Evolutionary game theory and replicator dynamics (draft version 4)

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Contents

1 introduction

Ordinary non-cooperative game theory is usually presented as a (strategic) game where hyper rational players (companies or persons) meet, and they will reach an equilibrium by abstract reasoning. Those hyper rational players can, in an abstract and rational way, reason about their pay-off (profit) when they play a certain strategy and the other players play some other strategies. Moreover the players, in some games, can predict how the other players will play when they play a certain strategy. All of this is possible because all the players in the game are assumed to be rational and when the players can assume that the other players are rational their behaviour can be predicted.

The Evolutionary game theory has a very different point of view on the games and on the rationality of the players. Here the players are not called players, instead they are called individuals because they do not choose any strategy, they just play the strategy that they are pre-programmed to play. In this model many pre-programmed individuals are randomly and repeatedly played against each other and the result of each of those games determine how the next phase of the game will be configured. If a type of individual, pre-programmed to play a certain strategy, gets a higher then average pay-off, its share of the total population will increase in the next phase of the game. If, on the other hand, it gets a lower then the average pay-off, then its share of the population will decrease. The object of study of the evolutionary game theory is not the behaviour of the individuals, instead it is the dynamic system that controls the behaviour and the relative shares of different types of pre-programmed individuals.

Even if evolutionary game theory takes a different view and formulates a totally different model, it is still game theory - the strategic problems will be the same, but the way to solve them will be different. What strategies will be played? What equilibrium will be reached? How can we explain why the game reached a certain solution and not another solution?

Some questions that arise in evolutionary game theories are: How can games be formulated as population models? How can a system of differential equations occur in a game? How can those dynamic systems be solved and what does the solution to the systems say about the solution of the game? How can the mathematic theory on dynamic systems be used to say something about the solutions of the game? How are the solutions to the dynamic systems linked to the concept of Nash equilibrium in ordinary game theory? What concepts from ordinary game theory are important in evolutionary game theory? All of those questions will be discussed in this thesis. Of course, in most cases, we will not give any straight answer because there are no straight answers or they are very complicated.

The material in this thesis is organized as follows:

In chapter 2 fundamental concepts of game theory are presented. The presentation is concentrated on concepts that are important for the analysis in later chapters. Of great importance are mixed strategies, how the (expected) pay-off is defined for mixed strategies, Nash equilibrium, bi-matrix games and symmetric two-player games.

In chapter 3 the replicator dynamics is instroduced. The central concepts of replicators and replicator dynamic models are presented. Pay-off as fitness and derivation of the replicator dynamic equations are presented. Population shares as a state in the dynamic process are treated. Finally solutions to system of ordinary differential equation and stability criteria are discussed.

In chapter 4 the concept of evolutionarily stable strategy and its connection to Nash equilibrium is discussed. And a summary and a conclusion are made about the subject of this thesis.

2 Game theory

In order to introduce the evolutionary game theory, it is necessary first to discuss the basic elements of the game theory. In this chapter I will introduce the fundamental definitions and concepts of game theory, which are prerequisite for understanding the evolutionary game theory. I will discuss only the definitions and concepts that are necessary for explaining the evolutionary game theory. I will keep the mathematics on a fairly high level since the evolutionary game theory uses a lot of mathematics. In general, the games in this thesis will not be complicated - it is the concepts of evolutionary game theory that are of most interest. Most of the examples are two person-games in matrix form. The types of games that will be treated are non-cooperative games.

2.1 Basic concepts and definitions

A game consists of three different parts:

- A set of n-players $I = \{I_1, \ldots, I_n\}$.
- A set of strategies for each player, $S_i = \{s_{i1}, \ldots, s_{im_i}\}$ where i indicate that the strategies are for player I_i and m_i is the number of strategies for player i. S_i is called the player's strategy space.
- A pay-off function π_i for each player I_i that assign a pay-off for each strategy combination. $\Pi = (\pi_1, \ldots, \pi_n)$ is a vector containing each player's pay-off for a certain strategy combination.

The set of players is usually composed of persons or companies but it can also be composed of something else. The set of strategies is the actions that are available for the players (persons or companies), and the pay-off is the utility that the player gets when a certain strategy combination is played. The pay-off can be in form of profit when the players are companies or it can be in form of utility when the players are persons. Let

$$S = S_1 \times \ldots \times S_n \tag{1}$$

that is the Cartesian product between the strategy space for each player and defines the set of all pure strategy combinations.

Formally the definition of a game is

Definition 1 *A game is a triplet $G = (I, S, \Pi)$ where I is the set of players, S is the set of strategy spaces and Π is a combined pay-off function.*

To make the definition and the discussion clearer, we start with one of the most famous games, the so-called Prisoner's dilemma. (For a popular science presentation of the dilemma see [7].)

Example 1 *Prisoner's dilemma.*
In Prisoner's dilemma there are two players I_1 and I_2. The two players are suspected to have committed two crimes: one big and one small crime. The police do not have enough evidence for the big crime, they only have evidence for the small crime. The only way that the players can be convicted for the bigger crime is to make any of them confess. If no one confesses, both players

will be convicted for the small crime (and free from the big crime). If one of the players confesses, the other player will be convicted for the big crime and the confessing player will be freed from charge for both crimes. If both players confess the big crime, both of them will be convicted for the big crime. The players' two strategies are: either to keep silence or to confess - $S = (s, c)$. The pay-off matrices for this game has the following form, where player 1 is the row player and player 2 is the column player. The first row (column) is to keep silence and the second row (column) is to confess.

$$A = \begin{pmatrix} -1 & -9 \\ 0 & -6 \end{pmatrix} \tag{2}$$

$$B = \begin{pmatrix} -1 & 0 \\ -9 & -6 \end{pmatrix} \tag{3}$$

The pay-off for this game can be the number of months in prison (-9 is 9 months in prison) or the fine to be paid (-9 is a fine of 9 dollars).

A pure strategy for a player is a strategy that he always chooses and always plays it. If a player always plays the same strategy it is said that he plays a pure strategy. The opposite of a pure strategy is when the player randomly (according to a probability distribution) chooses different strategies. Of course to play a pure strategy, for instance strategy s_i, is a special case of a mixed strategy: when the player plays strategy s_i with probability 1 and all the other strategies with probability 0. In example (1) a pure strategy is when player 1 always confesses and a mixed strategy is when the player, with probability 0.5, will keep silence and with probability 0.5 will confess.

For the case when only pure strategies are available, the pay-off function is

$$\pi_i : S \mapsto R^1. \tag{4}$$

The pay-off function for each player assigns a real value to each possible strategy in the combined strategy space, defined in (1).

By combining each of those individual pay-off functions, we get the combined pay-off function

$$\Pi : S \mapsto R^n. \tag{5}$$

This function assigns a pay-off to each player for each possible strategy in the combined strategy space. Of course this combined pay-off function is $(\pi_1, \ldots, \pi_1)$.

Example 2 *Prisoner's dilemma (continued) .*
In example (1) there are two pure strategies: to keep silence or to confess. The solution to this game is that both players will confess. To see this, first we should look at the case when player 1 keeps silence and player 2 will be free from any punishment if he confesses. In this case player 2's best reply is to confess, when player 1 keeps silence.

In the other case, when player 1 confesses, the best strategy, player 2 can use is to confess too. If player 2 doesn't confess, while player 1 confesses, he would get a lower pay-off -9, instead of -6 in case that he (player 2) also confessed. It is clear that his best reply when player 1 confesses is to confess too.

Therefore, the best strategy that player 2 can play, independent of what player 1 is doing, is to confess. By symmetric, the same reasoning is valid for player 1 and the best that player 1 can do is to confess independent of what player 2 does.

2.2 Mixed strategies

In many cases the players do not have a pure optimal strategy, instead they must, randomly according to a probability distribution, chose between different strategies to optimise their pay-off. For instance, a soccer player can not shoot the penalty in the same corner all the time, even if he is very good at just one corner. By always shooting in the same corner, the player is very predictable and it makes it much easier for the goal-keeper to save the penalty. Other example when a mixed strategy gives higher pay-off is when to bluff in poker-games.

Let I_i be a player with strategy space $S_i = \{s_{i1}, \ldots, s_{im_i}\}$ where m_i is the number of strategies in the space. If the player plays a pure strategy he will chose one strategy $s_{ij} \in S_i$ and always play the same strategy. In a mixed strategy, the player instead will chose each strategy in S_i with a probability $P_i = (p_{i1}, \ldots, p_{ij}, \ldots, p_{im_i})$. p_{ij} is the probability that the player choses strategy $s_{ij} \in S_i$. As mentioned earlier a pure strategy is a special case of the mixed strategy: if a player plays the pure strategy $s_{ij} \in S_i$, the player plays the mixed strategy $(p_{i1}, \ldots, p_{ij}, \ldots, p_{im_i}) = (0, \ldots, 1, \ldots, 0)$.

Definition 2 *Mixed strategy*
A mixed strategy is a probability distribution $P_i = (p_{i1}, \ldots, p_{ij}, \ldots, p_{im_i})$ over a strategy space $S_i = \{s_{i1}, \ldots, s_{im_i}\}$ where p_{ij} is the probability for strategy $s_{ij} \in S_i$. The probability distribution has the following two properties:

$$\sum_{j=1}^{m_i} p_{ij} = 1 \tag{6}$$

and

$$p_{ij} \geq 0. \tag{7}$$

The pay-off in the case of mixed strategies is a little bit more complicated. The Cartesian product, defined in (1), is not enough any longer for defining the pay-off function in the mixed strategy case. We need some more definitions to make the concept of mixed strategies clearer and to define the (expected) pay-off function for mixed strategies.

Definition 3 *Mixed strategy simplex $\triangle_i$*
A mixed strategy simplex is

$$\triangle_i = \{(p_{i1}, \ldots, p_{im_i}) \in R^{m_i} : \sum_{j=1}^{m_i} p_{ij} = 1\}. \tag{8}$$

A mixed strategy simplex $\triangle_i$ is all possible mixed strategies for a player I_i. Geometrically the $\triangle_i$ is a plane in R^{m_i}. Each player has its own $\triangle_i$ and the pay-off function must be a function from the Cartesian product over all those $\triangle_i$. The mixed strategy space for a game is

$$\theta = \triangle_1 \times \ldots \times \triangle_n \tag{9}$$

and its dimension is $m = \sum_{i=1}^{n} m_i$.

A $p = (p_{11}, \ldots, p_{1m_1}, \ldots, p_{i1}, \ldots, p_{im_i}, \ldots, p_{n1}, \ldots, p_{nm_n}) \in \theta$ is called a mixed strategy profile and is a combined probability distribution for each player.

The pay-off function for each player has the form:

$$\pi_i : \theta \mapsto R^1.$$

The interpretation of this function is more or less the same as for the pure strategy pay-off function. Here we assign a real value (the pay-off) to all combinations of mixed strategies for each player.

To be able to define the pay-off in the mixed strategy case, we must define the probability that a pure strategy $s = (s_1, \ldots, s_n)$ will be used when a mixed strategy profile $x \in \theta$ is played. The probability that player i plays strategy s_j is value x_{ij} in x, where the i index indicates the player and the j index indicates the strategy. (x is not a matrix but we use double index to stress which strategy that each player is playing.) Under the assumption that each player's random choice is independent, the probability that a pure strategy is played is simply the product

$$x(s) = x_{1s_1} \times \ldots \times x_{ns_n} = \Pi x_{is_i}.$$

This probability is used to define the (expected) pay-off for a mixed strategy $x \in \theta$

$$\pi_i(x) = \sum_{s \in S} \pi_i(s) x(s). \tag{10}$$

An interpretation of this function is: $\pi_i(s)$ is the pay-off for player I_i when strategy s is played and $x(s)$ is the probability that the pure strategy s will be played. The expected pay-off for a pure strategy s, given the mixed strategy x, is the pay-off for s times the probability that it will be played ($\pi(s) \times x(s)$). The (expected) pay-off for a mixed strategy x is when the sum over the expected pay-off over all strategies in S (as defined in (10)).

A special case of games is the so-called bi-matrix games. They can be presented in a simple way and we will use them extensively during the following presentation and in the chapter on evolutionary game theory.

Definition 4 *Bi-matrix game.*
A game containing two players where player I_1 is the row player with pay-off matrix A and player I_2 is the column player with pay-off matrix B.

That player 1 is a row player means that his strategies are represented by rows in the matrices - row i represent strategy s_{1i}. That player 2 is a column player means that his strategy is represented by the columns in the matrices - column j represent strategy s_{2j}.

The pay-off function for such a game is quite simple and for player I_1 the pay-off is

$$\pi_1(x) = x_1 A x_2^T \tag{11}$$

and for player I_2 the pay-off is

$$\pi_1(x) = x_1 B x_2^T. \tag{12}$$

In the above equations $x = (x_1, x_2)$, $x_1 = (x_{11}, \ldots, x_{1m_i})$ and x is an instance of the mixed strategy probability distribution.

The Prisoner's dilemma described in example (1) is a bi-matrix game. The pay-off for player I_1 is

$$\begin{aligned} \pi_1(x) &= x_1 A x_2^T \\ &= (x_{11}, x_{12}) \begin{pmatrix} -1 & -9 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} \\ &= -(x_{21} + 9x_{22})x_{11} - 6x_{22}x_{12} \end{aligned}$$

2.3 Nash equilibrium

One goal for game theory is to predict an outcome of a conflict. In other situations the goal for game theory can be to explain why certain strategies were chosen. In both cases the Nash equilibrium plays an important roll. Informally, a strategy s_i for a player I_i is a Nash equilibrium, if it is the best strategy he can chose, given that none of the other players change strategy and that this property holds simultaneously for all players. This implies that, given that a game reaches a Nash equilibrium, none of the players has any motivation to choose any other strategy.

Definition 5 *Nash equilibrium for pure strategies*
the strategies $(s_1^*, \ldots, s_n^*)$ in a game $G = (I, S, \Pi)$ where $s_j^* \in S_j$ constitute a Nash equilibrium if

$$\pi_i(s_1^*, \ldots, s_i, \ldots, s_n^*) \leq \pi_i(s_1^*, \ldots, s_i^*, \ldots, s_n^*) \tag{13}$$

for all $s_i \in S_i$ and for all $i = 1, \ldots, m$.

It is clear from the definition that a Nash equilibrium s^* is a solution, for each player i, to the equation

$$max_{s_i \in S_i}\{\pi_i(s_1^*, \ldots, s_i, \ldots, s_n^*)\} \tag{14}$$

that maximizes s_i when all the other strategies are treated as fixed. Taken all the other players' strategies as fixed and already determined, a Nash equilibrium is a best reply to the others' strategies.

In example (2) we discuss the solution of the game presented in example (1). It is clear from the discussion in example (2) and the definition of Nash equilibrium in definition (5) that the solution is also a Nash equilibrium.

It is straightforward to extend the definition of Nash equilibrium to the (general) mixed strategies case.

Definition 6 *Nash equilibrium for mixed strategies*
Let $p^* = (p_1^*, \ldots, p_i^*, \ldots, p_n^*)$ be a mixed strategy profile in θ, p is a mixed Nash equilibrium if

$$\pi_i((p_1^*, \ldots, p_i, \ldots, p_n^*)) \leq \pi_i(p^*)$$

for all $p_i \in \triangle_i$ and for $i = 1, \ldots, n$.

Informally the definition means that, given the other players strategies as fixed, there exists no mixed strategy that gives a higher pay-off. Given a mixed strategy Nash equilibrium, none of the players has any motivation to change their mixed strategy.

Example 3 *Matching pennies*
In this game there are two players - $I = \{I_1, I_2\}$ - with two strategies. Each player can choose one side of a coin - $S = \{Head, Tail\}$. Each player takes one penny simultaneously and puts one side up in front of him. If both put the same side - Head or Tail - up, player 1 wins the two coins. If they have put up different sides of the pennies, player 2 wins the two coins. The pay-off matrix for player 1 is

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{15}$$

and for player 2 is

$$B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{16}$$

It is clear that the game does not have any Nash equilibria in pure strategies.

2.4 Symmetric 2x2 Games

Definition 7 *Symmetric two-player game*
A game $G = (I, S, \Pi)$ is a symmetric two-player game if $I = \{1, 2\}, S1 = S2$ and $\pi_1(s_1, s_2) = \pi_2(s_2, s_1)$ for all $(s_1, s_2) \in S$

In the case of symmetric two-player games, with pay-off matrices A and B, following identity holds $A = B^T$. This means that the game can be presented with just the A matrix. A simple example of a symmetric two-player game with just two strategies is the Prisoner's dilemma in (1), the whole game can be presented just with the matrix A in (2)

Definition 8 *Positive affine transformation*
A positive affine transformation is a transformation

$$y = ax + b \tag{17}$$

$$a, b > 0 \tag{18}$$

(x can either be variable or a function $x(t)$.)

The set of Nash equilibrium is invariant to positive affine transformations (see [2] p. 112-115). This means that if there exist two numbers $\lambda_i \geq 0$ and $\mu_i \geq 0$ for each player i such that

$$\hat{\pi}_i = \lambda_i \pi_i + \mu_i \tag{19}$$

then the two games $G = (I, S, \pi)$ and $\hat{G} = (I, S, \hat{\pi})$ have the same Nash equilibrium and are equivalent (in some sense).

The set of Nash equilibrium is also local shift invariant. This means that for the row player with pay-off matrix A you can replace a column in A with another column $\hat{A}$ as long as the difference between the corresponding elements in A and $\hat{A}$ is the same. That is

$$\hat{a}_{ij} = a_{ij} + \lambda_j \tag{20}$$

for all i.

For the column player in a game in bi-matrix form the property is the same, but of course for the rows. For a more strict and general treatment of the local shift invariant property see [11] p. 17-19.

An interesting special case of this fact is the symmetric two-player game. The pay-off matrix for this game can be written in general form:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{21}$$

By subtracting certain values from each column, the matrix transforms into the following form

$$A = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix} \tag{22}$$

By setting $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$ the matrix takes the following form

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{23}$$

the process of transforming the pay-off matrix into this form is called normalization and the matrix is said to be normalized. It is clear that the pay-off matrices (21) and (23) have the same Nash equilibrium.

The Prisoner's dilemma - (in example 1) - represented by the pay-off matrix A in (2) turns into

$$A = \begin{pmatrix} -1 & -9+9 \\ 0 & -6+9 \end{pmatrix} \tag{24}$$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \tag{25}$$

after local shifting.

3 Replicator dynamics

3.1 Introduction

In evolutionary game theory the perspective on the games is from a different angle. In ordinary game theory rational players meet once in a game and come to a solution by using abstract reasoning about the behaviour of the other players.

The basic evolutionary game theory has the following setup:

- A large set of players.
- Each player is programmed to play a pure strategy from the strategy space.
- Two players are randomly paired to play a game. In this way a number of players are paired and the games are played.
- The probability for a player to survive and to produce a new player depends on the pay-off in its game, compared with the average pay-offs for the other player.

Another difference between traditional and evolutionary game theory is that the latter is not interested in the players' choices, because they can't choose any strategy. In evolutionary game theory the players are programmed to play just a pure strategy. Instead the evolutionary game theory focuses on the behaviour of this dynamic process, and especially in what will happen in the long-run (asymptotic behaviour).

3.2 Replicator

One fundamental part of replicator dynamics is the players in the game: the replicators. In most presentations the replicators are presented as some biological species. Each replicator is programmed to follow one strategy. It is often said that it is in their genes to follow that strategy. Each replicator can produce an offspring that is programmed - has the same gene - to follow the same strategy as its parent. The reproduction is an asexual reproduction, i.e. each replicator can produce its own offspring in isolation. To sum up a replicator has the following two properties:

- Pre-programmed strategy: its strategy in a game is determined.
- Asexual reproduction: it can replicate by itself.

The replicators can be viewed in a much wider sense. It does not need to be species with a strategy pre-programmed in the genes. Instead a lot of behaviours can be viewed as replication or at least almost satisfy the replicator definition. Fashion, life-style, the use of language phrases, rules of thumbs and scientific ideas are some examples. In all of those examples the reproduction is done in an asexual way and even in a non-biological way. Instead the reproduction is done by imitations and educations. It is also clear from the example that they can be viewed as replicators. It is also clear that there are aspects that make them less suitable to be viewed as replicators. For instance a person can change his life-style and does not need to stick with one all his life.

Even if human behaviours can be viewed as replicators, the example that best fits the replicator dynamics game model is the one that does not involve any thinking at all.

3.3 Fitness

The pay-off in replicator dynamics is usually called fitness and it has a special meaning. The meaning of fitness is of course the same as in evolutionary biology. Fitness refers to how well a individual is adapted to an (dynamic) environment. The (dynamic) environment is composed by: more or less static things, as the climate or, in economics, the number of persons living close to a store; and more dynamic things, as the behaviour of the other species in the environment, for instance the price strategy for the competing companies.

As in evolutionary biology the fitness in replicator dynamics refers to the ability to survive and to reproduce in this dynamic environment.

3.4 Replicator dynamics equations

The standard way to introduce replicator dynamics into game theory is by the example of Hawks and Doves ([2] p. 416-420, [5] p. 193-195, [11] p. 27-28, [6] p. 173-176 and [8] p. 49-51).

Example 4 *Hawk-Dove game*
In the Hawk-Dove game there are two types of replicators, the Hawks and the Doves. (Formally it is the same species programmed with different strategies, but they can be viewed as two different species.) The raptors live in an area that can feed n individuals. Each evening they reproduce asexually in a nest. There are two types of nest Good and Bad. In a Bad nest a raptor produces u offspring and in a Good nest the raptor produces $u+2$ offspring. There is only $2n$ Good nest available. If two Doves meet at a Good nest, one of the Doves will get the Good nest with equal probability. The expected number of offspring when two Doves meet is $u+1 = \frac{(u+2+u)}{2}$. If a Dove and a Hawk meet at a Good nest, the Dove will leave the Good nest to the Hawk. If two Hawks meet at a Good nest, they will fight about the nest until the expected number of offspring is $u-1$. This is a 2x2 symmetric game with pay-off matrix

$$A = \begin{pmatrix} u-1 & u+2 \\ u & u+1 \end{pmatrix}. \tag{26}$$

Let p be the fraction of Hawks in the population, while $1-p$ is the fraction of Doves in the population. Let $x = (p, 1-p)$ and let τ be a time interval then

$$f_d(p)\tau = (u+1-p)\tau \tag{27}$$

is the expected number of offspring for one Dove raptor in the time interval τ. There is $n(1-p)$ Doves for reproduction, so the total number of Doves after reproduction is

$$n(1-p)(1+f_d(p)\tau).$$

The number of offspring for one Hawk in the same time period is

$$f_h(p)\tau = (u+2(1-p)-p)\tau$$

and the number of Hawks after reproduction is

$$np(1+f_h(p)\tau).$$

Let

$$f(p) = pf_h(p) + (1-p)f_d(p)$$

then $f(p)\tau$ is the total number of offspring in the time period τ and $n(1+f(p)\tau)$ is total number of raptors after time period τ. The change of fraction of Hawks in a time interval τ is then

$$p(t+\tau) = p(t)\left(\frac{1+\tau f_h(p)}{1+\tau f(p)}\right).$$

Rewriting of the equation gives

$$\frac{p(t+\tau)-p(t)}{\tau} = p\left(\frac{f_h(p)-f(p)}{1+\tau f(p)}\right).$$

Letting $\tau \to 0$ (derivate) gives

$$\frac{\partial d}{\partial t} = p(f_h(p) - f(p))$$

which is the replicator equation.

The general case is a little bit more complicated and we need some more definitions in order to be able to derivate the general replicator equation. We will not give a truly general derivation of the equation, but only a more general derivation. (General or partly general derivation of the equation can be found in [11] p. 71-74, [5] p. 190-193 and [2] p. 427-429.)

Let $x_i(t)$ be the number of individuals playing strategy s_i at the time t, further let $x(t) = \sum x_i(t)$ the total number of individuals in the population at time t. The fraction individuals that play strategy s_i at time t is

$$p_i(t) = x_i(t)/x(t) \tag{28}$$

and the population state vector is defined as $p(t) = (p_1(t), \ldots, p_m(t))$. $p(t) \in \triangle$ defined in (3) and is formally a mixed strategy for any of the individuals in the population, further $(p(t), p(t)) \in \Theta = \triangle \times \triangle$. Because there are always two players in this type of games, we can write the expected pay-off for playing a pure strategy s_i when the other player plays the mixed strategy p as $\pi(s_i, p)$. The expected pay-off when both players play the mixed strategy p can be written as

$$\pi(p,p) = \sum_{i=1}^{m} p_i \pi(s_i, p) \tag{29}$$

Given that the pay-off (or in other words the fitness) measures the number of offspring in one time unit, we can now derivate the replicator dynamics equation. First we define the background fitness $\beta \geq 0$, that is the reproduction rate independent of the outcome of the game. We also define the death rate $\delta \geq 0$, that is also independent of the outcome of the game and the same for all individuals. The expected number of offspring reproduced in a time t, programmed to play strategy s_i is $\beta + \pi(s_i, p(t)) - \delta$, given the current population state $p(t)$.

Given the above definitions, the derivation of the number of individuals playing strategy s_i becomes

$$x_i' = (\beta + \pi(s_i, p) - \delta)x_i \tag{30}$$

where the time dependency has been omitted. The replicator dynamics equation is of course the derivation of the population state function $p_i(t)$. Rewriting the equation in (28) gives

$$x_i(t) = x(t)p_i(t)$$

taking the time derivate, using the product rule [9] p. 134, of both sides gives

$$x_i'(t) = x'(t)p_i(t) + x(t)p_i'(t)$$

re-arranging the equation gives

$$p_i'(t)x(t) = x_i'(t) - x'(t)p_i(t).$$

Using (30) gives

$$p_i'(t)x(t) = (\beta + \pi(s_i, p(t)) - \delta)x_i - (\beta + \pi(p(t), p(t)) - \delta)x(t)p_i(t)$$

dividing both sides with $x(t)$ gives

$$p_i'(t) = (\beta + \pi(s_i, p(t)) - \delta)p_i(t) - (\beta + \pi(p(t), p(t)) - \delta)p_i(t) \tag{31}$$

after simplification (suppressing the time variable), we have the replicator dynamics equation

$$p_i' = (\pi(s_i, p) - \pi(p, p))p_i. \tag{32}$$

This is the differential equation that controls the development of the game. How it will develop depends on this equation but also from its start position - its initial value. Those initial values are of course the relative frequencies $p_i(t)$ of the different strategies when the game starts. One common feature for replicator dynamics is that if the game starts with $p_i(t_0) = 0$, then the strategy will be $p_i(t) = 0$ for all t. The reason for this is that if it is no species from the start that play strategy s_i, then there is no species that can reproduce that strategy. (In evolutionary biology this means no mutation.)

Sometimes it is possible to find an explicit solution to this replicator dynamics equation, but in many cases it is impossible to solve those equations. Instead of finding an explicit solution, it can be possible to find the asymptotic behaviours of the system.

3.5 Ordinary differential equation

Differential equation is a very big and important field in mathematics. Any explicit treatment of the subject is beyond the limits of this thesis. In this thesis we will only discuss some fundamental properties and definitions that are necessary for solving the replicator dynamics equation. There exists a lot of good literature on ordinary differential equations. Two good books are [10] and [1](in swedish); good presentation with examples from economics can be found in [9] p. 729-802 and [5] p. 164-187.

An ordinary differential equation is an equation that contains a function $y(t)$ and one or more of its derivates $y'(t), y''(t), \ldots, y^n(t)$. Formally this can be written

$$F(t, y, y', \ldots, y^n) = 0. \tag{33}$$

The solution to this problem is a function $y(t)$, such that $F(t, y'(t), \ldots, y, y^n(t)) = 0$ for all $t \in [a, b]$. By solving the above equation (33) we get the general function satisfying the equation. This general solution usually contains n arbitrary constants. For a total determination of the function we also need n initial conditions. These initial conditions have the form $y^j(0) = a_j$ for $j = 0, \ldots, n-1$.

The variable t usually refers to the time in economics, physics and biology. And the differential equation tells how the system changes over time, according to the equation.

The order of a differential equation is the highest derivate that the equation contains. The order of the differential equation in the above definitions is n. The replicator dynamics only contains the first derivate and therefore it is of order 1.

3.5.1 Stationary state and stability

In general it is not possible to find an explicit function that satisfies the equation (33). As mentioned above, the variable t often refers to the time and the equation tells how the system changes over time. Thus it can be possible to tell something about the system even if no explicit solution can be found. In the case of the replicator dynamics we are often interested only in the long-run outcome of the system. The system's behaviour (asymptotic behaviour) in the long-run can sometimes be determined without first finding the explicit solution. Because replicator dynamics usually is of order 1, we only discuss that special case.

Definition 9 *stationary state (point)*
Let

$$x'(t) = F(x) \tag{34}$$

then a is a stationary state (point) if

$$F(a) = 0. \tag{35}$$

One fundamental property of the stationary state is that when the system starts in a stationary state it will never leave that state. Even if this property by definition is shared among all stationary states, the states may differ very much. Some stationary states will never be reached if a system does not start in that state. Other stationary states will attract starting points situated far away; and as t grows, we will move closer to the stationary point.

This is the concept of the stable stationary states and the unstable stationary states. The formal definition of a stable state is quite technical and it can be found theoretically explained and discussed in [10] p. 70 ff, [11] p. 243 ff and [1] p. 253 ff.

Informally the concept is not so difficult to understand. A stationary state is stable if it attracts points (from all directions) close to it. This means that if we move a little bit away from the state, we will return to the same stationary state. A stationary state is unstable if it is not stable. This means that if a point moves a bit away from the stationary state, it is not sure that it will move back to that stationary state. For an unstable stationary state it always exists at least one direction that will not lead the system back to the stationary state. (It should also be noted that some stable stationary states will never be reached, instead the system will get closer and closer to the state. This is an asymptotic stationary state for the system.)

To decide if a stationary point is stable or unstable, the following theorem can be used ([9] p. 786).

Theorem 1 *Stability condition on stationary states*
Let a be stationary point ($F(a) = 0$) then

- *if $F' < 0$ then a is a stable stationary state.*
- *if $F' > 0$ then a is a unstable stationary state.*

However, this theorem (1) does not say anything about the case, when $F'(a) = 0$.

Usually the stationary states for the replicator dynamics equation are of great interest. The reason for this is that we are not interested in what is happening in a specific time t, instead we are interested in what is happening in the long-run. In the long-run we will reach or be close to some of the stationary states and the population state at this stationary state is of great interest.

3.6 Symmetric 2x2 Games

As we showed earlier all symmetric 2x2 games can be written after transformation in form (23). It can be showed that the stationary states for the replicator dynamics equation is invariant to scaling and to local shift in the same way as ordinary games ([11] p. 73-74). Therefore it is enough to analyse the replicator dynamics equation that occurs in the matrix in (23).

Using (11) we can rewrite the replicator dynamics equation for the normalized pay-off matrix in the following way

$$p_1' = (\pi(s_1, p) - \pi(p, p))p_1 = ((1, 0)A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - (p_1, p_2)A \begin{pmatrix} p_1 \\ p_2 \end{pmatrix})p_1 \tag{36}$$

Using (23) in the above expression gives

$$((1, 0) \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - (p_1, p_2) \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix})p_1$$

$$= (a_1 p_1 - a_1 p_1^2 - a_2 p_2^2)p_1$$

To simplify the expression we can break out $a_1 p_1$, this gives

$$(a_1 p_1(1 - p_1) - a_2 p_2^2)p_1.$$

Using the fact that $p_1 + p_2 = 1$ gives

$$(a_1 p_1(1-(1-p_2)) - a_2 p_2^2)p_1$$

$$(a_1 p_1 p_2 - a_2 p_2^2)p_1.$$

Breaking out p_2 gives

$$(a_1 p_1 - a_2 p_2)p_1 p_2.$$

To sum up we have

$$p_1' = (a_1 p_1 - a_2 p_2)p_1 p_2. \tag{37}$$

this is the general replicator dynamics equation for the normalized symmetric 2x2 game. The equation for p_2' can be simplified in the same way and will result in the following expression

$$p_2' = (a_2 p_2 - a_1 p_1)p_1 p_2. \tag{38}$$

Those equations for p_1' and p_2' can be used to analyse any symmetric 2x2 game.

Example 5 *Prisoner's dilemma (continued)*
The game of prisoner's dilemma was introduced in (1). It is a 2x2 symmetric game and its normalized pay-off matrix can be found in (25). Substituting the values in (25) into equation 37 and using the fact that $p_1 + p_2 = 1$ gives

$$p_1' = (2p_1 - 3)(1-p_1)p_1.$$

The stationary point $p_1 = 1.5$ is not an acceptable solution because it is outside the interval $[0,1]$. The other two stationary points $(1,0)$ and $(0,1)$ are acceptable. To determine the stability of those stationary points, they will use theorem (1).

Derivations of the right-hand side of the equation gives

$$F' = D((2p_1-3)(1-p_1)p_1) = D(-2p_1^3 + 5p_1^2 - 3p_1) = -6p_1^2 + 10p_1 - 3.$$

$F'(1) = 1$ and $F'(0) = -3$. The only stable stationary state is therefore (0,1). Thus, the only stable stationary state for the prisoner's dilemma replicator dynamics equation is to always confess and this is the only Nash equilibrium for that game.

Example 6 *Hawk-Dove game(continued)*
The Hawk-Dove game in (4) is a symmetric 2x2 game. The pay-off matrix is (26) and after normalization

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Plugging those values into (37) and use the fact that $p_1 + p_2 = 1$ gives

$$p_1' = (1-2p_1)(1-p_1)p_1$$

setting $p' = 0$ gives the stationary states $(1,0),(0,1)$ and $(0.5,0.5)$. To find out if any of the stationary states are stable we will use theorem (1). Derivations of the right hand side of the equation gives

$$D((1-2p_1)(1-p_1)p_1) = 6p_1^2 - 6p_1 + 1$$

By plugging in the values at the stationary state in the derivation, we find that the only stable solution is $(0.5,0.5)$, (the only solution with $F'(0.5) \leq 0$).

Example 7 *General Hawk-Dove game*
We can make the Hawk-Dove in (4) more general in the following way. The extra off-spring produced in a Good nest is v and the cost for a fight is c. The background pay-off is still set to u. With the same motivation as for (26) we get the following expected pay-off matrix

$$A = \begin{pmatrix} u + \frac{(v-c)}{2} & u+v \\ u & u+\frac{v}{2} \end{pmatrix}.$$

After normalization this matrix transforms into

$$A = \begin{pmatrix} \frac{(v-c)}{2} & 0 \\ 0 & -\frac{v}{2} \end{pmatrix}.$$

Substituting those values into equation (37) and using $p_1 + p_2 = 1$ gives

$$p_1' = (\frac{v}{2} - \frac{c}{2}p_1)(1-p_1)p_1.$$

As in the special case of Hawk-Dove game, we have the two unstable solutions $(1,0)$ and $(0,1)$. We also have a new stable solution $(\frac{v}{c}, 1-\frac{v}{c})$. We can also verify that the special case, presented in (4) and (6), gives the same solution. In that example $v = 2$ and $c = 4$. The solution is $(\frac{2}{4}, 1-\frac{2}{4}) = (\frac{1}{2}, \frac{1}{2})$ and it is of course the same solution as we found in (6).

4 Extension and conclusions

The following chapter contains a short and informal discussion of the concept of Evolutionarily Stable Strategy (ESS). ESS is the fundamental stability criteria used in evolutionary game theory. ESS and its connection to other equilibrium definitions is a central part of evolutionary game theory. A major discussion of ESS and its connection to other equilibrium definitions is outside the scope of this thesis. Here we will only informally discuss the concept of ESS and some of its properties including some relations to Nash equilibrium. (Good and more formal treatment of ESS and its properties can be found in [11] p. 33-68, [5] p. 149 ff and [2] p. 422-427).

Finally we will discuss and make some conclusions about the subject for this thesis.

4.1 Evolutionarily stable strategy

The concept of evolutionarily stable strategy also comes from evolutionary biology. The setup is the same as for the dynamic replicator games: individuals are pair wise repeatedly randomly matched to play a game. The individuals are programmed to play a pure strategy. What will happen if a small fraction of the original individuals was replaced with individuals that were programmed to play another (mixed) strategy? Those new individuals are called mutants, and they are said to be mutated. Will the mutants be driven to extinction or will their share of the population grow stronger?

Let ϵ be the (small) fraction of mutated individuals in the (large) population then $1-\epsilon$ is the fraction of non-mutated individuals. Suppose that each individual in the population is drawn with equal probability and that the individuals are drawn independently, then the probability to draw a mutant is ϵ and to draw a non-mutant is $1-\epsilon$.

Let $x \in \triangle$ be the mixed strategy played by the non-mutated individuals (formally of course it is the population shares for the different pure strategies) and let $y \in \triangle$ be the mixed strategy played by the mutated individuals.

Let $\pi(p_1, p_2)$ be the (expected) pay-off function. The (expected) pay-off when a non-mutated individual is playing is

$$\pi(x, \epsilon y + (1-\epsilon)x) \tag{39}$$

and when a mutated individual is playing

$$\pi(y, \epsilon y + (1-\epsilon)x). \tag{40}$$

If there exists an ϵ_0 such that

$$\pi(x, \epsilon_0 y + (1-\epsilon_0)x) > \pi(y, \epsilon_0 y + (1-\epsilon_0)x). \tag{41}$$

for all mutations $y \in \triangle$ and for all $\epsilon < \epsilon_0$, x is said to be an evolutionarily stable strategy. The interpretation of the equation is that it is not possible to get higher pay-off by a change (called mutation) in a small part of the population.

Evolutionarily stable strategy only deals with one mutation at each time, it does not say anything about two different types of mutation injected in the population at the same time. The population will be injected by a fraction of one type of mutant. After that the system must get time to move back to its stability state before a new and different type of mutant is allowed to be injected into the population. (Informally it is a bit like the concept of Nash equilibrium; it is stable if only one of the players changes at the same time, it does not say anything about what will happen if two players change their strategies at the same time.)

The pay-off function must, in the same way as in replicator dynamics, measure an individual's fitness to a dynamic environment. The pay-off function measures, in some way, the possibility for an individual to reproduce. If a strategy always has lower (expected) pay-off then it will in the long run fade away and finally be exterminated.

4.2 Nash equilibrium and evolutionarily stable strategy

The relation between ESS and Nash equilibrium connects the evolutionary game theories with the traditional non-cooperative game theory. There exist a lot of definitions and theorems that connect the two concepts. We will not discuss this theory in this thesis, we will only state some fundamental relations between the different concepts.

If x is an evolutionarily stable strategy then x is also a Nash equilibria. This fact states that all ESS are also Nash equilibria, so if we have found one ESS, we also know that it is a Nash equilibrium. Stated more formally the set of ESS is a subset of the set of Nash equilibria.

The criterion for ESS is stronger then the criterion for Nash equilibrium. This fact implies that all Nash equilibria are not ESS. There are a lot of theories and theorems that treat the relation between the two equilibria and analyse the criteria for determining when a Nash equilibrium is also an ESS.

4.3 Conclusions and summary

In this thesis we show how some simple examples taken from non-cooperative game theory can be formulated in a population model. This model is inspired by evolutionary biology and is called replicator dynamics. Game theoretical problems formulated with replicator dynamics transform the problem into a dynamic system problem. The problem consists of a system of ordinary differential equations that is called replicator dynamics equations.

To solve replicator dynamics equations, methods taken from the mathematic theory of ordinary differential equations must be used. However in most cases it is impossible to solve those equations.

In the examples given in this thesis it was possible to find explicit solutions to the equations. Solving differential equations and finding the stationary states for the differential equations gives interesting information about the system. By using criteria for stability on those stationary states it was possible to determine the type of the stationary state. We saw that in the given examples the stable state solutions matched the Nash equilibria for the corresponding game problem.

Evolutionary game theory, and the replicator dynamics that are presented here, are formulated without any rationality assumption on the player. The rationality assumption has been replaced by many other assumptions in the model. As important assumptions in the model we can mention the pre-programmed replicators and that the pay-off, called fitness, must measure the ability to reproduce.

Evolutionary game theory removes some assumptions from ordinary game theory but instead it imposes new assumptions. Evolutionary game theory answers some of the questions raised in ordinary game theory but it also raises new questions. Incorporated with ordinary game theory, evolutionary game theory contributes to the understanding of the game theory with an interesting and important interpretation of a game situation.

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