Statistical Mechanics of Games
—— Evolutionary Game Theory ——

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This paper formulates evolutionary game theory with a new concept using statistical mechanics. This study analyzes the following situations: each player on the lattice plays a game with its nearest neighbor or with a randomly matched player. These situations are formulated using an analogy with the Ising model and the Sherrington-Kirkpatrick model, the simplest models in statistical mechanics. As a result, theoretical calculations agree with classical evolutionary game theory in terms of the parameter size. This paper shows that bifurcations occur in a quenched system with externalities, hence, this system has multiple equilibria.

This paper discusses the simplified Cont and Bouchaud model through our models. We extend the player's behavior and matching in Cont and Bouchaud model.

§1. Introduction

In evolutionary game theory, a large number of players is assumed to search at random for trading opportunities, and when they meet the terms of game are started. We have described the above situations with the classical approaches using the *replicator dynamics*,[1),*)] or a perturbed finite-state Markov process.[2)] In contrast to these approaches, our study formulates a large number of players playing games simultaneously using an analogy with the Ising model and the Sherrington-Kirkpatric model, the simplest models in statistical mechanics.

Numerous papers published recently have used statistical mechanics in evolutionary game theory, Blume,[3)] Diederich and Opper,[4)] McKelvey and Palfrey.[5),6)] However, these papers applied the Ising model[3)] and the standard Sherrington-Kirkpatrick model,[4)] vigorously researched in theoretical physics, in a straightforward manner. Furthermore, they paid very little attention to the basic elements. This paper presents a novel model using statistical mechanics for evolutionary game theory with basic elements.

This paper is organized as follows. In §2, we formulate a model with nearest-neighbor interaction, and compute the order parameter. In §3, we formulate a model for play with a randomly matched player in annealed and quenched systems, and compute the optimal order parameter for each system. In §4, we extend our model to add an externality. In §5, we comment on the econophysics model of Cont and

*) Replicator dynamics:

$$\frac{\dot{x}_i}{x_i} = \big((Ax)_i - x \cdot Ax\big), \quad i = 1, \cdots, n, \quad A : \textit{payoff matrix}.$$

means that if the player's payoff from the outcome i is greater than the expected utility $x \cdot Ax$, then the probability of the action i is higher than before.

Bouchaud[7)] in relation to our model. In §6, we present the conclusions and discuss future work.

§2. Nearest neighbor interaction (Ising model)

2.1. *Theoretical framework*

In this section, we construct a nearest-neighbor interaction model with reference to the Ising model, the simplest model in statistical mechanics.

Let $\boldsymbol{Z}^2$ be the plane square lattice and we refer to the vertex i as the *site*. Each site on the lattice is the address of one player. Every site $i \in \boldsymbol{Z}^2$ is directly connected to a finite number of other sites. The set of sites $B = \{(ij)\}$ directly connected to site i is the *neighbor* of i, j.

A player who has chosen an action strategy receives a payoff from his neighbor, which is determined by his strategy and his neighbor's choice of action.

EXAMPLE 2.1 (Two players and two strategies, symmetric strategic game)

The set of actions of row player 1 is {Action 1, Action 2} and that of column player 2 is {Action 1, Action 2}, and for instance, the row player's payoff from the outcome (Action 1, Action 1) is a, then the column player's payoff is also a.

If the set of actions' index is $\{+1, -1\}$ and payoff $a, b > 0$, then this model corresponds to the Ising model, where the payoff represents the energy.

1\2	Action 1(+1)	Action 2(+2)
Action 1(+1)	a,a	0,0
Action 2(+2)	0,0	b,b

Payoff Matrix 1

PROPOSITION 2.2[*)] We obtain the probability distributions of actions, $\{S_i\}, i = 1, \cdots, N$, and the player's payoff from the outcome is f,

$$P\big(\{S_i\}\big) = Z^{-1}\exp(\gamma f), \tag{2·1}$$

where $\{S_i\}$ is a player i's action, and γ is a non-negative constant ; for instance, γ is the optimal choice behavior,[8),**)] f is the player's payoff from the outcome $\{S_i\}$, and Z is the normalization parameter, with $\sum_{i=1}^{N} P\big(\{S_i\}\big) = 1$.

This implies that if payoff f is greater, then the probability of choosing the action is higher.

DEFINITION 2.3 We define an *order parameter* $m \in \boldsymbol{R}$, as how often a player

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*) We omit this proof. There exist many ways of proving this proposition, however, this form is derived from *the law of the conservation of energy* and *the principle of equal a priori probability.* In this model, the payoff represents the energy in theoretical physics, but it admits negative values. Of course, the total payoff $2f$ is constant. See statistical mechanics textbooks for details.

**) When parameter γ approaches infinity, the model of behavior approaches the best response model. When $\gamma = 0$, the behavior is essentially random, as all strategies are played with equal probability.

has chosen an action in this game.

$$m = \sum_{i}^{N} S_i P\big(\{S_i\}\big), \tag{2·2}$$

where N is the number of the actions.

EXAMPLE 2.4 Considering EXAMPLE 2.1, the actions' index $\{S_i\} = \{1, 2\}, N = 2$, and the order parameter for each case is computed as follows.

(i) If all the players' actions are {Action 1}, then we obtain $m = 1$.

(ii) If all the players' actions are {Action 2}, then we obtain $m = 2$.

(iii) If half of all the players' actions are {Action 1}, then we obtain $m = \frac{3}{2}$.

If the order parameter m is near 1, then we know that there are many more players choosing {Action 1} than {Action 2}. If the order parameter m is near 2, then we know that more players chose {Action 2} than {Action 1}.

If γ is sufficiently large, then the actions for all players are chosen. If γ is sufficiently small, then the actions for all players are essentially random as all strategies are played with equal probability, independent of the payoff size.

In particular, if the actions' index S_i is $\{-1, 1\}$, then the order parameter m is 1,0(random),-1 for the above cases.

DEFINITION 2.5 (Weibull[1]) $x \in \varDelta$ is an *evolutionary stable strategy* (ESS) if for every strategy $y \neq x$, there exists some $\bar{\epsilon}_y \in (0, 1)$ such that the following inequality holds for all $\varepsilon \in (0, \bar{\epsilon}_y)$

$$u\big[x, \varepsilon y + (1 - \varepsilon)x\big] > u\big[y, \varepsilon y + (1 - \varepsilon)x\big], \tag{2·3}$$

where $\varDelta = \big\{x \in \boldsymbol{R}_+^k : \sum_{i \in K} x_i = 1\big\}$, $K = \{1, 2, \cdots, k\}$.

PROPOSITION 2.6 $x \in \varDelta$ is an evolutionary stable strategy if and only if it meets these first-order and second-order best-repy:

$$u(y, x) \leq u(x, x), \qquad \forall y, \tag{2·4}$$

$$u(y, x) = u(x, x) \Rightarrow u(y, y) < u(x, y), \qquad \forall y \neq x. \tag{2·5}$$

PROOF For a proof, see Weibull.[1]

We characterize the evolutionary stable strategy with the order parameter m.

PROPOSITION 2.7 $x \in \varDelta$ is an evolutionary stable strategy in an evolutionary game with statistical mechanics, if there exists some m such that the inequality (2·7) holds for all m^*.

$$u(y, x) \leq u(x, x), \qquad \forall y, \qquad (\textit{Equilibrium Condition}) \tag{2·6}$$

$$|m - m^*| < \varepsilon. \qquad (\textit{Stability Condition}) \tag{2·7}$$

where m^* is the index of the equilibrium action.
PROOF Obvious.

PROPOSITION 2.6 implies that $x \in \varDelta$ is an evolutionary stable strategy, if and only if it meets Nash equilibrium and asymptotic stability conditions. On the other hand, PROPOSITION 2.7 implies that the *Lyapunov stable* condition is replaced by the stability condition in PROPOSITION 2.6.

Let this model add an order parameter; we can analyze an asymmetric two-person game in the same way. In conclusion, we formulate the simplest symmetric and asymmetric two-person games with statistical mechanics in evolutionary game theory.

2.2. *Spatial Pattern : Percolation*

We examine the relations, the order parameter, and the action's probability distribution on the lattice with *percolation.**)

First, we introduce some definitions and notation. For $S \in \varOmega$, let $S_i^{-1}(+1) = \{x \in \boldsymbol{Z}^2 \mid S_i = +1\}$. $S_i^{-1}(-1)$ is defined in the same way. $C_z^+(S)$ denotes the *connected component* of $S_i^{-1}(+1)$ containing the point z.**) $C_z^-(S)$ is defined in the same way.

If $S_i(z) = +1$,

$$C_z^+(S_i) = \Big\{x \in \boldsymbol{Z}^2 \;\Big|\; \textit{there exists the points } \{x_i\}_{i=1}^N \subset S_i^{-1}(+1), \textit{ such that}$$

$$\big|x_i - x_{i-1}\big| = 1, 1 \le i \le N+1, \textit{where } x_0 = z, x_{n+1} = x\Big\}. \tag{2·8}$$

If $S_i(z) = -1$, $C_z^+(S_i) = \emptyset$.

If z is the origin, then we deal with $C_0^+(S_i)$. For $W \in \boldsymbol{Z}^2$, $|W|$ is the cardinality of W, or the number of vertices of a graph W. We analyze the behavior of $\big\{S_i \,\big|\, \big|C_0^+(S_i)\big| = \infty\big\}$ on the pair (γ, h). The parameter h represents an effect of externality. In this section, we mainly deal with $h = 0$.

*) Percolation is known in the simplest models as phase transition. We define a typical percolation problem.

[**d-dimensional Percolation**] Let $Z^d (d \ge 2)$ be the plane cube lattice and p be a number satisfying $0 \le p \le 1$. We examine each edge of Z^d, and consider it to be *open* with probability p and *closed* otherwise, independent of all other edges. The edges of Z^d represent the inner passageways of the stone, and the parameter p is the proportion of passages that are broad enough to allow water to pass along them. Suppose we immerse a large porous stone in a bucket of water. What is the probability that the center of the stone is wetted ?

**) Here, we define connected and related matter.

DEFINITION A subset $A \subset B^2$ is called *connected* if and only if for every $x, y \in \bar{A}$, there exists a sequence $\{b_1, b_2, \cdots, b_n\} \in A$, such that

(a) $x \in b_1$ and $y \in b_n$.
(b) For every $1 \le i \le n-1$, there exists a point $x_i \in Z^2$, such that $b_i \cap b_{i+1} = x_i$.

DEFINITION For $A \subset B^2$, $C \subset A$ is called A's *connected component* if and only if
(a) C is connected,
(b) for every $b \in A \setminus C$, $C \cup \{b\}$ is not connected.

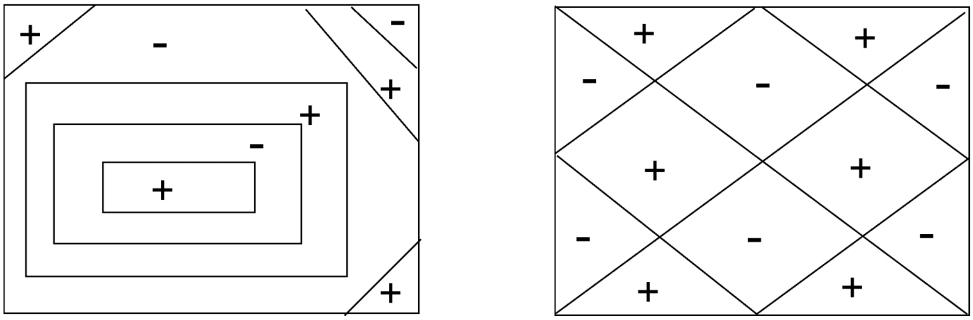

Fig. 1. (LEFT) Concentric Circle Pattern, (RIGHT) Chess Pattern.

Coniglio et al.[9] proves the fundamental relationship between percolation and phase transition.

THEOREM 2.8 (Coniglio et al.[9]) In the two-dimensional Ising model, we obtain
(i) if $\gamma > \gamma_c$, $\quad \mu^+_{\gamma,0}\big(\{|C^+_0| = \infty\}\big) > 0, \quad \mu^-_{\gamma,0}\big(\{|C^-_0| = \infty\}\big) > 0$,
where μ^s, $s = \{+, -\}$ is Gibbs measures.
(ii) if μ is external to the set of all Gibbs states $\mathcal{G}(\gamma, h)$,

$$\mu\big(|C^+_0| = \infty\big)\mu\big(|C^-_0| = \infty\big) = 0.$$

REMARK 2.9 If μ is external to the set of all Gibbs states $\mathcal{G}(\gamma, h)$, then $\mu\Big(\bigcup_{x \in Z^2} \big\{|C^+x(\omega)| = \infty\big\}\Big) = 0$ or 1.[10] If this value is 1, then there exists a.e., an infinite cluster of the corresponding sign and no infinite clusters of the opposite sign — this is called percolation.

The above theorem implies that for $\gamma > \gamma_c, h = 0$, there exists a.e., an infinite cluster of the corresponding sign and no infinite clusters of the opposite sign((i)). For $0 < \gamma < \gamma_c, h = 0$, there exists an infinite cluster for neither actions ((ii)).

For $0 < \gamma < \gamma_c$ and $h = 0$ (i.e., an infinite cluster exists for neither action), what kind of pattern do the actions' distribution on the lattice make ? We know two typical patterns : the concentric circle and chess patterns. The former is a cluster of $+$ actions surrounded by a bigger cluster of $-$ actions, which is surrounded by a bigger cluster of $+$ actions ,$\cdots$. The latter is a cluster of $+$ actions and $-$ actions placed alternately (Fig. 1). We definite the connectivity to characterize these patterns.

DEFINITION 2.10 A subset $A \subset \boldsymbol{Z}^2$ is called $(*)$ *connected* if and only if for every $x, y \in A$, there exists a sequence of points $\{x_1, x_2, \cdots, x_n\} \subset A$ such that $x_0 = x, x_{n+1} = y$ and for every $1 \leq i \leq n+1$,

$$\big\|x_i - x_{i+1}\big\| = 1,$$

where $x = (x^1, x^2) \in \boldsymbol{Z}^2$, $\ \big\|x\big\| = \max\big\{|x^1|, |x^2|\big\}$.

Using the above definition, we can find that the concentric circle pattern has finite $(*)$ connections and the chess pattern has infinite $(*)$ connections for each action. The latter is called the *coexistence of infinite* $(*)$*-clusters*.

THEOREM 2.11 (Higuchi[11]) For every sufficiently small $\gamma > 0$, there exists h such that $\gamma' h' < \frac{1}{2}\log\frac{p_c}{1-p_c} - 4\gamma'$, $\gamma h > \frac{1}{2}\log\frac{1-p_c}{p_c} + 4\gamma$, implying the coexistence of infinite $(*)$-clusters with respect to the Gibbs state for $\mu_{\gamma,h}$.
PROOF For details, Higuchi.[11]

To conclude this section, the condition of the existence of infinite clusters was computed. If infinite clusters do not exist, then we know the kind of patterns the distribution of actions makes on the lattice. These patterns are either a concentric circle or a chess pattern. If γ is sufficiently small and meets certain conditions, then infinite $(*)$-clusters coexist in a chess pattern.

§3. Random matching interaction (Sherrington-Kirkpatrick model)

In §2, we discussed a nearest-neighbor model based on the Ising model. In this section, the players are assumed to search at random for trading opportunities and when they meet the terms of game are started. This randomly matched model was formulated by Sherrington-Kirkpatrick.[12]

Each player's payoff from the outcome is as follows:

$$H\big(\{J_{ij}\}\big) = \sum_{i\neq j} J_{ij}S_iS_j, \text{ where } P(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}}\exp\Big\{-\frac{(J_{ij}-J_0)^2}{2J^2}\Big\}, \tag{3·1}$$

where i, j are players, and $S_k = \{-1, 1\}$, $k = i, j$, $P(J_{ij})$ are Gaussian random variables with a mean of J_0 and a variance of J^2.

3.1. *Annealed system*

We analyze two models, an *annealed system* and a *quenched system* in spin-glass physics. First, we analyze the annealed system, where J_{ij} is chosen randomly, but then each player moves to obtain a better payoff. Second, we analyze the quenched system, where J_{ij} is chosen randomly, but then is fixed.

A particular spin-glass will have a social welfare function[*)] and the partition function is defined by

$$F = \gamma \log\big\langle Z\big\rangle, \tag{3·2}$$

$$\begin{aligned}\big\langle Z\big\rangle &= \sum_{\{S_i\}}\int_{-\infty}^{\infty}\prod_{(ij)} dJ_{ij}P\{J_{ij}\}\exp\big(\gamma H\{J_{ij}\}\big)\\ &= \sum_{\{S_i\}}\exp\Big[\sum_{(ij)}\Big\{\gamma J_0S_iS_j + \frac{(\gamma J)^2}{2}(S_iS_j)^2\Big\}\Big]. \end{aligned}\tag{3·3}$$

*) A social welfare function is a mapping from allocations of goods or rights among people to the real numbers.

We obtain the following proposition.

PROPOSITION 3.1 In the annealed system, the order parameter is the points that maximize the social welfare function in the model. If there are infinite players on this lattice, then the order parameter is 0.
PROOF We maximize the social welfare for the order parameter m.

$$\frac{\partial F}{\partial m} = 2\gamma^2 J_0 n^2 m + 2\gamma^3 J^2 n^4 m^3 = 0, \quad m = 0 \quad \text{or} \quad \pm\sqrt{\frac{-J_0}{\gamma J^2 n^2}}. \tag{3·4}$$

We can understand $J_0 < 0$, because m is a real number. The limit of optimal order parameter m is 0, as n approaches to ∞.

This implies that the optimal order parameter is a point, like a replicator system.

3.2. *Quenched system*

We analyze the quenched system, where J_{ij} is chosen randomly, but then is fixed. Diederich and Opper[4)] analyzed such a quenched system.

In a quenched system, the social welfare function is given by

$$F = \gamma\big\langle \log Z \big\rangle. \tag{3·5}$$

The partition function is the same as (3·3). We obtain the next proposition.

PROPOSITION 3.2 In a quenched system, the order parameter maximizes the social welfare of the model.

$$m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}z^2\Big) \tanh\big(\gamma\tilde{J}\sqrt{q}z + \gamma\tilde{J}_0 n\big)dz. \tag{3·6}$$

PROOF We omit the detailed proof. The above equation computes the maximization of the social welfare for order parameter m by employing standard methods.

3.3. *Extension : TAP equation*

To this subsection, we compute the optimal order parameter in general case for J_{ij}. Here, if we take a example for $\{J_{ij}\}$, we analyze it. In detail, we find that the order parameter's equation (TAP equation[12)]) has the condition of a phase transition, using the property of the eigenvalues of the matrix. We compute the *Frobenius root* and the boundary condition between stability and instability from the *Perron-Frobenius theorem*. The player's payoff from the outcome varies randomly because the players are randomly matched and play a game. These situations can be expressed using the *random matrix theory*. This theory has several laws, because the elements of this matrix are varied randomly. Moreover, if we assume that $J_{ij} = J_{ji}$ for the elements of the random matrix, then this elements can be transformed into a Hermite matrix, since the payoff matrix is invariant under positive affine transformations of payoffs. As a result, we can compute the Frobenius root from *Wigner's semi-circle* law, and this condition from the Perron-Frobenius theorem.

Let a model add another parameter h_j (an effect of externality). We consider that the payoff is affected by around games. In this case, the payoff is defined as

$$H\big(\{J_{ij}\}\big) = \sum_{i\neq j} J_{ij} S_i S_j + \sum_j h_j S_j. \tag{3·7}$$

We obtain the following propositions for annealed and quenched systems.

PROPOSITION 3.3 In an annealed system with externality, no phase transition occurs.
PROOF We compute the social welfare in the same way. We obtain

$$h_j = 2\gamma m(1-N)(J_0 + J^2 m^2). \tag{3·8}$$

This implies that no phase transition occurs.

This proposition implies that no phase transition occurs because each player in an annealed system moves to obtain a better payoff.

Second, we analyze the variation in the order parameter in a quenched system. In this case, we obtain the following proposition.

PROPOSITION 3.4 In a quenched system with externality, there exist discontinuous variations in the order parameter. Bifurcations occur, hence, this system has multiple equilibria.
PROOF First, we compute the order parameter in the same manner, as mentioned earlier. The *Weiss approximation* is given by

$$m_i = \tanh\Big\langle \gamma\Big(h_i + \sum_j J_{ij} m_j\Big)\Big\rangle,$$

using the approximation $\langle f[s]\rangle \approx f\big[\langle s\rangle\big]$, i.e., by approximating the expected value of a function of s with the function of the expected values. This approximation neglects fluctuations.

If we expand this equation for $J_0 = 0$,

$$m_i = \gamma \sum_j J_{ij} m_i - \gamma \sum_j J_{ij}^2 m_i + \gamma h_i + \cdots .$$

We expand $N\times N$ J_{ij} matrices using the eigenvector. Let the eigenvector $\big\{\langle i|\lambda\rangle\big\}$ be a completely normalized orthogonal system and J_λ be the eigenvalue, $\sum_j J_{ij}\langle i|\lambda\rangle = J_\lambda \langle i|\lambda\rangle$. Let $m_\lambda = \sum_i m_i \langle i|\lambda\rangle$, i.e., the projection of the magnetization vector onto eigenvector $|\lambda\rangle$ of matrix J, with the corresponding eigenvalue J_λ and $h_\lambda = \sum_i h_i \langle i|\lambda\rangle$ in the same way. Thus let it add λ mode to parameters J_{ij}, m, h, then the order parameter is given by

$$m_\lambda = \frac{1}{T - J_\lambda} h_\lambda, \quad \text{where } T = \frac{1}{\gamma}.$$

On the other hand, according to the random matrix theory, the maximal eigenvalue of J_Λ is $2J$, the minimal eigenvalue is $-2J$, and the semi circle law is realized, i.e.,

$$\rho(J_\lambda) = \frac{2}{\pi J_\Lambda^2}\left(J z_\Lambda - J_\lambda^2\right)^{1/2}.$$

This implies that the critical point T_C is $2J_\lambda$. There exist discontinuous variations for the order parameter. Bifurcations occur, hence, this system has multiple equilibria.

§4. Implications

4.1. *Review: Cont and Bouchaud*[7)]

We consider the simplified Cont and Bouchaud[7)] model (hereafter, C-B model). The C-B model uses the percolation approach of multi-agent simulations of stock market fluctuations. Our model in §2 dealing with the problem about percolation is similar to the C-B model.

We consider a stock market with n agents, labeled by an integer $1 \leq i \leq n$, trading in a single asset whose price at time t is denoted as $x(t)$. During each time period, an agent may choose either to buy the stock, to sell it, or not to trade. The demand for the stock of agent i is represented by a random variable ϕ_i, which can take the values $0, -1, +1$. A positive value of ϕ_i represents a "bull", an agent willing to buy stock; a negative value represents a "bear", an agent eager to sell stock; and $\phi_i = 0$ means that agent i does not trade during the given period. The marginal distribution of agent i's individual demands is assumed to be symmetric,

$$P(\phi_i = +1) = P(\phi_i = -1) = a, \qquad P(\phi_i = 0) = 1 - 2a$$

such that the average aggregate excess demand is zero; i.e., the market is considered to fluctuate around its equilibrium. A value of $a(< \frac{1}{2})$ allows for a finite fraction of agents not to trade during a given period. This is called the *activity parameter.*

We assume that between price changes and excess demand: $\Delta x(t) = x(t+1) - x(t) = \dfrac{1}{\lambda}\sum_{i=1}^{n}\phi_i(t)$, where λ is the market depth. Excess demand is needed to move the price by one unit — it measures the sensitivity of the price to fluctuations in excess demand. The C-B model is known that for decrease in the activity parameter a showing its similarity with real stock market phenomena : the *heavy tails* observed in the distribution of stock market.

We compare the C-B model with the model in §2. Our model based on microeconomic representation of the market phenomenon, i.e., the players' actions depend on the payoff. If parameter γ, which represents optimal choice behavior, is sufficiently small, then the C-B model coincides with our model.

We find the distribution of action is a chess pattern under certain conditions, otherwise, it is a concentric circle pattern. If γ is sufficiently large, then one action is occupied mostly, i.e., the price is higher or lower than before.

If we consider the C-B model through our model, then we can understand the detailed structure, microfoundations, etc. Moreover, we understand the case of random matched players, presented in the next subsection.

4.2. *Random matching*

If each player is randomly matched in the C-B model, then we can understand these things caused by extension of our model in §3.

If an annealed system has no institutional constraints, each player is freely matched and the order parameter is independent of the size of γ. We can find which action is occupied, i.e., the price is higher or lower than before. However, if the number of players is infinite, then the order parameter is zero — the price remains the same as before. If a quenched system has institutional constraints, then the order parameter is similar to the Ising model.

In an annealed system with externality, the order parameter behaves in the same way. However, in a quenched system with the externality, the order parameter is different, because the number of equilibria is increased. The rate of price change is dependent on the size of γ.

§5. Concluding remarks

In this paper, a statistical framework is presented for modelling nearest-neighbor and random interactions in evolutionary game theory. This framework is different from classical evolutionary game theory. The limit behavior, as γ approaches infinity, is closely connected to the modelling of game theory with rational players. When $\gamma = 0$, behavior is essentially random, as all strategies are played with equal probability. We compute the optimal order parameter for each system. In a quenched system with externality, there are multiple equilibria. This paper discusses the simplified C-B model through our models. We extend the player's behavior and matching in C-B model.

This framework can be extended in various ways because of the simplicity of the models. For example, we will analyze the framework in the case the action number is more than three or infinity. We will let the important parameter γ be endogenous ; this is known as *superstatistics*. We will analyze the connection between our model and the real stock markets.

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