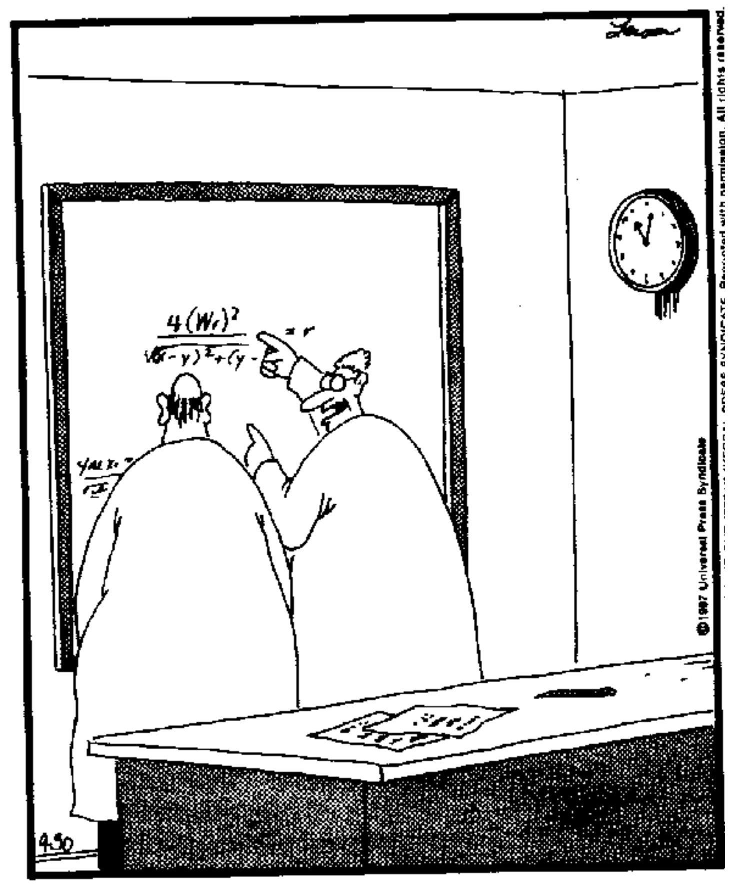
  Halle (Saale)



"Yes, yes, I know that, Sidney ... everybody knows that! ...But look: Four wrongs squared, minus two wrongs to the fourth power, divided by this formula, do make a right."