Introduction to Game Theory
Evolution Games Theory: Replicator Dynamics

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Outline

Motivation

- In previous lecture, we discussed the evolutionary stable strategy (ESS).
- Note the
  - ESS may *not* exist, in which the analysis tells us nothing about the evolution of the system.
  - The definition of an ESS deals only with monomorphic population (every individual uses the same strategy).
  - If the ESS is a mixed strategy, all strategies in the support of the ESS have the same payoff as the ESS strategy.
- We want to ask whether a polymorphic population with the same population profile as that generated by the ESS can also be stable.
- The aim of this lecture is to look at a specific type of evolutionary dynamics: replicator dynamics.

Mathematical Model

- Consider a population in which individuals, called replicator, exist in several different types.
- Each type of individual uses a pre-programmed strategy and passes this behavior to its descendants without modification.
- In the replicator dynamics, individuals use only pure strategies form a finite set $\boldsymbol{S} = \{s_1, s_2, \ldots, s_k\}$.
- Let n_i be the number of individual using s_i, then the total population size is

$$N = \sum_{i=1}^{k} n_i,$$

and the proportion of individuals using s_i is:

$$x_i = \frac{n_i}{N}.$$

Mathematical Model: continue

- The population state is $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$.
- Let β and δ be the background *per capita* birth and death rates in the population (or contributions to the rate of appearance or disappearance of individuals in the population).
- The rate of change of the number of individuals using s_i ($\dot{n}_i$) and rate of change of total population ($\dot{N}$):

$$\begin{aligned}
\dot{n}_i &= (\beta - \delta + \pi(s_i, \boldsymbol{x}))n_i = (\beta - \delta + \pi(s_i, \boldsymbol{x}))x_i N \\
\dot{N} &= \sum_{i=1}^{k} \dot{n}_i = (\beta - \delta)\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} \pi(s_i, \boldsymbol{x})n_i \\
&= (\beta - \delta)N + N\sum_{i=1}^{k} x_i\pi(s_i, \boldsymbol{x}) = (\beta - \delta + \bar{\pi}(\boldsymbol{x}))N,
\end{aligned}$$

where $\bar{\pi}(\boldsymbol{x}) = \sum_{i=1}^{k} x_i\pi(s_i, \boldsymbol{x})$ is the average payoff.

Mathematical Model: continue

- We are interested in how the proportion of each type change over time. Since $n_i = x_i N$, we have: $\dot{n}_i = N\dot{x}_i + x_i\dot{N}_i$. So

$$\begin{aligned} N\dot{x}_i &= \dot{n}_i - x_i\dot{N}_i \\ &= (\beta - \delta + \pi(s_i, \boldsymbol{x}))x_i N - x_i(\beta - \delta + \bar{\pi}(\boldsymbol{x}))N. \end{aligned}$$

- Cancelling N from both sides, we have

$$\dot{x}_i = (\pi(s_i, \boldsymbol{x}) - \bar{\pi}(\boldsymbol{x}))x_i. \tag{1}$$

- In other words, the proportion of individuals using strategy s_i increases (decreases) if its payoff is bigger (smaller) than the average payoff in the population.

HW: Exercise 9.1 and 9.2.

Definition

A fixed point of the replicator dynamics is a population that satisfies $\dot{x}_i = 0 \, \forall i$. Fixed points describe populations that are no longer evolving.

Example

- Consider a pairwise contest population game with action set $\boldsymbol{A} = \{E, F\}$ and payoffs: $\pi(E, E) = 1$, $\pi(E, F) = 1$, $\pi(F, E) = 2$, $\pi(F, F) = 0$.
- So $\pi(E, \boldsymbol{x}) = 1x_1 + 1x_2$ and $\pi(F, \boldsymbol{x}) = 2x_1$, which gives

$$\bar{\pi}(\boldsymbol{x}) = x_1(x_1 + x_2) + x_2(2x_1) = x_1^2 + 3x_1x_2.$$

- The replicator dynamics for this game:

$$\begin{aligned} \dot{x}_1 &= x_1(x_1 + x_2 - x_1^2 - 3x_1x_2) \\ \dot{x}_2 &= x_2(2x_1 - x_1^2 - 3x_1x_2). \end{aligned}$$

- The fixed points are $(x_1 = 0, x_2 = 1)$, $(x_1 = 1/2, x_2 = 1/2)$, and $(x_1 = 1, x_2 = 0)$

Exercise

- Consider the pairwise contest with payoffs given by (with $a < b$):

	A	B
A	$a-b$, $a-b$	$2a$, 0
B	0, $2a$	a, a

- Derive the replicator dynamics equations and find all fixed points.

Solution

- We have $\pi(A, \boldsymbol{x}) = (a-b)x_1 + 2ax_2 \quad ; \quad \pi(B, \boldsymbol{x}) = ax_2$.
- The average payoff is $\bar{\pi}(\boldsymbol{x}) = (a-b)x_1^2 + 2ax_1x_2 + ax_2^2$.
- Replicator dynamics is:
  $\dot{x}_1 = x_1((a-b)x_1 + 2ax_2 - \bar{\pi}(\boldsymbol{x})) \quad ; \quad \dot{x}_2 = x_2(ax_2 - \bar{\pi}(\boldsymbol{x}))$.
- Clearly $(x_1 = 1, x_2 = 0)$ and $(x_1 = 0, x_2 = 1)$ are fixed points.
- At the polymorphic fixed point, we must have $(a-b)x_1 + 2ax_2 - \bar{\pi}(\boldsymbol{x}) = 0 = ax_2 - \bar{\pi}(\boldsymbol{x})$, which gives $(a-b)x_1 = -ax_2$. Substituting this into the equation $ax_2 - \bar{\pi}(\boldsymbol{x}) = 0$ gives $x_1 = a/b$.

Two-strategy Pairwise Contests

- Let us consider a simplification of the game: a pairwise contest game that only has two pure strategies.
- Suppose $\boldsymbol{S} = \{s_1, s_2\}$ and $x \equiv x_1$, then $x_2 = 1 - x$ and $\dot{x}_2 = -\dot{x}_1$
- For this case, we can simplify Eq. (1) to

$$\dot{x} = (\pi(s_1, \boldsymbol{x}) - \bar{\pi}(\boldsymbol{x}))\, x.$$

- We can simplify it further, since $\bar{\pi}(\boldsymbol{x}) = x\pi(s_1, \boldsymbol{x}) + (1 - x)\pi(s_2, x)$, we have

$$\dot{x} = x(1 - x)(\pi(s_1, \boldsymbol{x}) - \pi(s_2, \boldsymbol{x})). \tag{2}$$

Let us show the applicability via an example.

Example

- Consider a pairwise contest Prisoner's Dilemma. The pure strategies are $\{C, D\}$ and the payoffs of the focal individual in the corresponding 2-player game are: $\pi(C, C) = 3$, $\pi(C, D) = 0$, $\pi(D, C) = 5$, and $\pi(D, D) = 1$. Let x be the proportion of individuals using C, then

$$\pi(C, \boldsymbol{x}) = 3x + 0(1 - x) = 3x \quad ; \quad \pi(D, \boldsymbol{x}) = 5x + 1(1 - x) = 1 + 4x.$$

- Applying Eq. (2), we have:

$$\begin{aligned} \dot{x} &= x(1 - x)\left(\pi(C, \boldsymbol{x}) - \pi(D, \boldsymbol{x})\right) \\ &= x(1 - x)(3x - (1 + 4x)) = -x(1 - x)(1 + x) \end{aligned}$$

- The fixed points are $x^* = 0$ and $x^* = 1$ (not $x^* = -1$).
- The unique NE for the Prisoners' Dilemma is for everyone to use D. This means $x^* = 0$ corresponds to a NE but $x^* = 1$ does not.
- Since $\dot{x} < 0$ for $x \in (0, 1)$, any population that is not at the fixed point will evolve towards the fixed point of the NE.

HW: Exercise 9.4.

Comment

- It seems that every NE corresponds to a fixed point in the replicator dynamics.
- But not every fixed point corresponds to a NE.

We formalize this in the following theorem.

Theorem

Let $\boldsymbol{S} = \{s_1, s_2\}$ and $\sigma^* = (p^*, 1 - p^*)$ be the strategy that uses s_1 with probability p^*. If (σ^*, σ^*) is a symmetric Nash equilibrium, then the population $\boldsymbol{x}^* = (x^*, 1 - x^*)$ with $x^* = p^*$ is a fixed point of the replicator dynamics $\dot{x} = x(1 - x)(\pi(s_1, \boldsymbol{x}) - \pi(s_2, \boldsymbol{x}))$.

Proof

- If σ^* is a pure strategy, then $x^* = 0$ or $x^* = 1$. In either case, we have $\dot{x} = 0$.
- If σ^* is a mixed strategy, then the Theorem of Equality of Payoffs says that $\pi(s_1, \sigma^*) = \pi(s_2, \sigma^*)$. Now for the pairwise contest,

$$\pi(s_i, \sigma^*) = p^* \pi(s_i, s_1) + (1 - p^*)\pi(s_i, s_2) = \pi(s_i, \boldsymbol{x}).$$

  So we have $\pi(s_1, \boldsymbol{x}^*) = \pi(s_2, \boldsymbol{x}^*)$.
- Given the replicator dynamics of $\dot{x} = x(1 - x)(\pi(s_1, \boldsymbol{x}) - \pi(s_2, \boldsymbol{x}))$, using the result above, we have $\dot{x} = 0$.

So NE in two-player games corresponds to a fixed point in a replicator dynamics. Is there a consistent relation between the ESSs in a population game and the fixed point?

Example

- Consider a pairwise contest with actions A and B and the following payoffs in the associated two-player game: $\pi(A, A) = 3$, $\pi(B, B) = 1$, $\pi(A, B) = \pi(B, A) = 0$.
- The ESSs are for everyone to play A, or everyone to play B.
- The mixed strategy $\sigma = (1/4, 3/4)$ is NOT an ESS.
- Let x be the proportion of individuals using A, we have

$$\begin{aligned} \dot{x} &= x(1-x)(\pi(A, \boldsymbol{x}) - \pi(B, \boldsymbol{x})) \\ &= x(1-x)(3x - (1-x)) = x(1-x)(4x-1). \end{aligned}$$

- fixed points are $x^* = 0$, $x^* = 1$ and $x^* = 1/4$.
- However, $\dot{x} > 0$ if $x > 1/4$ and $\dot{x} < 0$ if $x < 1/4$. So only pure strategies of either use A or B are evolutionary end points
- This means only the evolutionary end pts correspond to an ESS.
- Do Exercise 9.5.

Motivations

We like to seek answer to the following questions:

- Do all ESSs have a corresponding end point?
- Do all evolutionary end points have a corresponding ESS?

Let us first consider the special case of two-strategy pairwise contest game.

Definition

A fixed point of the replicator dynamics (or any dynamical system) is said to be asymptotically stable if any small deviations from that state are eliminated by the dynamics as $t \to \infty$.

Example

- Consider a pairwise contest with pure strategies A and B and the following payoffs: $\pi(A, A) = 3$, $\pi(B, B) = 1$, $\pi(A, B) = \pi(B, A) = 0$.
- The ESS for this game is for everyone to play A, or for everyone to play B. The mixed strategy $\sigma = (1/4, 3/4)$ is a NE but it is **not** an ESS.
- Let x be the proportion of individuals using A, the replicator dynamics is :

$$\begin{aligned} \dot{x} &= x(1-x)(\pi(A, \boldsymbol{x}) - \pi(B, \boldsymbol{x})) \\ &= x(1-x)(3x - (1-x)) = x(1-x)(4x-1), \end{aligned}$$

with fixed point $x^* = 0$, $x^* = 1$ and $x^* = 1/4$.

Example: continue

- Consider the fixed point at $x^* = 0$. Let $x = x^* + \epsilon = \epsilon$ where we must have $\epsilon > 0$ to ensure $x > 0$.
- Then $\dot{x} = \dot{\epsilon}$ because x^* is a constant. Thus, we have

$$\dot{\epsilon} = \epsilon(1 - \epsilon)(4\epsilon - 1).$$

- Since $\epsilon << 1$, we can ignore terms proportional to ϵ^n where $n > 1$. This is called linearization. Thus $\dot{\epsilon} \approx -\epsilon$, which has the solution of $\epsilon(t) = \epsilon_0 e^{-t}$.
- This states that the dynamics reduces small deviations from the population $\boldsymbol{x} = (0, 1)$ (i.e., $\epsilon \to 0$ and $t \to \infty$). In other words, the fixed point $x^* = 0$ is **asymptotically stable**.

Example: continue

- Now consider $x^* = 1$. Let $x = x^* - \epsilon = 1 - \epsilon$ with $\epsilon > 0$ (so $x < 1$). Using linearization, we have $\dot{\epsilon} \approx -3\epsilon$, which has the solution of $\epsilon(t) = \epsilon_0 e^{-3t}$. So $x^* = 1$ is asymptotically stable.
- Now consider $x^* = 1/4$. Let $x = x^* + \epsilon = \frac{1}{4} + \epsilon$ with no sign restriction on ϵ.
- Using linearization, we have $\dot{\epsilon} \approx \frac{3}{4}\epsilon$, which has the solution of $\epsilon(t) = \epsilon_0 e^{3t/4}$. So $x^* = 1/4$ is **not** asymptotically stable.

Lesson: In this example, we find that a strategy is ESS if and only if the corresponding point in the replicator dynamics is asymptotically stable.

Theorem

For any two-strategy pairwise contest, a strategy is an ESS if and only if the corresponding fixed point in the replicator dynamic is asymptotically stable.

Proof

Consider a pairwise contest with strategies A and B. Let x be the proportion of individuals using A, then based on Eq. (2), the replicator dynamics is

$$\dot{x} = x(1-x)\left[\pi(A, \boldsymbol{x}) - \pi(B, \boldsymbol{x})\right].$$

There are three possible cases to consider:

- A single pure-strategy ESS or stable monomorphic population;
- Two pure-strategy ESSs or stable monomorphic populations;
- One mixed strategy ESS or polymorphic population.

Proof: for case 1

- Let $\sigma^* = (1, 0)$. Then for $\sigma = (y, 1 - y)$ with $y \neq 1$, based on the definition of stability of ESS, σ^* is an ESS if and only if $\pi(A, \boldsymbol{x}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) > 0$.

$$
\begin{aligned}
&\iff \pi(A, \boldsymbol{x}_\epsilon) - y\pi(A, \boldsymbol{x}_\epsilon) - (1 - y)\pi(B, \boldsymbol{x}_\epsilon) > 0 \\
&\iff (1 - y)[\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon)] > 0 \\
&\iff \pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon) > 0.
\end{aligned}
$$

- Let $x = 1 - \epsilon$ with $\epsilon > 0$. So $\dot{x} = -\dot{\epsilon}$. Using linearization, we have:

$$
\dot{\epsilon} = -\epsilon[\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon)].
$$

- So $\sigma^* = (1, 0)$ is an ESS if and only if the corresponding population $x^* = 1$ is asymptotically stable.

Proof: for case 2

- Let $\sigma^* = (0, 1)$. Then for $\sigma = (y, 1 - y)$ with $y \neq 0$, based one the definition of stability of ESS, σ^* is an ESS if and only if $\pi(B, \boldsymbol{X}_\epsilon) - \pi(\sigma, \boldsymbol{x}_\epsilon) > 0$.

$$
\begin{aligned}
&\iff \pi(B, \boldsymbol{x}_\epsilon) - y\pi(A, \boldsymbol{x}_\epsilon) - (1 - y)\pi(B, \boldsymbol{x}_\epsilon) > 0 \\
&\iff -y(\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon) > 0 \\
&\iff (\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon) < 0.
\end{aligned}
$$

- Let $x = 0 + \epsilon$ with $\epsilon > 0$. So $\dot{x} = \dot{\epsilon}$. using linearization, we have:

$$\dot{\epsilon} = \epsilon[\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon)].$$

- So $\sigma^* = (0, 1)$ is an ESS if and only if the corresponding $\boldsymbol{x}^* = 0$ is asymptotically stable.

Proof: for case 3

- Let $\sigma^* = (p^*, 1-p^*)$ with $0 < p < 1$. Then σ^* is an ESS if and only if $\pi(\sigma^*, \sigma) > \pi(\sigma, \sigma)$.
- Taking $\sigma = A$ and $\sigma = B$ in turn, this becomes two conditions:

$$\pi(\sigma^*, A) > \pi(A, A) \quad ; \quad \pi(\sigma^*, B) > \pi(B, B).$$

  Based on equality of payoff, the above conditions translate to:

$$\pi(B, A) > \pi(A, A) \quad ; \quad \pi(A, B) > \pi(B, B).$$

- Let $x = x^* + \epsilon$. Then, for a pairwise contest, the replicator dynamics: $\dot{x} = x(1-x)[\pi(A, \boldsymbol{x}_\epsilon) - \pi(B, \boldsymbol{x}_\epsilon)]$ becomes:

$$\dot{\epsilon} = x^*(1-x^*)\epsilon([\pi(A,A) - \pi(B,A)] + [\pi(B,B) - \pi(A,B)])$$

  using the assumption that $\boldsymbol{x}^*$ is a fixed point. So $\boldsymbol{x}^*$ is asymptotically stable if and only if σ^* is an ESS.

Summary

Let $\boldsymbol{F}$ be the set of fixed points, $\boldsymbol{A}$ be the set of asymptotically stable fixed points in the replicator dynamics. Let $\boldsymbol{N}$ be the set of symmetric Nash equilibrium strategies and $\boldsymbol{E}$ be the set of ESSs in the symmetric game corresponding to the replicator dynamics. For any *two-strategy pairwise-contest game*, the following relationships hold for a strategy σ^* and the corresponding $\boldsymbol{x}^*$:

- $\sigma^* \in \boldsymbol{E} \Longleftrightarrow \boldsymbol{x}^* \in \boldsymbol{A}$;
- $\boldsymbol{x}^* \in \boldsymbol{A} \Longrightarrow \sigma^* \in \boldsymbol{N}$ (this follows from the first equivalence because $\sigma^* \in \boldsymbol{E} \Longrightarrow \sigma^* \in \boldsymbol{N}$).
- $\sigma^* \in \boldsymbol{N} \Longrightarrow \boldsymbol{x}^* \in \boldsymbol{F}$.

We can write the relations more concisely as $\boldsymbol{E} = \boldsymbol{A} \subseteq \boldsymbol{N} \subseteq \boldsymbol{F}$.
We will show that for pairwise-contest games with more than two strategies these relations become $\boldsymbol{E} \subseteq \boldsymbol{A} \subseteq \boldsymbol{N} \subseteq \boldsymbol{F}$.

HW: Exercise 9.6

Introduction

- If we increase the number of pure strategies to n, then we have n equations:

$$\dot{x}_i = f_i(\boldsymbol{x}) \quad i = 1, \ldots, n.$$

- Using the constraints $\sum_{i=1}^{n} x_i = 1$, we can introduce a reduced state vector $(x_1, x_2, \ldots, x_{n-1})$ and reduced it to $n-1$ equations:

$$\dot{x}_i = f_i(\boldsymbol{x}) \quad i = 1, \ldots, n-1.$$

- Rewrite the dynamic system in vector format as $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

Example

- Consider the following pairwise contest game:

	A	B	C
A	0,0	3,3	1,1
B	3,3	0,0	1,1
C	1,1	1,1	1,1

- The replicator dynamics for this game is:

$$\begin{aligned}\dot{x}_1 &= x_1(3x_2 + x_3 - \bar{\pi}(\boldsymbol{x}))\\ \dot{x}_2 &= x_2(3x_1 + x_3 - \bar{\pi}(\boldsymbol{x}))\\ \dot{x}_3 &= x_3(x_1 + x_2 + x_3 - \bar{\pi}(\boldsymbol{x})) = x_3(1 - \bar{\pi}(\boldsymbol{x}))\end{aligned}$$

with $\bar{\pi}(\boldsymbol{x}) = 6x_1x_2 + 2x_2x_3 + 2x_1x_3 + x_3^2$.

- Writing $x_1 = x$, $x_2 = y$ and $x_3 = 1 - x - y$, we have:

$$\dot{x} = x(1 - x + 2y - \bar{\pi}(\boldsymbol{x})) \quad ; \quad \dot{y} = y(1 + 2x - y - \bar{\pi}(\boldsymbol{x}))$$

with $\bar{\pi}(x, y) = 1 + 4xy - x^2 - y^2$.

Fixed pts: (a) (1,0,0) (b) (0,1,0) (c) (0,0,1).

HW: Exercise 9.7

Definition

The replicator dynamics is defined on the simplex

$$\triangle = \left\{ x_1, x_2, \ldots, x_n | 0 \leq x_i \leq 1 \, \forall i \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}.$$

An invariant manifold is a connected subset $M \subset \triangle$ such that if $\boldsymbol{x}(0) \in M$, then $\boldsymbol{x}(t) \in M$ for all $t > 0$.

Remark: it follows from the definition that fixed points of a dynamical system are invariant manifolds. Boundaries of the simplex $\triangle$ (subsets where one or more population types are absent) are also invariant because $x_i = 0 \Longrightarrow \dot{x}_i = 0$.

Example: continue

- For the previous dynamic system:

$$\dot{x} = x(1 - x + 2y - \bar{\pi}(\boldsymbol{x})) \quad ; \quad \dot{y} = y(1 + 2x - y - \bar{\pi}(\boldsymbol{x}))$$

- The obvious invariant manifolds (or fixed points) are:
  - Fixed point $(1, 0, 0)$.
  - Fixed point $(0, 1, 0)$.
  - Fixed point $(0, 0, 1)$.
  - The boundary line $x = 0$.
  - The boundary line $y = 0$.
  - The boundary line $x + y - 1 = 0$ because

$$\frac{d}{dt}(x + y) = \dot{x} + \dot{y} = (x + y - 1)(1 - \bar{\pi}(x, y)) = 0$$

    the last equality is based on $\bar{\pi}(\boldsymbol{x}) = 6x_1x_2 + 2x_2x_3 + 2x_1x_3 + x_3^2$, or $\bar{\pi}(x, y) = 6xy + 2y(1 - x - y) + 2x(1 - x - y) + (1 - x - y)^2$.
  - The line $x = y$ because $\dot{x} = \dot{y}$ on that line.

Summary of Results

Let $\boldsymbol{F}$ be the set of fixed points, $\boldsymbol{A}$ be the set of asymptotically stable fixed points in the replicator dynamics. Let $\boldsymbol{N}$ be the set of symmetric Nash equilibrium strategies and $\boldsymbol{E}$ be the set of ESSs in the symmetric game corresponding to the replicator dynamics. For any pairwise-contest game (may have more than two strategies), the following relationships hold for a strategy σ^* and the corresponding population state $\boldsymbol{x}^*$:

- $\sigma^* \in \boldsymbol{E} \Longrightarrow \boldsymbol{x}^* \in \boldsymbol{A}$;
- $\boldsymbol{x}^* \in \boldsymbol{A} \Longrightarrow \sigma^* \in \boldsymbol{N}$;
- $\sigma^* \in \boldsymbol{N} \Longrightarrow \boldsymbol{x}^* \in \boldsymbol{F}$.

Or more concisely:

$$\boldsymbol{E} \subseteq \boldsymbol{A} \subseteq \boldsymbol{N} \subseteq \boldsymbol{F}.$$

Proof for $\boldsymbol{N} \subseteq \boldsymbol{F}$

Theorem

If (σ^*, σ^*) is a symmetric Nash equilibrium, then the population state $\boldsymbol{x}^* = \sigma^*$ is a fixed point of the replicator dynamics.

Proof

- Suppose the NE strategy σ^* is a pure strategy s_j and every player uses s_j. Then $x_i = 0$ for $i \neq j$ and $\bar{\pi}(\boldsymbol{x}) = \pi(s_j, \boldsymbol{x}^*)$. Hence $\dot{x}_i = 0 \, \forall i$.

Proof: continue

- Suppose σ^* is a mixed strategy and let $\boldsymbol{S}^*$ be the support of σ^*. The equality of payoffs theorem states

$$\pi(s, \sigma^*) = \pi(\sigma^*, \sigma^*) \quad \forall s \in \boldsymbol{S}^*.$$

- This implies that in a polymorphic population with $\boldsymbol{x}^* = \sigma^*$, we must have all $s_i \in \boldsymbol{S}^*$:

$$\pi(s_i, \boldsymbol{x}^*) = \sum_{j=1}^{k} \pi(s_i, s_j) x_j = \sum_{j=1}^{k} \pi(s_i, s_j) p_j = \pi(s_i, \sigma^*) = \text{constant}$$

- For strategies $s_i \notin \boldsymbol{S}^*$, the condition $\boldsymbol{x}^* = \sigma^*$ gives us $x_i = 0$ and hence $\dot{x}_i = 0$. For strategies $s_j \in \boldsymbol{S}^*$, we have

$$\dot{x}_j = x_j \left[\pi(s_j, \boldsymbol{x}^*) - \sum_{i=1}^{k} x_i \pi(s_j, \boldsymbol{x}^*) \right] = x_j \left[\pi(s_j, \boldsymbol{x}^*) - \pi(s_j, \boldsymbol{x}^*) \sum_{i=1}^{k} x_i \right] = 0.$$

Remark

The above theorem shows that an evolutionary process can produce apparently rational (Nash equilibrium) behavior in a population composed of individuals who are not required to make consciously rational decisions. So individuals are no longer required to work through a sequence of optimizations, but merely evaluate the consequence of their actions, compare them to the results obtained by others who behaved differently and swap to a better (and not necessary the *best*) strategy for the current situation. The population is stable when, given what everyone else is doing, no individual world get a better result by adopting a different strategy. This is known as the mass action as stated by J. Nash.

Proof for $\boldsymbol{A} \subseteq \boldsymbol{N}$

Theorem

If $\boldsymbol{x}^*$ is an asymptotically stable fixed point of the replicator dynamics, then the symmetric strategy pair $[\sigma^*, \sigma^*]$ with $\sigma^* = \boldsymbol{x}^*$ is a Nash equilibrium.

Proof

- If $\boldsymbol{x}^*$ is a fixed point with $x_i > 0 \, \forall i$ (i.e., all pure strategy types are present in the population), then all pure strategies must earn the same payoff in the population.
- It follows from the consequence that σ^* and $\boldsymbol{x}^*$ that $\pi_i(s, \sigma^*) = \pi(s, \boldsymbol{x}^*)$ is also constant for all pure strategies s. Therefore $[\sigma^*, \sigma^*]$ is a Nash equilibrium.

Proof: continue

- Now consider stationary population with one more pure strategies are absent.
- Denote the set of present pure strategy by $\boldsymbol{S}^* \subset \boldsymbol{S}$ (i.e., $\boldsymbol{S}^*$ is the support of the fixed point $\boldsymbol{x}^*$ and the postulated NE strategy σ^*).
- Because $\boldsymbol{x}^*$ is a fixed point, we have $\pi(s, \boldsymbol{x}^*) = \bar{\pi}(\boldsymbol{x}^*) \forall s \in \boldsymbol{S}^*$ and $\pi_1(s, \sigma^*) = \pi(\sigma^*, \sigma^*) \forall s \in \boldsymbol{S}^*$.
- Now suppose $[\sigma^*, \sigma^*]$ is not a NE, there must be some strategy $s' \neq \boldsymbol{S}^*$ for which $\pi_1(s', \sigma^*) > \pi_1(\sigma^*, \sigma^*)$ and consequently $\pi(s', \boldsymbol{x}^*) > \bar{\pi}(\boldsymbol{x}^*)$.
- Consider a population $\boldsymbol{x}_\epsilon$ that is close to $\boldsymbol{x}^*$ but has a small population ϵ of s' players, then

$$\dot{\epsilon} = \epsilon(\pi(s', \boldsymbol{x}_\epsilon) - \bar{\pi}(\boldsymbol{x}_\epsilon) = \epsilon(\pi(s', \boldsymbol{x}^*) - \bar{\pi}(\boldsymbol{x}^*)) + O(\epsilon^2).$$

- So $x_{s'}$ increases, contradicting the assumption $\boldsymbol{x}^*$ is asymptotically stable.

Proof for $\boldsymbol{E} \subseteq \boldsymbol{A}$

Definition

Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ be a dynamical system with a fixed point at $\boldsymbol{x}^*$. Then a scalar function $V(\boldsymbol{x})$, defined for allowable states of the system close to $\boldsymbol{x}^*$, such that:

1. $V(\boldsymbol{x}^*) = 0$;
2. $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{x}^*$;
3. $\frac{dV(\boldsymbol{x})}{dt} < 0$ for $\boldsymbol{x} \neq \boldsymbol{x}^*$.

V is called a **Lyapounov function**. If such a function exists, then the fixed point $\boldsymbol{x}^*$ is asymptotically stable.

Theorem

Every ESS corresponds to an asymptotically stable fixed point in the replicator dynamics. That is, if σ^* is an ESS, then the population with $\boldsymbol{x}^* = \sigma^*$ is asymptotically stable.

Proof

- If σ^* is an ESS, then by definition, there exists an $\bar{\epsilon}$ such that for all $\epsilon < \bar{\epsilon}$

$$\pi(\sigma^*, \sigma_\epsilon) > \pi(\sigma, \sigma_\epsilon) \quad \forall \sigma \neq \sigma^*$$

  where $\sigma_\epsilon = (1 - \epsilon)\sigma^* + \epsilon\sigma'$.
- This holds for $\sigma = \sigma_\epsilon$, so $\pi(\sigma^*, \sigma_\epsilon) > \pi(\sigma_\epsilon, \sigma_\epsilon)$.
- This implies in the replicator dynamics we have, for $\boldsymbol{x}^* = \sigma^*$, $\boldsymbol{x} = (1 - \epsilon)\boldsymbol{x}^* + \epsilon\boldsymbol{x}'$ and for all $\epsilon < \bar{\epsilon}$

$$\pi(\sigma^*, \boldsymbol{x}) > \bar{\pi}(\boldsymbol{x}).$$

- Now consider the relative entropy function

$$V(\boldsymbol{x}) = -\sum_{i=1}^{k} x_i^* \ln\left(\frac{x_i}{x_i^*}\right).$$

Proof: continue

- We have $V(\boldsymbol{x}^*) = 0$ (by applying Jensen's inequality $E[f(x)] \geq f(E[x])$ for any convex function, such as logarithm):

$$
\begin{aligned}
V(\boldsymbol{x}) &= -\sum_{i=1}^{k} x_i^* \ln\left(\frac{x_i}{x_i^*}\right) \geq -\ln\left(\sum_{i=1}^{k} x_i^* \frac{x_i}{x_i^*}\right) \\
&= -\ln\left(\sum_{i=1}^{k} x_i\right) = -\ln(1) = 0.
\end{aligned}
$$

- The time derivative of $V(\boldsymbol{x})$ along solution trajectories of the replicator dynamics is:

$$
\begin{aligned}
\frac{d}{dt}V(\boldsymbol{x}) &= \sum_{i=1}^{k} \frac{\partial V}{\partial x_i} x_i = -\sum_{i=1}^{k} \frac{x_i^*}{x_i} \dot{x}_i \\
&= -\sum_{i=1}^{k} \frac{x_i^*}{x_i} x_i(\pi(s_i, \boldsymbol{x}) - \bar{\pi}(\boldsymbol{x})) = -[\pi(\sigma^*, \boldsymbol{x}) - \bar{\pi}(\boldsymbol{x})].
\end{aligned}
$$

Proof: continue

- If σ^* is an ESS, then we established above that there is a region near $\boldsymbol{x}^*$ where $[\pi(\sigma^*, \boldsymbol{x}) - \bar{\pi}(\boldsymbol{x})] > 0$ for $\boldsymbol{x} \neq \boldsymbol{x}^*$. Hence

$$dV/dt < 0$$

  for population states sufficiently close to the fixed point.
- $V(\boldsymbol{x})$ is therefore a strict Lyapounov function in this region.
- And the fixed point $\boldsymbol{x}^*$ is asymptotically stable.

The previous three theorems established

$$\boldsymbol{E} \subseteq \boldsymbol{A} \subseteq \boldsymbol{N} \subseteq \boldsymbol{F}.$$

HW: Exercise 9.9 and Example 9.10.